On the aperiodic avoidability of binary patterns with variables and reversals

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Abstract

In this work we present a characterisation of the avoidability of all unary and binary patterns, that do not only contain variables but also reversals of their instances, with respect to aperiodic infinite words. These types of patterns were studied recently in either more general or particular cases.

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1. Introduction

The *pattern unavoidability* concept was introduced by Bean, Ehrenfeucht and McNulty in [1] and by Zimin in [32]. A pattern consisting of variables is said to be unavoidable over a k-letter alphabet, if every infinite word over such an alphabet contains an instance of the pattern. That is, there exists a factor of the infinite word which is obtained from the pattern through an assignment of non-empty words to the variables (each occurrence of a variable is substituted with the same word).

The unary patterns, or powers of a single variable α, were investigated by Thue [30, 31]: α is unavoidable, $\alpha\alpha$ is 2-unavoidable but 3-avoidable, and α^m with $m \geq 3$ is 2-avoidable. Schmidt proved that there are only finitely many binary patterns, or patterns over $E = \{\alpha, \beta\}$, that are 2-unavoidable [28, 29]. Later on, Roth showed that there are no binary patterns of length six or more that are 2-unavoidable [27]. The classification of unavoidable binary patterns was completed by Cassaigne [7] who showed that $\alpha\alpha\beta\beta\alpha$ is 2-avoidable.

In time, the concept of unavoidability was investigated in several other contexts. The ternary patterns were fully characterised in [8, 23], the binary patterns in the setting of partial words in [19, 3, 4, 5, 6], several variations of avoidability of patterns with restrictions on the length of the instances can be found in [26], while the binary patterns avoidable by cube-free words were characterised in [21] together with their growth rates. However, the topic of our work is mostly inspired by [25], where the authors look at the avoidability of words and their reversals, by [14] where the authors show that the pattern $\alpha\alpha\alpha^R$ is avoidable over a binary alphabet, and by the work in [2, 10, 20], where a more generalised form of avoidability, that of pseudo-repetitions, is investigated.

In this work, we investigate the avoidability of binary patterns, when some of the variables might be reversed. For example, instead of looking only at the pattern $\alpha\alpha$, we shall also investigate the pattern $\alpha\alpha^R$, which does not occur in the word `0101`, while the former does (take $\alpha = $ `01`); this is obviously enough for length 2 unary patterns as other variations only consist of complements or mirror images. However, as most of these patterns are avoidable by trivial periodic words (as shown in [9, 22]), we extend a bit our interest and focus on the cases when infinite aperiodic words which do not meet these patterns exist.

Our work is structured as follows. In the next section we present basic definitions and notations, as well as some preliminary observations. In Section 3 we give a characterisation for unary patterns with reversals,

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where the aperiodic constraint is also considered. Section 4 considers the current state of the art regarding avoidability of binary patterns with reversals. Finally, in Section 5 our focus is on the aperiodic avoidability version of the problem in the case of binary patterns with reversals (this section represents the main novelty of this work in comparison to [9, 22]).

2. Definitions and Preliminaries

Cassaigne's Chapter 3 of [18] provides background on unavoidable patterns, while the handbook itself contains detailed definitions on words.

Let Σ be a non-empty finite set of symbols called an *alphabet*. Each element $0 \in \Sigma$ is called a *letter*. A *word* is a sequence of letters from Σ. The *empty word* is the sequence of length zero, denoted by ε. The set of all finite words (respectively, non-empty finite words) over Σ is denoted by Σ^* (respectively, Σ^+).

A word u is a *factor* of a word v if there exist x, y such that $v = xuy$ (the factor u is *proper* if $u \neq \varepsilon$ and $u \neq v$). We say that u is a *prefix* of v if $x = \varepsilon$ and a *suffix* of v if $y = \varepsilon$. The *length* of u is denoted by $|u|$ and represents the number of symbols in u. We denote by $u[i..j]$, where $0 \leq i \leq j < |u|$, the factor of u starting at position i in u and ending at position j, inclusive. By $|u|_v$ we denote the number of distinct, possibly overlapping, occurrences of a factor v in u. We denote by $u^R = u[|u|-1] \cdots u[1]u[0]$, the *reversal* or *mirror image* of a word u. A word u is said to be a *palindrome* if $u = u^R$. In this work we only consider palindromes of length greater than 1, as letters are just trivial instances of such.

For a word u, the powers of u are defined recursively by $u^0 = \varepsilon$ and for $n \geq 1$, $u^n = uu^{n-1}$. Furthermore, $\lim_{n\to\infty} u^n$ is denoted by u^ω. For legibility, the 2-powers of words are called *squares*, while 3-powers are called *cubes*. Furthermore, if $u = v^k v'$, where v' is a prefix of v, we say that u is a $\frac{k|v|+|v'|}{|v|}$-power.

A *period* of a word u is an integer p, such that for every defined positions i and $i+p$ of u, we have $u[i] = u[i+p]$. Furthermore, the minimal such p associated to some word is called *the (minimal) period* of the word. This can obviously be extended to infinite words, where the existence condition is dropped. An infinite word for which no such period exists is called *non-periodic*. Observe that in the case of non-periodic infinite words the period will increase the longer a prefix of the word is considered. Finally, if for an infinite word there exists no suffix of it which is periodic, the word is called *aperiodic*. In the case when such a suffix exists, thus the word is of the form uv^ω, the word is called *ultimately periodic*.

Let E be a non-empty finite set of symbols, distinct from Σ, whose elements are denoted by α, β, γ, etc. Symbols in E are called *variables*, and words in E^* are called *patterns*. The *pattern language* over Σ associated with a pattern $p \in E^*$, denoted by $p(\Sigma^+)$, is the subset of Σ^* containing all words of $\varphi(p)$, where φ is any non-erasing morphism that maps each variable in E to an arbitrary non-empty word from Σ^+. A word $w \in \Sigma^*$ *meets* the pattern p (or p *occurs* in w) if for a factorisation $w = xuy$, we have $u \in p(\Sigma^+)$. Otherwise, w *avoids* p.

More precisely, let $p = \alpha_0 \cdots \alpha_m$, where $\alpha_i \in E$ for $i \in \{0, \ldots, m\}$. Define an *occurrence* of p in a word w as a factor $u_0 \cdots u_m$ of w, where for $i, j \in \{0, \ldots, m\}$, if $\alpha_i = \alpha_j$, then $u_i = u_j$. Stated differently, for all $i \in \{0, \ldots, m\}$, $u_i \in \varphi(\alpha_i)$, where φ is any non-erasing morphism from E^* to Σ^* as described earlier. These definitions extend to infinite words w over Σ which are functions from $\mathbb{N}$ to Σ.

Considering the pattern $p = \alpha\beta\beta\alpha$, the language associated with p over the alphabet $\{\mathtt{0}, \mathtt{1}\}$ is $p(\{\mathtt{0},\mathtt{1}\}^+) = \{uvvu \mid u, v \in \{\mathtt{0},\mathtt{1}\}^+\}$. The word $\mathtt{001100}$ meets p (take $\varphi(\alpha) \in \{\mathtt{0}, \mathtt{00}\}$ and $\varphi(\beta) = \mathtt{1}$), while the word $\mathtt{01011}$ avoids p.

Let p and p' be two patterns. If p' meets p, then p *divides* p', which we denote by $p \mid p'$. For example, $\alpha\alpha \nmid \alpha\beta\alpha$, but $\alpha\alpha \mid \alpha\beta\alpha\beta$. When both $p \mid p'$ and $p' \mid p$ hold, the patterns p and p' are *equivalent*, and this happens if and only if they differ by a permutation of E. For instance, $\alpha\alpha$ and $\beta\beta$ are equivalent.

A pattern $p \in E^*$ is *k-avoidable* if in Σ^* there are infinitely many words that avoid p, where Σ is a size k alphabet. On the other hand, if every long enough word in Σ^* meets p, then p is *k-unavoidable* (unavoidable over Σ). Finally, a pattern $p \in E^*$ which is k-avoidable for some k is simply called *avoidable*, and one which is k-unavoidable for every k is called *unavoidable*. The *avoidability index* of p is the smallest k such that p is k-avoidable, or it is ∞ if p is unavoidable.

In the rest of this work, we only consider binary patterns, hence we fix $E = \{\alpha, \beta\}$. Moreover, we define $\overline{\alpha} = \beta$ and $\overline{\beta} = \alpha$, and, similarly, $\overline{\mathtt{0}} = \mathtt{1}$ and $\overline{\mathtt{1}} = \mathtt{0}$ if Σ is binary, as complementing variables and, respectively, letters. Furthermore, when we talk about reversals we will refer to images of variables, while the term of mirror image will be used to refer to patterns, which might contain variables that are reversed or not, and factors of a word.

Preliminaries. In this paper we are interested in the avoidability of binary patterns in a more general setting. That is, we look at patterns formed not only from variables, but also from their reversals. As it can be seen, the word 0011001 has three occurrences of the pattern $\alpha\alpha$, but also has no fewer than six occurrences of the pattern $\alpha\alpha^R$, when $\alpha \in \{\mathtt{0}, \mathtt{01}, \mathtt{001}, \mathtt{1}, \mathtt{10}\}$. Furthermore, it has no occurrence of $\alpha\alpha\alpha$, but has one occurrence of $\alpha\alpha^R\alpha$ for $\alpha = \mathtt{01}$.

Remark 1. *Every even length palindrome meets the pattern $\alpha\alpha^R$ and its complement.*

Since for every pattern its avoidability or unavoidability induces also the avoidability or unavoidability of its mirror images and its complements, we shall restrict our investigation to one of its forms, as the others follow trivially. In [13, 14, 15] some results regarding the avoidability of palindromes under certain conditions have already been provided.

When considering a four letter alphabet, following a result of Pansiot [24], there exist infinite words that avoid palindromes. This is due to the fact that over a four letter alphabet there exists an infinite word that has the repetitive threshold 7/5, thus does not contain any factors of the form 00 or 010, for 0, 1 letters, since these would create a 2, respectively, a 3/2-power.

When analysing ternary alphabets as to avoid all palindromes, therefore also factors of forms 00 and 010, for any letters 0, 1 of the alphabet, we get that the only infinite words that avoid palindromes are isomorphic to $(\mathtt{012})^\omega$.

For binary alphabets the avoidability of palindromes is not possible as every word of length 3 would contain one.

However, since $\alpha\alpha^R$ is an even length palindrome, the following is immediate:

Remark 2. *Any square-free word will avoid all even length palindromes.*

Therefore, we already have an upper limit on our avoidability indices.

3. Unary patterns avoidability

In this section we overview the avoidability of patterns formed from a single variable and its reversal, considering also the aperiodicity argument.

Obviously, when considering a unary alphabet no pattern is avoidable. The results of Thue [30, 31] give us precise bounds for the cases when reversals do not occur. Squares are avoidable on a ternary alphabet, while for powers of at least three a binary alphabet is enough.

For the case of reversals, as seen above, a ternary alphabet is enough to avoid any unary pattern containing a variable and its reversal, that is a pattern divisible by $\alpha\alpha^R$. On further investigation, we see that this is also the case for a binary alphabet, whenever we consider for example the word $(\mathtt{01})^\omega$. Therefore, a first straightforward result is the following:

Remark 3. *Every pattern p that has both α and α^R as symbols is 2-avoidable.*

However, both previously given words, $(\mathtt{01})^\omega$ and $(\mathtt{012})^\omega$, are periodic, thus not that interesting (within the community). Moreover, all infinite binary or ternary words avoiding such patterns are in fact obtained via a bijection from these two words. Our investigation shall deal with the avoidability of these patterns in aperiodic words, e. g., words that are not of the form uv^ω.

A first step in this direction was made in [14], where the authors show that the pattern $\alpha\alpha\alpha^R$, which can be found in the English word *bepepper* by taking $\alpha = ep$, is avoidable on a binary alphabet, see [14, Theorem 37]. Furthermore, the same work conjectures that every binary aperiodic word avoiding this pattern has critical exponent $\geq 2+\varphi$ (the golden ratio), while the insightful recent work of [12] shows that the number of these words grows between polynomial and exponential relative to their lengths.

It is also not that difficult to find a binary infinite aperiodic word that avoids the pattern $\alpha\alpha^R\alpha$ (see [11, 22]). For this consider the binary word $\tau = (\mathtt{01})^\omega$. Next we "double" in τ a $\mathtt{1}$ at positions exponentially far away from the first, and denote the newly obtained word by τ'. That is, if we inserted a $\mathtt{1}$ at position k in τ, then at position $k-1$ we have a $\mathtt{1}$, and the next $\mathtt{1}$ will be inserted at some position greater than $2k$ after an occurrence of another $\mathtt{1}$. We have:

$$\tau' = \mathtt{01\underline{1}0101\underline{1}01010101\underline{1}0101010101010101\underline{1}}\cdots,$$

where the new inserted characters are depicted as underlined.

Following the above discussion, we have a characterisation of all unary patterns with reversals, even when aperiodicity is required.

Theorem 1. *Let $p \in \{\alpha, \alpha^R\}^+$ be a pattern. Then*

i) p is unavoidable, whenever $p \in \{\alpha, \alpha^R\}$;

ii) there exist infinite aperiodic ternary words that avoid p, whenever $|p| > 1$;

iii) p is avoidable over a binary alphabet, whenever $p \notin \{\alpha, \alpha^R, \alpha\alpha\}$;

iv) there exist infinite aperiodic binary words that avoid p, whenever $|p| > 2$.

4. Binary patterns

We already discussed the avoidability indexes of all unary patterns in the previous section. To start the investigation of binary patterns with reversals, we have to first recall the results characterising the classical avoidability of binary patterns. For more details see [18, Chapter 3].

Theorem 2 ([18])**.** *Regarding avoidability, binary patterns fall into three categories:*

1. *the binary patterns $\varepsilon, \alpha, \alpha\beta, \alpha\beta\alpha$, their mirror images, and their complements are unavoidable (or have avoidability index ∞);*
2. *the binary patterns $\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\alpha\beta\alpha$, $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\beta$, $\alpha\beta\beta\alpha$, $\alpha\alpha\beta\alpha\alpha$, $\alpha\alpha\beta\alpha\beta$, their mirror images, and their complements have avoidability index 3;*
3. *all other binary patterns, and in particular all binary patterns of length six or more, have avoidability index 2.*

Using further the results of Theorem 1, we shall establish the avoidability of some binary patterns also in the case when reversals are present.

First, for **item 1** of Theorem 2, all of the patterns and their variations that include reversals are trivially unavoidable on a finite alphabet, since any factor of the form axa, where a is a letter and x is any non-empty word, occurs in every infinite word.

Now, from Theorem 1.(ii), we conclude that for all of the patterns at **item 2** of Theorem 2, except for $\alpha\beta\alpha\beta$, there exists an aperiodic ternary word avoiding them, no matter how we replace α by α^R or β by β^R. We just have to see now if 3 is in fact the smallest index possible.

Remark 4. *Since $\alpha\alpha^R$ is avoidable by $(\mathtt{01})^\omega$, all patterns $\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\alpha\beta\beta$, $\beta\alpha\alpha\beta$, $\alpha\alpha\beta\alpha\alpha$, their mirror images, and their complements, have avoidability index 2, whenever one α is replaced by α^R. This is also true for $\alpha^R\alpha\beta\alpha$, $\alpha\alpha^R\beta\alpha$, $\alpha\alpha^R\beta\alpha^R\alpha$, $\alpha^R\alpha\beta\alpha^R\alpha$, and all variations of $\alpha\alpha\beta\alpha\beta$ with one of the first two α's reversed, their mirror images, and their complements.*

In this context of avoidability, when periodicity is allowed, we still have to analyse the patterns $\alpha\alpha\beta\alpha^R$, $\alpha\beta\beta\alpha^R$, $\alpha\alpha\beta\alpha^R\alpha^R$, the variations with reversals of $\alpha\beta\alpha\beta$, of $\alpha\alpha\beta\alpha\beta$, all their mirror images, and their complements. Moreover, for the aperiodic case, since β can be chosen to be an arbitrary word, none of the first three patterns of **item 2** of Theorem 2 is avoidable by an infinite aperiodic binary word wherever α^R occurs.

For $\alpha\alpha\beta\alpha^R$ and $\alpha\alpha\beta\alpha^R\alpha^R$, it is immediate that since every infinite binary word contains `01010` or `00` as a recurring factor (or their complements), they, their mirror images, and their complements will occur in every binary infinite word (take `0` or `01` as the image of α) . Thus none of these is avoidable by either an ultimately periodic or aperiodic infinite binary word.

Following [9, Theorem 9] and [22, Lemmas 2 and 3], the patterns $\alpha^R\beta^R\alpha\beta$ and $\alpha^R\beta\alpha\beta$, their mirror images, and their complements have avoidability index 3 and there exist infinite aperiodic ternary words avoiding the patterns.

For the pattern $\alpha\beta\beta\alpha$, we know it has avoidability index 3. In every infinite aperiodic binary word we have either `0110`, `1111`, $\mathtt{10}^i\mathtt{1110}^i$, $\mathtt{0}^i\mathtt{1110}^i\mathtt{1}$, or one of their complements as factors, for some $i > 0$. It immediately follows that $\alpha\beta\beta^R\alpha$ is met by every aperiodic infinite binary word. Furthermore, a binary word avoiding $\alpha\beta\beta\alpha^R$ or $\alpha\beta\beta^R\alpha^R$, would have to be of the form $w = \prod \mathtt{0}^{\{1,3\}}\mathtt{1}^{\{1,3\}}$. But since every such aperiodic word contains `101011` as a factor, we have that $\alpha\beta\beta\alpha^R$ is met by every aperiodic infinite binary word. For the word $(\mathtt{0111})^\omega$, observe that the pattern occurs in it as the factor `1101110111`, where α goes to `1` and β to `1011`. However, $\alpha\beta\beta\alpha^R$ is not met by $(\mathtt{01})^\omega$. This is straightforward, as the image of $\beta\beta$ would have even length, and thus would always be preceded and followed by different characters. As the image of α ends with the same letter as the image of α^R begins with, the conclusion follows.

Lemma 1. *The pattern $\alpha\beta\beta^R\alpha^R$, its mirror image, and their complements have avoidability index* 2 *and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Let us apply a strategy similar to before and triple `1`'s, at positions exponentially apart from the beginning, in the word $\tau = (\mathtt{10})^\omega$. Moreover, in order to make later on further use of this constructed word, we shall also impose the condition that between every two consecutive factors `111` there is an odd number of `0`'s. We have the word

$$\tau''' = \mathtt{01}\underline{\mathtt{11}}\mathtt{010101}\underline{\mathtt{11}}\mathtt{01010101010101}\underline{\mathtt{11}}\mathtt{010101010101010101010101010101}\underline{\mathtt{11}}\cdots .$$

Observe that in fact, our pattern is an even length palindrome. However, since τ''' contains none of `00`, `0110`, nor `1111`, as a factor, it follows immediately that no even length palindrome of length greater than 3 can exist in τ'''. ☐

Let us now consider the variations of the pattern $\alpha\alpha\beta\beta$. We know that when we reverse one variable, the pattern is 2-avoidable according to Theorem 1.(*iii*). This is also the strategy used in [9] to prove this fact. Thus we only need to consider the variations of this pattern in the context of aperiodic infinite words. Obviously any variation of the pattern is met by every word that has `0011` or `1111` as factors.

Lemma 2. *The pattern $\alpha\alpha^R\beta^R\beta$, its mirror image, and their complements have avoidability index* 2 *and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Let us again consider the above constructed word τ'''.

Obviously, the only unary square that occurs in the word is `11`. Thus, the last letter of the images of α and β^R has to be `1`. If the image of any of these has length 1, then the image of the other, has to have length greater than 1. However, τ'' contains no even length palindromes of length greater than 3. ☐

Lemma 3. *The only infinite binary word avoiding $\alpha\alpha^R\beta\beta$ has $(\mathtt{01})^\omega$ as a suffix.*

Proof. Let us consider towards a contradiction that there exists an infinite binary word that avoids the pattern. Obviously such a word contains no unary 4-power.

First assume that this word contains `00` as a factor (the case when it has `11` is symmetrical). We consider the first occurrence of `00` in this word, starting after position 1; this position is preceded by `1`. It is easy to check that every word starting with `100` and having length 11 contains an occurrence of the pattern.

Hence, our word has to be ultimately periodic with `01` as period. To see that this word avoids our pattern it is straightforward, as it contains no unary square that would be created by the image of α and its reversal. ☐

The only patterns left from **item 2** of Theorem 2 are variations of $\alpha\alpha\beta\alpha\beta$. However, the 2-avoidability of $\alpha\alpha\beta\alpha^R\beta$ is proved in [9, Corollary 13], while that of $\alpha\alpha^R\beta\alpha\beta$ and $\alpha^R\alpha\beta\alpha\beta$ is a straightforward consequence of pattern divisibility, following Remark 3 (see also [22, Lemma 5]).

For the variations of the pattern $\alpha\alpha\beta\alpha\beta$ we will show in the next section that all of them that have one of the β's reversed are avoidable by an aperiodic infinite binary word. For the others, we have the following result.

Lemma 4. *There do not exist infinite aperiodic binary words avoiding any variations of the patterns $\alpha\alpha\beta\alpha^R\beta$, $\alpha\alpha^R\beta\alpha\beta$, $\alpha^R\alpha\beta\alpha\beta$, their mirror images, or their complements.*

Proof. We assume that an infinite aperiodic binary word avoiding one of the patterns exists. A first observation is that such an aperiodic word must contain $\mathtt{00}$ or $\mathtt{11}$ as factors. Furthermore, the word cannot contain a unary power of length 5, as in this case we could take each variable to be represented by a letter. Next, we consider the largest $i > 1$ such that $\mathtt{0}^i$ is a factor of the word (the same works for $\mathtt{1}^i$) and the next group of 0's will have a lower exponent. Then, we have a factor $\mathtt{0}^i\mathtt{1}^j\mathtt{0}^k\mathtt{1}^\ell\mathtt{0}^m\mathtt{1}^p$, such that $i > k$ and all powers are less than 5. Observe that this is possible, as otherwise the word would either be ultimately periodic, or we could in fact look at the exponents of the blocks of 1's.

If $j \leq \ell$ and $k < 3$, then we can easily chose $\alpha = 0$ and $\beta = 1^j$, or $\alpha = 0$ and $\beta = 01^j$, respectively. In all cases, we would obtain that the word meets the pattern, which is a contradiction. If $j \leq \ell$ and $k = 3$, then it must be that $i = 4$. However, in this case choosing $\alpha = 0$ and $\beta = 001^j$ gives us an occurrence of the pattern in the word.

If $j > \ell$, we can obviously iterate the above argument starting now from the block 1^j. This will either lead to a contradiction, or will lead to even smaller exponents, which in the end will also render a contradiction. $\square$

Finally, we consider the variations of the patterns of **item 3** of Theorem 2. First, observe that all patterns of length six or more that have only one occurrence of one of the variables, either normal or as a reversal, contain a unary factor of length 3 or longer. Following Theorem 1.(iv), we conclude that for each of these patterns there exists an aperiodic infinite binary word avoiding them.

Therefore, the only patterns that we still have to investigate are the variations with reversals of $\alpha\alpha\beta\alpha\beta$, $\alpha\alpha\beta\beta\alpha$, $\alpha\alpha\beta\beta\alpha\alpha$, $\alpha\beta\alpha\beta\alpha$, $\alpha\beta\alpha\alpha\beta$, the patterns $\alpha\beta\alpha\beta\beta\alpha^R$, $\alpha\beta\alpha\beta\beta^R\alpha$, $\alpha\beta\alpha\beta\beta^R\alpha^R$, $\alpha\beta\alpha\beta^R\beta\alpha$, $\alpha\beta\alpha\beta^R\beta\alpha^R$, $\alpha\beta^R\alpha\beta\beta\alpha$, $\alpha\beta^R\alpha\beta\beta\alpha^R$, $\alpha^R\alpha\beta\alpha\beta\beta$, their mirror images, and their complements. Please observe that the avoidability for the variations of the pattern $\alpha\alpha\beta\alpha\beta$ where β is not reversed will also induce avoidability for the same variations of $\alpha\alpha\beta\alpha\beta\beta$ and $\alpha\beta\alpha\beta\beta\alpha$, the latter having the mirror image of the pattern as a prefix. Moreover, failing to show the avoidability of these patterns will, in some cases, imply a deeper analysis of other patterns that meet these (contain them or some mirror image or complement of them as factors).

The result in [9, Corollary 13] also shows that all the variations of the pattern $\alpha\alpha\beta\beta\alpha$ that include reversals, their mirror images, and their complements have avoidability index 2 (see also [22, Lemma 9]).

Moreover, it is not difficult to see that since $\mathtt{11}$ is the only even length palindrome in τ', the image of β when considering the pattern $\alpha\alpha\beta\beta^R\alpha$ must be $\mathtt{1}$. However, there exists no suitable image for α in this case, that would make the pattern occur in τ', which makes it avoidable by binary infinite aperiodic words.

Following the observations after Lemma 3, regarding the 2-avoidability of the patterns $\alpha\alpha\beta\alpha^R\beta$, $\alpha\alpha^R\beta\alpha\beta$, $\alpha^R\alpha\beta\alpha\beta$, the above ones regarding the binary avoidability of all variations of $\alpha\alpha\beta\beta\alpha$, and the 2-avoidability of $\alpha\beta\beta\alpha^R$ respectively, due to the division of the patterns, the next results are straightforward.

Lemma 5. *All variations of the patterns $\alpha\alpha\beta\alpha\beta\beta$, $\alpha\alpha\beta\beta\alpha\alpha$, and $\alpha\beta\alpha\beta\beta\alpha$ that include reversals, their mirror images, and their complements have avoidability index 2.*

Using a binary sequence from [16] that avoids all squares but $00, 11, 0101$, [9, Theorem 11] and [22, Lemma 11] show that the pattern $\alpha^R\beta\alpha\beta\alpha$, its mirror image, and their complements are avoidable by infinite aperiodic words.

Since $\alpha\beta\alpha\alpha\beta$ is avoidable by Theorem 2, due to the pattern divisibility property, $\alpha^R\beta\alpha\beta\beta\alpha$ is also avoidable by the same binary infinite aperiodic word. The same stands for $\alpha\alpha\beta\beta\alpha\alpha^R$, that is divisible by the pattern $\alpha\alpha\beta\beta\alpha$ which was shown to be avoidable by a binary infinite aperiodic word in [7] .

Lemma 6. *The patterns $\alpha\alpha\beta\beta\alpha^R\alpha$, $\alpha^R\alpha\beta\beta\alpha\alpha^R$, $\alpha^R\alpha\beta\beta\alpha^R\alpha$, $\alpha^R\alpha^R\beta\beta\alpha\alpha$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the previously defined word τ'''. We note that the only even length palindrome of it is `11`, while all squares have the form `11`, $(\mathtt{01})^{2\ell}$, $(\mathtt{10})^{2\ell}$, $(\mathtt{11}(\mathtt{10})^{2\ell+1})^2$ and $(\mathtt{11}(\mathtt{01})^{2\ell+1})^2$, for some positive integer ℓ.

For the pattern $\alpha^R\alpha^R\beta\beta\alpha\alpha$, if the image of α starts with `1`, then that of α^R either is 1, or it ends in $(\mathtt{01})^{2k}$. In both cases we get a contradiction with the possible images for β. The same goes for the case when the image of α starts with `0`.

Since the only even length palindrome in τ''' is `11`, we conclude that for the rest of the patterns, the image of α must be `1`. However, β cannot go to neither $(\mathtt{01})^k$ nor $(\mathtt{10})^k$, since between each factor `11` there is an odd number of `0`'s. ☐

Lemma 7. *The patterns $\alpha^R\alpha\beta\alpha\beta\beta$ and $\alpha\alpha\beta\alpha\beta\beta^R$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. If we consider again the word τ''', just as in the previous proof again α must be replaced by `1`. For $\alpha^R\alpha\beta\alpha\beta\beta$, if the image of β starts or ends with 1 we get a contradiction as neither `0110` nor `1111` are factors of τ''', and if it starts and ends with `0` we reach a contradiction as `00` is not a valid factor of the word. The other one is just the mirrored complement of it. ☐

As a consequence of pattern divisibility and Lemmas 1 and 2, respectively, we have the following results:

Lemma 8. *The patterns $\alpha\alpha\beta^R\beta\alpha^R$, $\alpha\alpha^R\beta^R\beta\alpha\alpha$, $\alpha\beta\alpha\beta\beta^R\alpha^R$, $\alpha\beta\alpha\beta^R\beta\alpha^R$, $\alpha^R\alpha^R\beta^R\beta\alpha\alpha$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Lemma 9. *The patterns $\alpha\alpha^R\beta^R\beta\alpha$, $\alpha^R\alpha\beta^R\beta\alpha$, $\alpha^R\alpha\beta^R\beta\alpha\alpha$, $\alpha^R\alpha\beta^R\beta\alpha\alpha^R$, $\alpha^R\alpha\beta^R\beta\alpha^R\alpha$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Following the results in [9, Theorem 11] and [22, Lemma 11], all variations with reversals of the pattern $\alpha\beta\alpha\beta\alpha$, their mirror images, and their complements are avoided by infinite aperiodic binary words. In fact [9] provides a comprehensible characterisation of the avoidability of all binary patterns with reversals.

5. Aperiodic avoidability

In [9] a full characterisation of the avoidability of binary patterns with reversals is given (alternative proofs for some of the results can be found in [22]). To this end, note that while in [9] the patterns in the set

$$S_{2,4} = \{\alpha\beta\alpha\beta^R\alpha^R, \alpha\beta\alpha^R\beta^R\alpha, \alpha\beta\alpha\beta^R\alpha, \alpha\beta\alpha^R\beta\alpha\}$$

are proven to be avoidable by aperiodic infinite binary words (see [9, Section 5.2]), the work [22] does not even characterise the avoidability of these patterns in the periodic case. However,[9, Corollary 13] says that all the patterns in the set

$$\{\alpha\alpha^R, \alpha\alpha\beta\alpha\beta^R, \alpha\alpha\beta\alpha^R\beta, \alpha\alpha\beta\alpha^R\beta^R, \alpha\alpha\beta\beta\alpha^R, \alpha\beta\alpha^R\alpha^R\beta, \alpha\beta\beta\alpha^R\}$$

are avoided by $(\mathtt{01})^\omega$ or its complement. While this is obviously true, the word avoiding these patterns is strongly periodic. Furthermore, since patterns such as $\alpha\alpha^R\beta\beta$ meet the pattern $\alpha\alpha^R$, these were also automatically concluded as 2-avoidable, thus their aperiodicity property still needs investigation.

The unavoidability by binary infinite aperiodic words for $\alpha\alpha\beta\alpha^R\beta$, $\alpha\alpha^R\beta\alpha\beta$, and $\alpha^R\alpha\beta\alpha\beta$ is proven in Lemma 4. Then $\alpha\alpha^R$ is avoidable by only the two periodic binary words $(\mathtt{01})^\omega$ and its complement. The pattern $\alpha\alpha^R\beta\beta$ is proven not to be avoidable by any binary infinite aperiodic word in Lemma 3, while the reasoning why $\alpha\beta\beta\alpha^R$ is not avoidable by any infinite aperiodic binary word is presented right before Lemma 1.

As a result of the above analysis we are presented with the following question:

Question 1. *Do there exist infinite aperiodic binary words avoiding the following patterns, their mirror images, and their complements?*

- $\alpha\alpha\beta\beta\alpha^R$, $\alpha\alpha^R\beta\beta\alpha$, *and* $\alpha^R\alpha\beta\beta\alpha$, *which are variations of* $\alpha\alpha\beta\beta\alpha$;
- $\alpha\alpha\beta^R\alpha\beta$, $\alpha\alpha\beta^R\alpha^R\beta$, $\alpha\alpha^R\beta^R\alpha\beta$, *and* $\alpha^R\alpha\beta^R\alpha\beta$, *which are variations of* $\alpha\alpha\beta\alpha\beta$;
- $\alpha\alpha\beta\alpha^R\beta\beta$, $\alpha\alpha\beta^R\alpha^R\beta\beta$, $\alpha\alpha^R\beta^R\alpha\beta\beta$, *and* $\alpha^R\alpha\beta^R\alpha\beta\beta$, *which are variations of* $\alpha\alpha\beta\alpha\beta\beta$;
- $\alpha\beta\alpha\alpha^R\beta$, $\alpha\beta\alpha\alpha^R\beta$, $\alpha\beta\alpha^R\alpha\beta$, $\alpha\beta\alpha^R\alpha^R\beta$, $\alpha\beta^R\alpha\alpha\beta$, *and* $\alpha\beta^R\alpha^R\alpha^R\beta$, *which are variations of* $\alpha\beta\alpha\alpha\beta$;
- $\alpha\beta\alpha\beta\beta\alpha^R$ *and* $\alpha\beta^R\alpha\beta\beta\alpha^R$, *which are variations of* $\alpha\beta\alpha\beta\beta\alpha$.

In the rest of this section we will prove the avoidability by binary infinite aperiodic words of the last of our patterns, the ones from Question 1. We do this by applying some convenient morphisms to infinite words with certain properties, already known from the literature. To this end, consider the Thue-Morse [30, 31] overlap-free infinite word $\Gamma = \Gamma^\omega(\mathtt{0})$, where $\Gamma(\mathtt{0}) = \mathtt{01}$ and $\Gamma(\mathtt{1}) = \mathtt{10}$, and the square-free infinite word $\Omega = \Omega^\omega(\mathtt{0})$ attributed to Hall [17], where $\Omega(\mathtt{0}) = \mathtt{012}$, $\Omega(\mathtt{1}) = \mathtt{02}$, and $\Omega(\mathtt{2}) = \mathtt{1}$. We provide morphisms that applied to either Γ or Ω render infinite words that do not meet our considered patterns. Since none of these known words are ultimately periodic, and the images of our morphisms are not conjugates of one another, it is straightforward that the infinite words obtained are aperiodic. Thus, for the remainder of this section, whenever proving our results we will only focus on the avoidability part.

Lemma 10. *The patterns $\alpha\beta^R\alpha\beta\beta\alpha^R$, $\alpha\beta\alpha\beta\beta\alpha^R$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism δ with $\delta(\mathtt{0}) = \mathtt{0111}$ and $\delta(\mathtt{1}) = \mathtt{1100}$. Let us now define the infinite word $\Delta = \delta(\Gamma)$ and prove that Δ does not meet any of the above patterns.

Claim 1. All squares that are factors of Δ either have length less than 4 or their length is a multiple of 4 and they are a factor of $\{\delta(\mathtt{0}), \delta(\mathtt{1})\}^+$.

Proof (Claim 1). Assume that for some x with $|x| > 4$ and $|x|$ not divisible by 4, we have a factor xx in Δ. Consider an occurrence of such a factor. Since $|x| > 4$, it must be that either x has $\mathtt{1000}$ as a prefix, in which case our result easily follows, or at some position i in xx, for $i \leq 4$, we have a first occurrence of the factor $\mathtt{11}$ (this factor does not occur beforehand in x). Therefore, at position $i + |x|$ we have again $\mathtt{11}$. However, since either $x[i-2]x[i-1] = \mathtt{00}$ or $x[i+2]x[i+3] \in \{\mathtt{10}, \mathtt{11}\}$ we immediately get a contradiction. (Claim 1) □

Following Claim 1, we conclude that for all patterns the image of β is either smaller than 4, or both it and the image of α (or α^R) are multiples of 4. This is due to the fact that in all of these patterns we have an occurrence of $\beta\beta$ preceded by $\beta\alpha$ or $\beta^R\alpha$.

It is easy to check that for no unary images does Δ meet the above patterns (the images have length at most 2). Using a computer it can also be checked that the sum of the images of α and β must be longer than 8. Since there exist no length 4 factors of $\{\delta(\mathtt{0}), \delta(\mathtt{1})\}^2$ starting at the same position such that one is the reversal of the other, we conclude that no occurrence of $\alpha\beta^R\alpha\beta\beta\alpha^R$, where the image of β is of length at least 4, occurs in Δ. The same conclusion is easily drawn also in the case of $\alpha\beta\alpha\beta\beta\alpha^R$ by considering α and α^R.

For the cases when the image of β is shorter than 4, we only need to consider the cases when this image is in the set $\{\mathtt{0}, \mathtt{1}, \mathtt{11}\}$, the only words whose squares are factors of the word. Moreover, the images of α must be in all cases longer than 4. We immediately conclude that Δ avoids the patterns since they both have $\alpha\beta\beta\alpha^R$ as factor. This is true since the first (last) four letters from the image of α are different from the last (first) four letters from the image of α^R.

This concludes our proof. □

Lemma 11. *The patterns $\alpha^R\alpha\beta\beta\alpha$, $\alpha\alpha^R\beta\beta\alpha$, $\alpha\alpha\beta\beta\alpha^R$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism δ' with $\delta'(\texttt{0}) = \texttt{11101}$ and $\delta'(\texttt{1}) = \texttt{00010}$. Let us now define the infinite word $\Delta' = \delta'(\Gamma)$ and prove that Δ' does not meet any of the above patterns. The proof is similar to that of Lemma 10.

Following the same strategy as the one in Claim 1 of the above proof, we can easily show that all squares of words with length greater than 4 have their length a multiple of 5. Thus, by looking at the length 3 factors preceding and following the image of $\beta\beta$, we draw the conclusion that the patterns $\alpha\alpha^R\beta\beta\alpha$ and $\alpha\alpha\beta\beta\alpha^R$ are avoidable on Δ', whenever the length of the image of β is longer than 4. For shorter lengths, since $\alpha\alpha$ also occurs in $\alpha\alpha\beta\beta\alpha^R$, we check by computer that the pattern is avoidable on Δ' for all possible images of α and β of length at most 4. Moreover, the only even length palindromes in Δ' are $\{\texttt{00}, \texttt{0000}, \texttt{11}, \texttt{1111}, \texttt{011110}, \texttt{100001}, \texttt{01000010}, \texttt{10111101}\}$. Thus we can restrict the images of $\alpha\alpha^R$ and $\alpha^R\alpha$, for the first two patterns, to be the ones in the set. While for the second pattern we can check by computer that Δ' avoids it (considering the images of β of length at most 4), for the first pattern we can check that no squares (images of $\beta\beta$) can be preceded and followed by the restrictive factors given by the images of α without creating overlaps in Γ. ☐

Lemma 12. *The patterns $\alpha\beta\alpha^R\alpha^R\beta$, $\alpha\beta\alpha\alpha^R\beta$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ with $\psi(\texttt{0}) = \texttt{110}$, $\psi(\texttt{1}) = \texttt{1000}$ and $\psi(\texttt{2}) = \texttt{01011}$. Let us now define the infinite word $\Psi = \psi(\Omega)$ and prove that Ψ does not meet any of the above patterns.

Using a computer it can be checked that whenever the sum of the images of α and β is less than 8, the patterns do not occur in Ψ.

The only unary images of α that would make the patterns be met by ψ are from $\{\texttt{0}, \texttt{1}, \texttt{00}, \texttt{11}\}$. Hence, in all of these cases the image of β must be of length at least 5. Nevertheless, looking at all possibilities of prefixes of length 5 for the image of β following the image of the two consecutive occurrences of α, we can easily check that these would strictly determine the factor preceding the possible image of $\alpha\beta$, and would impose the occurrence of a square after the application of ψ to Ω. This is a contradiction.

The other cases that need to be considered are when α has its image in $\{\texttt{01}, \texttt{10}\}$ for the first pattern (the only other two words in Ψ that create squares), or a suffix of one of the words $\{\texttt{10}, \texttt{001}, \texttt{001011}, \texttt{011110100}\}$, which are the first halves of the only even palindromes in Ψ that cannot be extended, for the second pattern (by first halves of the only even palindromes we mean that concatenating to these their mirror image would generate all even length palindromes that cannot be extended as factors of the word). In both cases the proof is similar to the one of the above unary case, which concludes our result. ☐

Lemma 13. *The patterns $\alpha\beta\alpha^R\alpha\beta$, $\alpha\beta\alpha\alpha^R\beta$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ' with $\psi'(\texttt{0}) = \texttt{110}$, $\psi'(\texttt{1}) = \texttt{0101}$ and $\psi'(\texttt{2}) = \texttt{00011}$. Let us now define the infinite word $\Psi' = \psi'(\Omega)$ and prove that Ψ' does not meet any of the above patterns.

The first halves of the only even length palindromes in Ψ' which cannot be further extended are $\{\texttt{01}, \texttt{10}, \texttt{00011}, \texttt{11100}\}$. The proof follows the same idea as the one of Lemma 12. ☐

Lemma 14. *The pattern $\alpha\beta^R\alpha^R\alpha^R\beta$, its mirror image, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ'' with $\psi''(\texttt{0}) = \texttt{0}$, $\psi''(\texttt{1}) = \texttt{0}$ and $\psi''(\texttt{2}) = \texttt{101110}$. Let us now define the infinite word $\Psi'' = \psi''(\Omega)$ and prove that Ψ'' does not meet the above pattern.

A computer investigation of the word Ψ'', shows that the number of squares that it contains is quite big. Thus we follow the strategy above, but using a different observation. Note that the factors of Ψ'' whose mirror images are also factors of Ψ are quite limited. In particular, these are represented by all the suffixes of the words in the following set:

$$\{\texttt{100}, \texttt{010}, \texttt{101}, \texttt{01110}, \texttt{0111}, \texttt{011}, \texttt{1001}, \texttt{100001}, \texttt{10000}, \texttt{1000}, \texttt{10001}\}.$$

Since our pattern contains both variables also as reversals, it follows that the images of α and β are bounded in length by 6. Using a computer we can check that no substitution of the variables with such images results in an occurrence of the pattern in Ψ''. $\square$

Lemma 15. *The pattern $\alpha\beta^R\alpha\alpha\beta$, its mirror image, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ''' with $\psi'''(\mathtt{0}) = \mathtt{0}$, $\psi'''(\mathtt{1}) = \mathtt{1}$ and $\psi'''(\mathtt{2}) = \mathtt{0}$. Let us now define the infinite word $\Psi''' = \psi'''(\Omega)$ and prove that Ψ''' does not meet the above pattern. It is straightforward that Ψ''' does not contain 11 as a factor, since Ω is square free. The following claim is quite surprising (in our opinion):

Claim 2. The word `1001` is also not a factor of Ψ'''.

Proof (Claim 2). Assume the contrary and consider the factor of Ω to which applying ψ''' would generate such a word, `21021` (this is in fact the only one). However, in order to avoid squares in Ω, it must be that this factor is preceded by `01`. This in turn implies that this is in fact the image of 0212, an earlier factor of the word, since Ω is the fixed point of $\Omega^\omega(\mathtt{0})$. However, it is well known that Ω avoids all factors of the form `212`, thus the contradiction. (Claim 2) $\square$

Observe that if `1001` would be a factor of Ψ''', since `11` does not occur in the word, considering the letter preceding this factor we would in fact have an occurrence of our considered pattern, where α goes to `0` and β goes to `1`.

Since there is no bound on the number of squares that Ψ''' contains, and that the factors whose mirror images are also factors of Ψ''' can have any length, we will prove our result by looking at the form of α.

It is straightforward that the image of α is not unary, as neither `0000`, `11` nor `1001` are factors of Ψ'''.

Assume that the image of α starts with `01`. Then it must be the case that it also ends with `01` or `100`. Indeed, since the image of α starts with `01`, and since there is another image of α following it at times, it must be that it ends with either `01` or `100`, as to avoid the factor `01001`. At the same time, the image of β^R ends in the same factors, and thus, the image of β starts with either `10` or `001`. However, in both cases, looking at the image of $\alpha\beta$, we get a contradiction since all possibilities for these will create words which are not factors of Ψ'''.

If the image of α starts with `001`, then the image of α must end with `010`, while the image of β must start with `01`. Joining together these two factors, we get again a word which is not a factor in Ψ'''.

If the image of α starts with `0001`, then the image of α must end with `01`, while the image of β must start with `10`. Since `11` is a forbidden factor in Ψ''', we get a contradiction also in this case.

Finally, assume that the image of α starts with `10`. In this case, the image of α must end with `10` or `1000`, while the image of β must start with `01` or `0001`. Again joining together in all possible way these factors, we get a contradiction with possible factors of Ψ'''. $\square$

Lemma 16. *The pattern $\alpha\alpha^R\beta^R\alpha\beta$, its mirror image, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ'''' with $\psi''''(\mathtt{0}) = \mathtt{0}$, $\psi''''(\mathtt{1}) = \mathtt{110000111}$ and $\psi''''(\mathtt{2}) = \mathtt{0}$. Let us now define the infinite word $\Psi'''' = \psi''''(\Omega)$ and prove that Ψ'''' does not meet any of the above patterns.

It is not difficult to observe that the only even length palindromes that Ψ'''' contains are factors of `11000011` or `00011000`. Since for the word to meet the pattern we have that $\alpha\alpha^R$ matches such an even length palindrome, we conclude that the length of α is less than 5 and have all possible images of it. Furthermore, the longest factor of Ψ'''' whose mirror image is also a factor of the word is `000111000`. Thus, we also conclude that the maximal length of β is 9 and we can easily find all of its possible images.

A computer check that considers all possible images of α and all possible images of β confirms that the word does not meet the pattern. $\square$

Lemma 17. *The patterns $\alpha\alpha\beta^R\alpha\beta$, $\alpha\alpha\beta^R\alpha^R\beta$, $\alpha^R\alpha\beta^R\alpha\beta$, their mirror images, and their complements have avoidability index 2 and there exist infinite aperiodic binary words avoiding the patterns.*

Proof. Consider the morphism ψ''''' with $\psi'''''(\texttt{0}) = \texttt{11}$, $\psi'''''(\texttt{1}) = \texttt{00001}$ and $\psi'''''(\texttt{2}) = \texttt{10001001}$. Let us now define the infinite word $\Psi''''' = \psi'''''(\Omega)$ and prove that Ψ''''' does not meet any of the above patterns.

The longest factor of Ψ''''' that also occurs as a mirror image is the word 100001110000. Hence we can limit the possible length of the image of β to 10. Moreover, for the last two of the patterns, this upper bound on the length is also valid for the image of α.

The only squares in Ψ''''' are those of the words in the set {1, 0, 00, 001, 11, 010, 100, 1000011}. For these possible images of α and all possible images of β (110 of them) we verified via a computer check that none of these would map either $\alpha\alpha\beta^R\alpha\beta$ or $\alpha\alpha\beta^R\alpha^R\beta$ to a factor of Ψ'''''.

In the same way, since on occurrence of $\alpha^R\alpha\beta^R\alpha\beta$ in the word would determine an even length palindrome, we can easily identify all possible images of α and β and conclude via a computer check that for none of these does the word meet the pattern (Ψ''''' contains 16 even length palindromes, namely, the factors of the words 0001001000, 00011000, 0001111000, 11000011). ☐

We are now ready to state our main result is:

Theorem 3. *Regarding avoidability, the binary patterns that include variables and reversals, except for the variations of $\alpha\alpha\beta\alpha\beta$ that have one of the α's reversed, their mirror images and complements, fall into the following categories:*

1. *all variations with reversals of the binary patterns ε, α, $\alpha\beta$, $\alpha\beta\alpha$, their mirror images, and their complements, are unavoidable (or have avoidability index ∞);*
2. *the binary patterns $\alpha\alpha^R$, $\alpha\alpha^R\beta$, $\alpha\alpha^R\beta\alpha$, $\alpha^R\alpha\beta\alpha$, $\alpha\alpha\beta\alpha^R\beta$, $\alpha\alpha^R\beta\alpha\beta$, $\alpha^R\alpha\beta\alpha\beta$, $\alpha\alpha^R\beta\beta$, $\alpha\beta\beta^R\alpha$, $\alpha\beta\beta\alpha^R$, their mirror images, and their complements have avoidability index 2, and are unavoidable by binary aperiodic words;*
3. *the binary patterns $\alpha\alpha$, $\alpha\alpha\beta$, $\alpha\alpha\beta\alpha$, $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\beta$, $\alpha\beta\beta\alpha$, $\alpha\alpha\beta\alpha\alpha$, $\alpha\alpha\beta\alpha\beta$, $\alpha\alpha\beta\alpha^R$, $\alpha\alpha\beta\alpha^R\alpha^R$, $\alpha^R\beta^R\alpha\beta$, $\alpha^R\beta\alpha\beta$, their mirror images, and their complements have avoidability index 3, and are avoidable by infinite ternary aperiodic words;*
4. *all other binary patterns have avoidability index 2 and are avoidable by binary infinite aperiodic words.*

6. Conclusion

This work presents a survey regarding the avoidability of binary patterns that also include reversals of variables. Apart from the results of Section 5, which as far as we know are new, the others have been previously investigated by several authors, see, e. g., [9, 11, 12, 14, 25].

As future work, one of the most attractive topics of investigation would be an analysis of the growth functions of the number of words that avoid all these variations of patters. These functions would describe the ratio between the number of words having such property relative to their different lengths. The best starting point in this direction would be [12], where the authors show that, surprisingly, the number of words avoiding the pattern $\alpha\alpha\alpha^R$ relative to their length is between polynomial and exponential. Recently in [11], in a not so tedious and involved manner as in [12], the same authors showed similar results for the pattern $\alpha\alpha^R\alpha$ (for this same pattern, Shallit and Du found the lexicographically least sequence avoiding it). On the same page, Currie and Rampersad also shown that the number of binary words avoiding $\alpha\beta\beta\alpha^R$ grows exponentially with length. To this end, we mention that a variety of proving techniques regarding the handling of these growth functions is also present in [21].

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