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On the Einstein-Weyl and conformal self-duality equations

M. Dunajski,[1,2,a)] E. V. Ferapontov,[3,a)] and B. Kruglikov[4,a)]
[1]*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[2]*Department of Computer Science, Faculty of Physics and Applied Informatics, University of Lodz, Pomorska 149/153, 90-236 Lodz, Poland*
[3]*Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom*
[4]*Institute of Mathematics and Statistics, NT-Faculty, University of Tromsø, Tromsø 90-37, Norway*

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The equations governing anti-self-dual and Einstein-Weyl conformal geometries can be regarded as "master dispersionless systems" in four and three dimensions, respectively. Their integrability by twistor methods has been established by Penrose and Hitchin. In this note, we present, in specially adapted coordinate systems, explicit forms of the corresponding equations and their Lax pairs. In particular, we demonstrate that any Lorentzian Einstein-Weyl structure is locally given by a solution to the Manakov-Santini system, and we find a system of two coupled third-order scalar partial differential equations for a general anti-self-dual conformal structure in neutral signature. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4927251]

I. INTRODUCTION

There exist two key "integrable" conformal geometries, namely, Einstein-Weyl (EW) geometry in three dimensions and anti-self-dual (ASD) geometry in four dimensions (see Ref. 7 for a comprehensive overview). In spite of their fundamental role in twistor theory, general relativity, and the theory of dispersionless integrable systems, these geometries remain largely unknown to the integrable system community due to a lack of explicit coordinate formulas for the underlying PDEs and Lax pairs. The aim of this paper is to present them in their simplest possible forms, in specially adapted coordinates.

In Section II, we discuss EW structures in three dimensions. Recall that an EW geometry on a three-dimensional manifold M^3 consists of a conformal structure $[g]$ and a symmetric connection $\mathbb{D}$ compatible with $[g]$ in the sense that, for any $g \in [g]$,

$$\mathbb{D}g = \omega \otimes g$$

for some covector ω, and such that the trace-free part of the symmetrized Ricci tensor of $\mathbb{D}$ vanishes. Using Cartan's approach relating EW structures to a special class of third-order ODEs, we shall demonstrate the following.

Theorem 1. *There exists a local coordinate system (x, y, t) on M^3 such that any Lorentzian Einstein–Weyl structure is locally of the form*

$$g = -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt, \quad \omega = -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt, \tag{1}$$

where the functions u and v on M^3 satisfy a coupled system of second-order PDEs,

$$P(u) + u_x^2 = 0, \quad P(v) = 0, \tag{2}$$

[a)]Electronic addresses: M.Dunajski@damtp.cam.ac.uk; E.V.Ferapontov@lboro.ac.uk; and boris.kruglikov@uit.no *where*

$$P = \partial_x\partial_t - \partial_y^2 + (u - v_y)\partial_x{}^2 + v_x\partial_x\partial_y.$$

System (2) is known as the Manakov-Santini system and was originally derived in Ref. 30 as a two-component generalization of the dispersionless Kadomtsev-Petviashvili (KP) equation. It was shown in Ref. 16 that any solution to (2) gives rise to an EW structure of form (1), but the question whether all EW structures arise in that way has remained open. System (2) possesses the Lax representation $[X_1, X_2] = 0$ where

$$X_1 = \partial_y - (\lambda + v_x)\partial_x - u_x\partial_\lambda, \quad X_2 = \partial_t - (\lambda^2 + v_x\lambda - u + v_y)\partial_x - (u_x\lambda + u_y)\partial_\lambda$$

are vector fields on the correspondence space $M^3 \times \mathbb{R}P^1$, where $\lambda \in \mathbb{R}P^1$. Projecting integral surfaces of the distribution spanned by X_1, X_2 from $M \times \mathbb{R}P^1$ to M^3 yields a two-parameter family of surfaces in M^3 which are null with respect to the conformal structure $[g]$ and totally geodesic in the Weyl connection $\mathbb{D}$ (the existence of such surfaces is equivalent to the EW property).[9] System (2) consists of 2 second-order PDEs for 2 functions of 3 independent variables, and its general solution in real-analytic category depends on 4 arbitrary functions of 2 variables: this confirms Cartan's count.[9]

The relation between EW geometry and dispersionless integrable systems in three dimensions has been known since.[38,17,16] It was observed recently in Ref. 21 that the dispersionless integrability of various classes of second-order PDEs is equivalent to the EW property of conformal structures defined by their principal symbols. Moreover, in many examples, the covector ω is expressed in terms of $g \in [g]$ by the universal explicit formula

$$\omega_k = 2g_{kj}\mathcal{D}_{x^s}(g^{js}) + \mathcal{D}_{x^k}(\ln\det g_{ij}). \tag{3}$$

Here, $\mathcal{D}_{x^s}$ denotes total derivative with respect to x^s and $(x^1, x^2, x^3) = (x, y, t)$. In three dimensions, this formula is invariant under the transformation,

$$g \to \varphi^2 g, \quad \omega \to \omega + 2d\ln\varphi, \quad \text{where} \quad \varphi : M^3 \to \mathbb{R}^+, \tag{4}$$

that keeps the Einstein-Weyl equations invariant. The Manakov-Santini system fits into this framework: the covector in (1) is given by formula (3), and the principal symbol of system (2) equals P^2 (it is doubly degenerate) where P, viewed as a symmetric bivector, gives rise to the EW metric g given by (1).

In Section III, we study ASD conformal structures in four dimensions. Recall that a conformal structure $[g]$ is called anti-self-dual if the self-dual (SD) part of the Weyl tensor of any $g \in [g]$ vanishes: $W_+ = \frac{1}{2}(W + *W) = 0$. We shall establish the following.

Theorem 2. *There exist local coordinates (w, z, x, y) such that any ASD conformal structure in signature $(2,2)$ is locally represented by a metric,*

$$g = dwdx + dzdy + F_y dw^2 - (F_x + G_y)dwdz + G_x dz^2, \tag{5}$$

where the functions $F, G : M^4 \to \mathbb{R}$ satisfy a coupled system of third-order PDEs,

$$\partial_x(Q(F)) - \partial_y(Q(G)) = 0, \quad (\partial_w - F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x - G_x\partial_y)Q(F) = 0, \tag{6}$$

where

$$Q = \partial_w\partial_x + \partial_z\partial_y - F_y\partial_x{}^2 - G_x\partial_y{}^2 + (F_x + G_y)\partial_x\partial_y.$$

System (6) arises as $[X_1, X_2] = 0$ from the dispersionless Lax pair,

$$X_1 = \partial_w - F_y\partial_x + G_y\partial_y + \lambda\partial_y + Q(F)\partial_\lambda, \quad X_2 = \partial_z + F_x\partial_x - G_x\partial_y - \lambda\partial_x - Q(G)\partial_\lambda. \tag{7}$$

Projecting integral surfaces of the distribution spanned by X_1, X_2 in the correspondence space $M^4 \times \mathbb{R}P^1$ to M^4 gives a 3-parameter family of totally null surfaces with self-dual tangent bi-vector. These are the α-surfaces of the corresponding conformal structure $[g]$. The existence of such surfaces is equivalent to the ASD property.[33] System (6) consists of 2 third-order PDEs for 2 functions of 4 independent variables, and its general solution depends on 6 arbitrary functions of 3 variables:

this agrees with the count of Ref. 24 based on the Cartan-Kähler theory. Lax pair (7) has appeared in Ref. 1, where the Riemann–Hilbert problem associated to (7) has been formulated. The fact that the system of 3 second order PDEs derived in Refs. 1 and 2 leads to system (6) has been recently pointed out in Ref. 40.

ASD equations and their reductions provide a number of key examples of dispersionless integrable systems in four dimensions.[34,15,19,39] It was conjectured in Ref. 21 that the dispersionless integrability of some four-dimensional PDEs is equivalent to the requirement that the principal symbol of the equation defines a conformal structure that must be ASD on every solution. In Ref. 21, this was demonstrated to be the case for integrable symplectic Monge-Ampère equations.[14] The example of ASD equations fits into this scheme: the principal symbol of system (6) equals Q^3 (it is triply degenerate), and the symmetric bivector Q gives rise to the ASD metric g given by (5).

Reality conditions. If all coordinates and functions in Theorems 1 and 2 are assumed to be real, then the corresponding conformal structures in three and four dimensions have Lorentzian or $(2,2)$ (also called neutral or Kleinian) signatures, respectively. Alternatively, one can assume real analyticity and work in the complexified settings where all structures are assumed to be holomorphic. We shall make this additional assumption whenever we rely on the Cauchy–Kovalevskaya theorem to assert that a general solution depends on m functions of n variables.

II. EINSTEIN-WEYL GEOMETRY

The twistor integrability of EW equations was established in Ref. 25. It was demonstrated in Ref. 17 that EW equations possess a Lax pair given by two vector fields that form an integrable distribution and may contain derivatives with respect to the spectral parameter. Integral manifolds of this distribution provide the 2-parameter family of null totally geodesic surfaces, and as shown by Cartan,[9] the EW property is equivalent to the existence of such family. However, the explicit coordinate form of the Lax pair has not been exhibited in the general case. Below, we list various forms of EW equations, as well as their Lax pairs, in specially adapted coordinates.

A. Einstein-Weyl equations in Cartan's approach

Our proof of Theorem 1 builds on Cartan's approach to Einstein-Weyl geometry via special third-order ordinary differential equations (ODEs).[10] We shall briefly review it following the paper of Tod.[37] Consider an equivalence class of third-order ODEs,

$$Y''' = F(X,Y,Y',Y''), \tag{8}$$

modulo point transformations $(X,Y) \to (\bar{X}(X,Y),\bar{Y}(X,Y))$. Here, $' = d/dX$. Let the general solution of (8) be of the form

$$Y = Z(X,x^j), \tag{9}$$

where x^j are coordinates on the three-dimensional solution space M^3. The necessary and sufficient conditions for the solution space M^3 to carry an Einstein–Weyl structure such that the 2-parameter family of surfaces in M^3 corresponding to fixing (X,Y) in (9) is null and totally geodesic are given by the vanishing of the Wunshmann and Cartan invariants W and C. These invariants are given by

$$W = \tfrac{1}{6}\mathcal{D}^2F_Q - \tfrac{1}{3}F_Q\mathcal{D}F_Q - \tfrac{1}{2}\mathcal{D}F_P + \tfrac{2}{27}F_Q^3 + \tfrac{1}{3}F_QF_P + F_Y,$$
$$C = (\tfrac{1}{3}\mathcal{D}F_Q - \tfrac{1}{9}F_Q^2 - F_P)F_{QQ} + \tfrac{2}{3}F_QF_{QP} - 2F_{QY} + F_{PP} + 2W_Q,$$

where $\mathcal{D} = \partial_X + P\partial_Y + Q\partial_P + F\partial_Q$ is the total derivative.

The above W is actually a relative contact invariant, while C is a relative point invariant (so their vanishing is an invariant condition for respective pseudogroups).

Following the approach of Tod[37] (see Refs. 31 and 29 for other approaches), the conformal structure and the covector are given by

$$\begin{aligned} g &= 2\,dY\,dQ - \tfrac{2}{3}F_Q\,dY\,dP + (\tfrac{1}{3}\mathcal{D}F_Q - \tfrac{2}{9}F_Q^2 - F_P)\,dY^2 - dP^2,\\ \omega &= \tfrac{2}{3}(F_{QP} - \mathcal{D}F_{QQ})\,dY + \tfrac{2}{3}F_{QQ}dP, \end{aligned} \tag{10}$$

where in these expressions X is fixed. Both g and ω depend on X explicitly, but a change in X corresponds to a gauge transformation of form (4). Thus, as long as $W = C = 0$, the resulting Einstein-Weyl structure is independent of X.

In this approach, the EW equations, $W = C = 0$, constitute an overdetermined system of two PDEs for a scalar function F of four variables. One can show that this system is compatible (formally integrable), which follows from the vanishing of the Mayer bracket $[W, C] = 0$.[27] In other words, this system of third- and second-order PDEs is in involution (after three prolongations). The characteristic variety is a complete intersection, so the general solution is parametrized by $6 = 3 \cdot 2$ functions of 2 variables (we refer to Refs. 11, 6, and 28 for the general dimension theory of solution spaces). However, the system is point invariant, so the diffeomorphism freedom is 2 functions of 2 variables, and henceforth, the actual solution space is parametrized by $4 = 6 - 2$ functions of 2 variables. This re-proves Cartan's count.

Proof of Theorem 1. Setting $A = -\frac{2}{3}F_Q,\ B = \frac{1}{3}\mathcal{D}F_Q - \frac{2}{9}F_Q^2 - F_P$, one can rewrite (10) in the form

$$g = 2\,dY\,dQ + A\,dY\,dP + B\,dY^2 - dP^2,$$
$$\omega = (A_P - 2B_Q - \frac{1}{2}AA_Q)\,dY - A_Q dP.$$

To bring the corresponding EW equations to the desired form, we fix $X = 0$ and set the variables as follows: $Q(0) = x,\ P(0) = y,\ Y(0) = 2t,\ A|_{X=0} = a,\ B|_{X=0} = -b - \frac{1}{4}a^2$, where now (x, y, t) are local coordinates on M^3 and $a, b : M^3 \to \mathbb{R}$. This results in

$$\begin{aligned} g &= 4dt\,dx + 2a\,dt\,dy - (a^2 + 4b)\,dt^2 - dy^2,\\ \omega &= (aa_x + 2a_y + 4b_x)\,dt - a_x dy. \end{aligned} \tag{11}$$

The EW equations reduce to a pair of second-order conservative PDEs,

$$(a_t + aa_y + ba_x)_x = (a_y)_y, \qquad (b_t + bb_x - ab_y)_x = (b_y - 2ab_x)_y, \tag{12}$$

which coincide with Manakov-Santini system (2) upon substitution $a = v_x,\ b = u - v_y$, see also Ref. 32. Note that system (12) allows one to uniquely reconstruct g and ω in (11): the conformal structure g comes from the principal symbol of system (12), and ω is given by formula (3). Since the construction directly follows from Cartan's approach, we can conclude that the Manakov-Santini system gives *all* EW structures. The general solution of system (12) depends on 4 arbitrary functions of 2 variables which agrees with Cartan's result. □

The Lax representation of (12) has the form $[X_1, X_2] = 0$, where

$$X_1 = \partial_t - (\lambda^2 - a\lambda - b)\,\partial_x + m\,\partial_\lambda, \qquad X_2 = \partial_y - \lambda\,\partial_x + n\,\partial_\lambda,$$

and

$$m = -a_x\lambda^2 + (aa_x - a_y - b_x)\lambda + (ab_x - b_y), \qquad n = -a_x\lambda - b_x.$$

We point out that this Lax pair transforms to the one of the Manakov-Santini system presented in the Introduction via the change of variables $a = v_x,\ b = u - v_y,\ \lambda = \tilde{\lambda} + v_x$. Taking a linear transformation of the Lax vector fields results in a Lax pair linear in the parameter λ. A further affine translation of λ with non–constant coefficients can be used to bring the Lax pair to the canonical form used in Refs. 17 and 16.

Projecting integral surfaces of the distribution spanned by X_1 and X_2 in the 4D space with coordinates (x, y, t, λ) to the space M^3 with coordinates (x, y, t), one obtains a 2-parameter family of null totally geodesic surfaces of the corresponding EW structure. There is an $\mathbb{R}P^1$–worth of such surfaces through any point in M^3.

The constraint $a = 0,\ b = u$ reduces system (12) to the dispersionless KP equation, $(u_t + uu_x)_x = u_{yy}$, while the corresponding EW structure reduces to the one from[17]

$$g = 4dt\,dx - 4u\,dt^2 - dy^2, \quad \omega = 4u_x\,dt.$$

Any EW structure which admits a parallel vector field can locally be put in this form. Another possible reduction is $u = 0$. This corresponds to the most general hyper-CR Einstein–Weyl structure.[18]

1. Translationally non-invariant version of the Manakov-Santini system

Here, our starting point is the general ansatz for a metric in the conformal class and a covector.[16] Using the diffeomorphism and conformal freedom, the representative metric can be put in form (11). Set $\omega = \omega_1\, dt + \omega_2\, dx + \omega_3\, dy$. Imposing the Einstein-Weyl conditions, we obtain a system of 5 PDEs for a, b, ω_i, which is not presented here due to its complexity. The corresponding Lax pair has the form $[X_1, X_2] = 0$, where

$$X_1 = \partial_t - (\lambda^2 - a\lambda - b)\,\partial_x + m\,\partial_\lambda, \qquad X_2 = \partial_y - \lambda\,\partial_x + n\,\partial_\lambda,$$

and m and n are the following cubic and quadratic polynomials in λ:

$$\begin{aligned} m = &-\frac{1}{2}\omega_2\lambda^3 + \frac{1}{4}(a\omega_2 + 4\omega_3)\lambda^2 - \frac{1}{2}(\omega_1 + b\omega_2 + 2a\omega_3 - aa_x - 2b_x)\lambda \\ &+\frac{1}{4}(a\omega_1 + ab\omega_2 + a^2\omega_3 - 2aa_y - 4b_y), \\ n = &-\frac{1}{4}\omega_2\lambda^2 + \frac{1}{2}(\omega_3 - a_x)\lambda - \frac{1}{4}(\omega_1 + b\omega_2 + a\omega_3 - 2a_y). \end{aligned}$$

One of the five EW equations has the simple form $(\omega_2)_x + \omega_2^2/2 = 0$. This leads to the natural branching.

Case 1: $\omega_2 = 0$. Up to further elementary integration and changes of variables, this case can be reduced to form (11), with Manakov-Santini system (12) for a, b.

Case 2: $\omega_2 = 2/x$ (strictly speaking, $\omega_2 = 2/(x + f(y,t))$, however, $f(y,t)$ can be removed by a transformation $x \to x + f(y,t)$, which preserves the form of the metric after appropriate redefinition of a and b). This branch can be viewed as a translationally non-invariant (x-dependent) version of the Manakov-Santini system.

In view of Theorem 1, both branches are equivalent, but we have been unable to find a combination of a conformal rescaling and a coordinate transformation which reduces Case 2 to Case 1.

B. Einstein-Weyl equations via Bogdanov's system

The following system was proposed by Bogdanov[3] as a two-component generalization of the dispersionless Toda equation,

$$(e^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{yt}, \quad m_{tt}e^{-\phi} = m_x m_{yt} - m_t m_{xy}.$$

It possesses a Lax representation $[X_1, X_2] = 0$, where

$$X_1 = \partial_x - \left(\lambda + \frac{m_x}{m_t}\right)\partial_t + \lambda\left(\phi_t\frac{m_x}{m_t} - \phi_x\right)\partial_\lambda, \quad X_2 = \partial_y + \frac{1}{\lambda}\frac{e^{-\phi}}{m_t}\partial_t + \frac{(e^{-\phi})_t}{m_t}\partial_\lambda.$$

It was observed in Ref. 21 that, for any solution of the Bogdanov system, the metric

$$g = (m_x dx + m_t dt)^2 + 4e^{-\phi}m_t dx dy$$

and the covector

$$\omega = \left(\frac{m_{tt}}{m_t^2} - 2\frac{\phi_t}{m_t}\right)(m_x\, dx + m_t\, dt) + 2\frac{m_{yt}}{m_t}\, dy$$

satisfy the EW equations. Note that g comes from the principal symbol of the system, and ω is given by formula (3). The general solution of the Bogdanov system depends on 4 arbitrary functions of 2 variables. It is natural to expect that this gives (locally) a *generic* EW structure.

Setting $m = t$, one obtains the $SU(\infty)$ Toda equation,[4,39] $(e^{-\phi})_{tt} = \phi_{xy}$, while the corresponding EW structure reduces to the one from[38]

$$g = dt^2 + 4e^{-\phi}dxdy, \quad \omega = -2\phi_t dt.$$

C. Einstein-Weyl equations in diagonal coordinates

Note that any three-dimensional metric possesses diagonal coordinates depending locally on 3 arbitrary functions of 2 variables.[11,13] We can therefore use conformal freedom $g \to \varphi g$, $\omega \to \omega + d\ln\varphi$ to set

$$g = a^2dt^2 - dx^2 + b^2dy^2, \qquad \omega = \omega_1 dx + \omega_2 dy + \omega_3 dt.$$

In this case, the EW equations give rise to a system of five PDEs for the five functions a, b, ω_i, which are second-order in a, b and first-order in ω_i (this system is not presented explicitly due to its complexity). It possesses the Lax pair $[X_1, X_2] = 0$,

$$X_1 = \partial_t - a\cos\lambda\,\partial_x + m\,\partial_\lambda, \quad X_2 = \partial_y - b\sin\lambda\,\partial_x + n\,\partial_\lambda,$$

where

$$m = -\frac{a}{2b}\omega_2\sin^2\lambda - \frac{1}{2}\omega_3\sin\lambda\cos\lambda + \left(\frac{1}{2}a\omega_1 - a_x\right)\sin\lambda + \frac{a_y}{b},$$
$$n = \tfrac{b}{2a}\omega_3\cos^2\lambda + \tfrac{1}{2}\omega_2\sin\lambda\cos\lambda - \left(\tfrac{1}{2}b\omega_1 - b_x\right)\cos\lambda - \tfrac{b_t}{a}.$$

The general solution of the system for a, b, ω_i depends locally on $7 = 2\cdot 2 + 3\cdot 1$ arbitrary functions of 2 variables (recall the order of PDEs). Since diagonal coordinates exist with the freedom of 3 arbitrary functions of 2 variables, this again confirms that EW structures depend on $4 = 7 - 3$ arbitrary functions of 2 variables.

The above system possesses a reduction,[21]

$$g = (1 - e^{-u})dt^2 - dx^2 + (e^u - 1)dy^2, \quad \omega = \frac{e^u + 1}{e^u - 1}u_x dx - u_y dy + u_t dt,$$

for which the EW equations reduce to the scalar second-order PDE,

$$u_{xx} + u_{yy} - (\ln(e^u - 1))_{yy} - (\ln(e^u - 1))_{tt} = 0.$$

This is the dispersionless limit of the "gauge-invariant" Hirota equation.[20]

III. ANTI-SELF-DUALITY EQUATIONS

A conformal structure g on a four-dimensional manifold is called ASD if the SD part of its conformal Weyl tensor vanishes: $W_+ = \frac{1}{2}(W + *W) = 0$. The twistor-theoretic integrability of the ASD condition was established in Ref. 33. It was shown in Ref. 24 that generic ASD structure depends on 6 arbitrary functions of 3 variables. The existence of a Lax pair is implicitly built in the fact that any ASD structure possesses a 3-parameter family of totally null α-surfaces. Below, we present several forms of ASD equations and their Lax pairs in specially adapted coordinates.

A. Anti-self-duality equations in Plebański-Robinson coordinates

Here, we present explicit formulas, including the corresponding Lax pair, in Plebański-Robinson coordinates (w, z, x, y) where the metric g in the ASD conformal class on an open set $M^4 \subset \mathbb{R}^4$ takes the hyper-heavenly form,

$$g = dwdx + dzdy + pdw^2 + 2qdwdz + rdz^2, \tag{13}$$

where p, q, r are functions of all four variables. We assume that all coordinates are real, so that the signature of g is $(2,2)$.

Proposition 3.1. Metric (13) has ASD Weyl tensor if the functions p, q, r satisfy the system of three second order PDEs,

$$\begin{aligned} &p_{xx} + 2q_{xy} + r_{yy} = 0, \\ &m_x + n_y = 0, \\ &m_z - qm_x - rm_y + (q_x + r_y)m = n_w - pn_x - qn_y + (p_x + q_y)n, \end{aligned} \tag{14}$$

where

$$m := p_z - q_w + pq_x - qp_x + qq_y - rp_y, \quad n := q_z - r_w + qr_y - rq_y + pr_x - qq_x. \tag{15}$$

Conversely, any ASD conformal structure is locally of form (13), where (p, q, r) satisfy system (14).

Proof. We will make use of the isomorphisms $\Lambda^2{}_+ = S' \odot S'$ and $TM^4 = S \otimes S'$, where S and S' are real rank-two symplectic vector bundles (spin bundles), $\Lambda^2{}_+$ is the rank-three bundle of self-dual two-forms on M^4, and $\odot$ is the symmetrized tensor product. The seminal result of Penrose[33] asserts that the ASD condition is equivalent to the existence of a 3-parameter family of α-surfaces (totally null surfaces in M^4 with SD tangent bi-vector). This means that any section of S' corresponds to a SD two-form defining a 2D distribution integrable in the Frobenius sense. To arrive at canonical form (13), we select a 2-parameter family of α-surfaces corresponding to a section $\iota \in \Gamma(S')$. Let $\Sigma \in \Gamma(\Lambda^2{}_+)$ be a SD two-form corresponding to this section. It is Frobenius-integrable so there exist independent functions x and y on M^4 such that $\mathrm{Ker}(\Sigma) = \mathrm{Span}(\partial/\partial x, \partial/\partial y)$. We can moreover rescale the spinor ι so that the corresponding two-form Σ is closed and proportional to $dw \wedge dz$. Therefore, w and z are constant on each α-surface in the 2-parameter family, and (x, y) are coordinates on the surface. The α-surfaces are totally null so that the conformal structure is represented by

$$g = \mathbf{e}^{00'}\mathbf{e}^{11'} - \mathbf{e}^{01'}\mathbf{e}^{10'}, \tag{16}$$

where

$$\mathbf{e}^{00'} = a\,dz, \quad \mathbf{e}^{10'} = b\,dw, \quad \mathbf{e}^{01'} = -dx - p\,dw - q\,dz, \quad \mathbf{e}^{11'} = dy + q\,dw + r\,dz; \tag{17}$$

here, (a, b, p, q, r) are so far unspecified functions (we have set the $dzdx$ and $dwdy$ coefficients in g to 0 by exploiting the coordinate freedom in the choice of (x, y)). To examine the ASD condition, we choose a basis of S' consisting of two spinors (o, ι). The self-dual Weyl spinor W' is a section of $\mathrm{Sym}^4(S')$ given by

$$W' = W_0\, o\, o\, o\, o + 4W_1\, o\, o\, o\, \iota + 6W_2\, o\, o\, \iota\, \iota + 4W_3\, o\, \iota\, \iota\, \iota + W_4\, \iota\, \iota\, \iota\, \iota,$$

where the symmetrised tensor product is implicit in this formula. We find that W_4 vanishes identically and that $W_3 = \frac{1}{4}\partial_x\partial_y \ln(a/b)$. Therefore, $a = b\exp(\alpha + \beta)$, where $\alpha = \alpha(x, w, z), \beta = \beta(y, w, z)$.

Now we make a coordinate transformation $x \to \tilde{x}(x, w, z), y \to \tilde{y}(y, w, z)$ such that $\partial\tilde{y}/\partial y = \exp(-\beta)$ and $\partial\tilde{x}/\partial x = \exp(\alpha)$. Finally, we redefine (p, q, r) and conformally rescale the resulting metric by $b^{-1}\exp(-\alpha)$. This puts the metric in form (13), with the corresponding null tetrad given by (17) with $a = b = 1$. So far our proof has more or less followed the construction of Plebański and Robinson,[35] but now we shall proceed differently. Instead of imposing the Einstein equations, we shall assume that the remaining three components of W' vanish. This gives coupled system (14). □

System (14) possesses the Lax pair $[X_1, X_2] = 0$, where X_1 and X_2 are λ-dependent vector fields,

$$\begin{aligned} X_1 &= \partial_w - p\partial_x - q\partial_y + \lambda\partial_y + [m - \lambda(p_x + q_y)]\partial_\lambda, \\ X_2 &= \partial_z - q\partial_x - r\partial_y - \lambda\partial_x + [n - \lambda(q_x + r_y)]\partial_\lambda, \end{aligned} \tag{18}$$

where m, n are given by expressions (15). This Lax pair is a coordinate realization of the general twistor distribution $L_A = (X_1, X_2)$ on the projectivized spin bundle S' given by

$$L_A = \pi^{A'}\mathbf{e}_{AA'} - \pi^{A'}\pi^{B'}\pi^{C'}\Gamma_{AA'B'C'}\frac{\partial}{\partial\lambda}, \tag{19}$$

where the indices $A, B, A', B', \ldots$ take values 0 or 1, the vector fields $\mathbf{e}_{AA'}$ are dual to one forms (17), $\Gamma_{AA'B'C'}$ are components of the spin connection, and $\pi^{A'} = (1, \lambda)$ are homogeneous coordinates on the fibres of $\mathbb{P}S'$. Projecting integral surfaces of the distribution spanned by X_1 and X_2 from the correspondence space $M^4 \times \mathbb{R}P^1$ with coordinates (w, z, x, y, λ) to M^4, we obtain a 3-parameter family of null surfaces (α-surfaces) of the conformal structure g. The spectral parameter λ on $\mathbb{R}P^1$ is a coordinate on the circle of α-surfaces at each point of M^4. Conformal structure (13) can be read off the principal symbol of system (14). Indeed, the principal symbol of (14) equals Q^3, where

$$Q = \partial_w \partial_x + \partial_z \partial_y - p\, \partial_x^2 - 2q\, \partial_x \partial_y - r\, \partial_y^2,$$

and the inverse matrix of the symmetric bivector Q defines conformal structure (13).

Theorem 2 states that a further simplification is possible, so that ASD conditions reduce to a system of 2 third-order PDEs for 2 functions. The proof below uses one of the equations from Proposition 3.1 as integrability conditions.

Proof of Theorem 2. Rewrite the first equation in (14) as $(p_x + q_y)_x + (q_x + r_y)_y = 0$, which implies the existence of a function s such that $p_x = s_y - q_y$ and $r_y = -s_x - q_x$. These two equations can again be regarded as the integrability conditions for the existence of two functions F and G on M^4 such that

$$p = F_y, \quad q = -(F_x + G_y)/2, \quad r = G_x.$$

The remaining two equations in (14) now yield (6).

To exhibit a simple Lax pair for (6), we shall make a linear transformation (null rotation) of the frame of S' which does not change metric (16)

$$\mathbf{e}^{11'} \to \mathbf{e}^{11'} + \gamma \mathbf{e}^{10'} = dy - G_y dw + G_x dz, \quad \mathbf{e}^{01'} \to \mathbf{e}^{01'} + \gamma \mathbf{e}^{00'} = -dx - F_y dw + F_x dz,$$

where $\gamma = (F_x - G_y)/2$. In this spin frame, Lax pair (19) gives (7). □

B. Anti-self-duality equations and torsion-free ODE systems

In the spirit of Cartan, it was shown by Grossman[24] that there is a one-to-one correspondence between ASD conformal structures in signature (2,2) and systems of second-order ODEs with vanishing generalized Wilczynski invariants (torsion-free systems in his terminology). In particular, Grossman has shown that a generic torsion-free system depends on 6 arbitrary functions of 3 variables. Canonical form (5) of the ASD metric can be directly derived from Grossman's approach:[8] if a torsion-free system of 2 ODEs is of the form

$$W'' = G(X, W, Z, W', Z'), \quad Z'' = F(X, W, Z, W', Z'),$$

(here prime denotes differentiation by X), then the solution space M^4 can be parametrized by fixing X, say, $X = 0$, and defining (w, z, x, y) to be the initial conditions: $w = W(0)$, $z = Z(0)$, $y = W'(0)$, $x = -Z'(0)$. The conformal ASD structure on M^4 is then defined by demanding that points in the (X, W, Z) space correspond to totally null α-surfaces in M^4. In the chosen coordinates, this leads to formula (5), where F, G are evaluated at $X = 0$, see Ref. 8 for details of this construction.

C. Anti-self-duality equations in doubly biorthogonal coordinates

It was demonstrated in Ref. 23 that any (analytic) four-dimensional metric can be brought into block-diagonal form,

$$g = \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_2 & b_3 \end{pmatrix}.$$

Coordinates of this type are known as doubly biorthogonal. They depend locally on 4 arbitrary functions of 3 variables. Using the conformal freedom to set $\det g = 1$, one can show that the equations of self-duality reduce to a (complicated) system of 5 second-order PDEs for the 5 (independent) functions among a_i and b_i. The general solution of this system depends locally on 10 arbitrary functions of 3 variables. Since biorthogonal coordinates exist with the freedom of 4 arbitrary functions of 3 variables, this is again in agreement with the fact that self-dual structures depend locally on $10 - 4 = 6$ arbitrary functions of 3 variables.

D. Reductions of self-duality equations

ASD equations possess several geometric reductions of interest.

Hyper-Hermitian case: this case is characterized by the existence of a Lax pair that does not contain derivatives with respect to the spectral parameter.[15,7] Taking into account (7), this leads to a pair of second-order PDEs,

$$Q(F) = 0, \quad Q(G) = 0. \tag{20}$$

This system was first derived in Ref. 22, where the corresponding conformal structures were referred to as "weak heavenly spaces." The dispersionless Lax pair for system (20) is

$$X_1 = \partial_w - F_y\partial_x + G_y\partial_y + \lambda\partial_y, \quad X_2 = \partial_z + F_x\partial_x - G_x\partial_y - \lambda\partial_x.$$

In Ref. 15, it was shown that all (pseudo) hyper-Hermitian conformal structures locally arise from solutions to (20). The general solution to this system depends on 4 arbitrary functions of 3 variables. In the special case where $F = \theta_y, G = \theta_x$, the hyper-Hermitian system reduces to Plebanski's 2nd heavenly equation,

$$\theta_{yz} + \theta_{xw} + \theta_{xy}^2 - \theta_{xx}\theta_{yy} = 0, \tag{21}$$

and the metric g is Ricci–flat. It depends on 2 arbitrary functions of 3 variables.

Null Kähler case: the ansatz $F = \theta_y, G = \theta_x$ reduces ASD equations (6) to a single fourth-order PDE for θ,[19]

$$\begin{aligned} Q(f) = 0, \quad f = \theta_{yz} + \theta_{xw} + \theta_{xy}^2 - \theta_{xx}\theta_{yy}, \\ \text{where} \quad Q = \partial_w\partial_x + \partial_z\partial_y - \theta_{yy}{\partial_x}^2 - \theta_{xx}{\partial_y}^2 + 2\theta_{xy}\partial_x\partial_y. \end{aligned} \tag{22}$$

In this case, the self-dual two form $\Sigma = dw \wedge dz$ corresponding to the two-parameter family of α-surfaces from the proof of Theorem 3.1 is covariantly constant. Conversely, it was demonstrated in Ref. 19 that any ASD metric g that admits a self-dual covariantly constant two-form Σ such that $\Sigma \wedge \Sigma = 0$ is locally given by a solution to (22). The dispersionless Lax pair for (22) is

$$\begin{aligned} X_1 &= \partial_w - \theta_{yy}\partial_x + \theta_{xy}\partial_y + \lambda\partial_y + f_y\partial_\lambda, \\ X_2 &= \partial_z + \theta_{xy}\partial_x - \theta_{xx}\partial_y - \lambda\partial_x - f_x\partial_\lambda. \end{aligned}$$

In the special case $f = 0$, we recover second heavenly equation (21).

Other reductions: The coordinate system introduced in Proposition 3.1 is adapted to a choice of a preferred two-parameter family of α-surfaces determined by a section $\iota \in \Gamma(S')$ or equivalently by a Frobenius-integrable simple two form Σ. There are other possibilities which single out a non-degenerate two form Σ such that $\Sigma \wedge \Sigma \neq 0$. This requires a choice of two independent sections of S' and leads to PDEs generalising Plebanski's 1st heavenly equation.[34,5] In particular, the Przanowski equation[36] describing all ASD Einstein metrics with non-vanishing cosmological constant is written down in such coordinates. A Lax pair for this equation has recently been found in Ref. 26. Its 2nd heavenly form analogous to (21) has been given in Ref. 12.

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