Lag Synchronization of Switched Neural Networks via Neural Activation Function and Applications in Image Encryption

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Abstract—This paper investigates the problem of global exponential lag synchronization of a class of switched neural networks with time-varying delays via neural activation function and applications in image encryption. The controller is dependent on the output of the system in the case of packed circuits, since it is hard to measure the inner state of the circuits. Thus, it is critical to design the controller based on the neuron activation function. Comparing the results, in this paper, with the existing ones shows that we improve and generalize the results derived in the previous literature. Several examples are also given to illustrate the effectiveness and potential applications in image encryption.

Index Terms—Exponential stability, lag synchronization, switched neural networks.

I. INTRODUCTION

HYBRID systems have been investigated extensively with the rapid development of intelligent control. As a special case of hybrid systems, switched systems consist of a family subsystems, which are controlled by a switching rule. In reality, many systems can be modeled as switched systems, switched circuits, switched networks, and so on. Considerable attention has been drawn to the theoretical analysis of switched systems [1].

Meanwhile, synchronization of neural networks has attracted great attention due to its potential applications in many fields such as secure communications, biological systems, information science, image encryption,

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pseudorandom number generator, and adaptive dynamic programmer [2]–[14]. Synchronization phenomena including complete synchronization [15]–[17], generalized synchronization [18], phase synchronization [19], and lag synchronization [20] have been investigated. In connected electronic networks, time delays are unavoidable due to finite signal transmission times, switching speeds, and the complete synchronization of neural networks with time delays hard to implement effectively, but we can implement lag synchronization.

Several control methods have been proposed for the lag synchronization of delayed neural networks, such as periodically intermittent control in [21]–[23]. Exponential stability criteria are derived for the synchronization error systems with constant time delays in [21] and [22], however, these criteria are not applicable for systems with time-varying delays. Meanwhile, only asymptotical stability criteria are derived for synchronization error systems in [23]. Li and Bohnerb [24] investigated the exponential synchronization of chaotic neural networks via linear matrix inequality techniques. However, there are a few results on global exponential lag synchronization of switched neural networks.

Motivated by the above discussion, in this paper, we investigate the problem of globally exponential lag synchronization for a class of switched neural networks with time-varying delays. It is worth pointing out that, the proposed problem is nontrivial because of the difficulties such as the controller is designed via the neuron activation function.

The rest of this paper is organized as follows. In Section II, preliminaries are given. In Section III, a new model of synchronization error system is formulated within a unified framework. In Section IV, synchronization of switched neural networks is discussed by the controller based on the neuron activation function. Several sufficient conditions are derived to ensure the synchronization of switched neural networks. Analysis has been made on results in this paper and the previous ones. In Section V, two illustrative examples are discussed to demonstrate the effectiveness of the theoretical analysis. Finally, the conclusion is drawn in Section VI.

II. PRELIMINARIES

Denote $u = (u_1, \ldots, u_n)^T$, $|u|$ as the absolute-value vector, i.e., $|u| = (|u_1|, |u_2|, \ldots, |u_n|)^T$, $||x||_p$ as the p-norm of vector x, $1 \leq p < \infty$. $||x||_\infty = \max_{i \in \{1,2,\ldots,n\}} |x_i|$ is the

Control Gain

Response System

System States

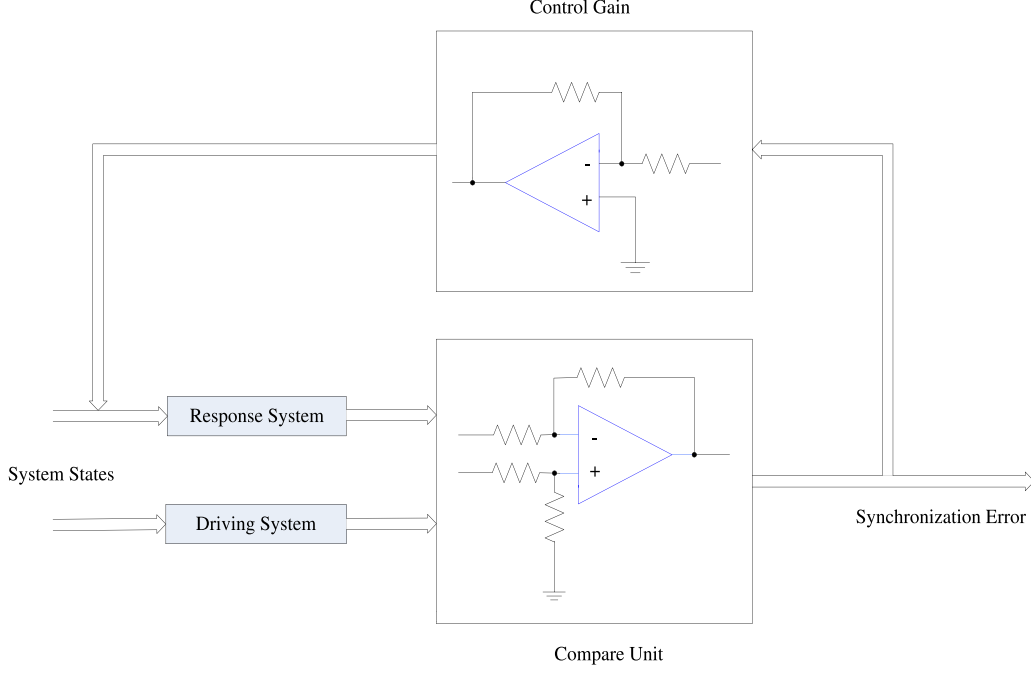Driving System

Synchronization Error

Compare Unit

Fig. 1.   Synchronization scheme of coupled memristive neural networks.

infinity norm of vector x. Denote $||D||_p$ as the p-norm of the matrix D. Denote $\mathcal{C}$ as the set of continuous functions.

A set of neural networks is considered as the individual subsystems of the switched neural network. The driving switched neural network is described as follows:

$$\dot{x}(t) = -C_l x(t) + A_l f(x(t)) + B_l f(x(t - \tau(t))) + I \quad (1)$$

where l is a switching signal taking its value in the finite set $\mathcal{I} = \{1, \dots, N\}$, which means that the matrices (A_l, B_l, C_l) are allowed to take values in the finite set $\{(A_1, B_1, C_1), \dots, (A_N, B_N, C_N)\}$. The parameters of system (1) are utilized to reflect the switched property of the electronic elements in neural networks, such as switched resistors and so on.

Throughout this paper, we assume that the switching rule l is known priori to the receiver and its instantaneous value is available in real time. The initial condition of system (1) is in the form of $x(t) = \phi(t) \in \mathcal{C}([-\chi, 0], \mathbb{R}^n)$, $\chi = \max\{\bar{\tau}\}$, where $\bar{\tau} = \max_{1 \le l \le N} \bar{\tau}_l$, $0 \le \tau_l(t) \le \bar{\tau}_l$.

Consider the following response system:

$$\dot{x}(t) = -C_l x(t) + A_l f(x(t)) + B_l f(x(t - \tau(t))) + I. \quad (2)$$

Define an indicator function $\Pi(t) = (\Pi_1(t), \dots, \Pi_N(t))^T$, where

$$\Pi_l(t) = \begin{cases} 1, & \text{when the switched system is described} \\ & \text{by the } l\text{th mode}(A_l, B_l, C_l) \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

with $l = 1, \dots, N$. Then, the driving switched neural network (1) can be represented by

$$\dot{x}(t) = \sum_{l=1}^{N} \Pi_l(t)(-C_l x(t) + A_l f(x(t))$$
$$+ B_l f(x(t - \tau(t))) + I). \quad (4)$$

It follows that $\sum_{l=1}^{N} \Pi_l(t) = 1$ under any switching rules. Assume the response system has the same switching law as the driving system

$$\dot{y}(t) = \sum_{l=1}^{N} \Pi_l(t)(-C_l y(t) + A_l f(y(t))$$
$$+ B_l f(y(t - \tau(t))) + I + u_l(t)) \quad (5)$$

where $u_l(t)(l = 1, \dots, N)$ are the controllers. The initial condition of system (5) is in the form of $y(t) = \varphi(t) \in \mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n)$.

The synchronization scheme of coupled switched neural networks can be presented as in Fig. 1, in which, we have the synchronization error state via the compare units and the control input.

In this paper, we assume the following.

A1: For $i \in \{1, 2, \dots, n\}$, the activation function f_i is Lipchitz continuous; and $\forall r_1, r_2 \in \mathbb{R}$, there exists real number ι_i such that

$$0 \le \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \le \iota_i.$$

A2: For $i \in \{1, 2, \dots, n\}$, $\tau_i(t)$ satisfies

$$0 \le \tau_i(t) \le \bar{\tau}_i, \quad \dot{\tau}_i(t) \le \mu_i < 1.$$

To derive sufficient conditions for the global exponential lag synchronization of system (4) with system (5), we will need the following lemmas.

Lemma 1 [25]: Given any real matrices X, Z, P of appropriate dimensions and a scalar $\varepsilon_0 > 0$, where $P > 0$, the following inequality holds:

$$X^T Z + Z^T X \le \varepsilon_0 X^T P X + \varepsilon_0^{-1} Z^T P^{-1} Z.$$

In particular, if X and Z are vectors, $X^T Z \leq 1/2$ $(X^T X + Z^T Z)$.

III. NEW MODEL FOR THE SYNCHRONIZATION ERROR SYSTEM

It is hard to obtain real-time inner states of the integrated and packed circuit, the output of this circuit can be utilized to measure such packed circuit. Therefore, we aim to design a controller of the circuit based on its output function, which is also called activation function in neuromorphic circuit, and to reach lag synchronization $[y(t) \rightarrow x(t-\xi)$ for some constant lag time $\xi > 0]$. The error system can be obtained as

$$
\dot{e}(t) = \sum_{l=1}^{N} \Pi_l(t)(-C_l e(t) + A_l \Phi(e(t))
$$
$$
+ B_l \Phi(e(t - \tau(t))) + u_l(t)) \qquad (6)
$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T$ is the lag-synchronization error, and

$$
e_i(t) = y_i(t) - x_i(t - \xi_i). \qquad (7)
$$

The neural activation functions with/without delays are

$$
\Phi(e(t)) = (\Phi_1(e_1(t)), \ldots, \Phi_n(e_n(t)))^T
$$
$$
= f(e(t) + x(t - \xi)) - f(x(t - \xi))
$$
$$
\Phi(e(t-\tau(t))) = (\Phi_1(e_1(t - \tau_1(t))), \ldots, \Phi_n(e_n(t - \tau_n(t))))^T
$$
$$
= f(e(t - \tau(t)) + x(t - \tau(t) - \xi))
$$
$$
- f(x(t - \tau(t) - \xi)).
$$

In the case of packed circuits, it is hard to measure the inner state of the circuits, the controller is dependent on the output of the systems. An output controller is designed in this paper as follows:

$$
u_l(t) = K_l \Phi(e(t)) \qquad (8)
$$

where $K_l = (k_{lij})_{n \times n}$ is a constant gain matrix to be determined to synchronize the drive and response systems, and $\Phi(e(t))$ is the output function without delays.

With controller (8), the error system (6) is transformed into

$$
\dot{e}(t) = \sum_{l=1}^{N} \Pi_l(t)(-C_l e(t) + \hat{A}_l \Phi(e(t)) + B_l \Phi(e(t - \tau(t))))
$$
$$
\qquad (9)
$$

where $\hat{A}_L = (\hat{a}_{lij})_{n \times n} = (a_{lij} + k_{lij})_{n \times n}$.

IV. MAIN RESULTS

To gain the main results, the following lemma is introduced. The initial condition of system (9) is in the form of $e(t) = \varphi(t) - \phi(t - \xi) \in \mathcal{C}([-\bar{\tau} + \xi, \xi], \mathbb{R}^n)$.

Lemma 2: For the lag-synchronization error system (9), if there exist a positive number λ and positive definite diagonal matrices $L = \text{diag}\{\iota_1, \iota_2, \ldots, \iota_n\}$, $F = \text{diag}(F_1, F_2, \ldots, F_n)$, $R = \text{diag}(r_1, r_2, \ldots, r_n)$, and

$$
\Theta_l = -2\lambda F L^{-1} - \daleth_l + R + 2||F||||B_l||_2 I < 0
$$

where $\lambda = \min_{l=1,2,\ldots,N}\{\lambda_{\min}\{C_l\}\}$, $\daleth_l = (\daleth_{lij})_{n \times n}$ with $\daleth_{lii} = -2F_i \hat{a}_{lii}$, and $\daleth_{lij} = -(F_i \hat{a}_{lij} + F_j \hat{a}_{lji})$, for $i \neq j$, such that $\dot{V}(t)|_{(9)} \leq -\beta\lambda/2 e^T(t)e(t)$, where

$$
V(t) = \frac{\beta}{2}e^T(t)e(t) + 2\sum_{i=1}^{n} F_i \int_0^{e_i(t)} \Phi_i(s)ds
$$
$$
+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \Phi_i^2(e_i(s))r_i ds. \qquad (10)
$$

Proof: Let $V_1(t) = 1/2 e^T(t)e(t)$, $V_2(t) = 2\sum_{i=1}^{n} F_i \int_0^{e_i(t)} \Phi_i(s)ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \Phi_i^2(e_i(s))r_i ds$. Then

$$
V(t) = \beta V_1(t) + V_2(t)
$$

where the scalar $\beta > 0$. Then

$$
\dot{V}(t) = \beta \dot{V}_1(t) + \dot{V}_2(t)
$$

where

$$
\dot{V}_1(t) = \sum_{l=1}^{N} \Pi_l(t)\big(-e^T(t)C_l e(t) + e^T(t)\hat{A}_l \Phi(e(t))
$$
$$
+ e^T(t)B_l \Phi(e(t - \tau(t)))\big)
$$

$$
\dot{V}_2(t) = 2\sum_{i=1}^{n} F_i \Phi_i^T \dot{e}_i(t)
$$
$$
+ \sum_{i=1}^{n} r_i \big(\Phi_i^T(e_i(t))\Phi_i(e_i(t)) - (1 - \dot{\tau}(t))\Phi_i^T
$$
$$
(e_i(t - \tau(t)))\Phi_i(e_i(t - \tau(t)))\big)
$$
$$
= 2\Phi^T(e(t))F\dot{e}(t) + \Phi^T(e(t))R\Phi(e(t))
$$
$$
- (1 - \dot{\tau}(t))\Phi^T(e(t - \tau(t)))R\Phi(e(t - \tau(t)))
$$
$$
= 2\sum_{l=1}^{N} \Pi_l(t)\big(-\Phi^T(e(t))FC_l e(t)
$$
$$
+ 2\Phi^T(e(t))F\hat{A}_l \Phi(e(t))
$$
$$
+ 2\Phi^T(e(t))F B_l \Phi(e(t - \tau(t)))\big)
$$
$$
+ \Phi^T(e(t))R\Phi(e(t))
$$
$$
- (1 - \dot{\tau}(t))\Phi^T(e(t - \tau(t)))R\Phi(e(t - \tau(t))).
$$

As $\dot{V}_1(t)$ can be presented as

$$
\dot{V}_1(t) = \sum_{l=1}^{N} \Pi_l(t)\left(-e^T(t)C_l e(t) + \left(e^T(t)\frac{(C_l)^{\frac{1}{2}}}{\sqrt{2}}\right)\right.
$$
$$
\times \sqrt{2}(C_l)^{\frac{-1}{2}}\hat{A}_l \Phi(e(t)) + \left(e^T(t)\frac{(C_l)^{\frac{1}{2}}}{\sqrt{2}}\right)
$$
$$
\left.\times \sqrt{2}(C_l)^{\frac{-1}{2}}B\Phi(e(t - \tau(t)))\right).
$$

By Lemma 1

$$
\begin{aligned}
&\dot{V}_1(t) \\
&\leq \frac{1}{2}\sum_{l=1}^{N}\Pi_l(t)\Big(-e^T C_l e(t) + 2\Phi^T(e(t))\hat{A}_l^T C_l^{-1}\hat{A}_l\Phi(e(t)) \\
&\qquad\qquad + 2\Phi^T(e(t-\tau(t)))B_l^T C_l^{-1}B_l\Phi(e(t-\tau(t)))\Big) \\
&\leq \frac{1}{2}\sum_{l=1}^{N}\Pi_l(t)\Big(-e^T(t)C_l e(t) + 2||C_l^{-1}||||\hat{A}_l||_2^2 \\
&\qquad\qquad \times \Phi^T(e(t))\Phi(e(t)) + 2||C_l^{-1}||||B_l||_2^2 \\
&\qquad\qquad \times \Phi^T(e(t-\tau(t)))\Phi(e(t-\tau(t)))\Big) \\
&\leq -\frac{\lambda}{2}\sum_{l=1}^{N}\Pi_l(t)e^T(t)e(t) \\
&\quad + \sum_{l=1}^{N}\Pi_l(t)\Big(||C_l^{-1}||||\hat{A}_l||_2^2\Phi^T(e(t))\Phi(e(t)) + ||C_l^{-1}||||B_l||_2^2 \\
&\qquad\qquad \times \Phi^T(e(t-\tau(t)))\Phi(e(t-\tau(t)))\Big) \\
&\leq -\frac{\lambda}{2}e^T(t)e(t) + M\Phi^T(e(t))\Phi(e(t)) \\
&\quad + M\Phi^T(e(t-\tau(t)))\Phi(e(t-\tau(t)))
\end{aligned}
$$

where

$$
M = \max_{l=1,\ldots,N}\left\{\lambda^{-1}||\hat{A}_l||_2^2, \lambda^{-1}||B_l||_2^2\right\} \geq 0.
$$

As $e_i(t)\Phi_i(t) \geq \iota_i^{-1}(\Phi_i(t))^2$, we have

$$
\begin{aligned}
&-2\Phi^T(e(t))F\sum_{l=1}^{N}\Pi_l(t)C_l e(t) \\
&\leq -2\Phi^T(e(t))F\sum_{l=1}^{N}\Pi_l(t)C_l L^{-1}\Phi(e(t)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\dot{V}_2(t) &\leq 2\Phi^T(e(t))F \\
&\quad \times \sum_{l=1}^{N}\Pi_l(t)\Big(-C_l L^{-1}\Phi(e(t)) \\
&\qquad\qquad + 2\Phi^T(e(t))F\hat{A}_l\Phi(e(t)) \\
&\qquad\qquad + 2\Phi^T(e(t))F B_l\Phi(e(t-\tau(t)))\Big) \\
&\quad + \Phi^T(e(t))R\Phi(e(t)) \\
&\quad - (1-\mu)\Phi^T(e(t-\tau(t)))R\Phi(e(t-\tau(t))) \\
&\leq -\Phi^T(e(t))(2\lambda F L^{-1})\Phi(e(t)) \\
&\quad - \sum_{l=1}^{N}\Pi_l(t)\Big(\Phi^T(e(t))\daleth_l\Phi(e(t)) - 2||F||||B_l||_2 \\
&\qquad\qquad \times \Phi^T(e(t))||||\Phi(e(t-\tau(t)))||\Big) \\
&\quad + \Phi^T(e(t))R\Phi(e(t)) \\
&\quad - (1-\mu)\Phi^T(e(t-\tau(t)))R\Phi(e(t-\tau(t))). \quad (11)
\end{aligned}
$$

Furthermore, we can obtain

$$
\begin{aligned}
&2||F||||B_l||_2||\Phi^T(e(t))||||\Phi(e(t-\tau(t)))|| \\
&\qquad \leq ||F||||B_l||_2\Big(\Phi^T(e(t))\Phi(e(t)) \\
&\qquad\qquad + \Phi^T(e(t-\tau(t)))\Phi(e(t-\tau(t)))\Big). \quad (12)
\end{aligned}
$$

From (11) and (12), we have

$$
\begin{aligned}
&\dot{V}_2(t) \\
&\leq \sum_{l=1}^{N}\Pi_l(t)\Big(\Phi^T(e(t))\big(-2\lambda F L^{-1} - \daleth_l + R + ||F||||B_l||_2\big) \\
&\qquad\qquad \times \Phi(e(t)) - \Phi^T(e(t-\tau(t)))\big((1-\mu)R \\
&\qquad\qquad - ||F||||B_l||_2\big)\Phi(e(t-\tau(t)))\Big).
\end{aligned}
$$

As $\Theta_l < 0$, there exists $\nu > 0$ such that $\Theta_l + 2\nu I < 0$. Define

$$
R = \max_{l=1,\ldots,N}\left\{\frac{1}{1-\mu}\big(||F||||B_l||_2 + \nu\big)\right\}I > 0.
$$

Then, we can obtain

$$
\dot{V}_2(t) \leq -\nu\Phi^T(e(t))\Phi(e(t)) - \nu\Phi^T(e(t-\tau(t)))\Phi(e(t-\tau(t))).
$$

Let

$$
\beta = \begin{cases} \frac{\nu}{\varpi M}, & M > 0 \\ 1, & M = 0 \end{cases}
$$

where $\varpi \geq 1$. Then, $\dot{V}(t) \leq -\beta\lambda/2 e^T(t)e(t)$.

Moreover, we can obtain the following theorem.

Theorem 1: Assume that the conditions in Lemma 2 hold, then the driving system (4) is globally exponentially lag synchronized with the response system (5).

Proof: Let $\lambda = \min_{l=1,\ldots,N}\{\lambda_{\min}\{C_l\}\}$. As $V(t)$ defined in Lemma 2 is a positive definite and radially unbounded Lyapunov functional. We can choose a positive number $\epsilon > 0$ to satisfy

$$
\epsilon\beta - \lambda\beta + 2\epsilon||LF|| + 2\epsilon\bar{\tau}e^{\epsilon\bar{\tau}}||L^2 R|| < 0. \quad (13)
$$

By Lemma 2, we can obtain

$$
\begin{aligned}
&\frac{d}{dt}\{e^{\epsilon t}V(t)\} \\
&= e^{\epsilon t}(\epsilon V(t) + \dot{V}(t)) \\
&\leq \sum_{l=1}^{N}\Pi_l(t)e^{\epsilon t}\bigg(\epsilon\bigg(\frac{\beta}{2}e^T(t)e(t) + 2\sum_{i=1}^{n}F_i\int_0^{e_i(t)}\Phi_i(s)ds \\
&\qquad\qquad + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\Phi_i^2(e_i(s))r_i ds\bigg) - \frac{\beta\lambda}{2}e^T(t)e(t)\bigg) \\
&\leq \frac{1}{2}\sum_{l=1}^{N}\Pi_l(t)e^{\epsilon t}\bigg(\epsilon\beta e^T(t)e(t) - \beta\lambda e^T(t)e(t) \\
&\qquad\qquad + 4\epsilon\sum_{i=1}^{n}F_i\int_0^{e_i(t)}\Phi_i(s)ds\bigg) \\
&\quad + \epsilon e^{\epsilon t}\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\Phi_i^2(e_i(s))r_i ds.
\end{aligned}
$$

Since

$$\sum_{i=1}^{n} F_i \int_0^{e_i(t)} \Phi_i(s) ds$$

$$\leq \sum_{i=1}^{n} F_i \int_0^{e_i(t)} \iota_i s ds \leq \frac{1}{2} e^T(t) L F e(t)$$

then

$$\frac{d}{dt}(e^{\epsilon t} V(t))$$

$$\leq \frac{1}{2} \sum_{l=1}^{N} \Pi_l(t) e^{\epsilon t} \Bigg(\epsilon \beta e^T(t) e(t) - \beta \lambda e^T(t) e(t) + 2\epsilon e^T(t) L F e(t)$$

$$+ \epsilon e^{\epsilon t} \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \Phi_i^2(e_i(s)) r_i ds \Bigg)$$

$$\leq \frac{1}{2} e^{\epsilon t} \Big(\epsilon \beta - \lambda \beta + 2\epsilon ||LF|| \Big) e^T(t) e(t)$$

$$+ \epsilon e^{\epsilon t} \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \Phi_i^2(e_i(s)) r_i ds. \tag{14}$$

Changing the integral of the second term on the right-hand side of (14)

$$\epsilon \int_0^s e^{\epsilon t} \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \Phi_i^2(e_i(\varsigma)) r_i d\varsigma dt$$

$$= \epsilon \sum_{i=1}^{n} \int_0^s e^{\epsilon t} \int_{t-\tau_i(t)}^{t} \Phi_i^2(\varsigma) r_i d\varsigma dt$$

$$\leq \epsilon \sum_{i=1}^{n} \int_{-\bar{\tau}}^{s} \int_{\max\{\varsigma,0\}}^{\min\{\varsigma+\bar{\tau},s\}} e^{\epsilon t} dt \Phi_i^2(e_i(\varsigma)) r_i d\varsigma$$

$$\leq \epsilon \sum_{i=1}^{n} \int_{-\bar{\tau}}^{s} \left(\int_{\varsigma}^{\varsigma+\bar{\tau}} e^{\epsilon t} dt \right) \Phi_i^2(e_i(\varsigma)) r_i d\varsigma$$

$$\leq \sum_{i=1}^{n} \int_{-\bar{\tau}}^{s} \bar{\tau} e^{\epsilon(\varsigma+\bar{\tau})} \Phi_i^2(e_i(\varsigma)) r_i d\varsigma$$

$$\leq \sum_{i=1}^{n} \int_{-\bar{\tau}}^{s} \bar{\tau} e^{\epsilon(\varsigma+\bar{\tau})} e_i^2(\varsigma) \iota_i^2 r_i d\varsigma$$

$$\leq \bar{\tau} e^{\epsilon \bar{\tau}} ||L^2 R|| \int_{-\bar{\tau}}^{s} e^{\epsilon \varsigma} e^T(\varsigma) e(\varsigma) d\varsigma$$

$$\leq \bar{\tau} e^{\epsilon \bar{\tau}} ||L^2 R|| \Bigg(\int_{-\bar{\tau}}^{0} e^{\epsilon \varsigma} e^T(\varsigma) e(\varsigma) d\varsigma$$

$$+ \int_0^s e^{\epsilon \varsigma} e^T(\varsigma) e(\varsigma) d\varsigma \Bigg). \tag{15}$$

By (13)–(15), we have

$$e^{\epsilon s} V(s) - V(0)$$

$$\leq \frac{1}{2} \Big(\epsilon \beta - \lambda \beta + 2\epsilon ||LF|| + 2\epsilon \bar{\tau} e^{\epsilon \bar{\tau}} ||L^2 R|| \Big)$$

$$\times \int_0^s e^{\epsilon t} e^T(t) e(t) dt + \epsilon \bar{\tau} e^{\epsilon \bar{\tau}} ||L^2 R|| \int_{-\bar{\tau}}^{0} e^{\epsilon t} e^T(t) e(t) dt$$

$$\leq \left(\epsilon \bar{\tau} ||L^2 R|| \int_{-\bar{\tau}}^{0} e^{\epsilon t} dt \right) ||\psi||^2 \equiv H_1 ||\psi||^2.$$

Thus

$$V(t) \leq (V(0) + H_1 ||\psi||^2) e^{-\epsilon t} \quad \forall t > 0 \tag{16}$$

where

$$V(0) = \frac{\beta}{2} e^T(0) e(0) + 2 \sum_{i=1}^{n} q_i \int_0^{e_i(0)} \Phi_i(s) ds$$

$$+ \sum_{i=1}^{n} \int_{-\tau_i(t)}^{0} \Phi_i^2(e_i(s)) r_i ds$$

$$\leq \frac{1}{2} (\beta + 2||FL|| + 2\bar{\tau}||L^2 R||) ||\psi||^2$$

$$\equiv H_2 ||\psi||^2.$$

By (10) and (16), we have

$$\frac{\beta}{2} e^T(t) e(t) \leq V(t) \leq (H_1 + H_2) ||\psi||^2 e^{-\epsilon t} \quad \forall t > 0.$$

Thus, we have

$$||e(t)|| \leq \sqrt{\frac{2}{\beta} (H_1 + H_2)} ||\psi|| e^{-\frac{\epsilon}{2} t} \tag{17}$$

which implies the drive system (4) is globally exponentially lag synchronized with the response system (5). This completes the proof.

By Theorem 1, we can obtain the following corollary.

Corollary 1: The drive system (4) is globally exponentially lag synchronized with the response system (5), if there exist positive definite matrices $L = \text{diag}\{\iota_1, \iota_2, \ldots, \iota_n\}$, $F = \text{diag}\{F_1, F_2, \ldots, F_n\}$, $R = \text{diag}\{r_1, r_2, \ldots, r_n\}$ and a positive definite symmetric matrix M, such that

$$\hat{\Pi}_l = -2\lambda F L^{-1} - \daleth_l + R + M + ||F||^2 ||M^{-1}|| ||B_l||_2^2 I < 0 \tag{18}$$

where $\lambda = \min\limits_{l=1,\ldots,N} \{D_l\}$, $\daleth_l = (\daleth_{lij})_{n \times n}$ with $\daleth_{lii} = -2F_i \hat{a}_{lii}$, and $\daleth_{lij} = -(F_i \hat{a}_{lij} + F_j \hat{a}_{lji})$, for $i \neq j$.

Proof: By Lemma 1

$$\frac{1}{||F|| ||B_l||_2} M + ||F|| ||B_l||_2 M^{-1} \geq 2I$$

then

$$M + ||F||^2 ||M^{-1}|| ||B_l||_2^2 I \geq M + ||F||^2 ||B_l||_2^2 M^{-1}$$

$$\geq 2||F|| ||B_l||_2^2 I.$$

Thus

$$-2\lambda F L^{-1} - \daleth + R + 2||F|| ||B_l||_2 I$$

$$\leq -2\lambda F L^{-1} - \daleth_l + R + M + ||F||^2 ||M^{-1}|| ||B_l||_2^2 I$$

$$< 0. \tag{19}$$

This proof is complete.

Let $F = M = I$ in Corollary 1, we obtain the following corollary.

Corollary 2: The drive system (4) is globally exponentially lag synchronized with the response system (5), if $\daleth_l = (\daleth_{lij})_{n \times n}$ is positive definite, and $||B_l||_2 \leq (2\pi - 1)^{1/2}$, where $\pi = \min\limits_{1 \leq i \leq n} (\lambda / \iota_i)$, and $\daleth_{lii} = -2F_i \hat{a}_{lii}$, $\daleth_{lij} = -(F_i \hat{a}_{lij} + F_j \hat{a}_{lji})$, for $i \neq j$.
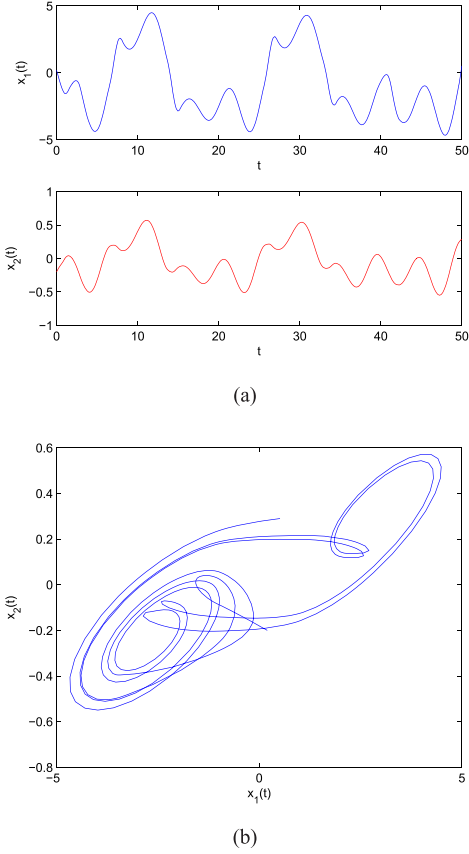
(a)

(b)

Fig. 2. Transient behavior of the switched system (4) with the initial value $[0.2 - 0.2]^T$.

(a)

(b)

Fig. 3. (a) and (b) State trajectories of driving system (4) with the initial value $[0.2 - 0.2]^T$ and slave system (5) with the initial value $[0.5 - 0.5]^T$, when lag time $\xi = 1.5$.

Some criteria about globally exponential stability can be derived for the switched lag-synchronization error system (9) with the controller of the neural activation function [26]–[31]. The globally asymptotical stability conditions for for the switched lag-synchronization error system in [26, eq. (9)] and [27, eq. (9)] are presented as follows.

Corollary 3: The drive system (4) is globally exponentially lag synchronized with the response system (5), if $\daleth_l = (\daleth_{lij})_{n \times n}$ is positive definite, and $||B_l||_2 \le \pi$, where $\pi = \min_{1 \le i \le n}(\lambda/\iota_i)$, and $\daleth_{lii} = -2F_i \hat{a}_{lii}$, $\daleth_{lij} = -(F_i \hat{a}_{lij} + F_j \hat{a}_{lji})$, for $i \ne j$.

Corollary 4: The synchronization error system (9) is globally asymptotically stable if there exist a positive definite diagonal matrices $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\}$, $R = \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$ such that

$$\hat{\Pi}_l = -2\omega I - \daleth_l + R + 2||F||^2||B_l||_2^2 I < 0 \qquad (20)$$

where $\omega = \min_{1 \le i \le n}(F_i \lambda/\iota_i)$, $\daleth_l = (\daleth_{lij})_{n \times n}$ with $\daleth_{lii} = -2F_i \hat{a}_{lii}$, $\daleth_{lij} = -(F_i \hat{a}_{lij} + F_j \hat{a}_{lji})$, for $i \ne j$.

The following inequality holds:

$$\begin{aligned}\tilde{\Pi}_l &= -2\omega I - \daleth_l + R + 2||F||^2||B_l||_2^2 I \\ &\ge -2\lambda F L^{-1} - \daleth_l + R + ||F||^2||B_l||_2^2 I \qquad (21)\end{aligned}$$

where $\omega = \min_{1 \le i \le n}(F_i \lambda/\iota_i)$. This means the conditions about stability criteria derived for switched neural networks in [26] and [27] are more restrictive than those in Theorem 1. Meanwhile, these results can only guarantee the globally
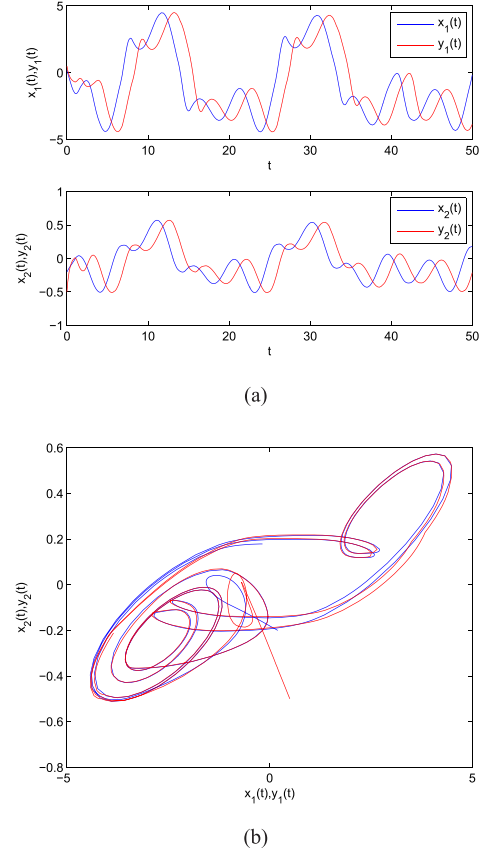
asymptotical stability of the switched lag synchronization error system (9).

V. ILLUSTRATIVE EXAMPLES

To show the effectiveness of the obtained results, two illustrative examples are presented as follows.

Example 1: Consider a switched system (4) with

$$A = \begin{bmatrix} 1.8 & 10 \\ 0.1 & 1.8 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & 0.1 \\ 0.1 & -1.5 \end{bmatrix}$$

$$f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|)$$

$$\tau_i(t) = 0.97, i = 1, 2.$$

If $x_1(t) \le 0$

$$C = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.2 \end{bmatrix}.$$

Else

$$C = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.0 \end{bmatrix}.$$

The initial values of driving system (4) is set to be $[0.2 - 0.2]^T$. In addition, the dynamical behaviors of this system is shown as in Fig. 2, which is chaotic and can be used in secure communications.

As

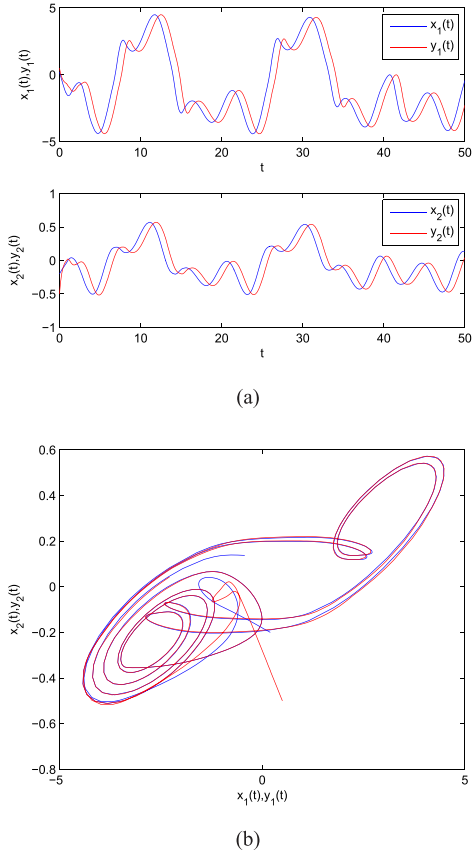$$\lambda = \min_{l=1,2}\{\lambda_{\min}\{C_l\}\} = 1.$$

(a)

(b)

Fig. 4. (a) and (b) State trajectories of driving system (4) with the initial value $[0.2 - 0.2]^T$ and slave system (5) with the initial value $[0.5 - 0.5]^T$, when lag time $\xi = 0.8$.

Obviously, there exists a positive definite diagonal matrix $F = \text{diag}\{0.5, 0.5\}$, $\nu = 0.1$, $R = 1/(1 - \mu)\{(\|F\| \|B\|_2 + \nu)I\}$, such that

$$\daleth_l = -\hat{A}_l$$

to make

$$\Theta_l = -2\lambda F L^{-1} - \daleth_l + R + \|F\|\|B_l\|_2 I$$
$$= K_l + \begin{bmatrix} 3.3 & 11 \\ 1 & 3.3 \end{bmatrix} < 0$$

therefore

$$K_l < \begin{bmatrix} -3.3 & -11 \\ -1 & -3.3 \end{bmatrix}$$

can make the lag-synchronization error system (9) globally exponentially stable. However, $\|B\|_2 = 1.6 > \varpi$, which means that the results in [26] cannot be used for the lag-synchronization error system (9). To simulate the obtained result, let

$$K_l = \begin{bmatrix} -3.6 & -11 \\ -1 & -3.6 \end{bmatrix}.$$

Set the initial states of slave system (9) is $[0.5 - 0.5]^T$. The state trajectories of driving system and slave system are presented in Figs. 3–5 with the lag times $\xi = 1.5$, $\xi = 0.8$, and $\xi = 0$, respectively, which illustrate the effectiveness of the obtained results.

Algorithm 1 Transformation of Chaotic Signals

Initialization:
　　Set $i \leftarrow 1$; $j \leftarrow 1$; $k \leftarrow 1$;
1: **while** $i \neq m$ **do**
2: 　**while** $j \neq n$ **do**
3: 　　$z_1(i, j) \leftarrow 1000 * \Big(z_1(k) - \text{floor}(z_1(k))\Big)$;
　　　$z_1(i, j) \leftarrow \text{mod}(z_1(i, j), 256)$;
　　　$z_2(i, j) \leftarrow 1000 * \Big(z_2(k) - \text{floor}(z_2(k))\Big)$;
　　　$z_2(i, j) \leftarrow \text{mod}(z_2(i, j), 256)$;
　　　$z_3(i, j) \leftarrow 1000 * \Big(z_3(k) - \text{floor}(z_3(k))\Big)$;
　　　$z_3(i, j) \leftarrow \text{mod}(z_3(i, j), 256)$;
　　　$k \leftarrow k + 1$;
　　　$j \leftarrow j + 1$;
4: 　**end while**
　　$i \leftarrow i + 1$;
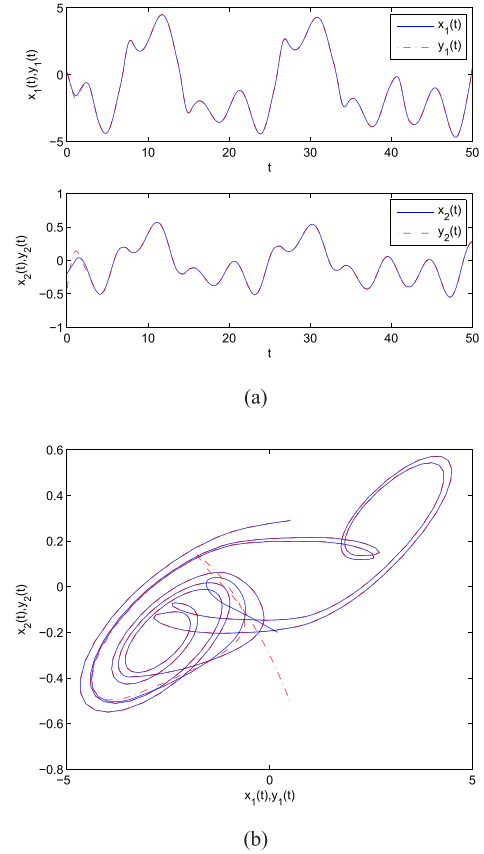5: **end while**

(a)

(b)

Fig. 5. (a) and (b) State trajectories of driving system (4) with the initial value $[0.2 - 0.2]^T$ and slave system (5) with the initial value $[0.5 - 0.5]^T$, when lag time $\xi = 0$.

Example 2: Based on Example 1, the obtained results can be applied in the field of digital signal processing, and the algorithm is presented for a color picture F with a size $m \times n \times 3$, as follows.

1) Separating color image F into three gray ones with red, green, and blue, respectively, and via sort function to rearrange the pixels in each gray image,
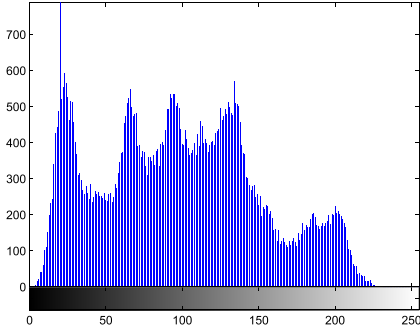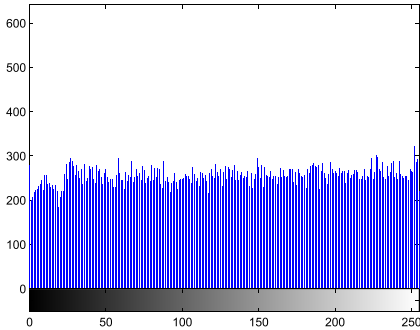
(a)
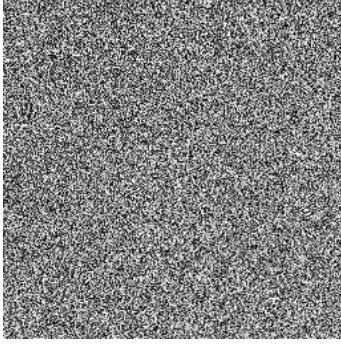
(b)

Fig. 6.   (a) Original image. (b) Encrypted image.

(a)

(b)

Fig. 7.   (a) Histogram of the original gray image. (b) Histogram of the encrypted gray image.
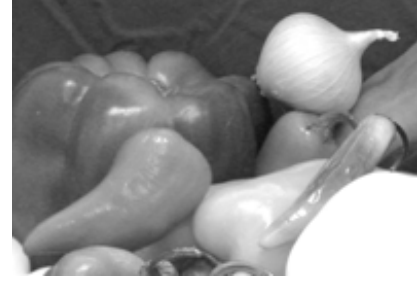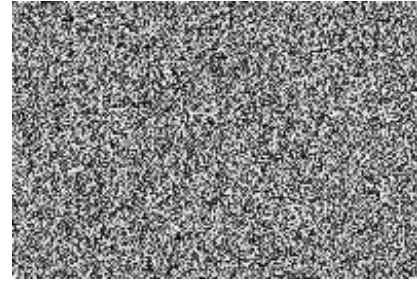

(a)

(b)

Fig. 8.   (a) Original image. (b) Encrypted image.

(a)

(b)

Fig. 9.   (a) Gray image of the red channel of the original image. (b) Same plots for the encrypted image.

therefore, three new ordered pixel series are obtained as $R(i, j), G(i, j), B(i, j), i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.

2) Through driving system (4), two groups of time-series chaotic signals obtained as $z_l(i, j) = x_l(k)$,

$k \in \{1, \ldots, mn\}, l \in \{1, 2\}$. As there are three gray images needed to be encrypted, the third chaotic signal can be set as $z_3(k) = 0.5(x_1(k) + x_2(k))$. After certain transformation, the chaotic signals can be presented as in Algorithm 1.

3) Based on the proceeded chaotic signals and gray images, the encrypted gray images can be obtained as the
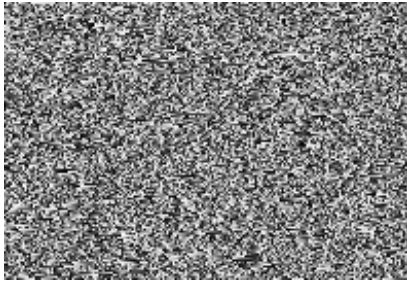
(a)
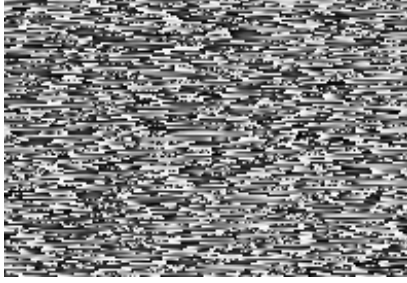
(b)

Fig. 10. (a) Gray image of the green channel of the original image. (b) Same plots for the encrypted image.

(a)

(b)

Fig. 11. (a) Gray image of the blue channel of the original image. (b) Same plots for the encrypted image.

4) Reorganizing $R(i, j), G(i, j),$ and $B(i, j)$, we can obtain the encrypted color image.

As the decryption process is the same as the encryption process, is omitted, we notice that as the existence of lag, the decryption chaotic signal should be employed after ξ/h signals, where h is the length of the iterative step. In addition, two simulations about gray and color image encryption have been provided in Figs. 6–11, which illustrate the application potential of the lag synchronization of switched neural networks in signal encryption. As the existence of the lag between the coupled switched neural networks, decryption chaotic signal should be adopted after certain iterative steps.

VI. CONCLUSION

As the applications of switched neural networks become more and more popular, lag synchronization of such networks becomes necessary. On the other hand, after the switched neural networks have been packed, it is very hard to measure their inner states. Therefore, the authors investigate the problem of global exponential lag synchronization of a class of switched neural networks with time-varying delays via the neural activation controller. Two numerical examples were provided to demonstrate the effectiveness and improvement of the obtained results in this paper.

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following operation:

$$R(i, j) \leftarrow \mod(R(i, j) * 1000, 256) \oplus z_1(i, j)$$
$$G(i, j) \leftarrow \mod(G(i, j) * 1000, 256) \oplus z_2(i, j)$$
$$B(i, j) \leftarrow \mod(G(i, j) * 1000, 256) \oplus z_3(i, j).$$

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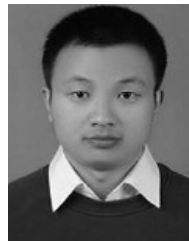
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