Painlevé equations and applications

by

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Abstract

The theme running throughout this thesis is the Painlevé equations, in their differential, discrete and ultra-discrete versions. The differential Painlevé equations have the Painlevé property. If all solutions of a differential equation are meromorphic functions then it necessarily has the Painlevé property. Any ODE with the Painlevé property is necessarily a reduction of an integrable PDE.

Nevanlinna theory studies the value distribution and characterizes the growth of meromorphic functions, by using certain averaged properties on a disc of variable radius. We shall be interested in its well-known use as a tool for detecting integrability in difference equations — a difference equation may be integrable if it has sufficiently many finite-order solutions in the sense of Nevanlinna theory. This does not provide a sufficient test for integrability; additionally it must satisfy the well-known singularity confinement test.

Reductions of the self-dual Yang-Mills equations to many integrable systems are well known. They can be reduced directly to each of the Painlevé equations $\mathrm{P_I}$ to $\mathrm{P_{VI}}$. Often an integrable system can be reduced to one or more of the Painlevé equations. In particular we will work with a reduction called the Ernst equation, which arises in the description of spacetimes admitting two commuting Killing vectors by general relativity. In particular we look at the Bianchi models of types I-VII as such spacetimes. Furthermore the Ernst equation has reductions to $\mathrm{P_{III}}$ and $\mathrm{P_{VI}}$. For some Bianchi types the particular reduction is believed to be original.

Roots and poles of a classical meromorphic function are mapped by a limiting process called ultra-discretization to our definitions of roots and poles of a (max,+) meromorphic function. This is used to derive (max,+) Nevanlinna theory, which could be useful in detecting integrability of ultra-discrete equations. Analogously to the difference equations, it appears that an integrable ultra-discrete equation has sufficiently many finite order (max,+) meromorphic solutions. Numerical simulations of ultra-discrete equations are produced to study their integrability.

Contents

Chapter 1

Introduction

The central theme running throughout this thesis is the Painlevé equations. They arise in describing many physical systems, of which we shall work with one such example in detail, producing some results believed to be new. On another note, we shall consider generalizations of the classical Painlevé (differential) equations to both discrete and ultra-discrete Painlevé equations. In the analysis of ultra-discrete equations we produce new results, mainly a new version of classical Nevanlinna theory, in this case applicable to a certain class of piecewise linear functions. Such functions solve ultra-discrete equations. We also make attempts to classify ultra-discrete equations, using a scheme that is believed to be new, which we also present in the preprint by Halburd and Southall [79].

Integrable systems of equations shall feature heavily in this text. They are significant in that they combine tractability with nonlinearity. In contrast, it is difficult to characterise a large class of nonlinear equations. A number of general methods for solving integrable systems begin with forming a linear system, whose compatibility conditions (constraints that mixed second derivatives commute) are the original integrable system [19].

The Painlevé property is possessed by an ODE[1] if all movable singularities (ones whose positions are dependent on the solution chosen) of its solutions are poles. This implies that all solutions are single-valued about all movable singularities. However

[1]Throughout the text, the abbreviations ODE and PDE shall be used for ordinary and partial differential equations respectively.

the latter statement does not necessarily imply the former: for instance $e^{1/z}$ is single-valued but does not have a pole at $z = 0$. The Painlevé property is a strong indicator (some would take it as a definition) of integrability. The Painlevé equations have the Painlevé property. The Painlevé test is a useful tool for identifying equations with the Panlevé property. Only nonlinear equations can have movable singularities [18]. We note that not all solutions of an equation with the Painlevé property are necessarily meromorphic.

The six Painlevé equations, conventionally labeled by Roman numerals I to VI as $\mathrm{P_I}$ to $\mathrm{P_{VI}}$, are solved by a previously unknown class of functions called the Painlevé transcendents. They were classified between the late nineteenth and early twentieth century by Painlevé and colleagues [13, 14, 18]. They are related to each other by a sequence of reductions called coalescence limits, starting from $\mathrm{P_{VI}}$ (which is in some sense the most general Painlevé equation) and reducing towards $\mathrm{P_I}$. The solutions of $\mathrm{P_{VI}}$ may not be meromorphic; they may have branch points.

There are many discrete equations with continuum limits to the Painlevé differential equations. However, most of these equations do not inherit the integrability properties of the Painlevé equations — such as the existence of associated linear problems. The discrete equations which do have the integrability properties are known as discrete Painlevé equations [55]. These usually look quite different from the naive discretizations of the Painlevé equations. In turn, the ultra-discrete Painlevé equations are obtained from certain discrete Painlevé equations by a limiting process called ultra-discretization, which was introduced in [69]. We note that not every discrete equation can be ultra-discretized.

We shall progress from defining discrete Painlevé equations to the more general problem of finding a discrete analogue of the Painlevé property as an indicator of integrability. The singularity confinement test was introduced in [55] to give such a property. Those discrete equations which pass the singularity confinement test, and also satisfy a condition of zero algebraic entropy [58], are believed to be integrable.

A rational function is a ratio of two polynomials, which can be taken to have no common factors. The degree of a rational function is the maximum degree of these polynomials. A set of functions $\{y_n\}$ which are each rational in an auxiliary variable $z \in \mathbb{C}$ may occur as a solution to a discrete equation. Then we have a set of degrees $q_n := \deg y_n(z)$. The algebraic entropy of a discrete equation quantifies the growth of the degrees of such iterates of that equation. The algebraic entropy of a generic discrete equation is nonzero (corresponding to exponential degree growth) but the algebraic entropy of a large class of integrable discrete equations is zero (corresponding to polynomial degree growth) [74].

The self-dual Yang-Mills (SDYM) equations are called the master integrable system, since it is conjectured in [2] that any integrable differential equation is obtainable by reduction of the SDYM equations (or their generalizations). The SDYM equations arise as a special case of Yang-Mills theory [50] in a four-dimensional space with complex-valued coordinates. We are free to specify the Lie algebra in which the Yang-Mills connection is valued - the so-called gauge algebra. Where the gauge algebra is chosen to be $\mathfrak{sl}(2,\mathbb{C})$, there exist reductions of the SDYM equations by certain three-dimensional symmetry groups to each of the Painlevé equations. Instead, reduction by a certain two dimensional symmetry yields an integrable equation called the Ernst equation. Furthermore it has been shown [21] that the Ernst equation can be reduced to $\mathrm{P_{III}}$ and to $\mathrm{P_{VI}}$. Many integrable systems such as soliton equations possess symmetry reductions to one or more Painlevé equations; see for example [11, 15]. The process of reducing an integrable system always yields an integrable system [20]. It follows that the Painlevé equations themselves are each one-dimensional integrable systems.

The Ernst equation finds many applications in the context of general relativity such as to gravitational solitons [31]. The vacuum Einstein field equations of general relativity for a geometry which admits two commuting Killing vectors include a form of the Ernst equation [35]. This class of geometries includes the Bianchi cosmological models of types I-VII.

In 1898 Bianchi [39] classified the distinct three-parameter Lie groups as types I to IX. Such groups are equivalent to three-dimensional Bianchi manifolds G. In 1951 Taub [41] systematically extended the Bianchi classification to general relativity, by introducing corresponding Bianchi spacetime manifolds $M = \mathbb{R} \times G$ [6]. The Bianchi models are spatially homogeneous models in cosmology (invariant under space translations), where the widely used FRW models are subclasses [6]. A significant feature of Bianchi spacetime models is that they belong to the class of co-homogeneity one class of manifolds, meaning that the local topology of M consists of a one-parameter family of three-dimensional submanifolds. The Einstein field equations on this class of manifolds reduce to a system of ODEs, which means that the rich body of analysis of dynamical systems theory is applicable [23]. We choose a coordinate system such that the surfaces of symmetry are the surfaces of constant time, by defining a time coordinate $t \in \mathbb{R}$ as in [6], which becomes the independent variable of the ODEs.

The reductions of the Ernst equation to P_V was given by Persides and Xanthopoulos [10], together with a further reduction to P_{III}. Also, a reduction to P_{VI} was first given by Calvert and Woodhouse [21]. Here we shall perform such reductions using discrete groups of variables that are each relevant to a particular Bianchi model. The Bianchi models fall into either class A or class B depending on a property of their particular symmetries [37]. In particular the Painlevé equation obtained from a Bianchi model is a special case of either P_{III} for class A models, or P_{VI} for class B models. The class A reductions were already known [32] but the class B reductions are believed to be new.

As well as the applications detailed above, we shall be concerned with Nevanlinna theory which studies meromorphic functions in the complex plane. If all solutions of a differential equation are meromorphic, then the equation has the Painlevé property. Given a meromorphic function of one variable $f(z)$, the Nevanlinna characteristic $T(r, f)$ is a non-negative non-decreasing function of $r > 0$. Here $T(r, f)$ is a measure of the "affinity" of f for infinity on the disc $|z| \leq r$. The behaviour of $T(r, f)$ as $r \to \infty$ encodes a lot of information about f. An important class of meromorphic functions are those for which $T(r, f)$ is bounded by r^σ for some constant σ. Such a function is said to be of finite order.

Since Nevanlinna theory is a theory of meromorphic functions it can only be used on solutions of those differential equations which necessarily have the Painlevé property and are therefore integrable. In [54] Ablowitz, Halburd and Herbst described a natural interpretation of discrete equations as difference equations in the complex domain. This is similar to the interpretation of the factorial function $n!$ as discrete points of the Gamma function $\Gamma(z)$, which has the property $\Gamma(z+1) = z\Gamma(z)$. They found that Nevanlinna theory provides many of the tools necessary to detect integrability in a large class of difference equations. In particular, if a difference equation has sufficiently many solutions of finite order then the equation is integrable — see Ablowitz, Halburd and Herbst [54] and also the review by Halburd and Korhonen [78].

The (max,+) semiring is $(\mathbb{R}\cup\{-\infty\}, \oplus, \otimes)$ where the binary operators are defined by

$$a \oplus b := \max(a, b), \qquad a \otimes b := a + b, \qquad a, b \in \mathbb{R} \cup \{-\infty\}.$$

The (max,+) semiring has no additive inverse.

An ultra-discrete equation can be defined naturally on the (max,+) semiring. Take a discrete variable which satisfies a particular discrete equation (we note that the following only works on certain types of discrete equations). Ultra-discretization of such a variable leads to a new discrete variable X_n which satisfies an ultra-discretized form of the original discrete equation. It follows that the ultra-discrete Painlevé equations are derived by ultra-discretization of certain discrete Painlevé equations, see [68]. As such they serve as prototypes of integrability in the ultra-discrete sense. An example of an ultra-discrete equation often called u-$\mathrm{P_I}$ is

$$X_{n+1} \otimes X_n \otimes X_{n-1} = 0 \oplus X_n \otimes K, \tag{1.1}$$

where K is an arbitrary constant. An ultra-discrete equation such as (1.1) is a generalized *cellular automata*. This means that the values of solutions may be represented as discrete points in a finite dimensional, infinitely sized grid, whose evolution over the grid is governed by the ultra-discrete equation.

Another interpretation of ultra-discrete equations is to take the independent variable as continuous, $x \in \mathbb{R}$, and then the dependent variable $X(x)$ can be any real number. The ultra-discrete equation (1.1) is then reinterpreted as

$$X(x+1) \otimes X(x) \otimes X(x-1) = 0 \oplus X(x) \otimes \pi_1(x), \tag{1.2}$$

where $\pi_1(x)$ is an arbitrary period 1 function. The (max,+) meromorphic functions we have defined may be admitted as solutions to such equations. The process of going from ultra-discrete equation (1.1) to (1.2) is analogous to that for discrete equations on the complex plane introduced in [54].

An original extension of Nevanlinna theory to piecewise linear functions is presented in this thesis. Here the role of meromorphic functions is played by piecewise linear functions of integer slopes, in the real domain. We name functions with these properties (max,+) meromorphic, since they can be described naturally on the (max,+) semiring and they admit natural analogues of the zeros and poles of classical meromorphic functions. This theory is applicable to ultra-discrete equations. Preliminary results suggest that the ultra-discrete Painlevé equations admit finite order (max,+) meromorphic solutions. Results are derived in this thesis which suggest that this property can be used to detect those certain ultra-discrete equations of Painlevé type.

An integrable ultra-discrete equation is one obtained by ultra-discretization of one of a certain subclass of the integrable discrete/difference equations. Joshi and Lafortune [73] have described an analogue of singularity confinement for ultra-discrete equations as a test of integrability.

We derive a new theory of the value distribution of (max,+) meromorphic functions on the real line. In many ways this is analogous to Nevanlinna theory on the complex plane which concerns the value distribution of meromorphic functions. In this light we shall refer to the theory described here as (max,+) Nevanlinna theory. We define analogues of the Nevanlinna characteristic, proximity and counting functions. Analogues of some — but not all — of the results from classical Nevanlinna theory are proved, such as the first main theorem of Nevanlinna and the lemma on the logarithmic derivative.

Some ultra-discrete equations admit (max,+) meromorphic solutions. We conjecture that in the sense of the (max,+) Nevanlinna theory we have introduced, the ultra-discrete Painlevé equations (and in general all integrable ultra-discrete equations) admit finite-order (max,+) meromorphic solutions on $\mathbb{R}$.

As a further, numerical study of ultra-discrete equations we shall work with those ultra-discrete equations in a dependent variable X_n where $n \in \mathbb{Z}$ such as equation (1.1). A solution of such an equation is a sequence of iterates. Moreover we shall let each iterate be a (max,+) rational function of an auxiliary variable $x \in \mathbb{R} \cup \{-\infty\}$. Then we shall define the degree q_n of $X_n(x)$.

The evolution of the degrees of successive iterates is investigated for different ultra-discrete equations. The equations are grouped according to whether they are integrable or not, determined by whether they satisfy the singularity confinement test for ultra-discrete equations introduced in [73]. Our aim in doing this is to look for an analogue of the concept of algebraic entropy from section 6.3, which can be used as a detector of integrability in discrete equations. We conclude that zero algebraic entropy appears to be a necessary condition for integrability of an ultra-discrete equation.

Chapter 2

Complex analysis

In this chapter we review those parts of complex analysis which shall form a basis for original work presented later in the thesis, but shall provide no original work here. Specifically we review the Painlevé equations, their properties and some derivations. Also we review meromorphic functions and their value distribution in the complex plane, as studied by classical Nevanlinna theory. We do not apply classical Nevanlinna theory in this thesis, but shall later introduce an original generalization of it, for a special class of piecewise linear functions on the real line.

The (differential and difference) Painlevé equations live in the complex domain. Therefore a study of them will necessitate tools from complex analysis, which is the study of complex functions, especially of analytic functions — those that can be written locally as a power series. We then study Nevanlinna theory, which is concerned with the value distribution of meromorphic functions on the complex plane [13]. A meromorphic function can be written $\forall z$ in terms of two entire functions $f(z)$ and $g(z) \not\equiv 0$ as $f(z)/g(z)$ [26]. A function is meromorphic if all of its singularities are poles. If all solutions of a differential equation are meromorphic, then the equation has the Painlevé property which is a strong indicator of integrability. Tools from Nevanlinna theory may be used as a detector of integrability in difference equations.

We shall study the different types of singularities a complex function can have. In particular a type in singularity called a pole must be defined in order to state the Painlevé property. An equation satisfies the Painlevé property if all movable singularities of its solutions are poles.

Since the nineteenth century, the study of integrable equations has been an active area of research in nonlinear analysis. The six Painlevé equations, nonlinear ODEs from complex analysis, have special properties that will be described in this chapter.

2.1 Holomorphic functions

Definition 2.1.1 *Given a set S that is a subset of the complex plane $S \subseteq \mathbb{C}$, a mapping $w : S \to \mathbb{C}$ which assigns to each $z \in S$ a unique complex number $w(z)$, is called a complex valued function on S. A complex function w is differentiable at z if*

$$\lim_{\delta z \to 0} \frac{w(z+\delta z) - w(z)}{\delta z}, \tag{2.1}$$

exists. When the limit exists it is denoted by $w'(z)$ or $\frac{dw}{dz}$. If the function is differentiable at every point of S, then it is said to be a holomorphic function in S.

Let w be differentiable at $z = x + iy$. Then we write

$$w(z) = u(x, y) + iv(x, y), \tag{2.2}$$

where u, v are real-valued functions defined on an open[1] subset of $\mathbb{C}$. There, we can obtain the first derivatives u_x, u_y, v_x, v_y by calculating $w'(z)$ using the definition (2.1) with h taken as either pure real, or pure imaginary. We find that $w'(z) = u_x + iv_x$ and $w'(z) = -iu_y + v_y$. Equating the real and imaginary parts separately gives the Cauchy-Riemann equations,

$$u_x = v_y, \qquad u_y = -v_x, \tag{2.3}$$

which do not hold at a point (x, y) if $w(z)$ is not differentiable there. They provide a necessary, but not necessarily sufficient condition for a function to be holomorphic. It is a sufficient condition if, for example, u_x, u_y, v_x, v_y are continuous everywhere.

Theorem 2.1.2 *Let $w(z) = u(x, y)+iv(x, y)$, where u and v have continuous partial derivatives throughout a region Ω. Then w is holomorphic throughout Ω if and only if u and v satisfy the Cauchy-Riemann equations (2.3) there.*

[1] A set is said to be open if and only if it contains no points on its boundary.

Note that the Cauchy-Riemann equations can be written compactly as $w_{\bar{z}} = 0$. The terms analytic function and holomorphic function may be used interchangeably. In this text we shall mainly use the former.

Example 2.1.3 *Consider $w(z) = x^2 + iy^2$. Then the first derivatives are $u_x = 2x$, $v_y = 2y$, $u_y = v_x = 0$. The Cauchy-Riemann equations give that $w'(z)$ only exists where $x = y$.*

Varying x gives the limit

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x, \tag{2.4}$$

while varying y gives the limit

$$\lim_{ik\to 0} \frac{(y+k)^2 - y^2}{k} = \lim_{ik\to 0}(2y+k) = 2y, \tag{2.5}$$

which can be equal to $w'(z)$ only on the line $x = y$. The Cauchy-Riemann equations do not hold on any open disc in the complex plane, so w is nowhere analytic.

2.2 Singularities and zeros

Singularities of a complex-valued function $w(z)$ are points z_0 in the domain where w fails to be holomorphic/analytic. The point $z = z_0$ is called regular if w is holomorphic there. A singularity may be, for example, a pole or a branch point. A critical point is a singularity (a point at which the solution is not holomorphic) which is not a pole. Therefore a critical point may be, for example, a branch point or an essential singularity.

Definition 2.2.1 *A pole of a function $w(z)$ of order/multiplicity m is a point $z_0 \in \mathbb{C}$ such that the Laurent series $w(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ has $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$, where m is a positive integer. Therefore it behaves like $1/(z-z_0)^m$ at $z = z_0$. A pole of order 1, 2, 3 is referred to as a simple pole, double pole, triple pole and so on.*

Corollary 2.2.2 *Let z_0 be a pole of $w(z)$ of order n. Then $(z-z_0)^m w(z)$ is non-singular, and $(z-z_0)^k w(z)$ is singular for $k = 0, 1, \cdots, m-1$.*

Definition 2.2.3 *A zero of a function of order m is a point $z_0 \in \mathbb{C}$ such that the function takes the form $w(z) = (z - z_0)^m f(z)$ where f is a holomorphic function such that $f(z_0) \neq 0$.*

We note that a zero is not a singularity.

A removable singularity of a function is a point at which the function is not defined (a singularity) but at which the function can be so defined that it is analytic at the singularity. An isolated singularity $z = z_0$ of a function f is called an essential singularity if $(z - z_0)^m f(z)$ is also singular at $z = z_0, \forall m \in \mathbb{Z}$.

An essential singularity $z = z_0$ of a function $f(z)$ is one also exhibited by $(z - z_0)^m f(z)$, for any finite m. In some way, an essential singularity behaves like a pole of order ∞.

Singularities of ODEs are either fixed or movable. The location of a fixed singularity does not vary with the particular solution chosen but can only occur at special points. The location of a movable singularity depends on the constant(s) of integration in the solution. Only nonlinear equations can have movable singularities.

Example 2.2.4 *As an example of an equation which possesses fixed singularities, consider the linear ODE*

$$w'' + p(z)w' + q(z)w = 0. \tag{2.6}$$

A solution $w(z)$ can only have singularities when p or q is singular. Therefore they are fixed singularities.

Example 2.2.5 *Consider the nonlinear ODE,*

$$w' + w^2 = 0, \tag{2.7}$$

which admits a solution $w(z) = \frac{1}{z-c}$ where $c \in \mathbb{C}$ is an arbitrary constant. Therefore $z = c$ is a movable singularity.

Example 2.2.6 *The function $\frac{e^{2z}-1}{z^4} = \frac{2}{z^3} + \cdots$ has a triple pole at $z = 0$. The function $\frac{z^3+2z^2-1}{z+1}$ has a removable singularity at $z = -1$, since the function can be rewritten as $z^2 + z + 1$ which is analytic at $z = -1$.*

2.3 Cauchy's integral theorem

Definition 2.3.1 *Let $f(z)$ be holomorphic inside and on a positively oriented contour γ. For a inside γ, Cauchy's integral formula is*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw. \tag{2.8}$$

Liouville's theorem states that if f is an entire function that is bounded in the complex plane $\mathbb{C}$, then f is constant. To prove this, s

Example 2.3.2 *Suppose $|f(w)| \leq M$ for all $w \in \mathbb{C}$. Fix $a, b \in \mathbb{C}$. Take $R \geq 2\max\{|a|, |b|\}$, so that $|w-a| \geq \frac{1}{2}R$ and $|w-b| \geq \frac{1}{2}R$ whenever $|w| \geq R$. Applying Cauchy's integral formula with the contour $\gamma(0; R)$,*

$$f(a) - f(b) = \frac{1}{2\pi i}\int_\gamma f(w)\left(\frac{1}{w-a} - \frac{1}{w-b}\right)dw, \tag{2.9}$$

and its magnitude is

$$|f(a) - f(b)| \leq \frac{4M|a-b|}{R}. \tag{2.10}$$

We can make this distance arbitrarily small by letting $R \to \infty$. Then we can take $f(a) = f(b)$, in which case f is constant. □

Let $p(z)$ be a non-constant polynomial with constant coefficients. Then there exists $\zeta \in \mathbb{C}$ such that $p(\zeta) = 0$. This is the fundamental theorem of algebra. We prove this by contradiction, i.e. by supposing that $p(z) \neq 0$ for all z. Since $|p(z)| \to \infty$ as $|z| \to \infty$, there exists R such that $1/|p(z)| < 1$ for $|z| > R$. On the compact set $\bar{D}(0; R)$, $1/p(z)$ is continuous and hence bounded on $\mathbb{C}$. It is also holomorphic. Therefore by Liouville's theorem, it is constant.

2.4 Painlevé property

Definition 2.4.1 *An ODE is said to possess the Painlevé property if all movable singularities are poles.*

In this section we will describe Painlevé's α-test and use it to classify first order equations with the Painlevé property.

We consider a rational equation of the form

$$u' = \frac{P(z,u)}{Q(z,u)}, \tag{2.11}$$

where P and Q are polynomials in u. Then $Q(z,u) = (u(z) - a_1(z))^{m_1}(u(z) - a_2(z))^{m_2} \cdots (u(z) - a_n(z))^{m_n}$, where the $a_i(z)$ are distinct functions. Making the transformation $u \to u + a_1$, after renaming m_1 as m, locally equation (2.11) takes the form

$$u^m u' = F(z,u), \tag{2.12}$$

where F is analytic in a neighbourhood of $(z = z_0, u = 0)$ and $F(z_0, 0) \neq 0$. Applying the coordinate transformations $u = \alpha U$, $z - z_0 = \alpha^{m+1} Z \Rightarrow \frac{d}{dz} = \alpha^{-m-1}\frac{d}{dZ}$, and taking the limit as $\alpha \to 0$,

$$U^m \frac{dU}{dZ} = F(z_0 + \alpha^{m+1}Z, \alpha U) \to F(z_0, 0), \tag{2.13}$$

which takes the constant value $F(z_0, 0) = \kappa$ for some z_0. The solution of this equation is $U(Z) = ((m+1)\kappa Z + C)^{(m+1)^{-1}}$. For $m > 0$ the solution is branched. In order for equation (2.13) to have the Painlevé property $U(Z)$ must not be branched and therefore $m = 0 \Rightarrow U(Z) = \kappa Z + C$. This implies that $Q(z,u) = Q(z)$ only. In this case we can write

$$u' = a_0(z) + a_1(z)u + a_2(z)u^2 + \cdots + a_n(z)u^n. \tag{2.14}$$

The transformation $w(z) = \frac{1}{u(z)}$ preserves the Painlevé property and gives

$$\begin{aligned} w' &= -\tfrac{1}{w^{n-2}}(a_0(z)w^n + a_1(z)w^{n-1} + a_2(z)w^{n-2} + \cdots + a_n(z)) \\ &= -a_0(z)w^2 - a_1(z)w - a_2(z) + \cdots - a_n(z)w^{2-n} \end{aligned} \tag{2.15}$$

Note that (2.15) is a rational equation of the form (2.11). We have just shown that the right hand side of such an equation with the Painlevé property is in fact a polynomial. Therefore $n \leq 2$, and in terms of u,

$$u' = a_0(z) + a_1(z)u + a_2(z)u^2. \tag{2.16}$$

Equation (2.16) is the Riccati equation, see for example [13]. It is the most general first order ODE that possesses the Painlevé property.

Example 2.4.2 *The general solution of the equation*

$$w'^2 = 4w^3 + w + 1 \tag{2.17}$$

is solved by a Weierstrass function, whose only singularities are poles, and therefore this equation possesses the Painlevé property.

Example 2.4.3 *The general solution of the equation*

$$w'^2 = 4w^5 + w + 1 \tag{2.18}$$

can be shown to have branch points. Therefore it does not possess the Painlevé property.

2.5 The Painlevé equations

Here we consider only second order equations that have the Painlevé property. Specifically we consider equations of the form

$$w'' = f(z; w, w'), \tag{2.19}$$

where f is analytic in z, and rational in w and in w'. There are fifty equivalence classes[2] of equations of the form (2.19) that possess the Painlevé property. The solutions of forty four of these equations which satisfy the Painlevé property may be expressed in terms of known functions (trigonometric, elliptic, solutions of linear ODEs etc.) The other six are the Painlevé equations, labeled $\mathrm{P_I} - \mathrm{P_{VI}}$, classified by Painlevé and colleagues (1887-1909), see for example [11, 14].

The Painlevé equations require the introduction of new transcendental functions, Painlevé transcendents, for their solution. The first three, $\mathrm{P_I} - \mathrm{P_{III}}$, were discovered by Painlevé; $\mathrm{P_{IV}}$ and $\mathrm{P_V}$ were later added by Gambier; and $\mathrm{P_{VI}}$ was added by Fuchs [11]. We note that a special case of $\mathrm{P_{VI}}$ with fixed parameters can be transformed to $\mathrm{P_{III}}$.

[2]We consider equivalence classes each consisting of equations that are transformed into each other under Möbius (bilinear rational) transformations,

$$W(\zeta) = \frac{a(z)w + b(z)}{c(z)w + d(z)}, \qquad \zeta = \phi(z), \tag{2.20}$$

where a, b, c, d and ϕ are all locally analytic functions of z [14].

The Painlevé equations are [14, 13]

$$\mathrm{P_I}: \ w'' = 6w^2 + z, \tag{2.21}$$

$$\mathrm{P_{II}}: \ w'' = 2w^3 + zw + \alpha, \tag{2.22}$$

$$\mathrm{P_{III}}: \ w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{(\alpha w^2 + \beta)}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{2.23}$$

$$\mathrm{P_{IV}}: \ w'' = \frac{w'^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{2.24}$$

$$\mathrm{P_V}: \ w'' = \left(\tfrac{1}{2w} + \tfrac{1}{w-1}\right) w'^2 - \tfrac{w'}{z} + \tfrac{(w-1)^2}{z^2}\left(\alpha w + \tfrac{\beta}{w}\right) + \tfrac{\gamma w}{z} + \tfrac{\delta w(w+1)}{w-1}, \tag{2.25}$$

$$\begin{aligned} \mathrm{P_{VI}}: \ w'' &= \tfrac{1}{2}\left(\tfrac{1}{w} + \tfrac{1}{w-1} + \tfrac{1}{w-z}\right) w'^2 - \left(\tfrac{1}{z} + \tfrac{1}{w-1} + \tfrac{1}{w-z}\right) w' \\ &+ \tfrac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \tfrac{\beta z}{w^2} + \tfrac{\gamma(z-1)}{(w-1)^2} + \tfrac{\delta z(z-1)}{(w-z)^2}\right) \end{aligned} \tag{2.26}$$

where α, β, γ and δ are constant parameters.

An integrable equation is a nonlinear equation which is solvable by an associated linear problem, in that the nonlinear equation is the compatibility condition of the linear problem, so any solution of the linear problem can be used to construct a solution of the nonlinear equation. We shall present examples of methods of reducing integrable equations to one or more than one equations which possess the Painlevé property. Such reductions led to the formulation of the following conjecture by Ablowitz, Ramani and Segur, the ARS conjecture [11]:

Any ODE which arises as a reduction of an integrable PDE possesses the Painlevé property, possibly following a transformation of variables.

The singular points of $\mathrm{P_I} - \mathrm{P_{VI}}$ are tabulated as follows [14],

	z	w
$\mathrm{P_I}, \mathrm{P_{II}}$	∞	∞
$\mathrm{P_{III}}$	$0, \infty$	$0, \infty$
$\mathrm{P_{IV}}$	∞	$0, \infty$
$\mathrm{P_V}$	$0, \infty$	$0, 1, \infty$
$\mathrm{P_{VI}}$	$0, 1, w, \infty$	$0, 1, z, \infty$

The Painlevé equations have asymptotic limits to other equations with the Painlevé property. Such limits are called coalescence limits because singularities of the equation merge under them. Coalescence limits usually take one Painlevé equation to another, as represented by (2.27), but these limits also occur for integrable PDEs [14].

$$\begin{tikzcd}[column sep=small] & & \mathrm{P_{III}} \arrow[dr] & & \\ \mathrm{P_{VI}} \arrow[r] & \mathrm{P_V} \arrow[ur] \arrow[dr] & & \mathrm{P_{II}} \arrow[r] & \mathrm{P_I}. \\ & & \mathrm{P_{IV}} \arrow[ur] & & \end{tikzcd} \tag{2.27}$$

2.6 Solutions of the Painlevé equations

We shall check a necessary condition for the Painlevé property, by using a method due to Ablowitz, Ramani and Segur considering the asymptotic behaviour [15]. The method was based on the work of Kowalevskaya [13]. We consider equations of the form

$$w'' = 6w^2 + f(z), \tag{2.28}$$

which reduces to $\mathrm{P_I}$ (2.21) if $f(z) = z$. Equation (2.28) has a solution in the limit as $w \to \infty$, with the leading order behaviour

$$w(z) = A(z - z_0)^n. \tag{2.29}$$

Substituting (2.29) in (2.28) gives

$$n(n-1)A(z-z_0)^{n-2} = 6A^2(z-z_0)^{2n} + f(z). \tag{2.30}$$

Supposing that $n < 0$, we may neglect the last term on the right in the limit, since the other terms will tend to infinity. Then by comparing coefficients we find that $n = -2$ and $A = 1$.

We look for series solutions of the form $w(z) = \sum_{i=0}^{\infty} a_i(z-z_0)^{i-2}$ where the above analysis gives $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$. In this case equation (2.28) may be expanded as

$$\sum_{i=0}^{\infty}(i-2)(i-3)a_i(z-z_0)^{i-4} = 6\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j (z-z_0)^{i+j-4} + \sum_{i=0}^{\infty} \frac{f^{(i)}(z_0)}{i!}(z-z_0)^i.$$

Now we compare the coefficients of $(z-z_0)^{n-4}$ and substitute $a_0 = 1$ to find

$$(n+1)(n-6)a_n = 6\sum_{m=1}^{n-1} a_m a_{n-m} + \frac{1}{(n-4)!} f^{(n-4)}(z_0), \qquad n \geq 4. \tag{2.31}$$

The index n of a coefficient a_n is called a resonance if the value of that coefficient is free. The above expansion fails to define a_6: equation (2.31) for $n = 6$ is

$$0 = 6(2a_1a_5 + 2a_2a_4 + a_3^2) + \frac{1}{2}f''(z_0),$$

therefore since $a_1 = a_2 = a_3 = 0$ there is a resonance when $n = 6$ that gives $f''(z_0) = 0$. Since this is true for all z_0 we get

$$w'' = 6w^2 + Az + B \tag{2.32}$$

where A and B are constants, and z can be scaled and shifted to get P_I provided that $A \neq 0$. If A does vanish then the equation can be solved in terms of elliptic functions [16].

For special values of the parameters, $P_{II} - P_{VI}$ possess rational solutions and solutions expressible in terms of special functions.

The solutions of P_I, P_{II} and P_{IV} are meromorphic functions of the independent variable z. If the substitution $z = e^t$ is made in P_{III} and P_V, then the solutions become meromorphic functions of t. For P_{VI}, 0, 1 and ∞ are fixed critical points which are branch points. Hence its general solution is not meromorphic throughout the finite complex plane. Since there exist three branch points, no transformation can remove all of them from the finite complex plane. In fact P_{VI} contains the other five Painlevé equations, which may be obtained from it by taking appropriate limits [11]

2.7 Nevanlinna theory

Nevanlinna theory studies the value distribution of meromorphic functions in the complex plane. A meromorphic function can always be written as $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z) \neq 0$ are entire functions. The function $\tan z = \frac{\sin z}{\cos z}$ is meromorphic. A further example of a meromorphic function is a rational function, which takes the form of a quotient of two polynomials multiplied by a constant factor $c \in \mathbb{C}$,

$$f(z) = c\frac{\prod_{\mu=1}^{M}(z-a_\mu)}{\prod_{\nu=1}^{N}(z-b_\nu)}. \tag{2.33}$$

In particular, $f(a_\mu) = 0$ and $f(b_\nu) = \infty$, so that a_μ and b_ν are respectively called the zeros and poles of f. The number of times a zero or a pole appears in the appropriate product is its multiplicity. A zero or pole of multiplicity one is said to be simple, a double pole or zero has multiplicity two and so on [26].

Basics of Nevanlinna theory

In the 1920s the Nevanlinna brothers developed an extensive theory of the value distribution of meromorphic functions [18]. Nevanlinna theory associates to each meromorphic function $f(z)$ the real valued functions $m(r, f)$, $N(r, f)$ and $T(r, f)$. These functions can be used indirectly to obtain information on the growth and value distribution of $f(z)$ more efficiently than a direct analysis.

In the complex plane, consider the open disc $D = \{z : |z| < r\}$. Let $f(z)$ be a function which is analytic and nowhere zero on D. Using polar coordinates we have that the boundary of the disc is the circle $|z| = r$, called ∂D. We use Cauchy's integral theorem (2.8) to calculate $\log|f|$ at the origin in terms of its value distribution on ∂D, choosing to traverse ∂D in the anti clockwise direction,

$$\log|f(0)| = \frac{1}{2\pi i}\oint_{\partial D}\frac{\log|f(z)|}{z}dz = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\phi})|d\phi. \tag{2.34}$$

However we wish to generalize to the case where f has finitely many zeros $a_1, \ldots, a_M$ and poles $b_1, \ldots, b_N$, counting multiplicities, in the punctured disc $D \setminus \{0\}$. We suppose that the Laurent series expansion of f at $z = 0$ is

$$f(z) = \beta z^\gamma + \ldots, \tag{2.35}$$

where $\beta \neq 0$. If $\gamma < 0$ then f has a pole of order $-\gamma$ at $z = 0$, or if $\gamma > 0$ then f has a pole of order γ there. Otherwise if $\gamma = 0$ then $z = 0$ is a regular point of f.

Then we define the function

$$g(z) = z^{-\gamma}\frac{\prod_{\mu=1}^{M}B(a_\mu, z)}{\prod_{\nu=1}^{N}B(b_\nu, z)}f(z) \tag{2.36}$$

where $B(\alpha, z) = \frac{r^2 - \bar{\alpha}z}{r(z-\alpha)}$. Then $g(z)$ has no zeros or poles in D. To see why, we note that if f has a zero a then f has a factor $(z-a)$, while a pole b introduces a factor $\frac{1}{(z-b)}$. Then $\frac{B(a,z)}{B(b,z)}f(z)$ does not have these factors. This argument is particularly clear if f is a rational function (2.33).

Jensen's formula results from writing (2.34) for g, see for example [27],

$$\log|\beta| + \gamma \log r = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\phi})|d\phi}_{\text{boundary of disc}} - \underbrace{\sum_{|a_\mu|<r} \log\frac{r}{|a_\mu|}}_{\text{zeros of } f \text{ on disc}} + \underbrace{\sum_{|b_\nu|<r} \log\frac{r}{|b_\nu|}}_{\text{poles of } f \text{ on disc}}. \quad (2.37)$$

In the case where f is regular at $z = 0$, the left hand side becomes $\log|f(0)|$, which gives the form of Jensen's formula that is usually presented in textbooks [18, 26, 27].

Solutions of $f(z) = a$, called a-points, are counted, with due count of multiplicities, by a counting function $n(t, a, f)$, $t \in \mathbb{R}$, which gives the number of a-points on the disc $|z| \leq t \leq r$. Defining there to be $n(r, a, f) = n$ such a-points $z = \alpha_i, i = 1, 2, \ldots, n$ on the disc, ordered by $0 \leq |\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_n| \leq r$. In particular

$$n(t, a, f) = \begin{cases} 0 & (0 \leq t \leq |\alpha_1|) \\ 1 & (|\alpha_1| \leq t \leq |\alpha_2|) \\ i & (|\alpha_i| \leq t \leq |\alpha_{i+1}|) \\ n & (|\alpha_n| \leq t \leq r). \end{cases} \quad (2.38)$$

It could equivalently be said that $n(r, a, f)$ counts the poles of $g(z) = \frac{1}{f(z)-a}$ (counting multiplicities). Also, a pole of $f(z)$ is a zero of $g(z)$. For example $f(z) = z + a$ leads to $g(z) = \frac{1}{z}$ which has a simple pole at $z = 0$ so $f(z)$ has a simple a-point there.

We define the *integrated counting function* as the integral of $n(t, a, f)$ over the disc $|z| \leq r$ with respect to the logarithmic measure dt/t,

$$N(r, a, f) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r. \quad (2.39)$$

It can be shown that these solutions α_i satisfy

$$\sum_{0<|\alpha_i|\leq r} \log\frac{r}{|\alpha_i|} = \int_0^r \frac{n(t, a, f)}{t} dt. \quad (2.40)$$

Applying this to the poles and zeros of f respectively,

$$\sum_{|a_\mu|<r} \log \frac{r}{|a_\mu|} = N(r,0,f), \qquad \sum_{|b_\nu|<r} \log \frac{r}{|b_\nu|} = N(r,\infty,f). \tag{2.41}$$

We define the function $\log^+$ of a positive real argument $x \in \mathbb{R}^+$ by

$$\log^+ x = \max(0, \log x), \tag{2.42}$$

and note that $\log x = \log^+ x - \log^+ \frac{1}{x}$, $\forall x > 0$. Now we are in a position to define the proximity function of a function f as

$$m(r,\infty,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi. \tag{2.43}$$

This is the mean value of $\log^+|f|$ on the circle $|z| = r$. Roughly speaking, it describes how close on average the values of $f(z)$ are to infinity on this circle. Another useful variant of the proximity function is

$$m(r,a,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\phi}) - a|} d\phi. \tag{2.44}$$

Its growth correlates with the proximity of the values of $f(z)$ to the value $a \in \mathbb{C}$. The closer the values of $f(z)$ are to a on the circle $|z| = r$, the larger the function $m(r,a,f)$ is.

The *characteristic function* is

$$T(r,f) = m(r,\infty,f) + N(r,\infty,f), \tag{2.45}$$

which provides a measure of the affinity of a meromorphic function f for the value ∞. In a similar way,

$$T\left(r, \frac{1}{f-a}\right) = m(r,a,f) + N(r,a,f) \tag{2.46}$$

is the Nevanlinna characteristic giving the affinity of f for the value a. In particular $T\left(r, \frac{1}{f}\right)$ gives the affinity of f for 0. In terms of meromorphic functions $f_i, i = 1, \ldots, n$, some properties of the Nevanlinna characteristic are given for example in [27],

$$T\left(r, \prod_{i=1}^{n} f_i\right) \le \sum_{i=1}^{n} T(r, f_i), \tag{2.47}$$

$$T\left(r, \sum_{i=1}^{n} f_i\right) \le \sum_{i=1}^{n} T(r, f_i) + \log n. \tag{2.48}$$

We shall prove these results by noting that

$$m\left(r, \infty, \sum_{i=1}^{n} f_i\right) \le \frac{1}{2\pi}\int_0^{2\pi} \log^+(n \max_{i=1}^{n} |f_i(re^{i\phi})|)d\phi \le \sum_{i=1}^{n} m(r, f_i) + \log n, \tag{2.49}$$

$$m\left(r, \infty, \prod_{i=1}^{n} f_i\right) \le \sum_{i=1}^{n} m(r, \infty, f_i), \tag{2.50}$$

where we use the inequality $\log^+ |\prod_{i=1}^n f_i| \le \sum_{i=1}^n \log^+ |f_i|$. The number of poles, with due count of multiplicity, of the sum or product of the functions f_i at a point is at most equal to the sum of the multiplicities of the poles at that same point. Therefore

$$\begin{aligned} &N\left(r, \infty, \textstyle\sum_{i=1}^n f_i\right) \le \textstyle\sum_{i=1}^n N(r, \infty, f_i), \\ &N\left(r, \infty, \textstyle\prod_{i=1}^n f_i\right) \le \textstyle\sum_{i=1}^n N(r, \infty, f_i). \end{aligned} \tag{2.51}$$

We now combine the results according to equation (2.45) to prove the inequalities [26].

In order to relate the Nevanlinna characteristic of f to that of $f^n, n \in \mathbb{N}$ we observe that $\log^+ |f^n| = \log^+ |f|^n = n \log^+ |f|$, since $|f| \le 1 \Rightarrow |f^n| \le 1$, so $m(r, \infty, f^n) = nm(r, \infty, f)$. Also the multiplicities of all the poles of f are multiplied by a factor of n, and therefore so is the counting function, so we have

$$T(r, f^n) = nT(r, f). \tag{2.52}$$

To end this section, we state the Poisson-Jensen formula for $f(z) \ne 0, \infty$, which is a generalization of the Jensen formula (2.37) at any point $|z| < r$,

$$\begin{aligned} &\log |f(z)| = \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{r^2-\rho^2}{r^2-2r\rho\cos(\theta-\phi)+\rho^2} \log |f(re^{i\phi})| d\phi \\ &+ \textstyle\sum_{|a_\mu|<r} \log \left|\tfrac{r(z-a_\mu)}{r^2-\bar{a}_\mu z}\right| \\ &- \textstyle\sum_{|b_\nu|<r} \log \left|\tfrac{r(z-b_\nu)}{r^2-b_\nu z}\right|. \end{aligned} \tag{2.53}$$

Fundamental theorems of Nevanlinna

Substituting the identity $\log |f| = \log^+ |f| - \log^+ \frac{1}{|f|}$ in the Jensen formula (2.37) and using results of the previous section gives (provided that $f(0) \ne 0$ and $f(0) \ne \infty$)

$$\begin{aligned} \log |f(0)| &= m(r, \infty, f) - m(r, 0, f) + N(r, \infty, f) - N(r, 0, f) \\ &= T(r, f) - T\left(r, \tfrac{1}{f}\right). \end{aligned} \tag{2.54}$$

That is, the affinity of any meromorphic function for ∞ and its affinity for 0 differ only by a constant. Replacing f by $f-a$ and comparing with the original, we have that

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \tag{2.55}$$

where $O(1)$ is bounded as $r \to \infty$. This relation is Nevanlinna's first main theorem [26, 18]. It implies that if $f(z)$ takes a value $a \in \mathbb{C} \cup \{\infty\}$ fewer times than average, i.e. the counting function $N(r,a,f)$ is relatively small, then the proximity function $m(r,a,f)$ must be large, and vice versa. Loosely speaking, if a meromorphic function takes a certain value a relatively few times, then the values of $f(z)$ are "near" the value a in a large part of the complex plane.

As an example, we shall investigate the effect of a Möbius transformation of f on the Nevanlinna characteristic. Specifically we want a relation between $T(r,f)$ and $T\left(r, \frac{\alpha f+\beta}{\gamma f+\delta}\right)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are such that $\alpha\delta - \beta\gamma \neq 0$.[3] Then, the Möbius transformed function can be written as $\frac{c}{f-a} + b$, where a, b, c are constant. Then application of Nevanlinna's first fundamental theorem together with properties of the characteristic give

$$T\left(r, \frac{\alpha f + \beta}{\gamma f + \delta}\right) = T(r,f) + O(1). \tag{2.56}$$

Nevanlinna's second main theorem depends on an estimate for the proximity function $m(r, f'/f)$, where f'/f is the logarithmic derivative of f. Given a meromorphic function f, we denote by $S(r,f)$ any quantity that is of growth $o(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set E of finite linear measure; that is $\int_E dr < \infty$. An important result of Nevanlinna theory is the lemma on the logarithmic derivative,

$$m\left(r, \frac{f'}{f}\right) = S(r,f), \tag{2.57}$$

which holds for all transcendental meromorphic functions f.

[3] In the case $\alpha\delta - \beta\gamma = 0$ then the transformed function is

$$\frac{\alpha f + \beta}{\gamma f + \delta} = \frac{\alpha(f + \delta/\gamma)}{\gamma(f + \delta/\gamma)} = \frac{\alpha}{\gamma}$$

which is not allowed since it transforms f to a constant.

Now we state Nevanlinna's second theorem. Let $q \geq 2$, and let $z_1, \ldots, z_q$ be distinct points in the complex plane, then the value distribution of f satisfies [27]

$$m(r,f) + \sum_{n=1}^{q} m\left(r, \frac{1}{f - z_n}\right) \leq 2T(r,f) + S(r,f). \tag{2.58}$$

It is a generalization of Picard's great theorem, which states that if a meromorphic function f takes each of three distinct values in $\mathbb{C} \cup \{\infty\}$ at most finitely many times, then f is a constant [18].

Chapter 3

Geometry and integrable systems

In this chapter we shall not introduce any original work, but shall present examples of reductions of integrable differential equations to Painlevé equations, by imposing either an appropriate symmetry group or a transformation of variables. In turn, the integrable equations we consider may be obtained as reductions of the SDYM equations. We also review the reductions of the integrable Ernst equation to either $\mathrm{P_{III}}$ or $\mathrm{P_{VI}}$, which will be the basis of original work in a later chapter.

Integrable systems are systems of equations (differential or difference) that combine nontrivial nonlinearity with unexpected tractability. Often such systems admit large families of exact solutions [19]. There are deep links between integrability and geometry.

The SDYM equations are a system of equations for Lie algebra-valued functions on a complex manifold $\mathbb{C}^4$. They play a central role in the field of integrable equations, amongst other areas of mathematics and physics [29]. Many examples of integrable equations may be obtained from the SDYM equations by symmetry reduction. This is suggested by the fact that their associated linear problems arise as reductions of the linear problem for the SDYM equations. The Painlevé equations are further examples of integrable systems that are reductions of the SDYM equations [11, 20]. In general, Ward (1985) conjectured in [2] that:

Many (possibly all) of the ODEs and PDEs that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction (that is, imposing more symmetry conditions on the equations).

3.1 Symmetry reductions of integrable PDEs to Painlevé equations

Some examples were given in [11, 15] where known integrable PDEs were reduced to one of the Painlevé equations, after a possible change of variables.

KdV equation and extensions

The KdV (Korteweg-de Vries) equation is a nonlinear wave equation of the form

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.1}$$

in 1+1 dimensions [11, 15, 17]. We choose to effect the following transformation,

$$u(t,x) = (3t)^{-2/3}F(z), \qquad z = x(3t)^{-1/3}, \tag{3.2}$$

where the numerical factors are included for later convenience, and we find that

$$-2F - zF' + 6FF' + F''' = 0. \tag{3.3}$$

Equation (3.3) may be transformed such that it can be expressed in terms of solutions of $\mathrm{P_{II}}$. There exists a correspondence between solutions of this equation and those of $\mathrm{P_{II}}$ (2.22) given by [14]

$$F = -w' - w^2, \qquad w = \frac{F' + \alpha}{2F - z}, \tag{3.4}$$

where α is the constant that appears in $\mathrm{P_{II}}$. It can also be shown that the modified KdV equation and the cylindrical KdV equation both reduce to $\mathrm{P_{II}}$ via the method explained above [17].

Sine-Gordon equation

Another example of an integrable equation is provided by the sine-Gordon equation,

$$u_{tx} = \sin u, \tag{3.5}$$

which admits the scaling symmetry $(\bar{t}, \bar{x}) = (\lambda^{-1}t, \lambda x)$. We restrict to the subspace of solutions invariant under this scaling symmetry by introducing

$$y(z) = u(t,x), \qquad z = xt. \tag{3.6}$$

Then we get [16]

$$zy'' + y' = \sin y. \tag{3.7}$$

We note that equation (3.7) does not have the Painlevé property. However making the transformation $w = e^{iy}$ puts the reduction of the sine-Gordon equation in a rational form. It is

$$z(ww'' - w'^2) + ww' = \frac{1}{2}w(w^2 - 1), \tag{3.8}$$

which is a particular case of $\mathrm{P_{III}}$. We can conclude that the sine-Gordon equation has a reduction to $\mathrm{P_{III}}$. We have therefore performed a change of variables which takes an equation which does not have the Painlevé property to one which does.

3.2 Self-dual Yang-Mills (SDYM) equations

Yang-Mills theory was introduced in the seminal work of Yang and Mills [50], as a generalization of Maxwell's Abelian gauge theory of electromagnetism to include non-Abelian gauges. Here we shall introduce Yang-Mills theory using the language of differential forms; see for example [3] for an introduction.

We introduce a four dimensional manifold M, over some neighbourhood of which is defined a connection one-form A which is valued in a Lie algebra $\mathfrak{g}$; see appendix A for a brief introduction to Lie algebras and Lie groups. The Lie group generated by this algebra is the structure group of A. Then we define a two-form F called the curvature of A, and a covariant derivative $D = d + A$,

$$F[A] = dA + A \wedge A =: DA. \tag{3.9}$$

The definition of $F[A]$ given by [11, 29] differs from our definition (3.9) which can be reached by the transformations $A \longrightarrow -A$, $F \rightarrow -A$.

The Hodge dual of F (see again [3]) is called ${}^*F$ which on our four dimensional manifold is a two-form. Then we define the Yang-Mills equation of motion,

$$D^*F = 0. \tag{3.10}$$

Additionally, any solution F of equation (3.10) must satisfy the Bianchi identity

$$DF \equiv 0. \tag{3.11}$$

The Yang-Mills equation gives second order PDEs for A and is in general very difficult to solve [29]. However if we impose the constraint

$$^{*}F = \lambda F, \tag{3.12}$$

where λ is a constant scalar, then equation (3.10) reduces to an identity due to (3.11). Note that we can impose this condition (3.12) in the case when we work on a four dimensional manifold, in which case $^{*}F$ and F are both two-forms. In any $D \neq 4$ dimensions it does not work, owing to the fact that $^{*}F$ is a $(D-2)$-form.

We shall work with that special case of (3.12) where $\lambda = 1$, in which the curvature is equal to its dual. Therefore it is known as the self-dual Yang-Mills (SDYM) equation. Since we have shown solutions of the SDYM equation necessarily satisfy the Yang-Mills equation (3.10), all solutions of the SDYM equation form a subset of the solutions of Yang-Mills theory [29].

Now define a set of four local coordinates $\{x^\mu\}$ on a neighbourhood of M. Then the potential and curvature take the respective forms

$$A = A_\mu dx^\mu, \qquad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{3.13}$$

We shall now write the curvature (3.14) in component form,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.14}$$

The covariant derivative $D = d + A$ has components $D_\mu = \partial_\mu + A_\mu$, in terms of which curvature as $F_{\mu\nu} = [D_\mu, D_\nu]$. We define a column vector $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)^T$ for arbitrary N, from which it follows that the compatibility conditions of the linear equations

$$D_\mu \Phi := \partial_\mu \Phi + A_\mu \Phi = 0 \tag{3.15}$$

are equivalent to the vanishing of the Yang-Mills curvature. In this way the Yang-Mills curvature arises naturally in linear analysis [19]. The connection components A_μ may be either scalars or $N \times N$ matrices, but we consider only the latter, more yielding case.

We consider an element g of a Lie group G. In terms of this we define the transformation

$$A_\mu \to \hat{A}_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g, \qquad g \in G. \tag{3.16}$$

We now wish to determine the resulting curvature from transformation (3.16). Using (3.14),

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - [\hat{A}_\mu, \hat{A}_\nu] = \partial_\mu(g^{-1}A_\nu g - g^{-1}\partial_\nu g) - \partial_\nu(g^{-1}A_\mu g - g^{-1}\partial_\mu g)$$

$$-[g^{-1}A_\mu g - g^{-1}\partial_\mu g, g^{-1}A_\nu g - g^{-1}\partial_\nu g]$$

$$= \partial_\mu(g^{-1}A_\nu g) - \partial_\nu(g^{-1}A_\mu g) - [g^{-1}A_\mu g, g^{-1}A_\nu g] + [g^{-1}A_\mu g, g^{-1}\partial_\nu g] + [g^{-1}\partial_\mu g, g^{-1}A_\nu g]$$

$$= g^{-1}F_{\mu\nu}g - g^{-1}(\partial_\mu g)g^{-1}A_\nu g + g^{-1}A_\nu\partial_\mu g + g^{-1}(\partial_\nu g)g^{-1}A_\mu g - g^{-1}A_\mu\partial_\nu g$$

$$+[g^{-1}A_\mu g, g^{-1}\partial_\nu g] + [g^{-1}\partial_\mu g, g^{-1}A_\nu g]$$

$$= g^{-1}F_{\mu\nu}g.$$

The action of this conjugation is by the structure group G only, so symmetries on the manifold M remain unaffected. In this light equation (3.16) is called a gauge transformation of Yang-Mills theory[1]

Following [29], we now give the component form of the Hodge dual $^*F$. On a four dimensional manifold $^*F$ is a two-form

$$^*F = \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu}{}^{\gamma\delta}F_{\gamma\delta}dx^\mu \wedge dx^\nu,$$

where the antisymmetric tensor $\epsilon_{\mu\nu\lambda\kappa}$ is defined with $\epsilon_{0123} = +1$, and $g := \det g_{\mu\nu}$. Indices are raised and lowered using a metric $g_{\mu\nu}(x)$ defined locally on M.

If both F and its dual are defined in the same coordinate system, then the self-dual equation $^*F = F$ is

$$F_{\mu\nu} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu}{}^{\gamma\delta}F_{\gamma\delta} \tag{3.17}$$

We will proceed in the specific case where $M = \mathbb{R}^{2,2}$, on which we define the metric

$$ds^2 = -(dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{3.18}$$

The SDYM equations on $\mathbb{R}^{2,2}$, (and also on $\mathbb{R}^4$, which we shall not consider here) are, with the metric defined in (3.18) substituted in (3.17)

$$F_{01} = F_{23}, \qquad F_{02} = F_{13}, \qquad F_{03} = F_{21}. \tag{3.19}$$

[1]We consider the example of electromagnetism. The structure group is $G = U(1)$, meaning in this case the connection is Abelian. Therefore the curvature as defined by (3.9) is $F = dA$ and a general element is $g = e^\chi$ where χ is a pure imaginary scalar. Using this Lie group in (3.16) we see that a gauge transformation in electromagnetism is $\hat{A}_\mu = A_\mu + \partial_\mu\chi \Rightarrow \hat{F}_{\mu\nu} = F_{\mu\nu}$.

It shall be convenient to transform to the complex null coordinates $(\tau, \bar{\tau}, \sigma, \bar{\sigma})$ as

$$\tau = x^0 + ix^1, \qquad \bar{\tau} = x^0 - ix^1, \qquad \sigma = x^2 + ix^3, \qquad \bar{\sigma} = x^2 - ix^3. \tag{3.20}$$

Here an overhead bar denotes complex conjugation. The metric (3.18) is hence transformed to

$$ds^2 = -d\tau d\bar{\tau} + d\sigma d\bar{\sigma}. \tag{3.21}$$

Under a local coordinate transformation $x^\mu \to \bar{x}^\mu$, the curvature tensor transforms to $\bar{F}_{\kappa\lambda} = F_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\kappa}\frac{\partial x^\nu}{\partial \bar{x}^\lambda}$. In terms of the coordinates (3.21) the SDYM equations are

$$F_{\tau\sigma} = 0, \qquad F_{\bar{\tau}\bar{\sigma}} = 0, \qquad F_{\tau\bar{\tau}} - F_{\sigma\bar{\sigma}} = 0, \tag{3.22}$$

where it is noted that the Greek indices here label components of the curvature tensor. We shall define the connection components using a similar convention,

$$A = A_\tau d\tau + A_{\bar{\tau}} d\bar{\tau} + A_\sigma d\sigma + A_{\bar{\sigma}} d\bar{\sigma}.$$

The SDYM equations (3.22) in terms of the connection components are

$$\partial_\sigma A_\tau - \partial_\tau A_\sigma + [A_\sigma, A_\tau] = 0, \tag{3.23}$$

$$\partial_{\bar{\sigma}} A_{\bar{\tau}} - \partial_{\bar{\tau}} A_{\bar{\sigma}} + [A_{\bar{\sigma}}, A_{\bar{\tau}}] = 0, \tag{3.24}$$

$$\partial_\sigma A_{\bar{\sigma}} - \partial_{\bar{\sigma}} A_\sigma - \partial_\tau A_{\bar{\tau}} + \partial_{\bar{\tau}} A_\tau + [A_\sigma, A_{\bar{\sigma}}] - [A_\tau, A_{\bar{\tau}}] = 0. \tag{3.25}$$

We present the isospectral linear problem [29],

$$(\partial_\tau + \zeta\partial_{\bar{\sigma}})\Phi = (A_\tau + \zeta A_{\bar{\sigma}})\Phi, \qquad (\partial_\sigma + \zeta\partial_{\bar{\tau}})\Phi = (A_\sigma + \zeta A_{\bar{\tau}})\Phi, \tag{3.26}$$

where ζ is the spectral parameter, and $\Phi(\tau, \bar{\tau}, \sigma, \bar{\sigma}) \in SL(2, \mathbb{C})$. The SDYM equations (3.22) are the compatibility conditions $[\partial_\tau + \zeta\partial_{\bar{\sigma}}, \partial_\sigma + \zeta\partial_{\bar{\tau}}]\Phi = 0$ which we calculate explicitly,

$$\begin{aligned}
&[\partial_\tau + \zeta\partial_{\bar{\sigma}}, \partial_\sigma + \zeta\partial_{\bar{\tau}}]\Phi \\
&= (\partial_\tau A_\sigma + A_\sigma\partial_\tau + \zeta(\partial_\sigma A_{\bar{\sigma}} + A_{\bar{\sigma}}\partial_\sigma + \partial_{\bar{\tau}} A_\tau + A_\tau\partial_{\bar{\tau}}) + \zeta^2(\partial_{\bar{\sigma}} A_{\bar{\tau}} + A_{\bar{\tau}}\partial_{\bar{\sigma}}))\Phi \\
&-(\partial_\sigma A_\tau + A_\tau\partial_\sigma + \zeta(\partial_{\bar{\sigma}} A_\sigma + A_\sigma\partial_{\bar{\sigma}} + \partial_\tau A_{\bar{\tau}} + A_{\bar{\tau}}\partial_\tau) + \zeta^2(\partial_{\bar{\tau}} A_{\bar{\sigma}} + A_{\bar{\sigma}}\partial_{\bar{\tau}}))\Phi \\
&= (F_{\tau\sigma} + \zeta(F_{\sigma\bar{\sigma}} - F_{\tau\bar{\tau}}) + \zeta^2 F_{\bar{\sigma}\bar{\tau}})\Phi,
\end{aligned}$$

where we must substitute equations (3.26) back in to obtain the final step. We see that each of the SDYM equations (3.22) must be satisfied in order for the isospectral linear problem (3.26) to hold.

We define two maps $H, \bar{H} : \mathbb{R}^{2,2} \to SL(2,\mathbb{C})$, and therefore the maps are each complex-valued 2×2 matrices with determinant $+1$. The maps are defined up to $H \to \bar{P}(\bar{\tau},\bar{\sigma})H$, $\bar{H} \to P(\tau,\sigma)\bar{H}$, where $P, \bar{P} \in SL(2,\mathbb{C})$. Following Pohlmeyer [47] and Yang [48] we have the freedom to give the connection components as

$$A_\sigma = H^{-1}\partial_\sigma H, \qquad A_\tau = H^{-1}\partial_\tau H,$$

$$A_{\bar{\sigma}} = \bar{H}^{-1}\partial_{\bar{\sigma}}\bar{H}, \qquad A_{\bar{\tau}} = \bar{H}^{-1}\partial_{\bar{\tau}}\bar{H},$$

since after these substitutions the SDYM equations (3.23-3.25) all reduce to triviality. Since the gauge group has been chosen as $SL(2,\mathbb{C})$ the connection is valued in the algebra $\mathfrak{sl}(2,\mathbb{C})$, whose elements are complex-valued traceless 2×2 matrices.

We present Yang's matrix $J = H\bar{H}^{-1} \in SL(2,\mathbb{C})$, which is defined up to $J \to \bar{P}JP^{-1}$. Introducing an element $g \in SL(2,\mathbb{C})$, recall that the gauge transformation (3.16) sends $F \to g^{-1}Fg$ and therefore leaves the SDYM equations invariant. We have from [48] that A is gauge-equivalent to

$$\hat{A} = J^{-1}\partial_\tau J d\tau + J^{-1}\partial_\sigma J d\sigma.$$

We will show the gauge equivalence of A and $\hat{A}$ by setting $g = \bar{H}^{-1}$. Then

$$\hat{A}_\tau = \bar{H}A_\tau\bar{H}^{-1} + \bar{H}\partial_\tau(\bar{H}^{-1}) = \bar{H}H^{-1}(\partial_\tau H)\bar{H}^{-1} - (\partial_\tau\bar{H})\bar{H}^{-1} = J^{-1}\partial_\tau J,$$

$$\hat{A}_{\bar{\tau}} = \bar{H}A_{\bar{\tau}}\bar{H}^{-1} + \bar{H}\partial_{\bar{\tau}}(\bar{H}^{-1}) = (\partial_{\bar{\tau}}H)\bar{H}^{-1} - (\partial_{\bar{\tau}}\bar{H})\bar{H}^{-1} = 0,$$

and the other two components may be obtained in similar ways. The first two SDYM equations (3.23,3.24) are then trivial, while the other SDYM equation (3.25) is

$$\partial_{\bar{\tau}}(J^{-1}\partial_\tau J) - \partial_{\bar{\sigma}}(J^{-1}\partial_\sigma J) = 0, \tag{3.27}$$

which is named Yang's equation (see for example [47].) Up to the freedom $J \to \bar{P}JP^{-1}$, Yang's equation is therefore equivalent to the SDYM equations.

3.3 An isomonodromy problem

We introduce a complex vector $\Phi = (\Phi_1, \Phi_2)^T$. Each of the six Painlevé equations can be expressed as compatibility conditions $(\Phi_\lambda)_z = (\Phi_z)_\lambda$ of the following linear system [14]

$$\Phi_\lambda = M(z; \lambda)\Phi, \tag{3.28}$$

$$\Phi_z = N(z; \lambda)\Phi. \tag{3.29}$$

Equations (3.28,3.29) are collectively known as an isomonodromy problem. The parameter λ is independent of z, and the particular 2×2 matrices M and N depend on the particular Painlevé equation. The compatibility condition $(\Phi_z)_\lambda = (\Phi_\lambda)_z$ of this system is

$$M_z - N_\lambda + [M, N] = 0. \tag{3.30}$$

Isomonodromy problems for Painlevé equations are not unique [14]. We will show how the particular equation (3.30) may be obtained from the SDYM equations (3.23-3.25). We define a connection on $\mathbb{R}^{2,2}$ as

$$A = -Md\lambda - Ndz,$$

and in terms of the coordinates $(\tau, \bar{\tau}, \sigma, \bar{\sigma})$ we define $\tau := \lambda$ and $\sigma := z$. Then the SDYM equation (3.23) reduces to (3.30) while the other SDYM equations (3.24,3.25) each reduce to triviality.

3.4 The Painlevé equations as reductions of the SDYM equations

In 1907, Fuchs considered the linear monodromy problem [11]

$$w_{yy} = p(y; t)w, \tag{3.31}$$

where $p(y; t)$ is a rational function of y and t, with four regular singular points of y; three of which can be located at the fixed points 0, 1, ∞, and the fourth at the variable point t. Fuchs showed that the monodromy matrices are independent of t if and only if $y(t)$ satisfies $\mathrm{P_{VI}}$. Historically, this is how the sixth Painlevé equation was discovered [11]. It is the compatibility condition of equation (3.31) and the equation

$$w' = M(y;t)w_y + N(y;t)w. \tag{3.32}$$

It is well known that the six Painlevé equations are equivalent to reductions of the SDYM equations [20], with certain three dimensional Abelian conformal symmetry groups giving the different equations [20]. The Yang-Mills connection is valued in $\mathfrak{sl}(2,\mathbb{C})$. The Painlevé transcendents are gauge invariants of the Yang-Mills fields, and determine the fields uniquely up to gauge and choice of constants of integration.

We recall that the orbits of a group generated by a given set of vectors consist of the union of the points of the manifold obtained by the operation of the vectors on the manifold. In terms of a four dimensional spacetime M, for a reduction to one dimension, the orbits of M must be three dimensional. For this the Abelian symmetry G group is generated by three independent commutating Killing vectors,

$$X = \partial_p, \qquad Y = \partial_q, \qquad Z = \partial_r. \tag{3.33}$$

We shall use the notation for the contraction of a one-form Z with a vector V, $V\rfloor Z := V^\mu Z_\mu$. Since there is no dependence on t in (3.33) it is permissible to make a gauge transformation to eliminate the dt component (since $X\rfloor dt = Y\rfloor dt = Z\rfloor dt = 0$). Then the Yang-Mills potential in component form is

$$A = A_p dp + A_q dq + A_r dr, \tag{3.34}$$

and we define $\mathfrak{sl}(2,\mathbb{C})$-values functions $P := X\rfloor A = A_p, Q := Y\rfloor A = A_q, R := Z\rfloor A = A_r$, where P, Q, R are each functions of t only. If the orbits of G are null, that is the tangent vector to every orbit has vanishing norm, then the SDYM equations for the connection (3.34) are singular. In all other cases they reduce to ODEs in t [19].

We recall that P, Q, R, being valued in $\mathfrak{sl}(2,\mathbb{C})$ are each zero trace 2×2 matrices with complex-valued components.

There is a gauge freedom in the matrices P, Q, R in the SDYM equations up to conjugation by a constant 2×2 matrix K. For instance $P \to KPK^{-1}$ is a gauge transformation. We may exploit this by reducing P to either

$$P = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.35}$$

according to whether $\kappa \neq 0$ (where P is semisimple[2]), or $\kappa = 0$ (where P is nilpotent), since $\kappa^2 := \frac{1}{2}\mathrm{Tr}\ P^2$ is an invariant.

. We note that with this choice, one of the SDYM equations is fixed to $P' = 0$ [19]. This is achieved by choosing an appropriate single independent variable $t = t(z; \lambda)$ in equation (3.30).

The other two $\mathfrak{sl}(2, \mathbb{C})$ matrices Q, R take the component forms,

$$Q = \begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix}, \tag{3.36}$$

$$R = \begin{pmatrix} \rho & \sigma \\ \tau & -\rho \end{pmatrix}. \tag{3.37}$$

In each case, the equations for the unknown components $\lambda, \mu, \nu, \rho, \sigma, \tau$ come down to a single ODE of second order — which is the corresponding Painlevé equation.

We shall use the manifold $M = \mathbb{R}^{2,2}$, as in the previous section using complex null coordinates, as in metric (3.21) with three dimensional symmetry groups G of conformal transformations. These are generated by three commuting conformal Killing vectors X, Y, Z. Up to conjugation there are five possible choices of G. One corresponds to $\mathrm{P_I}$ and to $\mathrm{P_{II}}$, the others to one each of $\mathrm{P_{III}} - \mathrm{P_{VI}}$ [20]. The list of groups is

Reduction(s)	X	Y	Z
$\mathrm{P_I}, \mathrm{P_{II}}$	$\partial_{\bar\tau}$	$\tau(\partial_\sigma - \partial_{\bar\sigma}) + (\bar\sigma - \sigma)\partial_{\bar\tau} + \partial_\tau$	$\partial_\sigma + \partial_{\bar\sigma}$
$\mathrm{P_{III}}$	$\partial_{\bar\tau}$	$\sigma\partial_\sigma - \bar\sigma\partial_{\bar\sigma}$	∂_τ
$\mathrm{P_{IV}}$	$\partial_{\bar\tau}$	$2(\sigma\partial_\sigma + \tau\partial_\tau)$	$\tau\partial_\sigma + \bar\sigma\partial_{\bar\tau} + \partial_{\bar\sigma}$
$\mathrm{P_V}$	$\partial_{\bar\tau}$	$\tau\partial_\tau + \bar\sigma\partial_{\bar\sigma}$	$\sigma\partial_\sigma + \tau\partial_\tau$
$\mathrm{P_{VI}}$	$-\tau\partial_\tau - \sigma\partial_\sigma$	$\sigma\partial_\sigma + \bar\tau\partial_{\bar\tau}$	$\tau\partial_\tau + \bar\sigma\partial_{\bar\sigma}$

[2]A square matrix M is called nilpotent if there exists some integer q such that $M^q = 0$. Otherwise if no such integer exists then M is calld semisimple

The full list of Painlevé equations in matrix form is given below [19]

$$
\begin{array}{|c|cc c|}
\hline
\mathrm{P_I} & P'=0 & Q'=[R,P] & R'=[tP+R,Q] \\
\mathrm{P_{II}} & P'=0 & Q'=[R,P] & R'=[tP+R,Q] \\
\mathrm{P_{III}} & P'=0 & tQ'=2[Q,R] & R'=2t[Q,P] \\
\mathrm{P_{IV}} & P'=0 & Q'=[P,R+tQ] & R'=[Q,R] \\
\mathrm{P_V} & P'=0 & Q'=[P,R] & tR'=[R,tP+Q] \\
\mathrm{P_{VI}} & P'=0 & tQ'=[R,Q] & t(1-t)R'=[tP+Q,R] \\
\hline
\end{array} \tag{3.38}
$$

We will now present the derivation that leads to the result that the Painlevé equations are equivalent to reductions of the $\mathfrak{sl}(2,\mathbb{C})$ SDYM equations (3.23-3.25) with the above groups [20]. In further work, we shall find it necessary to work with only the matrix forms of $\mathrm{P_{III}}$ and $\mathrm{P_{VI}}$, so we shall study only these cases separately.

$\mathrm{P_{III}}$

We shall explicitly derive the matrix form of $\mathrm{P_{III}}$ as a reduction of the SDYM equations, assuming that the symmetry group G is generated by two translations and a rotation, as above. In coordinates $(\tau,\tilde{\tau},\sigma,\tilde{\sigma})$ the Killing vectors are

$$X=(0,1,0,0), \qquad Y=(0,0,\sigma,-\tilde{\sigma}), \qquad Z=(1,0,0,0). \tag{3.39}$$

Then we introduce an $\mathfrak{sl}(2,\mathbb{C})$-valued potential 1-form $A=A_\mu dx^\mu = A_\tau d\tau + A_{\tilde{\tau}} d\tilde{\tau} + A_\sigma d\sigma + A_{\tilde{\sigma}} d\tilde{\sigma}$, on which we use the gauge invariance of the SDYM equations under $A\to g^{-1}Ag+g^{-1}dg$, $g\in\mathfrak{sl}(2,\mathbb{C})$ to set $A_{\tilde{\sigma}}=0$, and then have

$$P=X\rfloor A=A_{\tilde{\tau}}, \qquad Q=\sigma A_\sigma, \qquad R=Z\rfloor A=A_\tau. \tag{3.40}$$

Now we substitute in the SDYM equations (3.23-3.25),

$$0=F_{\sigma\tau}=\partial_\sigma R-\frac{\partial_\tau Q}{\sigma}+\frac{1}{\sigma}[Q,R], \tag{3.41}$$

$$0=F_{\tilde{\sigma}\tilde{\tau}}=\partial_{\tilde{\sigma}}P, \tag{3.42}$$

$$0=F_{\sigma\tilde{\sigma}}-F_{\tau\tilde{\tau}}=-\frac{\partial_{\tilde{\sigma}}Q}{\sigma}-\partial_\tau P+\partial_{\tilde{\tau}}R-[R,P]. \tag{3.43}$$

We next make the substitution $(Q,R)\to(R,-Q)$ in order to make this system equivalent to that given in table (3.38). Insisting now that P, Q and R are functions only of $t=\sqrt{\sigma\tilde{\sigma}}$, we reduce to the ODEs

$$P'=0, \qquad tQ'=2[Q,R], \qquad R'=2t[Q,P], \tag{3.44}$$

which is equivalent to P_{III}. The second equation of system (3.44), for both semisimple and nilpotent P, has components

$$t\lambda' = 2(\mu\tau - \sigma\nu), \qquad t\mu' = 4(\lambda\sigma - \rho\mu), \qquad t\nu' = 4(\rho\nu - \lambda\tau). \tag{3.45}$$

The third equation of system (3.44), for semisimple matrices P has components

$$\rho' = 0, \qquad \sigma' = -4\kappa t\mu, \qquad \tau' = 4\kappa t\nu, \tag{3.46}$$

while for nilpotent matrices P it has components

$$\rho' = -2t\nu, \qquad \sigma' = 4t\lambda, \qquad \tau' = 0. \tag{3.47}$$

Also we have the three conserved quantities $\ell^2 = \frac{1}{2}\mathrm{Tr}\ Q^2$, $m = \mathrm{Tr}\ PR$ and $n = \mathrm{Tr}\ QR$. In both semisimple and nilpotent cases, $\ell^2 = \lambda^2 + \mu\nu$ and $n = 2\rho\lambda + \mu\tau + \nu\sigma$. In the semisimple case $m = 2\kappa\rho$ while in the nilpotent case $m = \tau$.

We note that a P_{III} transcendent is $y(t) = t^{-1}u(t)$ [19, 20, 21] which solves P_{III},

$$y'' = \frac{y'^2}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 - \frac{\delta}{y}. \tag{3.48}$$

where $\{\alpha, \beta, \gamma, \delta\}$ is a set of constant parameters to be determined.

Provided that $\gamma\delta \neq 0$, we can rescale y and t such that without loss of generality, equation (3.48) can be transformed to

$$y'' = \frac{y'^2}{y} - \frac{y'}{t} + \frac{\hat{\alpha} y^2 + \hat{\beta}}{t} + y^3 - \frac{1}{y}, \tag{3.49}$$

which is P_{III} with $\hat{\gamma} = 1$ and $\hat{\delta} = -1$ [14]. Rational solutions of the transformed equation exist if

$$\hat{\alpha} + \epsilon\hat{\beta} = 4n, \qquad n \in \mathbb{Z}, \tag{3.50}$$

where $\epsilon = \pm 1$ [22].

P_{VI}

We will write P_{VI} as

$$P' = 0, \qquad tQ' = [R, Q], \qquad t(1-t)R' = [tP + Q, R], \tag{3.51}$$

The second equation of system (3.51), for both semisimple and nilpotent P, has components

$$t\lambda' = \sigma\nu - \mu\tau, \qquad t\mu' = 2(\rho\mu - \lambda\sigma), \qquad t\nu' = 2(\lambda\tau - \rho\nu). \tag{3.52}$$

The third equation of system (3.51), for semisimple P has components

$$(1-t)\rho' + \lambda' = 0, \qquad (1-t)\sigma' + \mu' = 2\kappa\sigma, \qquad (1-t)\tau' + \nu' = -2\kappa\tau, \tag{3.53}$$

while for nilpotent P it has components

$$(1-t)\rho' + \lambda' = \tau, \qquad (1-t)\sigma' + \mu' = -2\rho, \qquad (1-t)\tau' + \nu' = 0. \tag{3.54}$$

Also we have the three conserved quantities $\ell^2 = \frac{1}{2}\mathrm{Tr}\ Q^2$, $m^2 = \frac{1}{2}\mathrm{Tr}\ R^2$ and $n^2 = \frac{1}{2}\mathrm{Tr}\ (P+Q+R)^2$. In both semisimple and nilpotent cases, $\ell^2 = \lambda^2 + \mu\nu$ and $m^2 = \rho^2 + \sigma\tau$. In the semisimple case $n^2 = (\kappa + \lambda + \rho)^2 + (\mu + \sigma)(\nu + \tau)$ while in the nilpotent case $n^2 = (\lambda + \rho)^2 + (1 + \mu + \sigma)(\nu + \tau)$.

Finally, we note that a Painlevé transcendent can be obtained in all cases given, in terms of a root $u(t)$ of the equation

$$\det[P, uQ - R] = 0. \tag{3.55}$$

In the semisimple case, the roots are $u_1 = \frac{\tau}{\nu}$ and $u_2 = \frac{\sigma}{\mu}$, while in the nilpotent case we have the single repeated root $u = \frac{\tau}{\nu}$.

If we consider an Abelian group, then $P = 0$ which implies that the SDYM equations are trivial [20]. We shall deal with such cases separately.

We note that a $\mathrm{P_{VI}}$ transcendent is $y(t) = t(1 + (1-t)u(t))^{-1}$ [19, 20, 21] which solves $\mathrm{P_{VI}}$ (3.56),

$$\begin{aligned} y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right), \end{aligned} \tag{3.56}$$

where $\{\alpha, \beta, \gamma, \delta\}$ is a set of constant parameters to be determined.

3.5 The Ernst equation

We have seen that the SDYM equations are equivalent to Yang's equation (3.27) up to a gauge transformation. We shall reduce (3.27) to the Ernst equation, which has two independent variables (ρ, z) defined as $\rho = 2\sigma^{1/2}\bar{\sigma}^{1/2}$ and $z = \tau - \bar{\tau}$. Then Yang's equation for $J = J(\rho, z)$ is

$$\begin{aligned} 0 &= \partial_{\bar{\tau}}(J^{-1}\partial_\tau J) - \partial_{\bar{\sigma}}(J^{-1}\partial_\sigma J) \\ &= -\partial_z(J^{-1}\partial_z J) - \sigma^{1/2}\bar{\sigma}^{-1/2}\partial_\rho(J^{-1}\sigma^{-1/2}\bar{\sigma}^{1/2}\partial_\rho J) \end{aligned}$$

Substituting the above definition of ρ, we arrive at

$$\rho^{-1}\partial_\rho(\rho J^{-1}\partial_\rho J) + \partial_z(J^{-1}\partial_z J) = 0. \tag{3.57}$$

Where J is real, the components of equation (3.57) can also be obtained as reductions of the Einstein vacuum field equations for cases when the spacetime geometry admits a pair of commuting Killing vectors [19, 21, 31] which we shall study in chapter 4. All solutions of the Einstein vacuum equations with such symmetry were shown in [12] to correspond to solutions of the SDYM equations with gauge group $SU(2)$.

More specifically, to obtain the Ernst equation as originally derived in [35], we parameterize Yang's matrix as

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ g & f^2 + g^2 \end{pmatrix} \qquad f, g \in \mathbb{R}$$

On substitution of this Yang's matrix in (3.57) we obtain the system

$$f\nabla^2 f = \nabla f \cdot \nabla f - \nabla g \cdot \nabla g, \qquad f\nabla^2 g = 2(\nabla f) \cdot (\nabla g).$$

Introducing a complex-valued Ernst potential $\mathcal{E} = f + ig$, we can write the above system as the Ernst equation,

$$(\mathrm{Re}\ \mathcal{E})\nabla^2\mathcal{E} = \nabla\mathcal{E} \cdot \nabla\mathcal{E},$$

where the gradient $\nabla := (\partial_\rho, \partial_z)$ and Laplacian $\nabla^2 := \rho^{-1}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{\partial^2}{\partial z^2}$ are defined on $\mathbb{R}^3$ in cylindrical polar coordinates, acting only on functions that are independent of azimuth.

The Ernst equation is found [21] to have reductions to P_{III} and to P_{VI}. Referring to the table of Killing vectors given in the previous section which generate the five distinct three dimensional conformal symmetry groups, we have that in the P_{III} case, reduction of the SDYM equations by Y and $X + Z$ gives the Ernst equation. In the

P_{VI} case, reduction by $X+Y$ and $Y+Z$ also gives the Ernst equation [11, 21]. We then may determine solutions to the Ernst equation from the third or sixth Painlevé transcendents. In order that a solution be physical, $J(\rho, z)$ should be real. These reductions were originally presented in [21], and we shall present such reductions in chapter 5, made by postulating a separation of the two variables in the Ernst equation.

Chapter 4

Spacetimes with two commuting Killing vectors

In this chapter we shall work within the theory of general relativity, which is outlined in appendix B. We do not present any original work here, but shall consider four-dimensional solutions (spacetimes) that admit two commuting Killing vectors, which are said to be hypersurface orthogonal. The vacuum Einstein field equations are thus reduced to a system which includes the Ernst equation (or an analytic continuation of it). As we saw in chapter 3 the Ernst equation is integrable [11, 31], and has known reductions to certain Painlevé equations [21]. In the next chapter we apply these methods in the context of particular Bianchi spacetime models that admit two commuting Killing vectors, which is believed to be original for some such models.

An integrable equation, in the context of an inverse scattering method, yields well-behaved solutions via a related linear problem. The Ernst equation in general relativity is a nonlinear PDE which is a special case of the vacuum Einstein field equations, when the spacetime geometry admits two commuting Killing vectors. The compatibility condition of this linear system is the Ernst equation, which in turn can be reduced to either $\mathrm{P_{III}}$ or $\mathrm{P_{VI}}$.

4.1 Killing vectors

Consider a Riemannian manifold M endowed locally with a metric tensor $g_{\mu\nu}$. A Killing vector field K is said to be defined on a manifold M where its operation leaves the metric invariant; it is said to generate an isometry. By virtue of this it solves Killing's equation

$$\nabla_{(\mu}K_{\nu)} := \frac{1}{2}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0, \tag{4.1}$$

where ∇_μ is the covariant derivative compatible with $g_{\mu\nu}$ (in that we demand $\nabla_\mu g_{\nu\kappa} = 0$ over the subset of M on which the metric is defined.)

In terms of the isometry group of M, the orbit of a spacetime point $x \in M$ is the set of points to which x can be moved to by the elements of the isometry group, see for example [4].

4.2 Hypersurface orthogonality

We shall consider those local solutions of general relativity $(M, g_{\mu\nu})$ which admit two commuting, hypersurface orthogonal Killing vectors. It is known that Bianchi type I-VII models are amongst such spacetimes [31]. In a coordinate system labeled (t, x^1, x^2, z) we choose the commuting, spacelike Killing vectors $K_1 = \partial_1$ and $K_2 = \partial_2$ which generate translational symmetries in the x^1 and x^2 coordinates respectively, along a given hypersurface $h(x^1, x^2)$ = constant. We have the freedom to choose $K_a = \partial_a$, where $a = 1, 2$. In order for the metric to admit these Killing vectors, we must have $g_{\mu\nu} = g_{\mu\nu}(t, z)$ only.

A Killing vector is $K_a = K_a{}^b\partial_b$. We chose $K_a{}^b = \delta_a{}^b$, from which we have that $g_{a3} = g_{b3}\delta_a{}^b = (K_a)_3 = 0$ and similarly $g_{a0} = 0$ [8, 9]. The vacuum field equations together with the requirement that the metric signature should be $(-+++)$ lead to

$$ds^2 = f(t, z)(-dt^2 + dz^2) + g_{ab}(t, z)dx^a dx^b, \tag{4.2}$$

where $a, b = 1, 2$.

4.3 Light cone coordinates

We shall define two dimensional submanifolds of the spacetime manifold with metric (4.2) by the constraints x^a = constant. On any such submanifold the metric is

$$ds^2 = f(t,z)(-dt^2 + dz^2).$$

On this submanifold a null geodesic satisfies $dt^2 - dz^2 = 0$. We define the light cone (null) coordinates (ζ, η) by $t = \zeta - \eta$ and $z = \zeta + \eta$, in terms of which the equation of a null geodesic is $d\zeta d\eta = 0$. Then the metric (4.2) becomes [11]

$$ds^2 = 4f(\zeta,\eta)d\zeta d\eta + g_{ab}(\zeta,\eta)dx^a dx^b. \tag{4.3}$$

We shall use a coordinate system $(x^0, x^1, x^2, x^3) = (\zeta, x, y, \eta)$. We require that the conformal factor $f(\zeta,\eta)$ be a positive definite function in order to preserve metric signature $(-+++)$. Therefore the coordinate freedom $(\zeta,\eta) \to (\tilde{\zeta},\tilde{\eta})$ is allowed provided that $\tilde{\zeta}_\zeta \tilde{\eta}_\eta > 0$.

4.4 Vacuum Einstein field equations

In order to calculate the curvature tensor, we define the 2×2 matrix $g = (g_{ab})$,

$$g(\zeta,\eta) = \begin{pmatrix} a(\zeta,\eta) & b(\zeta,\eta) \\ b(\zeta,\eta) & c(\zeta,\eta) \end{pmatrix} \tag{4.4}$$

and define $\alpha^2 = \det g = ac - b^2$. It shall also be convenient to define the matrices

$$A = -\alpha g_\zeta g^{-1}, \qquad B = \alpha g_\eta g^{-1}. \tag{4.5}$$

The first step is to write the spacetime metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ defined by $g_{\mu\kappa} g^{\kappa\nu} = \delta^\nu{}_\mu$,

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & 2f \\ 0 & a & b & 0 \\ 0 & b & c & 0 \\ 2f & 0 & 0 & 0 \end{pmatrix}, \qquad (g^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2f} \\ 0 & \frac{c}{\alpha^2} & -\frac{b}{\alpha^2} & 0 \\ 0 & -\frac{b}{\alpha^2} & \frac{a}{\alpha^2} & 0 \\ \frac{1}{2f} & 0 & 0 & 0 \end{pmatrix}.$$

The affine connection coefficients are

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}), \tag{4.6}$$

from which we see that ${\Gamma^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\mu\nu}(\zeta, \eta)$ only and similarly for ${R^\lambda}_{\mu\kappa\nu}$. This, and the requirement that everything be symmetric under interchange of ζ and η, simplifies the calculation of the curvature tensor. In displaying the curvature we have substituted our definition of α where convenient.

$$\begin{array}{lll} {\Gamma^0}_{00} = (\ln f)_\zeta, & {\Gamma^0}_{11} = -\frac{a_\eta}{4f}, & {\Gamma^0}_{12} = -\frac{b_\eta}{4f}, \\ {\Gamma^0}_{22} = -\frac{c_\eta}{4f}, & {\Gamma^1}_{01} = \frac{1}{2\alpha^2}(ca_\zeta - bb_\zeta) & {\Gamma^1}_{02} = \frac{1}{2\alpha^2}(cb_\zeta - bc_\zeta), \\ {\Gamma^2}_{01} = \frac{1}{2\alpha^2}(-ba_\zeta + ab_\zeta), & {\Gamma^2}_{02} = \frac{1}{2\alpha^2}(-bb_\zeta + ac_\zeta). & \end{array} \tag{4.7}$$

The non vanishing components of the Ricci tensor are derived from the Riemann curvature tensor as the contraction $R_{\mu\nu} = {R^\lambda}_{\mu\lambda\nu}$, giving

$$R_{03} = -(\ln\alpha)_{\zeta\eta} - (\ln f)_{\zeta\eta} + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB,$$

$$R_{00} = -(\ln\alpha)_{\zeta\zeta} + (\ln f)_\zeta(\ln\alpha)_\zeta - \frac{1}{4\alpha^2}\mathrm{Tr}\ A^2,$$

$$R_{33} = -(\ln\alpha)_{\eta\eta} + (\ln f)_\eta(\ln\alpha)_\eta - \frac{1}{4\alpha^2}\mathrm{Tr}\ B^2,$$

$$R_{11} = -\frac{a_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(a(4b_\zeta b_\eta - a_\zeta c_\eta - a_\eta c_\zeta) - 2b(a_\zeta b_\eta + b_\zeta a_\eta) + 2ca_\zeta a_\eta),$$

$$R_{22} = -\frac{c_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(2ac_\zeta c_\eta - 2b(b_\zeta c_\eta + b_\eta c_\zeta) + c(4b_\zeta b_\eta - a_\zeta c_\eta - a_\eta c_\zeta)),$$

$$R_{12} = -\frac{b_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(a(b_\zeta c_\eta + b_\eta c_\zeta) - 2b(a_\zeta c_\eta + a_\eta c_\zeta) + c(a_\zeta b_\eta + a_\eta b_\zeta)).$$

Then the Ricci curvature scalar is

$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= -\frac{1}{f}\left(-(\ln\alpha)_{\eta\zeta} - (\ln f)_{\eta\zeta} + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB\right) \\ &+\frac{a}{\alpha^2}\left(\frac{c_{\eta\zeta}}{2f} - \frac{1}{8f\alpha^2}(2ac_\zeta c_\eta - 2b(b_\zeta c_\eta + b_\eta c_\zeta) + c(4b_\eta b_\zeta - a_\zeta c_\eta - a_\eta c_\zeta)\right) \\ &-\frac{2b}{\alpha^2}\left(\frac{b_{\eta\zeta}}{2f} - \frac{1}{8f\alpha^2}(a(b_\zeta c_\eta + b_\eta c_\zeta) - 2b(a_\zeta c_\eta + a_\eta c_\zeta) + c(a_\zeta b_\eta + a_\eta b_\zeta)\right) \\ &+\frac{c}{\alpha^2}\left(\frac{a_{\eta\zeta}}{2f} - \frac{1}{8f\alpha^2}(2ca_\zeta a_\eta - 2b(a_\zeta b_\eta + a_\eta b_\zeta) + a(4b_\eta b_\zeta - a_\eta c_\zeta - a_\eta c_\zeta)\right). \end{aligned}$$

We impose $R_{\mu\nu} = 0$, i.e. the vacuum field equations in the case $\Lambda = 0$. The dynamical equation is the Ernst equation, a matrix equation formed from the equations $R_{11} = R_{12} = R_{22} = 0$. Therefore we have [9]

$$A_\eta - B_\zeta = 0, \tag{4.8}$$

which governs the dynamics of $g(\zeta,\eta)$ [9]. It is an integrable equation [11, 15, 29, 31].

For any square matrix M, $\det e^M = e^{\mathrm{Tr}\ M}$. Making use of this identity and the cyclical property of the trace [1]

$$\mathrm{Tr}\ B_\zeta = \mathrm{Tr}\ (\alpha(\ln g)_\eta)_\zeta = (\alpha(\mathrm{Tr}\ \ln g)_\eta)_\zeta = (\alpha(\ln\alpha^2)_\eta)_\zeta = 2\alpha_{\zeta\eta} = -\mathrm{Tr}\ A_\eta.$$

With this result, taking the trace of equation (4.8) yields [31, 32]

$$\alpha_{\zeta\eta} = 0. \tag{4.9}$$

The other non-trivial vacuum field equations $R_{03} = R_{00} = R_{33} = 0$ are

$$(\ln f)_{\zeta\eta} = -(\ln\alpha)_{\zeta\eta} + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB, \tag{4.10}$$

$$(\ln f)_\zeta = \frac{(\ln\alpha)_{\zeta\zeta}}{(\ln\alpha)_\zeta} + \frac{1}{4\alpha\alpha_\zeta}\mathrm{Tr}\ A^2, \tag{4.11}$$

$$(\ln f)_\eta = \frac{(\ln\alpha)_{\eta\eta}}{(\ln\alpha)_\eta} + \frac{1}{4\alpha\alpha_\eta}\mathrm{Tr}\ B^2, \tag{4.12}$$

where equations (4.11) and (4.12) may be solved for $f(\zeta,\eta)$ using numerical methods, upon which (4.10) is trivialized [9].

For convenience in later calculations we shall write the reduced Einstein field equations in terms of the original coordinates (t,z) in the metric (5.26). The matrices $A = -\alpha(g_z + g_t)g^{-1}$ and $B = -\alpha(g_z - g_t)g^{-1}$.

We consider the Ricci-flat metrics of the above form - satisfying the equations $R_{\mu\nu} = 0$ for such a metric. In terms of a matrix $g = (g_{ab})$, the equations $R_{ab} = 0$ give

$$(\alpha g_t g^{-1})_t - (\alpha g_z g^{-1})_z = 0. \tag{4.13}$$

The trace equation (4.9) transforms to

$$\alpha_{tt} - \alpha_{zz} = 0. \tag{4.14}$$

The remaining nontrivial Einstein vacuum field equations, $R_{00} = 0$, $R_{33} = 0$ and $R_{03} = 0$ respectively, transform to

$$(\ln f)_z + (\ln f)_t = \frac{(\ln\alpha)_{zz} + 2(\ln\alpha)_{zt} + (\ln\alpha)_{tt}}{(\ln\alpha)_z + (\ln\alpha)_t} + \frac{\mathrm{Tr}\ A^2}{4\alpha(\alpha_z + \alpha_t)}, \tag{4.15}$$

[1] It is not clear whether the matrix logarithmic derivative $(\ln g)_\eta$ is equal to $g_\eta g^{-1}$ or to $g^{-1}g_\eta$. However using the cyclical property of the trace, $\mathrm{Tr}\ g_\eta g^{-1} = \mathrm{Tr}\ g^{-1}g_\eta$, so we can define $\mathrm{Tr}\ (\ln g)_\eta$ unambiguously.

$$(\ln f)_z - (\ln f)_t = \frac{(\ln\alpha)_{zz} - 2(\ln\alpha)_{zt} + (\ln\alpha)_{tt}}{(\ln\alpha)_z - (\ln\alpha)_t} + \frac{\operatorname{Tr} B^2}{4\alpha(\alpha_z - \alpha_t)}, \tag{4.16}$$

$$(\ln f)_{zz} - (\ln f)_{tt} = \frac{\operatorname{Tr} AB}{4\alpha^2} - (\ln\alpha)_{zz} + (\ln\alpha)_{tt}, \tag{4.17}$$

4.5 Vacuum Einstein field equations with cosmological constant

We consider the Einstein field equations (B.2) in vacuum $T_{\mu\nu} = 0$ with arbitrary cosmological constant Λ,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{4.18}$$

Contraction of (4.18) with the inverse metric $g^{\mu\nu}$ yields the trace equation $R = 4\Lambda$ in four spacetime dimensions.

We shall continue to work with the metric admitting two commuting Killing vectors (4.3)

$$ds^2 = 4f(\zeta,\eta)d\zeta d\eta + g_{ab}(\zeta,\eta)dx^a dx^b,$$

and will adopt the definitions of section 4.4.

It is convenient to define a quantity $X := -A_\eta + B_\zeta$. Then the dynamical equation (4.13) when $\Lambda = 0$ becomes $X = 0$. In order to generalize for any Λ, by reducing some of equations (4.18) to an analogous system, we aim to write the matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \tag{4.19}$$

in terms of components of the Ricci curvature tensor. We calculate

$$\begin{aligned} X_{11} &= 2\alpha^{-1}(a_{\eta\zeta}c - bb_{\eta\zeta}) + \tfrac{1}{2}\alpha^{-3}[(a_\eta c_\zeta + a_\zeta c_\eta)(ac - 2b^2) \\ &-4b_\zeta b_\eta ac - 2a_\zeta a_\eta c^2 + 3bc(a_\zeta b_\eta + a_\eta b_\zeta) + ab(b_\zeta c_\eta + b_\eta c_\zeta)] \end{aligned} \tag{4.20}$$

$$\begin{aligned} X_{12} &= 2\alpha^{-1}(b_{\eta\zeta}a - ba_{\eta\zeta}) + \tfrac{1}{2}\alpha^{-3}[(a_\eta c_\zeta + a_\zeta c_\eta)ab \\ &+4b_\zeta b_\eta ab + 2a_\zeta a_\eta bc - (ac + 2b^2)(a_\zeta b_\eta + a_\eta b_\zeta) - a^2(b_\zeta c_\eta + b_\eta c_\zeta)] \end{aligned} \tag{4.21}$$

$$\begin{aligned} X_{21} &= 2\alpha^{-1}(b_{\eta\zeta}c - bc_{\eta\zeta}) + \tfrac{1}{2}\alpha^{-3}[(a_\eta c_\zeta + a_\zeta c_\eta)bc \\ &+4b_\zeta b_\eta bc + 2c_\zeta c_\eta ab - c^2(a_\zeta b_\eta + a_\eta b_\zeta) - (ac + 2b^2)(b_\zeta c_\eta + b_\eta c_\zeta)] \end{aligned} \tag{4.22}$$

$$X_{22} = 2\alpha^{-1}(c_{\eta\zeta}a - bb_{\eta\zeta}) + \tfrac{1}{2}\alpha^{-3}[(a_\eta c_\zeta + a_\zeta c_\eta)(ac - 2b^2)$$
$$-4b_\zeta b_\eta ac - 2c_\zeta c_\eta a^2 + bc(a_\zeta b_\eta + a_\eta b_\zeta) + 3ab(b_\zeta c_\eta + b_\eta c_\zeta)] \tag{4.23}$$

Comparing these results with the relevant Riemann curvature tensor components,

$$R_{11} = -\frac{a_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(a(4b_\zeta b_\eta - a_\zeta c_\eta - a_\eta c_\zeta) - 2b(a_\zeta b_\eta + b_\zeta a_\eta) + 2ca_\zeta a_\eta),$$

$$R_{22} = -\frac{c_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(2ac_\zeta c_\eta - 2b(b_\zeta c_\eta + b_\eta c_\zeta) + c(4b_\zeta b_\eta - a_\zeta c_\eta - a_\eta c_\zeta)),$$

$$R_{12} = -\frac{b_{\zeta\eta}}{2f} + \frac{1}{8f\alpha^2}(a(b_\zeta c_\eta + b_\eta c_\zeta) - 2b(a_\zeta c_\eta + a_\eta c_\zeta) + c(a_\zeta b_\eta + a_\eta b_\zeta)).$$

we deduce that $X_{11} = \frac{4f}{\alpha}(-cR_{11} + bR_{12}) = \frac{4\Lambda f}{\alpha}(-ac + b^2) = -4\Lambda f\alpha$, $X_{12} = \frac{4f}{\alpha}(-aR_{12} + bR_{11}) = 0$, $X_{21} = \frac{4f}{\alpha}(-cR_{12} + bR_{22}) = 0$ and $X_{22} = \frac{4f}{\alpha}(-aR_{22} + bR_{12}) = -4\Lambda f\alpha$. Then

$$A_\eta - B_\zeta = 4\Lambda f\alpha I_2. \tag{4.24}$$

Taking the trace of equation (4.24) yields $\alpha_{\zeta\eta} = -2\Lambda f\alpha$.

It remains to calculate the other components of the Einstein field equations. They are $R_{00} = R_{33} = 0$ and $R_{03} = 2\Lambda f$,

$$2\Lambda f = -(\ln\alpha)_{\zeta\eta} - (\ln f)_{\zeta\eta} + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB,$$

$$(\ln f)_\zeta = \frac{(\ln\alpha)_{\zeta\zeta}}{(\ln\alpha)_\zeta} + \frac{1}{4\alpha\alpha_\zeta}\mathrm{Tr}\ A^2,$$

$$(\ln f)_\eta = \frac{(\ln\alpha)_{\eta\eta}}{(\ln\alpha)_\eta} + \frac{1}{4\alpha\alpha_\eta}\mathrm{Tr}\ B^2,$$

We can now substitute to find the following equation for $g(\zeta,\eta)$,

$$A_\eta - B_\zeta = 2\alpha\left(-(\ln\alpha)_{\zeta\eta} - \left(\frac{(\ln\alpha)_{\zeta\zeta}}{(\ln\alpha)_\zeta} + \frac{1}{4\alpha\alpha_\zeta}\mathrm{Tr}\ A^2\right)_\eta + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB\right) I_2. \tag{4.25}$$

and its trace,

$$\alpha_{\zeta\eta} = -\left(-(\ln\alpha)_{\zeta\eta} - \left(\frac{(\ln\alpha)_{\zeta\zeta}}{(\ln\alpha)_\zeta} + \frac{1}{4\alpha\alpha_\zeta}\mathrm{Tr}\ A^2\right)_\eta + \frac{1}{4\alpha^2}\mathrm{Tr}\ AB\right)\alpha. \tag{4.26}$$

We note that equations (4.25,4.26) while dependent only on g and its determinant α^2, are both of third order in α. We could reduce them to second order equations by substituting $\alpha_{\zeta\eta} = -2\Lambda f\alpha$, however this would unavoidably introduce the conformal factor f into the equations. We shall look at a way to remove f from the equations, by means of an extra constraint on the metric, in the next section.

Transforming back to the coordinates (t, z) equation (4.24) becomes

$$(\alpha g_z g^{-1})_z - (\alpha g_t g^{-1})_t = -2\Lambda f \alpha I_2 \tag{4.27}$$

and its trace yields $-\alpha_{tt} + \alpha_{zz} = -2\Lambda f \alpha$.

4.6 Generalization of the elliptic Ernst equation

We shall modify the metric (5.26) by introducing the spacelike coordinate $\rho = it$. Then

$$ds^2 = f(\rho, z)(d\rho^2 + dz^2) + g_{ab}(\rho, z)dx^a dx^b, \qquad a = 1, 2. \tag{4.28}$$

Such a metric admits one spacelike and one timelike Killing vector, and describes a stationary, axially symmetric gravitational field. This means the field does not change in time, and it admits a single spatial axis of symmetry.

In order to give the metric (4.28) signature $(- + ++)$, then $\alpha^2 := \det g$ must be negative, hence α is imaginary in this case. We define a real variable β by $\alpha = i\beta$, in terms of which equation (4.24) reduces to

$$\beta^{-1}(\beta g_z g^{-1})_z + \beta^{-1}(\beta g_\rho g^{-1})_\rho = -2\Lambda f I_2, \tag{4.29}$$

The trace of (4.24) in the present coordinates yields

$$\beta_{\rho\rho} + \beta_{zz} = -2\Lambda f \beta. \tag{4.30}$$

We note that $\beta(\rho, z)$ is a harmonic function if and only if the cosmological constant $\Lambda = 0$.

From equation (4.24), we obtain

$$-2\Lambda f g = \Delta g - g_z g^{-1} g_z - g_\rho g^{-1} g_\rho, \tag{4.31}$$

where we define a two dimensional gradient $\nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial z})$ and the Laplacian we shall use[2] is $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{\beta_\rho}{\beta}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{\beta_z}{\beta}\frac{\partial}{\partial z}$.

[2]The Laplacian is not on our defined spacetime manifold [4] with metric (4.33), but is on the metric

$$ds^2 = d\rho^2 + \beta^2 d\varphi^2 + dz^2. \tag{4.32}$$

We shall consider the specific metrics

$$ds^2 = h^{-1}(k(d\rho^2 + dz^2) + \beta^2 d\varphi^2) - h(dt + ad\varphi)^2 \tag{4.33}$$

with

$$(x^1, x^2) = (t, \varphi), \qquad f = h^{-1}k, \qquad g = \begin{pmatrix} -h & -ah \\ -ah & -a^2h + \beta^2 h^{-1} \end{pmatrix}, \tag{4.34}$$

where h, k, a, β are each functions of (ρ, z) only. We note that setting $\beta(\rho, z) = \rho$ yields the case, using Weyl-Papapetrou coordinates, in which case the Ernst equation was originally derived from some of the symmetry reduced Einstein field equations [35]. We see from substituting for β in equation (4.30) that in this case $\Lambda = 0$.

Writing the matrix g in component form as above, and defining $h := e^{2U}$, the 11 and 12 matrix components of equation (4.31) respectively are

$$\begin{aligned} -2\Lambda f h^2 &= h\Delta h - (\nabla h)^2 + \beta^{-2}h^4(\nabla a)^2, \\ -2\Lambda f a h &= a\Delta h - ah^{-1}(\nabla h)^2 + h(a_{\rho\rho} + a_{zz}) + 2\nabla a \cdot \nabla h \\ &\quad +\beta^{-2}ah^3(\nabla a)^2 - \beta^{-1}h\nabla\beta \cdot \nabla a. \end{aligned} \tag{4.35}$$

Substituting these equations into one another gives

$$0 = a^{-1}h^2(a_{\rho\rho}+a_{zz})+2a^{-1}h\nabla a\cdot\nabla h-\beta^{-1}a^{-1}h^2\nabla\beta\cdot\nabla a = \nabla\cdot(\beta^{-1}h^2\nabla a). \tag{4.36}$$

The 22 component of the matrix equation (4.31) yields

$$\begin{aligned} &0 = a^2\Delta h + \beta^2 h^{-2}\Delta h + 2ah(a_{\rho\rho} + a_{zz}) + h(\nabla a)^2 + 4a\nabla a \cdot \nabla h \\ &-2\beta^{-1}ah\nabla\beta \cdot \nabla a - 2\beta h^{-1}(\beta_{\rho\rho} + \beta_{zz}) + 2\Lambda f a^2 h - 2\Lambda f \beta^2 h^{-1} \\ &+\beta^{-2}a^2h^3(\nabla a)^2. \end{aligned} \tag{4.37}$$

Substituting the 11 component equation yields

$$0 = ah(a_{\rho\rho}+a_{zz})+2a\nabla a\cdot\nabla h-\beta^{-1}ah\nabla\beta\cdot\nabla a-\beta h^{-1}(\beta_{\rho\rho}+\beta_{zz}+2\Lambda f\beta), \tag{4.38}$$

which proves the wave equation for β (4.30) after substitution of equation (4.36).

The remaining function k is determined by solving the following two equations,

$$\begin{aligned} &\tfrac{k_\rho}{k} = \tfrac{1}{\beta_\rho^2+\beta_z^2}[2\beta_z\beta_{\rho z} + \beta_\rho(\beta_{\rho\rho} - \beta_{zz}) + h^{-2}\beta\beta_z(h_\rho h_z + b_\rho b_z) \\ &+\tfrac{1}{2}h^{-2}\beta\beta_\rho(h_\rho^2 - h_z^2 + b_\rho^2 - b_z^2)] \end{aligned} \tag{4.39}$$

$$\begin{aligned}\frac{k_z}{k} = \frac{1}{\beta_\rho^2+\beta_z^2}[2\beta_\rho\beta_{\rho z} + \beta_z(\beta_{zz} - \beta_{\rho\rho}) + h^{-2}\beta\beta_\rho(h_\rho h_z + b_\rho b_z) \\ -\tfrac{1}{2}h^{-2}\beta\beta_z(h_\rho^2 - h_z^2 + b_\rho^2 - b_z^2)].\end{aligned} \tag{4.40}$$

It is notable that the cosmological constant does not enter either of the equations (4.39,4.40).

We consider the complex variable, called an Ernst potential and first introduced in [35],

$$\mathcal{E}(\rho, z) = h(\rho, z) + ib(\rho, z), \tag{4.41}$$

where $b(\rho, z)$ is defined in terms of its first derivatives,

$$b_\rho = -\frac{h^2}{\beta}a_z, \qquad b_z = \frac{h^2}{\beta}a_\rho, \tag{4.42}$$

since the compatibility condition $(b_\rho)_z = (b_z)_\rho$ is then equivalent to the equation of motion (4.36). Then the matrix equation (4.31) can be written as the equation

$$(\mathrm{Re}\ \mathcal{E})(\Delta\mathcal{E} + 2\Lambda k) = (\nabla\mathcal{E})^2. \tag{4.43}$$

Here we have used the same operator notation as above. We note that when Λ vanishes equation (4.43) is the Ernst equation [35]. With general Λ, the real a nd imaginary parts of this equation respectively are

$$h(\Delta h + 2\Lambda k) = (\nabla h)^2 - (\nabla b)^2, \qquad h\Delta b = 2\nabla h \cdot \nabla b. \tag{4.44}$$

The former equation is the 11 component of the Einstein field equations, and the latter becomes an identity after imposing the conditions (4.42).

Chapter 5

Bianchi models

In this chapter we review the Bianchi classification of the distinct three dimensional Lie algebras and corresponding Lie groups into types I-IX, and the systematic application of this classification to Bianchi spacetime models in general relativity.

It is known that the Bianchi models of types I-VII admit two commuting Killing vectors, and in these cases we recall that the vacuum Einstein field equations are reduced to a system including the Ernst equation. Then a separation of variables is postulated under which the Ernst equation is reduced to a particular case of $\mathrm{P_{III}}$ or $\mathrm{P_{VI}}$, in accordance with the reductions obtained in [21]. Each Bianchi model falls into either class A or B; the class A reductions to particular cases of $\mathrm{P_{III}}$ are already known [33], but the class B reductions to particular cases of $\mathrm{P_{VI}}$ — following a change of independent variables — are believed to be new.

Bianchi models are four dimensional spacetime manifolds having local topology $M = \mathbb{R} \times \Sigma$, where Σ is a three dimensional space manifold. The Bianchi models are spatially homogeneous in the sense that they each admit a three dimensional group of isometries G, which acts simply transitively on each manifold Σ of the foliation by $\mathbb{R}$. The spatial coordinates are defined on Σ, which is foliated by a time coordinate. For a recent survey of Bianchi models in the context of general relativity, see [49]. The time evolution of the metric is determined by the Einstein field equations, which are reduced to a system of ODEs. For Bianchi types I-VII some of these ODEs are Painlevé equations, after a possible change of independent variable. Techniques from dynamical systems analysis can also be readily applied to systems of ODEs. The evolution of a universe described by a Bianchi model is determined solely from the time dependence of the metric, once we have specified the Bianchi type [36]. The Bianchi type of a model cannot change [37].

Physical motivations for the study of various Bianchi models are that they may describe local anisotropic regions of the observed universe, or large scale anisotropy in its early evolution. They are also useful to study the singularity structure of general relativity [6, 43].

5.1 Bianchi algebra classification

In 1898, Bianchi published a list of the nine types of three parameter Lie algebra [39]. He labeled them as Bianchi types I to IX, where types VI and VII are each one-parameter families of algebras, while the other types are each single algebras [6]. It is not possible to convert between one Bianchi type and another by means of a change of basis. We shall derive the Bianchi classification of structure constants, following [30].

Let $\mathfrak{g}$ be a three parameter Lie algebra, which admits a basis $\{e_i\}$, $i = 1, 2, 3$ where $e_i = e_i{}^j\partial_j$. Recall from appendix A that the algebra is defined, up to global topological considerations, by its structure constants determined by the commutation relations $[e_i, e_j] = C^k{}_{ij}e_k$ [34]. For the case of the Bianchi classification, the structure constants may be decomposed as [37]

$$C^k{}_{ij} = \epsilon_{ijl}N^{lk} + 2A_{[i}\delta^k{}_{j]}, \tag{5.1}$$

where $N^{ij} = N^{(ij)}$.

Taking the trace of the structure constants (5.1) we find [4, 36]

$$A_i = \frac{1}{2}C^j{}_{ij},$$

$$N^{ij} = \frac{1}{2}\epsilon^{ikl}(C^j{}_{lk} - \delta^j{}_k A_l).$$

which completely define a three dimensional Lie algebra. Here ϵ^{ijk} is the totally antisymmetric tensor.

Any Lie algebra is required to satisfy the Jacobi identity $C^i{}_{[jk}C^l{}_{m]i} = 0$, which gives the condition

$$0 = (A_j\epsilon_{kit} + A_i\epsilon_{jkt} + A_k\epsilon_{ijt})N^{tm} + (\epsilon_{ijs}\delta^m_k + \epsilon_{kis}\delta^m_j + \epsilon_{jks}\delta^m_i)N^{sl}A_l$$

$$+(\epsilon_{ijs}\epsilon_{klt} + \epsilon_{kis}\epsilon_{jlt} + \epsilon_{jks}\epsilon_{ilt})N^{sl}N^{tm}.$$

The first term and the third term vanish identically, since they are the product of a symmetric part and an antisymmetric part, but the second term does not. Therefore we require that a three dimensional algebra of the above form must satisfy

$$N^{ij}A_j = 0. \tag{5.2}$$

It follows from the definition of a Lie bracket that the structure constants behave as a tensor. Therefore Lie algebras are defined only up to a change of basis $e_i \rightarrow e'_i = R_i{}^j e_j$ where $R \in GL(3,\mathbb{R})$. The transformations of the structure constants are [30]

$$C'^i_{jk} = (R^{-1})_l{}^i R_j{}^m R_k{}^n C^l{}_{mn}, \tag{5.3}$$

$$N'^{ij} = (\det R)(R^{-T}NR^{-1})^{ij}, \tag{5.4}$$

$$A'_i = R_i{}^j A_j, \tag{5.5}$$

where R^{-T} denotes the transpose of the inverse of R or equivalently, the inverse of the transpose. Without loss of generality, the above rotations allow us to take $N^{ij} = \text{diag }(N_1, N_2, N_3)$ and $A_i = (0, 0, A)$ [36]. Class A Bianchi algebras are those for which the identity part of the structure constants $A = 0$, while class B Bianchi algebras have $A \neq 0$ and therefore a nontrivial identity part [37].

We shall enumerate the Bianchi classification by noting that the invariants of N^{ij} are its rank (number of non vanishing diagonal components) and its signature. Nonvanishing eigenvalues may be normalised as $\lambda_i = \pm 1$. With such a normalisation the signature is given by the modulus of the trace of N^{ij}. We shall use Bianchi's classification scheme. The possible cases are [30]:

Rank $N^{ij} = 3$: $N^{ij} = \text{diag }(\lambda_1, \lambda_2, \lambda_3)$. The Jacobi identity (5.2) tells us that in this case all components of A_i necessarily vanish. Therefore the possible algebras are $N^{ij} = \text{diag }(1, 1, 1)$, Tr $N^{ij} = 3$ (Bianchi IX) and $N^{ij} = \text{diag }(1, -1, 1)$, Tr$N^{ij} = 1$ (Bianchi VIII).

Rank $N^{ij} = 2$: $N^{ij} = \text{diag}\ (\lambda_1, \lambda_2, 0)$. In this case the Jacobi identity gives the most general form $A_i = (0, 0, A)$, where A is a further invariant parameter which gives families of algebras, including special cases we shall return to later. The families are $N^{ij} = \text{diag}\ (1, 1, 0)$, Tr $N^{ij} = 2$ (Bianchi VII$_h$) and $N^{ij} = \text{diag}(1, -1, 0)$, Tr $N^{ij} = 0$ (Bianchi VI$_h$). Here we find it convenient to define $A = \sqrt{hN^{11}N^{22}}$. However for Bianchi type VI$_h$ there is a restriction $h \neq -1$. This case turns out to be unique and we classify it as Bianchi III [18].

The invariant h which arises in the above rank 2 cases is defined by

$$(1 - h)C^k_{ik}C^l_{\ l} = -2hC^k_{\ il}C^l_{\ jk}. \tag{5.6}$$

Rank $N^{ij} = 1$: $N^{ij} = \text{diag}\ (\lambda_1, 0, 0)$. In this case, the Jacobi identity together with arbitrary rotations give that either $A_i = 0$ or $A_i = (0, 0, 1)$. Here, the only possibility is $N^{ij} = \text{diag}\ (1, 0, 0)$, with $A_i = 0$ (Bianchi II) or with $A_i = (0, 0, 1)$ (Bianchi IV).

Rank $N^{ij} = 0$: $N^{ij} = 0$. As in the previous case we find that either $A_i = 0$ (Bianchi I) or $A_i = (0, 0, 1)$ (Bianchi V).

A summary of the results is [34, 41]

Bianchi type	A	N_1	N_2	N_3	Class
I	0	0	0	0	A
II	0	1	0	0	A
III=VI$_{-1}$	1	1	-1	0	B
IV	1	1	0	0	B
V	1	0	0	0	B
VI$_0$	0	1	-1	0	A
VI$_h$ $h \neq 0, -1$	$\sqrt{-h}$	1	-1	0	B
VII$_0$	0	1	1	0	A
VII$_h$ $h \neq 0$	$\sqrt{h}$	1	1	0	B
VIII	0	1	-1	1	A
IX	0	1	1	1	A

An important property of Bianchi I to VII algebras which results from the observation $C^i_{\ 12} = 0$ is that two of the elements $\{e_i\}$ commute. Therefore they admit an Abelian invariant subalgebra [7]. The linearly independent structure constants corresponding to the Bianchi I-VII algebras are

Bianchi type	${C^1}_{31}$	${C^1}_{23}$	${C^2}_{31}$	${C^2}_{32}$
I	0	0	0	0
II	0	1	0	0
III	1	1	-1	1
IV	1	1	0	1
V	1	0	0	1
VI_h	$\sqrt{-h}$	1	-1	$\sqrt{-h}$
VII_h	$\sqrt{h}$	1	1	$\sqrt{h}$

We can use the trace to distinguish a class A for which the trace of the structure constant matrix, ${C^i}_{ij}$ vanishes, for which $A_i = 0$ (including Bianchi I, II, VI_0 and VII_0), and a class B for which the trace does not vanish, for which $A_i \neq 0$ (including Bianchi III, IV, V, VI_h and VII_h) [6, 36, 37].

In this article we choose to focus on Bianchi algebras of types I-VII, which share the property $N_3 \equiv 0$. These types may be parameterized by the 2×2 matrix $K = ({C^a}_{b3})$ where $a = 1, 2$. Explicitly

$$K = \begin{pmatrix} A & N_1 \\ -N_2 & A \end{pmatrix}. \tag{5.7}$$

Defining $k = \mathrm{Tr}\ K$, we note that class A Bianchi algebras of types I-VII have $k = 0$, while class B Bianchi algebras have $k \neq 0$. In what follows we shall find it necessary to consider the two cases separately.

We shall introduce, for later convenience, the zero trace constant matrix $\hat{K} = K - \frac{k}{2} I_2$, which is valued in a Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$, and generates a Lie group $\hat{H} \subset SL(2, \mathbb{R})$. Explicitly

$$\hat{K} = \begin{pmatrix} 0 & N_1 \\ -N_2 & 0 \end{pmatrix}. \tag{5.8}$$

We note that this matrix does not distinguish between class A and class B algebras due to the subtraction of the trace.

A further classification distinguishing the Bianchi I-VII algebras involves asking whether the corresponding matrix $\hat{K}$ is either nilpotent or semisimple.[1] The classifications are shown in the following table.

	Class A	Class B
Nilpotent	I,II	IV,V
Semisimple	VI_0, VII_0	III, VI_h, VII_h

5.2 Bianchi groups

A Bianchi group G of type I-IX is generated by a Bianchi algebra $\mathfrak{g}$ of the same type [39] as explained in Appendix A. In this thesis we choose to focus on Bianchi groups of types I-VII, which take the local form

$$G = \mathbb{R} \times H,$$

where the defining subgroup $H \subset GL(2,\mathbb{R})$. In contrast we note that the Bianchi VIII and IX groups are isomorphic to $SO(2,1)$ and $SO(3)$ respectively.

The exterior derivative of a 1-form $\omega(X)$ is

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]), \tag{5.9}$$

in terms of a second arbitrary vector field Y.

A Maurer-Cartan 1-form ω_G is defined globally on a Lie group G, which like any Lie group may be identified with a manifold. Let T_gG be the tangent space to G at $g \in G$. In particular if e denotes the identity element of G then T_eG is the Lie algebra $\mathfrak{g}$ which generates G, see [46] for more details. Then the Maurer-Cartan form is a linear mapping

$$\omega_G : e_i|_g \longrightarrow e_i|_e, \tag{5.10}$$

where the set of left invariant vector fields $\{e_i|_g\}$ comprise a basis for T_gG, and it follows that $\{e_i|_e\}$ are the generators of the Lie algebra.

[1] A square matrix M is called nilpotent if there exists some integer q such that $M^q = 0$. Otherwise if no such integer exists then M is called semisimple.

Let e_i be a dual basis to the Maurer-Cartan forms ω^i_G such that $\omega^i_G(e_j) = \delta^i{}_j$. Returning to equation (5.9) and defining $X = e_i$ and $Y = e_j$, in component form

$$d\omega^k_G(e_i, e_j) = e_i\omega^k_G(e_j) - e_j\omega^k_G(e_i) - \omega^k_G([e_i, e_j]) = -C^k{}_{ij},$$

where the structure constants have been introduced by $[e_i, e_j] = C^k_{ij}e_k$. Henceforth we neglect the G subscript, and arrive at the Maurer-Cartan structure equations for the components $(d\omega^k)_{ij} = d\omega^k(e_i, e_j)$,

$$d\omega^i = \frac{1}{2}(d\omega^i)_{jk}\omega^j \wedge \omega^k = -\frac{1}{2}C^i{}_{jk}\omega^j \wedge \omega^k. \tag{5.11}$$

which states that the Maurer-Cartan derivative forms $d\omega^i$ are each 2-forms having components $-C^i_{jk}$. We find it convenient to impose the boundary conditions $\omega^i{}_j|_{z=0} = \delta^i{}_j$. We proceed by substituting for each form of $C^k{}_{ij}$ given in the above table. Motivated by the form of the metric resulting from the hypersurface orthogonality condition, we choose a basis in each case of a Bianchi I-VII group of the form [33],

$$\omega^a(z) = \omega^a{}_b(z)dx^b, \qquad \omega^3 = dz. \tag{5.12}$$

Then $d\omega^3 = d^2z = 0$ in all cases. Our aim is to derive the 1-forms $\omega^a(z)$ from the structure constants for each Bianchi algebra, as classified in section 5.1. In the Bianchi I case the Maurer-Cartan structure equations reduce to $d\omega^1 = 0, d\omega^2 = 0$, from which $\omega^1 = Adx + Bdy$ and $\omega^2 = Cdx + Ddy$, where upper case letters A, B, C and D are arbitrary constants to be determined from the boundary conditions $\omega^i{}_j|_{z=0} = \delta^i{}_j$. Then we find $A = D = 1$ and $B = C = 0$.

In the Bianchi II case we have $d\omega^1 = \omega^3 \wedge \omega^2, d\omega^2 = 0$, so $\omega^2 = Edx + Fdy$ and

$$\frac{d\omega^1{}_1}{dz}dz \wedge dx + \frac{d\omega^1{}_2}{dz}dz \wedge dy = \omega^2{}_1 dz \wedge dx + \omega^2{}_2 dz \wedge dy = Edz \wedge dx + Fdz \wedge dy.$$

Therefore we have two independent equations that give $\omega^1{}_1 = Ez + G$ and $\omega^1{}_2 = Fz + H$. Using the boundary conditions, we find that $\omega^1 = dx + zdy$ and $\omega^2 = dy$.

For the remaining Bianchi IV-VII groups we shall give the appropriate reductions of the Maurer-Cartan equations for $\omega^a{}_b(z)$,

For Bianchi IV groups,

$$\frac{d\omega^1{}_a}{dz} = \omega^2{}_a - \omega^1{}_a, \qquad \frac{d\omega^2{}_a}{dz} = -\omega^2{}_a.$$

For Bianchi V groups,

$$\frac{d\omega^1{}_a}{dz} = -\omega^1{}_a, \qquad \frac{d\omega^2{}_a}{dz} = -\omega^2{}_a.$$

For Bianchi VI$_h$ groups,

$$\frac{d\omega^1{}_a}{dz} = \omega^2{}_a - \sqrt{-h}\omega^1{}_a, \qquad \frac{d\omega^2{}_a}{dz} = \omega^1{}_a - \sqrt{-h}\omega^2{}_a.$$

For Bianchi VII$_h$ groups,

$$\frac{d\omega^1{}_a}{dz} = \omega^2{}_a - \sqrt{h}\omega^1{}_a, \qquad \frac{d\omega^2{}_a}{dz} = -\omega^1{}_a - \sqrt{h}\omega^2{}_a.$$

Specifying the boundary conditions $\omega^a{}_b(0) = \delta^a{}_b$, we find that the exact form of each dual basis is

Bianchi type	$\omega^1{}_1$	$\omega^1{}_2$	$\omega^2{}_1$	$\omega^2{}_2$
I	1	0	0	1
II	1	z	0	1
III	$e^{-z}\cosh z$	$e^{-z}\sinh z$	$e^{-z}\sinh z$	$e^{-z}\cosh z$
IV	e^{-z}	ze^{-z}	0	e^{-z}
V	e^{-z}	0	0	e^{-z}
VI$_0$	$\cosh z$	$\sinh z$	$\sinh z$	$\cosh z$
VI$_h$	$e^{-\sqrt{-h}z}\cosh z$	$e^{-\sqrt{-h}z}\sinh z$	$e^{-\sqrt{-h}z}\sinh z$	$e^{-\sqrt{-h}z}\cosh z$
VII$_0$	$\cos z$	$\sin z$	$-\sin z$	$\cos z$
VII$_h$	$e^{-\sqrt{h}z}\cos z$	$e^{-\sqrt{h}z}\sin z$	$-e^{-\sqrt{h}z}\sin z$	$e^{-\sqrt{h}z}\cos z$

The tabulated one-forms for Bianchi types I, II, IV and V are the same as those in [6].

5.3 Bianchi spacetime models

A spatially homogeneous spacetime is one which possesses a three dimensional group of isometries whose orbits are a one-parameter family of spacelike hypersurfaces which foliate (pass through every point of) the spacetime, such that for each value of the time coordinate t and for arbitrary points $p, q \in \Sigma_t$, there exists an isometry of $g_{\mu\nu}$ which takes p into q. In a spatially homogeneous spacetime $(M, g_{\mu\nu})$ there exists a family of spacelike hypersurfaces Σ_t such that for arbitrary $p, q \in \Sigma_t$ there exists an element $g \in G$ of the Lie group of isometries, $g : M \to M$ such that $g(p) = q$. In the case where each g is unique in this way, G is said to act simply transitively on each Σ_t. This is the case for all Bianchi models, and implies that $\dim \Sigma_t = \dim G = 3$ [4].

Considering a simply transitive action, we can put the elements of G into correspondence with arbitrary points $p \in \Sigma_t$ by the association $g \to g(p)$. The action of Σ_t of the isometry $g \in G$ corresponds to left multiplication by g on G [4].

Bianchi models each admit a three dimensional isometry Lie group G that acts simply transitively on each leaf Σ_t of the homogeneous foliation [32]. Consequently, there exists a set of three left invariant vector fields $\{e_i\}$ on Σ_t which form the Lie algebra of G. It is found that the basis $\{e_i\}$ in all cases of Bianchi models depend only on the spatial coordinates $\vec{x} = (x, y, z)$ [36]. We define a dual basis $\{\omega^i\}$ by $\omega^i(e_j) = \delta^i{}_j$, having components $\omega^i = \omega^i{}_j dx^j$. The condition of spatial homogeneity implies that the Bianchi models take the form [4]

$$ds^2 = -dt^2 + \gamma_{ij}(t)\omega^i(\vec{x})\omega^j(\vec{x}). \tag{5.13}$$

We introduce a coordinate basis $\{dx^i\}$, in terms of which the metric transforms as

$$g_{ij}(t, \vec{x})dx^i dx^j = \gamma_{kl}(t)\omega^k(\vec{x})\omega^l(\vec{x}). \tag{5.14}$$

The manifold structure of a spatially homogeneous spacetime on which the isometry group G acts in a simply transitive manner is $M = \mathbb{R} \times G$. We can make a change of coordinate $t \to T(t)$ which scales the 00 component to any desired function of t. We define the timelike vector $e_0 = \partial_t$ which generates translations along the real line $\mathbb{R}$ in the topological product. It commutes with the elements of the basis $\{e_i\}$.

In the case of a homogeneous model, we may introduce a coordinate system in spacetime such that the dependence of the g_{ij} on the spatial coordinates is determined by the symmetry requirements, as determined from the particular structure constants. The Einstein field equations then determine the time dependence of the g_{ij}, and also the allowed forms of $g_{00}(t)$ given $g_{ij}(t, \vec{x})$.

Three spacelike Killing vectors $\{K_i\}$ form a basis of the isometry Lie group G admitted by a Bianchi type metric. Killing vectors correspond to the right invariant vector fields on G. Their components are given by $K_i = K_i{}^j \partial_j$.

The Lie derivatives of the invariant basis $\{K_i\}$ with respect to the Killing vectors vanish,

$$\mathcal{L}_{K_i} e_j = [K_i, e_j] = 0. \tag{5.15}$$

Due to the simple transitivity the Killing vectors also form a basis of a spatial hypersurface Σ_t, as does $\{e_i\}$. In light of this the elements of the bases are related by linear transformations. Furthermore we define $e_i|_p = K_i|_p$ at a fixed spatial point $p \in \Sigma_t$. This is consistent with equation (5.15). Then

$$e_i = a_i^{\ j} K_j, \tag{5.16}$$

$$a_i^{\ j}|_p = \delta_i^{\ j}. \tag{5.17}$$

By substituting (5.16) in (5.15), and comparing this evaluated at p with the Lie algebra definition (A) we find that

$$K_i a_j^{\ k}|_p = C^k_{\ ij}. \tag{5.18}$$

We may evaluate

$$[e_i, e_j] = [a_i^{\ k} K_k, a_j^{\ l} K_l] = a_i^{\ j}(K_k a_j^{\ l})K_l - a_j^{\ l}(K_l a_i^{\ k})K_k + a_i^{\ j} a_j^{\ l}[K_k, K_l].$$

at p, and comparison with the Lie algebra definition of appendix A gives the definition

$$[K_i, K_j] = -C^k_{\ ij} K_k, \tag{5.19}$$

which tells us that since the $C^k_{\ ij}$ are independent of position, the Killing vectors satisfy a Lie algebra at all points in the spacelike hypersurface Σ_t

Once we have a basis of three Killing vectors $\{K_i\}$, we can generate a Lie group G which is the isometry group of M by means of exponentiation, to obtain a group element $g \in G$ as

$$g = \exp(\theta^i K_i), \qquad \text{(no sum)}, \tag{5.20}$$

where the $\{\theta^i\}$ are a set of three arbitrary parameters [6].

Although the condition that the Bianchi types each admit simply transitive groups requires there to exist three Killing vectors, the full group of isometries may well be bigger [30].

We consider the reduced metric $\gamma_{ij}(t)$ which was defined in (5.13) as a three-metric. For Bianchi I-VII models we have from the condition of hypersurface orthogonality that $\gamma_{3i} = \delta_{3i}$, so we consider the two-metric γ_{ab}. In the first instance, we shall choose $\gamma_{ab}(t) = \mathrm{diag}(a^2(t), b^2(t))$ as the reduced metric [33]. Using such a basis, the metrics of Bianchi spacetimes as defined in (5.23) take the forms $g_{ab}(t,z) = \gamma_{cd}(t)\omega^c{}_a(z)\omega^d{}_b(z)$.

Now that we have developed the theory of Bianchi models, we shall specialize it to Bianchi types I-VII, naming $x^3 = z$ in all such cases. Substituting the appropriate structure constants into the Maurer-Cartan structure equations (5.11), we find that $\omega^3 = dz$, and $\omega^a(z) = \omega^a{}_b(z)dx^b$ where $a = 1,2$, and the 2×2 matrix $\omega = (\omega^a{}_b)$ satisfies the equation

$$\omega_z = K\omega, \tag{5.21}$$

with K the matrix defined by (5.7) describing the relevant Bianchi type I-VII algebra; this is solved by $\omega(z) = e^{Kz}$. Then (5.14) reduces to the matrix equation

$$g(t,z) = e^{K^T z}\gamma(t)e^{Kz}, \tag{5.22}$$

where the 2×2 matrices $g = (g_{ab})$ and $\gamma = (\gamma_{ab})$.

We see that the metric (4.2) derived under the condition of hypersurface orthogonality is only of Bianchi I-VII type if an obstruction condition is satisfied by the conformal factor $f(t,z) = f(t)$. Therefore

$$ds^2 = f(t)(-dt^2 + dz^2) + g_{ab}(t,z)dx^a dx^b. \tag{5.23}$$

and we find from the above 1-forms the following classes of left invariant metrics $g_{ab}(t,z)$.

Bianchi type	$g_{ab}(t,z)dx^a dx^b$
I	$a^2(t)dx^2 + b^2(t)dy^2$
II	$a^2(t)(dx + zdy)^2 + b^2(t)dy^2$
III	$e^{-2z}(a^2(t)(\cosh zdx + \sinh zdy)^2 + b^2(t)(\sinh zdx + \cosh zdy)^2)$
IV	$e^{-2z}(a^2(t)(dx + zdy)^2 + b^2(t)dy^2)$
V	$e^{-2z}(a^2(t)dx^2 + b^2(t)dy^2)$
VI_0	$a^2(t)(\cosh zdx + \sinh zdy)^2 + b^2(t)(\sinh zdx + \cosh zdy)^2$
VI_h $(h < 0, h \neq -1)$	$e^{-2\sqrt{-h}z}(a^2(t)(\cosh zdx + \sinh zdy)^2 + b^2(t)(\sinh zdx + \cosh zdy)^2)$
VII_0	$a^2(t)(\cos zdx + \sin zdy)^2 + b^2(t)(\sin zdx - \cos zdy)^2$
VII_h $(h > 0)$	$e^{-2\sqrt{h}z}(a^2(t)(\cos zdx + \sin zdy)^2 + b^2(t)(\sin zdx - \cos zdy)^2)$

As we noted, the analysis is greatly simplified for Bianchi type I-VII models compared with Bianchi type VIII and IX models. For completeness, we shall give some explicit forms of these metrics, found using the Maurer-Cartan equations with the appropriate structure constants. We find the Bianchi type VIII metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(\cosh zdx - \sinh z \sinh xdy)^2 \\ &+b^2(t)(\sinh zdx - \cosh z \sinh xdy)^2 + c^2(t)(dz + \cosh xdy)^2, \end{aligned} \tag{5.24}$$

and the Bianchi type IX metric

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(\cos zdx + \sin z \sin xdy)^2 \\ &+b^2(t)(-\sin zdx + \cos z \sin xdy)^2 + c^2(t)(dz + \cos xdy)^2. \end{aligned} \tag{5.25}$$

5.4 Einstein field equations for Bianchi I-VII models

In the current chapter we have seen that the Bianchi I-VII spacetime models admit two commuting Killing vectors. The reductions of the Einstein field equations for these spacetimes are described in chapter 4. The spacetime metric to be considered is

$$ds^2 = f(t,z)(-dt^2 + dz^2) + g_{ab}(t,z)dx^a dx^b. \tag{5.26}$$

For strict compliance with the Bianchi models we should choose $f = f(t)$ only in (5.26). However since we have already assumed the more general case, we shall not impose this restriction until necessary.

Here we shall only consider the case in which the cosmological constant vanishes. Then equations (4.13-4.17) are the appropriate reductions [33]. We will reduce these further by imposing the following separation of variables postulate, in line with the preceding analysis,

$$g(t,z) = e^{K^T z}\gamma(t)e^{Kz},$$

in accordance with (5.22). Taking its determinant,

$$\alpha^2 := \det g = (\det e^{K^T z})(\det \gamma)(\det e^{Kz}) = \sigma^2 e^{2kz},$$

where we define $\sigma^2 := \det \gamma$ and $k := \mathrm{Tr}\ K$ [2]. Then we must also require that

$$\alpha(t,z) = \sigma(t)e^{kz}. \tag{5.27}$$

With the definition (5.27), the wave equation (4.14) reduces to the ODE in t,

$$\ddot{\sigma} = k^2\sigma, \tag{5.28}$$

where an overhead dot always denotes a total derivative with respect to the metric time coordinate t.

We find that $\hat{g} = \alpha^{-1}g$ also solves (4.13), with $\det \hat{g} = 1$. The transformed metric $\hat{g}$ can be written as

$$\hat{g}(t,z) = e^{\hat{K}^T z}J(t)e^{\hat{K}z} \tag{5.29}$$

with $\hat{K} = K - \frac{k}{2}I_2$ being the tracefree part of K, and $\det J = 1$. Since $\hat{g}$ solves (4.13) we can substitute (5.29) in it to find the ODE in t,

$$\frac{d}{dt}(\sigma \dot{J}J^{-1}) = \sigma[\hat{K}^T, J\hat{K}J^{-1}] + k\sigma(\hat{K}^T + J\hat{K}J^{-1}). \tag{5.30}$$

We shall consider separately those cases where $k = 0$ (Bianchi class A spacetimes), and those where $k \neq 0$ (Bianchi class B spacetimes).

[2]For any square matrices A and B,

$$\det e^A \equiv e^{\mathrm{Tr}\ A}, \qquad \det B \equiv \det B^T$$

5.4.1 Bianchi class A models

We recall that Bianchi class A models are types I, II, VI_0 and VII_0, VIII and IX, of which we shall only work with the first four. The vacuum Einstein field equations for these four types are reduced by a separation of variables to a system including a form of $\mathrm{P_{III}}$, first given in [33].

Here the linear wave equation for $\alpha(t,z)=\sigma(t)$ simplifies to $\ddot{\sigma}=0$, whose general solution is $\sigma = At+B$. With the assumption that $A\neq 0$, we shift and rescale to get $\sigma = t$ [32]. Noting $k=0$ and $\hat{K}=K$ for Bianchi class A groups, equation (5.30) then reduces to

$$\frac{d}{dt}(t\dot{J}J^{-1}) = t[K^T, JKJ^{-1}]. \tag{5.31}$$

If we now define $P=K^T$, $Q=\frac{1}{4}JKJ^{-1}$ and $R=-\frac{1}{2}t\dot{J}J^{-1}$, then we obtain the system of equations,

$$\dot{P}=0, \qquad t\dot{Q}=2[Q,R], \qquad \dot{R}=2t[Q,P]. \tag{5.32}$$

where the first two are consequences of the parameterization whilst the third is equivalent to equation (5.31). For general $P,Q,R\in\mathfrak{sl}(2,\mathbb{C})$, system (5.32) is equivalent to $\mathrm{P_{III}}$ [19]. The matrix P can be transformed to either

$$P=\begin{pmatrix}\kappa & 0\\ 0 & -\kappa\end{pmatrix} \qquad \text{or} \qquad P=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \tag{5.33}$$

according to whether $\kappa\neq 0$ (when P is semisimple) or $\kappa=0$ (when P is nilpotent), see [19] for details. Equivalently $\kappa^2=\frac{1}{2}\mathrm{Tr}\,P^2$.

We define the components of J as

$$J(t)=\begin{pmatrix}A(t) & B(t)\\ B(t) & C(t)\end{pmatrix} \tag{5.34}$$

and have that $\det J = AC-B^2=1$. Writing Q,R in terms of this with P in semisimple form gives the components defined in (3.36,3.37),

$$\begin{aligned} \lambda &= \tfrac{1}{4}\kappa(AC+B^2), & \mu &= -\tfrac{1}{2}\kappa AB, & \nu &= \tfrac{1}{2}\kappa BC,\\ \rho &= \tfrac{t}{2}(\dot{B}B-\dot{A}C), & \sigma &= \tfrac{t}{2}(\dot{A}B-A\dot{B}), & \tau &= \tfrac{t}{2}(B\dot{C}-\dot{B}C). \end{aligned} \tag{5.35}$$

Considering the nilpotent case, ρ, σ, τ are the same as in the semisimple case, but instead

$$\lambda = \tfrac{1}{4}BC, \quad \mu = -\tfrac{1}{4}B^2, \quad \nu = \tfrac{1}{4}C^2. \tag{5.36}$$

This parameterization on the nilpotent case gives $\ell^2 = \lambda^2 + \mu\nu = 0$.

Now that we have P, Q, R written in terms of (components of) $J(t)$, we wish to use equation (3.55) to write them in terms of a Painlevé transcendent $y(t)$. In the nilpotent case, we find it necessary to consider separately the cases $m := \tau = 0$ and $m \neq 0$. With $m = 0$ the equations reduce to triviality. With $m \neq 0$ we find that the independent components of $J(t)$ are

$$B(t) = -m^{1/2}t^{-1/2}y^{-1/2}Y, \tag{5.37}$$

$$C(t) = 2m^{1/2}t^{-1/2}y^{-1/2}, \tag{5.38}$$

where $Y(t) = \int dt y(t)$.

In the semisimple case, the independent components are

$$B(t) = \exp\left(\tfrac{1}{2}\int dt\left(\kappa y - \tfrac{\dot{y}+4\kappa}{y} - \tfrac{1}{t}\right)\right), \tag{5.39}$$

$$C(t) = \exp\left(\tfrac{1}{2}\int dt\left(\kappa y + \tfrac{\dot{y}+4\kappa}{y} + \tfrac{1}{t}\right)\right) + \exp\left(\tfrac{1}{2}\int dt\left(3\kappa y - \tfrac{\dot{y}+4\kappa}{y} - \tfrac{1}{t}\right)\right). \tag{5.40}$$

Dynamical equation (5.31) is equivalent to $\mathrm{P_{III}}$ in the following form

$$\ddot{y} = \frac{\dot{y}^2}{y} - \frac{\dot{y}}{t} + \frac{-2my^2 + 8(m-\kappa)}{t} + \kappa^2\left(y^3 - \frac{16}{y}\right), \tag{5.41}$$

where we have applied the equations $m = 4n$ and $\ell^2 = \frac{1}{16}\kappa^2$ that are implied by both types of parameterizations. This equation describes the semisimple cases, namely Bianchi types $\mathrm{VI_0}$ and $\mathrm{VII_0}$. The nilpotent case, namely Bianchi type II, is obtained by setting $\kappa = 0$. We note that the Bianchi I model is a special case for which the present analysis does not apply, since the Painlevé transcendent defined by equation (3.55) is trivial. We shall return to it later.

We shall consider the semisimple case $\kappa \neq 0$. Then we can substitute rescaled variables by $y = 2w$ and $t = \frac{1}{2\kappa}s$ in (5.41) such that

$$w'' = \frac{w'^2}{w} - \frac{w'}{s} + \frac{-2mw^2 + 2(m-\kappa)}{\kappa s} + w^3 - \frac{1}{w}, \tag{5.42}$$

where $w' = \frac{dw}{ds}$ [14]. Substituting in (3.50) gives $\hat{\alpha} + \epsilon\hat{\beta} = -2, -\frac{4m}{\kappa} + 2$. We saw that rational solutions exist provided that this quantity equals $4n, n \in \mathbb{Z}$, which it does not, so we conclude that rational solutions do not exist in the semisimple case [22].

Using the fact that the trace of a matrix is invariant under conjugation, Tr $C =$ Tr DCD^{-1}, we find that $A := -\alpha(g_z + g_t)g^{-1}$ and $B := -\alpha(g_z - g_t)g^{-1}$ give

$$\text{Tr } A^2 = 2 + \text{Tr } (t(P+4Q)-2R)^2, \qquad \text{Tr } B^2 = 2 + \text{Tr } (t(P+4Q)+2R)^2. \tag{5.43}$$

Substituting $\alpha = t$ in equations (4.15, 4.16) and taking appropriate linear combinations gives

$$(\ln f)_t = -\frac{1}{t} + \frac{1}{8t}(\text{Tr } A^2 + \text{Tr } B^2) = \frac{1}{8t}(-4 + 2t^2\text{Tr } (P+4Q)^2 + 8\text{Tr } R^2), \tag{5.44}$$

$$(\ln f)_z = \frac{1}{8t}(\text{Tr } A^2 - \text{Tr } B^2) = -\text{Tr } (P+4Q)R := -(m+4n) = -2m, \tag{5.45}$$

where we have substituted (5.43). Solving (5.45) gives $f(t,z) = F(t)e^{-2mz}$. We can therefore only impose the obstruction condition $f(t,z) = f(t)$ to obtain a spatially homogeneous class A Bianchi model in cases where $m = 0$. In the semisimple case, the conformal factor in terms of the appropriate Painlevé transcendent is given by

$$(\ln f)_t = -\frac{3}{8}t^{-1} + 2my^{-1} + \frac{1}{4}y^{-1}\dot{y} + \frac{1}{8}ty^{-2}\dot{y}^2 + \frac{1}{2}my - \frac{1}{2}\kappa^2 t\left(\frac{1}{2}y - 2y^{-1}\right)^2, \tag{5.46}$$

whereas the conformal factor in the nilpotent case is obtained by setting $\kappa = 0$.

Example 5.4.1 *We shall now present the particular results for the Bianchi type VI_0 model. The relevant structure constant matrices from (5.7) in this case are*

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P = K^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

from which $k = \text{Tr } K = 0$ and $\kappa = \pm\sqrt{\frac{1}{2}\text{Tr } P^2} = \pm 1$. We also require the homogeneity condition $m = 0$. Then the reduction of $\mathrm{P_{III}}$ (5.41) is

$$\ddot{y} = \frac{\dot{y}^2}{y} - \frac{\dot{y}}{t} - \frac{8\epsilon}{t} + \left(y^3 - \frac{16}{y}\right), \tag{5.47}$$

where $\epsilon = -1, 1$ depending on the particular choice of κ. Equation (5.46) reduces to

$$(\ln f)_t = -\tfrac{3}{8}t^{-1} + \tfrac{1}{4}y^{-1}\dot{y} + \tfrac{1}{8}ty^{-2}\dot{y}^2 - \tfrac{1}{2}t\left(\tfrac{1}{2}y - 2y^{-1}\right)^2. \tag{5.48}$$

5.4.2 Bianchi class B models

We recall that Bianchi class B models are types III, IV, V, $\mathrm{VI}_{h\neq 0}$ and $\mathrm{VII}_{h\neq 0}$. The reductions of the vacuum Einstein equations for these models to a system including a Painlevé equation, involving a change of variable, are original. Here $\alpha(t,z)=\sigma(t)e^{kz}$ with $k\neq 0$.

We wish to write (5.30) in terms of commutators. In order to do this we shall eliminate its final term. To this end we let

$$J(t)=e^{h(t)\hat{K}^T}H(t)e^{h(t)\hat{K}}, \tag{5.49}$$

for a function $h(t)$ to be determined such that the result may be written purely in terms of commutators. We have $\det J=\det H=1$. We define its components

$$H(t)=\begin{pmatrix} D(t) & E(t) \\ E(t) & F(t) \end{pmatrix}. \tag{5.50}$$

Substitution in equation (5.30) gives

$$\begin{aligned}&\tfrac{d}{dt}(\sigma\dot{H}H^{-1})=\dot{h}[\sigma\dot{H}H^{-1},\hat{K}^T-H\hat{K}H^{-1}]+\sigma(1-\dot{h}^2)[\hat{K}^T,H\hat{K}H^{-1}]\\ &+\left(k\sigma-\tfrac{d}{dt}(\sigma\dot{h})\right)(\hat{K}^T+H\hat{K}H^{-1}).\end{aligned} \tag{5.51}$$

We will eliminate the last term in order to get the equation in commutator form. Therefore we impose

$$k\sigma=\frac{d}{dt}(\sigma\dot{h}). \tag{5.52}$$

Substituting the reduced wave equation (5.28) in equation (5.52) and integrating, we get

$$\dot{\sigma}=k(\sigma\dot{h}+c), \tag{5.53}$$

where c is an arbitrary constant. Using the general solution of (5.28) $\sigma(t)=Ae^{kt}+Be^{-kt}$, we can calculate

$$c=A(1-\dot{h})e^{kt}-B(1+\dot{h})e^{-kt}. \tag{5.54}$$

We then substitute the solution $\sigma(t)$ and expression (5.54) into (5.53), upon which we find that

$$e^{kt}=\frac{1}{2A}((1+\dot{h})\sigma+c), \tag{5.55}$$

which may be substituted back into our expression for $\sigma(t)$ to yield a quadratic expression in σ. After defining $\dot{h}$ by applying the constraint $c^2 + 4AB = 0$, solving the quadratic gives the repeated root

$$\sigma = \frac{2c\dot{h}}{1-\dot{h}^2}, \tag{5.56}$$

With this result equation (5.51) now simplifies to

$$\frac{d}{dt}(\sigma \dot{H}H^{-1}) = \dot{h}([\sigma \dot{H}H^{-1}, \hat{K}^T - H\hat{K}H^{-1}] + 2c[\hat{K}^T, H\hat{K}H^{-1}]) \tag{5.57}$$

We define the matrices $X = \frac{1}{2c}\sigma \dot{H}H^{-1}$ and $Y = H\hat{K}H^{-1}$. This parameterization automatically satisfies $\sigma\dot{Y} = 2c[X, Y]$. Then equation (5.57) can be rearranged to

$$\dot{X} = \dot{h}[X - Y, \hat{K}^T - X]. \tag{5.58}$$

We define $s = \dot{h}^2$ as a natural new independent variable. Substituting this definition into (5.52) and expanding the derivative, we find that

$$\dot{s} = ks^{1/2}(1-s) \Rightarrow \sigma\frac{d}{dt} = 2cks\frac{d}{ds}, \tag{5.59}$$

thus $s(t) = \tanh^2 \frac{kt}{2}$. Then

$$\sigma = \frac{s^{1/2}}{1-s} = 2c\tanh\frac{kt}{2}\cosh^2\frac{kt}{2} = 2c\sinh\frac{kt}{2}\cosh\frac{kt}{2} = \frac{c}{2}(e^{kt} + e^{-kt}), \tag{5.60}$$

which is a particular solution of the reduced wave equation (5.28).

We define three matrices P, Q, R, which are functions of t, as

$$kP = -\hat{K}^T, \qquad kQ = Y, \qquad kR = X - Y, \tag{5.61}$$

which have the component forms (3.36-3.35).

In terms of s, we obtain the form of P_{VI} given by

$$P' = 0, \qquad sQ' = [R, Q], \qquad s(1-s)R' = [sP + Q, R], \tag{5.62}$$

where $f' = \frac{df}{ds}$. For general $P, Q, R \in \mathfrak{sl}(2, \mathbb{C})$, system (5.62) is equivalent to P_{VI} [19]. The matrix P can be transformed to either

$$P = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} \qquad \text{or} \qquad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.63}$$

according to whether $\kappa \neq 0$ (when P is semisimple) or $\kappa = 0$ (when P is nilpotent), see [19] for details. Equivalently $\kappa^2 = \frac{1}{2}\mathrm{Tr}\ P^2$.

In terms of s, $X = \frac{k}{2c} s H' H^{-1}$. We wish to write the components of H in terms of the zero trace matrices P, Q, R. From its definition (5.49) we have that $\det H = DF - E^2 = 1$. Writing Q, R in terms of this with P in semisimple form gives the components defined in (3.36,3.37),

$$\begin{aligned} &\lambda = -\kappa(DF + E^2), && \mu = 2\kappa DE, \\ &\nu = -2\kappa EF, && \rho = s(D'F - E'E) + \kappa(DF + E^2), \\ &\sigma = s(-D'E + E'D) - 2\kappa DE, && \tau = s(E'F - F'E) + 2\kappa EF. \end{aligned} \tag{5.64}$$

Considering the nilpotent case, we find instead that

$$\begin{aligned} &\lambda = -EF, && \mu = E^2, \\ &\nu = -F^2, && \rho = s(D'F - E'E) + EF, \\ &\sigma = s(-D'E + E'D) - E^2, && \tau = s(E'F - F'E) + F^2. \end{aligned} \tag{5.65}$$

This parameterization on the nilpotent case gives $\ell^2 = \lambda^2 + \mu\nu = 0$.

We wish to write the matrix $H(s)$ in terms of a Painlevé transcendent $y(s)$ defined by equation (3.55). In the semisimple case, we find that the independent components of $H(s)$ are

$$E(s) = \exp\left(\frac{1}{2}\int ds \left(\frac{y'}{y} - \frac{1}{1-s}\left(\frac{1-y}{s} - 2\kappa y\left(\frac{(1-y)^2}{y^2} + 2\frac{y-1}{y^2} + \frac{1-s}{s}\right)\right)\right)\right), \tag{5.66}$$

$$F(s) = \exp\left(\frac{1}{2}\int ds \left(\frac{y'}{y} - \frac{1}{1-s}\left(\frac{1-y}{s} - 2\kappa y\left(\frac{(y-1)^2}{y^2} + \frac{1-s}{s}\right)\right)\right)\right), \tag{5.67}$$

which implies that the constant of motion $\ell^2 = \kappa^2$, while in the nilpotent case, the independent components are

$$E(s) = \left(\int ds \frac{(1-y)}{y(s-1)}\right) \exp\left(\frac{1}{2}\int ds \left(\frac{y'}{y} + \frac{(1-y)}{s(s-1)}\right)\right), \tag{5.68}$$

$$F(s) = \exp\left(\frac{1}{2}\int ds \left(\frac{y'}{y} + \frac{(1-y)}{s(s-1)}\right)\right). \tag{5.69}$$

In the semisimple case, the dynamical equation (5.62) is equivalent to $\mathrm{P_{VI}}$ in the following form

$$\begin{aligned} y'' = &\frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-s}\right) y'^2 - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{y-s}\right) y' \\ &+ \frac{y(y-1)(y-s)}{s^2(s-1)^2}\left(2\kappa^2 + 2\kappa + \frac{1}{2} - \frac{2\kappa^2 s}{y^2} + \frac{2m^2(s-1)}{(y-1)^2} + \left(-2m^2 + \frac{1}{2}\right)\frac{s(s-1)}{(y-s)^2}\right), \end{aligned} \tag{5.70}$$

and we find that $m^2 = n^2$ in both nilpotent and semisimple cases. This equation describes the semisimple case, namely Bianchi types III, VI_h and VII_h. The nilpotent case, namely Bianchi type IV, is obtained by setting $\kappa = 0$. We note that the Bianchi V model is a special case for which the present analysis does not apply, since the Painlevé transcendent defined by equation (3.55) is trivial. We shall return to it later.

Using the fact that the trace of a matrix is invariant under conjugation, $\mathrm{Tr}\ C = \mathrm{Tr}\ DCD^{-1}$, we find that $A := -\alpha(g_z + g_t)g^{-1}$ and $B := -\alpha(g_z - g_t)g^{-1}$ as defined earlier give, in terms of the variable s,

$$k^{-2}e^{-2kz}\mathrm{Tr}\ A^2 = \frac{1}{2}\left(\frac{1+s^{1/2}}{1-s^{1/2}}\right)^2 + \mathrm{Tr}\ \left(\frac{s^{1/2}}{1-s^{1/2}}(-P+Q)+Q+R\right)^2 \quad (5.71)$$

$$k^{-2}e^{-2kz}\mathrm{Tr}\ B^2 = \frac{1}{2}\left(\frac{1-s^{1/2}}{1+s^{1/2}}\right)^2 + \mathrm{Tr}\ \left(\frac{s^{1/2}}{1+s^{1/2}}(P-Q)+Q+R\right)^2. \quad (5.72)$$

Substituting $\alpha = e^{kz}s^{1/2}(1-s)^{-1}$ in equations (4.15, 4.16) and taking appropriate linear combinations gives

$$(\ln f)_z = k + \frac{(1+s)(\mathrm{Tr}\ A^2 - \mathrm{Tr}\ B^2)}{4ks^{1/2}e^{2kz}} - \frac{\mathrm{Tr}\ A^2 + \mathrm{Tr}\ B^2}{2ke^{2kz}} = k\left(\frac{3}{2} - 2m^2\right), \quad (5.73)$$

$$\begin{aligned}(\ln f)_s &= -\tfrac{1+s}{2s(1-s)} - \tfrac{\mathrm{Tr}\ A^2 - \mathrm{Tr}\ B^2}{2k^2s^{1/2}(1-s)e^{2kz}} + \tfrac{(1+s)(\mathrm{Tr}\ A^2+\mathrm{Tr}\ B^2)}{4k^2s(1-s)e^{2kz}}\\ &= \tfrac{1}{4(1-s)^3s}\left((y-s)(s+1)(y-1+s-1) - (y-s)(y-1) - \tfrac{s(y-1)^3}{y-s}\right)\\ &+\tfrac{1}{(1-s)sy}((y-1)(y-s)+y(1+s))\kappa^2 + \tfrac{1}{1-s}\left(\tfrac{y-s}{s(y-1)} + \tfrac{y-1}{y-s}\right)m^2\\ &-\tfrac{1}{2(1-s)^2y}\left(y-s+\tfrac{s(y-1)^2}{y-s} - (y-1)(1+s)\right)y'\\ &+\tfrac{s}{4(1-s)y^2}\left(1+s-\tfrac{y-s}{y-1} - \tfrac{s(y-1)}{y-s}\right)y'^2\end{aligned} \quad (5.74)$$

From equation (5.73), which holds in nilpotent and semisimple cases, we find that $f(t,z) = F(t)e^{k(\frac{3}{2}-2m^2)z}$, so the condition for a model to be spatially homogeneous (of Bianchi type) is $m = \pm\frac{\sqrt{3}}{2}$. Equation (5.74) holds for the semisimple case, and with $\kappa = 0$ for the nilpotent case.

Example 5.4.2 *We shall now present the particular results for the Bianchi type III model. The relevant structure constant matrices from (5.7) and (5.8) in this case are*

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = -\frac{1}{2}\hat{K}^T = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

from which $k = \mathrm{Tr}\ K = 2$ *and* $\kappa = \pm\sqrt{\frac{1}{2}\mathrm{Tr}\ P^2} = \pm\frac{1}{2}$. *We also require the homogeneity condition* $m = \pm\frac{\sqrt{3}}{2}$. *Then the reduction of* $\mathrm{P_{VI}}$ *(5.70) is*

$$\begin{aligned} y'' &= \tfrac{1}{2}\left(\tfrac{1}{y} + \tfrac{1}{y-1} + \tfrac{1}{y-s}\right)y'^2 - \left(\tfrac{1}{s} + \tfrac{1}{s-1} + \tfrac{1}{y-s}\right)y' \\ &+ \tfrac{y(y-1)(y-s)}{s^2(s-1)^2}\left(2\delta - \tfrac{s}{2y^2} + \tfrac{3(s-1)}{2(y-1)^2} - \tfrac{s(s-1)}{(y-s)^2}\right), \end{aligned} \tag{5.75}$$

where $\delta = 0, 1$ *depending on the particular choice of* κ. *Equation (5.74) reduces to*

$$\begin{aligned} (\ln f)_s &= \tfrac{1}{4(1-s)^3 s}\left((y-s)(s+1)(y-1+s-1) - (y-s)(y-1) - \tfrac{s(y-1)^3}{y-s}\right) \\ &+ \tfrac{1}{4(1-s)sy}((y-1)(y-s) + y(1+s)) + \tfrac{3}{4(1-s)}\left(\tfrac{y-s}{s(y-1)} + \tfrac{y-1}{y-s}\right) \\ &- \tfrac{1}{2(1-s)^2 y}\left(y - s + \tfrac{s(y-1)^2}{y-s} - (y-1)(1+s)\right)y' \\ &+ \tfrac{s}{4(1-s)y^2}\left(1 + s - \tfrac{y-s}{y-1} - \tfrac{s(y-1)}{y-s}\right)y'^2. \end{aligned} \tag{5.76}$$

5.4.3 Bianchi models with two-parameter Abelian reduced subgroups

When the two parameter isometry subgroup $\hat{H}$ is Abelian, as in the Bianchi models of types I and V, the corresponding spacetime metric (5.26) can always be written in the diagonal form [41]

$$ds^2 = f(t,z)(-dt^2 + dz^2) + e^{2\epsilon z}(a^2(t)dx^2 + b^2(t)dy^2). \tag{5.77}$$

If we consider the reduction to a spatially homogeneous (Bianchi) model, namely where $f(t,z) = f(t)$ only, then $\epsilon = 0$ corresponds to a Bianchi I model, while $\epsilon = 1$ corresponds to a Bianchi V model.

We found in the above cases that there was one model in each case - Bianchi I, V respectively - for each of which $\hat{K} = 0$ and the self-duality equations reduced to triviality. Therefore a different analysis is required for these cases. The metric $\hat{g} = \alpha^{-1}g$ for both models becomes

$$\hat{g}(t,z) = J(t) = \frac{\gamma(t)}{\sigma(t)} = \frac{1}{\sigma(t)}\begin{pmatrix} a^2(t) & 0 \\ 0 & b^2(t) \end{pmatrix} \tag{5.78}$$

where $\sigma^2 = \det\gamma = a^2b^2$. We have used the diffeomorphism invariance of general relativity which allows us to write the metric in diagonal form. This is only possible for Bianchi types I and V. The field equations (4.13) in both cases are

$$\frac{d}{dt}(\sigma a^{-1}\dot{a}) = 0 \tag{5.79}$$

For the Bianchi I model, as previously for class A models we choose $\sigma = t$,

$$\ddot{a} = \frac{\dot{a}^2}{a} - \frac{\dot{a}}{t}, \tag{5.80}$$

so that $a(t)$ satisfies $\mathrm{P_{III}}$. Integrating gives $a(t) = At^m$, where $A, m \in \mathbb{R}$ are arbitrary constants. We choose to scale the coordinate x such that $A = 1$.

For the Bianchi V model, like previously for class B models we define a new independent variable s and determine the function $\sigma(s)$. Then, noting that the trace $k = 2$, equation (5.79) with the substitutions (5.59) becomes

$$2s^{1/2}(1-s)\frac{d}{ds}(4csa^{-1}a') = 0 \Rightarrow a'' = \frac{a'^2}{a} - \frac{a'}{s}, \tag{5.81}$$

where $a' = \frac{da}{ds}$, so that $a(s)$ satisfies $\mathrm{P_{III}}$. Integrating gives $a(s) = Bs^n$, where $B, n \in \mathbb{R}$ are arbitrary constants. We choose to scale the coordinate x such that $B = 1$.

For a Bianchi I model, equation (5.44) becomes

$$(\ln f)_t = 2m(m-1)t^{-1}, \tag{5.82}$$

and equation (5.45) implies that $f(t,z) = f(t)$ only. Therefore the model is spatially homogeneous, with $f(t) = t^{2m(m-1)}$ (with (t,z) scaled appropriately), and $b(t) = t^{1-m}$.

For a Bianchi V model, in terms of the (s,z) variables and after substituting $a(s) = s^n$ we find

$$(\ln f)_s = \frac{1+s}{1-s}\left(\frac{1}{(1-s)^2} - 2n\frac{1+s}{s(1-s)} + \frac{4n^2}{s}\right), \tag{5.83}$$

and

$$(\ln f)_z = \frac{2}{(1-s)^2}(1-4s+s^2) + 8n\left(\frac{1+s}{1-s}\right) - 16n^2. \tag{5.84}$$

We solve the homogeneity condition $(\ln f)_z = 0$ with (5.84) as a quadratic equation in n, on which we find the solutions $n = \frac{1+s}{4(1-s)} \pm \frac{\sqrt{3}}{4}$. However these values are not allowed since we require n to be a constant, so we conclude that our analysis leads to no Bianchi V solutions.

Chapter 6

Discrete Painlevé equations

In this chapter we review some known methods for determining whether a given complex discrete equation is integrable, as extensions of criteria for integrability of differential equations presented in earlier chapters. We perform numerical tests using integrable and nonintegrable discrete equations with particular initial conditions to investigate the degree growth, to confirm known results. Also we present Nevanlinna theory for difference equations (discrete equations where the independent variable varies over the complex plane). In following chapters we shall discover analogous concepts to these in the context of ultra-discrete equations.

A discrete equation is a recurrence relation between an iterate y_n — where n is a discrete independent variable taking only integer values — and other iterates which are shifted along the discrete variable. A jth order discrete equation relates y_{n-j} to y_n. In the usual sense a solution of a discrete equation is a set of iterates $\{y_i : i \in \mathbb{Z}\}$. Alternatively, in section 6.3 we consider each iterate y_n to be a rational function in an auxiliary, continuous variable z.

There are many discrete equations with continuum limits to the Painlevé differential equations. Some cases of this are considered in section 6.1. However, most of these equations do not inherit the integrability properties of the Painlevé equations — such as the existence of associated linear problems. The discrete equations which do have the integrability properties are known as discrete Painlevé equations [55]. These usually look quite different from the naive discretizations of the Painlevé equations.

We shall progress from defining discrete Painlevé equations to the more general problem of finding a discrete analogue of the Painlevé property, which is a strong indicator (some would take it as a definition) of integrability. The singularity confinement test was introduced in [55] to find such a property. Those discrete equations which pass the singularity confinement test, and also satisfy a condition of zero algebraic entropy [58], are believed to be integrable.

Discrete Painlevé equations do occur in physics. For example the Ising model in statistical field theory [56], and in quantum gravity where a discrete Painlevé equation often called d-P_I has arisen,

$$y_{n+1} + y_n + y_{n-1} = \frac{an+b}{y_n} + c, \tag{6.1}$$

where $\{a, b, c\}$ are arbitrary constants. Specifically, it has arisen from the calculation of a certain partition function in a model of 2 dimensional quantum gravity [52, 53]. It is the compatibility condition for a linear problem, and it has a simple continuum limit to P_I, hence its name. However it is not unique in possessing such a limit.

In order to find a complex analytic analogue of the Painlevé property for discrete equations, it was necessary in [54] to reinterpret discrete equations as difference equations. For example (6.1) is reinterpreted as the difference equation

$$y(z+1) + y(z) + y(z-1) = \frac{az+b}{y(z)} + c. \tag{6.2}$$

Here the discrete independent variable $n \in \mathbb{Z}$ of equation (6.1) has been replaced by a continuous variable $z \in \mathbb{C}$. In this way the independent variable is allowed much greater freedom. Another example of a generalization of a discrete to a continuous variable is the replacement of the factorial $n!$ by the gamma function $\Gamma(z)$.

Nevanlinna theory, which we studied in section 2.7, may be applied to meromorphic solutions of difference equations such as (6.2). Of particular interest is the order of solutions. It was suggested in [54] that a difference equation is integrable if it has sufficiently many finite order solutions. This usefulness of Nevanlinna theory should be contrasted with the situation for differential equations. In that case, if all solutions of an equation are meromorphic then the equation has the Painlevé property. However since Nevanlinna theory is a theory of meromorphic functions it can only be used on solutions of those equations which necessarily have the Painlevé property, and are therefore integrable.

A rational function is a ratio of two polynomials, which are taken to have no common factors. The degree of a rational function is the maximum degree of these polynomials. A set of functions $\{y_n\}$ which are each rational in an auxiliary variable $z \in \mathbb{C}$ may occur as a solution to a discrete equation such as (6.1). Then we have a set of degrees $q_n := \deg y_n(z)$. The algebraic entropy of a discrete equation quantifies the growth of the degrees of such iterates of that equation. The algebraic entropy of a generic discrete equation is nonzero (corresponding to exponential degree growth) but the algebraic entropy of a large class of integrable discrete equations is zero (corresponding to polynomial degree growth) [74].

6.1 The continuous limit

We start from the discrete equation, with n as discrete independent variable and y_n as dependent variable,

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2}, \tag{6.3}$$

where a, b and c are constants. We will show that it has $\mathrm{P_{II}}$ as a continuous limit. First we define an independent variable $z = nh$, where h is a small number[1]. Then we identify $y_n \equiv w(z)$, and

$$y_{n+1} = w(z+h) = w(z) + hw'(z) + \frac{h^2}{2}w''(z) + \frac{h^3}{6}u'''(z) + O(h^4), \tag{6.4}$$

using a Taylor series expansion. We obtain a similar result for y_{n-1} with h replaced by $-h$. Substituting in the discrete equation (6.3) gives

$$2w + h^2 w'' = ((az/h + b)w + c)(1 + w^2) + O(h^4), \tag{6.5}$$

where we have used the truncated series $(1 - w^2)^{-1} = 1 + w^2 + \ldots$. Rescaling $w \longrightarrow hw$,

$$2hw + h^3 w'' = azw + bhw + ah^2 zw^3 + bh^3 w^3 + c + ch^3 w^3 + O(h^4 w^4) \tag{6.6}$$

[1]We used a similar method in going from the discrete equation (6.1) to the difference equation (6.2) where $h = 1$, but there $z = n$ only on the integers.

Now we choose $a = h^3$, $b = 2$ and $c = h^3\alpha$ and omit terms of order $(hw)^4$, to give the continuous form of $\mathrm{P_{II}}$,

$$w'' = 2w^3 + zw + \alpha. \tag{6.7}$$

As a second example, we consider the discrete equation we referred to as d-$\mathbf{P}_\mathrm{I}$ (6.1). By applying $y_n = 1 - 3h^2w(z)$, setting $(a, b, c) = (-3h^5, -2, 4)$ and taking the limit $h \to 0$, a similar procedure to above yields $\mathrm{P_I}$,

$$w'' = 6w^2 + z. \tag{6.8}$$

Many other discrete Painlevé equations exist that can be transformed to one or more of the continuous Painlevé equations using the methods we have outlined.

6.2 Singularity confinement

The singularity confinement test, first introduced in [55], provides a test for integrability of discrete equations. It is not a perfect test of integrability, having to be augmented by a condition of zero algebraic entropy for this. The algebraic entropy of a discrete equation shall be defined in section 6.3.

We consider the behaviour of a sequence of iterates y_n of a discrete equation. We shall consider equation (6.1) with initial conditions $y_{n-1} = K$ and $y_n = \epsilon$, where K is finite and ϵ is small. After calculating a list of iterates, we take $\epsilon \to 0$ to find $y_{n+1} = \infty$, therefore the solution has a singular point. We might expect all further iterates to be singular, and for generic equations they would be, but for equation (6.1) there is cancellation such that future iterates are finite, such that the singularity y_n is confined. It is found that in particular, all discrete Painlevé equations possess the singularity confinement property.

Discrete-time integrable systems take one of two forms; as lattices or mappings. In lattices the spatial and time variables have been discretized, while mappings are finite-degrees-of-freedom systems in discrete time. We shall present a way to determine the integrability of a discrete time system, based on the singularity structure [55]. We consider the integrable lattice KdV equation [61],

$$y_j^{i+1} = y_{j+1}^{i-1} + \frac{1}{y_j^i} - \frac{1}{y_{j+1}^i} \tag{6.9}$$

which is a partial discrete equation, having two discrete independent variables (i, j). Equation (6.9) is of second order in i; to this end we specify two pieces of initial data at $i-2$ and $i-1$. Evolution takes place as the value of i increases. Let us assume that as evolution takes place $y_j^i \to 0$. The point where it vanishes depends on the initial data, so the singularity induced is movable, or confined.

We have seen that integrability in discrete-time systems is related to the existence of confined (movable) singularities.

Example 6.2.1 *We shall show how singularity confinement works by explicitly calculating the first few iterates of a solution of a discrete Painlevé equation, namely equation (6.1),*

$$y_{m+1} = -y_m - y_{m-1} + \frac{am+b}{y_m} + c.$$

The idea is to follow the sequence of iterates to see whether we can eventually "get through" the singularity and return to finite values. Equation (6.1) is singular when $y_n = 0$. *We let* $y_n = \epsilon$ *and* $y_{n-1} = k$, *arbitrary. After we have calculated "enough" values, we will take the limit as* $\epsilon \to 0$.

$m = n:$

$$\begin{aligned}
y_{n+1} &= -y_n - y_{n-1} + \frac{an+b}{y_n} + c \\
&= \frac{an+b}{\epsilon} + (c-k) - \epsilon \\
&= \frac{an+b}{\epsilon}\left(1 + \frac{c-k}{an+b}\epsilon - \frac{1}{an+b}\epsilon^2\right) \\
\frac{1}{y_{n+1}} &= \frac{\epsilon}{an+b}\left(1 - \frac{c-k}{an+b}\epsilon + \cdots\right)
\end{aligned}$$

$m = n+1:$

$$\begin{aligned}
y_{n+2} &= -y_{n+1} - y_n + \frac{an+a+b}{y_{n+1}} + c \\
&= -\left(\frac{an+b}{\epsilon} + (c-k) - \epsilon\right) - \epsilon + \epsilon\frac{an+a+b}{an+b}\left(1 - \frac{c-k}{an+b}\epsilon + \cdots\right) + c \\
&= -\frac{an+b}{\epsilon} + k + \frac{an+a+b}{an+b}\epsilon + \cdots \\
&= -\frac{an+b}{\epsilon}\left(1 - \frac{k}{an+b}\epsilon + \cdots\right) \\
\frac{1}{y_{n+2}} &= -\frac{\epsilon}{an+b}\left(1 + \frac{k}{an+b}\epsilon + \cdots\right)
\end{aligned}$$

$m = n+2:$

$$\begin{aligned}
y_{n+3} &= -y_{n+2} - y_{n+1} + \frac{an+2a+b}{y_{n+2}} + c \\
&= -\left(-\frac{an+b}{\epsilon} + k + \frac{an+a+b}{an+b}\epsilon + \cdots\right) - \left(\frac{an+b}{\epsilon} + (c-k) - \epsilon\right) \\
&\quad -\epsilon\frac{an+2a+b}{an+b}\left(1 + \frac{k}{an+b}\epsilon + \cdots\right) + c \\
&= -\frac{an+3a+b}{an+b}\epsilon + \cdots
\end{aligned}$$

$m = n+3:$

$$\begin{aligned}
y_{n+4} &= -y_{n+3} - y_{n+2} + \frac{an+3a+b}{y_{n+3}} + c \\
&= O(\epsilon) - \left(-\frac{an+b}{\epsilon} + k + \frac{an+a+b}{an+b}\epsilon + \cdots\right) \\
&\quad +(an+3a+b)\left(-\frac{an+b}{(an+3a+b)\epsilon} + \cdots\right) + c \\
&= O(1)
\end{aligned}$$

Taking the limit $\epsilon \to 0$ gives the sequence of iterates:

$$y_{n-1} = k, \quad y_n = 0, \quad y_{n+1} = \infty, \quad y_{n+2} = \infty, \quad y_{n+3} = 0, \quad y_{n+4} = \mathit{finite}$$

6.3 Evolution of degree of solutions of discrete equations

In this section we will reinterpret a discrete equation such that each iterate is a rational function of an auxiliary variable. We consider a sequence of rational functions $\{y_n : n \in \mathbb{Z}\}$ in a complex variable z,

$$y_n(z) = \frac{\sum_{i=0}^{s_n} a_{ni} z^i}{\sum_{j=0}^{t_n} b_{nj} z^j}, \tag{6.10}$$

where (a_{ni}, b_{nj}) are complex-valued arbitrary constants. We define the degree of y_n as $q_n = \max(s_n, t_n)$.

Examples

The two families of equations we shall consider are

$$y_{n+1} + y_{n-1} = \frac{P_n}{y_n} + Q_n, \qquad y_{n+1} + y_{n-1} = \frac{R_n + S_n y_n}{1 - y_n^2}. \tag{6.11}$$

where P_n, Q_n, R_n and S_n are each polynomial functions in n.

We shall consider the second order discrete equation

$$y_{n+1} + y_{n-1} = \frac{3y_n}{1 - y_n^2}. \tag{6.12}$$

Substituting the definition (6.10), computer experiments suggest that $q_n = p_n q_0 + r_n q_1$, which gives the boundary conditions $(p_0, p_1, r_0, r_1) = (1, 0, 0, 1)$. The subsequent evolution of p_n appears to be governed by the discrete equation

$$p_{n+1} - 2p_n + p_{n-1} = 1 - (-1)^n, \qquad n \geq 1, \tag{6.13}$$

and for the r_n we find

$$r_{n+1} - 2r_n + r_{n-1} = 1 - (-1)^{n-1}, \qquad n \geq 1. \tag{6.14}$$

We have that $r_n = p_{n+1}$, and equation (6.13) admits the general solution

$$p_n = \alpha n^2 + \beta n + \gamma + \delta(-1)^n. \tag{6.15}$$

We can substitute this solution back in (6.13) and substitute the initial conditions $p_0 = 1$ and $p_1 = 0$ to find that $\alpha = \frac{1}{2}$, $\beta = -1$, $\gamma = \frac{3}{4}$ and $\delta = \frac{1}{4}$. If we define the discrete operator $\triangle$ by $\triangle y_n = y_{n+1} - y_n$ and the second order discrete operator $\triangle^2 y_n = \triangle(\triangle y_{n-1}) = \triangle y_n - \triangle y_{n-1} = y_{n+1} - 2y_n + y_{n-1}$, then equations (6.13) and (6.14) can be written as

$$\triangle^2 p_n = 1 - (-1)^n, \qquad p_n = r_{n-1}, \qquad n \geq 1. \tag{6.16}$$

Next we consider the equation

$$y_{n+1} + y_{n-1} = \frac{3}{y_n} - 76. \tag{6.17}$$

Again we find a relation $q_n = p_n q_0 + r_n q_1$, where the discrete variables are

n	p_n	r_n
0	1	0
1	0	1
2	1	1
3	1	2
4	2	3
5	3	5
6	5	6
7	6	9
8	9	11
9	11	14

We note as before that $r_n = p_{n+1}$.

From here on we shall summarize our results for further discrete equations. As in all previous cases, the relation $q_n = p_n q_0 + p_{n+1} q_1$ is found to hold in all cases, so it suffices to list the first few p_n in each case.

	$y_{n+1} + y_{n-1} = \frac{n^2}{y_n} - 1$	$y_{n+1} + y_{n-1} = \frac{n^2}{y_n}$	$y_{n+1} + y_{n-1} = \frac{n^{20}}{y_n}$
n	p_n	p_n	p_n
2	1	1	1
3	1	1	1
4	2	2	2
5	3	2	2
6	5	3	3
7	7	3	3
8	11	4	4
9	16	4	4

The degree growth of the two right equations in the above table is linear in each case (indicating therefore integrability). However, these two equations are not considered to be integrable. Both of the these equations are of the form

$$y_{n+1} + y_{n-1} = \frac{a_n}{y_n}. \tag{6.18}$$

Equations of the form (6.18) are in fact linearisable and would therefore normally be considered to be integrable. Specifically, let $x_n = y_n y_{n+1}$. Then equation (6.18) becomes the first-order linear equation

$$x_{n-1} + x_n = a_n. \tag{6.19}$$

In the case in which $a_n = n^{20}$, finding a solution involves calculating a polynomial of degree 20. In the case in which $a_n = n^2$, a particular solution is $x_n = n(n+1)/2$. To find the general solution, we substitute $x_n = n(n+1)/2 + u_n$ into equation (6.19) with $a_n = n^2$, giving

$$u_{n-1} + u_n = 0.$$

So $u_n = c(-1)^n$, for some constant c. Hence

$$y_n y_{n+1} = x_n = \frac{n(n+1)}{2} + c(-1)^n. \tag{6.20}$$

This is a Riccati equation for y_n. Substituting $y_n = w_n/w_{n-1}$ in equation (6.20) gives

$$w_{n+1} = \left(\frac{n(n+1)}{2} + c(-1)^n\right) w_{n-1},$$

which is linear (and therefore, slow growth).

	$y_{n+1} + y_{n-1} = \frac{1}{y_n} + n$	$y_{n+1} + y_{n-1} = \frac{n^2}{y_n} + n + 1$	$y_{n+1} + y_{n-1} = \frac{n+1}{y_n}$
n	p_n	p_n	p_n
2	1	1	1
3	1	1	1
4	2	2	2
5	3	3	2
6	5	5	3
7	8	8	3
8	13	13	4
9	21	21	4

Next we consider the equation

$$y_{n+1} + y_{n-1} = \frac{n+1}{y_n} + 1 \tag{6.21}$$

and calculate the first few p_n and $\triangle^2 p_n$,

n	p_n	$\Delta^2 p_n$
2	1	-1
3	1	1
4	2	0
5	3	1
6	5	-1
7	6	2
8	9	-1
9	11	1
10	14	0
11	17	1
12	21	-1
13	24	2
14	29	-1
15	33	1
16	38	0
17	43	1
18	49	-1
19	54	2
20	61	-1

We find in this case that p_n satisfies the difference equation

$$\Delta^2 p_n = \frac{1}{2}(1 - (-1)^{n(\text{mod } 3)}) - (-1)^{n(\text{mod } 6)} \tag{6.22}$$

and the second discrete operator on y_n is of period 6, that is $\Delta^2 p_{n+6} = \Delta^2 p_n$.

It would seem that as far as the behaviour of p_n is concerned, the most general equations of the two families we have analyzed are

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c, \qquad y_{n+1} + y_{n-1} = \frac{d + (en+f)y_n}{1 - y_n^2}. \tag{6.23}$$

for arbitrary constants $\{a, b, c, d, e, f\}$. These are known to be integrable, discrete Painlevé equations. Here the growth of the degree is polynomial rather than exponential.

6.4 Algebraic entropy

Hietarinta and Viallet found an example [57] of a simple equation that appears to possess the singularity confinement property but is chaotic (non-integrable). Therefore the singularity confinement property must be augmented by some other property to give a condition for integrability, which they suggested is that the algebraic entropy of the discrete equation should vanish.

The algebraic entropy quantifies the growth of the degree of the nth iterate of a discrete equation as a function of the initial conditions, see for example [58]. Here the initial conditions are rational functions of one variable: however the same theory may be applied for an arbitrary number of such variables. We consider the rational functions (6.10) where the degree of the nth function is q_n. Then the definition of algebraic entropy is

$$s := \lim_{n\to\infty} \frac{\log q_n}{n}. \tag{6.24}$$

For exponential degree growth, set $q_n = d^n$ where d is a constant. Then the algebraic entropy is

$$s = \lim_{n\to\infty} \frac{n \log d}{n} = \log d.$$

which is nonzero provided that $d \neq 1$. For polynomial growth, set $q_n = n^\sigma$ where σ is constant,

$$s = \sigma \lim_{n\to\infty} \frac{\log n}{n} = 0.$$

Therefore the algebraic entropy quantifies the growth type of the degree according to whether it vanishes (which means polynomial growth) or not (which means exponential growth).

The idea of using the growth in the degree of iterates to determine whether a mapping is integrable was formulated by Veselov [59] and independently by Falqui and Viallet [60]. When an integrable mapping is used the degree growth is polynomial, while when an non-integrable mapping is used the degree growth is exponential.

The algebraic entropy of a generic discrete equation is nonzero (corresponding to exponential degree growth) but the algebraic entropy of a large class of integrable discrete equations is zero (corresponding to polynomial degree growth) [74]. For example equation (6.12) — which is well known to be integrable — gave rise to degree growth of the form (6.15). We see that the asymptotic degree in this case is $q_n \propto n^2$.

6.5 Nevanlinna theory

It is known that large classes of difference equations admit meromorphic solutions [54]. Therefore the existence of meromorphic solutions is alone not a good indicator of integrability, and for a better indicator we turn to Nevanlinna theory. In section 2.7 we gave an introduction to Nevanlinna theory, which studies the value distribution of meromorphic functions in the complex domain. Here, the meromorphic functions we consider solve difference equations [71].

For a meromorphic function $f(z)$, the Nevanlinna characteristic $T(r, f) := N(r, f) + m(r, f)$ is defined in terms of an open disc $|z| < r$, and measures the growth of f as r varies. We may take the limit $r \to \infty$ to consider the behaviour of f at infinity, in terms of its value distribution on the finite plane. Nevanlinna theory provides a number of important concepts and tools that can be used as detectors of equations that are integrable. In particular, the order of equations plays a central role. The order of a meromorphic function f is

$$\sigma(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

The order is a natural measure of the growth of a function at infinity. It was suggested in [54] that a useful analogue of the Painlevé property for difference equations is the existence of sufficiently many meromorphic solutions of finite order.

We recall the lemma on the logarithmic derivative (2.57), which puts an upper bound on the proximity function of the logarithmic derivative f'/f of a meromorphic function. If the function has discrete dependence on an independent variable, then it is not possible to define its derivative. However we may instead define a logarithmic difference as $\frac{f(z+c)}{f(z)}$ where $c \in \mathbb{C}$. Then we state the result for a non-constant meromorphic function f [71]

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)^{1+\epsilon}}{r^\delta}\right), \qquad \delta < 1, \epsilon > 0, \tag{6.25}$$

for all r outside of a possible exceptional set of finite logarithmic measure, that is $\int_E \frac{dr}{r} < \infty$.

Chapter 7

(Max,+) semiring and ultra-discrete equations

The language of this chapter, and its application to ultra-discrete equations, shall be used in the preprint of Halburd and Southall [79].

We begin this chapter by reviewing the (max,+) semiring and the notion of a (max,+) polynomial. Then we present a known limiting procedure on certain discrete equations to obtain ultra-discrete equations, which may be written in a concise form on the (max,+) semiring. We present a singularity confinement property for integrability in ultra-discrete equations. If the discrete independent variable is allowed to be continuous, then the ultra-discrete equations admit piecewise linear functions. In original work we define (max,+) meromorphic functions as piecewise linear with integer slopes everywhere, and we define roots and poles of such functions at points where the slope changes.

A (max,+) semiring is $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ where the binary operators are defined by

$$a \oplus b := \max(a, b), \qquad a \otimes b := a + b, \qquad a, b \in \mathbb{R} \cup \{-\infty\}.$$

The (max,+) semiring has no additive inverse. Note that we do not work with complex numbers in this presentation, since the maximum of complex numbers cannot be defined.

A (max,+) polynomial is a non-decreasing piecewise linear function with integer slopes. It may be defined using the $(\oplus, \otimes)$ operators in the same way that standard additive and multiplicative operators are used to define a standard polynomial. We only consider (max,+) polynomials in one independent variable $x \in \mathbb{R} \cup \{-\infty\}$. We extend this to define a (max,+) rational function, which is the difference of two (max,+) polynomials. It does not have the restriction on a (max,+) polynomial of being non-decreasing.

A (max,+) meromorphic function is a piecewise linear function with integer slopes. Note that the number of distinct linear pieces may be infinite. A (max,+) rational function is necessarily (max,+) meromorphic. We shall define zeros and poles of (max,+) meromorphic functions at points of discontinuity in their first derivatives.

Ultra-discrete equations are obtained from certain discrete/difference equations, in a limiting procedure called ultra-discretization of which we shall present an example [67, 68].

An ultra-discrete equation may be written naturally on the (max,+) semiring. We may choose the independent variable to be discrete, in which case an example of an ultra-discrete equation often called u-$\mathrm{P_I}$ is

$$X_{n+1} \otimes X_n \otimes X_{n-1} = 0 \oplus X_n \otimes K. \tag{7.1}$$

An ultra-discrete equation such as (7.1) is a generalized *cellular automata.* This means that the values of solutions may be represented as discrete points in a finite dimensional, infinitely sized grid, whose evolution over the grid is governed by the ultra-discrete equation. Alternatively, the independent and dependent variables can be taken as continuous, being valued on the real line. As an example we take

$$X(x+1) \otimes X(x) \otimes X(x-1) = 0 \oplus X(x) \otimes \pi_1(x), \tag{7.2}$$

where $\pi_1(x)$ is an arbitrary period 1 function. The (max,+) meromorphic functions we have defined may be admitted as solutions to such equations. The process of going from ultra-discrete equation (7.1) to (7.2) is analogous to that for discrete equations on the complex plane introduced in [54].

An ultra-discrete equation may be written concisely on the (max,+) semiring. We note that not every discrete equation can be ultra-discretized. A (max,+) meromorphic function may be admitted as a solution to an ultra-discrete equation in which the independent variable is continuous.

An integrable ultra-discrete equation is one obtained by ultra-discretization of an integrable discrete/difference equation. Joshi and Lafortune [73] have described an analogue of singularity confinement for ultra-discrete equations as a test of integrability. The ultra-discrete Painlevé equations - many of which were first given in [68] - are obtained by ultra-discretization of particular discrete Painlevé equations. As such they serve as prototypes of integrability in the ultra-discrete sense.

We want to find out whether the existence of (max,+) meromorphic solutions to an ultra-discrete equation is linked to its integrability properties.

7.1 (Max,+) semiring

We define an addition and a multiplication operation between two elements $a, b \in \mathbb{R} \cup \{-\infty\}$ by [62, 63, 64, 65, 66]

$$a \oplus b = \max(a, b), \qquad a \otimes b = a + b. \tag{7.3}$$

The standard convention that $a \otimes b \oplus c \otimes d = (a \otimes b) \oplus (c \otimes d)$, i.e. multiplication takes priority over addition, is applicable. The additive identity $\mathbb{O} = -\infty$ while the multiplicative identity $\mathbb{I} = 0$. The addition and multiplication tables are

$\oplus$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	1	2	3	4	5	6	7	8	9
2	2	2	2	3	4	5	6	7	8	9
3	3	3	3	3	4	5	6	7	8	9
4	4	4	4	4	4	5	6	7	8	9
5	5	5	5	5	5	5	6	7	8	9
6	6	6	6	6	6	6	6	7	8	9
7	7	7	7	7	7	7	7	7	8	9
8	8	8	8	8	8	8	8	8	8	9
9	9	9	9	9	9	9	9	9	9	9

(7.4)

$\otimes$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

(7.5)

This structure, $(S, \otimes, \oplus)$ where in this case $S = \mathbb{R} \cup \{-\infty\}$, is a *semiring.* A semiring is a set together with two binary operators $(\oplus, \otimes)$ satisfying the following conditions:

- Additive associativity: $\forall a, b, c \in S$, $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
- Additive commutativity: $\forall a, b \in S$, $a \oplus b = b \oplus a$,
- Multiplicative associativity: $\forall a, b, c \in S$, $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,
- Left and right distributivity: $\forall a, b, c \in S$, $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$.

The multiplicative inverse of a is $-a$ $(a \otimes -a = \mathbb{I})$. The operation $\oplus$ has an identity element but not all elements have an additive inverse. For example, the equation $a \oplus 3 = 1$ has no solution. Commutativity of both operations follows from their definitions. The operations are each associative,

$$(a \oplus b) \oplus c = \max(\max(a, b), c) = \max(a, b, c) = \max(a, \max(b, c)) = a \oplus (b \oplus c), \tag{7.6}$$

$$(a \otimes b) \otimes c = (a + b) + c = a + (b + c) = a \otimes (b \otimes c), \tag{7.7}$$

and are distributive,

$$a \otimes (b \oplus c) = a + \max(b, c) = \max(a + b, a + c) = a \otimes b \oplus a \otimes c. \tag{7.8}$$

The operation $\oplus$ is *idempotent*, that is $a \oplus a = a$. A semiring which carries this property is called a *dioid*.

We note that some authors, in a similar context, define min in place of max. That this is a matter of personal choice is demonstrated by noting that one can be transformed into the other by means of

$$\min(a, b) + \max(a, b) = a + b, \tag{7.9}$$

and in the min case the real line must be augmented with $+\infty$ rather than $-\infty$, and the additive identity $\mathbb{O} = +\infty$. In particular, the semiring $(\mathbb{N} \cup \{+\infty\}, \otimes, \oplus)$ is named the tropical semiring [62, 66].

We tabulate some quantities written in conventional notation and write the same quantities using the $(\oplus, \otimes)$ notation.

$(\max, +)$	$(\oplus, \otimes)$
ra	$a^{\otimes r} = a \otimes a \otimes \cdots \otimes a$ (r times)
$a - b$	$a \oslash b = a \otimes b^{-1}$
none	$a \ominus b$
$\sum_{i=1}^{n} a_i$	$\bigotimes_{i=1}^{n} a_i$
$\max_{i=1}^{n} a_i$	$\bigoplus_{i=1}^{n} a_i$

Considering the 2-vector transformation $\vec{v}' = A\vec{v}$, the $(\max, +)$ version of matrix multiplication is given in component form by [62]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a \otimes v_1 \oplus b \otimes v_2 \\ c \otimes v_1 \oplus d \otimes v_2 \end{pmatrix} = \begin{pmatrix} \max(a + v_1, b + v_2) \\ \max(c + v_1, d + v_2) \end{pmatrix}. \tag{7.10}$$

From this we deduce that the (max,+) 2×2 identity matrix is

$$\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}$$

7.2 (Max,+) polynomials

A $(\max, +)$ polynomial [63] in a variable x of degree n is an expression of the form

$$p(x) = \bigoplus_{i=0}^{n} c_i \otimes x^{\otimes i}, \tag{7.11}$$

which in conventional notation takes the form $p(x) = \max_{i=1}^{n}(ix + c_i)$, where $i \in \mathbb{N}$.

The (max,+) binomial theorem is, for $n \in \mathbb{N}$ [63],

$$(a \oplus b)^{\otimes n} = a^{\otimes n} \oplus b^{\otimes n}. \tag{7.12}$$

We shall consider the particular case $n = 2$ where

$$\begin{aligned} &(a \oplus b)^{\otimes 2} = a^{\otimes 2} \oplus a \otimes b \oplus b \otimes a \oplus b^{\otimes 2} \\ &= \max(2a, a + b, b + a, 2b) = 2\max(a, b) = a^{\otimes 2} \otimes b^{\otimes 2}. \end{aligned} \tag{7.13}$$

The middle two terms in the expansion are redundant since either $2a \geq a + b$ or $2b \geq a + b$. This brings us to a way to determine inessential terms in a polynomial. The condition for the rth term to be an inessential term, in the polynomial notation of (7.11) is [63]

$$c_r \otimes x^{\otimes r} \leq \bigoplus_{i=0, i \neq r}^{n} c_i \otimes x^{\otimes i}. \tag{7.14}$$

Analogously to the definition of a complex rational function (6.10), a (max,+) rational function $f(x)$ is a ratio (in a consistent sense) of two (max,+) polynomials. Namely

$$f(x) = \left(\bigoplus_{i=0}^{p} A_i \otimes x^{\otimes i}\right) \oslash \left(\bigoplus_{j=0}^{q} B_j \otimes x^{\otimes j}\right) = \max_{i=0}^{p}(A_i + ix) - \max_{j=0}^{q}(B_j + jx). \tag{7.15}$$

A (max,+) rational function is more general than a (max,+) polynomial in the sense that the latter is a strictly non-decreasing function in x.

7.3 Ultra-discretization

Ultra-discrete equations are obtained from discrete/difference equations, in a limiting procedure called ultra-discretization of which we shall present an example [67, 68].

Ultra-discretization is a technique used to obtain ultra-discrete equations from discrete equations. As we shall see, ultra-discretization is a procedure that requires definition of a certain limit.

We define real variables $a, b, c \geq 0$, in terms of which the new variables $A, B, C \in \mathbb{R} \cup \{-\infty\}$ are

$$a = e^{A/\epsilon}, \qquad b = e^{B/\epsilon}, \qquad c = e^{C/\epsilon}. \tag{7.16}$$

If we consider the equation $c = a \times b$, we deduce that $C = A + B$. Recalling that our definitions of new binary operations gives $C = A \otimes B$, we deduce that "ordinary" multiplication of a with b corresponds to our redefined multiplication of A with B.

Next we consider $c = a + b$, in which case we cannot so easily relate A, B and C. However

$$C = \lim_{\epsilon \to 0^+} \epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A, B). \tag{7.17}$$

This limiting process is referred to as ultra discretization. With our redefined notion of addition it can be written as $0C = A \oplus B$. We shall now prove that this limit is correct. If $A \geq B$, we have

$$\epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \epsilon \log e^{A/\epsilon}(1 + e^{(B-A)/\epsilon}) = A + \epsilon \log(1 + e^{(B-A)/\epsilon}). \tag{7.18}$$

Now taking the limit as $\epsilon \to 0^+$, the second term on the right tends to zero since $B - A \leq 0$. Then we get the correct answer of $\max(A, B) = A$. The proof for the case $A \geq B$ follows by symmetry.

We consider the generalization of equation (7.17) to n dependent variables $a_i = e^{A_i/\epsilon}, i = 1, \ldots, n$. We see that

$$\lim_{\epsilon \to 0^+} \left(\epsilon \log \sum_{i=1}^{n} a_i \right) = \bigoplus_{i=1}^{n} A_i, \qquad \epsilon \log \prod_{i=1}^{n} a_i = \bigotimes_{i=1}^{n} A_i. \tag{7.19}$$

As an example of carrying out ultra discretization, we start with a discrete equation called d-$\mathrm{P_{III}}$ (see for example [70]), as there exists a certain continuum limit in which the equation can be cast as the $\mathrm{P_{III}}$ differential equation. It is, with discrete variable $y_n = y(q^n)$,

$$y_{n+1}y_{n-1} = \frac{\alpha q^n + y_n^2}{1 + \alpha q^n y_n^2}, \qquad \alpha = \frac{(q-1)^2}{q}, \qquad y_n, q, \alpha \in \mathbb{R}. \tag{7.20}$$

Note that we must restrict all variables to the real line. We define the ultra-discrete variables Y_n, Q, A by

$$y_n = e^{Y_n/\epsilon}, \qquad q = e^{Q/\epsilon}, \qquad \alpha = e^{A/\epsilon}, \tag{7.21}$$

in terms of which equation (7.20) is

$$\begin{aligned} Y_{n+1} + Y_{n-1} &= \epsilon \log(e^{(A+nQ)/\epsilon} + e^{2Y_n/\epsilon}) - \epsilon \log(e^{0/\epsilon} + e^{(A+nQ+2Y_n)/\epsilon}) \\ &\to \max(A + nQ, 2Y_n) - \max(0, A + nQ + 2Y_n) \end{aligned} \tag{7.22}$$

in the limit as $\epsilon \to 0^+$. Also we find $A = \max(2Q, 0) - Q = |Q|$. With the circled binary operators and the associated notation we have

$$Y_{n+1} \otimes Y_{n-1} = (A \otimes Q^{\otimes n} \oplus Y_n^{\otimes 2}) \oslash (\mathbb{I} \oplus A \otimes Q^{\otimes n} \otimes Y_n^{\otimes 2}), \qquad A = (Q \oplus \mathbb{I})^{\otimes 2} \oslash Q, \tag{7.23}$$

which bears a striking, if slightly superficial, resemblance to the form of (7.20). We needed to use the notation for the multiplicative identity $0 = \mathbb{I}$. We refer to it as a form of u-P_{III} (ultra discrete P_{III}) because of its relation to the continuous P_{III} [67]. The independent variable $n \in \mathbb{Z}$ in equation (7.23). We wish to generalize it to $x \in \mathbb{R} \cup \{-\infty\}$, in a similar way to how we generalized the independent variable in the discrete equation (6.1) to obtain the difference equation (6.2). Then we obtain the ultra-discrete equation

$$\begin{gathered} Y(x+1) \otimes Y(x-1) = (A \otimes Q^{\otimes x} \oplus Y(x)^{\otimes 2}) \oslash (\mathbb{I} \oplus A \otimes Q^{\otimes x} \otimes Y(x)^{\otimes 2}), \\ A = (Q \oplus \mathbb{I})^{\otimes 2} \oslash Q, \end{gathered} \tag{7.24}$$

A (max,+) rational function (7.15) - or equivalently a piecewise linear function - may arise as a solution to (7.24).

7.4 (Max,+) meromorphic functions

Here we shall produce original ideas in defining a class of piecewise linear functions, and concepts of roots and poles of such functions, in a way that seems to be the most natural for extendeing the well-known definitions given in previous chapters.

Definition 7.4.1 *A piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is said to be* (max,+) meromorphic *if its first derivative $f'(x), x \in \mathbb{R}$ is an integer at each point $x \in \mathbb{R}$.*

Note that it is a generalization of a (max,+) rational function (7.15) to a function which may have an infinite number of distinct linear segments. For example a non-constant periodic (max,+) meromorphic function is not (max,+) rational.

In terms of a particular (max,+) meromorphic function f, define the function $\omega_f : \mathbb{R} \to \mathbb{Z}$ by

$$\omega_f(x) = \lim_{\epsilon \to 0^+} [f'(x+\epsilon) - f'(x-\epsilon)], \tag{7.25}$$

and if $\omega_f(x) > 0$ then x is called a *root* of f with multiplicity $\omega_f(x)$. If $\omega_f(x) < 0$ then x is called a *pole* of f with multiplicity $-\omega_f(x)$. Otherwise if $\omega_f(x) = 0$ then x is called an *ordinary point* of f.

Now we define the root/pole structure of f at $x = \pm\infty$. Define x_1, x_2 such that $f'(x) = m_1 \forall x < x_1$ and $f'(x) = m_2 \forall x > x_2$. Then $-\infty$ is a root of multiplicity m_1 if $m_1 > 0$, but is a pole of multiplicity $-m_1$ if $m_1 < 0$. Conversely $+\infty$ is a pole of multiplicity m_2 if $m_2 > 0$, but is a root of multiplicity $-m_2$ if $m_2 < 0$.

Let f and g be (max,+) meromorphic functions; then $h = f \circ g$ is piecewise linear. Moreover, on any interval on which the first derivative h' is defined, h is a composition of two linear functions with integer slopes. It follows that h is (max,+) meromorphic. To prove this, we work with an interval $\Omega \subseteq \mathbb{R}$ in which $f(x) = Ax+B$ and $g(x) = Cx + D$. Then

$$h = f \circ g = A(Cx + D) + B, \qquad x \in \Omega,$$

for which $h' = AC \neq 0$. We can extend this result to prove that if f and g are (max,+) meromorphic on $\Omega \subseteq \mathbb{R}$, then so are $f \oplus g$, $f \otimes g$ and $f \circ g$. Furthermore, if $g \neq -\infty$, then $f \oslash g$ is also (max,+) meromorphic.

Let $R(x, y)$ be a (max,+) rational function in y, with coefficients (A_i, B_j) that are (max,+) meromorphic functions of x. Then it can be written as a (max,+) ratio of (max,+) polynomials, as

$$R(x,y) = \left(\bigoplus_{i=0}^{p} A_i(x) \otimes y^{\otimes i} \right) \oslash \left(\bigoplus_{j=0}^{q} B_j(x) \otimes y^{\otimes j} \right). \tag{7.26}$$

If $y(x)$ is (max,+) meromorphic then so is $R(x, y(x))$.

Examples of (max,+) meromorphic functions

Reverting temporarily to the definitions of complex analysis, the complex function $f(z) = z^n, n \in \mathbb{N}$ has a zero of multiplicity n at the origin of the complex plane. It has a pole of multiplicity n at ∞. We restrict this function to $\mathbb{R}$; $f(x) = x$ and define ultra-discrete variables $s, F(s)$ by $x = e^{s/\epsilon}$ and $f = e^{F/\epsilon}$. Then the function transforms to

$$F(s) = s^{\otimes n} = ns, \qquad n \in \mathbb{N},$$

which, in our (max,+) language, has a root of multiplicity n at $s = -\infty$, and has a pole of multiplicity n at $s = +\infty$. We see that we have mapped the origin of the complex plane to $-\infty$ adjacent to the real line.

Using similar techniques on $f(z) = 1/z^n, n \in \mathbb{N}$ which has a pole of multiplicity n at $z = 0$ and a zero of multiplicity n at $z = \infty$ we obtain

$$F(s) \doteq \mathbb{I} \oslash s^{\otimes n} = -ns, \qquad n \in \mathbb{N},$$

which, in our (max,+) language, has a pole of multiplicity n at $s = -\infty$, and has a root of multiplicity n at $s = +\infty$.

7.5 A test for integrability of ultra-discrete equations

As in [73] we consider the ultra discrete equation

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0), \tag{7.27}$$

which is obtainable as the limit of an integrable discrete equation. The equation (7.27) has a critical point at $X_n = -K$, where the right hand side is discontinuous. Introducing a small number ϵ, we perturb from the critical point to $X_n = -K + \epsilon$. With this choice equation (7.27) will vary depending on the sign of ϵ. We demand that $X_{n-1} > 2|K|$. Then we get the iterates tabulated below.

	$\epsilon > 0$	$\epsilon < 0$
X_n	$-K + \epsilon$	$-K + \epsilon$
X_{n+1}	$K - X_{n-1}$	$K - X_{n-1} - \epsilon$
X_{n+2}	$X_{n-1} - \epsilon$	X_{n-1}
X_{n+3}	X_{n-1}	$X_{n-1} + \epsilon$
X_{n+4}	$K - X_{n-1} + \epsilon$	$K - X_{n-1}$
X_{n+5}	$-K - \epsilon$	$-K - \epsilon$
X_{n+6}	X_{n-1}	X_{n-1}

(7.28)

From table (7.28) it is seen that while we have imposed the initial condition that X_n is the same for $\epsilon < 0$ and $\epsilon > 0$, the coefficients of ϵ in X_i do not match in the range $n+1 \leq i \leq n+4$. This implies they are not differentiable at $X_i = -K$ in that range, but are outside it, meaning the discontinuity in the derivative is confined. For generic ultra-discrete equations we would expect a discontinuity to persist in future iterates once it has formed. This type of singularity confinement is argued in [73] to be a test for integrability of ultra-discrete equations.

Chapter 8

Nevanlinna theory on the (max,+) semiring

We shall work with (max,+) meromorphic functions as defined in the previous chapter, and derive an original theory of the value distribution of such functions on the real line. The theory is presented in the preprint by Halburd and Southall [79]. In many ways this is analogous to Nevanlinna theory on the complex plane which concerns the value distribution of meromorphic functions, as presented in sections 2.7 and 6.5. In this light we shall refer to the theory described here as (max,+) Nevanlinna theory.

In terms of a (max,+) meromorpic function f, we define the Nevanlinna characteristic $T(r, f)$, proximity function $m(r, f)$ and counting function $N(r, f)$. Analogues of some — but not all — of the results from classical Nevanlinna theory are proved, such as the first main theorem of Nevanlinna and the lemma on the logarithmic derivative.

Some ultra-discrete equations admit (max,+) meromorphic solutions. We conjecture that in the sense of the (max,+) Nevanlinna theory we have introduced, the ultra-discrete Painlevé equations (and in general all integrable ultra-discrete equations) admit finite-order (max,+) meromorphic solutions on $\mathbb{R}$. Our definition of a finite-order (max,+) meromorphic function f is that there exist positive numbers σ and r_0 such that $T(r, f) \leq r^\sigma$, $\forall r > r_0$.

It will be shown that many ultra-discrete equations admit infinite-order (max,+)

meromorphic solutions but the ultra-discrete Painlevé equations appear to admit finite-order (max,+) meromorphic solutions. The general solutions of both difference equations and ultra-discrete equations contain arbitrary period one functions. One significant difference, however, is that many meromorphic period one functions have infinite order in the complex setting, while all non-constant (max,+) meromorphic periodic functions have order two.

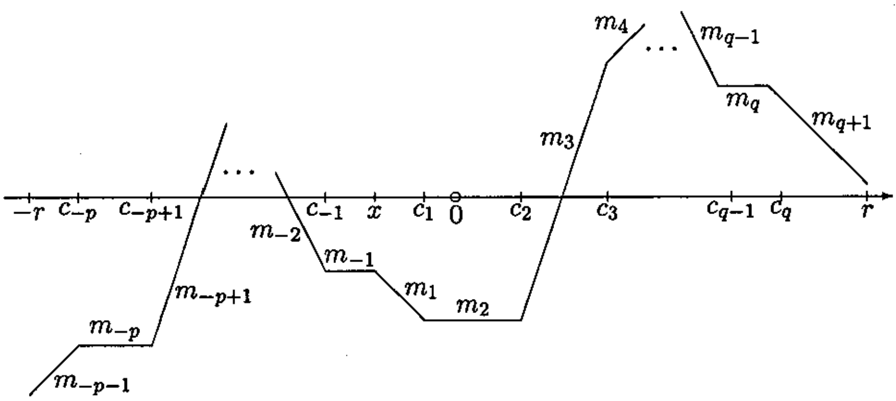
8.1 (Max,+) Poisson-Jensen formula

Lemma 8.1.1 *Suppose that f is a (max,+) meromorphic function defined for $x \in [-r, r]$, $r > 0$. The roots of f in this interval are denoted by $a_\mu, \mu = 1, \ldots, M$ and the poles are denoted by $b_\nu, \nu = 1, \ldots N$. They are denoted according to their multiplicities; for example a double pole - meaning of multiplicity two - is counted twice in the appropriate summation. In this interval we have the* (max,+) Poisson-Jensen formula *[79],*

$$\begin{aligned} f(x) &= \tfrac{1}{2}(f(r)+f(-r)) + \tfrac{x}{2r}(f(r)-f(-r)) \\ &\quad -\tfrac{1}{2r}\sum_{\mu=1}^{M}(r^2-|a_\mu - x|r - a_\mu x) + \tfrac{1}{2r}\sum_{\nu=1}^{N}(r^2-|b_\nu - x|r - b_\nu x). \end{aligned} \tag{8.1}$$

Proof. Define a finite increasing series of points $\{c_k\}, k = -p, \ldots, q$ as follows. Let $c_0 = x$ and let the other elements in the series denote the points in $\gamma \in (-r, r)$ at which $f'(\gamma)$ does not exist. We denote by m_k the gradients of the line segments in the graph of f.

Figure 8.1: Notation used in the proof of the (max,+) Poisson-Jensen formula



Specifically, for $k = 0, \ldots, p$ we set $m_{k+1} = \lim_{\epsilon \to 0^+} f(c_k + \epsilon)$, and deduce that

$$f(x) = f(r) - m_1(r - x) + \sum_{j=1}^{q} (m_j - m_{j+1})(r - c_j). \tag{8.2}$$

and for $k = -q, \ldots, 0$ we set $m_{k-1} = \lim_{\epsilon \to 0^+} f(c_k - \epsilon)$. It follows that

$$f(x) = f(-r) + m_{-1}(r + x) + \sum_{i=1}^{p} (m_{-i-1} - m_{-i})(r + c_{-i}). \tag{8.3}$$

Adding equation $(r + x) \times (8.2)$ to $(r - x) \times (8.3)$ gives

$$\begin{aligned} 2rf(x) &= r(f(r) + f(-r)) + x(f(r) - f(-r)) + (r^2 - x^2)(m_{-1} - m_1) \\ &\quad + \textstyle\sum_{i=1}^{p} (m_{-i-1} - m_{-i})(r^2 + (c_{-i} - x)r - c_{-i}x) \\ &\quad + \textstyle\sum_{j=1}^{q} (m_j - m_{j+1})(r^2 + (x - c_j)r - c_j x) \end{aligned} \tag{8.4}$$

We now wish to write the summations in terms of the roots a_μ and poles b_ν. In particular, if c_{-i} is a root of multiplicity $\omega_{-i} \in \mathbb{Z}$, then $m_{-i-1} - m_{-i} = \omega_{-i}$, but if it is a pole of multiplicity ω_{-i} then $m_{-i-1} - m_{-i} = -\omega_{-i}$. After performing the same procedures on the sum over j we obtain the Poisson-Jensen formula (8.1). □

We define a real independent variable $x \in [-r, r]$, and a function $f : \mathbb{R} \to \mathbb{R}$, and define in terms of this the nonnegative function

$$f^+(x) = \max(f(x), 0), \tag{8.5}$$

from which we find the property $f^+ + (-f)^+ = |f| \forall f \in \mathbb{R}$. The *(max,+) proximity function* is

$$m(r, f) = \frac{f^+(r) + f^+(-r)}{2}. \tag{8.6}$$

The *(max,+) counting function* $n(r, f)$ gives the number of poles b_ν of f in the interval $(-r, r)$, counting multiplicities. This integrates to yield

$$N(r, f) = \frac{1}{2} \int_0^r n(t, f)dt = \frac{1}{2} \sum_{\nu=1}^{N} (r - |b_\nu|). \tag{8.7}$$

Putting together the above results, we define the *(max,+) Nevanlinna characteristic function* as

$$T(r, f) = m(r, f) + N(r, f). \tag{8.8}$$

8.2 An analogue of Nevanlinna's first main theorem for piecewise linear functions

With $x = 0$ the (max,+) Poisson-Jensen formula reduces to the (max,+) Jensen formula,

$$f(0) = \frac{1}{2}(f(r) + f(-r)) - \frac{1}{2}\sum_{\mu=1}^{M}(r - |a_\mu|) + \frac{1}{2}\sum_{\nu=1}^{N}(r - |b_\nu|). \tag{8.9}$$

We note that inverting f about the x-axis is equivalent to replacing f with $-f$, in which case the roots and poles swap. As equation (8.7) counts the number of poles $n(r, f)$ of f, it follows that $n(r, -f)$ counts the roots of f. Integrating,

$$N(r, -f) = \frac{1}{2}\sum_{\mu=1}^{M}(r - |a_\mu|). \tag{8.10}$$

Since $\max(f, 0) - \max(-f, 0) = f$,

$$m(r, f) - m(r, -f) = \frac{f(r) + f(-r))}{2}. \tag{8.11}$$

Substituting these results in equation (8.9) yields

$$T(r, f) - T(r, -f) = f(0). \tag{8.12}$$

8.3 Properties of the Nevanlinna characteristic

Lemma 8.3.1 *$T(r, f)$ is a continuous non-decreasing piecewise linear function of r.*

Proof. Choose $r > 0$ such that f does not have a root or a pole at either $x = r$ or $x = -r$. Differentiation yields

$$\frac{dT(r, f)}{dr} = \frac{1}{2}\left(\frac{df^+(r)}{dr} - \frac{df^+(-r)}{d(-r)}\right) + \frac{1}{2}n(r, f). \tag{8.13}$$

In the case where both $f(r) < 0$ and $f(-r) < 0$, then the non negativity of $n(r, f)$ implies $\frac{dT(r,f)}{dr} \geq 0 \forall r$ which is the required result.

Before evaluating the case where $f(-r) \geq 0$ and $f(r) \geq 0$, we differentiate the (max,+) Jensen formula,

$$0 = \frac{f'(r) - f'(-r)}{2} + \frac{1}{2}n(r, f) - \frac{1}{2}n(r, -f) \tag{8.14}$$

where a prime denotes a derivative of a function with respect to its argument. Then

$$\begin{aligned}\tfrac{dT(r,f)}{dr} &= \tfrac{1}{2}(f'(r) - f'(-r) + n(r,f)) \\ &= \tfrac{1}{2}(n(r,-f) - n(r,f) + n(r,f)) \\ &= \tfrac{1}{2}n(r,-f) \\ &\geq 0.\end{aligned} \tag{8.15}$$

Next we consider the case in which $f(-r) < 0$ and $f(r) \geq 0$. There must be a sub-interval of $(-r, r)$ on which the graph of f has strictly positive slope. Therefore its slope at $x = r$ is strictly greater than $-n(r,f)$ - $f'(r) > -n(r,f)$. It follows that

$$\frac{dT(r,f)}{dr} = \frac{1}{2}(f'(r) + n(r,f)) > 0. \tag{8.16}$$

Lastly we consider the case in which $f(-r) \geq 0$ and $f(r) < 0$. Similar reasoning to that in the previous case shows that $f'(-r) < n(r,f)$. Hence

$$\frac{dT(r,f)}{dr} = \frac{1}{2}(-f'(-r) + n(r,f)) > 0. \tag{8.17}$$

□

We introduce a finite set of piecewise linear functions $f_i(x) \in \mathbb{R} \cup \{-\infty\}$, $i = 1, \ldots, n$ which satisfy, in the (max,+) notation,

$$T\left(r, \bigoplus_{i=1}^{n} f_i\right) \leq \bigotimes_{i=1}^{n} T(r, f_i), \tag{8.18}$$

$$T\left(r, \bigotimes_{i=1}^{n} f_i\right) \leq \bigotimes_{i=1}^{n} T(r, f_i). \tag{8.19}$$

We shall prove these properties in the case $n = 2$ with $(f_1, f_2) = (f, g)$, from which the general results follow by induction. Then $\oplus_{i=1}^{n} f_i = \max(f, g)$ and $\otimes_{i=1}^{n} f_i = f + g$. In particular

$$\begin{gathered} m(r, f+g) = \tfrac{(f+g)^+(r) + (f+g)^+(-r)}{2} \\ \leq \tfrac{f^+(r) + f^+(-r)}{2} + \tfrac{g^+(r) + g^+(-r)}{2} = m(r,f) + m(r,g). \end{gathered} \tag{8.20}$$

Also

$$\begin{gathered} m(r, \max(f,g)) = \tfrac{(\max(f,g))^+(r) + (\max(f,g))^+(-r)}{2} \\ \leq \tfrac{f^+(r) + f^+(-r)}{2} + \tfrac{g^+(r) + g^+(-r)}{2} = m(r,f) + m(r,g). \end{gathered} \tag{8.21}$$

The above inequalities can be proved using general arguments. Elements $a, b \in \mathbb{R} \cup \{-\infty\}$ satisfy $(a+b)^+ \leq a^+ + b^+$, The equality is satisfied if $a \geq 0, b \geq 0$ since then $a + b \geq 0$. Otherwise the left hand side is strictly less than the right. Also $(\max(a,b))^+ \leq a^+ + b^+$. Here, if $a \geq 0 \geq b$, or if $b \geq 0 \geq a$ then the equality is satisfied. Otherwise the left hand side is strictly less than the right. Then we may replace a and b by $f(x)$ and $g(x)$ respectively, and the same inequalities will hold $\forall x \in [-r, r]$.

Next we consider the poles of f and g. If all poles of f differ from those of g, then $N(r, f+g) = N(r, f) + N(r, g)$. However in general their poles may coincide in which case $N(r, f+g) \leq N(r, f) + N(r, g)$. The function $\max(f, g)$, as the upper envelope of f and g will have at most the sum of the roots of f and of g. Therefore the counting functions satisfy $N(r, \max(f, g)) \leq N(r, f) + N(r, g)$.

Combining the results into $T(r, f) = m(r, f) + N(r, f)$ and switching to (max,+) notation,

$$T(r, f \oplus g) \leq T(r, f) \otimes T(r, g), \qquad T(r, f \otimes g) \leq T(r, f) \otimes T(r, g). \tag{8.22}$$

These are easily generalized by induction to equations (8.18,8.19).

Theorem 8.3.2 *If f is a (max,+) meromorphic function, then $T(r, f) = O(r)$ if and only if f is a (max,+) rational function.*

Proof. If f is a (max,+) rational function then there exists $R > 0$ such that $f(r) = A_1 r + A_2$, $f(-r) = A_3 r + A_4$ and $n(r, f) = A_5$ where the A_i are constants and $r > R$. The result then follows from our definitions of $N(r, f)$ and $T(r, f)$.

Next we assume that f is a meromorphic function satisfying $T(r, f) \leq Kr$, for some K and all sufficiently large r. From Lemma 8.3.3 we have, for any $k > 1$,

$$n(r, f) \leq \frac{2}{(k-1)r} N(kr, f) \leq \frac{2}{(k-1)r} T(kr, f) \leq \frac{2kK}{(k-1)}.$$

Hence f has a finite number of poles. Similarly, using equation (??), we find that $n(r, -f)$ is also bounded and so f has a finite number of roots. It follows from Lemma ?? that f is rational. □

Let f be a non-constant periodic function with period 2ω. Then there exist positive constants K_1, K_2 such that

$$K_1 r^2 \leq T(r, f) \leq K_2 r^2, \qquad \forall r \geq \omega. \tag{8.23}$$

Lemma 8.3.3 *For any $k > 1$ we have that*

$$n(r,f) \leq \frac{2}{(k-1)r} N(kr,f). \tag{8.24}$$

Proof.

$$n(r,f)r = n(r,f)\int_r^{2r} dt \leq \int_r^{2r} n(t,f)dt \leq \int_0^{2r} n(t,f)dt = 2N(2r,f). \tag{8.25}$$

□

8.4 Borel-Nevanlinna lemma

Lemma 8.4.1 *Let $T(r,f)$, $\xi(x)$ and $\phi(r)$ be positive, nondecreasing, continuous functions with $r \in [r_0, \infty)$. Also assume that $T(r,f) \geq e$, and $x \in [e, \infty)$. Then the Borel-Nevanlinna lemma (see [72]) states that*

$$T\left(r + \frac{\phi(r)}{\xi(T(r,f))}, f\right) \leq 2T(r,f), \tag{8.26}$$

$\forall r$ outside of a set E which satisfies, for $R < \infty$,

$$\int_{E\cap[r_0,R]} \frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)} + \frac{1}{\log 2}\int_e^{T(R,f)} \frac{dx}{x\xi(x)}. \tag{8.27}$$

In particular, by choosing $\phi(r) = 1$ and $\xi(x) = x$ this implies the standard Borel Lemma (see for example [26]) which states that

$$T\left(r + \frac{1}{T(r,f)}, f\right) \leq 2T(r,f) \tag{8.28}$$

outside an exceptional set E of finite linear measure, that is $\int_E dt < \infty$.

Theorem 8.4.2 *If f is (max,+) meromorphic, then the (max,+) counting function satisfies*

$$n(r,f) \leq \frac{4}{r} N(r,f)^{1+\epsilon} \tag{8.29}$$

where $\epsilon > 0$, outside of a possible exceptional set E of finite logarithmic measure, that is $\int_E \frac{dt}{t} < \infty$.

Proof. If $n(r,f) \equiv 0$, there is nothing to prove. From Lemma 8.3.3 with $k = 1 + N(r,f)^{-\epsilon}$, we have

$$n(r,f) \leq \frac{2}{r} N(r,f)^{\epsilon} N\left(r + \frac{r}{N(r,f)^{\epsilon}}, f\right).$$

Now we apply Lemma 8.26 with $T(r) = N(r,f)$, $\phi(r) = r$ and $\xi(x) = x^{\epsilon}$, which shows that

$$N\left(r + \frac{r}{N(r,f)^{\epsilon}}, f\right) \leq 2N(r,f),$$

outside an exceptional set E satisfying

$$\int_{E\cap[r_0,R]} \frac{dr}{r} \leq \frac{1}{e^{\epsilon}} + \frac{1}{\log 2}\int_{e}^{N(R,f)} \frac{dx}{x^{1+\epsilon}} \leq \left(1 + \frac{1}{\epsilon \log 2}\right) e^{-\epsilon}.$$

□

Definition 8.4.3 *A (max,+) meromorphic function is said to be of finite order if there exist positive numbers σ and r_0 such that $T(r,f) \leq r^{\sigma}$, for all $r > r_0$.*

Corollary 8.4.4 *Let f be a finite-order (max,+) meromorphic function. Then for all $\delta < 1$, $n(r,f) \leq r^{-\delta} N(r,f)$, outside an exceptional set E of finite logarithmic measure.*

Proof. Now $N(r,f) \leq T(r,f) \leq r^{\sigma}$. Choose $\epsilon < (1-\delta)/\sigma$. Then for sufficiently large r, $4N(r,f)^{\epsilon} < r^{1-\delta}$. Now apply Theorem 8.4.2. □

The finite-order condition is important in Corollary 8.4.4. Consider the infinite-order function f such that $f(x) = 0$ for all $x < 0$, f has no roots and its only poles occur at each non-negative integer n with multiplicity 2^n. In this case $N(r,f) = O(n(r,f))$.

8.5 An analogue of the lemma on the logarithmic derivative

In standard Nevanlinna theory we work with a meromorphic function f of a single complex variable z. One useful result in Nevanlinna theory is the lemma on the logarithmic derivative $f'(z)/f(z)$. In [71] an analogue of this lemma was proved for the logarithmic difference $f(z+c)/f(z),\ f,z,c \in \mathbb{C}$. This result was important in the classification of difference equations admitting finite-order meromorphic solutions by Halburd and Korhonen [78].

The (max,+) analogue of the logarithmic difference is $f(x+c) \oslash f(x) = f(x+c) - f(x),\ f,x,c \in \mathbb{R}$.

Lemma 8.5.1 *Let f be a (max,+) meromorphic function. Then*

$$m(r, f(x+c) \oslash f(x)) \leq \frac{2^{1+\epsilon} \cdot 14|c|}{r} \left\{T(r+|c|,f)^{1+\epsilon} + o(T(r+|c|,f)\right\}, \quad (8.30)$$

for any $\epsilon > 0$, outside an exceptional set of finite logarithmic measure.

Proof. We define a function $g(x) = f(x+c) \oslash f(x)$. For any $\rho > r+|c|$ and $x \in [-r,r]$, the (max,+) Poisson-Jensen formula gives

$$\begin{aligned} &g(x) = f(x+c) - f(x) = \tfrac{c}{2\rho}(f(\rho) + f(-\rho)) \\ &+\tfrac{1}{2\rho}\textstyle\sum_{\mu}[(|a_\mu - x - c| - |a_\mu - x|)\rho + ca_\mu] \\ &-\tfrac{1}{2\rho}\textstyle\sum_{\nu}[(|b_\nu - x - c| - |b_\nu - x|)\rho + cb_\nu], \end{aligned} \quad (8.31)$$

with $|a_\mu| < \rho$ and $|b_\nu| < \rho$. We shall next prove (8.30). We first deduce a relation between x^+ and $|x|$ where $x \in \mathbb{R}$,

$$|x| = \max(x,0) + \max(-x,0) = x^+ + (-x)^+, \quad (8.32)$$

which implies $x^+ \leq |x|$. Using the results (8.22) we deduce that

$$\begin{aligned} &m(r, g(x)) \leq m\left(r, \tfrac{c}{2\rho}(f(\rho) + f(-\rho))\right) \\ &+m\left(r, \tfrac{1}{2\rho}\textstyle\sum_{\mu}[(|a_\mu - x - c| - |a_\mu - x|)\rho + ca_\mu]\right) \\ &+m\left(r, -\tfrac{1}{2\rho}\textstyle\sum_{\nu}[(|b_\nu - x - c| - |b_\nu - x|)\rho + cb_\nu]\right). \end{aligned} \quad (8.33)$$

We note that the first term on the right is independent of x; therefore,

$$\begin{aligned} m\left(r, \tfrac{c}{2\rho}(f(\rho) + f(-\rho))\right) &= \left(\tfrac{c}{2\rho}(f(\rho) + f(-\rho))\right)^+ \\ &\leq \left|\tfrac{c}{2\rho}(f(\rho) + f(-\rho))\right| \\ &\leq \tfrac{|c|}{2\rho}(|f(\rho)| + |f(-\rho)|) \\ &= \tfrac{|c|}{2\rho}(f^+(\rho) + (-f)^+(\rho) \\ &\quad +f^+(-\rho) + (-f)^+(-\rho)) \\ &= \tfrac{|c|}{\rho}(m(\rho, f) + m(\rho, -f)). \end{aligned} \quad (8.34)$$

Also

$$\begin{aligned}
& m\left(r, \frac{1}{2\rho}\left\{\left(|a_\mu - x - c| - |a_\mu - x|\right)\rho + ca_\mu\right\}\right) \\
\leq\ & \frac{1}{2}\left(\left\{\left||a_\mu - r - c| - |a_\mu - r|\right| + \left||a_\mu + r - c| - |a_\mu + r|\right|\right\} + \frac{|ca_\mu|}{\rho}\right) \leq \frac{3}{2}|c|,
\end{aligned}$$

since $|a_\mu| < \rho$. From the above estimates and Theorem 8.4.2, for any $\epsilon > 0$,

$$\begin{aligned}
& m(r, f(x+c) - f(x)) \\
\leq\ & |c|\left\{\frac{1}{\rho}\left(m(\rho, f) + m(\rho, -f)\right) + \frac{3}{2}\left(n(\rho, f) + n(\rho, -f)\right)\right\} \\
\leq\ & \frac{|c|}{\rho}\left\{T(\rho, f) + T(\rho, -f) + 6T(\rho, f)^{1+\epsilon} + 6T(\rho, -f)^{1+\epsilon}\right\} \\
\leq\ & \frac{7|c|}{\rho}\left\{T(\rho, f)^{1+\epsilon} + T(\rho, -f)^{1+\epsilon}\right\} \\
\leq\ & \frac{14|c|}{\rho}\left\{T(\rho, f)^{1+\epsilon} + o\left(T(\rho, f)\right)\right\},
\end{aligned}$$

outside an exceptional set of finite logarithmic measure. Choosing $\rho = r + |c| + 1/T(r+|c|, f)$ and using Lemma 8.26 with r replaced by $r+|c|$, we obtain $T(\rho, f) \leq 2T(r+|c|, f)$, outside a set of finite linear measure. □

Lemma 8.5.2 *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing continuous function, $s > 0$, $\alpha < 1$, and let $F = \{r \in \mathbb{R}^+ : T(r) \leq \alpha T(r+s)\}$. If the logarithmic measure of F is infinite, that is, $\int_F \frac{dt}{t} = \infty$, then $\limsup_{r\to\infty} \log T(r)/\log r = \infty$.*

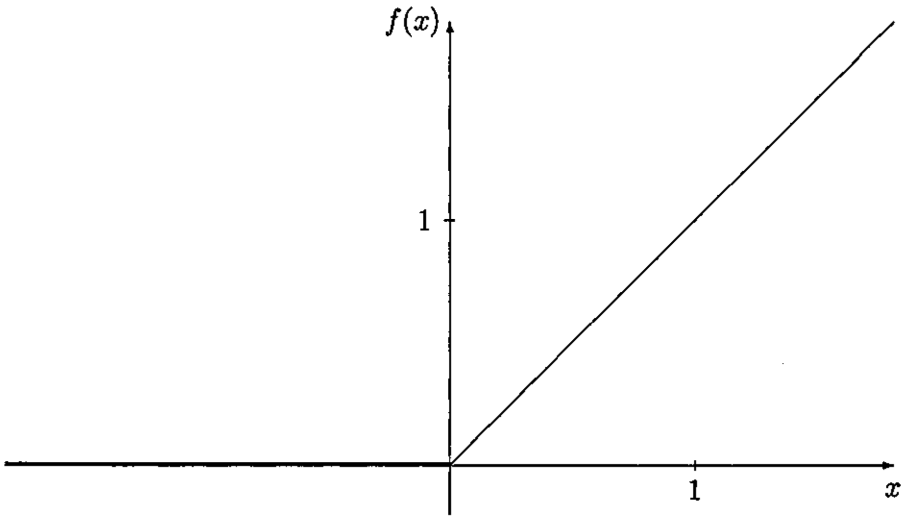
Theorem 8.5.3 *Given $\delta < 1$, any finite-order (max,+) meromorphic function f satisfies the lemma on the logarithmic difference,*

$$m(r, f(x+c) \oslash f(x)) = O\left(r^{-\delta} T(r, f)\right), \tag{8.35}$$

outside an exceptional set of finite logarithmic measure.

Proof. Since f has finite order, Lemma 8.5.2 implies that $T(r+|c|, f) \leq 2T(r, f)$ outside an exceptional set of finite logarithmic measure. Also, there exist positive constants σ and r_0 such that $T(r, f) \leq r^\sigma$ for all $r > r_0$. Choose $\epsilon = (1-\delta)/\sigma$. Then $T(r, f)^\epsilon / r \leq r^{-\delta}$. □

Figure 8.2: Graph showing function (8.36)



8.6 Applications to (max,+) rational functions

We consider a (max,+) rational function of degree 1 with $A_0 = 0, A_1 = 1, B_0 = 0$,

$$f(x) = \max(0, x), \tag{8.36}$$

which has a simple root at $x = 0$ and no poles. Its graph is shown in Figure 8.2. Its inverse with respect to the $\otimes$ operation is $-f(x) = -\max(0, x)$, which has no roots and a simple pole at $x = 0$. We substitute this in the (max,+) Poisson-Jensen formula (8.1) to give

$$\begin{aligned} f(x) &= \tfrac{1}{2}(\max(r,0) + \max(-r,0)) + \tfrac{x}{2r}(\max(r,0) - \max(-r,0)) \\ &\quad -\tfrac{1}{2r}(r^2 - |x|r) \\ &= \tfrac{r}{2} + \tfrac{x}{2} - \tfrac{1}{2}(r - |x|) \\ &= \tfrac{1}{2}(x + |x|), \end{aligned} \tag{8.37}$$

which is an alternative expression of function (8.36). The (max,+) proximity function (8.6) becomes

$$\begin{aligned} m(r,f) &= \tfrac{\max(\max(r,0),0)+\max(\max(-r,0),0)}{2} = \tfrac{r}{2}, \\ m(r,-f) &= \tfrac{\max(-\max(r,0),0)+\max(-\max(-r,0),0)}{2} = 0. \end{aligned} \tag{8.38}$$

The (max,+) counting function $n(r,f)$ gives the number of poles of f in the interval $(-r,r)$, which is zero $\forall r$. It follows that $n(r,-f)$ gives the number of roots, which equals one, with due count of multiplicities $\forall r$. The integrated (max,+) counting function (8.7) becomes

$$N(r,f) = \frac{1}{2}\sum_{\nu=1}^{N}(r-|b_\nu|) = 0, \qquad N(r,-f) = \frac{r}{2}. \tag{8.39}$$

Then the (max,+) characteristic function (8.8) is

$$T(r,f) = m(r,f)+N(r,f) = \frac{r}{2}, \qquad T(r,-f) = m(r,-f)+N(r,-f) = \frac{r}{2}, \tag{8.40}$$

so $T(r,f) - T(r,-f) = 0 =: f(0)$ as required.

We shall see that inequality (8.24) holds for our current definition of f. There is nothing to prove for $f(x) = \max(0,x)$ since both sides vanish. For $-f(x) = -\max(0,x)$ we have

$$1 \leq \frac{2}{(k-1)r}\frac{kr}{2} = \frac{k}{k-1}, \qquad k > 1. \tag{8.41}$$

Similarly we shall see that inequality (8.29) is satisfied. Again it holds trivially for $f(x) = \max(0,x)$. For $-f(x) = -\max(0,x)$ we have

$$1 \leq 2\left(\frac{r}{2}\right)^{\epsilon} \tag{8.42}$$

which is true for sufficiently large r. We next consider the difference function, assuming $c > 0$,

$$f(x+c) \oslash f(x) = \max(0,x+c) - \max 0,x) = \begin{cases} 0, & x < -c \\ x+c, & -c \leq x < 0 \\ c, & x \geq 0, \end{cases} \tag{8.43}$$

while if $c \leq 0$ then the difference function vanishes.

$$f(x+c) \oslash f(x) = \max(0,x+c) - \max 0,x) = \begin{cases} 0, & x < 0 \\ -x, & 0 \leq x < -c \\ c, & x \geq -c. \end{cases} \tag{8.44}$$

The (max,+) proximity function (8.6) is, for arbitrary c and $r > |c|$,

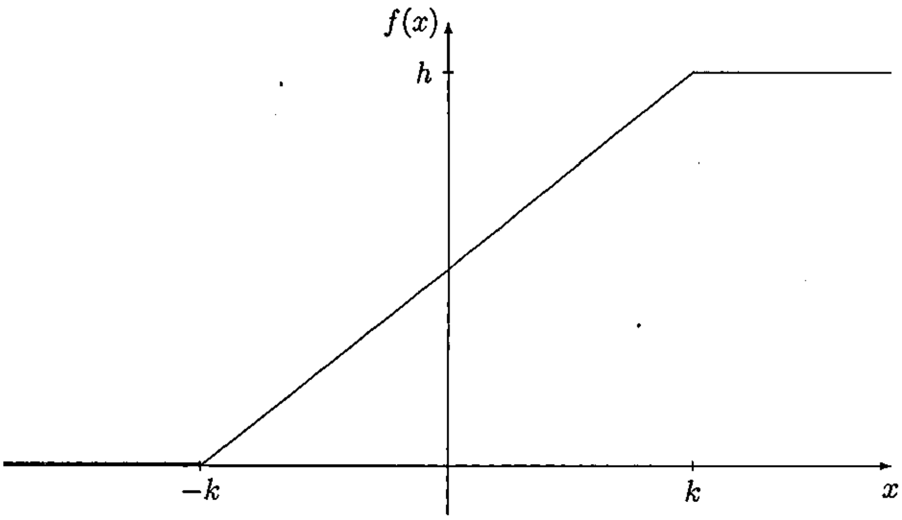
$$m(r, f(x+c) \oslash f(x)) = \frac{c}{2}. \tag{8.45}$$

We next substitute into the lemma on the logarithmic difference (8.30),

$$\frac{c}{2} = o\left(\frac{r}{2}\right). \tag{8.46}$$

Single ramp (max,+) rational function

Figure 8.3: Graph showing function (8.47)



The next piecewise linear function we shall consider has $f(x) = 0$ as $x \to -\infty$, and $f(x) = h > 0$ as $x \to \infty$. It is non constant only within a range centred on the origin,

$$\begin{aligned} f(x) &= \begin{cases} 0, & x \le -k, \\ \frac{h}{2k}(x+k), & -k < x \le k, \\ h, & k < x. \end{cases} \\ &= \tfrac{1}{2}(\max(0, h(x/k+1)) - \max(0, h(x/k-1))) \end{aligned} \tag{8.47}$$

Its graph is shown in Figure 8.3. We note that $f(x)$ has a root of multiplicity $h/2k$ at $x = -k$ and a pole of multiplicity $h/2k$ at $x = k$. In order for $f(x)$ to be (max,+) rational we require $h/2k \in \mathbb{N}$.

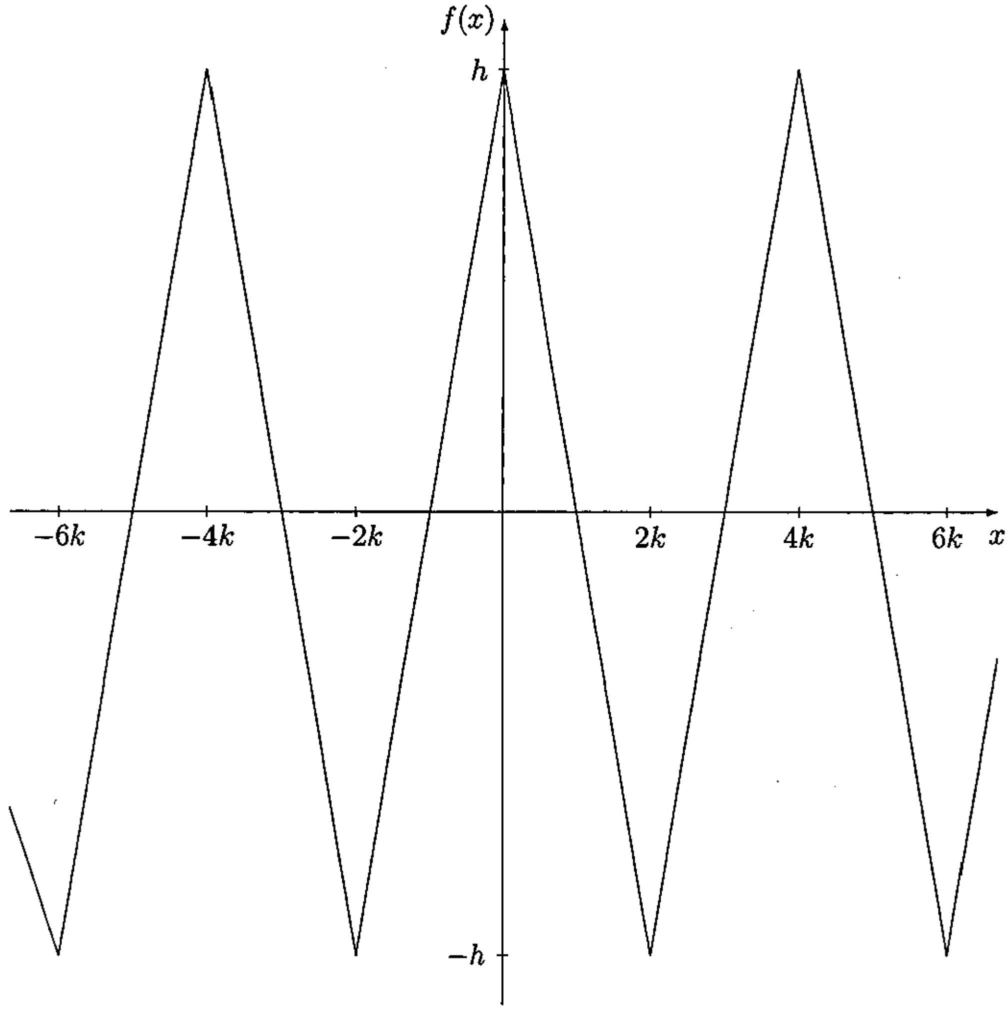
We shall apply our findings to the (max,+) Poisson-Jensen formula (8.1) to check we recover the original function. We let $r > k$, where $x \in (-r, r)$. In fact we may let $r \to \infty$ in which case $x \in \mathbb{R}$. Then

$$\begin{aligned} f(x) &= \tfrac{h}{2r}(r+x) - \tfrac{1}{2r}\tfrac{h}{2k}(r^2 - |k+x|r + kx) + \tfrac{1}{2r}\tfrac{h}{2k}(r^2 - |k-x|r - kx) \\ &= \tfrac{h}{4k}(2k + |k+x| - |k-x|), \end{aligned}$$

which may be shown by working out a few specific cases to be equivalent to the original form (8.47).

8.7 Application to a (max,+) meromorphic function

Figure 8.4: Graph showing function (8.48)



We shall consider a periodic (max,+) meromorphic function $f(x)$, $x \in [-\infty, +\infty)$.

It oscillates between $\pm h$ with period $4k$, with $h/k \in \mathbb{N}$, and $f(0) = h$. Then

$$f(x) = \begin{cases} \vdots & \vdots \\ h(x/k+5), & -6k \le x \le -4k, \\ h(-x/k-3), & -4k \le x \le -2k, \\ h(x/k+1), & -2k \le x \le 0, \\ h(-x/k+1), & 0 \le x \le 2k, \\ h(x/k-3), & 2k \le x \le 4k, \\ h(-x/k+5), & 4k \le x \le 6k, \\ \vdots & \vdots . \end{cases} \tag{8.48}$$

Its graph is shown in Figure 8.4. Substitution of f and $-f$ respectively in (8.6) give

$$m(r,f) = \begin{cases} h(-r/k+1), & 0 \le r \le k, \\ 0, & k \le r \le 3k, \\ h(r/k-3), & 3k \le r \le 4k, \\ h(-r/k+5), & 4k \le r \le 5k, \\ 0, & 5k \le r \le 7k, \\ \vdots & \vdots , \end{cases} \tag{8.49}$$

$$m(r,-f) = \begin{cases} 0, & 0 \le r \le k, \\ h(r/k-1), & k \le r \le 2k, \\ h(-r/k+3), & 2k \le r \le 3k, \\ 0, & 3k \le r \le 5k, \\ h(r/k-5), & 5k \le r \le 6k, \\ \vdots & \vdots . \end{cases} \tag{8.50}$$

We see that the function (8.48) has roots at $x = 4k(\mu + \frac{1}{2}), \mu \in \mathbb{Z}$ and poles at $x = 4k\nu, \nu \in \mathbb{Z}$, and each root or pole has multiplicity $2h/k$, where $h/k \in \mathbb{Z}$. We note that each root or pole $x = \alpha$ is repeated at $x = -\alpha$, except for the root or pole at the origin which is not repeated. Then the (max,+) counting function (8.7) for our cases is

$$N(r,f) = \begin{cases} hr/k, & 0 \le r < 4k, \\ h(3r/k-8), & 4k \le r < 8k, \\ \vdots & \vdots , \end{cases} \tag{8.51}$$

$$N(r,-f)=\begin{cases} 0, & 0\le r<2k,\\ h(2r/k-4), & 2k\le r<6k,\\ h(4r/k-16), & 6k\le r<10k,\\ \vdots & \vdots. \end{cases} \tag{8.52}$$

We can now calculate the Nevanlinna characteristic $T(r,f)=m(r,f)+N(r,f)$ for f and $-f$ respectively,

$$T(r,f)=\begin{cases} h, & 0\le r<k,\\ hr/k, & k\le r<3k,\\ h(2r/k-3), & 3k\le r<5k,\\ h(3r/k-8), & 5k\le r<7k,\\ \vdots & \vdots, \end{cases} \tag{8.53}$$

$$T(r,-f)=\begin{cases} 0, & 0\le r<k,\\ h(r/k-1), & k\le r<3k,\\ h(2r/k-4), & 3k\le r<5k,\\ h(3r/k-9), & 5k\le r<7k,\\ \vdots & \vdots. \end{cases} \tag{8.54}$$

Then we calculate $T(r,f)-T(r,-f)=h=:f(0)$ as expected from the (max,+) analogue of Nevanlinna's first main theorem (8.12), for $r\in[0,7k)$. Since $f(x)$ was defined to repeat to infinity it follows that the result holds for arbitrary values of r.

8.8 Applications to ultra-discrete equations

The main application that we have in mind for (max,+) Nevanlinna theory is as a measure of the complexity of solutions of ultra-discrete equations. In particular, the aim is to use ideas from (max,+) Nevanlinna theory to classify equations that are natural ultra-discrete analogues of the Painlevé equations. Many such equations have been considered in the literature recently [67, 70, 68, 76, 73]. Most of these equations have been obtained directly as ultra-discretizations of known discrete Painlevé equations. Examples of such equations include

$$y_{n+1}+y_{n-1} = \max\{y_n+n,0\}-y_n, \tag{8.55}$$

$$y_{n+1} + y_{n-1} = a + \max\{y_n, n\} - \max\{y_n + n, 0\} - y_n, \tag{8.56}$$

$$\begin{aligned} y_{n+1} + y_{n-1} &= \max\{n + a, y_n\} + \max\{n - a, y_n\} \\ &\quad - \max\{y_n + n + b, 0\} - \max\{y_n + n - b, 0\}, \end{aligned} \tag{8.57}$$

where a and b are constants.

Conventionally, only solutions of ultra-discrete equations that are functions from $\mathbb{Z}$ to itself are considered. However, equations such as (8.55–8.57) can be reinterpreted as equations for a continuous piecewise linear real function y of a real variable x. In particular, instead of equation (8.55), we consider the "extended" equation

$$y(x + 1) + y(x - 1) = \max\{y(x) + x, 0\} - y(x), \qquad x \in \mathbb{R}. \tag{8.58}$$

It now makes sense to ask about the existence and general properties of (max,+) meromorphic solutions of equations such as (8.58). Based on analogous considerations of (genuine) meromorphic solutions of difference equations in the complex domain in [54, 68, 78], it is natural to begin by considering ultra-discrete equations admitting finite-order (max,+) meromorphic solutions. We will present evidence that this property can be thought of as an ultra-discrete analogue of the Painlevé property. The Painlevé property is property is closely associated with the integrability of differential equations.

We begin by addressing some simple questions on the existence of (max,+) meromorphic solutions.

Lemma 8.8.1 *The equation*

$$y(x + 1) = y(x)^{\otimes n} := ny(x) \tag{8.59}$$

admits a non-constant (max,+) meromorphic solution on $\mathbb{R}$ *if and only if* $n = \pm 1$.

Proof. If $n = 0$ then $y \equiv 0$ is the only solution. Recall that any periodic (max,+) meromorphic function is of finite order. If $n = 1$ then y is any (max,+) meromorphic period one function. If w is any period two (max,+) meromorphic function, then $y(x) := w(x + 1) - w(x)$ is a (max,+) meromorphic solution of equation (8.59) with $n = -1$.

If y is non-constant then $\exists x_0 \in \mathbb{R}$ such that y' exists and is a non-zero integer m at x_0. It follows from equation (8.59) that for all $\nu \in \mathbb{Z}$, $y'(x_0 - \nu) = m/n^\nu$. Therefore if $\nu \neq \pm 1$ then for sufficiently large ν, $0 < |y'(x_0 - \nu)| < 1$, and hence the slope is not an integer. □

Note that the (max,+) Nevanlinna characteristic can be defined for arbitrary continuous piecewise linear functions (not necessarily with integer slopes) if we allow the counting function $n(r, f)$ to count poles of non-integer multiplicites (i.e., the differences in slopes). For now we remark that allowing for non-integer multiplicities, the extra condition of finite-order needs to be added to the assumptions in equation (8.59) in order to reach the same conclusion.

Apart from the analogue of Clunie's lemma which we derive in section 8.9, we shall restrict our attention to ultra-discrete equations of the form

$$y(x+1) \otimes y(x-1) = R(x, y(x)), \tag{8.60}$$

where R is (max,+) rational in x and y. We remark that all such equations admit infinitely many (max,+) meromorphic solutions. To see this, choose $y(0)$ and $y(1)$ to be any real numbers and calculate $y(2) := R(1, y(1)) - y(0)$. Now define y on $(0,1) \cup (1,2)$ such that y is a continuous piecewise-linear function on $[0,2]$ with integer slopes wherever y' is defined. Then the equation itself extends y uniquely to a (max,+) meromorphic solution on $\mathbb{R}$.

We will show that large classes of equations of the form (8.60) admit infinite-order solutions. In the simplest cases, this can be achieved by showing that there is a sequence of integers (ν_n) such that $|\nu_n| \to \infty$ and $y(x_0 + \nu_n) \geq C^{\nu_n}$ for some $C > 1$.

Lemma 8.8.2 *Let $y \not\equiv 0$ be a (max,+) meromorphic solution of*

$$y(x+1) \otimes y(x-1) = y(x)^{\otimes n}, \tag{8.61}$$

for some $n \in \mathbb{Z}$. If y is of finite order then $|n| \leq 2$.

Proof. Let

$$\lambda_\pm = \frac{n \pm \sqrt{n^2 - 4}}{2}.$$

$\exists x_0 \in \mathbb{R}$ such that $y(x_0) \neq 0$. Therefore, for at least one choice of "+" or "−", we have that $y(x_0+1) \neq \lambda_\pm y(x_0)$. Then for each $\nu \in \mathbb{Z}$,

$$y(x_0+\nu) = \alpha\lambda_+^\nu + \beta\lambda_-^\nu, \tag{8.62}$$

where

$$\alpha = \frac{y(x_0+1) - \lambda_- y(x_0)}{\lambda_+ - \lambda_-} \quad \text{and} \quad \beta = \frac{y(x_0+1) - \lambda_+ y(x_0)}{\lambda_- - \lambda_+}$$

are not both zero.

Now if $n > 2$, then $\lambda_+ > 1$ and $\lambda_-^{-1} > 1$, while if $n < -2$ then $-\lambda_- > 1$ and $-\lambda_+^{-1} > 1$. Hence either $y(x_0+\nu)$ or $y(x_0-\nu)$ grows exponentially as ν tends to infinity on the even integers. So $T(r,y)$ is not bounded by a power of r. □

Theorem 8.8.3 *Let $P(y) = \max\{a_0, a_1 + y, \ldots, a_p + py\}$ and $Q(y) = \max\{b_0, b_1 + y, \ldots, b_q + qy\}$ be two (max,+) polynomials with no common roots and neither a_p nor b_q is $-\infty$. If $|p-q| > 2$, then the equation*

$$y(x+1) + y(x-1) = P(y(x)) - Q(y(x)) \tag{8.63}$$

admits infinitely many infinite-order (max,+) meromorphic solutions.

Proof. If y is sufficiently large for all x larger than some number ξ, then equation (8.63) reduces to

$$y(x+1) + y(x-1) = (p-q)y(x) + a_p - b_q,$$

for all $x > \xi$. If $p - q > 2$ then, given $\alpha > 0$, there is a family of solutions such that any member evaluated at an integer $\nu > \xi$ has the form

$$y(\nu) = \frac{b_q - a_p}{p-q-2} + \alpha\left(\frac{(p-q) + \sqrt{(p-q)^2 - 4}}{2}\right)^\nu.$$

So y and hence $T(r,y)$ grow exponentially. If $p - q < -2$ then the same argument works with a minus sign in front of the square root. □

In [73], Joshi and Lafortune consider the equation

$$y_n + 3y_n + y_{n-1} = \max\{n + K, 0\},$$

where K is a constant, as an example of an ultra-discrete equation that does not possess their singularity confinement property. Analogously, we have the following.

Lemma 8.8.4 *Let K be a positive constant and let y be a (max,+) meromorphic solution of*

$$y(x+1)+3y(x)+y(x-1)=\max\{y(x)+K,0\} \tag{8.64}$$

such that $y(0)>0$, $y(1)<-K$ and $y(1)<-y(0)$. Then y has infinite order.

Proof. It is straightforward to show by induction that for all $n\geq 1$, if n is odd then $y(n+1)\geq -2y(n)>0$ and if n is even then $-y(n+1)\geq y(n)>0$. Hence y grows exponentially on $\mathbb{N}$. □

In [76], Joshi and Lafortune considered the ultra-discrete equation

$$X_{n-1}+X_n+X_{n+1}=\max\{X_n+\phi_n,0\}.$$

and showed that the condition for singularity confinement is

$$\phi_{n+5}-\phi_{n+3}-\phi_{n+2}+\phi_n=0.$$

That is,

$$\phi_n=\alpha+\beta n+\gamma(-1)^n+\delta\cos\left(\frac{2\pi n}{3}\right)+\omega\sin\left(\frac{2\pi n}{3}\right).$$

We consider the analogous equation

$$y(x-1)+y(x)+y(x+1)=\max\{y(x)+\phi(x),0\}. \tag{8.65}$$

The confinement criterion now becomes

$$\phi(x)=\pi_2(x)+\pi_3(x)+Nx+C,$$

where π_2 and π_3 are arbitrary periodic (max,+) meromorphic functions of period 2 and 3 respectively, and N is an integer and C is a real number.

We note some important observations of equation (8.65). Analytically it can be shown that the solutions of equation (8.65) are of finite-order if ϕ is a linear function. Furthermore, numerical studies suggest that if ϕ is a periodic function of order 2 or 3 (or a sum of such functions) then the order of y is finite. If ϕ is chosen to be a (max,+) meromorphic function of period 4 or 5 then y appears to have infinite order. However, when $\phi(x)$ is chosen to have the form $x+\psi(x)$, where ψ is bounded, then numerical studies suggest that y is finite order, regardless of the precise form of ψ. However, in the cases studied, for sufficiently large x, the solutions of equation (8.65) are identical to (not merely asymptotic to) solutions of simpler "integrable" ultra-discrete equations. This is quite unlike the complex analytic setting in which we have uniqueness of analytic continuation.

In [78], the classification of difference equations admitting finite-order meromorphic solutions in the complex domain relied on estimating the relative distribution of the points at which the solution, y, hits one of the finite singular values of the equation and the distribution of the poles of y. The method used naturally led to a variant of the usual singularity confinement method. An analogue of this part of the argument exists relating the distribution of the singular values where $y(x) = -\phi(x)$ to the poles of y, using method related to singularity confinement. In order to deduce that non-confinement implies that the solution has infinite order, we need to show that there are "many" points at which the solution takes a singular value. In the complex analytic case [78], this is guaranteed by using a difference version of Clunie's lemma and Nevanlinna's first main theorem. Below we present an ultra-discrete version of Clunie's lemma, however, it is the absence of a strong (max,+) version of Nevanlinna's first main theorem that prevents the same argument going through. Indeed, when ϕ grows sufficiently fast it appears from numerical studies that any solution only hits a singular point a finite number of times.

In [77], Grammaticos, Ramani, Tamizhmani, Tamizhmani and Carstea show that the equation

$$y(x+1) = y(x-1) + |y(x)| \tag{8.66}$$

does not possess the ultra-discrete singularity confinement property. From the (max,+) Nevanlinna point of view, equation (8.66) possesses infinite-order (max,+) meromorphic solutions. In particular, if $y(0) = y(1) = 1$ then $(y(n))_{n\in\mathbb{N}}$ is the Fibonacci sequence, which grows exponentially.

8.9 Clunie's lemma for (max,+) meromorphic functions

We will present an analogue of Clunie's lemma for ultra-discrete equations. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m)$, where the λ_js are non-negative integers, be a multi-index with respect to the shifts $(0, c_1, \ldots, c_m) \in \mathbb{R}^{m+1}$. Let

$$f^{\otimes\lambda}(x) := \lambda_0 f(x) + \lambda_1 f(x+c_1) + \cdots \lambda_m f(x+c_m).$$

An expression of the form

$$\sum_{\lambda\in\Lambda} a_\lambda(x) \otimes f^{\otimes\lambda}(x),$$

where Λ is a finite set of indices, is called a (max,+) polynomial in f and its shifts. We will say that the coefficients are small if $T(r, a_\lambda) = o(T(r,f))$ outside a set of finite logarithmic measure.

The following is a natural analogue of Clunie's lemma.

Theorem 8.9.1 *Let $P(x,f)$ and $Q(x,f)$ be (max,+) polynomials in f and its shifts with small coefficients. If f is a finite-order (max,+) meromorphic function satisfying $f^{\otimes n}(x)P(x,f) = Q(x,f)$, where the degree of Q in f and its shifts is less than or equal to n, then for any $\delta < 1$,*

$$m\left(r, P(x,f)\right) = O\left(r^{-\delta}T(r,f)\right) + o\left(T(r,f)\right),$$

outside an exceptional set of finite logarithmic measure.

Proof. Given $r > 0$, let $S_+ := \{s : f(s) \geq 0 \text{ and } |s| = r\}$ and $S_- := \{s : f(s) < 0 \text{ and } |s| = r\}$. Then

$$m(r, P(x,f)) = \frac{1}{2}\left(\sum_{s\in S_+} P(s,f)^+ + \sum_{s\in S_-} P(s,f)^+\right).$$

Let $P(x,f) = \sum_{\lambda\in\Lambda_P} a_\lambda(x) \otimes f^{\otimes\lambda}(x)$ and $Q(x,f) = \sum_{\lambda\in\Lambda_Q} b_\lambda(x) \otimes f^{\otimes\lambda}(x)$. For any $x \in S_-$,

$$\begin{aligned} P(x,f) &= \sum_{(\lambda_0,\ldots,\lambda_m)\in\Lambda_P} a_\lambda(x) \otimes f^{\otimes\lambda_0}(x) \otimes f^{\otimes\lambda_1}(x+c_1) \otimes \cdots \otimes f^{\otimes\lambda_m}(x+c_m) \\ &\leq \max_{\substack{(\lambda_0,\ldots,\lambda_m)\\ \in\Lambda_P}} \left\{a_\lambda(x) + \lambda_1[f(x+c_1) - f(x)] + \cdots + \lambda_m[f(x+c_m) - f(x)]\right\}. \end{aligned}$$

So using the lemma on the logarithmic difference (8.35), we see that

$$\sum_{s\in S_-} P(s,f)^+ = O\left(r^{-\delta}T(r,f)\right) + o\left(T(r,f)\right),$$

outside an exceptional set of finite logarithmic measure. For $x \in S_+$, we note that $P(x,f) = Q(x,f) - nf$ and the degree of Q is at most n. Hence

$$P(x,f) \leq \max_{\substack{(\lambda_0,\ldots,\lambda_m)\\ \in\Lambda_Q}} \{b_\lambda(x) + \lambda_1[f(x+c_1) - f(x)] + \cdots + \lambda_m[f(x+c_m) - f(x)]\}.$$

So again using the lemma on the logarithmic difference (8.35) we find that

$$\sum_{s\in S_+} P(s,f)^+ = O\left(r^{-\delta}T(r,f)\right),$$

outside an exceptional set of finite logarithmic measure. □

Chapter 9

(Max,+) algebraic entropy

In this chapter we shall work with numerical simulations of various integrable and nonintegrable ultra-discrete equations. The ultra-discrete Painlevé equations are prototypal integrable equations, while perturbations of their coefficients are believed to lead to nonintegrable equations. The conclusions we reach are believed to be original.

We consider those ultra-discrete equations in a dependent variable X_n where $n \in \mathbb{Z}$ such as

$$X_{n+1} \otimes X_n \otimes X_{n-1} = 0 \oplus X_n \otimes K,$$

where K is an arbitrary constant. A solution of such an equation is a sequence of iterates. Moreover we shall let each iterate be a (max,+) rational function of an auxiliary variable $x \in \mathbb{R} \cup \{-\infty\}$. Then we shall define the degree q_n of $X_n(x)$. As initial conditions we specify the (max,+) rational functions $X_0(x)$ and $X_1(x)$.

The evolution of the degrees of successive iterates is investigated for different ultra-discrete equations. The equations are grouped according to whether they are integrable or not, determined by whether they satisfy the singularity confinement test for ultra-discrete equations introduced in [73]. Our aim in doing this is to look for an analogue of the concept of algebraic entropy from section 6.3, which can be used as a detector of integrability in discrete equations. We conclude that zero algebraic entropy appears to be a necessary condition for integrability of an ultra-discrete equation.

9.1 Theory

We shall work with ultra-discrete equations in a similar way to which we worked with discrete equations in section 6.3. That is we consider the iterates of an ultra-discrete equation to be (max,+) rational functions of an auxiliary independent variable $x \in \mathbb{R} \cup \{-\infty\}$. This function $X_n(x)$ takes the form

$$\begin{aligned} X_n(x) &= (\textstyle\bigoplus_{i=0}^{s_n} A_{ni} \otimes x^{\otimes i}) \oslash (\bigoplus_{j=0}^{t_n} B_{nj} \otimes x^{\otimes j}) \\ &= \max_{i=0}^{s_n}(A_{ni} + ix) - \max_{j=0}^{t_n}(B_{nj} + jx) \end{aligned} \tag{9.1}$$

where on the right we have switched from $(\otimes, \oplus)$ notation to conventional (max,+) notation. The degree of $X_n(x)$ is $q_n := \max(s_n, t_n)$. We must ensure that a (max,+) rational function is represented such that there are no additive factors that might affect the degree; for example $f(x) = \max(6x, 7x) - \max(3x, 5x)$ might naively be said to be of degree 7, but it may be simplified to $f(x) = 3x - \max(0, x)$; therefore its degree is 3.

Our intention is to study the evolution of the degree q_n of a sequence of (max,+) rational functions $X_n(x)$, which are related by a second-order ultra-discrete equation. We define the algebraic entropy of an ultra-discrete equation in the way we defined it for a discrete equation, namely as equation (6.24),

$$s := \lim_{n\to\infty} \frac{\log q_n}{n}.$$

We recall that an (nonintegrable) equation with exponential degree growth has nonzero algebraic entropy whereas an (integrable) equation with polynomial degree growth has zero algebraic entropy.

We will describe how to calculate the degree of a (max,+) rational function (9.1) from its value distribution. We see that $X_n(x)$ has s_n roots and t_n poles in $\mathbb{R} \cup \{-\infty\}$, counting multiplicities. It is necessary also to consider the behaviour of $X_n(x)$ at $x = +\infty$. We see that for large x (9.1) is

$$X_n(x) = k_n + (s_n - t_n)x,$$

where each k_n is an arbitrary constant. If $s_n = t_n$ then it follows that $X_n(x)$ has no roots or poles at infinity. If $s_n < t_n$ then $X_n(x)$ has a root at infinity of multiplicity $\max(0, t_n - s_n)$, and the degree is the number of roots in $\mathbb{R} \cup \{\pm\infty\}$ counting multiplicities, $s_n + \max(0, t_n - s_n) = \max(s_n, t_n) =: q_n$. If $s_n > t_n$ then $X_n(x)$ has a pole at infinity of multiplicity $\max(0, s_n - t_n)$, and the degree is the number of poles in $\mathbb{R} \cup \{\pm\infty\}$ counting multiplicities, $t_n + \max(0, s_n - t_n) = \max(s_n, t_n) =: q_n$.

Next we describe a way to calculate the number of roots or poles of $X_n(x)$. We recall the function (7.25) which calculates the multiplicities of any roots or poles at given x,

$$\omega_{X_n}(x) = \lim_{\epsilon \to 0^+} [X_n'(x+\epsilon) - X_n'(x-\epsilon)].$$

In practice we will work with the approximation $\omega_{X_n}(x) \approx \frac{X_n(x+\nu) - 2X_n(x) + X_n(x-\nu)}{\nu}$ where $\nu \in \mathbb{R}$ is small but finite. In order for this to suffice we must take ν to be sufficiently small such that $X_n(x)$ does not "wiggle" up and down between integer points.

The numbers of roots or poles in $\mathbb{R} \cup \{-\infty\}$ counting multiplicities are then given by

$$\sum_{\forall x \in \nu\mathbb{Z}} \max\left(0, \alpha \frac{X_n(x+\nu) - 2X_n(x) + X_n(x-\nu)}{\nu}\right),$$

where $\alpha = +1$ counts only roots while $\alpha = -1$ counts only poles. The parameter ν is again taken to be small so that $\nu\mathbb{Z} \approx \mathbb{R}$. In practice we take the domain of x to be finite and large, compared with the orders of the coefficients in our (max,+) rational functions, so that we may assume that there are no roots or poles outside the domain.

Because the domain must be finite we will also have to deduce whether the function has roots/poles at $-\infty$. Define $x_1 \in \mathbb{R}$ such that $f'(x) = m \forall x < x_1$. Then $-\infty$ is a root of multiplicity m if $m > 0$, but is a pole of multiplicity $-m$ if $m < 0$.

9.2 Examples of ultra-discrete equations

We shall work with the following ultra-discrete Painlevé equation, which is a form of u-$\mathrm{P_I}$,

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \alpha + n, 0), \tag{9.2}$$

where $\alpha \in \mathbb{R}$ is constant. This example is taken from [68] where many examples of ultra-discrete Painlevé equations are given. That paper extends the definition of a Painlevé equation to ultra-discrete equations.

It was first reported in [73] that a generalization of equation (9.2) which also has the property of singularity confinement in the sense of section 7.5 and is therefore believed to be integrable, is

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \Phi_n, 0), \qquad (9.3)$$
$$\Phi_n = \alpha + \beta(-1)^n + \gamma \cos\left(\frac{2\pi n}{3}\right) + \delta \sin\left(\frac{2\pi n}{3}\right) + \lambda n.$$

Note that this is equivalent to the Painlevé equation (9.2) with $\beta = \gamma = \delta = 0$ and $\lambda = 1$. We shall make the change of variables $X_n \to X_n - \gamma\cos(\frac{2\pi n}{3}) - \delta\sin(\frac{2\pi n}{3})$, and using the trigonometric addition formulas we also have

$$\begin{aligned} X_{n+1} &\to X_{n+1} \\ &\quad +\gamma\left(\tfrac{1}{2}\cos\left(\tfrac{2\pi n}{3}\right) + \tfrac{\sqrt{3}}{2}\sin\left(\tfrac{2\pi n}{3}\right)\right) + \delta\left(\tfrac{1}{2}\sin\left(\tfrac{2\pi n}{3}\right) - \tfrac{\sqrt{3}}{2}\cos\left(\tfrac{2\pi n}{3}\right)\right), \\ X_{n-1} &\to X_{n-1} \\ &\quad +\gamma\left(\tfrac{1}{2}\cos\left(\tfrac{2\pi n}{3}\right) - \tfrac{\sqrt{3}}{2}\sin\left(\tfrac{2\pi n}{3}\right)\right) + \delta\left(\tfrac{1}{2}\sin\left(\tfrac{2\pi n}{3}\right) + \tfrac{\sqrt{3}}{2}\cos\left(\tfrac{2\pi n}{3}\right)\right). \end{aligned} \qquad (9.4)$$

Substitution of this transformation in equation (9.3) cancels the trigonometric terms in Φ_n. It follows that without loss of generality we can set $\gamma = \delta = 0$ there.

Next, we wish to consider an ultra-discrete equation which is not integrable, that is does not satisfy singularity confinement. To obtain one we change one of the coefficients in u-$\mathrm{P_I}$ to obtain

$$X_{n+1} + 3X_n + X_{n-1} = \max(X_n + \alpha + n, 0). \qquad (9.5)$$

The next ultra-discrete equation we shall work with is a form of u-$\mathrm{P_{II}}$, which is also taken from [68],

$$X_{n+1} + X_{n-1} - X_n = \max(-X_n + n + a, 0) - \max(X_n + n + b, 0). \qquad (9.6)$$

Also from [73] a generalization of (9.6) which has confined singularities is

$$X_{n+1} + X_{n-1} - X_n = \Psi_n + \max(-X_n + \Phi_n, 0) - \max(X_n + \Phi_n, 0), \qquad (9.7)$$
$$\Phi_n = \alpha + \beta n + \gamma\cos\left(\frac{2\pi n}{3}\right) + \delta\sin\left(\frac{2\pi n}{3}\right),$$
$$\Psi_n = \mu(-1)^n + \nu,$$

and a non-Painlevé form of (9.6) is

$$X_{n+1} + X_{n-1} - 3X_n = \max(-X_n + n + a, 0) - \max(X_n + n + b, 0). \qquad (9.8)$$

9.3 Numerical results

As with any second order equation, to determine a particular solution we must specify two initial conditions, which in our case are (max,+) rational functions. We choose

$$\begin{aligned} X_0(x) &= \max(-15, x-5, 2x-2, 3x, 4x+4) \\ &\quad - \max(0, x, 2x-4, 3x-15), \\ X_1(x) &= \max(-19, x-8, 2x-5, 3x-3, 4x) \\ &\quad - \max(1, x-3, 2x-6, 3x-16, 4x-22). \end{aligned} \tag{9.9}$$

Also, to investigate the effect of varying the initial conditions we choose a second set,

$$X_0(x) = q_0 x, \qquad X_1(x) = q_1 x, \tag{9.10}$$

These are (max,+) rational functions of the form (9.1) with $(s_0, t_0) = (q_0, 0)$ and $(s_1, t_1) = (q_1, 0)$. The coefficients are $A_{s0} = A_{s1} = 0$ and $A_{i0} = -\infty, i < s_0$, $A_{i1} = -\infty, i < s_1$.

We see that the initial conditions (9.9) have $(q_0, q_1) = (4, 4)$, where the degrees can be reduced by deleting appropriate terms.

In all simulations we shall work with the step size $\nu = 0.01$, which seems to be sufficiently small to account for "wiggling" of our sample functions. Then the presence of roots or poles at x is given by

$$\omega_{X_n}(x) \approx \frac{X_n(x+0.01) - 2X_n(x) + X_n(x-0.01)}{0.01}.$$

We shall calculate this at all points $x \in 0.01\mathbb{Z}$ in the range $[-100, 100]$. All roots and poles in our analysis seem to be well within this range and none were found outside it, excepting those expected at $\pm\infty$.

Tables 9.1 and 9.2 show the sequences of degrees obtained from iterates of the non-integrable equation $X_{n+1} + 3X_n + X_{n-1} = \max(X_n + n, 0)$. The plots of q_n against n show in all cases distinctly exponential evolution. To confirm this, plots of $\log q_n$ against n are approximately linear. Although the exact numbers q_n differ between the tables, the graphs are nearly the same and this is independent of initial conditions besides their degrees.

(q_0, q_1)	(4,4)	(4,3)	(4,2)	(4,1)	(4,0)	(3,4)	(2,4)	(1,4)	(0,4)	(0,1)	(1,0)
q_2	16	13	10	7	4	15	14	13	12	3	1
q_3	39	35	27	15	12	36	33	30	28	5	3
q_4	90	80	62	36	28	83	76	69	64	12	7
q_5	203	179	138	77	64	172	156	140	144	19	16
q_6	424	367	284	169	132	361	328	295	300	45	33
q_7	877	759	585	335	276	708	639	570	620	71	69
q_8	1734	1484	1148	692	540	1429	1294	1159	1224	168	135
q_9	3457	2971	2290	1325	1088	2740	2468	2196	2440	265	272
q_{10}	6688	5701	4410	2671	2084	5447	4926	4405	4716	627	521

Table 9.1: Evolution of degree using equation (9.5) with $\alpha = 0$, with initial conditions (9.9).

(q_0, q_1)	(4,4)	(4,3)	(4,2)	(4,1)	(4,0)	(3,4)	(2,4)	(1,4)	(0,4)	(0,1)	(1,0)
q_2	16	13	10	7	4	15	14	13	12	3	1
q_3	40	33	26	19	12	37	34	33	28	7	3
q_4	92	76	60	44	28	85	78	71	64	16	7
q_5	196	163	130	97	64	180	164	148	132	33	16
q_6	408	339	270	201	132	375	342	309	276	69	33
q_7	816	681	546	411	276	747	678	609	620	71	69
q_8	1628	1356	1084	812	540	1493	1358	1223	1088	272	135
q_9	3172	2651	2130	1609	1088	2900	2628	2356	2084	521	272
q_{10}	6224	5189	4154	3119	2084	5703	5182	4661	4140	1035	521

Table 9.2: Evolution of degree using equation (9.5) with $\alpha = 0$, with initial conditions (9.10).

Evolution equation	Degree growth
$X_{n+1} + X_n + X_{n-1} = \max(0, X_n + n)$	Slow/linear
$X_{n+1} + X_n + X_{n-1} = \max(0, X_n + 3 + 2n + 5(-1)^n)$	Slow/linear
$X_{n+1} + X_n + X_{n-1} = \max(0, X_n + n + \cos(\frac{2\pi n}{3}))$	Slow/linear
$X_{n+1} + 2X_n + X_{n-1} = \max(0, X_n + n)$	Slow/linear
$X_{n+1} - X_n + X_{n-1} = 1 + (-1)^n + \max(0, X_n + n) - \max(0, X_n)$	Slow/linear

Table 9.3: Evolution of degree of iterates $X_n(x)$ according to particular integrable discrete equations.

Evolution equation	Degree growth
$X_{n+1} + 3X_n + X_{n-1} = \max(0, X_n + n)$	Exponential
$X_{n+1} + X_n + X_{n-1} = \max(0, X_n + n + \cos(\frac{\pi n}{2}) + \sin(\frac{\pi n}{2}))$	Slow/linear
$X_{n+1} + X_n + X_{n-1} = \max(0, X_n + n^5 + \cos(\frac{2\pi n}{3}))$	Slow/linear
$X_{n+1} + X_n/2 + X_{n-1} = \max(0, X_n + n)$	Slow/linear
$X_{n+1} + X_n + 2X_{n-1} = \max(0, X_n + n)$	Exponential
$X_{n+1} + X_n + X_{n-1} = 3\max(0, X_n + n)$	Slow/linear
$X_{n+1} - 3X_n + X_{n-1} = 1 + (-1)^n + \max(0, X_n + n) - \max(0, X_n)$	Exponential

Table 9.4: Evolution of degree of iterates $X_n(x)$ according to particular non-integrable discrete equations.

We see from table 9.3 that for all integrable equations, the evolution of the degree with increasing n is slow and not exponential. This is also the conclusion reached from plots of $\log q_n$ against n. As for the non-integrable equations considered in table 9.4, the type of degree growth seems dependent only on the coefficients (a, b) in the term $X_{n+1} + aX_n + bX_{n-1}$ on the left hand sides of the equations. Namely if $|ab| \leq 1$ then the degree growth is slow, but if $|ab| > 1$ then a multiplicative factor of $|ab|^n$ may be expected in the nth iterate, leading to exponential degree growth.

According to the definition of algebraic entropy provided by equation (6.24), exponential degree growth corresponds to nonzero algebraic entropy while slow/linear (polynomial) degree growth corresponds to zero algebraic entropy. We conclude that zero algebraic entropy appears to be a necessary condition for integrability of an ultra-discrete equation.

Appendix A

Lie algebras and Lie groups

We consider a Lie algebra $\mathfrak{g}$ which has a basis of N elements $\{e_i\}$, $i = 1, 2, \ldots N$, defined locally by the commutation relations

$$[e_i, e_j] = C^k{}_{ij} e_k, \tag{A.1}$$

where the constant components of the tensor $C^k{}_{ij} = C^k{}_{[ij]}$ are the structure constants of the Lie algebra. The elements $\{e_i\}$ of $\mathfrak{g}$ are said to be the generators of an element of a Lie group G. They generate the group elements by means of exponentiation. An element $g \in G$ is

$$g = \exp(\theta^i e_i), \tag{A.2}$$

where $\{\theta^i\}$ is a set of N arbitrary parameters which give a particular group element.

We will work with a Lie algebra valued Yang-Mills connection on a manifold M given by

$$A = A^i_\mu e_i dx^\mu. \tag{A.3}$$

The e_i are generators of a Lie group G, and are therefore elements of the corresponding Lie algebra $\mathfrak{g}$. The x^μ are coordinates chosen on a part of the manifold M. The connection is a function of these coordinates.

Appendix B

Outline of general relativity

We consider a D dimensional pseudo-Riemannian manifold M whose points are spacetime events; see for example [4]. We describe a local patch of M by means of a set of D spacetime coordinates $x = (x^\mu)$. Then we give the invariant spacetime interval

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu,$$

where $g_{\mu\nu}(x)$ is called a metric. The metric describes the local gravitational field which is, according to general relativity, synonymous with the geometry of spacetime.

We proceed in our aim by defining an object called the affine connection,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}),$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

We define the Riemann tensor,

$$R^\lambda{}_{\mu\kappa\nu} = \partial_\mu \Gamma^\lambda{}_{\kappa\nu} - \partial_\nu \Gamma^\lambda{}_{\kappa\mu} + \Gamma^\rho{}_{\kappa\mu}\Gamma^\lambda{}_{\rho\nu} - \Gamma^\rho{}_{\kappa\nu}\Gamma^\lambda{}_{\rho\mu}, \tag{B.1}$$

which is of second order in derivatives of the metric. Defining the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, where the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, we are now in a position to write the Einstein field equations [4, 5, 7, 8],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{B.2}$$

We shall work in units in which $c = \hbar = 1$. The stress-energy tensor $T_{\mu\nu}$ quantifies any present external sources of the field. The cosmological constant Λ acts as an intrinsic pressure source [5]. A local solution $(M, g_{\mu\nu})$ of (B.2) exhibits diffeomorphism invariance, under coordinate transformations $x^\mu \to \tilde{x}^\mu(x)$.

Considering the case where $\Lambda = 0$, and when there is also no matter to act as a source hence $T_{\mu\nu} = 0$, we have the vacuum field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

The trace of these equations is $R(1 - \frac{1}{2}D) = 0$, yielding $R = 0$ provided that $D \neq 2$. Solutions $g_{\mu\nu}$ of this reduced equation are said to be Ricci flat. However, such solutions are not necessarily flat (in which case the Riemann curvature tensor would vanish) since a gravitational field can act as its own source [5].

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