Disjoint Hamilton cycles in transposition graphs

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Abstract

Most network topologies that have been studied have been subgraphs of transposition graphs. Edge-disjoint Hamilton cycles are important in network topologies for improving fault-tolerance and distribution of messaging traffic over the network. Not much was known about edge-disjoint Hamilton cycles in general transposition graphs until recently Hung produced a construction of 4 edge-disjoint Hamilton cycles in the 5-dimensional transposition graph and showed how edge-disjoint Hamilton cycles in $(n + 1)$-dimensional transposition graphs can be constructed inductively from edge-disjoint Hamilton cycles in n-dimensional transposition graphs. In the same work it was conjectured that n-dimensional transposition graphs have $n - 1$ edge-disjoint Hamilton cycles for all n greater than or equal to 5. In this paper we provide an edge-labelling for transposition graphs and, by considering known Hamilton cycles in labelled star subgraphs of transposition graphs, are able to provide an extra edge-disjoint Hamilton cycle at the inductive step from dimension n to $n + 1$, and thereby prove the conjecture.

Keywords: transposition graphs; star graphs; edge-disjoint Hamilton cycles; automorphisms.
AMS Subject Classification: 05C38; 05C45; 68R10.

1. Introduction

Given a finite group G and a set S of elements of G, such that S contains inverses of its own elements and does not contain the identity element of G, the Cayley graph (G, S) has vertex set $V((G, S)) = G$ and edges $\{g_1, g_2\} \in E((G, S))$ if and only if there is some $s \in S$ such that $g_2 = g_1 s$. In this paper, we choose G to be the symmetric group of permutations of $\{1, \dots, n\}$, and sets S to be sets of transpositions. The transpositions in S can be depicted by a graph with vertex set $\{1, \dots, n\}$ where there is an edge between i and j $(1 \leq i, j \leq n)$ if and only if the transposition (i, j) belongs to S. This latter graph is sometimes called the 'transposition generating' graph. We consider, in particular, the cases of complete transposition graphs where S is the set of all transpositions, and star graphs where the transposition generating graph is the 'star' of transpositions $\{(1, 2), (1, 3), \dots, (1, n)\}$.

Numerous properties of transposition graphs have been studied by several authors. The most important properties for interconnection networks are low degree and high connectivity, which star graphs have as well as symmetry and low diameter [1]. The bisection width, i.e. the size of the smallest edge-cut of a graph that divides it into two equal parts, has been determined for complete transposition graphs [9], [12]. In [7] the author considers routings of graphs G, i.e. sets R of $n(n-1)$ paths between all pairs of vertices where G is of order n, for which all edges of G belong to almost the same number of paths in R, and calculates the number of paths passing through edges in routings for star graphs and complete transposition graphs. An algorithm for finding a collection of vertex-disjoint paths connecting a given source vertex s and a given set of destination vertices D in complete transposition graphs is given in [5]. In [3] an orientation for transposition graphs where the transposition generating graphs are trees, is given and shown to have good connectivity properties. The orientation produces maximum connectivity and low diameter in the resulting directed graph, something that is difficult to achieve in general graphs. Ganesan studies the automorphism group of complete transposition graphs and proves non-normality of complete transposition graphs [6].

It has been known for some time that complete transposition graphs of all dimensions are hamiltonian [2], but not much was known about how multiple edge-disjoint Hamilton cycles might be constructed in any dimension until Hung [8] produced 4 edge-disjoint Hamilton cycles in TN_5 and gave an inductive method of constructing 4 edge-disjoint Hamilton cycles in TN_{n+1} from 4 edge-disjoint Hamilton cycles in TN_n for n greater than or equal to 5. In the same paper Hung conjectured that the n-dimensional complete transposition graph has $n-1$ edge-disjoint Hamilton cycles. In this paper we prove the conjecture.

This paper is structured as follows. We give basic notations, terminology and results for general undirected and transposition graphs in Section 2. In Section 3 we define a labelling for the edges of TN_n and consider its n-dimensional spanning subgraph St_n. We give properties of automorphisms of TN_n and St_n. In Section 4 we show that $n-1$ edge-disjoint Hamilton cycles can be constructed in TN_{n+1} from $n-1$ edge-disjoint Hamilton cycles in TN_n. This provides the basis of an inductive proof that TN_n has $n-1$ edge-disjoint Hamilton cycles if an extra edge-disjoint Hamilton cycle can be found at the inductive step. We give a proof in which the inductive step from $n = 5$ to $n = 6$ differs from the inductive step from $n = k$ to $n = k + 1$ when $k > 5$. The former is given in Section 5 and the latter, which then proves the conjecture, in Section 6. We draw conclusions in Section 7. Henceforth, when we refer to transposition graphs we shall mean complete transposition graphs.

2. Graphs and transposition graphs

A *graph* G is a pair $G = (V, E)$ where V is a set of vertices and E is a set of edges each of which is an unordered pair $\{u, v\}$ of distinct vertices $u, v \in V$. A *subgraph* F of G is a graph whose vertices are a subset of V, and whose edges are a subset E. If F is a subgraph of G, $V(F)$ and $E(F)$ will denote the set of vertices and edges respectively. A subgraph F of G is a *spanning* subgraph if $V(F) = V(G)$. Two distinct points $u, v \in V(F)$ are *adjacent* in F if $\{u, v\} \in E(F)$. A *path* P in F denoted $v_1 \to \ldots \to v_l$ where $l \geq 2$, is a sequence $(v_1, \ldots, v_l)$ of distinct vertices of F such that $\{v_i, v_{i+1}\} \in E(F)$ for $1 \leq i \leq l-1$. The first and last vertices of P, v_1 and v_l, are denoted *start*(P) and *end*(P) respectively. The path $v_l \to \ldots \to v_1$ is called the *reverse* path of P and is denoted by *rev*(P). A path P in F is a *cycle* if $|V(P)| \geq 3$ and *start*(P) and *end*(P) are adjacent in F. The set of vertices $V(P)$ and edges $E(P)$ of a path P are:

$$V(P) = \{v_1, \ldots, v_l\}, \quad E(P) = \{\{v_1, v_2\}, \ldots, \{v_{l-1}, v_l\}\}$$

A cycle is a *Hamilton* cycle in F if $V(P) = V(F)$. Two cycles P_1 and P_2 are *edge-disjoint* if $(E(P_1) \cup \{start(P_1), end(P_1)\}) \cap (E(P_2) \cup \{start(P_2), end(P_2)\}) = \emptyset$. If paths P_1 and P_2 are also *vertex-disjoint*, i.e. $V(P_1) \cap V(P_2) = \emptyset$ and *end*(P_1) is adjacent to *start*(P_2), then $P_1 \Rightarrow P_2$ denotes the concatenated path of P_1 followed by P_2. An *automorphism* Γ of F is a bijection $\Gamma : V(F) \to V(F)$ such that $\{u, v\} \in E(F)$ if and only if $\{\Gamma(u), \Gamma(v)\} \in E(F)$.

Let $p = p_1 \ldots p_n$ be a permutation of the set $\{1, \ldots, n\}$. A *transposition* $\phi_{i,j}$ $(1 \leq i, j \leq n)$ is a function that interchanges the digits in the i-th and j-th position of a permutation, so that if $p = p_1 \ldots p_n$ and $i < j$ then

$$\phi_{i,j}(p) = p_1 \ldots p_{i-1} p_j p_{i+1} \ldots p_{j-1} p_i p_{j+1} \ldots p_n$$

The *n-dimensional transposition graph* TN_n is the graph with set of vertices $V(TN_n)$ equal to all the permutations of $\{1, \ldots, n\}$ and set of edges

$$E(TN_n) = \{\{p, \phi_{i,j}(p)\} \mid p \in V(TN_n) \text{ and } 1 \leq i < j \leq n\}$$

Given a vertex $p = p_1 \ldots p_n \in TN_k$, $\psi_i^{k+1}(p)$ will denote the vertex in TN_{k+1} formed by inserting digit $k+1$ in the i-th position $(1 \leq i \leq k+1)$, i.e.

$$\psi_i^{k+1}(p_1 \ldots p_n) = p_1 \ldots p_{i-1}(k+1)p_i \ldots p_k$$

Below are some basic properties of ψ_i^{k+1}.

Lemma 2.1. *Let $k \geq 3$ and $1 \leq i \leq k+1$. Then, the following hold:*

(i) if $\{p, q\} \in E(TN_k)$, then $\{\psi_i^{k+1}(p), \psi_i^{k+1}(q)\} \in E(TN_{k+1})$,

(ii) $p^1 \to \ldots \to p^l$ is a path (cycle) in TN_k, if and only if $\psi_i^{k+1}(p^1) \to \ldots \to \psi_i^{k+1}(p^l)$ is a path (cycle) in TN_{k+1},

(iii) if paths (cycles) $p^1 \to \ldots p^l$ and $q^1 \to \ldots \to q^l$ have disjoint sets of vertices (edges), then $\psi_i^{k+1}(p^1) \to \ldots \to \psi_i^{k+1}(p^l)$ and $\psi_i^{k+1}(q^1) \to \ldots \to \psi_i^{k+1}(q^l)$ also have disjoint sets of vertices (edges).

3. Labelled transposition graphs, star graphs and automorphisms

We define a labelling for the edges of TN_n to correspond to the distance between the two digits that are interchanged by the edge, on the cyclic graph $1 \to 2 \to \ldots \to n \to 1$.

Definition 3.1. *The n-dimensional labelled transposition graph TN_n has edge labels defined by $L : E(TN_n) \to \{1, \ldots, \lfloor n/2 \rfloor\}$ where*

$$L(\{p, \phi_{i,j}(p)\}) \;=\; min\{|p_i - p_j|, n - |p_i - p_j|\}$$

The set of edges with label l will be denoted by $E^l(TN_n)$.

The n-dimensional transposition graph TN_n has a spanning subgraph which is a star graph [1], i.e. corresponding to the edges where the digit in the first position of a permutation is interchanged with the digit in some other position. For technical convenience, we consider instead the isomorphic star spanning subgraph St_n comprising edges where the digit in the n-th position of a permutation is interchanged with the digit in some other position.

Definition 3.2. *The n-dimensional star subgraph St_n of TN_n has vertex set $V(St_n) = V(TN_n)$ and edge set*

$$E(St_n) = \{\{p, \phi_{i,n}(p)\} \mid p \in V(St_n) \text{ and } 1 \leq i \leq n-1\}$$

It is known that every n-dimensional star graph has a Hamilton cycle whose edges are labelled by either 1 or 2 according to the labelling above.

Lemma 3.3. *([11]) For all $n \geq 5$, there exists a Hamilton cycle H_n^{12} of St_n (which is therefore a Hamilton cycle of TN_n) such that H_n^{12} has $n! - (n-2)!$ edges with label 1 and $(n-2)!$ edges with label 2.*

Given a Hamilton cycle H in TN_n, one way of obtaining another Hamilton cycle is as the image of H under an automorphism of TN_n. We shall consider the following automorphisms.

Lemma 3.4. *Let $\gamma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijection. The following hold:*

(i) $\Gamma : V(TN_n) \to V(TN_n)$ given by $\Gamma(p_1 \ldots p_n) = \gamma(p_1) \ldots \gamma(p_n)$, for all vertices $p = p_1 \ldots p_n \in V(TN_n)$, is an automorphism of the graph TN_n,

(ii) for all edges $\{p, p'\} \in E(TN_n)$, $e = \{p, p'\} \in E(St_n)$ if and only if $\Gamma(e) = \{\Gamma(p), \Gamma(p')\} \in E(St_n)$, and so Γ is also an automorphism of St_n,

(iii) if $H = (p^1, \ldots, p^{n!})$ is a Hamilton cycle in TN_n (St_n), then $\Gamma(H) = (\Gamma(p^1), \ldots, \Gamma(p^{n!}))$ is a Hamilton cycle in TN_n (St_n).

Proof Routine. ■

Lemma 3.5. *Let $inv : V(TN_n) \to V(TN_n)$ be given by $inv(p) = p^{-1}$, for all $p \in V(TN_n)$, where p^{-1} is the inverse of the permutation p, i.e. if $p = p_1 \ldots p_n$ then $p^{-1} = p_1^{-1} \ldots p_n^{-1}$ where $p_i^{-1} = j$ iff $p_j = i$ ($1 \leq i, j \leq n$). We have that:*

(i) inv is an automorphism of the graph TN_n,

(ii) if $H = (p^1, \ldots, p^{n!})$ is a Hamilton cycle in TN_n, then $inv(H) = (inv(p^1), \ldots, inv(p^{n!}))$ is a Hamilton cycle in TN_n.

Proof Routine. ■

4. Inductive construction of disjoint Hamilton cycles

In this section we show how $k-1$ edge-disjoint Hamilton cycles Q_j $(1 \leq j \leq k-1)$ in TN_{k+1} can be constructed from $k-1$ corresponding edge-disjoint Hamilton cycles P_j in TN_k. First of all, a $k! \times (k+1)$ matrix M_j (see Definition 4.4 below) of all the vertices of TN_{k+1} is formed from P_j as follows: insert digit $k+1$ at the first position in all vertices of P_j to produce the first row, insert digit $k+1$ at the second position in all vertices of P_j to produce the second row, and so on until the $(k+1)$-th row results from inserting digit $k+1$ at the $(k+1)$-th position in all vertices of P_j. Hamilton cycle Q_j is then constructed by visiting all vertices of TN_{k+1} by starting at the top row and travelling vertically from row to row down a 'shaft' S_j, which is a 3-columned submatrix of M_j, until the bottom row is reached, and then returning back up the shaft S_j row-by-row whilst horizontally cycling through all the vertices at each row, until the top row and starting vertex is reached. To ensure that the different Q_j are edge-disjoint, the corresponding shafts S_j are chosen to be pairwise disjoint (Lemma 4.5). The overall construction of the Q_j is given in the main theorem Theorem 4.6.

The Hamilton cycles Q_j $(1 \leq j \leq k-1)$ are further transformed in Theorem 4.6 by the automorphism *inv* to produce Hamilton cycles that only intersect edges of St_{k+1} at edges labelled 1. This property is used in Sections 5 and 6 to construct a k-th edge-disjoint Hamilton cycle in TN_{k+1}. The first lemma below considers the labels of edges that appear in Hamilton cycles in Theorem 4.6.

Lemma 4.1. *Let $k \geq 3$. The following hold:*

(i) if $\{p, q\} \in E(TN_k)$ and $i \in \{1, \ldots, k+1\}$, then

$$\{inv(\psi_i^{k+1}(p)), inv(\psi_i^{k+1}(q))\} \in E(TN_{k+1}) \backslash E(St_{k+1}),$$

(ii) if $p \in V(TN_k)$ and $i \in \{1, \ldots, k\}$, then

$$\{inv(\psi_i^{k+1}(p)), inv(\psi_{i+1}^{k+1}(p))\} \in E^1(St_{k+1}).$$

Proof In case (i), by the definition of ψ_i^{k+1}, $\psi_i^{k+1}(p)$ and $\psi_i^{k+1}(q)$, both have digit $k+1$ in the i-th position. Thus, $inv(\psi_i^{k+1}(p))$ and $inv(\psi_i^{k+1}(q))$ both have digit i in the $(k+1)$-th position. Hence, $\{inv(\psi_i^{k+1}(p)), inv(\psi_i^{k+1}(q))\}$ cannot belong to $E(St_{k+1})$, by Definition 3.2, but does belong to $E(TN_{k+1})$ by Lemma 2.1(i).

For (ii), clearly $\{\psi_i^{k+1}(p), \psi_{i+1}^{k+1}(p)\} \in E(TN_{k+1})$. As *inv* is an automorphism of TN_{k+1} (by Lemma 3.5(i)), $\{inv(\psi_i^{k+1}(p)), inv(\psi_{i+1}^{k+1}(p))\} \in E(TN_{k+1})$. Put $p = p_1 \ldots p_k$. Then, $\psi_i^{k+1}(p)$ and $\psi_{i+1}^{k+1}(p)$ insert digit $k+1$ at the i-th and $(i+1)$-th positions respectively:

$$\psi_i^{k+1}(p) = p_1 \ldots p_{i-1}(k+1)p_i \ldots p_k, \quad \psi_{i+1}^{k+1}(p) = p_1 \ldots p_{i-1}p_i(k+1) \ldots p_k$$

Therefore, $inv(\psi_i^{k+1}(p))$ will have digit i at the $(k+1)$-th position and digit $i+1$ at the p_i-th position, and $inv(\psi_{i+1}^{k+1}(p))$ will have digit $i+1$ at the $(k+1)$-th position and digit i at the p_i-th position. Thus, by Definitions 3.2 and 3.1, $\{inv(\psi_i^{k+1}(p)), inv(\psi_{i+1}^{k+1}(p))\} \in E^1(St_{k+1})$. ∎

Before we define the matrices M_j $(1 \leq j \leq k-1)$ of all vertices of TN_{k+1} and associated shafts S_j, corresponding to Hamilton cycles P_j in TN_k, we find pairwise disjoint 'sections' of 3 consecutive vertices of different P_j from which pairwise disjoint shafts S_j will be constructed.

Definition 4.2. *If P is a cycle in TN_k ($k \geq 3$), then a section of P is a subpath of P comprising 3 consecutive vertices.*

Lemma 4.3. *Let $k \geq 5$ and suppose that*

$$\begin{array}{lllllll} P_1 & = & p^{1,1} & \rightarrow & \ldots & \rightarrow & p^{1,k!} \\ \ldots & & \ldots & & & & \\ P_{k-1} & = & p^{k-1,1} & \rightarrow & \ldots & \rightarrow & p^{k-1,k!} \end{array}$$

are k-1 Hamilton cycles in TN_k. Then, for all $j \in \{1, \ldots, k-1\}$, we can choose a section

$$p^{j,s_j} \rightarrow p^{j,s_j+1} \rightarrow p^{j,s_j+2} \quad (s_j \in \{1, \ldots, k!-2\})$$

of P_j, such that the chosen sections are pairwise disjoint, i.e. for distinct $j, j' \in \{1, \ldots, k-1\}$

$$\{p^{j,s_j}, p^{j,s_j+1}, p^{j,s_j+2}\} \cap \{p^{j',s_{j'}}, p^{j',s_{j'}+1}, p^{j',s_{j'}+2}\} = \emptyset \tag{1}$$

Proof We generate $s_1, \ldots, s_{k-1}$ satisfying (1) inductively. Suppose that $s_1, \ldots, s_h$ have been generated, for some $h \in \{1, \ldots, k-2\}$, such that (1) is satisfied for all distinct $j, j' \in \{1, \ldots, h\}$. We find a s_{h+1} such that (1) is satisfied for all distinct $j, j' \in \{1, \ldots, h+1\}$. Firstly, partition P_{h+1} into $k!/3$ pairwise disjoint sections thus:

$$\{p^{h+1,1}, p^{h+1,2}, p^{h+1,3}\} \cup \ldots \cup \{p^{h+1,k!-2}, p^{h+1,k!-1}, p^{h+1,k!}\} \tag{2}$$

Consider the union of the sections that have been generated inductively so far:

$$\{p^{1,s_1}, p^{1,s_1+1}, p^{1,s_1+2}\} \cup \ldots \cup \{p^{h,s_h}, p^{h,s_h+1}, p^{h,s_h+2}\} \tag{3}$$

Clearly, (3) has $3h$ vertices in total. As there are $k!/3$ disjoint sets in (2) and as $3h < 3(k-2) < k!/3$ for all $k \geq 5$, there must be a set in (2)

$$\{p^{h+1,s_{h+1}}, p^{h+1,s_{h+1}+1}, p^{h+1,s_{h+1}+2}\} \quad (\text{where} \quad s_{h+1} \in \{1, \ldots, k!-2\})$$

which does not contain any element in (3). This yields the required s_{h+1} and completes the induction. ∎

Definition 4.4. *Let $k \geq 5$ and let*

$$P_j = p^{j,1} \rightarrow \ldots \rightarrow p^{j,k!}$$

be a Hamilton cycle in TN_k. Then, the matrix M_j of P_j in TN_{k+1} is the $(k+1) \times k!$ matrix of all the vertices of TN_{k+1} obtained from P_j below:

$$\begin{pmatrix} \psi_1^{k+1}(p^{j,1}) & \ldots & \psi_1^{k+1}(p^{j,k!}) \\ & \ldots & \\ \psi_{k+1}^{k+1}(p^{j,1}) & \ldots & \psi_{k+1}^{k+1}(p^{j,k!}) \end{pmatrix}$$

A shaft S_j of M_j is a $(k+1) \times 3$ submatrix of the form:

$$\begin{pmatrix} \psi_1^{k+1}(p^{j,s_j}) & \psi_1^{k+1}(p^{j,s_j+1}) & \psi_1^{k+1}(p^{j,s_j+2}) \\ & \ldots & \\ \psi_{k+1}^{k+1}(p^{j,s_j}) & \psi_{k+1}^{k+1}(p^{j,s_j+1}) & \psi_{k+1}^{k+1}(p^{j,s_j+2}) \end{pmatrix}$$

The set of vertices in S_j will be denoted by $V(S_j)$. Shafts S_j and $S_{j'}$ of matrices M_j and $M_{j'}$ respectively are disjoint if $V(S_j) \cap V(S_{j'}) = \emptyset$.

$$\begin{pmatrix} \psi_1^{k+1}(p^{j,s_j}) & \psi_1^{k+1}(p^{j,s_j+1}) & \psi_1^{k+1}(p^{j,s_j+2}) \\ & \dots & \\ \psi_{k+1}^{k+1}(p^{j,s_j}) & \psi_{k+1}^{k+1}(p^{j,s_j+1}) & \psi_{k+1}^{k+1}(p^{j,s_j+2}) \end{pmatrix} \text{and} \begin{pmatrix} \psi_1^{k+1}(p^{j',s_{j'}}) & \psi_1^{k+1}(p^{j',s_{j'}+1}) & \psi_1^{k+1}(p^{j',s_{j'}+2}) \\ & \dots & \\ \psi_{k+1}^{k+1}(p^{j',s_{j'}}) & \psi_{k+1}^{k+1}(p^{j',s_{j'}+1}) & \psi_{k+1}^{k+1}(p^{j',s_{j'}+2}) \end{pmatrix}$$

Figure 1: Disjoint shafts S_j and $S_{j'}$

It is clear that M_j has all the vertices of TN_{k+1} as the first row adds digit $k+1$ at the first position of all vertices of TN_k, the second row adds digit $k+1$ at the second position, and so on until the $(k+1)$-th row adds digit $k+1$ at the $(k+1)$-th position. The next lemma shows that shafts can be chosen to be pairwise disjoint.

Lemma 4.5. *Let $k \geq 5$ and $P_1, \dots, P_{k-1}$ be Hamilton cycles in TN_k with corresponding matrices $M_1, \dots, M_{k-1}$ respectively in TN_{k+1}. Then, there exist pairwise disjoint shafts $S_1, \dots, S_{k-1}$ of $M_1, \dots, M_{k-1}$ respectively.*

Proof By Lemma 4.3, there exist pairwise disjoint sections of the Hamilton cycles P_j $(1 \leq j \leq k-1)$ in TN_k

$$p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2} \quad (1 \leq j \leq k-1)$$

where each $s_j \in \{1, \dots, k!-2\}$. Then, the two shafts S_j and $S_{j'}$ of P_j and $P_{j'}$, where $j \neq j'$, displayed in Figure 1, are disjoint as two common vertices would need digit $k+1$ in the same i-th position and so would both be on the i-th row of both shafts, but then the intersection of the i-th rows $\{\psi_i^{k+1}(p^{j,s_j}), \psi_i^{k+1}(p^{j,s_j+1}), \psi_i^{k+1}(p^{j,s_j+2})\} \cap \{\psi_i^{k+1}(p^{j',s_{j'}}), \psi_i^{k+1}(p^{j',s_{j'}+1}), \psi_i^{k+1}(p^{j',s_{j'}+2})\} = \emptyset$ as $\{p^{j,s_j}, p^{j,s_j+1}, p^{j,s_j+2}\} \cap \{p^{j',s_{j'}}, p^{j',s_{j'}+1}, p^{j',s_{j'}+2}\} = \emptyset$ ■

The main theorem of this section constructing the $n-1$ edge-disjoint Hamilton cycles Q_j $(1 \leq j \leq k-1)$ in TN_{k+1} from corresponding edge-disjoint Hamilton cycles P_j in TN_k is now given, along with the properties of their images under the automorphism *inv*.

Theorem 4.6. *Let $k \geq 5$ and suppose that TN_k has k-1 edge-disjoint Hamilton cycles $P_1, \dots, P_{k-1}$. Then, TN_{k+1} has k-1 edge-disjoint Hamilton cycles $Q_1, \dots, Q_{k-1}$ such that:*

(i) $inv(Q_1), \dots, inv(Q_{k-1})$ are edge-disjoint Hamilton cycles, and

(ii) all edges of $inv(Q_1), \dots, inv(Q_{k-1})$ that are edges of St_{k+1} have label 1, i.e.

$$(E(inv(Q_1)) \cup \dots \cup E(inv(Q_{k-1}))) \cap E(St_{k+1}) \subseteq E^1(St_{k+1})$$

Proof Given $k-1$ edge-disjoint Hamilton cycles $P_1, \dots, P_{k-1}$ in TN_k, where

$$P_j = p^{j,1} \to \dots \to p^{j,k!} \quad (1 \leq j \leq k-1)$$

let $M_1, \dots, M_{k-1}$ respectively be the corresponding matrices in TN_{k+1} as in Definition 4.4. By Lemma 4.5 , we can choose pairwise disjoint shafts $S_1, \dots, S_{k-1}$ respectively of $M_1, \dots, M_{k-1}$, where S_j $(1 \leq j \leq k-1)$ is the matrix:

$$\begin{pmatrix} \psi_1^{k+1}(p^{j,s_j}) & \psi_1^{k+1}(p^{j,s_j+1}) & \psi_1^{k+1}(p^{j,s_j+2}) \\ & \dots & \\ \psi_{k+1}^{k+1}(p^{j,s_j}) & \psi_{k+1}^{k+1}(p^{j,s_j+1}) & \psi_{k+1}^{k+1}(p^{j,s_j+2}) \end{pmatrix}$$

$$
\begin{array}{ccccccccccccc}
\psi_1^{k+1}(p^{j,1}) & \rightarrow & \dots & \rightarrow & \psi_1^{k+1}(p^{j,s_j}) & \rightarrow & \boxed{\psi_1^{k+1}(p^{j,s_j+1})} & & \psi_1^{k+1}(p^{j,s_j+2}) & \rightarrow & \dots & \rightarrow & \psi_1^{k+1}(p^{j,k!}) \\
& & & & & & \downarrow & & \uparrow & & & & \\
\psi_2^{k+1}(p^{j,1}) & \leftarrow & \dots & \leftarrow & \psi_2^{k+1}(p^{j,s_j}) & & \psi_2^{k+1}(p^{j,s_j+1}) & & \psi_2^{k+1}(p^{j,s_j+2}) & \leftarrow & \dots & \leftarrow & \psi_2^{k+1}(p^{j,k!}) \\
& & & & \uparrow & & \downarrow & & & & & & \\
& & & & & & \dots & & & & & & \\
& & & & \uparrow & & \downarrow & & & & & & \\
\psi_{k-1}^{k+1}(p^{j,1}) & \rightarrow & \dots & \rightarrow & \psi_{k-1}^{k+1}(p^{j,s_j}) & & \psi_{k-1}^{k+1}(p^{j,s_j+1}) & & \psi_{k-1}^{k+1}(p^{j,s_j+2}) & \rightarrow & \dots & \rightarrow & \psi_{k-1}^{k+1}(p^{j,k!}) \\
& & & & & & \downarrow & & \uparrow & & & & \\
\psi_k^{k+1}(p^{j,1}) & \leftarrow & \dots & \leftarrow & \psi_k^{k+1}(p^{j,s_j}) & & \psi_k^{k+1}(p^{j,s_j+1}) & & \psi_k^{k+1}(p^{j,s_j+2}) & \leftarrow & \dots & \leftarrow & \psi_k^{k+1}(p^{j,k!}) \\
& & & & \uparrow & & \downarrow & & & & & & \\
\psi_{k+1}^{k+1}(p^{j,1}) & \rightarrow & \dots & \rightarrow & \psi_{k+1}^{k+1}(p^{j,s_j}) & & \psi_{k+1}^{k+1}(p^{j,s_j+1}) & \rightarrow & \psi_{k+1}^{k+1}(p^{j,s_j+2}) & \rightarrow & \dots & \rightarrow & \psi_{k+1}^{k+1}(p^{j,k!})
\end{array}
$$

Figure 2: Hamilton cycle Q_j in TN_{k+1} (k even)

and $s_j \in \{1, \dots, k! - 2\}$. We construct Hamilton cycle Q_j in TN_{k+1} from the paths:

$$
\begin{array}{lcl}
C_j^1 & = & \psi_1^{k+1}(p^{j,s_j+2}) \rightarrow \dots \rightarrow \psi_1^{k+1}(p^{j,k!}) \rightarrow \psi_1^{k+1}(p^{j,1}) \rightarrow \dots \rightarrow \psi_1^{k+1}(p^{j,s_j}) \\
& & \dots \\
C_j^{k+1} & = & \psi_{k+1}^{k+1}(p^{j,s_j+2}) \rightarrow \dots \rightarrow \psi_{k+1}^{k+1}(p^{j,k!}) \rightarrow \psi_{k+1}^{k+1}(p^{j,1}) \rightarrow \dots \rightarrow \psi_{k+1}^{k+1}(p^{j,s_j})
\end{array}
$$

by a route which connects C_j^i ($1 \leq i \leq k$) to the path below C_j^{i+1} down the shaft S_j via the edge $\psi_i^{k+1}(p^{j,s_j+1}) \rightarrow \psi_{i+1}^{k+1}(p^{j,s_j+1})$ and which connects from C_j^{i+1} back up to C_j^i in a similar way along the shaft S_j. The cases of k even and k odd are distinguished.

Case k is even. Hamilton cycle Q_j is best described with reference to the depiction in Figure 2 below. It starts at vertex $\psi_1^{k+1}(p^{j,s_j+1})$ and follows downward arrows along edges in TN_{k+1} of the form $\psi_i^{k+1}(p^{j,s_j+1}) \rightarrow \psi_{i+1}^{k+1}(p^{j,s_j+1})$ which correspond to a transposition that exchanges digit $k+1$ in the i-th position with the digit in the $(i+1)$-th position of $\psi_i^{k+1}(p^{j,s_j+1})$ to produce $\psi_{i+1}^{k+1}(p^{j,s_j+1})$. When the vertex $\psi_{i+1}^{k+1}(p^{j,s_j+1})$ is reached on the bottom row, Q_j follows the path C_j^{k+1} until it reaches vertex $\psi_{i+1}^{k+1}(p^{j,s_j})$ which has an upward arrow to the row above. At each row i $(1 \leq i \leq k+1)$ the path C_j^i is followed in an alternating forward or reverse direction to connect an incoming upward arrow to an outgoing upward arrow. Following all arrows in this way, leads to the vertex $\psi_1^{k+1}(p^{j,s_j})$ which is adjacent to the starting vertex $\psi_1^{k+1}(p^{j,s_j+1})$. This produces a Hamilton cycle as all vertices of TN_{k+1} are visited exactly once. Formally, Q_j is the Hamilton cycle:

$$\psi_1^{k+1}(p^{j,s_j+1}) \rightarrow \dots \rightarrow \psi_{k+1}^{k+1}(p^{j,s_j+1}) \Rightarrow C_j^{k+1} \Rightarrow rev(C_j^k) \Rightarrow C_j^{k-1} \Rightarrow \dots \Rightarrow rev(C_j^2) \Rightarrow C_j^1$$

Case k is odd. If k is odd, there are an even number $(k+1)$ of rows. Hamilton cycle Q_j starts at the same vertex $\psi_1^{k+1}(p^{j,s_j+1})$ as in the case when k is even, but finishes at the vertex $\psi_1^{k+1}(p^{j,s_j+2})$. Formally, Q_j is the Hamilton cycle:

$$\psi_1^{k+1}(p^{j,s_j+1}) \rightarrow \dots \rightarrow \psi_{k+1}^{k+1}(p^{j,s_j+1}) \Rightarrow C_j^{k+1} \Rightarrow rev(C_j^k) \Rightarrow C_j^{k-1} \Rightarrow \dots \Rightarrow C_j^2 \Rightarrow rev(C_j^1)$$

We show that $Q_1, \dots, Q_{k-1}$ are edge-disjoint. Consider Q_j and $Q_{j'}$ where $1 \leq j < j' \leq k-1$. Each of Q_j and $Q_{j'}$ comprises 'horizontal' edges corresponding to horizontal arrows and 'vertical' edges corresponding to vertical arrows in Figure 2. Horizontal edges of either are disjoint from vertical edges of the other as digit $k + 1$ remains in the same position along horizontal edges but changes position along vertical edges. So, the only possibilities of edge clashes are between horizontal edges of both or vertical edges of both. Suppose that Q_j and $Q_{j'}$ have a common horizontal edge e belonging to paths C_j^i and $C_{j'}^{i'}$, where $1 \leq i, i' \leq k+1$, in Q_j and $Q_{j'}$ respectively. If $i \neq i'$ then C_j^i and $C_{j'}^{i'}$ are vertex-disjoint as digit $k + 1$ is in the i-th position of vertices of C_j^i and in the i'-th position of vertices of $C_{j'}^{i'}$. Thus, $i = i'$ and e must belong to paths C_j^i and $C_{j'}^i$ where $1 \leq i \leq k + 1$. But, any edge in C_j^i is produced by inserting digit $k + 1$ in the i-th position of the vertices of an edge of P_j and similarly for any edge in $C_{j'}^i$. Hence, if edge e is common to both, and we remove digit $k + 1$ from its vertices, the resulting edge will be an edge of P_j as e belongs to C_j^i and also an edge of $P_{j'}$ as e belongs to $C_{j'}^i$. This cannot happen as P_j and $P_{j'}$ are edge-disjoint. Thus, horizontal edges cannot clash. The only other possibility - that of a vertical edge clash - is impossible as vertices of vertical edges of Q_j and $Q_{j'}$ belong to shafts S_j and $S_{j'}$ respectively and these are chosen to be disjoint.

To prove (i) for the images of the Q_j $(1 \leq j \leq k-1)$ under *inv* we have, by Lemma 3.5(ii), that $inv(Q_1), \ldots, inv(Q_{k-1})$ are Hamilton cycles as $Q_1, \ldots, Q_{k-1}$ are Hamilton cycles and are edge-disjoint as $Q_1, \ldots, Q_{k-1}$ are edge-disjoint. For (ii), let

$$e \in (E(inv(Q_1)) \cup \ldots \cup E(inv(Q_{k-1}))) \cap E(St_{k+1}),$$

$e \in E(inv(Q_j))$ say, where $j \in \{1, \ldots, k-1\}$. Then, e is the image under *inv* of either a horizontal or a vertical edge in Q_j. In the case of a horizontal edge, $e = inv(\psi_i^{k+1}(p^{j,h})) \to inv(\psi_i^{k+1}(p^{j,h+1}))$ for some $i \in \{1, \ldots, k+1\}$, $h \in \{1, \ldots, k!\}$ (where $k! + 1 \equiv 1$). However, this is impossible as then (by the definitions of *inv* and ψ_i^{k+1}) both vertices $inv(\psi_i^{k+1}(p^{j,h}))$ and $inv(\psi_i^{k+1}(p^{j,h+1}))$ of e have digit i in the $(k + 1)$-th position and so the edge e does not exchange the digit in the $(k + 1)$-th position and so e is not a star graph edge, contradicting the assumption that $e \in E(St_{k+1})$. Thus, e must be the image under *inv* of a vertical edge $e = \{inv(\psi_i^{k+1}(p^{j,h})), inv(\psi_{i+1}^{k+1}(p^{j,h}))\}$, where $h \in \{s_j, s_j + 1, s_j + 2\}$ and $1 \leq i \leq k$. By Lemma 4.1(ii), $e \in E^1(St_{k+1})$. ■

5. Inductive step for n equals 6

Theorem 4.6 shows that, for all $k \geq 5$, if TN_k has $k - 1$ edge-disjoint Hamilton cycles, then TN_{k+1} has $k - 1$ edge-disjoint Hamilton cycles $inv(Q_1), \ldots, inv(Q_{k-1})$ whose edges are either not edges of the star graph St_{k+1} or are star graph edges with label 1. Thus, if it can be shown that, for all $k \geq 5$, St_{k+1} has a Hamilton cycle without edges labelled 1, then this would provide an additional edge-disjoint Hamilton cycle in TN_{k+1} and it would follow, by induction, from the construction in [8] of 4 edge-disjoint Hamilton cycles for TN_5 as the base case, that TN_n has $n - 1$ edge-disjoint Hamilton cycles for all $n \geq 5$. We show in Section 6 below that St_{k+1} has a Hamilton cycle without edges labelled 1 if $k + 1 > 6$. However, it is not known whether such a Hamilton cycle exists in St_6. As such, we prove the inductive step for TN_6 separately in this section and then use $n = 6$ as the base case for the induction proof for general n in Section 6.

For the case $n = 6$, we construct edge-disjoint Hamilton cycles Q_1, Q_2, Q_3, Q_4 in TN_6 from the 4 known [8] edge-disjoint Hamilton cycles P_1, P_2, P_3, P_4 in TN_5 as in Theorem 4.6. By Theorem 4.6, the edge-disjoint cycles $inv(Q_1), inv(Q_2), inv(Q_3), inv(Q_4)$ will only have edges labelled 1 in common with edges of St_6. We then apply another automorphism Γ to TN_6 which maps edges of St_6 to edges of St_6 and edges labelled 1 to edges labelled 2 or 3. By choosing shafts carefully in the construction of Q_1, Q_2, Q_3, Q_4, we can ensure that edges labelled 2 in the edge-disjoint Hamilton cycles $\Gamma(inv(Q_1)), \Gamma(inv(Q_2)), \Gamma(inv(Q_3)), \Gamma(inv(Q_4))$ do not meet edges labelled 2 in the Hamilton cycle H_6^{12} of Lemma 3.3 which then provides the additional edge-disjoint Hamilton cycle needed.

Theorem 5.1. *The transposition graph TN_6 has 5 edge-disjoint Hamilton cycles.*

Proof Let P_1, P_2, P_3 and P_4 be 4 edge-disjoint Hamilton cycles in TN_5, for example those in [8]. Define the bijection $\gamma : \{1, \dots, 6\} \to \{1, \dots, 6\}$ by

$$\gamma(1) = 1,\ \gamma(2) = 4,\ \gamma(3) = 2,\ \gamma(4) = 6,\ \gamma(5) = 3,\ \gamma(6) = 5$$

and let Γ be the corresponding automorphism of TN_6 which is given by $\Gamma(p_1p_2p_3p_4p_5p_6) = \gamma(p_1)\gamma(p_2)\gamma(p_3)\gamma(p_4)\gamma(p_5)\gamma(p_6)$ for all vertices $p = p_1p_2p_3p_4p_5p_6 \in V(TN_6)$. By Lemma 3.4(ii) we have that, for all $\{u, v\} \in E(TN_6)$,

$$\{u, v\} \in E(St_6) \text{ if and only if } \{\Gamma(u), \Gamma(v)\} \in E(St_6) \tag{4}$$

We show that Γ maps edges labelled 1 to edges labelled 2 or 3. Let $p = p_1p_2p_3p_4p_5p_6 \in V(TN_6)$ and $e = \{p, \phi_{i,i'}(p)\} \in E(TN_6)$ where the transposition $\phi_{i,i'}$ exchanges digits p_i and $p_{i'}$ in the i-th and i'-th positions respectively. If $L(e) = 1$ then, by Definition 3.1,

$$L(e) = min\{|p_i - p_{i'}|, 6 - |p_i - p_{i'}|\} = 1 \tag{5}$$

From (5), $\{p_i, p_{i'}\}$ is one of the following sets:

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\} \tag{6}$$

Now,

$$L(\Gamma(e)) = min\{|\gamma(p_i) - \gamma(p_{i'})|, 6 - |\gamma(p_i) - \gamma(p_{i'})|\} \tag{7}$$

From (6), $\{\gamma(p_i), \gamma(p_{i'})\}$ is one of the following sets:

$$\{1, 4\}, \{4, 2\}, \{2, 6\}, \{6, 3\}, \{3, 5\}, \{5, 1\} \tag{8}$$

Calculating $L(\Gamma(e))$ given in (7) for each of the cases (8), we have that $L(\Gamma(e))$ can have corresponding labels:

$$3,\ 2,\ 2,\ 3,\ 2,\ 2 \tag{9}$$

To summarize, (4) and (9) show that, for all $e \in E(TN_6)$,

$$e \notin E(St_6) \text{ iff } \Gamma(e) \notin E(St_6) \text{ and } e \in E^1(St_6) \text{ implies } \Gamma(e) \in E^2(St_6) \cup E^3(St_6) \tag{10}$$

We now construct Hamilton cycles Q_1, Q_2, Q_3 and Q_4 as in Theorem 4.6 from the matrices M_1, M_2, M_3 and M_4 of P_1, P_2, P_3 and P_4. This means choosing disjoint shafts S_1, S_2, S_3 and S_4 for M_1, M_2, M_3 and M_4 which, in turn, means choosing disjoint sections, i.e. subpaths of the form: $p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2}$,where $s_j \in \{1, \dots, 5! - 2\}$, for each P_j $(1 \le j \le 4)$. Rather than choosing disjoint sections arbitrarily as in Lemma 4.5, we choose sections satisfying the following condition.

Claim Let A be the set of edges in H_6^{12} labelled 2. Then, there exist disjoint sections

$$p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2}$$

of P_j ($1 \leq j \leq 4$) such that the following corresponding 3×6 matrices N_j ($1 \leq j \leq 4$) of vertices of TN_6:

$$\begin{pmatrix} \Gamma(inv(\psi_1^6(p^{j,s_j}))) & \Gamma(inv(\psi_1^6(p^{j,s_j+1}))) & \Gamma(inv(\psi_1^6(p^{j,s_j+2}))) \\ & \dots & \\ \Gamma(inv(\psi_6^6(p^{j,s_j}))) & \Gamma(inv(\psi_6^6(p^{j,s_j+1}))) & \Gamma(inv(\psi_6^6(p^{j,s_j+2}))) \end{pmatrix}$$

do not contain both vertices of any edge in A.

Proof of Claim We construct $p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2}$, inductively. Suppose that for some case $h \in \{1, 2, 3\}$, sections $p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2}$ have been constructed for P_j, for all $j \leq h$, such that:

$$\{p^{j,s_j}, p^{j,s_j+1}, p^{j,s_j+2}\} \cap \{p^{j',s_{j'}}, p^{j',s_{j'}+1}, p^{j',s_{j'}+2}\} = \emptyset \ (1 \leq j < j' \leq h) \text{ and } N_j \cap A = \emptyset \qquad (11)$$

where by $N_j \cap A = \emptyset$ we mean that N_j does not contain both vertices of an edge in A. We find $\{p^{h+1,s_{h+1}}, p^{h+1,s_{h+1}+1}, p^{h+1,s_{h+1}+2}\}$ such that (11) continues to hold for the case $h + 1$. Partition Hamilton cycle $P_{h+1} = p^{h+1,1} \to \dots \to p^{h+1,5!}$ into 40 disjoint sections thus:

$$\{p^{h+1,1}, p^{h+1,2}, p^{h+1,3}\} \cup \dots \cup \{p^{h+1,5!-2}, p^{h+1,5!-1}, p^{h+1,5!}\} \qquad (12)$$

The sections for $P_1, \dots, P_h$ will have used at most $3h \leq 9$ vertices in total. Thus, at least 40-9=31 sets in (12) are clear of vertices of sections already created. Now, by Lemma 3.3, there are (6-2)!=24 edges in A. Also, as the 40 sets

$$section^x_{h+1} = \{p^{h+1,x}, p^{h+1,x+1}, p^{h+1,x+2}\} \quad (x = 1, 4, \dots, 5! - 2) \qquad (13)$$

in (12) are disjoint, their 40 corresponding matrices N^x_{h+1} given by:

$$\begin{pmatrix} \Gamma(inv(\psi_1^6(p^{h+1,x}))) & \Gamma(inv(\psi_1^6(p^{h+1,x+1}))) & \Gamma(inv(\psi_1^6(p^{h+1,x+2}))) \\ & \dots & \\ \Gamma(inv(\psi_6^6(p^{h+1,x}))) & \Gamma(inv(\psi_6^6(p^{h+1,x+1}))) & \Gamma(inv(\psi_6^6(p^{h+1,x+2}))) \end{pmatrix}$$

are disjoint. Thus, at most 24 of the sets $section^x_{h+1}$ have a matrix N^x_{h+1} which has both vertices of an edge in A. This leaves 31-24=7 sections $section^x_{h+1}$ in (13) clear of vertices of sections already constructed and whose matrix N^x_{h+1} does not contain both vertices of an edge in A. Putting $s_{h+1} = x$ and $N_{h+1} = N^x_{h+1}$ for any of these 7 choices of x, we obtain a section $p^{h+1,s_{h+1}} \to p^{h+1,s_{h+1}+1} \to p^{h+1,s_{h+1}+2}$ of P_{h+1} disjoint from each section $p^{j,s_j} \to p^{j,s_j+1} \to p^{j,s_j+2}$ ($1 \leq j \leq h$) and such that $N_{h+1} \cap A = \emptyset$. Thus (11) holds for case $h + 1$. ■

We can now construct 5 edge-disjoint Hamilton cycles in TN_6. Construct 4 edge-disjoint Hamilton cycles Q_1, Q_2, Q_3 and Q_4 in TN_6 as in Theorem 4.6 from the 4 edge-disjoint Hamilton cycles P_1, P_2, P_3 and P_4 in TN_5 respectively, using shafts S_1, S_2, S_3 and S_4 sourced from sections of P_1, P_2, P_3 and P_4 as in the Claim above. Automorphic images $\Gamma(inv(Q_1))$, $\Gamma(inv(Q_2))$,$\Gamma(inv(Q_3))$ and $\Gamma(inv(Q_4))$ are then edge-disjoint Hamilton cycles in TN_6. An edge in any $\Gamma(inv(Q_j))$ ($1 \leq j \leq 4$) that belongs to $E(St_6)$ must be the image under Γ of an edge in $inv(Q_j)$ that belongs to $E(St_6)$, by (10). By Theorem 4.6(ii), edges in $inv(Q_j)$ and $E(St_6)$ have label 1 and so, by (10), edges in $\Gamma(inv(Q_j))$ and $E(St_6)$ have label 2 or 3. Furthermore, we saw in the proof of Theorem 4.6 that edges in $inv(Q_j)$ that belong to $E(St_6)$ were the images of vertical edges in Figure 2. From that and (10), it is easy to see that edges in $\Gamma(inv(Q_j))$ and $E(St_6)$ are edges corresponding to vertically adjacent vertices in the matrix N_j in the Claim. Thus, edges in $\Gamma(inv(Q_j))$ and $E(St_6)$

have label 2 or 3 and avoid edges of A, i.e. edges of H_6^{12} with label 2, because N_j is constructed in the Claim to avoid both vertices of edges in A. Therefore, $\Gamma(inv(Q_j))$ does not have any edges that are in H_6^{12} of label 2 and hence is edge-disjoint from H_6^{12} as all other edges of H_6^{12} have label 1. It follows that H_6^{12} is edge-disjoint from all $\Gamma(inv(Q_j))$ $(1 \leq j \leq 4)$ and is therefore a 5th edge-disjoint Hamilton cycle in TN_6. ■

6. Inductive step for n greater than 6

In the case of TN_6 we applied an automorphism Γ in Theorem 5.1 to the Hamilton cycles $inv(Q_1)$, $inv(Q_2)$, $inv(Q_3)$, and $inv(Q_4)$ of Theorem 4.6 to obtain Hamilton cycles that were edge-disjoint from H_6^{12} which provided the extra edge-disjoint Hamilton cycle. In the case of TN_{k+1}, for $k+1 > 6$, we apply an automorphism Γ_{k+1}^H to H_{k+1}^{12} instead, in Theorem 6.1 below, to produce a Hamilton cycle edge-disjoint from the Hamilton cycles $inv(Q_1)$, $inv(Q_2)$, $inv(Q_3)$, and $inv(Q_4)$ as they are. Theorem 6.1 proves Hung's conjecture given in [8].

Theorem 6.1. *The n-dimensional transposition graph TN_n contains n-1 edge-disjoint Hamilton cycles for all $n \geq 5$.*

Proof The case of $n = 5$ is given in [8] and that of $n = 6$ in Theorem 5.1 above. Using $n = 6$ as the base case, we prove by induction all cases $n = k > 6$. Assume that case $n = k$ holds where $k \geq 6$, i.e. TN_k has $k-1$ edge-disjoint Hamilton cycles. We construct k edge-disjoint Hamilton cycles in TN_{k+1}. First of all, we obtain $k-1$ edge-disjoint Hamilton cycles $inv(Q_1), \ldots, inv(G_{k-1})$ in TN_{k+1} as in Theorem 4.6. Next, we define an automorphism Γ_{k+1}^H of St_{k+1} such that the Hamilton cycle $\Gamma_{k+1}^H(H_{k+1}^{12})$ has no edges with label 1. The automorphism Γ_{k+1}^H is defined in terms of a bijection $\gamma_{k+1}^H : \{1, \ldots, k+1\} \to \{1, \ldots, k+1\}$ given by:

$$\gamma_{k+1}^H(i) = (l * i)\ mod\ (k+1) \quad (1 \leq i \leq k+1),$$

where l is a chosen integer that is coprime to $k+1$ and $(k+1)\ mod\ (k+1) \equiv k+1$. Note that if $e = \{p, \phi_{i,j}(p)\} \in E(TN_{k+1})$ where $p = p_1 \ldots p_{k+1}$ then, as $L(e) = min\{|p_i - p_j|, (k+1) - |p_i - p_j|\}$,

$$L(\Gamma_{k+1}^H(e)) = min\{l * |p_i - p_j|, (k+1) - l * |p_i - p_j|\} \tag{14}$$

where multiplication $*$ is modulo $k+1$.
Case k even. In this case $k+1$ is odd and we choose $l = 2$. If $L(e) = 1$ then, by (14), $L(\Gamma_{k+1}^H(e)) = min\{2, k-1\} = 2$ as $k \geq 6$. If $L(e) = 2$ then $L(\Gamma_{k+1}^H(e)) = min\{4, k-3\}$ which equals 3 if $k = 6$ and equals 4 if $k > 6$. Thus, as H_{k+1}^{12} only has edges with label 1 or 2, $\Gamma_{k+1}^H(H_{k+1}^{12})$ has no edges with label 1.
Case k odd. In this case $k+1$ is even. By elementary number theory, we can choose an integer l that is coprime to $k+1$ and is not equal to 1 or k. If $L(e) = 1$ then $L(\Gamma_{k+1}^H(e)) = min\{l, k+1-l\} \neq 1$. If $L(e) = 2$ then $L(\Gamma_{k+1}^H(e)) = min\{l * 2, k+1 - l * 2\}$ which gives an even value as $k+1$ is even. Thus, $\Gamma_{k+1}^H(H_{k+1}^{12})$ has no edges with label 1.

It follows from the two cases above that, for all $k \geq 6$, Hamilton cycle $\Gamma_{k+1}^H(H_{k+1}^{12})$ which is comprised of edges in $E(St_{k+1})$ by Lemma 3.4(ii), has no edges with label 1. By Theorem 4.6(ii) as Hamilton cycles $inv(Q_1), \ldots, inv(Q_{k-1})$ only contain edges of St_{k+1} with label 1, $\Gamma_{k+1}^H(H_{k+1}^{12})$ will be edge-disjoint from those Hamilton cycles and the proof of this theorem is complete. ■

7. Conclusions

The method for constructing edge-disjoint Hamilton cycles in dimension $n + 1$ from edge-disjoint Hamilton in dimension n for 4 Hamilton cycles in [8] and $n - 1$ Hamilton cycles here in Theorem 4.6, can be further extended to $O(n^2)$ Hamilton cycles to obtain improved asymptotic bounds for the number of edge-disjoint in n-dimensional transposition graphs. In [11] it is noted that n-dimensional star graphs contain $\Omega(n/log\ log\ n)$ edge-disjoint Hamilton cycles of which only one has edges with labels 1 or 2 (see [11] or [4]). Thus, $\Omega(n/log\ log\ n)$ edge-disjoint Hamilton cycles can be added at the inductive step from dimension n to dimension $n+1$ in Theorem 6.1, giving asymptotic bounds of $\Omega(n^2/log\ log\ n)$ on the number of edge-disjoint Hamilton cycles in n-dimensional transposition graphs. Another aspect of the construction in [8] and Theorem 4.6 is that it shows that as dimension increases, so does the number of edge-disjoint Hamilton cycles in transposition graphs. It is not known whether star graphs share this monotonicity of number of edge-disjoint Hamilton cycles with respect to dimension. The construction of edge-disjoint Hamilton cycles for star graphs is not of the same inductive nature. It would be interesting to see if there is a method for constructing edge-disjoint Hamilton cycles in dimension $n+1$ from those in dimension n for star graphs as has been found for transposition graphs.

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