

GLOBAL EXISTENCE OF SMALL-NORM SOLUTIONS IN THE REDUCED OSTROVSKY EQUATION

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ABSTRACT. We use a novel transformation of the reduced Ostrovsky equation to the integrable Tzitzéica equation and prove global existence of small-norm solutions in Sobolev space $H^3(\mathbb{R})$. This scenario is an alternative to finite-time wave breaking of large-norm solutions of the reduced Ostrovsky equation. We also discuss a sharp sufficient condition for the finite-time wave breaking.

1. Introduction. *The reduced Ostrovsky equation*

$$(u_t + uu_x)_x = u, \quad (1)$$

is the zero high-frequency dispersion limit ($\beta \rightarrow 0$) of the Ostrovsky equation

$$(u_t + uu_x + \beta u_{xxx})_x = u. \quad (2)$$

The evolution equation (2) was originally derived by Ostrovsky [17] to model small-amplitude long waves in a rotating fluid of finite depth. Local and global well-posedness of the Ostrovsky equation (2) in energy space $H^1(\mathbb{R})$ was studied in recent papers [10, 12, 21, 26].

Corresponding rigorous results for the reduced Ostrovsky equation (1) are more complicated. Local solutions exist in Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ [20]. But for sufficiently steep initial data $u_0 \in C^1(\mathbb{R})$, local solutions break in a finite time [2, 8, 14] in the standard sense of finite-time wave breaking that occurs in the inviscid Burgers equation $u_t + uu_x = 0$.

However, a proof of global existence for sufficiently small initial data has remained an open problem up to now. In a similar equation with a cubic nonlinear term (called *the short-pulse equation*), the proof of global existence was recently developed with the help of a bi-infinite sequence of conserved quantities [18]. These global solutions for small initial data coexist with wave breaking solutions for large initial data [13]. Global existence and scattering of small-norm solutions to zero in the generalized

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short-pulse equation with quartic and higher-order nonlinear terms follow from the results of [20].

Rather different sufficient conditions on the initial data for wave breaking were obtained recently in [9] on the basis of asymptotic analysis and supporting numerical simulations (similar numerical simulations can be found in [2]). It was conjectured in [9] that initial data $u_0 \in C^2(\mathbb{R})$ with $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$ generate global solutions of the reduced Ostrovsky equation (1), whereas a sign change of this function on the real line inevitably leads to wave breaking in finite time.

This paper is devoted to the rigorous proof of the first part of this conjecture, that is, global solutions exist for all initial data $u_0 \in H^3(\mathbb{R})$ such that $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$. Note here that if $u_0 \in H^3(\mathbb{R})$, then $u_0 \in C^2(\mathbb{R})$, hence the function $1 - 3u_0''(x)$ is continuous for all $x \in \mathbb{R}$ and approaches 1 as $x \rightarrow \pm\infty$. The second part of the conjecture is also discussed and a weaker statement in line with this conjecture is proven.

Integrability of the reduced Ostrovsky equation was discovered first by Vakhnenko [23]. In a series of papers [16, 24, 25], Vakhnenko, Parkes and collaborators found and explored a transformation of the reduced Ostrovsky equation to the integrable *Hirota–Satsuma equation* with reversed roles of the variables x and t . As a particular application of the power series expansions [19], one can generate a hierarchy of conserved quantities for the reduced Ostrovsky equation (1). This hierarchy includes the first two conserved quantities

$$E_0 = \int_{\mathbb{R}} u^2 dx, \quad E_{-1} = \int_{\mathbb{R}} \left[(\partial_x^{-1} u)^2 + \frac{1}{3} u^3 \right] dx, \quad (3)$$

where the anti-derivative operator is defined by the integration of $u(x, t)$ in x subject to the zero-mass constraint $\int_{\mathbb{R}} u(x, t) dx = 0$.

Higher-order conserved quantities E_{-1}, E_{-2} , and so on involve higher-order anti-derivatives, which are defined under additional constraints on the solution u . Hence, these conserved quantities are not related to the H^s -norms for positive s and play no role in the study of global well-posedness of the reduced Ostrovsky equation (1) in Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$. Note in passing that the global well-posedness of the regular Hirota–Satsuma equation in the energy space $H^1(\mathbb{R})$ was considered recently in [6].

However, a different transformation has recently been discovered for the reduced Ostrovsky equation (1). This transformation is useful to generate a bi-infinite sequence of conserved quantities, which are more suitable for the proof of global existence.

The alternative formulation of the integrability scheme for the reduced Ostrovsky equation starts with the work of Hone and Wang [7], where the reduced Ostrovsky equation (1) was obtained as a short-wave limit of the integrable *Degasperis–Procesi equation*. As a result of the asymptotic reduction, these authors obtained the following Lax operator pair for the reduced Ostrovsky equation (1) in the original space and time variables:

$$\begin{cases} 3\lambda\psi_{xxx} + (1 - 3u_{xx})\psi = 0, \\ \psi_t + \lambda\psi_{xx} + u\psi_x - u_x\psi = 0, \end{cases} \quad (4)$$

where λ is a spectral parameter. Note that the function $1 - 3u_{xx}$ arises naturally in the third-order eigenvalue problem (4) in the same way as the function $m = u - u_{xx}$ arises in another integrable *Camassa–Holm equation* to determine if the global solutions or wave breaking will occur in the Cauchy problem [4, 5].

More recently, based on an earlier study of Manna & Neveu [15], Kraenkel *et al.* [11] found a transformation between the reduced Ostrovsky equation (1) and the integrable *Bullough–Dodd equation*, which is also widely known as the *Tzitzéica equation* after its original derivation in 1910 [22]. In new characteristic variables Y and T (see section 2), the Tzitzéica equation can be written in the form,

$$\frac{\partial^2 V}{\partial T \partial Y} = e^{-2V} - e^V. \tag{5}$$

Note that the Tzitzéica equation is similar to the *sine–Gordon equation* in characteristic coordinates, which arises in the integrability scheme of the short-pulse equation [18]. Similarly to the sine–Gordon equation, the Tzitzéica equation has a bi-infinite sequence of conserved quantities, which was discovered in two recent and independent works [1, 3]. Among those, we only need the first two conserved quantities

$$Q_1 = \int_{\mathbb{R}} (2e^V + e^{-2V} - 3) dY, \quad Q_2 = \int_{\mathbb{R}} \left(\frac{\partial V}{\partial Y} \right)^2 dY, \tag{6}$$

which were obtained from the power series expansions [1]. The conserved quantities (6) are related to the conserved quantities of the reduced Ostrovsky equation (1) in original physical variables

$$E_1 = \int_{\mathbb{R}} \left[(1 - 3u_{xx})^{1/3} - 1 \right] dx, \quad E_2 = \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx. \tag{7}$$

Note that the conserved quantities (7) also appeared in the balance equations derived in [11].

In Section 2, we shall use the conserved quantities E_0 in (3) and Q_1, Q_2 in (6), as well as the reduction to the Tzitzéica equation (5), to prove our main result, which is,

Theorem 1. *Assume $u_0 \in H^3(\mathbb{R})$ such that $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$. Then, the reduced Ostrovsky equation (1) admits a unique global solution $u \in C(\mathbb{R}_+, H^3(\mathbb{R}))$.*

It is natural to expect that the finite-time wave breaking occurs for any $u_0 \in H^3(\mathbb{R})$ such that $1 - 3u_0''(x)$ changes sign for some $x \in \mathbb{R}$. As shown in [9] in a periodic setting, this criterion of wave breaking is sharper than the previous criteria of wave breaking in [8, 14]. Although we are not able to give a full proof of this sharp criterion in the present work, we shall prove the following weaker statement in Section 3:

Theorem 2. *Assume that $u_0 \in H^3(\mathbb{R})$ is given and there is a finite interval $[X_-, X_+]$ and a point $X_0 \in (X_-, X_+)$ such that*

$$1 - 3u_0''(x) < 0, \quad x \in (X_-, X_+), \tag{8}$$

and

$$u_0'(x) < 0, \quad x \in (X_-, X_0), \quad u_0'(x) > 0, \quad x \in (X_0, X_+), \tag{9}$$

whereas $1 - 3u_0''(x) \geq 0$ for all $x \leq X_-$ and $x \geq X_+$. Then, a local solution $u \in C([0, t_0], H^3(\mathbb{R}))$ of the reduced Ostrovsky equation (1) breaks in a finite time $t_0 \in (0, \infty)$ in the sense

$$\limsup_{t \uparrow t_0} \|u(\cdot, t)\|_{H^3(\mathbb{R})} = \infty$$

if $u_x(x_-(t), t) < 0$ and $u_x(x_+(t), t) > 0$ hold for all $t \in [0, t_0)$ along the characteristics $x = x_{\pm}(t)$ originating from $x_{\pm}(0) = X_{\pm}$.

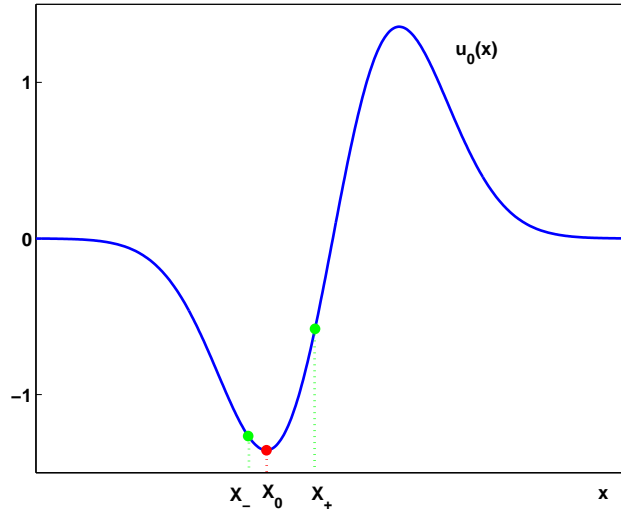


FIGURE 1. An example of the initial condition $u_0(x) = xe^{-ax^2}$ with $a = 0.1$, where positions of X_- , X_+ , and X_0 are shown.

A prototypical example of the initial data for the reduced Ostrovsky equation on the infinite line is the first Hermite function $u_0(x) = xe^{-ax^2}$, where $a > 0$ is a parameter. A straightforward computation of the maximum of $u_0''(x)$ shows that $u_0''(x) < \frac{1}{3}$ for all $x \in \mathbb{R}$ if $a \in (0, a_0)$, where

$$a_0 = \frac{e^{3-\sqrt{6}}}{108(3-\sqrt{6})} \approx 0.0292.$$

In this case, Theorem 1 implies global existence of solutions for such initial data. When $a > a_0$, condition (8) is satisfied. In addition, $u_0(x)$ has a global minimum at $x = -\frac{1}{\sqrt{2a}}$ so that condition (9) is satisfied for $a > a_* = \frac{e}{72} \approx 0.0378$. (Note that $a_* > a_0$.) Theorem 2 implies wave breaking in a finite time provided that *additional* constraints are satisfied, that is, $u_x(x_-(t), t) < 0$ and $u_x(x_+(t), t) > 0$ hold for all times before the wave breaking time along the characteristics $x = x_{\pm}(t)$ originating from $x_{\pm}(0) = X_{\pm}$. Although we strongly believe that these additional constraints as well as condition (9) are not needed for the statement of Theorem 2, we were not able to lift out these technical restrictions.

The initial function $u_0(x) = xe^{-ax^2}$ for $a > a_*$ is shown on Fig. 1, where the points X_- , X_+ , and X_0 introduced in Theorem 2 are also shown.

2. Proof of Theorem 1. We introduce characteristic coordinates for the reduced Ostrovsky equation (1) [9, 14, 23]:

$$x = X + \int_0^T U(X, T') dT', \quad t = T, \quad u(x, t) = U(X, T). \quad (10)$$

The coordinate transformation is one-to-one and onto if the Jacobian

$$\phi(X, T) = 1 + \int_0^T U_X(X, T') dT', \quad (11)$$

which is positive for $T = 0$ because $\phi(X, 0) = 1$, remains positive for all $(X, T) \in \mathbb{R} \times [0, T_0]$, where $T_0 > 0$ is the local existence time.

We note that the original equation (1) yields the relation

$$u = \frac{U_{XT}}{\phi} = \frac{\phi_{TT}}{\phi}, \tag{12}$$

whereas the transformation formulas (10) and (11) yield the relations

$$u_x = \frac{U_X}{\phi} = \frac{\phi_T}{\phi}, \tag{13}$$

and

$$u_{xx} = \frac{1}{\phi} \left(\frac{U_X}{\phi} \right)_X = \frac{\phi_{TX}\phi - \phi_X\phi_T}{\phi^3}. \tag{14}$$

Next, in accordance with [11], we introduce the variable

$$f = (1 - 3u_{xx})^{1/3}. \tag{15}$$

If u satisfies the reduced Ostrovsky equation (1), then f satisfies the balance equation

$$f_t + (uf)_x = \frac{[u - u_{xt} - (uu_x)_x]_x}{f^{2/3}} = 0. \tag{16}$$

In characteristic coordinates (10), we set $f(x, t) = F(X, T)$, use equation (13), and rewrite the balance equation (16) in the equivalent form

$$(F\phi)_T = 0 \quad \Rightarrow \quad F(X, T)\phi(X, T) = F_0(X), \tag{17}$$

where $F_0(X) = F(X, 0)$. Using equations (14) and (15), we obtain the evolution equation for $F(X, T)$:

$$\frac{\partial^2}{\partial T \partial X} \log(F) = -\frac{\partial^2}{\partial T \partial X} \log(\phi) = \frac{1}{3}\phi(F^3 - 1) = \frac{1}{3}F_0(X)(F^2 - F^{-1}). \tag{18}$$

We shall now consider the Cauchy problem for the reduced Ostrovsky equation (1) with initial data $u_0 \in H^3(\mathbb{R})$. By the local well-posedness result [20], there exists a unique local solution of the reduced Ostrovsky equation in class $u \in C([0, t_0], H^3(\mathbb{R}))$ for some $t_0 > 0$. By Sobolev embedding of $H^3(\mathbb{R})$ into $C^2(\mathbb{R})$, the function $f_0(x) := (1 - 3u_0''(x))^{1/3}$ is continuous, bounded, and satisfies $f_0(x) \rightarrow 1$ as $|x| \rightarrow \infty$.

To prove Theorem 1, we further require that $f_0(x) > 0$ for all $x \in \mathbb{R}$, which means from the above properties that $\inf_{x \in \mathbb{R}} f_0(x) > 0$. Because $x = X$ for $t = T = 0$, we have $F_0 \in C(\mathbb{R})$ such that $\inf_{X \in \mathbb{R}} F_0(X) > 0$. In this case, the transformation from X to Y defined by

$$Y := -\frac{1}{3} \int_0^X F_0(X') dX' \tag{19}$$

is one-to-one and onto for all $X \in \mathbb{R}$, because the Jacobian of the transformation is $-\frac{1}{3}F_0(X) < 0$ and $F_0(X) \rightarrow 1$ as $|X| \rightarrow \infty$. The change of variable,

$$F(X, T) = e^{-V(Y, T)}, \tag{20}$$

transforms the evolution equation (18) to the integrable Tzitzéica equation (5).

We can now transfer the well-posedness result for local solutions of the reduced Ostrovsky equation (1) to local solutions of the Tzitzéica equation (5).

Lemma 1. Assume $u_0 \in H^3(\mathbb{R})$ such that $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$. Let

$$V_0(Y) := -\frac{1}{3} \log(1 - 3u_0''(x)), \quad Y := -\frac{1}{3} \int_0^x (1 - 3u_0''(x'))^{1/3} dx'.$$

There exists a unique local solution of the Tzitzéica equation (5) in class $V \in C([0, T_0], H^1(\mathbb{R}))$ for some $T_0 > 0$ such that $V(Y, 0) = V_0(Y)$.

Proof. We rewrite transformations (15) and (20) into the equivalent form,

$$u_{xx}(x, t) = \frac{1}{3} (1 - f^3(x, t)) = \frac{1}{3} (1 - e^{-3V(Y, T)}).$$

The inverse of this transformation is

$$V(Y, T) = -\frac{1}{3} \log(1 - 3u_{xx}(x, t)).$$

For local solutions of the reduced Ostrovsky equation (1) in class $u \in C([0, t_0], H^3(\mathbb{R}))$, we have $u_{xx} \in C([0, t_0], H^1(\mathbb{R}))$ for some $t_0 > 0$ and $u_{xx} \rightarrow 0$ as $|x| \rightarrow \infty$. We have further assumed that $\sup_{x \in \mathbb{R}} u_0''(x) < \frac{1}{3}$, which implies that there is $T_0 \in (0, t_0)$ such that $\sup_{x \in \mathbb{R}} u_{xx}(x, t) < \frac{1}{3}$ for all $t \in [0, T_0]$. Under the same condition, the transformation from X to Y is one-to-one and onto for all $X \in \mathbb{R}$. Therefore, V is well-defined for all $(Y, T) \in \mathbb{R} \times [0, T_0]$ and $V(Y, T) \rightarrow 0$ as $|Y| \rightarrow \infty$. By construction, V is a solution of the Tzitzéica equation (5) and $V(Y, 0) = V_0(Y)$. It remains to show that V is in class $V \in C([0, T_0], H^1(\mathbb{R}))$.

The variables V and u_{xx} are related by $V = u_{xx}G(u_{xx})$, where

$$G(u_{xx}) := \frac{\log(1 - 3u_{xx})}{(-3u_{xx})}.$$

Both the function G and its first derivative G' remain bounded in L^∞ norm as long as

$$\sup_{x \in \mathbb{R}} u_{xx}(x, t) < \frac{1}{3},$$

which is satisfied for all $t \in [0, T_0]$. Note that $G(z)$ is analytic in z if $|z| < \frac{1}{3}$, but we only need boundedness of $G(z)$ and $G'(z)$, which is achieved if $z < \frac{1}{3}$.

Next recall the transformations (10) and (19) for any function $W(Y, T) = w(x, t)$,

$$\begin{aligned} \|W(\cdot, T)\|_{L^2}^2 &= \int_{\mathbb{R}} W^2(Y, T) dY = \frac{1}{3} \int_{\mathbb{R}} W^2(Y, T) F_0(X) dX \\ &= \frac{1}{3} \int_{\mathbb{R}} \frac{w^2(x, t) F_0(X)}{\phi(X, T)} dx = \frac{1}{3} \int_{\mathbb{R}} w^2(x, t) f(x, t) dx. \end{aligned}$$

Therefore,

$$\|V(\cdot, T)\|_{L^2} \leq \frac{1}{\sqrt{3}} \|G(u_{xx}(\cdot, t))\|_{L^\infty} \|f(\cdot, t)\|_{L^\infty} \|u_{xx}(\cdot, t)\|_{L^2},$$

which remains bounded as long as $\|u_{xx}(\cdot, t)\|_{L^\infty}$ and $\|u_{xx}(\cdot, t)\|_{L^2}$ remain bounded. Similarly, we can prove that $\|V_Y(\cdot, T)\|_{L^2}$ remains bounded as long as $\|u_{xx}(\cdot, t)\|_{L^\infty}$ and $\|u_{xxx}(\cdot, t)\|_{L^2}$ remain bounded. Thus, we have $V \in C([0, T_0], H^1(\mathbb{R}))$ for some $T_0 > 0$. □

Remark 1. The Jacobian of the transformation from (X, T) to (x, t) is given by (11) and controlled by the relation (13). Since $\phi(X, 0) = 1$ and

$$\phi(X, T) = \exp\left(\int_0^T u_x(x(X, T), T) dT\right), \tag{21}$$

we can see that there is $T_0 > 0$ such that $\phi(X, T) > 0$ for all $(X, T) \in \mathbb{R} \times [0, T_0]$. Because

$$\phi(X, T) = \frac{F_0(X)}{F(X, T)} = F_0(X)e^{V(Y, T)}, \tag{22}$$

the condition $\phi(X, T) > 0$ remains true as long as $V(Y, T)$ remains bounded in L^∞ -norm.

Lemma 2. *Let $V \in C([0, T_0], H^1(\mathbb{R}))$ for some $T_0 > 0$ be a unique local solution of the Tzitzéica equation (5). Then, in fact, $V \in C(\mathbb{R}_+, H^1(\mathbb{R}))$.*

Proof. We shall use Q_1 and Q_2 in (6). The quantities are well-defined for a local solution in class $V \in C([0, T_0], H^1(\mathbb{R}))$ and conserved in time for the Tzitzéica equation (5), according to the standard approximation arguments in Sobolev spaces.

To be able to use Q_1 for the control of $\|V(\cdot, T)\|_{L^2}$, we note that the function $H(V) := 2e^V + e^{-2V} - 3$ is convex with $H(0) = H'(0) = 0$ and

$$H''(V) = 2e^V + 4e^{-2V} \geq 2, \quad V \in \mathbb{R}.$$

Therefore, $H(V) \geq V^2$ for all $V \in \mathbb{R}$, so that

$$\|V\|_{H^1}^2 = \|V\|_{L^2}^2 + \|V_Y\|_{L^2}^2 \leq Q_1 + Q_2.$$

By a standard continuation technique, a local solution in class $V \in C([0, T_0], H^1(\mathbb{R}))$ is uniquely continued into a global solution in class $V \in C(\mathbb{R}_+, H^1(\mathbb{R}))$. \square

It remains to transfer results of Lemmas 1 and 2, as well as the L^2 conservation of E_0 in (3) for the proof of Theorem 1.

Proof of Theorem 1. It follows from the proof in Lemma 1 that $u_{xx} = Vg(V)$, where

$$g(V) := \frac{1 - e^{-3V}}{3V}.$$

Both the function g and its first derivative g' remain bounded as long as V remains bounded.

By Lemma 2, $V \in C(\mathbb{R}_+, H^1(\mathbb{R}))$ and hence $F(X, T) > 0$ for all $(X, T) \in \mathbb{R} \times \mathbb{R}_+$. Therefore, $\phi(X, T) > 0$ for all $(X, T) \in \mathbb{R} \times \mathbb{R}_+$, so that the transformation (10) is one-to-one and onto for all $(X, T) \in \mathbb{R} \times \mathbb{R}_+$. Using the bounded functions g and g' , we hence have $u_{xx} \in C(\mathbb{R}_+, H^1(\mathbb{R}))$.

Finally, conservation of E_0 in (3) and the elementary Cauchy-Schwarz inequality,

$$\|u_x\|_{L^2}^2 \leq \|u\|_{L^2} \|u_{xx}\|_{L^2},$$

implies that $u \in C(\mathbb{R}_+, H^3(\mathbb{R}))$. The proof of Theorem 1 is complete. \square

3. Proof of Theorem 2. We utilize the characteristic coordinates (10) and consider the evolution of the Jacobian ϕ defined by (11). Recall that $\phi(X, 0) = 1$ whereas $F(X, 0) = F_0(X) = (1 - 3u_0''(X))^{1/3}$. By conservation (17), assumption (8), and local existence in class $u \in C([0, t_0], H^3(\mathbb{R}))$, we have $F(X, T) < 0$ for all $X \in (X_-, X_+)$ at least for small $T \geq 0$, whereas $F(X, T) \geq 0$ for $X \leq X_-$ and $X \geq X_+$.

Using conservation (17) and evolution (18) for F , we obtain the evolution equation for $\phi(X, T)$:

$$\frac{\partial^2}{\partial T \partial X} \log(\phi) = \frac{1}{3} \phi \left(1 - \frac{F_0^3(X)}{\phi^3} \right). \tag{23}$$

Integrating this equation in T with the initial condition $\phi(X, 0) = 1$, we obtain

$$\frac{\partial \phi}{\partial X} = \frac{1}{3} \phi(X, T) \int_0^T \phi(X, T') \left(1 - \frac{F_0^3(X)}{\phi^3(X, T')} \right) dT'. \tag{24}$$

Because the right-hand side of (24) is positive for all $X \in (X_-, X_+)$, the function $\phi(X, T)$ is monotonically increasing for all $X \in (X_-, X_+)$ at least for small $T \geq 0$. Moreover, we obtain the following inequality.

Lemma 3. *Let $\psi(X, T) := \int_0^T \phi(X, T') dT'$. Under assumption (8) of Theorem 2, we have*

$$\frac{\partial \psi}{\partial X} \geq \frac{1}{6} \psi^2(X, T), \quad X \in (X_-, X_+), \tag{25}$$

as long as the solution remains in class $u \in C([0, t_0], H^3(\mathbb{R}))$.

Proof. Because $F_0(X) < 0$ for all $X \in (X_-, X_+)$, we have from (24):

$$\frac{\partial \phi}{\partial X} \geq \frac{1}{3} \phi(X, T) \int_0^T \phi(X, T') dT' = \frac{1}{6} \frac{\partial}{\partial T} \left(\int_0^T \phi(X, T') dT' \right)^2.$$

Integrating this inequality in T , we obtain the assertion of the lemma. □

It follows from Lemma 3 that

$$\frac{\partial}{\partial X} \left(-\frac{1}{\psi} \right) \geq \frac{1}{6} \Rightarrow \psi(X, T) \geq \frac{6\psi(\xi, T)}{6 - (X - \xi)\psi(\xi, T)}, \quad X \in (\xi, X_+), \tag{26}$$

for any $\xi \in (X_-, X_+)$, which may depend on T . Therefore, $\psi(X, T)$ becomes infinite near $X = X_+$ if there exists $T > 0$ such that $(X_+ - \xi)\psi(\xi, T) > 6$. To ensure that this is inevitable under assumptions of Theorem 2, we prove the following result.

Lemma 4. *Under assumptions (8) and (9) of Theorem 2, there exists a C^1 function $\xi(T)$ and T -independent constants ξ_{\pm} such that $\phi(\xi(T), T) = 1$, $\xi(0) = X_0 \in (X_-, X_+)$, and $\xi(T) \in [\xi_-, \xi_+] \subset (X_-, X_+)$ for all $T \geq 0$, as long as the solution remains in class $u \in C([0, t_0], H^3(\mathbb{R}))$ with $U_X(X_-, T) < 0$ and $U_X(X_+, T) > 0$.*

Proof. Under assumption (9), the function $\phi_T|_{T=0} = U_X|_{T=0} = u'_0(X)$ changes sign at $X = X_0$ from being negative for $X \in (X_-, X_0)$ to being positive for $X \in (X_0, X_+)$. Therefore, we can define $\xi(0) = X_0$ and consider the level curve $\phi(\xi(T), T) = 1$. It follows from the definition (11) that the function $\phi(X, T)$ is continuously differentiable in X and T as long as the solution remains in class $u \in C([0, t_0], H^3(\mathbb{R}))$ with

$$\frac{d\xi}{dT} = -\frac{\phi_T(\xi(T), T)}{\phi_X(\xi(T), T)} = -\frac{U_X(\xi(T), T)}{\phi_X(\xi(T), T)}. \tag{27}$$

Equation (24) implies that $\phi_X(\xi(T), T) > 0$ as long as $\xi(T)$ remains in the interval (X_-, X_+) . The differential equation (27) hence implies that if $U_X(X_-, T) < 0$ and $U_X(X_+, T) > 0$ for all $T \geq 0$, for which the solution is defined, then there exists T -independent constants ξ_{\pm} such that $\xi(T) \in [\xi_-, \xi_+] \subset (X_-, X_+)$. □

Remark 2. Since $\xi(0) = X_0$ is the point of minimum of $U(X, 0) = u_0(X)$ and $\phi(X, 0) = 1$, it follows from equations (12) and (27) that

$$\xi'(0) = -\frac{U_{XT}(X_0, 0) + \xi'(0)U_{XX}(X_0, 0)}{\phi_{XT}(X_0, 0)} = -\frac{u_0(X_0) + \xi'(0)u''(X_0)}{u''(X_0)}$$

and, therefore,

$$\xi'(0) = -\frac{u_0(X_0)}{2u_0''(X_0)}.$$

This equation shows that $\xi'(0) > 0$ if $u_0(X_0) < 0$ and $\xi'(0) < 0$ if $u_0(X_0) > 0$. Therefore, it is not apriori clear if $\xi(T)$ can reach X_- or X_+ in a finite time. The restrictions $u_X(X_-, T) < 0$ and $u_X(X_+, T) > 0$ serve as a sufficient condition that $\xi(T)$ does not reach X_- and X_+ in a finite time, for which the solution is defined.

With the help of Lemmas 3 and 4, we complete the proof of Theorem 2.

Proof of Theorem 2. We use estimate (26) with $\xi(T)$ defined by Lemma 4. Then, we have

$$\int_0^T \phi(X, T') dT' \geq \frac{6T}{6 - (X - \xi(T))T}, \quad X \in (\xi(T), X_+). \tag{28}$$

By Lemma 4, there are T -independent constants ξ_{\pm} such that $\xi(T) \in [\xi_-, \xi_+] \subset (X_-, X_+)$ as long as $U_X(X_-, T) < 0$ and $U_X(X_+, T) > 0$. The lower bound in (28) diverges at a point $X \in (\xi_+, X_+)$ if $T > \frac{6}{X_+ - \xi_+}$. However, divergence of $\int_0^T \phi(X, T') dT'$ implies divergence of $\phi(X, T)$ for some $X \in (\xi_+, X_+)$ also in a finite time $T_0 \in (0, \infty)$. Then, equation (21) shows that $u_x(x, t)$ cannot be bounded if $\phi(X, T)$ becomes infinite for some $X \in (\xi_+, X_+)$ and some $T = T_0$, hence the norm $\|u(\cdot, T)\|_{H^3(\mathbb{R})}$ diverges as $T \uparrow T_0$. The proof of Theorem 2 is complete. \square

Remark 3. Based on the asymptotic analysis and numerical simulations of [9], we anticipate that divergence of $\phi(X, T)$ near $X = X_+$ is related to the vanishing of $\phi(X, T)$ near $X = X_-$, such that equation (13) with $U_X(X_-, T) < 0$ would imply that u_x diverges in a finite time near $x = x_-(t)$. However, the best that can be obtained from equation (24) is

$$\phi(X_-, T) \leq \phi(X, T)e^{-\alpha(X)T}, \quad X \in [X_-, X_+], \tag{29}$$

where

$$\alpha(X) := \frac{1}{2^{2/3}} \int_{X_-}^X |F_0(X')| dX'.$$

This upper bound is obtained from the minimization of the integrand in (24) as follows:

$$\phi + \frac{|F_0(X)|^3}{\phi^2} \geq \frac{3}{2^{2/3}} |F_0(X)|.$$

If $X = \xi(T) \in (X_-, X_+)$ with $\phi(\xi(T), T) = 1$, the bound (29) only gives an exponential decay of $\phi(X_-, T)$ to zero as $T \rightarrow \infty$. The same difficulty appears in our attempts to use bound (29) in estimate (26).

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