Robust Portfolio Management with Multiple Financial Analysts

by

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Abstract

Portfolio selection theory, developed by Markowitz (1952), is one of the best known and widely applied methods for allocating funds among possible investment choices, where investment decision making is a trade-off between the expected return and risk of the portfolio. Many portfolio selection models have been developed on the basis of Markowitz's theory. Most of them assume that complete investment information is available and that it can be accurately extracted from the historical data. However, this complete information never exists in reality. There are many kinds of ambiguity and vagueness which cannot be dealt with in the historical data but still need to be considered in portfolio selection. For example, to address the issue of uncertainty caused by estimation errors, the robust counterpart approach of Ben-Tal and Nemirovski (1998) has been employed frequently in recent years. Robustification, however, often leads to a more conservative solution. As a consequence, one of the most common critiques against the robust counterpart approach is the excessively pessimistic character of the robust asset allocation.

This thesis attempts to develop new approaches to improve on the respective performances of the robust counterpart approach by incorporating additional investment information sources, so that the optimal portfolio can be more reliable and, at the same time, achieve a greater return. Among various methods developed in recent decades for improving on the performance of the classical portfolio selection approach in Markowitz (1952), the multi-expert approach of Lutgens and Schotman (2010) is of particular interest because this approach doesn't require the user to have any prior knowledge regarding the reliability of the chosen experts. However, the multi-expert approach cannot be applied directly in practice, because the approach doesn't account for the actual characteristics of the expert recommendations.

This thesis is based on the research framework developed in Lutgens and Schotman (2010), and it incorporates financial analysts' forecasts into the portfolio selection process. To deal with the ambiguities and vagueness associated with analysts' forecasts, fuzzy set theory is applied to modify the multi-expert approach so that it is capable of adopting ambiguous investment forecasts and expressing vague aspirations of the investor. On the basis of this, a multi-analyst approach to fuzzy portfolio selection is developed. Next, this multi-analyst approach is further extended using the concept of robust counterpart approach to account for the uncertainty in the return estimates. Finally, the developed approaches are tested by using real-world investment forecasts to assess the performances of the proposed approaches. It is shown that the proposed multi-analyst approaches outperformed the conventional investment strategies in terms of expected and realised returns for risk-loving investment. In addition, the advantage of employing multi-analyst approaches is more significant for shorter investment holding periods. This suggests that the proposed methods are more beneficial to risk-loving investors for short term investment.

Keywords: fuzzy variable, multi-analyst approach, mean-variance, portfolio selection, robust counterpart, uncertainties.

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Dedication

This work is dedicated to my parents, Alex and Lorcarrie.

Thank you for loving me and for everything you have done for me.

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List of Abbreviations

CAPM	The capital asset pricing model
CVaR	Conditional value-at-risk
TAIEX	The Taiwan Capitalisation Weighted Stock Index
TAIM100	The FTSE TWSE Taiwan Mid-Cap 100 Index
TAISE50	The FTSE TWSE Taiwan 50 Index
TWD	The official currency of Taiwan
TWSE	The Taiwan Stock Exchange
VaR	Value-at-risk

$\boldsymbol{1/N}$	Equally-weighted asset allocation
$\boldsymbol{E_{MV}}$	Multi-expert portfolio selection
$\boldsymbol{F_{MS}}$	Multi-analyst approach to mean-standard deviation portfolio selection
$\boldsymbol{F_{MV}}$	Multi-analyst portfolio selection with fuzzy aspiration
$\boldsymbol{M_{MV}}$	Multi-prior portfolio selection
$\boldsymbol{P}$	Optimisation problem with the uncertain parameter u
$\boldsymbol{P_{MV}}$	Mean-variance portfolio selection
$\boldsymbol{P_{MS}}$	Mean-standard deviation portfolio selection
$\boldsymbol{R_{MV}}$	Robust portfolio selection
$\boldsymbol{RE_{HP}}$	Robust multi-analyst portfolio selection with separate uncertainty sets
$\boldsymbol{RE_{MS}}$	Robust multi-analyst approach to mean-standard deviation portfolio selection with joint uncertainty set
$\boldsymbol{RE_{MV}}$	Robust multi-analyst approach to mean-variance portfolio selection with joint uncertainty set

List of Notations

$x \in \mathbb{R}^n$	Vector of portfolio weights
$x^* \in \mathbb{R}^n$	Optimal solution to portfolio selection problem
$x^0 \in \mathbb{R}^n$	Vector of portfolio weights generated from the realised parameter values μ^0 and Σ^0
$\mu \in \mathbb{R}^n$	Vector of expected returns
$\Sigma \in \mathbb{R}^n \times \mathbb{R}^n$	Covariance matrix of the asset returns
σ_i^2	Variance of the i^{th} asset
$\lambda \in [0, \infty)$	Risk aversion coefficient
θ_z	Credibility level of the z^{th} analyst prescribed by the investor
R^{Target}	Investment benchmark prescribed by the investor
$r_t \in \mathbb{R}^n$	Vector of historical asset returns at time t
$\bar{\mu}$, $\bar{\sigma}^2$	The sample moment estimates for μ and σ^2, respectively
$\hat{\mu}$, $\hat{\Sigma}$	The maximum likelihood estimates for μ and Σ, respectively
$\acute{\mu}$, $\acute{\Sigma}$	The chosen statistical estimates for μ and Σ, respectively
μ_z, Σ_z	Professional forecasts of μ and Σ given by the zth analyst
$\tilde{\mu}_z$, $\tilde{\Sigma}_z$	Fuzzy forecasts of μ and Σ given by the z^{th} analyst
$\check{\mu}_z$, $\check{\Sigma}_z$	Crisp possibilistic value of $\tilde{\mu}_z$ and $\tilde{\Sigma}_z$, respectively
μ^0, Σ^0	Realised parameter values (true values) for μ and Σ, respectively
$u \in \mathbb{R}^n$	Vector of uncertain input parameter
$x \in \mathbb{R}^n$	Vector of decision variables
$f(.)$	Objective function
$g(.)$	Constraint function
$\mathcal{L}(.)$	Lagrangian function

$\phi \in \mathbb{R}^Z$	Vector of the Lagrange multipliers
F	Feasible set
U	Uncertainty set
$U(\acute{\mu})$	Uncertainty set centred at a given parameter estimate $\acute{\mu}$
U^{Box}	Box uncertainty set
$U^{Ellipsoid}$	Ellipsoid uncertainty set
δ	Desired confidence level for the uncertainty set U

C_A	Characteristic function for indicating the degree of belonging of an element to a crisp variable $\tilde{A}$
$M_{\tilde{A}}$	Membership function for indicating the degree of belonging of an element to a fuzzy variable $\tilde{A}$
$\tilde{A}_\alpha$	$\alpha - cut$ set of fuzzy variable $\tilde{A}$
$\tilde{A}^{Tra}$	Trapezoidal fuzzy variable
$\tilde{A}^{Tri}$	Triangular fuzzy variable
$\tilde{A}^{Bell}$	Bell-shaped fuzzy variable
$[m_-, m_+]$	Peak tolerance interval of a fuzzy variable $\tilde{A}$
$\sigma_-,\ \sigma_+$	Negative and positive deviation from the tolerance interval

$\mathbf{1}$	Vector of ones
$\emptyset$	Empty set

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Chapter 1

Introduction

One of the major breakthroughs for modern finance is the portfolio selection theory developed by Harry Markowitz in 1952. This well-known investment theory employs two input parameters, the expected return and the risk of the underlying assets as measured by the corresponding variance, to determine the asset allocation and, on the basis of a trade-off between the expected return and the risk, the optimal portfolio allocation can then be obtained. However, to generate a satisfactory outcome from applying this theory, one needs to assess the accuracy of the input parameters, as this would influence the portfolio performance under any circumstance (Best & Grauer, 1991; Chopra & Ziemba, 1993; Schöttle & Werner, 2009).

In most practical applications it is very difficult, and can be impossible, to know the 'true' values of these parameters. In fact, the values of the parameters will only be realised in the future or, in any case, cannot be measured at the time that the portfolio selection problem needs to be solved. Therefore, some approximated or estimated values of the parameter are usually adopted. In addition, the framework of Markowitz's return-risk portfolio selection problem is very sensitive to even small changes in the input parameters, and thus, the optimal portfolio allocation generated by this model is not very reliable if the incorrect or inaccurate parameter values are adopted (Michaud, 1998; Schöttle & Werner, 2009).

Aside from using sophisticated statistical approaches to improve on the accuracy of the input parameter estimates, there have been many other methods proposed for eliminating or at least reducing the possibility of obtaining unwanted portfolio outcomes, such as the resampling approach and the fuzzy optimisation approach (Michaud, 1998; Liu, 2011). In particular, the robust counterpart approach solves an optimisation problem by including a wide possible range of input parameter values, hence it is guaranteed that its solution is good for all possible values of the input parameters. In addition, among all the different methods in the literature, the implementation of the robust counterpart approach is relatively straightforward, hence computationally cheaper (Ben-Tal & Nemirovski, 1998; Fa., 2007; Quaranta & Zaffaroni, 2008; Scherer, 2002). Due to these distinguishing features, the robust counterpart approach has attracted lots of attention in both academic research and practical application.

However, one must bear in mind that, when making decisions under uncertainty, there is a distinction between a good decision and a good outcome. The robust counterpart approach in general tends to give a conservative solution, hence the performance of the resulting robust portfolio is usually not ideal in practice, especially in terms of portfolio returns. Different suggestions and improvements have been proposed from various aspects, but none of them has appropriately incorporated additional investment information sources into the decision making process to improve robust portfolio performance. With this in mind, we outline the objectives of this thesis.

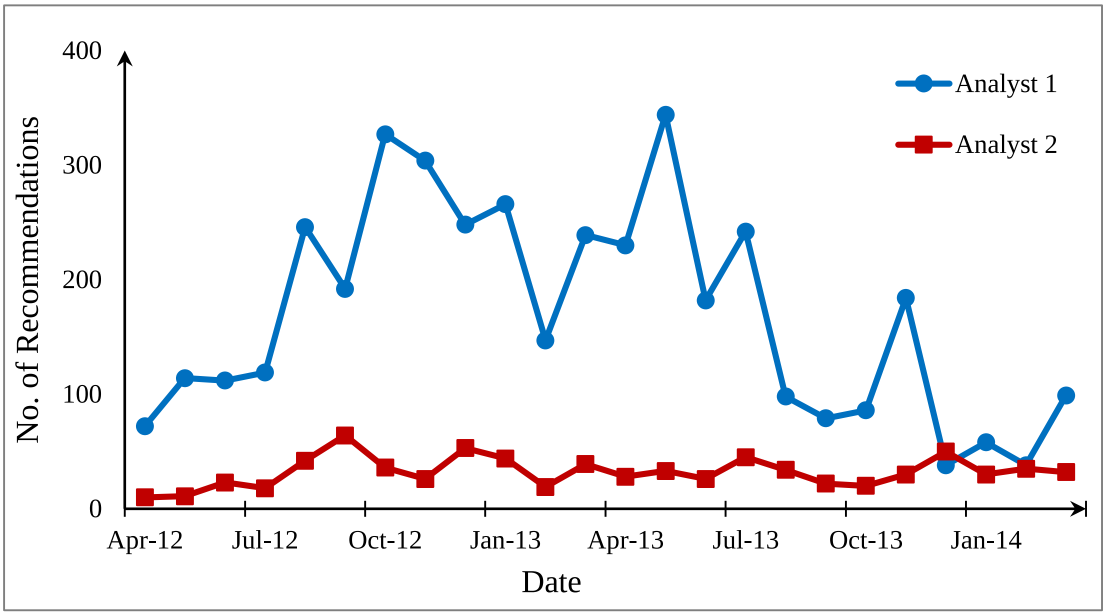
1.1 Objectives and Motivations of the Thesis

The purpose of applying the robust counterpart approach for asset allocation is to construct an optimal portfolio under the worst possible investment situation. This is achieved by including a set of possible values of parameters (i.e., an uncertainty set) in the optimisation framework and optimising the portfolio selection problem with the worst-case scenario. On the one hand, the robust effect provides protection against estimation errors and parameter uncertainties, but on the other hand, the robust effect can be too conservative and therefore lead to a rather pessimistic outcome.

To improve on this outcome, the existing studies of the robust counterpart approach to the portfolio selection problem have focused more on the structured restrictions or other parameter estimators for the uncertainty set (Fabozzi et al., 2010; Gabrel et al., 2014); only a few studies have considered adding extra elements to improve the quality of the robust portfolio allocation (Garlappi et al., 2007; Lutgens & Schotman, 2010). Without incorporating additional market information, no matter how sophisticated or specialised these suggested robust portfolio optimisation models are, they are all based on the historical data and are, therefore, restrictive and facing similar limitations to the classical portfolio selection theory of Markowitz. This is because the past performances of assets didn't contain information good enough for predicting future values of assets. Furthermore, the movement of the financial market could also be influenced by many other factors, such as new policies announced by the government that may affect the global markets. Retail investors often have limited time and resources to help them make investment decisions. In contrast, the professional analysts are well trained in researching and analysing market information for investment decision making purposes. Therefore, adopting investment forecasts from professionals would be beneficial in improving the performance outcome of the robust portfolios.

There are some existing studies in the literature on robust counterpart approaches that are developed for optimising portfolio problems with experts' recommendations (Garlappi et al., 2007; Lutgens & Schotman, 2010). However, they all have their weaknesses. The common issue is that they use return-generating models as the experts' recommendations. Unlike the return-generating model which obtains numerical estimates for asset returns, the investment forecasts provided by the financial analysts are seldom expressed in a precise and clear format. Despite the obvious difference in the format between the return models and the experts' recommendations, there is another issue regarding the volume of asset recommendations provided by the analysts. In reality, unlike the return models, which provide one estimate for every individual asset, a professional analyst usually provides a general market view of the entire stock market and then comments on a few specific stocks. This is illustrated by Figure 1.1, which displays the monthly volume of stock recommendations on the Taiwan Stock Exchange (TWSE) provided by two analysts during the sample period. Note that at the end of 2013, there are 809 stocks listed on the TWSE.

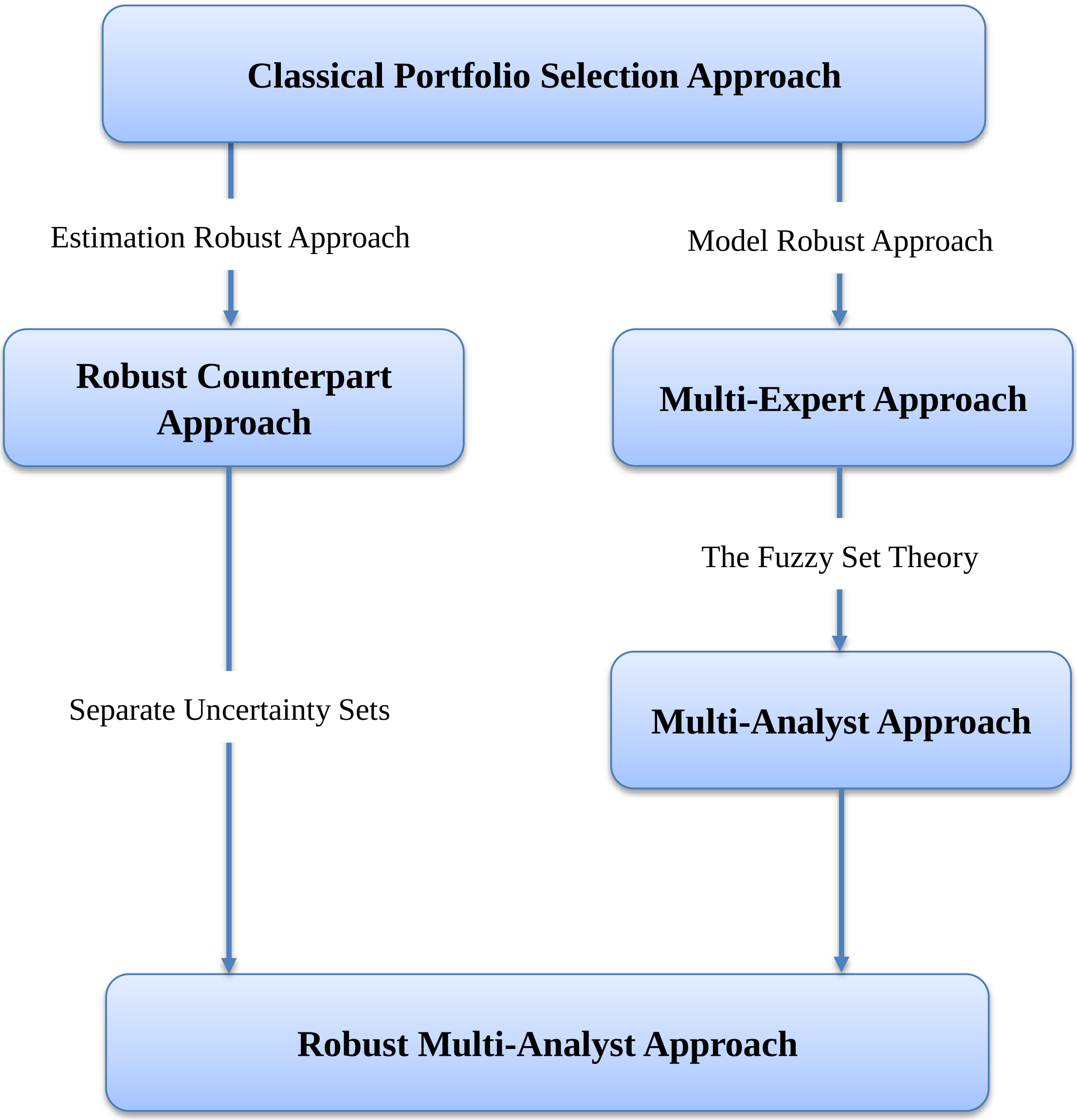
Figure 1.1 Monthly Volume of Stock Recommendations



Note: This figure reports the monthly volume of stock recommendations provided by two stock market analysts from April 2012 to April 2014. The data are obtained from the analysts selected for the empirical investigation in Chapter 6. Details of the analysts are contained in Chapter 5.

The aim of this study is to improve on the existing robust counterpart approach of the portfolio selection problem by incorporating additional investment information sources from stock market analysts. The following figure illustrates the portfolio selection models considered in this thesis. See Section 1.3 for detailed description of these approaches.

Figure 1.2 Diagram of Various Portfolio Selection Models



1.2 Contributions to Knowledge

This thesis aims to contribute to the literature of robust portfolio selection in the following aspects:

- We propose a portfolio selection approach that takes into account various professional investment forecasts, using fuzzy set theory. This is because the investment recommendations of financial analysts are usually expressed vaguely in words, and the existing studies of the robust multi-prior approach (Garlappi et al., 2007; Lutgens & Schotman, 2010) mainly focus on the fundamental structure of the optimisation framework, without paying attention to the nature of the investment forecasts of financial analysts. Unlike the other related studies in the existing literature, which use return-generating models or simulated data as the recommendations of experts (Garlappi et al., 2007; Huang et al., 2010; Lutgens & Schotman, 2010), this research studies the characteristics of investment forecasts and utilises fuzzy set theory to interpret these ambiguous forecasts, so that the proposed multi-analyst approach can apply to investment management in the real world.
- Following the inclusion of analysts' recommendations in our portfolio selection model, we further develop a robust counterpart approach to the multi-analyst portfolio selection problem to handle the estimation errors and parameter uncertainties of the input parameters. As shown previously, professional analysts are unlikely to provide investment forecasts for every individual asset. Therefore, historical data is required to generate parameter estimates for the assets without the analysts' recommendations. With the intention of considering the estimation errors and parameter uncertainties in different types of input data, i.e., investment forecasts of analysts and the historical data, separate uncertainty sets

are required in the portfolio selection process. Hence this robust multi-analyst framework combines the advantages of the model robust approach (Lutgens & Schotman, 2010) and the estimation robust approach (Ben-Tal & Nemirovski, 1998). To the best of our knowledge, the robust multi-analyst portfolio selection approach with the separate uncertainty sets has not yet been investigated in the literature.

- We undertake an empirical study to assess the performances of the proposed approaches. Instead of using simulated expert data, investment recommendations provided by professional analysts are adopted for examining the performances of the multi-analyst approaches. Our sample of the investment forecasts contains 2,133 investment newsletters, which are collected daily from four Taiwanese financial institutions over the period from April 2012 to April 2014. The chosen financial institutions are the top ten most active securities brokerage firms in Taiwan. Although there has been a notable increase in the application of the robust portfolio optimisation models (Gabrel et al., 2014), this empirical study is the first investigation conducted with practical analysts' investment forecasts.

1.3 Thesis Organisation

This thesis consists of seven chapters and the main body of the thesis is organised in two parts. The first part, which includes Chapters 2, 3 and 4, investigates the theoretical aspects of the portfolio optimisation problems with advice from multiple analysts and their associated robust counterpart approach. The second part, which consists of Chapters 5 and 6, illustrates the implementation of the multi-analyst approach and its robust counterpart approach, and the corresponding empirical studies.

Chapter 2 reviews the theories and research related to robust portfolio optimisation. It starts with the theoretical framework of the mean-variance portfolio selection approach of Harry Markowitz (1952) and addresses the weaknesses of this well-known portfolio selection model. Following this, a brief overview of possible solutions for improving on the mean-variance portfolio selection framework is given. Then the concept of the robust counterpart approach to the portfolio optimisation problem of Ben-Tal and Nemirovski (1998) is discussed in detail. Finally, this chapter provides the literature relating to robust portfolio optimisation approach with multiple experts and highlights the rationale for improving the existing multi-expert approach.

In Chapter 3 a new approach, the multi-analyst approach, is developed for asset allocation. This multi-analyst approach is built upon the multi-expert framework of Lutgens and Schotman (2010), where fuzzy set theory is incorporated into Lutgens and Schotman's multi-expert approach, to take into account the ambiguous nature of the investment recommendations. This chapter starts with all the necessary literature of fuzzy set theory, followed by the possibilistic interpretation of fuzzy parameters. Then the framework of the proposed multi-analyst approach is presented, with examples to illustrate our multi-analyst approach for asset allocation.

Chapter 4 presents the robust counterpart to the multi-analyst approach developed in Chapter 3. The robust multi-analyst approach is the second approach developed in this research for reducing the effect of estimation errors and parameter uncertainties. A standard framework of the robust counterpart to the multi-analyst approach with a joint uncertainty set is introduced first. This framework is then extended by adopting multiple uncertainty sets for handling different levels of parameter uncertainties of different datasets. Comparisons are given to illustrate the robust effect imposed on the robust multi-analyst approach at the end of Chapter 4.

Chapter 5 details the financial analysts' investment recommendations collected for the empirical study. Apart from the historical asset performances, the proposed multi-analyst approaches are designed to adopt investment forecasts provided by various professional analysts for generating potentially profitable asset allocations. This chapter discusses the process of data collection and the information of several Taiwanese financial institutions considered in this research. In addition, the procedure of converting the vague investment recommendations into ordinary numerical estimates will also be discussed in this chapter.

Chapter 6 provides an empirical investigation to examine the proposed multi-analyst approaches in the Taiwanese stock market. In order to present a comprehensive examination, the proposed multi-analyst approaches will be compared with other conventional investment strategies and examined under different scenarios. The analysis focuses on how the multi-analyst approaches have improved on portfolio performances and the effect of incorporating professional recommendations on asset allocation. In addition, this chapter investigates the impact of the risk preference, robustness preference, investor's preference for analysts and the duration of investment of the multi-analyst approaches.

Chapter 7 concludes the thesis. This final chapter begins with a summary of the main developments and findings to provide a full picture of this thesis, and then outlines the main contributions. Finally, limitations of this research and suggestions for future research are discussed at the end of this chapter.

Chapter 2

The Classical Portfolio Selection Theory and Extensions

Over the past few decades, the use of quantitative techniques in investment management has become more popular, especially after the major development that was the portfolio selection theory introduced by Harry Markowitz in the early 1950s. Markowitz suggests that investors should determine the allocation of funds based on a trade-off between the risks and the returns of assets. Compared to other sophisticated models, this risk-return theory is more widely used in practice today, mainly due to the simple and intuitive structure of the theory. However, this theory is also criticised by academic researchers and practitioners due to the possibility of unreliable solutions generated from incorrect input parameters. As a result, there is a substantial body of literature aiming to address these issues and expand the scope of the portfolio selection theory.

This chapter reviews the relevant underpinning portfolio selection theories carried out for this research. First, a summary of the classical portfolio selection theory proposed by Markowitz is provided, followed by an overview of the parameter estimation and a discussion of optimisation problems with estimation errors and parameter uncertainties. Then the concept of the robust counterpart approach of the portfolio selection problem is given in section 2.2, together with some fundamental features of the corresponding uncertainty set. Finally, the framework of robust portfolio optimisation with multiple experts is presented.

Throughout the thesis, the following are the assumptions (unless stated otherwise) related to the investment and its environment.

Assumptions 2.1

- There is no transaction cost and the market has perfect liquidity.
- The portfolio selection model only focuses on single-period problem.
- The investor is assumed to be rational and risk averse.
- The rate of expected returns and the corresponding risk measures are the only two types of input parameters required for making investment decisions.

2.1 The Mean-Variance Portfolio Selection Approach

Markowitz (1952) proposed the portfolio selection theory, which recommends to investors that a good portfolio is not just a collection of many good stocks and bonds, but it should also consider the risk and return of the investment according to the investors' objective. Based on the idea of the portfolio selection theory, Markowitz further developed the mean-variance optimisation model that only requires the expected performances of assets and the assumed investors' risk preference to determine the asset allocation. In addition, he suggested that the expected performance of the investment should be measured by the expected asset returns, and the risk be measured by the variances of the expected returns.

There are a few alternative formulations of the mean-variance optimisation model, and the most commonly used substitutions are the risk minimisation formulation, the return maximisation formulation and the risk aversion formulation. The risk minimisation formulation is aimed at investment which requires a target rate of portfolio return with the lowest risk. In contrast, the return maximisation formulation is adopted for investment which has to be kept under a prescribed level of risk with the highest portfolio return. The risk aversion formulation is based on the consideration of the trade-off between maximising the expected portfolio return and minimising the portfolio risk by introducing the risk aversion coefficient[1]. Those formulations are different, as they have dissimilar investment targets, but on the other hand, they are equivalent to each other due to the same efficient frontier that can be created when using the same inputs: expected return and the risk measure of the portfolio.

If not explicitly stated otherwise, the risk aversion formulation is adopted as the fundamental framework for the portfolio selection models throughout this research. The following literature about the mean-variance portfolio optimisation framework refers to the books of Cornuejols and Tütüncü (2007) and Capinski and Zastawniak (2003).

Suppose there is an investor who plans to invest in a financial market of n risky assets. The risk aversion formulation of the mean-variance portfolio optimisation model is defined as

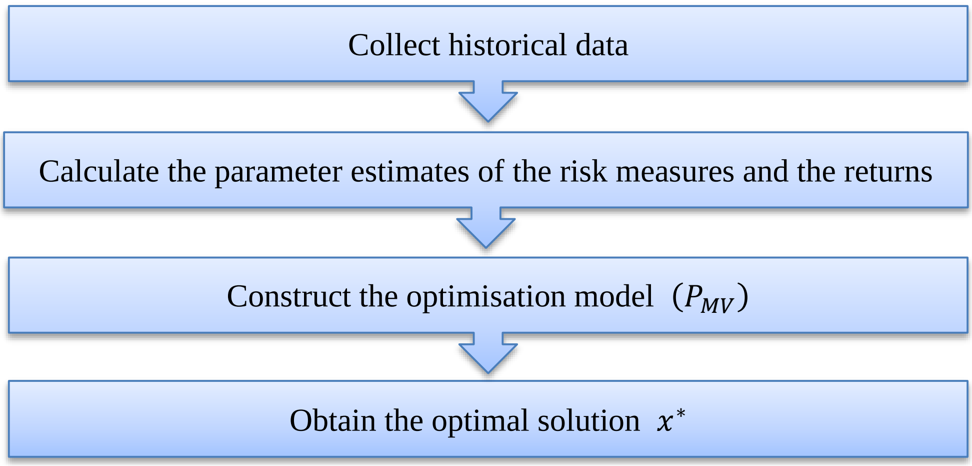
$$(\boldsymbol{P_{MV}}) \qquad \max_{x \in \mathbb{R}^n} \quad \mu^T x - \frac{\lambda}{2} x^T \Sigma x \quad , \tag{2.1}$$

where $\mu \in \mathbb{R}^n$ is the vector denoting the expected returns, $x \in \mathbb{R}^n$ is the n-dimensional decision vector denoting the weights of the portfolio, λ is the risk aversion coefficient prescribed by the investor, $\Sigma = [\sigma_{ij}] \in \mathbb{R}^n \times \mathbb{R}^n$ is the covariance matrix denoting the measure of risk with variance $\sigma_{ii} = \sigma_i^2$ for $i = j$ and covariance σ_{ij} for $i \neq j$.

[1]The risk aversion coefficient $\lambda \in [0, \infty)$ is also known as the Arrow-Pratt risk aversion index. If the aversion to risk is low, then the coefficient λ is small and which leads to a more risky portfolio with higher expected return. Likewise, if the aversion to risk is high, then the coefficient λ will be large and the optimisation problem will result in the portfolio with less risk and lower expected return.

As mentioned earlier, the risk measure and the expected returns play an important role in this portfolio optimisation method. Although it is impossible for anyone to know the true values of these parameters in advance, the investor would still prefer to use particular parameter estimators to forecast the future values of these input parameters, so that the resulting optimal asset allocation can be decided in an advantageous position and reduce the possibility of making losses in the investment. One of the most commonly used methods is to generate these input parameter estimates from the historical asset performances. The following diagram graphically illustrates the basic steps of the mean-variance portfolio selection approach.

Figure 2.1 The Diagram of Solving the Mean-Variance Portfolio



Although the theory of the mean-variance optimisation is very intuitive and the model itself can be applied easily in practice, it has been reported in academic research that the practitioners are still not confident enough to depend totally on the classical mean-variance portfolio selection model for achieving the optimal solution (Fabozzi et al., 2007). There are two main reasons for this. The first is the difficulty of having the correct and accurate values for the input parameters. The other reason is that the optimal solution obtained by solving the mean-variance portfolio selection problem is very sensitive to small changes in the input parameters: the mean-variance portfolio selection model usually tends to generate the optimal solution with extremely high weighting for a particular asset. Hence a small change in one asset's expected return may lead to a totally different asset allocation. See Best and Grauer (1991) and Britten-Jones (1999) for detailed discussion.

In order to improve on the reliability of the optimal solution provided by the mean-variance portfolio selection method, there are various suggestions based on different points of view. The type of suggested improvements can be divided into two possibilities, i.e., (a) how to improve on the accuracy of the input parameters; and (b) how to improve on the sensitive characteristic of the optimal solution. To enhance the reliability of the optimal solution, many researchers propose to use modified parameter estimators to reduce errors in input parameters. On the other hand, some scholars focus on improving the optimisation model by introducing new techniques. In the following sections, we first present some selected estimators of the input parameters for portfolio selection problems, and then briefly review various theoretical developments related to portfolio selection under uncertainty.

2.1.1 Parameter Estimation

The classical estimators for evaluating the values of the input parameters are the sample moment estimator and the maximum likelihood estimator. Other parameter estimators for the inputs of the portfolio selection problem, such as Bayesian estimators and shrinkage estimators are also suggested; see, e.g., Klein and Bawa (1976) and Jobson and Korkie (1981). Although both expected returns and risk measures are the fundamental parameters for the mean-variance portfolio selection model, the input parameter of the investment risk has less influence on the resulting optimal solution.

According to Chopra and Ziemba (1993) and Ziemba (2009), the errors in the expected returns are about ten times as important as errors in the risk measures, and consequently, the portfolio selection model suffers more from the problem of errors in the expected returns unless the optimisation model only focuses on minimising the portfolio risk. Therefore, we focus on the impact of the expected returns on the portfolio selection models in this thesis.

2.1.1.1 The Sample Moment Estimator

The most intuitive method for estimating the expected returns and the risk of an investment is to calculate the sample mean and the variance from the chosen historical data. However, if the historical data are adopted for the estimation purpose, there is always an assumption that the past records do provide a good estimate for the future.

Let $r_{i,1}, r_{i,2}, \dots, r_{i,T}$ denote the historical returns of asset i with T observations. The sample mean $\bar{\mu}_i$ and the sample variance $\bar{\sigma}_i{}^2$ for the i^{th} asset are defined to be

$$\bar{\mu}_i = \sum_{t=1}^{T} \frac{r_{i,t}}{T} \quad , \quad \bar{\sigma}_i{}^2 = \sum_{t=1}^{T} \frac{(r_{i,t} - \bar{\mu}_i)^2}{T-1} \quad . \tag{2.2}$$

The sample covariance between asset i and asset j is defined as

$$\bar{\sigma}_{ij} = \sum_{t=1}^{T} \frac{(r_{i,t} - \bar{\mu}_i)(r_{j,t} - \bar{\mu}_j)}{T-1} \quad . \tag{2.3}$$

By definition, the sample estimator of an asset’s expected return is simply the arithmetic average of the asset’s historical returns, hence the accuracy of this estimator is influenced by the size of the chosen sample. Some researchers suggest using more data from further back in time to generate more precise estimates rather than using higher frequency historical data, but practitioners usually believe that the best estimates are generated by combining historical data and financial theories with their own judgement; see, e.g., Bain & Engelhardt (2000) and Fabozzi et al. (2007).

2.1.1.2 The Maximum Likelihood Estimator

The maximum likelihood estimator (MLE) is one of the statistical estimators which has been widely adopted in many practical applications. The fundamental idea behind the maximum likelihood estimator is to determine the parameter value that can best describe the sample for a given distribution.

Specifically, let r_i be a random variable with the probability density function

$$f(r_i;\ \theta_1, \theta_2, \dots, \theta_k), \tag{2.4}$$

where $\theta_1, \theta_2, \dots, \theta_k$ are the parameters that need to be estimated. The maximum likelihood estimates of $\theta_1, \theta_2, \dots, \theta_k$ are obtained by maximizing the likelihood function

$$L(\theta) = L(\theta_1, \theta_2, \dots, \theta_k | \ r_i)\ . \tag{2.5}$$

Furthermore, if the likelihood function is differentiable, then the maximum likelihood estimates can be obtained by solving the maximum likelihood equation

$$\frac{d}{d\theta} \ln L(\theta) = 0. \tag{2.6}$$

Consider that the random variable r_i is normally distributed, $r_i \sim N(\mu_i, \sigma_i^2)$, with probability density function

$$f(r_i; \mu_i, \sigma_i^2) = \frac{1}{\sigma_i \sqrt{2\pi}} exp\left(-\frac{(r_i - \mu_i)^2}{2\sigma_i^2}\right), \tag{2.7}$$

where μ_i and σ_i^2 denote the mean and variance, respectively. For a random sample of size T from the normal distribution, $r_i \sim N(\mu_i, \sigma_i^2)$, the corresponding likelihood function is given by

$$L(\mu_i, \sigma_i^2) = f\left(r_{i,1}, r_{i,2}, \dots, r_{i,T}; \mu_i, \sigma_i^2\right)$$
$$= (2\pi\sigma_i^2)^{-\frac{T}{2}} exp\left(-\frac{\sum_{t=1}^{T}(r_{i,t} - \mu_i)^2}{2\sigma_i^2}\right). \tag{2.8}$$

It is easy to verify that the maximum likelihood estimates for the expected return μ_i are

$$\hat{\mu}_i = \sum_{t=1}^{T} \frac{r_{i,t}}{T} \tag{2.9}$$

by solving

$$\frac{d}{d\mu_i} \ln L(\mu_i, \sigma_i^2) = \frac{d}{d\mu_i} \ln\left((2\pi\sigma_i^2)^{-\frac{T}{2}} exp\left(-\frac{\sum_{t=1}^{T}(r_{i,t} - \mu_i)^2}{2\sigma_i^2}\right)\right) = 0. \tag{2.10}$$

Similarly, let $\sigma_i^2 = \theta$. By solving

$$\frac{d}{d\theta} \ln L(\mu_i, \theta) = \frac{d}{d\theta} \ln\left((2\pi\theta)^{-\frac{T}{2}} exp\left(-\frac{\sum_{t=1}^{T}(r_{i,t} - \mu_i)^2}{2\theta}\right)\right) = 0, \tag{2.11}$$

the maximum likelihood estimate for the variance of asset i is

$$\hat{\sigma}_i^2 = \sum_{t=1}^{T} \frac{(r_{i,t} - \hat{\mu}_i)^2}{T} \quad . \tag{2.12}$$

The formulations above are the maximum likelihood estimation of a single random variable r_i which is univariate normal distributed. In an n-dimensional setting, $r_1, \dots, r_n$, the random variables are multivariate normal distributed, $r_1, \dots, r_n \sim N(\mu, \Sigma)$, with the joint probability density function given as

$$f(r) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} exp\left(-\frac{1}{2}(r - \mu)^T \Sigma^{-1} (r - \mu)\right) \tag{2.13}$$

where $r = (r_1, \dots, r_n)^T$ is the vector of the random variables, $\mu = (\mu_1, \dots, \mu_n)^T$ is the mean vector, and $\Sigma = [\sigma_{ij}] \in \mathbb{R}^n \times \mathbb{R}^n$ is the covariance matrix. The maximum likelihood estimate for the expected return μ is

$$\hat{\mu} = \sum_{t=1}^{T} \frac{r_t}{T} \tag{2.14}$$

and the maximum likelihood estimate for the covariance matrix is given as

$$\hat{\Sigma} = \sum_{t=1}^{T} \frac{(r_t - \hat{\mu})(r_t - \hat{\mu})^T}{T} \ , \tag{2.15}$$

where $\hat{\sigma}_{ij} = \sum_{t=1}^{T} \frac{(r_{i,t}-\hat{\mu}_i)(r_{j,t}-\hat{\mu}_j)}{T}$ between asset i and asset j for $i \neq j$ (Bain & Engelhardt, 2000; Rencher & Schaalje, 2008).

It can be noticed that the maximum likelihood estimate of the expected return $\hat{\mu}_i$ coincides with the sample mean estimator $\bar{\mu}_i$, and the maximum likelihood estimate of the covariance $\hat{\sigma}_{ij}$ is a constant multiple of the sample covariance estimator $\bar{\sigma}_{ij}$, i.e., $\hat{\sigma}_{ij} = \frac{T-1}{T} \bar{\sigma}_{ij}$. Although the maximum likelihood estimator is one of the commonly used statistical estimators in practice, it still has some drawbacks that concerned its user. For instance, the maximum likelihood estimator can be developed for a large variety of estimation situations, but it can also be heavily biased for small samples. Moreover, the maximum likelihood estimator for generating the parameter estimates is based on the assumption that the chosen sample follows a particular distribution, and this estimation approach becomes difficult to conduct if the sample follows non-normal distributions (Bain & Engelhardt, 2000).

2.1.1.3 The Bayesian Estimator

The Bayesian approach is considered to be a more rational method for estimating the input parameters. Unlike the classical estimators that are mainly based on the information from historical data, the Bayesian estimator is generated by taking both the historical observations and subjective views about the investment into account.

In the Bayesian approach, the prior probability distribution is formed before the sample data are actually observed. This prior probability distribution represents the investors' knowledge about the parameters of interest. After receiving market information on the returns, the posterior probability distribution is then derived by considering both the prior probability distribution and the sample data. Finally, the Bayesian estimate of the input parameters for the portfolio selection problem can be evaluated from the posterior probability distribution (Satchell, 2007).

The advantage of using the Bayesian approach as the parameter estimator for the portfolio selection problem is that the decision makers can incorporate their own opinion into the decision-making process. This is particularly desirable for investors who may have some pretty clear ideas about the performances of certain assets. The additional investment information is combined with the sample data via the laws of probability.

A similar framework is proposed by Black and Litterman (1992), named the Black-Litterman model. This approach is based on a concept of combining the investors' views with the market equilibrium. The estimate of the expected return in the Black-Litterman approach is a weighted linear combination of the market equilibrium and the users' opinions, and the corresponding weight allocation is determined on the degree of confidence in the market equilibrium and in the investors' views. In other words, the users can adjust the confidence level to control the impact of the forecast on the optimal solution for the portfolio selection problem.

2.1.1.4 The Shrinkage Estimator

Jobson and Korkie (1981) proposed to use the shrinkage estimator to improve on the reliability of the optimal solution for the mean-variance portfolio selection approach. This estimator addresses the problem of the imprecise expected return by shrinking the sample mean $\bar{\mu}$ towards a targeted value. Hence the extreme observations from the sample data are less likely to affect the optimal solution.

There are various shrinkage estimator formulae for estimating the expected return in the financial literature, and according to Jorion (1986), all of them are generated from following three essential elements:

- A simple estimate of the expected return such as the sample mean $\bar{\mu}$.
- A shrinkage target μ^{Target}, which is the targeted value for the expected return.
- A shrinkage factor γ, which is derived from chosen theoretical properties or numerical simulations.

Basically, the shrinkage estimator is a weighted average of the simple estimate of the expected return and the shrinkage target μ^{Target}, where the shrinkage target μ^{Target} is computed based on the requirements that the shrinkage target needs to be robust and have some basic properties in common with the expected return. Although the use of the shrinkage estimators in the mean-variance portfolio selection approach has been supported by studies (Jorion, 1985; Michaud, 1989), this estimation approach is not flawless in the sense that it may convert the raw estimate into an improved but biased estimator if the chosen shrinkage target contains too much unnecessary and nonsensical information.

To conclude, using more robust statistical estimators for the input parameters is one possible way to improve the accuracy of the inputs, and hence provides a more reliable optimal solution for the portfolio selection problem. However, no matter how sophisticated these estimation approaches are, the estimation errors and parameter uncertainties can never be eliminated. Moreover, adding the new techniques and structures into the estimating approach for the input parameters may create other errors.

Furthermore, all those statistical estimators for the input parameter are point estimates[2], hence, the optimal solution to the portfolio selection problem is still based on two parameters: the vector of expected returns and the covariance matrix. Therefore, the fundamental concern that the outcome of the mean-variance portfolio selection approach is affected by the estimation errors and parameter uncertainties remains unsolved.

2.1.2 Portfolio Selection Under Uncertainty

In the classical portfolio selection theory, the investors are assumed to have complete and accurate investment information, which is required for solving the portfolio selection problem, and the aim of solving the portfolio selection problem is to obtain an optimal and also satisfying portfolio according to the available information and the investors' preferences. However, the procedure for deciding which assets should be included in the portfolio is always a tough task due to the difficulty of having full knowledge of the problem. In other words, investors face a situation in which they can only make their decisions based on limited investment information. Moreover, in addition to estimation errors and parameter uncertainties, the investment information also contains other non-probabilistic elements such as vagueness and ambiguities which influence the process of decision making and have great impact on the final result. Therefore, portfolio selection under uncertainty is an important topic in decision making and has attracted a lot of interest since Markowitz's seminal work. Many methods have been suggested in the literature for solving the portfolio selection problem under uncertainty, and the most well-known approaches are fuzzy programming, stochastic programming, resampling approach, and robust counterpart approach.

[2]A point estimate is a single number which represents the most likely value of the chosen target.

2.1.2.1 Fuzzy Programming

Zadeh (1965) introduced the notion of fuzzy sets as a solution to the problems of uncertainties, imprecision, and contradictions found in crisp sets, where the crisp sets theoretically have clear set boundaries and contain well-defined elements. Many researchers relate the concept of the fuzzy sets to probabilities and believe that fuzziness and probabilities can be treated similarly. However, the innate characters of fuzziness and probabilities are different. According to Espinosa et al. (2005), fuzziness describes the degree of belonging of the element to a specific set. On the other hand, probabilities describe the likelihood of certain elements being contained by the set.

Fuzzy set theory is an important achievement in decision theory and has been extensively employed in decision analysis, mainly used for modelling imperfect knowledge of the problem and describing the inexplicit preference of decision makers. Compared to all other aforementioned approaches that deal with the issue caused by the vagueness of the portfolio selection problem, fuzzy programming is probably the more intuitive and appropriate method in terms of the description of ambiguous input parameters (Parra et al., 2001; Dempe & Ruziyeva, 2012).

In optimising problems with fuzzy set theory, there are two main modelling approaches. The first approach is to adopt fuzzy set theory to integrate qualitative and quantitative investment information and model the uncertainty on returns. For example, Liu (2011) and Zhang et al. (2009) (2011) use interval fuzzy variables to express future return rates and risk for portfolio selection. Carlsson et al. (2002) suggest choosing portfolios with the highest utility score via a possibilistic approach under the assumption that asset returns are trapezoidal fuzzy variables. In addition, Tanaka and Guo (1999) propose a centrespread approach to handle portfolio selection problems, with asset returns described as exponential fuzzy variables. The second approach employs fuzzy set theory to formulate portfolio selection problems. To name a few: Watada (1997) considers fuzzy portfolio selection problems with vague goals of expected return and risk; Liern et al. (2002) investigate how fuzzy set theory can be applied to describe soft constraints and repair unfeasibility in portfolio selection problems. Ammar (2008) formulates the portfolio selection problem as a multi-objective quadratic model with fuzzy objectives and constraints. Furthermore, another fuzzy approach for the multi-objective portfolio selection problem with semi-absolute deviation is considered in Gupta et al. (2008).

2.1.2.2 Stochastic Programming

Similar to fuzzy programming, stochastic programming also addresses portfolio selection problems under uncertainty. The principal difference between fuzzy programming and stochastic programming is how the uncertain elements of the optimisation problem are modelled. In the fuzzy programming case, the random parameters are considered as fuzzy variables and the optimisation problem is formulated in terms of fuzzy sets. On the other hand, stochastic programming assigns discrete or continuous probability functions to the various unknown parameters, hence a particular uncertain parameter can be represented by a probabilistic estimation.

A typical critique of stochastic programming is that the probability distributions are usually unknown and the optimal solution of stochastic programming may perform badly if the chosen distribution of uncertainties is in fact different from the actual distribution (Ben-Tal et al., 2010; Goh & Sim, 2010). Furthermore, the estimation for the uncertain parameter may not satisfy the original constraints in the model of stochastic programming, but only the relaxation of those constraints. This feature of stochastic programming is not suitable for the portfolio selection problem, because the constraints of the portfolio selection problem are usually the hard constraints that need to be satisfied no matter what the realisation of the input data is. See Quaranta and Zaffaroni (2008) for further discussion.

2.1.2.3 Resampling Approach

In order to incorporate the estimation risk in the portfolio selection process, Michaud (1998) proposed another method based on the idea that the final solution should not contain much estimation risk if sufficient resampling procedures are conducted.

In the resampling approach, random samples are drawn from a given distribution to obtain the new expected return and covariance matrix, and then the new pair of input parameters is used to solve the portfolio selection problem and provide the corresponding optimal solution. After the resampling process has been repeated many times, the final optimal solution is then obtained by averaging the respective optimal solutions.

Although the resampling approach is a rather well known technique to reduce the estimation risk, this approach is not widely applied due to some limitations, such as the computational difficulty for larger portfolios and that the final optimal solution may not satisfy the imposed constraints of the portfolio selection problem (Fabozzi et al., 2007; Scherer, 2002).

2.1.2.4 Robust Portfolio Selection

Among all of the aforementioned techniques, robust portfolio selection uses a rather intuitive technique to incorporate uncertainties into the optimisation problems, and it is superior to other approaches in terms of both its simplicity and its efficiency of computation. The concept was introduced by Ben-Tal and Nemirovski (1998), and also independently by El-Ghaoui et al. (1998), based on the idea of providing the best outcome in the worst possible environment with uncertain input parameters lying in the corresponding uncertainty set U. To be more specific, the estimates of the input parameters in this approach are considered to be not totally reliable and to contain a certain level of ambiguity; hence a prescribed uncertainty set U that contains many possible values of the parameters is used for the portfolio selection problem instead of the point estimates. The optimal solution is then obtained by optimising the problem with the worst possible scenario in the pre-specified uncertainty set U.

Although the robust portfolio selection approach also has some undesirable features, such as the overly conservative optimal solution and the unclear definitions of the uncertainty set U, there are several advantages that attract researchers and practitioners. For instance, the robust portfolio selection model is computationally tractable and also more flexible than other approaches; the robust portfolio selection problem can be solved in about the same time as is required for the corresponding original problem; and the optimal solution of the robust model is less sensitive to estimation errors (Fabozzi et al., (2007)). A more detailed literature review of the robust portfolio selection approach is given in section 2.2.

2.2 The Robust Portfolio Selection Approach

Ben-Tal and Nemirovski (1998) have derived the robust counterpart approach to optimisation problem with uncertain input parameters. This approach is, in fact, the worst-case approach that transfers the original optimisation problem into the robust optimisation problem, because the modified optimisation model does not just solve the problem for every point within the prescribed uncertainty set U, but also provides the outcome that optimises the objective function even if the "worst" case occurs. However, the process of the robustification changes the original optimisation problem into a more difficult version. For instance, a linear optimisation problem converts to a second-order cone optimisation problem and a second-order cone problem turns into a semi-definite optimisation problem. Therefore, the robust optimisation problem can be unsolvable with standard techniques in some circumstances (see, e.g., Ben-Tal & Nemirovski, 2002).

In order to overcome this drawback of the robust counterpart approach, Ben-Tal and Nemirovski (2002) and Bertsimas and Sim (2006) suggest using tractable approximations for the robust formulation. A more comprehensive overview of the tractability and extensions for the robust optimisation model is provided by Ben-Tal and Nemirovski (2008). On the other hand, Fabozzi et al. (2010) summarised related research on robust portfolio selection strategies, especially the developments in solving the robust optimisation problem with down side risk measures. In addition, a more recent review of developments in robust optimisation is given by Gabrel et al. (2014), which focuses on both theoretical extensions of robust optimisation and real world applications. It also contains many references relating to the topic of robust optimisation.

2.2.1 The Robust Counterpart Approach

Let us consider a general optimisation problem $(\boldsymbol{P})$ with the uncertain input parameter u expressed in the form

$$(\boldsymbol{P}) \qquad \begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & f(x,u) \\ s.t. & g(x,u)\in K \end{array}, \tag{2.16}$$

where x denotes the decision variable, u denotes the input parameter that contains a certain level of uncertainties, the function f denotes the objective function, and the function g denotes the constraint function with the structure element set K.

If the uncertainty set U is a finite set of “scenarios” that consists of all possible values of input parameters, i.e., $U=\{u_1,u_2,\dots,u_m\}$, then the optimisation problem can be solved by transferring the function that contains the uncertain parameter u into finitely many functions for every single $u_i\in U$, and a similar process can be applied if there are finitely many points belonging to the uncertainty set U. Since the transferring process only duplicates the particular function, therefore, the original optimisation problem is turned into a larger but not more difficult version, because the structural properties of the original optimisation problem are preserved. That is, the general optimisation problem $(\boldsymbol{P})$ with the uncertain parameter $u \in U$ can be formulated as

$$\begin{array}{lll} \min\limits_{x,\zeta} & \zeta & \\ s.t. & \zeta - f(x,u_i) \geq 0 & i = 1, \dots, m \\ & g(x,u_i) \in K & i = 1, \dots, m \end{array} . \tag{2.17}$$

On the other hand, if the uncertainty set U is not finite, e.g., continuous sets in the shape of boxes, ellipsoids or the intersections of ellipsoids, then the original optimisation problem will be modified into a more complicated framework such that the function that contains the uncertain parameter has to be satisfied for all possible values in the uncertainty set U. The solution can be difficult to obtain in this situation, since the corresponding uncertainty set U can be quite large (Cornuejols & Tütüncü, 2007). Suppose the original optimisation problem is formulated as

$$(\boldsymbol{P}) \qquad \begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & f(x,u) \\ s.t. & x \in F \end{array} , \tag{2.18}$$

where $F(u) = \{x \mid g(x,u) \in K\}$ denotes the feasible set with $u \in U$. The robust counterpart of (2.18) is

$$\min_{x \in F} \max_{u \in U} \quad f(x,u) \ . \tag{2.19}$$

Note that the geometry of the uncertainty set U is not the only factor that influences the accessibility of the robust counterpart approach. The analytical structure complexity of the original optimisation problem is another possibility for increasing the difficulty in solving the robust optimisation problem. Furthermore, there are only guidelines and no strict formulations for the uncertainty set U of the robust counterpart approach. Therefore, uncertainty set U can be represented and formed according to different preferences and opinions on the future values of the input parameters.

2.2.2 The Uncertainty Set

The fundamental idea behind the uncertainty set U is to incorporate estimation errors and parameter uncertainties into a set, so that the uncertainty set U contains many possible values of the required input parameters.

In general, the shape of the uncertainty set U depends on the sources of uncertainties and also the sensitivity effect of the uncertainty. The most common shapes for the uncertainty set are box, ellipsoid and the intersections of the ellipsoids. There is no exact formulation for the uncertainty set, and every type of the uncertainty set is supported by various researchers. Cornuejols & Tütüncü (2007), El-Ghaoui et al. (2003) and Tütüncü and Koenig (2004), for example, adopt box uncertainty sets to define uncertain parameters. Ben-Tal and Nemirovski (1998) and El Ghaoui et al. (ElG981), on the other hand, describe uncertain parameters through ellipsoids or intersections of ellipsoids. However, Schöttle (2007) has shown that the ellipsoidal uncertainty set leads to a unique optimal solution that is continuous with respect to the uncertain parameter in most practical cases by investigating the impact of using uncertainty sets with different shapes on the continuity properties of the optimal solution set.

On the other hand, the size δ of the uncertainty set U depends on the desired robustness level for the parameter estimates. Different people may have different degrees of confidence in the parameter estimates. One may have 100% confidence in the estimated figures and decide to use point estimates with the size of the uncertainty set equal to zero, and others may feel unsure about the estimates and use some confidence intervals around the parameter estimates to handle estimation errors. As a general guideline, if the expected parameter values are assumed to belong to any possible probability distribution, then the desired robustness level δ of an ellipsoid uncertainty set, the size of the uncertainty set, can be defined as

$$\delta = \sqrt{\frac{1-\kappa}{\kappa}} \quad , \tag{2.20}$$

with κ denoting the probability that the true values of the parameter will fall in the uncertainty set (El Ghaoui et al., 2003; Fabozzi et al., 2007).

Despite the general descriptions of the size and the shape of the uncertainty set, the uncertainty set U is usually considered to be centred at the expected value of the parameter and the preferred level of confidence is denoted by the variance. Nevertheless, the procedure of obtaining statistically meaningful and precise estimates from available historical data is never an easy task, and these possible estimation errors may lead to an unreliable uncertainty set U that obtains an undesirable result for the robust optimisation problem. Many suggested improvements are focused on the substitution of the risk measures or the parameter estimators. Unlike others, Bertsimas and Brown (2009) proposed a prescriptive technique to build the uncertainty set U for the robust optimisation problem. In their approach, the starting point of the construction of the uncertainty set U is the framework of coherent risk measures. However, this approach is only applicable for the robust linear optimisation problem, but not viable for other more general robust optimisation problems such as convex optimisation problems.

Let us consider an optimisation problem with an n-dimensional vector of uncertain coefficients u. The box uncertainty set U^{Box} for the uncertain coefficients u is given by

$$U^{Box}(\acute{\mu}) = \{ \mu \in \mathbb{R}^n \, \big| |u_i - \acute{\mu}_i| \leq \delta, i = 1, \dots, n \} \quad , \tag{2.21}$$

where $\acute{\mu} = (\acute{\mu}_1, \acute{\mu}_2, \dots, \acute{\mu}_n)^T$ denotes the statistical parameter estimates of the uncertain coefficients u and $\delta \geq 0$ denotes the desired robustness level for the uncertainty set

U^{Box}. The box uncertainty set U^{Box} is also known as the interval uncertainty set. On the other hand, the ellipsoid uncertainty set $U^{Ellipsoid}$ for the uncertain coefficients u can be formulated in several ways, and the most common format is

$$U^{Ellipsoid}(\acute{\mu}) = \{\mu \in \mathbb{R}^n | (\mu - \acute{\mu})^T Q^{-1} (\mu - \acute{\mu}) \leq \delta^2\} \ , \tag{2.22}$$

where $Q \in \mathbb{R}^n \times \mathbb{R}^n$ is a carefully chosen matrix (Fabozzi et al., 2007).

2.2.3 The Robust Counterpart Approach of the Mean-Variance Portfolio Selection

In the previous sections we have discussed the framework of the robust counterpart approach and also notations of the corresponding uncertainty set. In the following we explain the structure of the robust counterpart approach of the mean-variance portfolio selection problem.

Generally speaking, the expected returns and the risk measures of the underlying assets are the only two possible uncertain input parameters for the portfolio selection problem. Therefore, these two uncertain parameters of the original portfolio selection problem are required to be described and expressed via the corresponding uncertainty set U. However, as mentioned in Section 2.1, the estimation errors of the risk measures have less impact on the portfolio selection. Hence, the uncertainty set U usually is defined only for the expected returns in most practical cases.

Note that the uncertainty set U is supposed to be non-empty, convex, and compact. Without considering the finite uncertainty set U, which is simply a collection of scenarios, the box uncertainty set U^{Box} and the ellipsoid uncertainty set $U^{Ellipsoid}$ for the parameter of the asset returns $\mu \in \mathbb{R}^n$ are defined as

1) The box uncertainty set U^{Box} for the parameter μ

$$\begin{aligned} U^{Box}(\acute{\mu}) &= \{\mu \in \mathbb{R}^n | \ |\mu_i - \acute{\mu}_i| \leq \delta, \ i = 1, \dots, n\} \\ &= \{\mu \in \mathbb{R}^n | \ \mu = \acute{\mu} + \delta\psi \ , \psi \in [-1,1]^{\ n} \} \end{aligned} \ , \tag{2.23}$$

2) The ellipsoid uncertainty set $U^{Ellipsoid}$ for the parameter μ

$$\begin{aligned} U^{Ellipsoid}(\acute{\mu}) &= \left\{\mu \in \mathbb{R}^n \middle| (\mu - \acute{\mu})^T \acute{\Sigma}^{-1} (\mu - \acute{\mu}) \leq \delta^2 \right\} \\ &= \left\{\mu \in \mathbb{R}^n \middle| \mu = \acute{\mu} + \delta \acute{\Sigma}^{\frac{1}{2}} \psi \,, \|\psi\| \leq 1 \right\} \end{aligned} \quad , \qquad (2.24)$$

where $\acute{\mu} = (\acute{\mu}_1, \acute{\mu}_2, \ldots, \acute{\mu}_n)^T$ denotes the statistical parameter estimates of asset returns μ and $\acute{\Sigma} \in \mathbb{R}^n \times \mathbb{R}^n$ denotes the covariance matrix of the asset returns.

Recall the risk aversion formulation of the mean-variance portfolio selection problem (2.1)

$$(\boldsymbol{P_{MV}}) \qquad \max_{x \in \mathbb{R}^n} \quad \mu^T x - \frac{\lambda}{2} x^T \Sigma x \ .$$

Then the robust counterpart approach of the mean-variance portfolio selection problem can be formulated as

$$(\boldsymbol{R_{MV}}) \qquad \max_{x \in \mathbb{R}^n} \min_{\mu \in U(\acute{\mu})} \quad \mu^T x - \frac{\lambda}{2} x^T \Sigma x \quad , \qquad (2.25)$$

with $U(\acute{\mu})$ denoting the prescribed uncertainty set for the parameter of asset returns μ. In addition to the already mentioned advantageous feature of the ellipsoid uncertainty set, i.e., the continuity of the optimal solution, the ellipsoid uncertainty set is generally a more suitable choice to describe the uncertain parameter, because the ellipsoid uncertainty set allows users to include second moment information about the distribution of the uncertain parameters. Therefore, we will adopt the ellipsoid uncertainty set $U^{Ellipsoid}$ for modelling the uncertain parameter of asset returns.

By using the formulation (2.24) of the ellipsoid uncertainty set $U^{Ellipsoid}$, the robust portfolio selection approach (R_{MV}) can be reformulated as follows

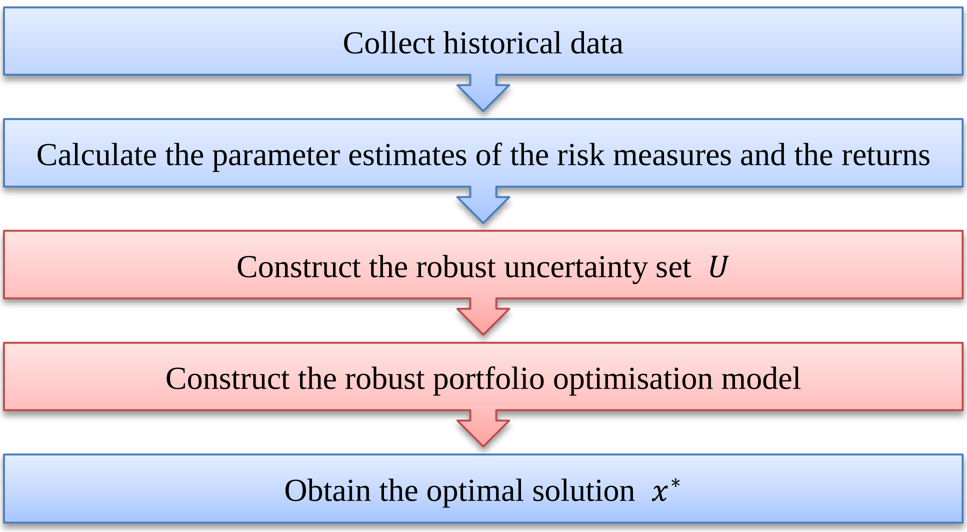
$$
\begin{aligned}
&\max_{x\in\mathbb{R}^n}\ \min_{\mu\in U(\hat{\mu})} && \mu^T x - \frac{\lambda}{2}x^T\Sigma x \\
&= \max_{x\in\mathbb{R}^n}\ \min_{\|\psi\|\le 1} && (\hat{\mu} + \delta\hat{\Sigma}^{\frac{1}{2}}\psi)^T x - \frac{\lambda}{2}x^T\Sigma x \\
&= \max_{x\in\mathbb{R}^n}\ \min_{\|\psi\|\le 1} && \hat{\mu}^T x + \delta\psi^T\hat{\Sigma}^{\frac{1}{2}^T} x - \frac{\lambda}{2}x^T\Sigma x \\
&= \max_{x\in\mathbb{R}^n} && \left(\hat{\mu}^T x - \frac{\lambda}{2}x^T\Sigma x + \delta\min_{\|\psi\|\le 1}\psi^T\hat{\Sigma}^{\frac{1}{2}}x\right)
\end{aligned}
$$

As the product of $\psi^T\hat{\Sigma}^{\frac{1}{2}}x$ is minimised at $\psi^T = -\frac{\hat{\Sigma}^{1/2}x}{\|\hat{\Sigma}^{1/2}x\|}$, the robust portfolio selection problem becomes

$$
\begin{aligned}
&= \max_{x\in\mathbb{R}^n}\quad \hat{\mu}^T x - \frac{\lambda}{2}x^T\Sigma x - \delta\,\frac{\hat{\Sigma}^{1/2}x}{\|\hat{\Sigma}^{1/2}x\|}\hat{\Sigma}^{\frac{1}{2}}x \\
&= \max_{x\in\mathbb{R}^n}\quad \hat{\mu}^T x - \frac{\lambda}{2}x^T\Sigma x - \delta\,\|\hat{\Sigma}^{1/2}x\|\,.
\end{aligned}
\tag{2.26}
$$

The above formulation is in fact a standard illustration of the robust counterpart approach derived from the framework of the classical mean-variance portfolio selection problem; see, e.g., Schöttle (2007). Based on this robust framework, there are many studies that focus on the possible extensions of the robust portfolio optimisation model. The following diagram graphically illustrates the basic steps of the robust counterpart approach.

Figure 2.2 The Diagram of Solving the Robust Counterpart Approach



Overall, the main research directions of the robust portfolio optimisation model can be divided into three research lines: robust optimisation model with different risk measures, investigation and construction of the uncertainty set for the robust optimisation problem, and the robust portfolio optimisation model with advice from multiple experts.

On the basis of robust mean-variance objective formulation, some researchers consider using risk measures other than variance, such as VaR (value-at-risk) and CVaR (conditional value-at-risk), since these risk measures seem to be more suitable for modelling the risk of an event, unlike variance, which is in fact a measure used to describe the dispersion of a random variable and which considers overperformance and underperformance to be equally important. El Ghaoui et al. (2003) and Natarajan et al. (2008) use VaR, and others like Huang et al. (2010), Quaranta and Zaffaroni (2008), and Zhu and Fukushima (2009) investigate the robust optimisation problem with CVaR.

A different approach for improving the robust optimisation model is to construct appropriate uncertainty sets for the input parameters. Various studies and investigations have been carried out in this specific field. In addition to the development mentioned previously, there are various studies, such as Natarajan et al. (2009) and Bertsimas and Sim (2004). Chen et al. (2007) introduced a new formulation of the uncertainty set that incorporates the asymmetric distributional behaviour of the uncertain parameter. In addition, Bertsimas et al. (2011) focused on the structure of the ellipsoid uncertainty set and proposed a method that controls the size of the ellipsoidal uncertainty set. It has the interpretation as the trade-off between the desired robustness and the performance of the robust optimisation model.

Another approach to improving robust portfolio selection is to incorporate different information sources about the uncertain parameters into the robust optimisation model.

Detailed descriptions and discussions of the robust optimisation model with multiple priors are given in the following section.

2.3 The Robust Portfolio Optimisation Model with Multiple Experts

There is no unique asset returns model that can satisfy every investor, and the specification of the returns model for the portfolio selection problem usually depends on the opinions of each individual investor. Garlappi et al. (2007) and Lutgens and Schotman (2010) proposed to incorporate additional information about the underlying assets directly into the framework of portfolio selection models rather than using existing techniques, such as the Bayesian approach.

2.3.1 The "Non-overlapping" Method for Robust Portfolio Selection with Multi-Prior

Garlappi et al. (2007) introduced a robust portfolio selection model that allows an investor to include multiple priors' knowledge into the portfolio selection process with ambiguity aversion. The multiple priors are characterised via a confidence interval around the estimate of expected returns and the ambiguity aversion is modelled by minimising over the priors. There are several interesting features of this approach that attract attention from both researchers and practitioners. Apart from the useful simplification to the mean-variance portfolio selection model with the adjusted estimate of expected returns, which reflects the investor's ambiguity about the chosen estimate, their approach captures more attention in the process for estimating the uncertain expected returns. In their approach, the expected returns can be estimated either jointly or via different non-overlapping subsets.

Garlappi et al. (2007) start by imposing an additional constraint on the standard mean-variance portfolio selection problem, to restrict possible parameter realisation so that it lies within a given confidence interval. Then they introduce an additional optimisation framework to minimise over different estimation choices of expected returns, subject to the additional constraint. This additional constraint represents the ambiguity aversion of the investor. By adding the two changes to the risk aversion formulation of the mean-variance portfolio selection problem with n assets, the model proposed by Garlappi et al. (2007) is expressed as follows

$$(\boldsymbol{M_{MV}}) \qquad \begin{array}{cc} \max\limits_{x\in\mathbb{R}^n} \min\limits_{\mu} & \mu^T x - \dfrac{\lambda}{2} x^T \Sigma x \\ s.t. & g(\mu, \acute{\mu}, \Sigma) \leq \varepsilon \end{array} \quad , \tag{2.27}$$

where $g(.)$ is a vector valued function with the vector ε reflecting the investor's aversion to the uncertain parameter. Note that Garlappi et al. (2007) are concerned more about the estimation errors in the expected returns, hence the value of the covariance matrix is assumed to be known and the uncertainty in the covariance matrix is ignored. There are three possible formulations to define the constraint $g(.)$, and they are distinguished from each other according to how the uncertainty about the expected returns is described.

1) The first possible formulation for the constraint $g(.)$ estimates the uncertainty of the expected returns individually, i.e., asset by asset

$$g_i(\mu, \acute{\mu}, \Sigma) = \frac{(\mu_i - \acute{\mu}_i)^2}{\sigma_i^2 / T_i} \quad , \tag{2.28}$$

where $i = 1,2, \dots, n$ and T_i denotes the number of observations in the sample for the i^{th} asset. By using formulation (2.28) for the constraint $g(.)$, the mean-variance portfolio selection (M_{MV}) becomes

$$\begin{array}{cc} \max\limits_{x\in\mathbb{R}^n} \min\limits_{\mu} & \mu^T x - \dfrac{\lambda}{2} x^T \Sigma x \\ s.t. & \dfrac{(\mu_i - \acute{\mu}_i)^2}{\sigma_i^2 / T_i} \leq \varepsilon_i \end{array} \quad , \tag{2.29}$$

where the constraint function $g(.)$ is actually the confidence intervals of parameter estimates.

2) The second possible formulation for the constraint $g(.)$ estimates the uncertainty of the expected returns jointly

$$g(\mu, \acute{\mu}, \Sigma) = \frac{T(T-n)}{(T-1)n} \, (\acute{\mu} - \mu)^T \, \Sigma^{-1} \, (\acute{\mu} - \mu) \ . \tag{2.30}$$

In this case, the mean-variance portfolio selection approach (M_{MV}) with respect to μ is formulated as

$$\begin{array}{ll} \max\limits_{x \in \mathbb{R}^n} \min\limits_{\mu} & \mu^T x - \frac{\lambda}{2} x^T \Sigma x \\ s.t. & (\acute{\mu} - \mu)^T \, \Sigma^{-1} \, (\acute{\mu} - \mu) \leq \varepsilon \end{array} \quad , \tag{2.31}$$

where ε is a non-negative number that describes the desired robustness level of the parameter estimate $\acute{\mu}$. This formulation of the portfolio selection approach (M_{MV}) is actually the robust counterpart approach of the mean-variance portfolio selection problem (R_{MV}) with the ellipsoid uncertainty set, which has been presented earlier in section 2.2.

3) The third possible formulation for the constraint $g(.)$ estimates the uncertainty of the expected returns separately via different subclasses of assets. Suppose there are Y non-overlapping subsets of n assets with $y = (1, \dots, Y)$, and let $S_y = (s_{1_y}, \dots, s_{n_y})$ with each element of S_y denoting a subset of assets, then the Y-valued constraint function $g(.)$ is

$$g_y(\mu, \acute{\mu}, \Sigma) = \frac{T_y(T_y - n_y)}{(T_y - 1)n_y} \, (\acute{\mu}_{S_y} - \mu_{S_y})^T \Sigma_{S_y}^{-1} \, (\acute{\mu}_{S_y} - \mu_{S_y}) \ . \tag{2.32}$$

Without loss of generality, a case of two non-overlapping subsets is considered for the purpose of illustrating the structure of the portfolio selection problem (M_{MV}) with the Y-valued constraint function $g(.)$. Let subset a and subset b represent the non-overlapping subsets, with set a consisting of Y_a assets and set b consisting of Y_b assets, and $Y_a + Y_b = n$. Thus, the portfolio selection problem (M_{MV}) is rearranged in the following form

$$\begin{aligned} \max_{x\in\mathbb{R}^n} \min_{\mu_a,\mu_b} \quad & \mu^T x - \frac{\lambda}{2} x^T \Sigma x \\ s.t. \quad & (\acute{\mu}_a - \mu_a)^T \Sigma_{aa}^{-1} (\acute{\mu}_a - \mu_a) \le \varepsilon_a \\ & (\acute{\mu}_b - \mu_b)^T \Sigma_{bb}^{-1} (\acute{\mu}_b - \mu_b) \le \varepsilon_b \end{aligned} \tag{2.33}$$

where $x = ({x_a}^T, {x_b}^T)^T$ is an n-dimensional decision vector with x_a denoting a Y_a-dimensional vector and x_b denoting a Y_b-dimensional vector. The factors of the expected returns μ and covariance matrix Σ are defined as

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}. \tag{2.34}$$

The parameters $\acute{\mu}_a$ and $\acute{\mu}_b$ denote a choice of statistical estimates of the expected returns obtained from subsets a and b, respectively. ε_a and ε_b represent the investor's ambiguity aversion for the subsets a and b.

2.3.2 The "Endogenous" Method for Robust Portfolio Selection with Multi-Expert

In the multiple experts approach proposed by Lutgens and Schotman (2010), an investor is assumed to be rational and does not depend on any particular investment information. This assumption is based on the hypothesis that "Although the investor has no knowledge about the credibility of each individual expert, the investor prefers to consider all recommendations provided by experts and treat all experts as equally important. Thus, the investor would have no regrets regarding the resulting portfolio, even if a particular expert turns out to be wrong". The investor first collects different return estimates suggested by experts, and then combines the distinct views together. Although this idea is similar to the Bayesian approach, there is a fundamental difference between these two approaches.

Compared to the Bayesian approach which assumed that the investor has a neutral attitude toward ambiguity, the investor is assumed to dislike the uncertainty generated from different opinions in Lutgens and Schotman's framework. Furthermore, these two approaches differ theoretically. The investor who adopts the Bayesian approach needs to assign prior probabilities to each expert. On the other hand, the investor who follows Lutgens and Schotman's approach doesn't need to assign a weighting to the various experts. Their approach simultaneously generates the optimal and robust portfolio in the worst case scenario, and the weights that are allocated to the parameter estimates provided by the experts are endogenously determined according to the objective function given by the investor. Hence, the asset allocation is considered to be robust in respect of differing advice. That is, the outcome is the best performance of the least favourable return model.

Unlike the robust counterpart approach proposed by Ben-Tal and Nemirovski (1998), the robust portfolio selection model derived by Lutgens and Schotman (2010) does not employ the framework of the uncertainty set for dealing the uncertainties within the optimisation problem. By considering the same risk aversion formulation as defined in section 2.2, the robust portfolio selection approach with Z experts is formulated as

$$(\boldsymbol{E_{MV}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \quad {\mu_z}^T x - \frac{\lambda}{2} x^T \Sigma_z x \quad , \tag{2.35}$$

where μ_z and Σ_z denote the professional forecasts of the expected returns and the variability of the expected returns given by the zth expert with $\mathrm{z} = 1,2,\ldots,Z$. The optimal asset allocation is given by

$$x^*_{E_{MV}} = \frac{1}{\lambda} \bar{\bar{\Sigma}}^{-1} \bar{\bar{\mu}} \tag{2.36}$$

with

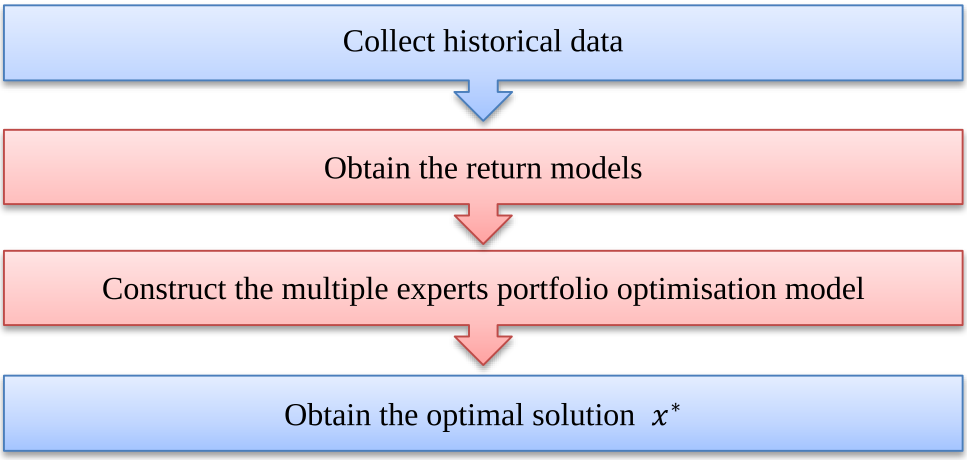
$$\begin{aligned} \bar{\bar{\mu}} &= \sum_{z=1}^{Z} w_z \mu_z \\ \bar{\bar{\Sigma}} &= \sum_{z=1}^{Z} w_z \Sigma_z \end{aligned} \tag{2.37}$$

where w_z denotes the weights assigned to expert z with $\mathrm{w}_z \geq 0$ and $\sum_{z=1}^{Z} \mathrm{w}_z = 1$.

Note that although Lutgens and Schotman (2010) provide an alternative framework to the Bayesian approach for allocating investment with suggestions from multiple experts, they do not pay attention to the fundamental issue of how to interpret these recommendations from experts. The experts in the multi-expert approach of Lutgens and Schotman (2010) are return models, such as the capital asset pricing model (CAPM) and the Fama & French factor model, which is substantially different from the experts' recommendations in reality. Despite the ambiguous and imprecise features of the investment recommendations, these real-life recommendations from stock market analysts are mostly expressed in words rather than numbers. Therefore, it is important to figure out how to convert the unclear and vague advice from experts into numbers, so that the additional investment information can be meaningful and incorporated into the portfolio selection model properly.

Finally, Figure 2.3 graphically illustrates the procedure of the multiple experts approach proposed by Lutgens and Schotman (2010).

Figure 2.3 The Diagram of Solving the Multiple Experts Approach



Although both “non-overlapping” and “endogenous” methods deal with estimation errors and parameter uncertainties by including additional information about the asset returns, they apply the multi-prior differently. The multiple priors in the former method are characterised via the confidence intervals around the estimate of the expected returns, and the degrees of confidence are adjusted for different subsets of assets to reflect the ambiguity about the estimated values. For instance, if the investor receives investment forecasts from two experts, then the estimate of the expected returns is calculated using classical methods such as the Bayesian approach and corresponding confidence intervals are specified as constraints on the expected returns. On the other hand, in the “endogenous” method, the multiple priors are expressed directly via the objective functions without stating the confidence intervals for the estimates. That is, when the investor has investment forecasts from two experts, two objective functions are formulated according to the forecasts of each individual expert.

The “endogenous” method is considered to possess a comparative advantage over the “non-overlapping” methods, as the investor is not required to assign prior probabilities to each expert or adjust the confidence intervals to reflect the uncertainty about the parameter estimates. Hence, the “endogenous” method proposed by Lutgens and Schotman (2010) may have a lower possibility of distorting the results. Nevertheless, the investor cannot apply the “endogenous” method directly, as the framework of the “endogenous” method is designed for utilising multiple return-generating models rather than investment forecasts provided by experts.

First, the investment forecasts provided by professional analysts, either published in print or online media, normally do not have a standard format, and are very likely to be expressed in words rather than numbers. Even if the analysts might be able to provide the estimated price of certain assets numerically, the statements are always vaguely expressed. In contrast, the return models provide parameter estimates in terms of numbers. Second, return models generate estimates of expected return for every asset, but the professional analysts only comment on a few assets, as shown in Chapter 1.

One may ask why we can't just use the return models and forget about the investment forecasts, so that the multi-expert approach can be applied directly without dealing with the issues addressed above. The main reason is that the investor does not have enough resources to collect and analyse all the investment information about the stock market. On the other hand, the analysts are trained to process investment information collected from a broad array of different sources, such as companies' annual reports, government announcements and major global events. Therefore, adopting professional investment recommendations can be beneficial for obtaining better asset allocation.

2.4 Summary

Optimising a portfolio selection problem is never an easy task to conduct, especially when the decision makers only have limited knowledge or uncertain information about the portfolio selection problem. Based on Markowitz's seminal work of portfolio selection theory, many improvements on either modelling frameworks or parameter estimations were developed in order to account for the uncertainty features and provide portfolios that perform better. In this chapter we have reviewed and discussed several approaches that deal with portfolio selection problems under uncertainty, and these approaches basically can be divided into two categories, namely the robust estimation approach and the robust modelling approach.

The robust estimators are widely adopted in portfolio selection problems since these parameter estimators are compatible with the formulation of the portfolio selection problem. The crux of the robust estimation approach is to obtain a meaningful parameter estimate that accounts for data uncertainties. In Section 2.1.1 we have discussed four of the most popular parameter estimators used in financial practice, followed by discussion of the potential weaknesses of employing robust estimators in the portfolio selection problem.

On the other hand, the robust modelling approach enhances portfolio selections by involving robust formulations in the optimisation framework. Instead of adopting a specific parameter estimator to deal with the uncertainties of the portfolio selection problem, the robust modelling approach tries to model or reduce uncertainties by utilising new techniques, such as ambiguity aversion formulation and extra constraints on input parameters. In Section 2.1.2 we have provided descriptions of four commonly used robust modelling approaches for the portfolio selection problem.

In the existing literature, all of the proposed approaches for improving the performance of the portfolio selection problem have their drawbacks. Among the various approaches, the robust counterpart approach performs better in terms of simplicity and efficiency of computation. Nevertheless, it is argued by many researchers that the robust portfolio allocation is too pessimistic in the way that it always assumes that the uncertainties of the portfolio selection problem will appear to be against the investor's benefits. Moreover, the robust portfolio allocation can be too conservative if the uncertainty set is too large. Furthermore, the advantages of applying the robust counterpart approach in the portfolio selection problem can only be realised if an appropriate uncertainty set is defined with careful planning for the underlying investment strategies.

In order to overcome the drawback of the existing robust portfolio selection model by providing a potentially profitable robust optimal asset allocation, we propose to incorporate additional investment information sources into the portfolio selection problem, so that the investor has opportunities to take on better quality investments and reduce the underlying uncertainty of the input parameters. Although theoretically the extra investment information would help to improve the reliability of the input parameters by providing further knowledge of the market environment, it is rare that the investment information provided by financial specialists is clear and definite. Usually these investment forecasts or market views are expressed linguistically. Therefore, it is difficult for the investor to make decisions based on this professional investment information. In the next chapter we will use fuzzy set theory and the multi-expert approach proposed by Lutgens and Schotman (2010) to develop a framework for solving portfolio selection problem with multiple investment information sources.

Chapter 3

Portfolio Selection with Fuzzy Advice from Multiple Analysts

There is no doubt that the mean-variance portfolio selection model is a remarkable approach which has made a great impact on the development of modern finance theory and also on practical financial decision making. However, unreliable portfolio allocation has lowered confidence in applying this theory practically. In addition, it is well known that even equally weighted portfolios can outperform Markowitz's mean-variance portfolios in many cases (DeMiguel et al., 2009). The major reason behind this is that the portfolio optimisation problem depends heavily on the input parameters, especially the parameters of the expected returns; those input parameters cannot be known *a priori* and are usually estimated with error. In other words, inaccurate or incorrect input parameters are one of the main problems that lead to undesirable outcomes. However, parameter uncertainty is not the only concern in the portfolio optimisation problem: the sensitivity feature of the return-risk portfolio model is another concern that influences the performance of the optimisation model, because the sensitivity feature will actually aggravate the effect caused by estimation errors.

On the other hand, it seems unrealistic that historical asset performances are the only type of the investment information source adopted in the portfolio selection problem for computing the input parameter estimates. From the investor's point of view, it will always be welcome to have as much stock market information as possible before making investment decisions. But there is a problem of how to decide which sources of information are more reliable. Lutgens and Schotman (2010) have proposed a robust portfolio optimisation framework that incorporates advice from multiple experts. In their model, the investor is supposed to be rational, and doesn't know the true value of the expected returns and the variances. The investment decision of the investor will be totally based on the recommendations offered by different experts without knowing how the experts arrived at their own estimates. Although the experts have different prior views on the parameters of the portfolio selection problem, they share and use the same sample data. The experts observe a sample with almost the same number of observations and then combine the results of the observations with their individual prior views to provide the posterior forecasts for the investor. Nevertheless, Lutgens and Schotman (2010) focus on the model structure of the multiple experts approach to the portfolio optimisation problem without considering the fundamental nature of the experts' suggestions.

As mentioned in the previous chapter, the investment recommendations provided by professional analysts are mostly expressed vaguely in words rather than in precise numerical formats. Therefore, we will modify the existing multiple experts framework of Lutgens and Schotman (2010) by adopting fuzzy set theory for the linguistic and imprecise experts' forecasts in this chapter. We will first provide all necessary definitions and notations of fuzzy set theory, followed by the possibilistic interpretation of fuzzy parameters. Then we will formulate the proposed portfolio selection approach with fuzzy advice from multiple analysts. Finally, we will present examples to illustrate our multiple analysts approach to the portfolio selection problem.

3.1 Fuzzy Set Theory

Generally speaking, it is common in real world applications that only a small portion of the knowledge about the problem under investigation can be considered as certain and useful information. The more uncertain the problem is, the less precise we can be with respect to understanding and solving the problem. Although making a good decision for solving the portfolio selection problem doesn't guarantee a good outcome, without a good decision based on a reasonable analysis for the problem, it is unlikely that the decision maker will have a satisfactory result.

There are many imprecise and ambiguous features which do not constitute classes or sets in the usual mathematical term, such as "the set of all real numbers which are much greater than 1" or "strong performance for Apple-related stocks". In order to deal with these uncertain and vague types of information, Zadeh (1965) developed a new mathematical tool named the fuzzy set theory. Instead of following the definition of an ordinary set with exact boundaries, Zadeh uses membership functions to describe mathematically the "grade of membership" of an element in a fuzzy set, so that there are no exact boundaries for a fuzzy set. The following definitions and statements mostly refer to Espinosa et al. (2005) and Ross (2004).

3.1.1 Fuzzy Sets

Let X be a universal set. If a subset $A \in X$ is an ordinary set, then an element $x \in X$ is either a member of the subset A or not. The subset A can be expressed as

$$A = \{x \in X \mid C_A(x) = 1\}$$

with characteristic function

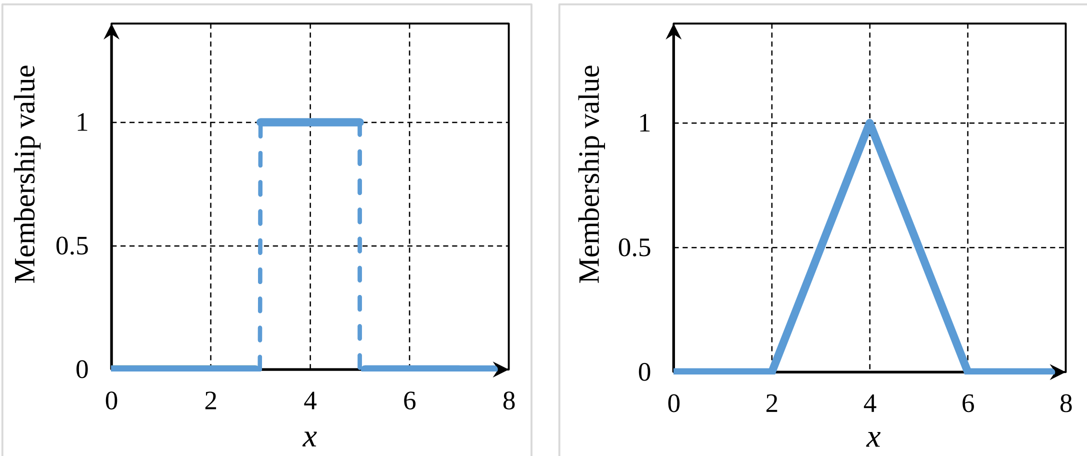
$$C_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

Alternatively, if the subset $\tilde{A} \in X$ is a fuzzy set, then the subset $\tilde{A}$ is defined by a membership function

$$M_{\tilde{A}} : X \to [0,1] \ .$$

This membership function $M_{\tilde{A}}$ describes the membership degree of an element $x \in X$, and the value of $M_{\tilde{A}}(x)$ closer to 1 indicates that the element x has a higher grade of membership towards the fuzzy subset $\tilde{A}$. More specifically, instead of deciding that an element $x \in X$ is either feasible or unfeasible in the subset A, fuzzy set theory describes the degree of belonging of this element x to the fuzzy subset $\tilde{A}$ by the membership function. The advantage of fuzzy set theory is the allowance for the intermediate membership degree $0 < M_{\tilde{A}}(x) < 1$, which provides the opportunity to deal with the problems of uncertainties, imprecision, and contradictions in crisp sets. A fuzzy set $\tilde{A}$ becomes an ordinary crisp set A when the membership function $M_{\tilde{A}}$ contains only two points, 0 and 1 . In other words, an ordinary crisp set is a special form of fuzzy set with sharp boundaries. The following figure graphically illustrates the difference between an ordinary set A and a fuzzy set $\tilde{A}$.

Figure 3.1 Illustration of the Characteristic Function and the Membership Function



(a) The characteristic function $C_A(x)$ (b) The membership function $M_{\tilde{A}}(x)$

For notational convenience in some of the subsequent formulations and discussions, we define the following basic features of fuzzy sets as stated in Ross (1995).

1) The support of a fuzzy set $\tilde{A}$, $Supp(\tilde{A}) \in \tilde{A}$, is a crisp subset that comprises elements having nonzero membership in the set $\tilde{A} \in X$:

$$Supp(\tilde{A}) = \{x \in \tilde{A} \mid M_{\tilde{A}}(x) > 0\}.$$

   In addition, a fuzzy set $\tilde{A}$ is said to be an empty set, $\tilde{A} = \emptyset$, if and only if the support of a fuzzy set $\tilde{A}$ does not exist for all $x \in X$:

$$M_{\tilde{A}}(x) = 0 \quad \forall\, x \in X \;.$$

2) The core of a fuzzy set $\tilde{A}$, $Core(\tilde{A}) \in \tilde{A}$, is a crisp subset that comprises elements having full and complete membership in the set $\tilde{A} \in X$:

$$Core(\tilde{A}) = \{x \in \tilde{A} \mid M_{\tilde{A}}(x) = 1\}.$$

3) The $\alpha - cut$ of a fuzzy set $\tilde{A}$, $\tilde{A}_\alpha \in \tilde{A}$, is a crisp subset that comprises elements having at least α degree of membership in the set $\tilde{A} \in X$:

$$\tilde{A}_\alpha = \{x \in \tilde{A} \mid M_{\tilde{A}}(x) \geq \alpha\}$$

   with $\alpha \geq 0$. The $\alpha - cut$ set $\tilde{A}_\alpha$ is a compact subset of X for all $\alpha \in [0,1]$.

4) The height of a fuzzy set $\tilde{A}$, $hgt(\tilde{A})$, is the maximum value of the membership function:

$$hgt(\tilde{A}) = \sup_{x \in \tilde{A}} M_{\tilde{A}}(x).$$

   The $hgt(\tilde{A})$ can be used to measure the level of validity or credibility of information expressed by the fuzzy set $\tilde{A}$, and a fuzzy set is "subnormal" if $hgt(\tilde{A}) < 1$ for all $x \in X$.

5) A fuzzy set $\tilde{A} \in X$ is said to be a convex fuzzy set if and only if the values of the corresponding membership function $M_{\tilde{A}}$ are monotonically increasing, or

monotonically decreasing, or monotonically increasing then monotonically decreasing as the values of the elements increased:

$$M_{\tilde{A}}(x_b) = \min\left[M_{\tilde{A}}(x_a), M_{\tilde{A}}(x_c)\right]$$

with $(x_a, x_b, x_c) \in X$ and $x_a < x_b < x_c$.

3.1.2 Features of the Membership Function

In fuzzy set theory, the membership function is employed as a measure to describe the relationship of an element from the universe to a particular set. The ambiguity, imprecision and paradox of the element can be represented by the values of the membership function. Although there is no unique formulation for the membership function, and different approaches to the membership function are constructed to serve different purposes, the most common structure of the membership functions adopted in practice is the one that preserves the desired properties: normality and convexity (see, e.g., Dombi, 1990).

Medaglia et al. (2002) further suggested that an efficient membership function should be able to reflect accurately our knowledge about the chosen data, easily calculate the corresponding membership value for a given element $x \in X$, and be computationally tractable with flexibility to adjust and tune the formulation of the membership function. Indeed, many meaningful and also useful parameterised membership functions have been proposed in the past, as nicely summarised by Dombi (1990). For instance, the membership functions based on probability density functions (Civanlar and Trussell, 1986) and the membership functions designed as the distance between an observation and the given benchmark (Zimmermann and Zysno, 1985). The trapezoidal and bell-shaped membership functions are the most commonly used formulations for expressing fuzzy sets in the literature. This is because the trapezoidal and bell-shaped membership functions not only fulfilled the desired features as above stated, but also are compatible with both symmetrical and asymmetrical fuzzy situations.

3.1.2.1 Trapezoidal Fuzzy Variable

A trapezoidal fuzzy variable $\tilde{A}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+)$ is a fuzzy set with tolerance interval $[m_-, m_+]$, left width σ_-, and right width σ_+, where the tolerance interval is also called the peak of the fuzzy variable, as the element $x \in [m_-, m_+]$ has full membership. The membership function of a trapezoidal fuzzy variable $\tilde{A}^{Tra}$ is formulated as

$$M_{\tilde{A}^{Tra}}(x) = \begin{cases} 1 & x \in [m_-, m_+] \\ L\left(\frac{m_- - x}{\sigma_-}\right) = 1 - \frac{m_- - x}{\sigma_-} & x \in [m_- - \sigma_-, m_-] \\ R\left(\frac{x - m_+}{\sigma_+}\right) = 1 - \frac{x - m_+}{\sigma_+} & x \in [m_+, m_+ + \sigma_+] \\ 0 & otherwise \end{cases} . \qquad (3.1)$$

The trapezoidal fuzzy variable $\tilde{A}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+)$ is one of the most generic classes of fuzzy variables with linear membership functions, which is superior to other linear and nonlinear membership functions in terms of conceptual and operational simplicity. For this reason, many researchers and practitioners have adopted trapezoidal formulations for modelling linear uncertain situations (see, e.g., Bansal, 2011). Moreover, the triangular fuzzy variable $\tilde{A}^{Tri} = (m, \sigma_-, \sigma_+)$ is a subclass of a trapezoidal fuzzy variable $\tilde{A}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+)$ with $m_- = m_+ = m$. On the other hand, an ordinary crisp interval $A = [m_-, m_+]$ is a special case of a trapezoidal fuzzy variable $\tilde{A}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+)$ with $\sigma_- = \sigma_+ = 0$.

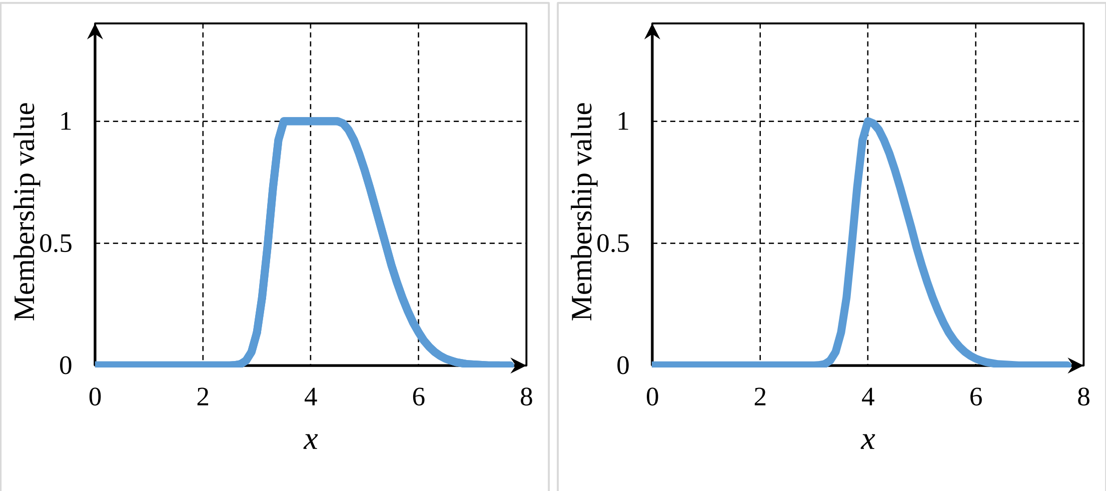
3.1.2.2 Bell-shaped Fuzzy Variable

A standard bell-shaped fuzzy variable $\tilde{A}^{Bell} = (m_-, m_+, \sigma_-, \sigma_+)$ is constructed by parts of two Gaussian functions with a peak tolerance interval $[m_-, m_+]$ in the middle, and σ_- and σ_+ are the negative and positive deviation, respectively. The membership function of the bell-shaped fuzzy variable $\tilde{A}^{Bell}$ is given by

$$M_{\tilde{A}^{Bell}}(x) = \begin{cases} L\left(\frac{m_- - x}{\sigma_-}\right) = exp\left(-\frac{1}{2}\left(\frac{m_- - x}{\sigma_-}\right)^2\right) & x \leq m_- \\ 1 & x \in [m_-, m_+]\,. \\ R\left(\frac{x - m_+}{\sigma_+}\right) = exp\left(-\frac{1}{2}\left(\frac{x - m_+}{\sigma_+}\right)^2\right) & m_+ \leq x \end{cases} \quad (3.2)$$

The bell-shaped fuzzy variable $\tilde{A}^{Bell}$ also has few subclasses. For instance, a bell-shaped fuzzy variable $\tilde{A}^{Bell} = (m_-, m_+, \sigma_-, \sigma_+)$ becomes a Pseudo-Gaussian fuzzy variable $\tilde{A}^{PG} = (m, \sigma_-, \sigma_+)$ if $m_- = m_+ = m$, and on the other hand, the bell-shaped fuzzy variable $\tilde{A}^{Bell}$ turns into a Gaussian fuzzy variable $\tilde{A}^{G} = (m, \sigma)$ if $m_- = m_+ = m$ and $\sigma_- = \sigma_+ = \sigma$. In addition to the desired properties of normality and convexity, the class of bell-shaped fuzzy variables has another advantage of being smooth and nonzero for all $x \in X$. Figure 3.2 shows the membership functions of the bell-shaped fuzzy variable $\tilde{A}^{Bell}$ and the Pseudo-Gaussian fuzzy variable $\tilde{A}^{PG}$.

Figure 3.2 Illustration of the Membership Functions for the Bell-Shaped Fuzzy Variables



(a) The bell-shaped fuzzy variable $\tilde{A}^{Bell}$ (b) The Pseudo-Gaussian fuzzy variable $\tilde{A}^{PG}$

3.1.3 The Defuzzification Process

Constructing an appropriate membership function for describing the underlying uncertain concept is the first step of the fuzzy set application. Converting a fuzzy quantity to a representative precise quantity by a suitable approach is the second step of the application and this procedure is often referred as defuzzification. Many different approaches for defuzzifying fuzzy outcomes have been proposed in the literature of fuzzy set theory and the most common defuzzification methods are the max membership method and the centroid method (Dubois, 2006; Ross, 1995), which are summarised as follows:

1) The max membership method uses the element with the highest membership degree to represent the fuzzy variable $\tilde{A}$:

$$M_{\tilde{A}}(x^*) \geq M_{\tilde{A}}(x) \qquad \forall x \in X\ ,$$

with $x^* \in X$ denoting the defuzzified value of fuzzy variable $\tilde{A}$. However, this method is limited to peak-shaped membership functions and doesn't consider other possible elements, except the element with the greatest degree of membership. The mean of maxima method extends the concept of the max membership method by using the middle point in the core interval of the fuzzy variable $\tilde{A}$. The defuzzified value of fuzzy variable $\tilde{A}$ is defined as

$$x^* = \frac{x_a + x_b}{2}$$

with $(x_a, x_b, x^*) \in X$ and elements x_a and x_b are the boundaries of the core of fuzzy variable $Core(\tilde{A})$. Similar to the max membership method, the mean of maxima method focuses on the core of the fuzzy variable $Core(\tilde{A})$ and ignores information about the rest of the fuzzy variable $\tilde{A}$.

2) The centroid method is also known as the centre of gravity method, which defines the defuzzified value $x^* \in X$ as the centre of mass in the support of the fuzzy variable $Supp(\tilde{A})$. In other words, the defuzzified value x^* obtained by the centroid method equally divides the area under the membership function $M_{\tilde{A}}$ into two parts. The centroid method equation for the defuzzified value x^* is based on algebraic integrations, and formulated as

$$x^* = \frac{\int M_{\tilde{A}}(x) \cdot x \, dx}{\int M_{\tilde{A}}(x) \, dx} \qquad \forall\, x \in Supp(\tilde{A}) .$$

A similar but more advanced defuzzification method has been given by Carlsson and Fuller (2001), which will be discussed in detail below.

3.1.4 The Crisp Possibilistic Interpretation of Fuzzy Variables

In order to account for the possibilistic nature of fuzzy intervals, Carlsson and Fuller (2001) proposed the crisp possibilistic interpretation of fuzzy variables based on the $\alpha - cut$ set for the fuzzy variable $\tilde{A}$. Their defuzzification method is a level-weighted function on $[0,1]$ that calculates the arithmetic mean of the lower and upper possibilistic mean values of the fuzzy variable $\tilde{A}$. The crisp possibilistic interpretation method can be applied to discrete or continuous and also symmetric or asymmetric membership functions. Moreover, this defuzzification approach is consistent with the fuzzy extension principle proposed by Zadeh (1978), as well as the definitions of the expected mean value and variance in probability theory.

Consider a normal and convex fuzzy variable $\tilde{A}$ with the corresponding $\alpha - cut$ set of $\tilde{A}$ denoted as $\tilde{A}_\alpha = [\tilde{A}^L(\alpha), \tilde{A}^R(\alpha)]$ for $\alpha \in [0,1]$. By using the $\alpha - cut$ set of fuzzy variable $\tilde{A}$, Carlsson and Fuller (2001) defined the possibilistic mean value $E(\tilde{A})$ of the fuzzy variable $\tilde{A}$ as the arithmetic mean of its lower possibilistic and upper possibilistic mean values; that is,

$$E(\tilde{A}) = \frac{E_*(\tilde{A}) + E^*(\tilde{A})}{2}$$

where $E_*(\tilde{A})$ denotes the lower possibilistic mean value with

$$E_*(\tilde{A}) = 2\int_0^1 \alpha \tilde{A}^L(\alpha)\ d\alpha,$$

and $E^*(\tilde{A})$ denotes the upper possibilistic mean value with

$$E^*(\tilde{A}) = 2\int_0^1 \alpha \tilde{A}^R(\alpha)\ d\alpha.$$

Equivalently, the crisp possibilistic mean value of the fuzzy variable $\tilde{A}$ given by Carlsson and Fuller (2001) is expressed as

$$E(\tilde{A}) = \int_0^1 \alpha(\tilde{A}^L(\alpha) + \tilde{A}^R(\alpha))\, d\alpha\ . \tag{3.3}$$

On the other hand, the notion of the crisp possibilistic variance of the fuzzy variable $\tilde{A}$ is based on the squared deviation between the arithmetic mean and the endpoints of the corresponding $\alpha - cut$ set $\tilde{A}_\alpha$, that is,

$$\begin{aligned} Var(\tilde{A}) &= \int_0^1 \alpha\left(\left[\frac{\tilde{A}^L(\alpha) + \tilde{A}^R(\alpha)}{2} - \tilde{A}^L(\alpha)\right]^2 + \left[\frac{\tilde{A}^L(\alpha) + \tilde{A}^R(\alpha)}{2} - \tilde{A}^R(\alpha)\right]^2\right) d\alpha \\ &= \frac{1}{2}\int_0^1 \alpha(\tilde{A}^R(\alpha) - \tilde{A}^L(\alpha))^2\, d\alpha \quad . \end{aligned} \tag{3.4}$$

Further descriptions for the crisp possibilistic interpretations of trapezoidal fuzzy variables $\tilde{A}^{Tra}$ and bell-shaped fuzzy variables $\tilde{A}^{Bell}$ are given in the following section (Carlsson & Fuller, 2001; Carlsson et al., 2002).

3.1.4.1. The Crisp Possibilistic Interpretation of Trapezoidal Fuzzy Variables

Suppose $\tilde{A}$ is a trapezoidal fuzzy variable $\tilde{A}^{Tra} = (m_-, m_+, \sigma_-, \sigma_+)$, and the $\alpha - cut$ set of $\tilde{A}^{Tra}$ is $\tilde{A}^{Tra}_\alpha = [\tilde{A}^L(\alpha), \tilde{A}^R(\alpha)]$ with $\alpha \in [0,1]$. By following the membership function (3.1) of the trapezoidal fuzzy variable $\tilde{A}^{Tra}$, the $\alpha - cut$ set of $\tilde{A}^{Tra}$ can be expressed as

$$\begin{aligned} \tilde{A}^{Tra}_{\alpha} &= \left[\tilde{A}^{L}(\alpha), \tilde{A}^{R}(\alpha)\right] \\ &= [m_{-} - \sigma_{-}\, L^{-1}(\alpha), m_{+} + \sigma_{+}\, R^{-1}(\alpha)] \\ &= [m_{-} - \sigma_{-}(1-\alpha), m_{+} + \sigma_{+}(1-\alpha)] \end{aligned} \tag{3.5}$$

for all $\alpha \in [0,1]$. Therefore, by the definition of the crisp possibilistic mean value of fuzzy variables (3.3) and equation (3.5), we have

$$\begin{aligned} E\left(\tilde{A}^{Tra}\right) &= \int_{0}^{1} \alpha(\,\tilde{A}^{L}(\alpha) + \tilde{A}^{R}(\alpha))\, d\alpha \\ &= \int_{0}^{1} \alpha(\,m_{-} - \sigma_{-}(1-\alpha) + m_{+} + \sigma_{+}(1-\alpha))\, d\alpha \\ &= \frac{m_{-} + m_{+}}{2} + \frac{\sigma_{+} - \sigma_{-}}{6}\ . \end{aligned} \tag{3.6}$$

Similarly, the crisp possibilistic variance of the trapezoidal fuzzy variable $\tilde{A}^{Tra}$ is given by (3.4) as

$$\begin{aligned} Var\left(\tilde{A}^{Tra}\right) &= \frac{1}{2}\int_{0}^{1} \alpha(\,\tilde{A}^{R}(\alpha) - \tilde{A}^{L}(\alpha))^{2}\, d\alpha \\ &= \frac{1}{2}\int_{0}^{1} \alpha(\,m_{+} + \sigma_{+}(1-\alpha) - (m_{-} - \sigma_{-}(1-\alpha)))^{2}\, d\alpha \\ &= \left[\frac{m_{+} - m_{-}}{2} + \frac{\sigma_{-} + \sigma_{+}}{6}\right]^{2} + \frac{(\sigma_{-} + \sigma_{+})^{2}}{72}\ . \end{aligned} \tag{3.7}$$

3.1.4.2. The Crisp Possibilistic Interpretation of Bell-shaped Fuzzy Variables

Let $\tilde{A}$ be a bell-shaped fuzzy variable $\tilde{A}^{Bell} = (m_{-}, m_{+}, \sigma_{-}, \sigma_{+})$ with the $\alpha - cut$ set of $\tilde{A}^{Bell}$ denoted as $\tilde{A}^{Bell}_{\alpha} = [\tilde{A}^{L}(\alpha), \tilde{A}^{R}(\alpha)]$ for all $\alpha \in [0,1]$. Then, the $\alpha - cut$ set $\tilde{A}^{Bell}_{\alpha}$ can be rearranged by following the membership function (3.2)

$$\begin{aligned} \tilde{A}^{Bell}_{\alpha} &= \left[\tilde{A}^{L}(\alpha), \tilde{A}^{R}(\alpha)\right] \\ &= [m_{-} - \sigma_{-}\, L^{-1}(\alpha), m_{+} + \sigma_{+}\, R^{-1}(\alpha)] \\ &= \left[m_{-} - \sigma_{-}\sqrt{-2\ln\alpha}\,, m_{+} + \sigma_{+}\sqrt{-2\ln\alpha}\right] \end{aligned} \tag{3.8}$$

with $\alpha \in [0,1]$. The crisp possibilistic mean value of the bell-shaped fuzzy variable $\tilde{A}^{Bell}$ can be expressed by using equations (3.3) and (3.8), i.e.,

$$E(\tilde{A}^{Bell}) = \int_0^1 \alpha(\tilde{A}^L(\alpha) + \tilde{A}^R(\alpha))\, d\alpha$$
$$= \int_0^1 \alpha(m_- - \sigma_-\sqrt{-2\ln\alpha} + m_+ + \sigma_+\sqrt{-2\ln\alpha})\, d\alpha \qquad (3.9)$$
$$= \frac{m_- + m_+}{2} - \sigma_- \int_0^1 \alpha\sqrt{-2\ln\alpha}\; d\alpha + \sigma_+ \int_0^1 \alpha\sqrt{-2\ln\alpha}\; d\alpha\ .$$

In addition, by following equations (3.4) and (3.8), the crisp possibilistic variance of the bell-shaped fuzzy variable $\tilde{A}^{Bell}$ is given by

$$Var(\tilde{A}^{Bell}) = \frac{1}{2}\int_0^1 \alpha(\tilde{A}^R(\alpha) - \tilde{A}^L(\alpha))^2\, d\alpha$$
$$= \frac{1}{2}\int_0^1 \alpha(m_+ + \sigma_+\sqrt{-2\ln\alpha} - (m_- - \sigma_-\sqrt{-2\ln\alpha}))^2\, d\alpha\ . \qquad (3.10)$$

3.2 Multi-Analyst Portfolio Selection with Fuzzy Aspiration

In this section we will develop a new approach to portfolio selection that takes into account analysts' forecasts expressed in vague linguistic statements.

To choose an appropriate model for optimising the portfolio selection problem with multiple analysts' recommendations, one must ensure that the professional advice is expressed and employed in a reasonable and also applicable manner. As already mentioned, the multiple experts approach proposed by Lutgens and Schotman (2010) outperforms other multi-prior approaches by providing an optimal portfolio selection which is robust to different advice without artificially assigning prior probabilities to the forecasts. Hence, we start with the multi-expert portfolio selection approach $(\boldsymbol{E_{MV}})$ as

$$\max_{x\in\mathbb{R}^n}\ \min_{z\in Z} \qquad \tilde{\mu}_z{}^T x - \frac{\lambda}{2} x^T \tilde{\Sigma}_z x \qquad , \qquad (3.11)$$

where x is the decision vector, $\tilde{\mu}_z \in \mathbb{R}^n$ is the fuzzy forecasts of the expected returns provided by the financial analyst $z \in Z$ with $z \in \{1, \ldots, Z\}$, and $\tilde{\Sigma}_z \in \mathbb{R}^n \times \mathbb{R}^n$ is the positive semi-definite matrix that denotes the fuzzy variability of the expected returns addressed by the analyst z. Note that, instead of assuming that the portfolio optimisation model only considers buy side analysis, and that short selling is restricted, a more general investment environment is considered without restrictions on the decision vector $x \in \mathbb{R}^n$.

The estimate of returns $\tilde{\mu}_z = (\tilde{\mu}_{z_1}, \tilde{\mu}_{z_2}, \dots, \tilde{\mu}_{z_n})$ suggested by the analyst z are fuzzy variables, and every fuzzy variable $\tilde{\mu}_{z_i}$, for $i = 1, \dots, n$, is characterised by a membership function $M_{\tilde{\mu}_{z_i}}$. Assume that the fuzzy variable $\tilde{\mu}_{z_i}$ is a trapezoidal fuzzy variable and denoted as $\tilde{\mu}_{z_i}^{Tra} = (\mu_{z_i}^{m-}, \mu_{z_i}^{m+}, \sigma_{z_i}^{-}, \sigma_{z_i}^{+})$ with tolerance interval $[\mu_{z_i}^{m-}, \mu_{z_i}^{m+}]$, left width $\sigma_{z_i}^{-}$, and right width $\sigma_{z_i}^{+}$. Then the crisp possibilistic mean value and the variance of asset i according to the analyst z's forecasts can be obtained via equations (3.6) and (3.7) as

$$E(\tilde{\mu}_{z_i}^{Tra}) = \check{\mu}_{z_i} = \frac{\mu_{z_i}^{m-} + \mu_{z_i}^{m+}}{2} + \frac{\sigma_{z_i}^{+} - \sigma_{z_i}^{-}}{6} \tag{3.12}$$

and

$$Var(\tilde{\mu}_{z_i}^{Tra}) = \check{\sigma}_{z_i}^2 = \left[\frac{\mu_{z_i}^{m+} - \mu_{z_i}^{m-}}{2} + \frac{\sigma_{z_i}^{-} + \sigma_{z_i}^{+}}{6}\right]^2 + \frac{(\sigma_{z_i}^{-} + \sigma_{z_i}^{+})^2}{72}. \tag{3.13}$$

In Lutgens and Schotman (2010), it is assumed that each of the experts provides his/her forecasts for all individual assets in the entire market. In reality, this is obviously unrealistic. As will be seen later in Chapter 5, usually financial analysts select only a few assets and comment on their future performances. This means that (3.12) and (3.13) can only be obtained for those assets which the analysts comment on. In addition, the financial analysts usually comment on individual assets but not on their relationships. Hence it is not possible to elicit the covariance structure of the asset returns using the financial analysts' forecasts.

We assume that when no forecasts are available from a financial analyst on one asset, the investor will use the historical data to work out the expected returns, variances, and underlying correlation structure. Consequently, instead of defuzzifying the fuzzy covariance matrix $\tilde{\Sigma}_z$ via the formulation proposed by Carlsson and Fuller (2001), we obtain the crisp possibilistic covariance matrix $Cov(\tilde{\mu}_z^{Tra})$ regarding to the z^{th} analyst's forecasts by combining the crisp possibilistic variance $Var(\tilde{\mu}_z^{Tra})$ with historical correlation coefficient matrix $Corr(\mu)$. In other words, the crisp possibilistic covariance matrices of all analysts are formulated with an identical correlation matrix $Corr(\mu)$, which is obtained from the historical data. That is,

$$Cov(\tilde{\mu}_z^{Tra}) = \check{\Sigma}_z = \left(\check{\sigma}_{z_{ij}}\right) = \begin{pmatrix} \check{\sigma}_{z_1}^2 & & & & \\ \rho_{12}\check{\sigma}_{z_1}\check{\sigma}_{z_2} & \check{\sigma}_{z_2}^2 & & & \\ \rho_{13}\check{\sigma}_{z_1}\check{\sigma}_{z_3} & \rho_{23}\check{\sigma}_{z_2}\check{\sigma}_{z_3} & \check{\sigma}_{z_3}^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \rho_{1n}\check{\sigma}_{z_1}\check{\sigma}_{z_n} & \rho_{2n}\check{\sigma}_{z_2}\check{\sigma}_{z_n} & \cdots & & \check{\sigma}_{z_n}^2 \end{pmatrix} \tag{3.14}$$

with $\check{\sigma}_{z_{ij}} = \check{\sigma}_{z_{ji}}$ denoting the crisp possibilistic covariance of asset i and asset j for $i,j = 1, \dots, n$ and $i \neq j$. $\check{\sigma}_{z_i}^2 = \check{\sigma}_{z_{ii}}$ and $\check{\sigma}_{z_i}$ are the crisp possibilistic variance and standard deviation of asset i, respectively. ρ_{ij} is the correlation coefficient between asset i and asset j for $i,j = 1, \dots, n$ and $i \neq j$. Therefore, the multi-analyst portfolio selection problem with fuzzy parameters (3.11) can be transformed into a quadratic optimisation problem by substituting the fuzzy parameters $\tilde{\mu}_z$ and $\tilde{\Sigma}_z$ with the crisp possibilistic interpretation of fuzzy expected returns $\check{\mu}_z$ and covariance matrix $\check{\Sigma}_z$,

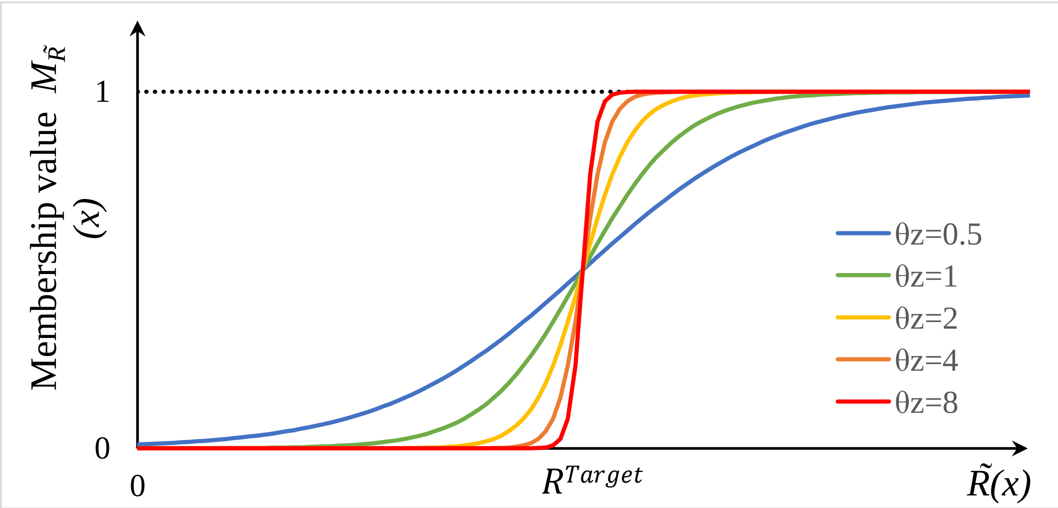
$$\max_{x \in \mathbb{R}^n} \min_{z \in \mathrm{Z}} \quad \check{\mu}_z{}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_z x \quad . \tag{3.15}$$

However, the investor does not know precisely how reliable the financial analysts are and hence the investor has rather uncertain confidence in each individual analyst. In order to take this vague credibility factor of analysts into account, a nonlinear logistic membership function, as described in Watada (1997) and Gupta et al. (2008), is introduced to express the ambiguous aspiration level of the investment $\tilde{R}_z$ for the investor with $\tilde{R}_z = \check{\mu}_z{}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_z x$, i.e.,

$$M_{\tilde{R}_z}(x) = \frac{1}{1 + \exp\left(-\theta_z(\tilde{R}_z - R^{Target})\right)} \ , \qquad (3.16)$$

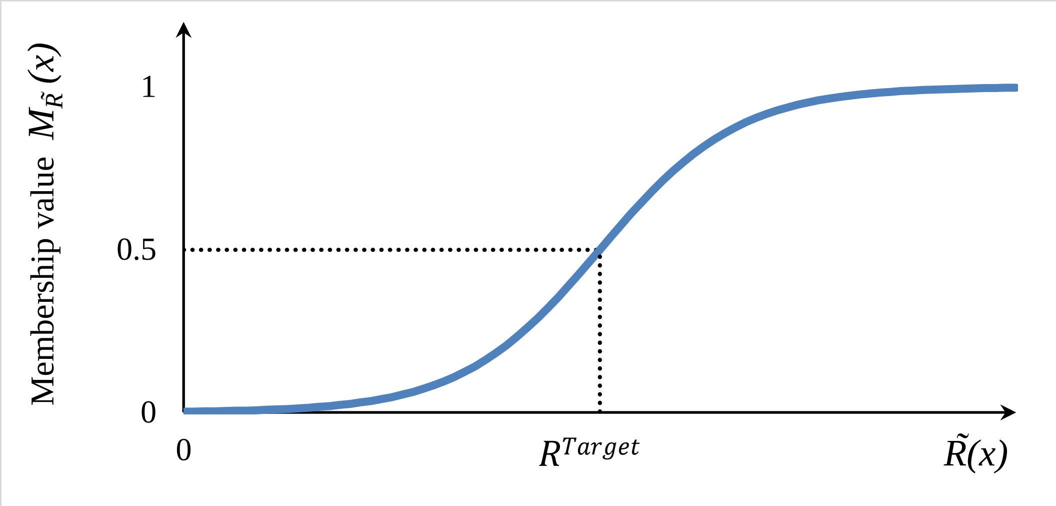
where $0 < \theta_z < \infty$ denotes the credibility level of the analyst z prescribed by the investor and determines the shape of the membership function $M_{\tilde{R}_z}$. The higher the value of θ_z, the more confidence the investor has in the analyst z. Although the aspiration level of an investor can be described more accurately by assigning appropriate values for the credibility level θ_z, there are no explicit guidelines for approaching these values (see Gupta et al., 2008). The value of credibility θ_z can only be given by the investor heuristically and experientially. Figure 3.3 graphically illustrates the effects on the shape of the membership function as the value of the parameter θ_z increased.

Figure 3.3 The Membership Function for Different Levels of Credibility



On the other hand, R^{Target} is the benchmark given by the investor to define the middle aspiration level for the portfolio performance of the investment. More specifically, R^{Target} is a fixed value located at the point that has 0.5 degree of membership, i.e., $M_{\tilde{R}_z}(R^{Target}) = 0.5$.

Figure 3.4 The Membership Function of the Ambiguous Investment Goal



Thus, instead of solving the portfolio optimisation problem (3.15), we formulate the portfolio selection as

$$(\boldsymbol{F_{MV}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+\exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)} \tag{3.17}$$

where $x \in \mathbb{R}^n$ is the decision vector. $\check{\mu}_z$ and $\check{\Sigma}_z$ are the crisp possibilistic expected return and covariance matrix according to recommendations provided by analyst $z \in Z$ with $z \in \{1, \ldots, Z\}$, respectively.

The portfolio allocation problem $(\boldsymbol{F_{MV}})$ is in fact equivalent to

$$\begin{aligned} &\max_{x\in\mathbb{R}^n,\ \zeta\in\mathbb{R}} && \zeta \\ &s.t. && \zeta \le \frac{1}{1+\exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)} \end{aligned} \tag{3.18}$$

for $z \in Z$ with $z \in \{1, \ldots, Z\}$. In order to transform the non-linear optimisation problem (3.18) into a simpler optimisation problem, we rewrite the constraints of (3.18) as

$$
\begin{aligned}
&\zeta \leq \frac{1}{1+\exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)} \\
\Rightarrow \quad & \exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right) \leq \frac{1}{\zeta} - 1 \\
\Rightarrow \quad & -\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right) \leq \log\left(\frac{1}{\zeta} - 1\right) \\
\Rightarrow \quad & \theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right) \geq -\log\left(\left(\frac{1}{\zeta} - \frac{\zeta}{\zeta}\right)\right) \\
\Rightarrow \quad & \theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right) \geq \log\left(\left(\frac{1-\zeta}{\zeta}\right)^{-1}\right) \\
\Rightarrow \quad & \theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right) \geq \eta
\end{aligned} \tag{3.19}
$$

with $\eta = \log\left(\frac{\zeta}{1-\zeta}\right)$. Since the value of the logistic function $\log\left(\frac{\zeta}{1-\zeta}\right)$ increases monotonically as the value of ζ increases, it follows immediately that maximising ζ is also maximising η. In this case, we have

$$
(\boldsymbol{F}^*_{\boldsymbol{MV}}) \qquad
\begin{aligned}
&\max_{x\in\mathbb{R}^n,\ \zeta\in\mathbb{R}} \quad \eta \\
&s.t. \quad \eta \leq \theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)
\end{aligned} \tag{3.20}
$$

for $z \in Z$ with $z \in \{1, \ldots, Z\}$ as an equivalent formulation of the multi-analyst portfolio selection problem with fuzzy aspiration $(\boldsymbol{F}_{\boldsymbol{MV}})$. By denoting $g_z(x) = \theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)$ as the ambiguous aspiration of the investment according to the forecasts provided by the z^{th} analyst, the Lagrangian function to portfolio selection problem $(\boldsymbol{F}^*_{\boldsymbol{MV}})$ is given for $\phi \in \mathbb{R}^Z$ by

$$
\mathcal{L}(\eta, x, \phi) = \eta - \sum_{z=1}^{Z} \phi_z\big(\eta - g_z(x)\big), \tag{3.21}
$$

where $\phi \in \mathbb{R}^Z$ is the vector of the Lagrange multipliers. We can easily verify that the partial derivatives of the Lagrangian function with respect to variables η and x are

$$\frac{\partial \mathcal{L}}{\partial \eta} = 1 - \sum_{z=1}^{Z} \phi_z$$
$$\frac{\partial \mathcal{L}}{\partial x} = \sum_{z=1}^{Z} \phi_z g_z{}'(x) \quad , \tag{3.22}$$

where

$$g_z{}'(x) = \theta_z \check{\mu}_z - \lambda \theta_z \check{\Sigma}_z x \, . \tag{3.23}$$

Thus, according to the Karush–Kuhn–Tucker conditions[3], the corresponding conditions for the optimal portfolio x^* of the portfolio selection problem $(\boldsymbol{F}_{\boldsymbol{MV}}^*)$ are

$$1 - \sum_{z=1}^{Z} \phi_z = 0 \quad , \tag{3.24}$$

$$\sum_{z=1}^{Z} \phi_z g_z{}'(x^*) = 0 \quad , \tag{3.25}$$

$$\phi_z\big(\eta - g_z(x^*)\big) = 0 \quad , \qquad z = 1, \ldots, Z \quad , \tag{3.26}$$

$$\phi_z \geq 0 \quad , \qquad z = 1, \ldots, Z \quad . \tag{3.27}$$

By using conditions (3.24) and (3.27), it can be easily seen that the Lagrange multipliers $\phi \in \mathbb{R}^Z$ must satisfy $0 \leq \phi_z \leq 1$ for every $z \in Z$ with $z \in \{1, \ldots, Z\}$. On the other hand, condition (3.25) can be rearranged by substituting $g_z{}'(x^*)$ with (3.23), that is,

$$\begin{aligned} & \sum_{z=1}^{Z} \phi_z g_z{}'(x^*) = 0 \\ \Rightarrow \quad & \sum_{z=1}^{Z} \phi_z\big(\theta_z \check{\mu}_z - \lambda \theta_z \check{\Sigma}_z x^*\big) = 0 \\ \Rightarrow \quad & \sum_{z=1}^{Z} \phi_z \theta_z \check{\mu}_z = \lambda \sum_{z=1}^{Z} \phi_z \theta_z \check{\Sigma}_z x^* \quad . \end{aligned} \tag{3.28}$$

Consequently, the optimal portfolio x^* of the portfolio selection problem $(\boldsymbol{F}_{\boldsymbol{MV}}^*)$ can be obtained and formulated as

[3] As in reference Bonnans et al. (2006).

$$x^* = \frac{1}{\lambda}\ \Sigma^{*-1}\mu^* \quad , \tag{3.29}$$

with

$$\Sigma^* = \sum_{z=1}^{Z} \varpi_z \check{\Sigma}_z \quad , \tag{3.30}$$

and

$$\mu^* = \sum_{z=1}^{Z} \varpi_z \check{\mu}_z \quad , \tag{3.31}$$

where $\varpi_z = \theta_z \phi_z$.

This result corresponds to the multi-expert approach for the robust portfolio optimisation model proposed by Lutgens and Schotman (2010). The structure of the optimal portfolio x^* (3.29) is similar to (2.36), in the way that both are based on the weighted average of forecasts given by different experts. On the other hand, the optimal portfolio x^* (3.29) is different to (2.36): the weighted average of the mean parameter (3.31) and the covariance parameter (3.30) are derived from the fuzzy membership functions elicited from the financial analysts' forecasts, and are influenced by the coefficient θ_z, the credibility level of the z^{th} analyst prescribed by the investor. Equation (2.37) in Lutgens and Schotman (2010), on the other hand, does not involve fuzzy set theory; rather, the experts' advice is assumed to be clear, without any vagueness or ambiguity.

We also note that although the above results are useful in analysing theoretical properties of the solution, they are not a complete solution for the optimisation problem $(\boldsymbol{F}^*_{\boldsymbol{MV}})$, because the Lagrange multipliers $\phi \in \mathbb{R}^Z$ cannot be formulated explicitly. Nevertheless, one can always use standard computer software, such as MATLAB, to solve the problem and obtain efficient numerical solutions.

The following example illustrates the impact on employing the coefficient θ_z for describing the confidence of the investor in different analysts by investigating a simple portfolio selection problem with investment recommendations provided by two analysts.

3.2.1 Illustrative Example

Consider a situation in which the investor only needs to distribute funds between two assets, one risky asset and one risk free asset, and the investor has no knowledge about the true value of the assets and would be satisfied as long as the investment doesn't make any loss, i.e., $R^{Target} = 0$. The investor takes professional advice from two analysts, analyst a and analyst b, for some investment information about the expected return and variance of the risky asset and sets the investment benchmark as $R^{Target} = 0$ in terms of expected portfolio return.

Let $\check{\mu}_z$ and $\check{\Sigma}_z$ denote the crisp possibilistic asset return and the covariance matrix obtained according to analyst z's recommendations for $z = (a, b)$. Recalling the multi-analyst portfolio selection problem $(\boldsymbol{F}_{\boldsymbol{MV}}^*)$, we have

$$\begin{aligned} \max_{x\in\mathbb{R}^n} \quad & \eta \\ s.t. \quad & \eta \le \theta_a\left(\check{\mu}_a{}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_a x\right) \\ & \eta \le \theta_b\left(\check{\mu}_b{}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_b x\right) \end{aligned} \quad . \tag{3.32}$$

There are few different outcomes that could happen for this situation. Without loss of generality, we consider two cases, $\theta_a = \theta_b$ or $\theta_a > \theta_b$.

3.2.1.1 Case I: Equal Preference for the Analysts $\boldsymbol{\theta_a = \theta_b}$

The first case is under the circumstance that the investor doesn't know which analyst is more reliable and decides to treat analysts' predictions as equally important, and hence the coefficients of the credibility level θ_z are assumed to be identical to each other, i.e., $\theta_a = \theta_b$. In this case, the multi-analyst approach with fuzzy aspiration

$(\boldsymbol{F}^*_{\boldsymbol{MV}})$ becomes the multi-expert approach, as proposed by Lutgens and Schotman (2010). According to Lutgens and Schotman (2010), there are only two possible scenarios:

1) Assume one of the analysts provides more optimistic predictions, for instance, the higher expected return with lower variance, $0 < \breve{\mu}_b < \breve{\mu}_a$ and $\breve{\sigma}_a^2 < \breve{\sigma}_b^2$. As a result, the forecast provided by this optimistic analyst doesn't influence the portfolio selection process and the structure of the corresponding optimal portfolio x^* depends only on the pessimistic analyst. That is,

$$x^* = x_b = \frac{\breve{\mu}_b}{\lambda \breve{\sigma}_b^2} \quad . \tag{3.33}$$

where $\breve{\mu}_b$ and $\breve{\sigma}_b^2$ denote the more pessimistic estimates according to recommendations provided by analyst b with $\breve{\mu}_b < \breve{\mu}_a$ and $\breve{\sigma}_a^2 < \breve{\sigma}_b^2$.

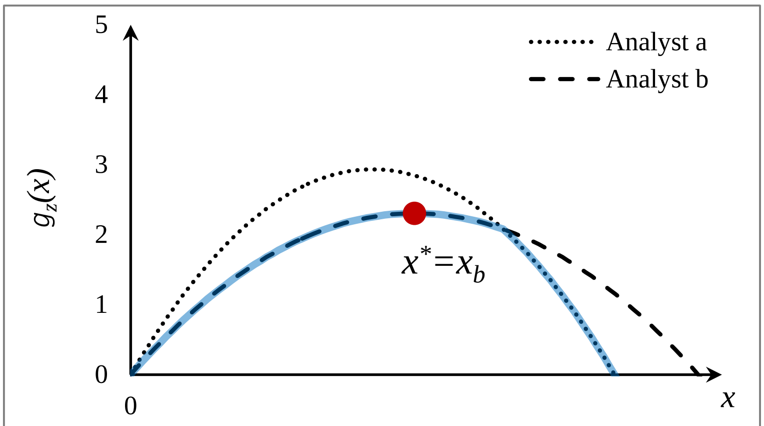
2) Assume neither analyst provides more optimistic predictions, for instance, the higher expected return with higher variance, $0 < \breve{\mu}_b < \breve{\mu}_a$ and $\breve{\sigma}_b^2 < \breve{\sigma}_a^2$. The optimal portfolio x^* is formulated as

$$x^* = \begin{cases} \dfrac{\breve{\mu}_b}{\lambda \breve{\sigma}_b^2} & \breve{\mu}_b \le \dfrac{2\breve{\mu}_a \breve{\sigma}_b^2}{\breve{\sigma}_a^2 + \breve{\sigma}_b^2} \\ \dfrac{2(\breve{\mu}_a - \breve{\mu}_b)}{\lambda(\breve{\sigma}_a^2 - \breve{\sigma}_b^2)} & \dfrac{2\breve{\mu}_a \breve{\sigma}_b^2}{\breve{\sigma}_a^2 + \breve{\sigma}_b^2} < \breve{\mu}_b < \dfrac{\breve{\mu}_a(\breve{\sigma}_a^2 + \breve{\sigma}_b^2)}{2\breve{\sigma}_a^2} \\ \dfrac{\breve{\mu}_a}{\lambda \breve{\sigma}_a^2} & \dfrac{\breve{\mu}_a(\breve{\sigma}_a^2 + \breve{\sigma}_b^2)}{2\breve{\sigma}_a^2} \le \breve{\mu}_b \end{cases} \quad . \tag{3.34}$$

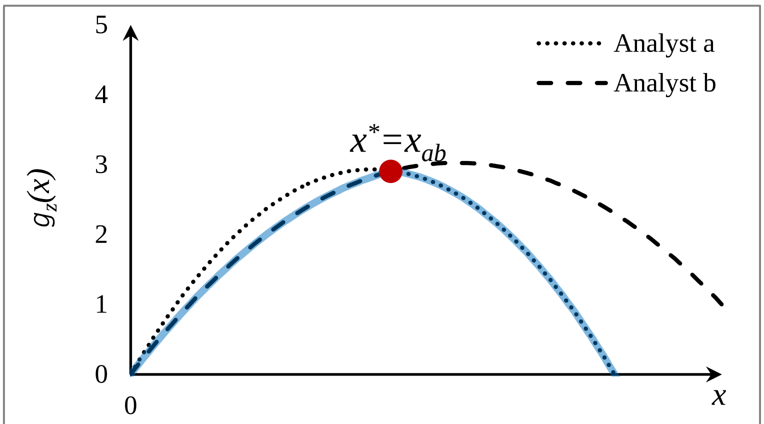
where $(\breve{\mu}_a, \breve{\sigma}_a^2)$ and $(\breve{\mu}_b, \breve{\sigma}_b^2)$ denote the estimates according to the recommendations given by analyst a and analyst b, respectively[4]. The three possible outcomes of the optimal portfolio x^* stated in equation (3.34) are graphically illustrated in Figure 3.5.

[4] For the proof, see Lutgens and Schotman (2010), Section 3.3.

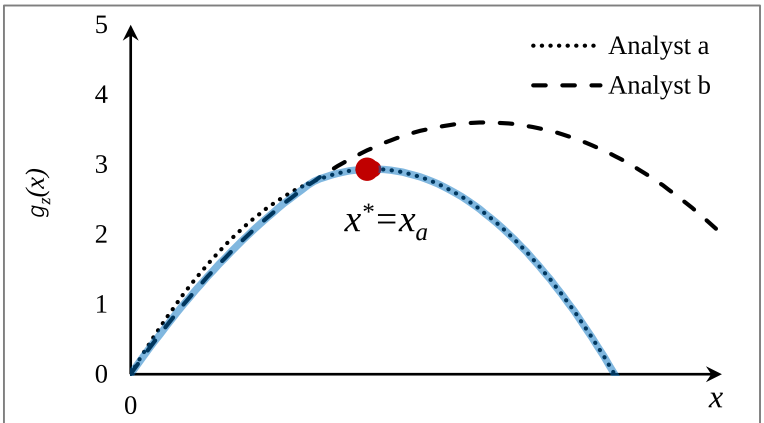
Figure 3.5 The Portfolio Selection with Two Analysts



(a) The optimal portfolio x^* depends only on analyst b



(b) The optimal portfolio x^* depends on both analysts



(c) The optimal portfolio x^* depends only on analyst a

Note: This figure displays the possible outcomes of the multi-expert approach for a simple case with two analysts (Lutgens & Schotman, 2010). The dotted lines and the dashed lines indicate the objective functions according to analyst a and analyst b, respectively. The blue solid line is the minimum of the objective functions, which indicates the robust objective function.

3.2.1.2 Case II: Unqual Preference for the Analysts $\theta_a > \theta_b$

The second case is under the circumstance that the investor does have a general view of which analyst is more reliable and decides to treat analysts' predictions differently. Therefore, the credibility level for the more reliable analyst (say analyst a) is greater than the other analyst, $\theta_a > \theta_b$. In this case, there are three possible scenarios. Before any further discussion, we define

$$\begin{aligned} g_a(x) &= \theta_a\left(\check{\mu}_a x - \frac{\lambda}{2}\check{\sigma}_a^2 x^2\right) \\ g_b(x) &= \theta_b\left(\check{\mu}_b x - \frac{\lambda}{2}\check{\sigma}_b^2 x^2\right) . \end{aligned} \tag{3.35}$$

1) Assume that the analyst a is considered to be more reliable by the investor and provides relatively more optimistic recommendations than the other, for instance, higher expected return with lower variance. That is, $\check{\mu}_a > \check{\mu}_b$ and $\check{\sigma}_b^2 > \check{\sigma}_a^2$. By following the assumptions $\theta_a > \theta_b$ together with $\check{\mu}_a > \check{\mu}_b$ and $\check{\sigma}_b^2 > \check{\sigma}_a^2$, one can easily notice that

$$\min(g_a(x), g_b(x)) = g_b(x) , \tag{3.36}$$

since

$$g_a(x) > \theta_b\left(\check{\mu}_a x - \frac{\lambda}{2}\check{\sigma}_a^2 x^2\right) > \theta_b\left(\check{\mu}_b x - \frac{\lambda}{2}\check{\sigma}_b^2 x^2\right) = g_b(x) . \tag{3.37}$$

In other words, the recommendations given by the more optimistic analyst a doesn't affect the decision making of the portfolio selection. Therefore, the optimal portfolio x^* depends only on the pessimistic analyst b and expresses as

$$x^* = x_b = \frac{\check{\mu}_b}{\lambda\check{\sigma}_b^2} . \tag{3.38}$$

2) Assume that the analyst a is considered to be more reliable by the investor, i.e., $\theta_a > \theta_b$, and provides relatively prudent recommendations that are more

pessimistic than the other, for instance, $\breve{\mu}_a < \breve{\mu}_b$ and $\breve{\sigma}_b^2 < \breve{\sigma}_a^2$. Therefore the optimal portfolio $x^* = x_a = \frac{\breve{\mu}_a}{\lambda \breve{\sigma}_a^2}$ if the ratio of the credibility levels is $\frac{\theta_a}{\theta_b} \leq \frac{\breve{\mu}_b}{\breve{\mu}_a}$, otherwise, the optimal portfolio x^* is formulated as

$$x^* = \begin{cases} x_b & \breve{\mu}_b \leq \dfrac{2\theta_a \breve{\mu}_a \breve{\sigma}_b^2}{\theta_a \breve{\sigma}_a^2 + \theta_b \breve{\sigma}_b^2} \\ x_{ab} & \dfrac{2\theta_a \breve{\mu}_a \breve{\sigma}_b^2}{\theta_a \breve{\sigma}_a^2 + \theta_b \breve{\sigma}_b^2} < \breve{\mu}_b < \dfrac{\breve{\mu}_a(\theta_a \breve{\sigma}_a^2 + \theta_b \breve{\sigma}_b^2)}{2\theta_b \breve{\sigma}_a^2} \\ x_a & \dfrac{\breve{\mu}_a(\theta_a \breve{\sigma}_a^2 + \theta_b \breve{\sigma}_b^2)}{2\theta_b \breve{\sigma}_a^2} \leq \breve{\mu}_b \end{cases} \tag{3.39}$$

Proof

First, assume that the ratio of credibility levels satisfies $\frac{\theta_a}{\theta_b} \leq \frac{\breve{\mu}_b}{\breve{\mu}_a}$. Then by following assumptions $\breve{\mu}_a < \breve{\mu}_b$ and $\breve{\sigma}_b^2 < \breve{\sigma}_a^2$, we have

$$\begin{aligned} g_b(x) &= \theta_b \left(\breve{\mu}_b x - \frac{\lambda}{2} \breve{\sigma}_b^2 x^2 \right) \\ &> \theta_b \breve{\mu}_b x - \frac{\lambda}{2} \theta_b \breve{\sigma}_a^2 x^2 \ . \end{aligned} \tag{3.40}$$

Since $\frac{\theta_a}{\theta_b} \leq \frac{\breve{\mu}_b}{\breve{\mu}_a}$, we have $\theta_b \geq \frac{\theta_a \breve{\mu}_a}{\breve{\mu}_b}$. Then,

$$\begin{aligned} \theta_b \breve{\mu}_b x - \frac{\lambda}{2} \theta_b \breve{\sigma}_a^2 x^2 &\geq \theta_a \breve{\mu}_a x - \frac{\lambda \theta_a \breve{\mu}_a}{2 \breve{\mu}_b} \breve{\sigma}_a^2 x^2 \\ &> \theta_a \breve{\mu}_a x - \frac{\lambda}{2} \theta_a \breve{\sigma}_a^2 x^2 \ , \end{aligned} \tag{3.41}$$

because $\frac{\breve{\mu}_a}{\breve{\mu}_b} < 1$. Therefore, we have $\min(g_a(x), g_b(x)) = g_a(x)$. In other words, the more optimistic recommendations given by the analyst b have no impact on the decision making of the portfolio selection if the ratio of the credibility levels is $\frac{\theta_a}{\theta_b} \leq \frac{\breve{\mu}_b}{\breve{\mu}_a}$, and the optimal portfolio x^* depends only on the more reliable but pessimistic analyst a and expresses as

$$x^* = x_a = \frac{\check{\mu}_a}{\lambda \check{\sigma}_a^2} \ .$$

Unlike the result obtained above that the optimal portfolio x^* relies entirely on one specific prediction, there are three possible outcomes if the ratio of the credibility levels is $\frac{\theta_a}{\theta_b} > \frac{\check{\mu}_b}{\check{\mu}_a}$. Since the optimal portfolio x^* is the maximum of the robust objective function $\min(g_a(x), g_b(x))$, the optimal portfolio x^* is either the maximum of function $g_a(x)$ or function $g_b(x)$, or the intersection point of the functions $g_a(x)$ and $g_b(x)$.

By expanding formulation (3.35), we have

$$\begin{aligned} g_a(x) &= \theta_a \check{\mu}_a x - \frac{\lambda}{2} \theta_a \check{\sigma}_a^2 x^2 \\ g_b(x) &= \theta_b \check{\mu}_b x - \frac{\lambda}{2} \theta_b \check{\sigma}_b^2 x^2 \ . \end{aligned} \tag{3.42}$$

Let $\theta_z \check{\mu}_z = \nu_z$ and $\theta_z \check{\sigma}_z^2 = \tau_z$ for $z = a, b$, then (3.42) becomes

$$\begin{aligned} g_a(x) &= \nu_a x - \frac{\lambda}{2} \tau_a x^2 \\ g_b(x) &= \nu_b x - \frac{\lambda}{2} \tau_b x^2 \ . \end{aligned} \tag{3.43}$$

Under assumptions $\check{\mu}_a < \check{\mu}_b$ and $\check{\sigma}_b^2 < \check{\sigma}_a^2$ with $\theta_a > \theta_b$ and $\frac{\theta_a}{\theta_b} > \frac{\check{\mu}_b}{\check{\mu}_a}$, we have $\theta_a \check{\mu}_a > \theta_b \check{\mu}_b$ and $\theta_a \check{\sigma}_a^2 > \theta_b \check{\sigma}_b^2$. That is, $\nu_a > \nu_b$ and $\tau_a > \tau_b$. Following directly from statement (3.34), the proposition 1 in Lutgens and Schotman (2010), the optimal portfolio x^* can be immediately formulated as

$$x^* = \begin{cases} \dfrac{\nu_b}{\lambda \tau_b} & \nu_b \le \dfrac{2\nu_a \tau_b}{\tau_a + \tau_b} \\ \dfrac{2(\nu_a - \nu_b)}{\lambda(\tau_a - \tau_b)} & \dfrac{2\nu_a \tau_b}{\tau_a + \tau_b} < \nu_b < \dfrac{\nu_a(\tau_a + \tau_b)}{2\tau_a} \\ \dfrac{\nu_a}{\lambda \tau_a} & \dfrac{\nu_a(\tau_a + \tau_b)}{2\tau_a} \le \nu_b \end{cases} \ . \tag{3.44}$$

By substituting $\nu_z = \theta_z \check{\mu}_z$ and $\tau_z = \theta_z \check{\sigma}_z^2$ for $z = a, b$, we have

$$x^* = \begin{cases} \dfrac{\theta_b \check{\mu}_b}{\lambda \theta_b \check{\sigma}_b^2} & \theta_b \check{\mu}_b \leq \dfrac{2\theta_a \check{\mu}_a \theta_b \check{\sigma}_b^2}{\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2} \\ \dfrac{2(\theta_a \check{\mu}_a - \theta_b \check{\mu}_b)}{\lambda(\theta_a \check{\sigma}_a^2 - \theta_b \check{\sigma}_b^2)} & \dfrac{2\theta_a \check{\mu}_a \theta_b \check{\sigma}_b^2}{\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2} < \theta_b \check{\mu}_b < \dfrac{\theta_a \check{\mu}_a (\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2)}{2\theta_a \check{\sigma}_a^2} \\ \dfrac{\theta_a \check{\mu}_a}{\lambda \theta_a \check{\sigma}_a^2} & \dfrac{\theta_a \check{\mu}_a (\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2)}{2\theta_a \check{\sigma}_a^2} \leq \theta_b \check{\mu}_b \end{cases}, \quad (3.45)$$

which is equivalent to

$$x^* = \begin{cases} x_b & \check{\mu}_b \leq \dfrac{2\theta_a \check{\mu}_a \check{\sigma}_b^2}{\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2} \\ x_{ab} & \dfrac{2\theta_a \check{\mu}_a \check{\sigma}_b^2}{\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2} < \check{\mu}_b < \dfrac{\check{\mu}_a (\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2)}{2\theta_b \check{\sigma}_a^2} \\ x_a & \dfrac{\check{\mu}_a (\theta_a \check{\sigma}_a^2 + \theta_b \check{\sigma}_b^2)}{2\theta_b \check{\sigma}_a^2} \leq \check{\mu}_b \end{cases} \quad (3.46)$$

with $x_a = \frac{\check{\mu}_a}{\lambda \check{\sigma}_a^2}$, $x_b = \frac{\check{\mu}_b}{\lambda \check{\sigma}_b^2}$, and $x_{ab} = \frac{2(\theta_a \check{\mu}_a - \theta_b \check{\mu}_b)}{\lambda(\theta_a \check{\sigma}_a^2 - \theta_b \check{\sigma}_b^2)}$.

3) Neither analyst provides more optimistic predictions. For instance, the lower expected return with lower variance or the higher expected return with higher variance. Assume that the more reliable analyst a provides the less conservative recommendations, $\check{\mu}_a > \check{\mu}_b$ and $\check{\sigma}_a^2 > \check{\sigma}_b^2$. Then the correlation between the ratio of the credibility levels and the parameters ratio are

$$\frac{\theta_a}{\theta_b} > \frac{\check{\mu}_b}{\check{\mu}_a} \quad \text{and} \quad \frac{\theta_a}{\theta_b} > \frac{\check{\sigma}_b^2}{\check{\sigma}_a^2} \ ,$$

which is equivalent to $\theta_a \check{\mu}_a > \theta_b \check{\mu}_b$ and $\theta_a \check{\sigma}_a^2 > \theta_b \check{\sigma}_b^2$, i.e., $\nu_a > \nu_b$ and $\tau_a > \tau_b$ as shown previously. Hence, the optimal portfolio x^* is formulated as statement (3.46)[5].

[5] The proof of this case is exactly the same as the previous proof for the case that the more reliable analyst provides relatively prudent advice with $\frac{\theta_a}{\theta_b} > \frac{\check{\mu}_b}{\check{\mu}_a}$.

3.3 Summary

Incorporating professional investment recommendations into the decision making process for portfolio selection can be beneficial in enabling investors to make better choices. However, the forecasts provided by the analysts are not written in clear numerical formats, and are usually expressed in vague linguistic statements. So far, there is no explicit or straightforward approach for constructing a portfolio with qualitative inputs. Therefore, adopting basic historical performance of assets together with additional investment information provided by professionals for constructing a portfolio without further implementations can be challenging.

In this chapter, a detailed literature review about the fuzzy set theory for interpreting imprecise and ambiguous situations is first provided. By following the idea of fuzzy logic, we have developed the multi-analyst portfolio selection approach with fuzzy aspiration $(\boldsymbol{F_{MV}})$ based on the portfolio optimisation frameworks proposed by Lutgens and Schotman (2010) and Gupta et al. (2008). The multi-analyst approach with fuzzy aspiration $(\boldsymbol{F_{MV}})$ possesses some good properties that allows more flexibility for its user, such as the choice of whether to use fuzzy variables or ordinary crisp variables and also the choice of whether to assign vague credibility levels θ to analysts' recommendations.

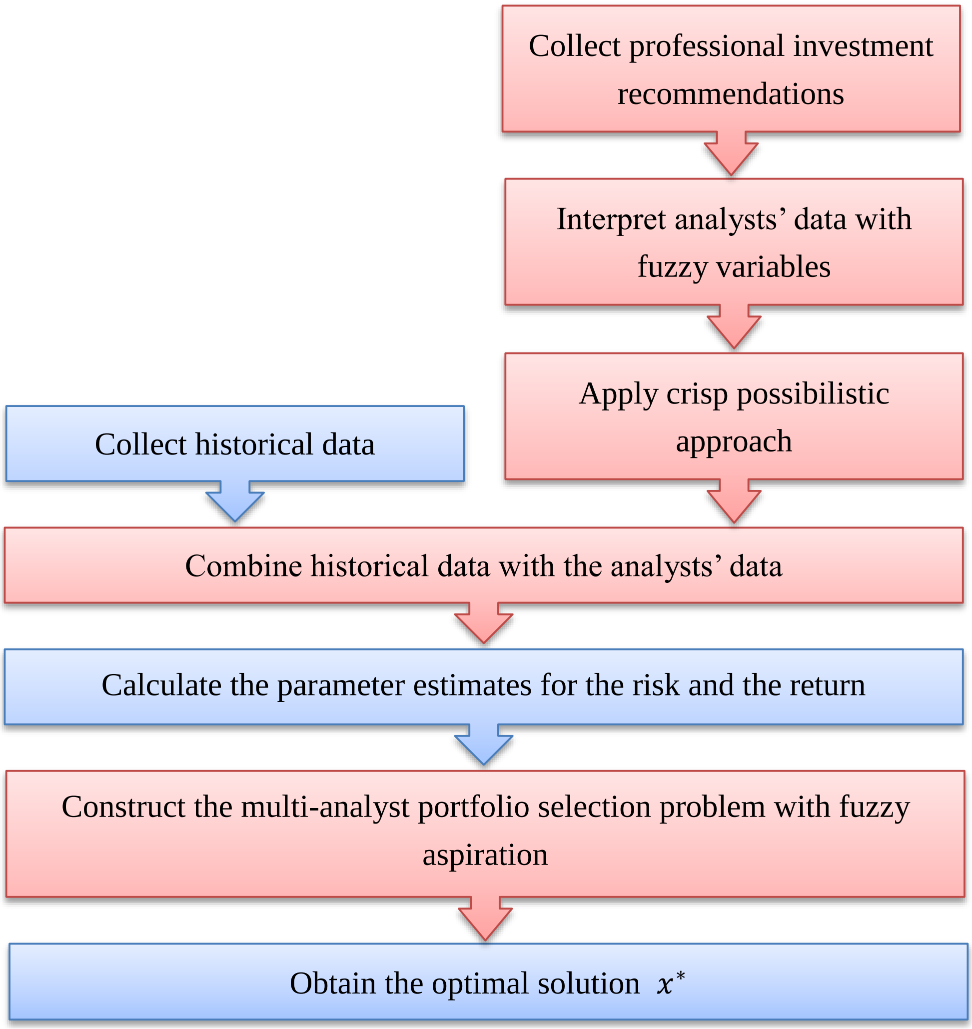
Compared to the multi-expert approach of Lutgens and Schotman (2010), where the expert input data are obtained from various return models, the multi-analyst approach $(\boldsymbol{F_{MV}})$ proposed in this chapter has been developed for adopting real investment forecasts from different financial analysts as input data. There is an obvious difference between the return models and the investment recommendations. In reality, the investment recommendations are vaguely expressed opinions or commentaries about future performance forecasts of assets, and on the other hand, the return models generate numerical estimates. Unlike Lutgens and Schotman (2010), we take the characteristics of investment forecasts into account and employ fuzzy set theory to deal with the analysts' vague recommendations. Gupta et al. (2008) proposed the multiple criteria approach via fuzzy programming. The portfolio selection problem is formulated as a multiple criteria optimisation problem and different vague investment goals are considered for each individual criterion. Although the portfolio selection framework of Gupta et al. (2008) has inspired the development of the multi-analyst approach $(\boldsymbol{F_{MV}})$ in this chapter, these two approaches differ substantially in terms of the information source. More specifically, Gupta et al. (2008) use historical asset performances as the only resource to generate estimates for input parameters. By contrast, we adopt investment recommendations from multiple professional analysts. That is, for every asset, their model only considers one estimate for each type of parameter. On the other hand, the multi-analyst approach $(\boldsymbol{F_{MV}})$ takes into account multiple estimates provided by different financial analysts for each type of parameter.

Finally, in order to illustrate the impact on employing the credibility coefficient θ for describing the confidence of the investor in different analysts, we have investigated the proposed approach $(\boldsymbol{F_{MV}})$ using an example where the investor receives suggestions from two analysts and only needs to allocate funds between a risky asset and a risk free asset.

Although the multi-analyst fuzzy approach $(\boldsymbol{F_{MV}})$ is able to incorporate multiple information sources into the optimisation model, it does not take into account estimation errors and parameter uncertainties when historical data are used. In the following chapter, we will extend the multi-analyst approach with fuzzy aspiration $(\boldsymbol{F_{MV}})$ by adopting the concept of the robust counterpart approach of Ben-Tal and Nemirovski (1998).

At the end of this chapter we summarise, in Figure 3.6, the basic steps for solving the multi-analyst portfolio selection problem with fuzzy aspiration $(\boldsymbol{F_{MV}})$.

Figure 3.6 The Diagram of Solving the Multiple Analysts Approach with Fuzzy Aspiration



Chapter 4

Robust Counterpart to Multi-Analyst Portfolio Selection Approach

The phrase "robust optimisation of portfolio selection problems" has been mentioned frequently in the literature over the last few decades. Broadly speaking, the idea is to provide an optimal asset allocation that performs well even if the worst possible scenario turns out to be true. This can be achieved via several different approaches, for instance, the model robust approach or the estimation robust approach. The former refers to the technique that incorporates multiple structured return models into the optimisation framework and selects portfolios with the most conservative perception, such as the multi-analyst approach $(\boldsymbol{F_{MV}})$ proposed in Chapter 3 and the multi-expert approach of Lutgens and Schotman (2010). The latter refers to an area of optimisation that explicitly takes estimation errors and parameter uncertainties into consideration and the most commonly used method is the robust counterpart approach of Ben-Tal and Nemirovski (1998), which has been mentioned earlier, in Section 2.2.

In this chapter we will extend the multi-analyst approach with fuzzy aspiration $(\boldsymbol{F_{MV}})$ by introducing the robust counterpart approach to deal with the uncertainties of the input parameters. Lutgens and Schotman (2006) proposed the model and estimation robust approach based on a joint uncertainty set for describing the parameter uncertainties, i.e., the uncertainties of parameter estimation for different assets are assumed to be identical to each other and estimated jointly for all assets. However, in reality, it is very unlikely that the estimation uncertainties of different assets will be the same across different assets. More importantly, in Lutgens and Schotman (2006), it is implicitly assumed that experts will provide forecasts for every individual asset. This is clearly not realistic; instead of providing forecasts for every single asset, the professional analysts or financial experts usually only make recommendations on a few assets (see Figure 1.1 for further details). Consequently, for those assets without any forecasts by the financial analysts, the investor has to rely on the historical data for portfolio selection and, in this case, sampling errors have to be taken into consideration. In order to deal with this issue, we propose to pool the robust counterpart approach with the multi-analyst approach $(\boldsymbol{F_{MV}})$ proposed in Chapter 3, where the assets are categorised into different subsets so that the estimation uncertainties of different subclasses of assets are characterised separately during portfolio selection. The proposed robust multi-analyst approach with separate uncertainty sets is the first portfolio selection approach that combines the advantages of the model robust approach (Lutgens & Schotman, 2010) and the estimation robust approach (Ben-Tal & Nemirovski, 1998) and that at the same time allows the user to consider parameter uncertainties of various datasets differently.

To start with, the robust multi-analyst approach under joint uncertainty set will be presented as an initial framework. Then we will develop the robust multi-analyst approach with multiple uncertainty sets. Finally, we will compare the multi-analyst approach with the robust multi-analyst approach to illustrate the impact of robustification.

4.1 Extension of the Multiple Analysts Approach

In Chapter 3, we developed the multi-analyst approach for portfolio selection problem with fuzzy aspiration $(\boldsymbol{F_{MV}})$ by extending the existing multi-expert approach in Lutgens and Schotman (2010). We have also illustrated that there is a strong dependence of the optimal solution x^* on the ratio of the respective credibility levels of analysts prescribed by the investor. However, it can be problematic to use analysts' recommendations directly without considering the uncertainties of the estimations, especially given that the financial analysts only make suggestions on a few assets. In this case, the investor has to use historical asset performances for those without further information provided by the analysts. Therefore, modification of the multi-analyst portfolio selection approach $(\boldsymbol{F_{MV}})$ for taking estimation uncertainties into account is indeed necessary for further improvements in constructing robust optimal portfolios.

Lutgens and Schotman (2006) proposed the model and estimation robust approach, which combines the multi-expert approach with the robust counterpart approach, based on a joint uncertainty set. In their framework, the professional forecasts provided by each expert are described through an uncertainty set, and each uncertainty set U_z represents the possible values of the asset returns according to the expert z's belief. Thus the model and estimation robust approach of the portfolio selection problem is formulated as

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu\in U_z} \quad f_z(\mu_z, x) \quad , \tag{4.1}$$

where $f_z(\mu_z, x) = {\mu_z}^T x - \frac{\lambda}{2} x^T \Sigma_z x$ for $z \in Z$ with $\mu_z \in U_z$ and Σ_z denoting the estimates of the expected returns and covariance matrix provided by expert z. Following the optimisation framework (4.1), we will propose the robust counterpart formulation of the multi-analyst approach $(\boldsymbol{F_{MV}})$ in the subsequent section.

4.2 The Robust Counterpart to Multi-Analyst Approach

Before going into more details of the development of the robust multi-analyst approach with fuzzy aspiration, we note that, from (4.1), the associated robust counterpart formulation of the multi-analyst portfolio selection problem $(\boldsymbol{F_{MV}})$ is given by

$$(\boldsymbol{RE_{MV}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_z\in U_z} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)}, \quad (4.2)$$

where $x \in \mathbb{R}^n$ is the decision vector, $\mu_z \in U_z$ is the parameter of the expected returns expressed via an uncertainty set U_z which is constructed according to the analyst z, $\check{\Sigma}_z$ is the crisp possibilistic covariance matrix according to recommendations provided by analyst z, and R^{Target} is the benchmark of the investment required by the investor.

As mentioned in the previous chapters, there is no universal format for the uncertainty set U_z and the uncertainty set U_z is formulated according to the requirements of the user. A general guideline for constructing the uncertainty set U_z is to set up the uncertainty set U_z to be centred at the estimate of the expected input value and then use the desired robustness of the optimisation problem to define the size of the uncertainty set U_z. Although the fluctuations in the input parameter, i.e., the vector of expected returns or the covariance matrix, are considered to be one of the reasons why the optimal portfolio performed disappointingly, the practitioners and researchers pay more attention to the uncertainties in the expected returns, because the covariance matrix is not as unstable as the expected returns and, in addition, the fluctuations in the covariance matrix do not influence the optimal solution crucially (Best & Grauer, 1991; Chopra & Ziemba, 1993; Michaud, 1998; Schöttle & Werner, 2009; Ziemba; 2009). Therefore it is common to define an uncertainty set only for the vector of the expected returns.

In the rest of this section, we will present two methods for describing estimation uncertainties in the expected returns and investigate their implications for the robust multi-analyst approach.

4.2.1 Uncertainty Set for All Assets

To define explicitly the estimation uncertainties of the expected returns for the portfolio selection problem, the conventional method is to estimate the uncertainties of the expected returns either individually for each asset or jointly for all assets. The former uses a confidence interval as the uncertainty set for each asset to describe the estimation uncertainty of the expected return and the latter uses a confidence ellipsoid or box around the vector of the expected returns as the uncertainty set to describe the estimation uncertainties of all assets[6]. It is more likely and also more realistic that the investor has a general confidence in the estimate of the expected returns for a certain group of assets or for the entire set of assets. Therefore we do not pursue the research line where the uncertainties of the expected returns are estimated individually.

4.2.1.1 Optimising the Portfolio Selection Problem $(\boldsymbol{RE_{MV}})$ via a Box Uncertainty Set

Consider the portfolio selection problem $(\boldsymbol{RE_{MV}})$ and assume the estimated uncertainties of the expected returns are expressed via a box uncertainty set U_z^{Box}, i.e.,

$$U_z^{Box}(\check{\mu}_z) = \left\{ \mu \in \mathbb{R}^n \ \middle| \left|\mu_i - \check{\mu}_{z_i}\right| \leq \delta_z \right\} \quad , \tag{4.3}$$

where $U_z^{Box} \subset \mathbb{R}^n$ is an non-empty, convex and compact uncertainty set formulated according to analyst z's forecasts, $\check{\mu}_z$ is the crisp possibilistic interpretation of fuzzy expected return, and $\delta_z \geq 0$ is the desired robustness level for the uncertainty set U_z^{Box} given by the investor. Following equation (2.22), the robust counterpart approach of the multi-analyst portfolio selection problem with box uncertainty set U_z^{Box} is then formulated as

[6] For further details of uncertainty set, readers should refer to Sections 2.2.2 and 2.2.3.

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_z\in U_z} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x-\frac{\lambda}{2}x^T\check{\Sigma}_z x-R^{Target}\right)\right)}$$
$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left((\check{\mu}_z-\delta_z\mathbf{1})^T x-\frac{\lambda}{2}x^T\check{\Sigma}_z x-R^{Target}\right)\right)} \quad , \qquad (4.4)$$

where $\check{\mu}_z-\delta_z\mathbf{1}$ explicitly denotes the worst case scenario of the asset returns, with $\mathbf{1}$ denoting the vector of ones. Note that formulation (4.4) of the portfolio selection problem $(\boldsymbol{RE_{MV}})$ can be transformed easily for the investor who wishes to specify the uncertainty about each expected return individually. Instead of prescribing a constant $\delta_z \geq 0$ as the desired level of robustness, the investor can use a vector $\boldsymbol{d}_z = \left(\boldsymbol{d}_{z_1},\dots,\boldsymbol{d}_{z_n}\right)^T$ to describe the robustness for the uncertainty set U_z^{Box}, i.e.,

$$U_z^{Box}(\check{\mu}_z) = \left\{\mu\in\mathbb{R}^n \,\middle|\, |\mu-\check{\mu}_z| \leq \boldsymbol{d}_z, \boldsymbol{d}_z = \left(\boldsymbol{d}_{z_1},\dots,\boldsymbol{d}_{z_n}\right)^T\right\} \ , \qquad (4.5)$$

with $\boldsymbol{d}_{z_i} \geq 0$ denoting the individual robustness level for the i^{th} asset where

$$\check{\mu}_{z_i}-\boldsymbol{d}_{z_i} < \mu_i < \check{\mu}_{z_i}+\boldsymbol{d}_{z_i}$$

representing the interval description of the expected return of asset i, $i = 1,2,\dots,n$.

4.2.1.2 Optimising the Portfolio Selection Problem $(\boldsymbol{RE_{MV}})$ via an Ellipsoid Uncertainty Set

Consider the portfolio selection problem $(\boldsymbol{RE_{MV}})$ and let the uncertainty set U_z for the parameter μ be given by a confidence ellipsoid, which is constructed according to the recommendations provided by the analyst z, i.e.,

$$\begin{aligned} U_z^{Ellipsoid}(\check{\mu}_z) &= \left\{\mu\in\mathbb{R}^n \,\middle|\, (\mu-\check{\mu}_z)^T{\check{\Sigma}_z}^{-1}(\mu-\check{\mu}_z) \leq \delta_z^2\right\} \\ &= \left\{\mu\in\mathbb{R}^n \,\middle|\, \mu = \check{\mu}_z+\delta_z\check{\Sigma}_z^{\frac{1}{2}}\psi \,, \|\psi\| \leq 1 \right\} \ , \end{aligned} \qquad (4.6)$$

where $U_z^{Ellipsoid} \subset \mathbb{R}^n$ is non-empty, convex and compact. $\check{\mu}_z$ and $\check{\Sigma}_z$ are the crisp possibilistic interpretations of the expected returns and covariance matrix, respectively.

$\delta_z \geq 0$ is the desired robustness level for the uncertainty set $U_z^{Ellipsoid}$ given by the investor. By following (4.6), the robust counterpart approach of the multi-analyst portfolio selection problem with ellipsoid uncertainty set $U_z^{Ellipsoid}$ can be formulated as

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_z\in U_z} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\|\psi\|\leq 1} \frac{1}{1+exp\left(-\theta_z\left(\left(\check{\mu}_z + \delta_z \check{\Sigma}_z^{\frac{1}{2}}\psi\right)^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\mu}_z}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target} + \delta_z \min_{\|\psi\|\leq 1} \psi^T\hat{\Sigma}^{\frac{1}{2}}x\right)\right)}. \tag{4.7}$$

Since the product $\psi^T\hat{\Sigma}^{\frac{1}{2}}x$ is minimised at $\psi^* = -\frac{\check{\Sigma}_z^{1/2}x}{\left\|\check{\Sigma}_z^{1/2}x\right\|}$, it follows immediately that the portfolio optimisation problem $(\boldsymbol{RE_{MV}})$ with the ellipsoid uncertainty set becomes

$$\left(\boldsymbol{RE_{MV}^{Ellipsoid}}\right) \quad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\mu}_z}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target} - \delta_z\left\|\check{\Sigma}_z^{\frac{1}{2}}x\right\|\right)\right)}, \tag{4.8}$$

with $\check{\mu}_z - \delta_z \frac{\check{\Sigma}_z x}{\left\|\check{\Sigma}_z^{1/2}x\right\|}$ explicitly denoting the worst case scenario of the expected returns.

4.2.2 Uncertainty Set for Subsets of Assets

In the setting of the robust multi-analyst approach $(\boldsymbol{RE_{MV}})$, it is assumed that the investor receives investment information from different professionals and decides the portfolio selection based on the collected information. In practice, however, financial analysts do not make suggestions on every asset. Instead, they provide forecasts on a few assets, usually less than 4% of the entire market (see Figure 1.1 for further details). In view of that, we assume that the analysts only make recommendations for the assets if they disagree with the historical performance of the assets. Moreover, for the assets without further information provided by the analysts, the historical asset performances are adopted to obtain parameter estimates for the modelling. Hence we have two types of input data, the historical data and the analysts' data, for every uncertainty set U_z, which should not be considered equally, nor applied with the same desired level of robustness. Therefore instead of stating one uncertainty set U_z for all assets, two non-overlapping subsets of assets are considered in order to distinguish the difference between the historical dataset and the analysts' dataset.

Specifically, suppose the analyst z only recommends on m assets, $0 \leq m \leq n$. Let the decision vector $x \in \mathbb{R}^n$ be partitioned into $x = ({x_H}^T, {x_P}^T)^T$ with $x_H \in \mathbb{R}^{n-m}$ denoting the column vector of weights in the assets of the historical dataset and $x_P \in \mathbb{R}^m$ denoting the column vector of weights in the assets of the analysts' dataset. Let us denote the expected returns μ_z and the covariance matrix Σ_z by

$$\mu_z = \begin{pmatrix} \mu_{H_z} \\ \mu_{P_z} \end{pmatrix} \qquad \text{and} \qquad \Sigma_z = \begin{pmatrix} \Sigma_{HH_z} & \Sigma_{HP_z} \\ \Sigma_{PH_z} & \Sigma_{PP_z} \end{pmatrix} , \tag{4.9}$$

where μ_{H_z} and μ_{P_z} represent the expected returns according to the historical asset performances and the analyst z's suggestions, respectively. Similarly, Σ_{HH_z} and Σ_{PP_z} are the covariance matrix obtained from the historical data and the analyst z's data. Therefore, instead of solving the portfolio selection problem $(\boldsymbol{RE_{MV}})$, we formulate the portfolio selection problem based on the non-overlapping method proposed by Garlappi et al. (2007). That is,

$$(\boldsymbol{RE_{HP}}) \quad \begin{array}{ll} \max\limits_{x\in\mathbb{R}^n} \min\limits_{z\in Z} \min\limits_{\mu_{H_z},\mu_{P_z}} & \dfrac{1}{1 + exp\left(-\theta_z\left({\mu_z}^T x - \dfrac{\lambda}{2} x^T \check{\Sigma}_z x - R^{Target}\right)\right)} \\ s.t. & \mu_{H_z} \in U_{H_z} \\ & \mu_{P_z} \in U_{P_z} \end{array} , \tag{4.10}$$

with U_{H_Z} and U_{P_Z} denoting the uncertainty sets for the historical dataset and the analyst's dataset, respectively. $\check{\acute{\Sigma}}_Z$ is the covariance matrix obtained from a chosen statistical estimate of variance $\acute{\sigma}_i{}^2$ of the historical dataset and the crisp possibilistic variance $\breve{\sigma}_{Z_i}^2$ of the analyst's dataset based on the historical correlation coefficient matrix $Corr(\mu)$, i.e.,

$$\check{\acute{\Sigma}}_Z = \begin{pmatrix} \acute{\Sigma}_{HH_Z} & \check{\acute{\Sigma}}_{HP_Z} \\ \check{\acute{\Sigma}}_{PH_Z} & \breve{\Sigma}_{PP_Z} \end{pmatrix} \quad , \tag{4.11}$$

where $\acute{\Sigma}_{HH_Z}$ is the covariance matrix for the historical dataset, $\breve{\Sigma}_{PP_Z}$ is the crisp possibilistic covariance matrix for the analysts' dataset according to equation (3.14), and the matrix $\check{\acute{\Sigma}}_{PH_Z} = \check{\acute{\Sigma}}_{HP_Z}$ is the combination of the historical data and the analysts' data, i.e.,

$$\check{\acute{\Sigma}}_{PH_Z} = \begin{pmatrix} \rho_{1(n-m+1)}\acute{\sigma}_{Z_1}\breve{\sigma}_{Z_{(n-m+1)}} & \cdots & \rho_{(n-m)(n-m+1)}\acute{\sigma}_{Z_{(n-m)}}\breve{\sigma}_{Z_{(n-m+1)}} \\ \vdots & \ddots & \vdots \\ \rho_{1n}\acute{\sigma}_{Z_1}\breve{\sigma}_{Z_n} & \cdots & \rho_{(n-m)n}\acute{\sigma}_{Z_{(n-m)}}\breve{\sigma}_{Z_n} \end{pmatrix} = \check{\acute{\Sigma}}_{HP_Z}{}^T, \tag{4.12}$$

where ρ_{ij} denotes the historical correlation coefficient between asset i and asset j for $i, j = 1, \dots, n$ and $i \neq j$. After setting up the robust counterpart to multi-analyst portfolio selection approach, the assets are divided into two separate subsets and the parameter estimates of assets are generated differently for each subset. We now consider the implementation of the multi-analyst portfolio selection approach $(\boldsymbol{RE_{HP}})$ with different uncertainty sets, i.e., the box uncertainty set and the ellipsoid uncertainty set.

4.2.2.1 Optimising the Portfolio Selection Problem $(\boldsymbol{RE_{HP}})$ via Box Uncertainty Sets

Suppose that the investor chooses the box uncertainty set for defining the estimated uncertainties for the portfolio selection problem $(\boldsymbol{RE_{HP}})$

$$U_{H_z}^{Box}(\acute{\mu}_{H_z}) = \left\{ \mu \in \mathbb{R}^{n-m} \; \middle\| \left|\mu_i - \acute{\mu}_{H_{z_i}}\right| \leq \delta_{H_z}, i = 1, \ldots, n-m \right\} \quad , \tag{4.13}$$

and

$$U_{P_z}^{Box}(\check{\mu}_{P_z}) = \left\{ \mu \in \mathbb{R}^{m} \; \middle\| \left|\mu_i - \check{\mu}_{P_{z_i}}\right| \leq \delta_{P_z}, i = n-m+1, \ldots, n \right\} \quad , \tag{4.14}$$

where $U_{H_z}^{Box}$ is the box historical uncertainty set with $\acute{\mu}_{H_{z_i}}$ denoting a statistical estimate of the expected return for asset i and δ_{H_z} denotes the desired robustness level for the historical dataset. $U_{P_z}^{Box}$ is referred to as the box uncertainty set for the expected returns of the analysts' dataset with $\check{\mu}_{P_{z_i}}$ denoting the crisp possibilistic return of the i^{th} asset and δ_{P_z} denoting the desired robustness level. Thus the robust multi-analyst portfolio selection approach with the box uncertainty sets $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ is formulated as

$$\left(\boldsymbol{RE_{HP}^{Box}}\right) \quad \begin{array}{ll} \max\limits_{x \in \mathbb{R}^n} \min\limits_{z \in Z} \min\limits_{\mu_{H_z}, \mu_{P_z}} & \dfrac{1}{1 + exp\left(-\theta_z \left({\mu_z}^T x - \frac{\lambda}{2} x^T \check{\Sigma}_z x - R^{Target}\right)\right)} \\ s.t. & \left|\mu_{H_Z} - \acute{\mu}_{H_z}\right| \leq \delta_{H_z}\mathbf{1} \\ & \left|\mu_{P_Z} - \check{\mu}_{P_z}\right| \leq \delta_{P_z}\mathbf{1} \end{array} \tag{4.15}$$

with $\mu_z = \begin{pmatrix} \mu_{H_Z} \\ \mu_{P_Z} \end{pmatrix}$.

Note that the desired robustness level δ of an uncertainty set is adopted to reflect the investor's aversion to estimation risk. As mentioned earlier, the return estimates generated from the analysts' forecasts are the crisp possibilistic interpretation of the fuzzy recommendations, which accounts for the uncertain and imprecise characteristics of the parameters. On the other hand, although the investor does not have support regarding the reliability of recommendations, it is assumed that the investor takes all advice given by the financial analysts into consideration in order to avoid disappointment that a particular recommendation is actually true; therefore, we assume the investor sets the desired robustness level for the uncertainty set of the analysts' dataset equal to zero, i.e., $\delta_{P_z} = 0$. Under this assumption, the robust multi-analyst portfolio selection approach with the box uncertainty sets $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ becomes

$$\left(\boldsymbol{RE_{HP}^{Box}}\right) \qquad \begin{array}{ll} \max\limits_{x\in\mathbb{R}^n}\ \min\limits_{z\in Z} & \dfrac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x-\dfrac{\lambda}{2}x^T\check{\acute{\Sigma}}_z x-R^{Target}\right)\right)} \\ s.t. & \mu_z = \begin{pmatrix} \acute{\mu}_{H_z} - \delta_{H_z}\mathbf{1} \\ \check{\mu}_{P_z} \end{pmatrix} \\ & \check{\acute{\Sigma}}_z = \begin{pmatrix} \acute{\Sigma}_{HH_z} & \check{\acute{\Sigma}}_{HP_z} \\ \check{\acute{\Sigma}}_{PH_z} & \check{\Sigma}_{PP_z} \end{pmatrix} \end{array}, \quad (4.16)$$

where $\acute{\mu}_{H_z} - \delta_{H_z}\mathbf{1} \in \mathbb{R}^{n-m}$ is the worst case scenario of the asset returns for the historical dataset, $\check{\mu}_{P_z} \in \mathbb{R}^m$ is the crisp possibilistic returns for the analysts' dataset according to analyst z's forecasts and $\check{\acute{\Sigma}}_z$ is the covariance matrix obtained from both historical and the analyst z's data. By denoting $\check{\acute{\mu}}_z = \begin{pmatrix} \acute{\mu}_{H_z} \\ \check{\mu}_{P_z} \end{pmatrix}$, the robust multi-analyst approach with the box uncertainty set $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ can be rearranged as

$$\left(\boldsymbol{RE_{HP}^{Box}}\right) \quad \max_{x\in\mathbb{R}^n}\ \min_{z\in Z} \quad \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x-\frac{\lambda}{2}x^T\check{\acute{\Sigma}}_z x-R^{Target}-\delta_{H_z}\mathbf{1}^T x_H\right)\right)} \qquad (4.17)$$

with $x_H \in \mathbb{R}^{n-m}$ denoting the weighting of the assets of the historical dataset.

Note that compared to the multi-analyst approach of the portfolio selection problem $(\boldsymbol{F_{MV}})$, the term $\delta_{H_z}\mathbf{1}^T x_H$ of the robust multi-analyst approach $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ is an additional term which can be interpreted as the penalty for investing in assets of the historical dataset. More explicitly, this penalty term of the robust multi-analyst approach $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ is a scalar product of the desired robustness of estimation and the weighting of assets for the historical dataset, therefore, it only penalised the investment in assets from the historical dataset. However, the penalty term here in the robust multi-analyst approach with the box uncertainty set $\left(\boldsymbol{RE_{HP}^{Box}}\right)$ penalises every asset from the historical dataset equally, without considering the historical performance or behaviour of the assets.

4.2.2.2 Optimising the Portfolio Selection Problem $(\boldsymbol{RE_{HP}})$ via Ellipsoid Uncertainty Sets

Suppose now that the investor uses the ellipsoid uncertainty sets for describing the estimated uncertainties for the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ as follows

$$\begin{aligned} U_{H_z}^{Ellipsoid}(\acute{\mu}_{H_z}) &= \left\{ \mu \in \mathbb{R}^{n-m} \;\middle|\; (\mu - \acute{\mu}_{H_z})^T {\acute{\Sigma}_{HH_z}}^{-1}(\mu - \acute{\mu}_{H_z}) \le \delta_{H_z}^2 \right\} \\ &= \left\{ \mu \in \mathbb{R}^{n-m} \middle| \mu = \acute{\mu}_{H_z} + \delta_{H_z}\acute{\Sigma}_{HH_z}^{\frac{1}{2}}\psi_{H_z}, \|\psi_{H_z}\| \le 1 \right\}, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} U_{P_z}^{Ellipsoid}(\check{\mu}_{P_z}) &= \left\{ \mu \in \mathbb{R}^{m} \;\middle|\; (\mu - \check{\mu}_{P_z})^T {\check{\Sigma}_{PP_z}}^{-1}(\mu - \check{\mu}_{P_z}) \le \delta_{P_z}^2 \right\} \\ &= \left\{ \mu \in \mathbb{R}^{m} \middle| \mu = \check{\mu}_{P_z} + \delta_{P_z}\check{\Sigma}_{PP_z}^{\frac{1}{2}}\psi_{P_z}, \|\psi_{P_z}\| \le 1 \right\}, \end{aligned} \tag{4.19}$$

where $U_{H_z}^{Ellipsoid}$ is the confidence ellipsoid of the historical dataset centred on a statistical estimate of the expected returns $\acute{\mu}_{H_z}$, and $U_{P_z}^{Ellipsoid}$ is referred to as the confidence ellipsoid for the analysts' dataset centred on the crisp possibilistic asset returns $\check{\mu}_{P_z}$. Consequently, the robust multi-analyst approach with the ellipsoid uncertainty sets $(\boldsymbol{RE_{HP}})$ is given as

$$\left(\boldsymbol{RE_{HP}^{Ellipsoid}}\right) \quad \begin{aligned} &\max_{x\in\mathbb{R}^n}\ \min_{z\in Z}\ \min_{\mu_{H_z},\mu_{P_z}} && \frac{1}{1 + exp\left(-\theta_z\left({\mu_z}^T x - \frac{\lambda}{2}x^T\check{\acute{\Sigma}}_z x - R^{Target}\right)\right)} \\ &\quad s.t. && (\mu_{H_z} - \acute{\mu}_{H_z})^T {\acute{\Sigma}_{HH_z}}^{-1}(\mu_{H_z} - \acute{\mu}_{H_z}) \le \delta_{H_z}^2 \\ & && (\mu_{P_z} - \check{\mu}_{P_z})^T {\check{\Sigma}_{PP_z}}^{-1}(\mu_{P_z} - \check{\mu}_{P_z}) \le \delta_{P_z}^2 \end{aligned} . \tag{4.20}$$

By following the assumption that the desired robustness level for the analysts' dataset equals zero, i.e., $\delta_{P_z} = 0$, the robust multi-analyst approach $\left(\boldsymbol{RE_{HP}^{Ellipsoid}}\right)$ reduces to

$$(\boldsymbol{RE_{HP}^{Ellipsoid}}) \quad \begin{array}{ll} \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_{H_z},\mu_{P_z}} & \dfrac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \dfrac{\lambda}{2} x^T \check{\acute{\Sigma}}_z x - R^{Target}\right)\right)}, \\ s.t. & \left(\mu_{H_z} - \acute{\mu}_{H_z}\right)^T {\acute{\Sigma}_{HH_z}}^{-1}\left(\mu_{H_z} - \acute{\mu}_{H_z}\right) \le \delta_{H_z}^2 \\ & \mu_{P_z} = \check{\mu}_{P_z} \end{array} \tag{4.21}$$

where the mean μ_z can be expressed as

$$\mu_z = \begin{pmatrix} \mu_{H_z} \\ \mu_{P_z} \end{pmatrix} = \begin{pmatrix} \acute{\mu}_{H_z} + \delta_{H_z} \acute{\Sigma}_{HH_z}^{\frac{1}{2}} \psi_{H_z} \\ \check{\mu}_{P_z} \end{pmatrix} \quad , \tag{4.22}$$

with $\|\psi_{H_z}\| \le 1$. Substituting equation (4.22) into (4.21), the formulation (4.21) of the framework can be transformed to

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_{H_z},\mu_{P_z}} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \frac{\lambda}{2} x^T \check{\acute{\Sigma}}_z x - R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\|\psi_{H_z}\|\le 1} \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x + (\delta_{H_z}\acute{\Sigma}_{HH_z}^{\frac{1}{2}}\psi_{H_z})^T x_H - \frac{\lambda}{2} x^T \check{\acute{\Sigma}}_z x - R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x - \frac{\lambda}{2} x^T \check{\acute{\Sigma}}_z x - R^{Target} + \delta_{H_z} \min_{\|\psi_{H_z}\|\le 1} (\acute{\Sigma}_{HH_z}^{\frac{1}{2}}\psi_{H_z})^T x_H\right)\right)} \tag{4.23}$$

where $\check{\acute{\mu}}_z = \begin{pmatrix} \acute{\mu}_{H_z} \\ \check{\mu}_{P_z} \end{pmatrix}$. Note that, the value of the scalar product $(\acute{\Sigma}_{HH_z}^{\frac{1}{2}}\psi_{H_z})^T x_H$ is minimised when ${\psi_{H_j}}^* = -\frac{{\acute{\Sigma}_{HH_z}}^{1/2} x_H}{\left\|{\acute{\Sigma}_{HH_z}}^{1/2} x_H\right\|}$. Consequently, we have

$$(\boldsymbol{RE_{HP}^{Ellipsoid}}) \quad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x - \frac{\lambda}{2} x^T \check{\acute{\Sigma}}_z x - R^{Target} - \delta_{H_z}\left\|\acute{\Sigma}_{HH_z}^{\frac{1}{2}} x_H\right\|\right)\right)}, \tag{4.24}$$

with $\delta_{H_z}\left(\acute{\Sigma}_{HH_z}^{\frac{1}{2}}\frac{\acute{\Sigma}_{HH_z}{}^{1/2}x_H}{\left\|\acute{\Sigma}_{HH_z}{}^{1/2}x_H\right\|}\right)^T x_H = \delta_{H_z}\frac{\left(\acute{\Sigma}_{HH_z}{}^{1/2}x_H\right)^2}{\left\|\acute{\Sigma}_{HH_z}{}^{1/2}x_H\right\|} = \delta_{H_z}\left\|\acute{\Sigma}_{HH_z}^{\frac{1}{2}}x_H\right\|.$

Similar to the robust multi-analyst approach with the box uncertainty set $(\boldsymbol{RE_{HP}^{Box}})$, the robust multi-analyst approach with the ellipsoid uncertainty set $(\boldsymbol{RE_{HP}^{Ellipsoid}})$ also has a penalty term, the scalar product $\delta_{H_z}\left\|\acute{\Sigma}_{HH_z}^{\frac{1}{2}}x_H\right\|$, compared with the multi-analyst approach $(\boldsymbol{F_{MV}})$. Unlike the robust multi-analyst approach $(\boldsymbol{RE_{HP}^{Box}})$, however, this penalty term of the robust multi-analyst approach $(\boldsymbol{RE_{HP}^{Ellipsoid}})$ contains the weighting and also the variance of assets which belongs to the historical dataset. In other words, the robust multi-analyst approach with the ellipsoid uncertainty set $(\boldsymbol{RE_{HP}^{Ellipsoid}})$ imposes higher penalties on the historical dataset, especially on those assets with large fluctuations in returns. This effect of the proposed robust multi-analyst approach with the ellipsoid uncertainty set $(\boldsymbol{RE_{HP}^{Ellipsoid}})$ increases the impact of the risk from the historical dataset on portfolio selection, which leads to a less risky portfolio selection for its user.

In the following, we first illustrate the effect of applying the uncertainty set for handling estimation uncertainties by comparing the portfolio selection frameworks of the multi-analyst approach and the robust multi-analyst approach with the joint uncertainty set. Then we show the impact of employing separate uncertainty sets for the robust multi-analyst portfolio selection problem by comparing the resulting robust portfolios obtained from the joint uncertainty set or the separate uncertainty sets.

4.3 Comparison of Multi-Analyst Approaches

Two examples are given in this subsection to help understand the robust effect of the proposed robust counterparts to the multi-analyst approach. Instead of considering the mean-variance portfolio selection framework as given in equation (2.1), we use the mean-standard deviation framework as Schöttle and Werner (2009) did to simplify the example:

$$(\boldsymbol{P_{MS}}) \qquad \max_{x\in\mathbb{R}^n} \quad \mu^T x - \frac{\lambda}{2}\sqrt{x^T \Sigma x} \quad . \tag{4.25}$$

The robust multi-analyst approach with the joint uncertainty set $(\boldsymbol{RE_{MS}})$ will be investigated first, followed by an examination of the robust multi-analyst approach with the uncertainty set based only on the historical dataset $(\boldsymbol{RE_{HP}})$. As already mentioned, the financial analysts usually only select a small proportion of assets and comment on their future performances. Hence, it is assumed that the historical data is adopted to work out the expected returns and variances for the assets which have no recommendations from the analysts. For notational ease in the subsequent results or explanations, we use $\check{\acute{\mu}}_z = \begin{pmatrix} \acute{\mu}_{H_z} \\ \check{\mu}_{P_z} \end{pmatrix}$ and $\check{\acute{\Sigma}}_z = \begin{pmatrix} \acute{\Sigma}_{HH_Z} & \check{\acute{\Sigma}}_{HP_Z} \\ \check{\acute{\Sigma}}_{PH_Z} & \check{\Sigma}_{PP_Z} \end{pmatrix}$ to denote the parameter estimates of the expected returns and covariance matrix provided by the z^{th} analyst. The ellipsoid uncertainty set has nicer properties in terms of continuity and contains more asset information than the box uncertainty set. Therefore we will describe the uncertainty set $\boldsymbol{U}$ of the robust counterpart to the multi-analyst approach in the shape of the ellipsoid, unless explicitly stated otherwise.

4.3.1 Comparison between the Multi-Analyst and the Robust Multi-Analyst Approaches

Consider the multi-analyst approach $(\boldsymbol{F_{MS}})$ and the robust multi-analyst approach with the joint uncertainty set $(\boldsymbol{RE_{MS}})$ for solving the mean-standard deviation portfolio selection problem:

$$(\boldsymbol{F_{MS}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \quad \frac{1}{1+\exp\left(-\theta_z\left(\check{\acute{\mu}}_z{}^T x - \frac{\lambda}{2}\sqrt{x^T \check{\acute{\Sigma}}_z x} - R^{Target}\right)\right)} \ ,$$

and

$$(\boldsymbol{RE_{MS}}) \quad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_z \in U_z} \frac{1}{1+exp\left(-\theta_z\left(\mu_z{}^T x - \frac{\kappa}{2}\sqrt{x^T \check{\Sigma}_z x} - R^{Target}\right)\right)} .$$

Let the uncertainty set for the return vector μ_z be given by a confidence ellipsoid as defined in equation (4.6), and denote $\lambda \in [0,\infty)$ and $\kappa \in [0,\infty)$ as the risk aversion coefficients for the portfolio selection problem $(\boldsymbol{F_{MS}})$ and $(\boldsymbol{RE_{MS}})$, respectively. As shown earlier, by using the formulation (4.6) for the ellipsoid uncertainty set, the worst case scenario of the asset returns can be obtained and expressed as $\check{\mu}_z - \delta_z \frac{\check{\Sigma}_z x}{\left\|\check{\Sigma}_z^{1/2} x\right\|}$. By substituting the parameter $\mu_z = \check{\mu}_z - \delta_z \frac{\check{\Sigma}_z x}{\left\|\check{\Sigma}_z^{1/2} x\right\|}$ into $(\boldsymbol{RE_{MS}})$, we have

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T \check{\Sigma}_z x} - R^{Target} - \delta_z\left(\frac{\check{\Sigma}_z x}{\left\|\check{\Sigma}_z^{1/2} x\right\|}\right)^T x\right)\right)}$$

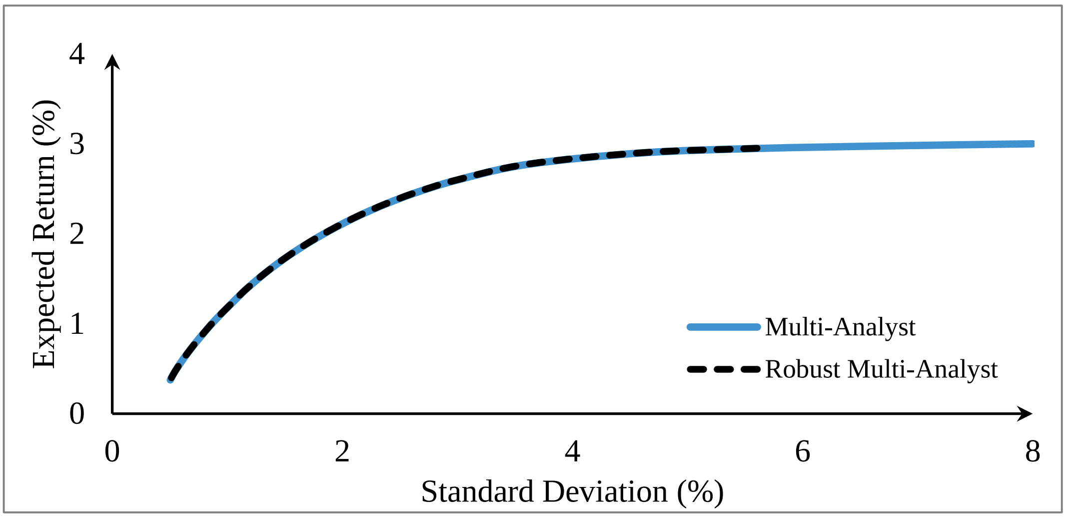
$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T \check{\Sigma}_z x} - R^{Target} - \delta_z \frac{\left(\check{\Sigma}_z^{1/2}\check{\Sigma}_z^{1/2} x\right)^T x}{\left\|\check{\Sigma}_z^{1/2} x\right\|}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T \check{\Sigma}_z x} - R^{Target} - \delta_z\sqrt{x^T \check{\Sigma}_z x}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \left(\frac{\kappa+2\delta_z}{2}\right)\sqrt{x^T \check{\Sigma}_z x} - R^{Target}\right)\right)} . \tag{4.26}$$

It can be noticed easily that the reformulation (4.26) of the robust multi-analyst approach $(\boldsymbol{RE_{MS}})$ is equivalent to the multi-analyst approach $(\boldsymbol{F_{MS}})$ by defining the risk aversion coefficient of $(\boldsymbol{F_{MS}})$ as $\lambda = \kappa + 2\delta_z$. Furthermore, since $\lambda \geq \kappa$ as both risk aversion coefficients are positive and $\delta_z \geq 0$, the efficient frontier of the robust multi-analyst approach $(\boldsymbol{RE_{MS}})$ is a shortened version of the efficient frontier of the multi-analyst approach $(\boldsymbol{F_{MS}})$. This result corresponds to what Schöttle (2007) has found as the efficient frontier of the robust counterpart approach coincides with the efficient frontier of the original optimisation problem up to a particular point. This is illustrated in Figure 4.1.

Figure 4.1 The Effect of Robustification in the Efficient Frontier of the Multi-Analyst Approach



Note: This figure shows the efficient frontiers of the multi-analyst approach (the solid line) and the robust multi-analyst approach (the dashed line). Under the assumption that the estimation errors and parameter uncertainties of expected returns, for both historical and analysts' dataset, are prescribed via a joint uncertainty set for the robust multi-analyst approach, the efficient frontier of the robust multi-analyst approach has the same curve as the efficient frontier of the multi-analyst approach, only shorter. This figure is for illustrative purpose only; the actual efficient frontiers depend on the input data.

4.3.2 Comparison between the Joint Uncertainty Set and the Separate Uncertainty Sets

Consider the robust multi-analyst mean-standard deviation portfolio selection approach with the separate ellipsoid uncertainty sets, and let the ellipsoid uncertainty sets for the parameter μ_z be given as formulae (4.18) and (4.19). Thus the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ for the mean-standard deviation portfolio selection problem can be formulated as

$$(\boldsymbol{RE_{HP}}) \quad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_z\in U_z} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\acute{\Sigma}}_z x} - R^{Target}\right)\right)} \quad (4.27)$$

with $\mu_z = \begin{pmatrix} \mu_{H_z} \\ \mu_{P_z} \end{pmatrix}$ and $\kappa \in [0, \infty)$ denoting the risk aversion coefficient. By assuming the desired robustness level for the analysts' dataset $\delta_{P_z} = 0$, the worst case scenarios for returns can be given as a vector $\begin{pmatrix} \acute{\mu}_{H_z} - \delta_{H_z}\frac{\acute{\Sigma}_{HH_z}x_H}{\left\|\acute{\Sigma}_{HH_z}^{1/2}x_H\right\|} \\ \check{\mu}_{P_z} \end{pmatrix}$.

Following the procedure of simplifying the robust counterpart (4.21)-(4.24), we replace the parameter μ_z with the vector of the worst case scenario $\begin{pmatrix} \acute{\mu}_{H_z} - \delta_{H_z}\frac{\acute{\Sigma}_{HH_z}x_H}{\left\|\acute{\Sigma}_{HH_z}^{1/2}x_H\right\|} \\ \check{\mu}_{P_z} \end{pmatrix}$ and obtain

$$\max_{x\in\mathbb{R}^n} \min_{z\in Z} \min_{\mu_{H_z},\mu_{E_z}} \frac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\acute{\Sigma}}_z x} - R^{Target}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\acute{\Sigma}}_z x} - R^{Target} - \delta_{H_z}\frac{\left(\acute{\Sigma}_{HH_z}^{1/2}\acute{\Sigma}_{HH_z}^{1/2}x_H\right)^T x_H}{\left\|\acute{\Sigma}_{HH_z}^{1/2}x_H\right\|}\right)\right)}$$

$$= \max_{x\in\mathbb{R}^n} \min_{z\in Z} \frac{1}{1+exp\left(-\theta_z\left({\check{\acute{\mu}}_z}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\acute{\Sigma}}_z x} - R^{Target} - \delta_{H_z}\sqrt{{x_H}^T\acute{\Sigma}_{HH_z}x_H}\right)\right)}$$

(4.28)

where $\check{\mu}_z = \begin{pmatrix} \acute{\mu}_{H_z} \\ \check{\mu}_{P_z} \end{pmatrix}$ and $\check{\Sigma}_z = \begin{pmatrix} \acute{\Sigma}_{HH_z} & \check{\Sigma}_{HP_z} \\ \check{\Sigma}_{PH_z} & \check{\Sigma}_{PP_z} \end{pmatrix}$. In order to illustrate the impact of adopting separated uncertainty sets in the robust multi-analyst approach, we compare formulations (4.26) and (4.28) and notice that the denominator of the last equation of (4.26) is larger than the denominator of the last equation from (4.28). That is,

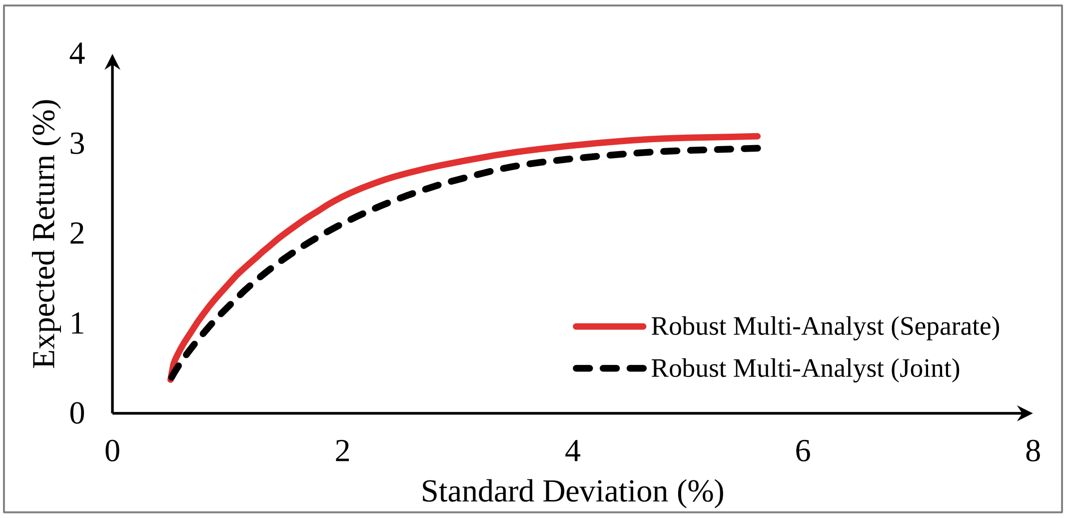
$$\begin{aligned} & exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \left(\frac{\kappa + 2\delta_z}{2}\right)\sqrt{x^T\check{\Sigma}_z x} - R^{Target}\right)\right) \\ & = exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\Sigma}_z x} - R^{Target} - \delta_z\sqrt{x^T\check{\Sigma}_z x}\right)\right) \\ & > exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\Sigma}_z x} - R^{Target} - \delta_{H_z}\sqrt{x_H{}^T\acute{\Sigma}_{HH_z}x_H}\right)\right) \end{aligned} \tag{4.29}$$

where $\delta_z = \delta_{H_z}$ and $x = (x_H{}^T, x_P{}^T)^T$ with $x_H \in \mathbb{R}^{n-m}$ and $x_P \in \mathbb{R}^m$. Consequently, we have

$$\begin{aligned} & \frac{1}{1 + exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \left(\frac{\kappa + 2\delta_z}{2}\right)\sqrt{x^T\check{\Sigma}_z x} - R^{Target}\right)\right)} \\ & < \frac{1}{1 + exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\kappa}{2}\sqrt{x^T\check{\Sigma}_z x} - \delta_{H_z}\sqrt{x_H{}^T\acute{\Sigma}_{HH_z}x_H} - R^{Target}\right)\right)} \end{aligned} \tag{4.30}$$

In other words, the robust multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$ obtains greater optimal values than the robust multi-analyst approach with the joint uncertainty set $(\boldsymbol{RE_{MS}})$, which indicates that the efficient frontier of the robust multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$ is located above the efficient frontier of the robust multi-analyst approach $(\boldsymbol{RE_{MS}})$. More specifically, for the same expected level of risk, the expected return is higher for the portfolio obtained via the robust-multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$. The following figure graphically illustrates this result.

Figure 4.2 Efficient Frontiers Constructed from Different Robust Multi-Analyst Approach



Note: This figure shows the efficient frontiers of the robust multi-analyst approach with the joint uncertainty set (the dashed line) and the robust multi-analyst approach with the separate uncertainty sets (the solid line). By adopting the separate uncertainty sets for the robust multi-analyst approach, the efficient frontier would have a higher level of return at every level of risk. This figure is for illustrative purpose only; the actual efficient frontiers depend on the input data.

4.4 Summary

In the previous chapter, we proposed the multi-analyst approach with fuzzy aspiration $(\boldsymbol{F_{MV}})$ by employing fuzzy set theory to handle the vague linguistic asset recommendations of various financial analysts. Although this model is more appropriate than other existing multi-prior models for solving portfolio selection problems with additional investment information, this model does not account for estimation errors and uncertainties, especially in the case that the historical data is adopted to generate parameter estimates for the assets which are not commented on by any analysts. Therefore, in this chapter, we presented the robust multi-analyst approaches, $(\boldsymbol{RE_{MV}})$

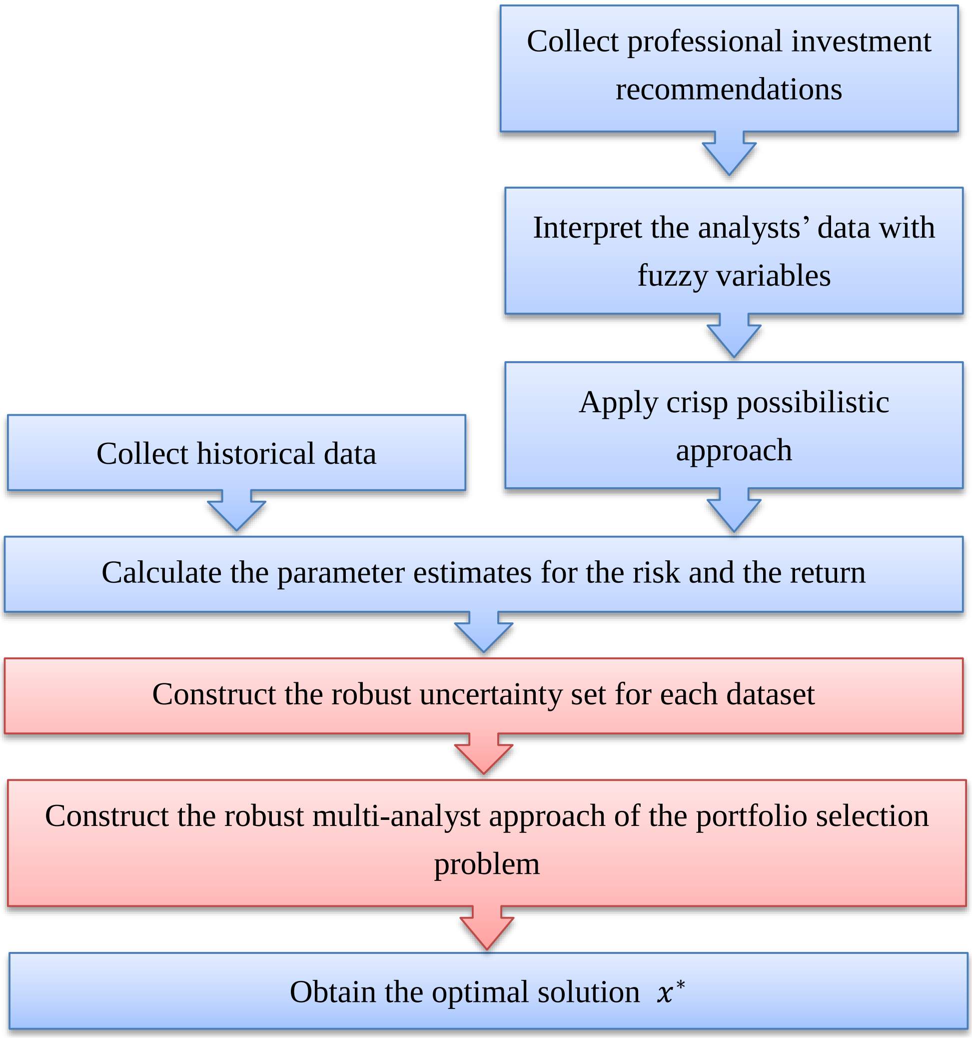
and $(\boldsymbol{RE_{HP}})$, to deal with estimation errors and uncertainties by using the robust counterpart approach in Ben-Tal and Nemirovski (1998). We further illustrated the impact of robustification on portfolio selection process by comparing the robust multi-analyst approaches with the multi-analyst approach $(\boldsymbol{F_{MV}})$.

On the base of the multi-analyst approach $(\boldsymbol{F_{MV}})$, we first constructed the standard robust counterpart of the multi-analyst approach by modifying the parameter of the expected returns to account for the estimation error and uncertainties. To do this, we followed the robust counterpart approach (Ben-Tal & Nemirovski, 1998) and defined an uncertainty set which contains most of the possible parameter values for every asset. This robust multi-analyst approach with the joint uncertainty set $(\boldsymbol{RE_{MV}})$ is a worst-case approach that generates the optimal portfolio based on the worst possible scenarios. A similar robust approach has been proposed by Lutgens and Schotman (2006) based on their multi-expert approach. However, in this thesis, as the multi-analyst approach $(\boldsymbol{F_{MV}})$ adopts two types of input data, the historical data and the analysts' data, to obtain parameter estimates, it is therefore unreasonable to define one uncertainty set for all assets with the same desired level of robustness. In view of this, we developed the robust multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$ based on the work of Garlappi et al. (2007) to distinguish the desired robustification for different types of input data.

In comparison to the robust counterpart approach in Lutgens and Schotman (2006), the proposed robust multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$ is more advanced in terms of: (a) providing a robust optimal portfolio which is theoretically less pessimistic; and (b) adopting non-overlapping uncertainty sets to specify the desired robustification for different types of input data. On the other hand, the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ is superior compared to the multi-prior approach suggested by Garlappi et al. (2007). First, Garlappi et al. (2007) uses a return-generating model as the expert's recommendations and doesn't consider the nature of recommendations; the method adopted for incorporating multiple investment sources is not clearly specified and the parameter estimates are obtained using classical methods such as the maximum likelihood estimator or Bayesian approach. Secondly, according to Garlappi et al. (2007), their framework can incorporate both parameter and model uncertainties by estimating expected returns with both sample returns and one particular return-generating model; nevertheless, the proposed robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ also possesses this advantage feature and, in addition, allows its user to incorporate multiple information sources.

Before proceeding to the implementation and examination of the multi-analyst approaches developed in Chapter 3 and Chapter 4, the practical analysts' data considered in this research is now detailed in the following chapter. At the end of this section, we summarise in Figure 4.3 the basic steps for the robust multi-analyst approach with the separate uncertainty sets $(\boldsymbol{RE_{HP}})$, which will later be applied for solving portfolio selection problems in real world applications.

Figure 4.3 The Diagram of Solving the Robust Multi-Analyst Approach with the Separate Uncertainty Set



Chapter 5

Analysts' Data and Fuzzification

The aim of this research is to develop a potentially profitable approach to robust optimal asset allocation by incorporating additional investment information from multiple stock market analysts into a robust portfolio selection problem. After developing portfolio selection frameworks in the previous chapters, the research effort now turns to the collection of data and the implementation of the proposed portfolio selection models.

So far, the existing research on the robust portfolio selection with multiple experts' recommendations has focused only on the construction of the optimisation framework. To the best of our knowledge, previous studies regarding the multi-expert approach of the robust portfolio selection problem have adopted either simulated expert data (Huang et al., 2010) or simply some particular return models (Garlappi et al., 2007; Lutgens and Schotman, 2010), such as the capital asset pricing model (CAPM) and the Fama & French factor model, instead of employing data from practical stock market analysts employed by financial institutions for investigating the performance of the models. As the main interest of this research is to develop the multi-analyst approach in financial practice, data obtained from financial analyst forecasts of the Taiwanese stock market will be used to investigate the performance of the proposed multi-analyst portfolio selection approaches, and to compare with the portfolio performances of some existing portfolio selection models in the literature.

This chapter will focus on discussing the data used in this thesis, and is organised as follows. First, an overview of the practical analysts' dataset together with information of the financial institution considered for this research will be provided in the data collection section, followed by Section 5.2 about data description. Then, in Section 5.3, the procedure for converting the fuzzy analysts' forecasts into clear numerical estimations will be presented. Finally, a brief summary will be given to conclude this chapter.

5.1 Data Collection

The practical analysts' data considered in this research are from the collection of daily investment advisory newsletters provided by Taiwanese financial institutions. A total of 2,133 newsletters, published from 1st February 2012 to 28th March 2014, were collected. These newsletters are only accessible by registered online traders who hold accounts with the financial institutions. Once the investors become registered customers, they can read the investment advisory newsletters from either their personal e-mail account or from the web page of the financial institution. As the newsletters are updated every trading day and the previous newsletters are usually automatically removed, therefore the newsletters need to be collected from each institutional investment firm before the website maintenance every trading day.

The practical analysts' dataset was obtained from the investment newsletters collected from Taiwanese securities brokerage firms. Due to time and resource limitations for completing this study, four out of the 87 securities brokerage firms were chosen for this study. All of the chosen securities brokerage firms have at least 25 years of experience in Taiwan's competitive securities industry. Furthermore, they all have overseas representative offices and subsidiaries in Asia. In addition, all of the chosen brokerage firms are members of the top ten most active securities firms in Taiwan, with at least 3% market share in brokerage since the year 2005[7]. See Appendix A for further details.

5.2 Data Description

The reporting formats of the newsletters are not consistent across all securities brokerage firms considered in this study. Normally, a newsletter starts with an overview of the Taiwanese stock market that summaries the recent market information such as the statistical figures of daily/weekly trading and highlights the news about different industries. Then a qualitative analysis of current market conditions is provided together with future prospects of the market movement, which is always expressed linguistically in statements like "Bull market in technology sector, especially the smartphone related industry" or "The result of the election could potentially lift both the market and economy, therefore, we do expect a bull market". Finally, detailed recommendations for some specific stocks are given with information such as the name and the stock identity number of the recommended stock and the reason for suggesting this particular stock in the newsletter. For a more comprehensive newsletter, further information on the recommended stock is included; for instance, the current and predicted price and the suggested action (i.e., buy, sell or neutral) for the recommended stock.

Although the newsletters may provide relatively useful information for allocating investment in the stock market, there are two potential disadvantages of the collected investment forecasts in addition to the fundamental ambiguity characteristics of the stock recommendations. First, the number of the recommended stocks varies between the securities brokerage firms, and it is very unlikely that different securities brokerage firms make detailed recommendations on the same stock. Second, none of these

[7]The information of the securities firm is available on the web page of TWSE (2005).

recommendations contains clear information about the predicted time frame for the targeted price movement.

We consider these practical analysts' data for stock recommendation as qualitative data (see Table 5.1 as an illustration). According to Herold (2003), there is only a limited amount of research regarding the constructions of asset allocation based on the market forecasts provided by analysts. In addition, the approaches adopted for optimising portfolios with experts' forecasts are mostly ad hoc or heuristic. To address the issue, Herold has proposed a portfolio selection approach for implementing qualitative market forecasts. In his framework, the Bayesian model is adopted for refining the expectations of future returns and additional diagnostic tools are implied for processing the recommendations and the resulting portfolio. Although Herold has made it possible for practitioners to use qualitative forecasts in asset allocation, there is one restriction on the format of the forecasts, i.e., the recommendations are always about pairs of assets but not one specific asset or asset class.

Only a few studies have been done on extending the literature of portfolio selection models with professional investment forecasts during the last ten years. To name a few, Fabozzi et al. (2006) adopted the Black-Litterman approach to combine cross-sectional momentum strategies with market equilibrium returns in the mean-variance framework. Chiarawongse et al. (2012) proposed a mean-variance portfolio optimisation model with qualitative ranking information of assets, where the qualitative forecasts are represented by linear inequalities. On the other hand, instead of following the conventional approaches, which quantify the uncertain experts' estimations via either the Bayesian approach or the concept of the fuzzy set, Huang (2012) incorporated the experts' forecasts into the mean-variance portfolio selection models with the experts' estimation of i^{th} asset return formulated as $\hat{\mu}_i = \frac{(p_i' + d_i - p_i)}{p_i}$, where p_i' is the expected price of the

i^{th} asset in the future, p_i is the current price of the asset and d_i is the estimated dividend of the i^{th} asset during the period.

Note that all the existing approaches for interpreting investment recommendations have their own advantages and disadvantages and there is no agreement in the literature regarding which particular approach provides better numerical expressions of these uncertain forecasts. Moreover, as previously noted, the formats of the newsletters are inconsistent across the different securities brokerage firms and every newsletter consists of many recommendations, which are usually expressed differently. Fuzzy set theory, the more generalised and intuitive method that has been supported by many authors for defining uncertain parameters (Ammar and Khalifa, 2003; Chen and Huang, 2009; Gupta et al., 2008; Liu, 2011; Zhang et al., 2009 & 2011), is considered an appropriate approach in this research for interpreting the analysts' forecasts. In addition to the favourable simplicity feature of fuzzy set theory mentioned earlier, it is also helpful in standardising the computation of the vague investment forecasts, rather than applying various specific approaches to deal with stock forecasts in different formats. In this regard, the procedure for generating estimates of input parameters can be more efficient and at the same time reduce the risk of misinterpreting analysts' suggestions caused by employing different data interpreting approaches. In the rest of this chapter, we discuss the forecast of the stock returns and deviations in terms of fuzzy variables.

5.3 Fuzzification

5.3.1 Trapezoidal Fuzzy Variables

In order to remain conceptually and operationally efficient with less demanding computation, the class of the trapezoidal fuzzy variables, whose membership function is normal and convex, is adopted for interpreting stock recommendations in this study. Recall from Section 3.1 that a typical trapezoidal fuzzy variable $\tilde{A}^{Tra} =$

$(m_-, m_+, \sigma_-, \sigma_+)$ with tolerance interval $[m_-, m_+]$, left width σ_- and right width σ_+ is defined via a corresponding membership function $M_{\tilde{A}^{Tra}}(x)$

$$M_{\tilde{A}^{Tra}}(x) = \begin{cases} 1 & x \in [m_-, m_+] \\ 1 - \dfrac{m_- - x}{\sigma_-} & x \in [m_- - \sigma_-, m_-] \\ 1 - \dfrac{x - m_+}{\sigma_+} & x \in [m_+, m_+ + \sigma_+] \\ 0 & otherwise \end{cases} . \tag{5.1}$$

Note that the triangular fuzzy variable $\tilde{A}^{Tri} = (m, \sigma_-, \sigma_+)$ and the ordinary crisp interval $A = [m_-, m_+]$ also belong to the class of the trapezoidal fuzzy variables, and the corresponding membership function $M_{\tilde{A}^{Tri}}(x)$ of the triangular fuzzy variable $\tilde{A}^{Tri}$ is formulated as

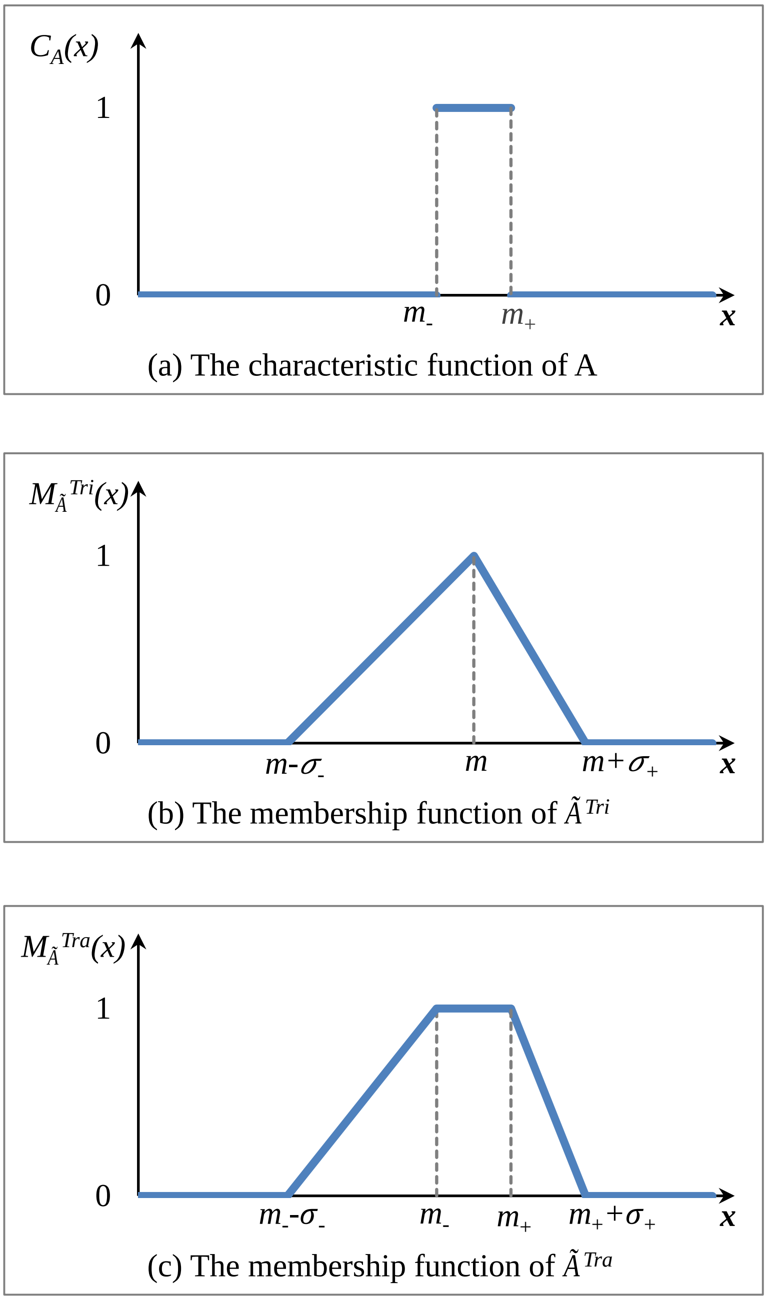
$$M_{\tilde{A}^{Tri}}(x) = \begin{cases} 1 & x = m \\ 1 - \dfrac{m - x}{\sigma_-} & x \in [m - \sigma_-, m] \\ 1 - \dfrac{x - m}{\sigma_+} & x \in [m, m + \sigma_+] \\ 0 & otherwise \end{cases} . \tag{5.2}$$

Although the ordinary crisp interval A is a special case of the trapezoidal fuzzy variable, it is expressed via a characteristic function $C_A(x)$ rather than a membership function, i.e.,

$$C_A(x) = \begin{cases} 1 & x \in [m_-, m_+] \\ 0 & otherwise \end{cases} . \tag{5.3}$$

The following figure graphically illustrates these trapezoidal fuzzy variables. The differences between these three fuzzy notations can be observed explicitly. For instance, the ordinary crisp interval A represents a full and equal belief in a range of estimations, i.e., $[m_-, m_+]$, with $Core(A) = Supp(A)$. On the other hand, the triangular fuzzy variable $\tilde{A}^{Tri}$ indicates that the future value is most likely to be at $x = m$ with the left spread σ_- and right spread σ_+ denoting the possible deviations from the prediction $x = m$. Finally, the trapezoidal fuzzy variable $\tilde{A}^{Tra}$ is the combination of the ordinary crisp interval A and the triangular fuzzy variable $\tilde{A}^{Tri}$ with full membership degree for every element in the interval $[m_{-}, m_{+}]$.

Figure 5.1 Illustration of Various Types of Trapezoidal Fuzzy Variable



(a) The characteristic function of A

(b) The membership function of $\tilde{A}^{Tri}$

(c) The membership function of $\tilde{A}^{Tra}$

5.3.2 Specific Formats of Stock Recommendation

In this subsection we discuss analysts' investment forecasts and the interpretation regarding the recommendations of the future stock prices. As previously noted, the investment newsletters collected for this study consist of various types of recommendations. Some of the recommendations are very unclear and do not provide sufficient information for making investment decisions. Instead of manipulating the forecasts from these unclear recommendations for creating estimates of stock returns and variances, we only accept stock recommendations which provide relatively clear investment information, such as the name of the recommended stocks, suggested investment actions, and the recommended price or the price range of the stock. Furthermore, there is a fundamental requirement on the vertices of the fuzzy variables. For instance, the triangular fuzzy variable $\tilde{A}^{Tri}$ requires at least two points and the trapezoidal fuzzy variable $\tilde{A}^{Tra}$ requires at least three points. Therefore, the mid-point of the predicted price range and the interpreted investment actions are adopted as the vertices in some occasions for constructing the fuzzy variables.

Now we focus on the following three format samples as found in the newsletters surveyed for the research, which give clearer information on stock recommendations considered in this study. As all the newsletters provided by the securities brokerage firms are written in Chinese, the samples presented below are the translated versions in English.

- The first typical format of the stock forecasts is the detailed recommendation for a specific stock, which clearly states the support and the resistance of the stock price with explanations. However, this particular format doesn't explicitly indicate the suggested investment action, hence the investor can only make investment decisions according to his/her own understanding of the recommendation. To convert this type of stock forecast into fuzzy variables, the first step is to compute the estimates of support and resistance in terms of stock return based on the predicted price support and resistance provided by the analyst. Although the analyst never clearly stated investment strategies for this particular format of recommendation, investment actions for the recommended stocks can be made according to the qualitative analysis. The following table shows a sample of the first typical format of stock forecasts.

Table 5.1 Sample of the First Basic Format for Stock Recommendations

<table>
<tr><td colspan="2">ILI TECHNOLOGY
<3598></td><td>1. In December last year, the company reported a 48.46% year-over-year increase in monthly revenue to NT$ 94.1 million.</td></tr>
<tr><td>Closing Price:</td><td>93.3</td><td rowspan="3">2. The company is now reaping the harvest of the touch panel IC products in the mainland China market, and the market share has been gradually increased. Therefore, the annual revenue is expected to reach another new high this year.</td></tr>
<tr><td>Resistance:</td><td>100</td></tr>
<tr><td>Support:</td><td>89</td></tr>
</table>

Source: Based on the newsletter provided by the analyst 2 considered for the empirical investigation in Chapter 6, Jan 2013. NT$ is the common abbreviation of the official currency of Taiwan, which is also indicated by the currency code TWD.

The stock recommendation displayed in Table 5.1 suggests a buying action on ILI Technology with the support and resistance of the stock return at -4.61% and 7.18%, respectively, i.e.,

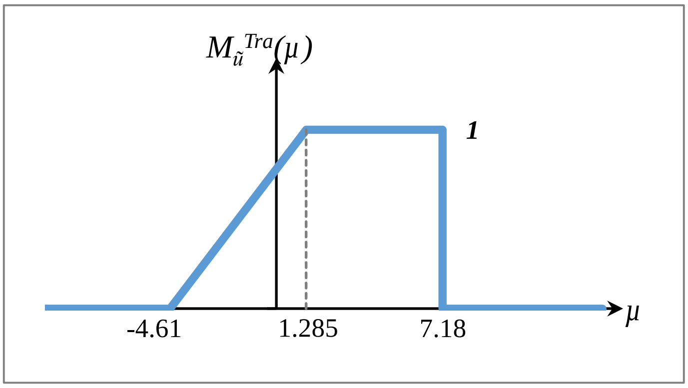
$$Support = \frac{89 - 93.3}{93.3} = -4.61\% \quad , \quad Resistance = \frac{100 - 93.3}{93.3} = 7.18\%$$

By adding the average of the support and resistance as one of the vertices, $\frac{-4.61\%+7.18\%}{2} = 1.285\%$, the fuzzy variable representing the forecast of ILI Technology is denoted as a trapezoidal fuzzy variable, i.e.,

$$\tilde{\mu}_{ILI}^{Tra} = (1.285\%,\ 7.18\%,\ 5.895\%,\ 0\%),$$

with tolerance interval $[1.285\%, 7.18\%]$, left deviation 5.895% and right deviation 0% from the tolerance interval. Figure 5.2 shows the fuzzy expression for the stock recommendations of ILI Technology.

Figure 5.2 The Membership Function of $\tilde{\mu}_{ILI}^{Tra}$



By following the defuzzification method of Carlsson and Fuller (2001) (see Section 3.1.4 for further information), the crisp possibilistic mean value and variance of the trapezoidal fuzzy variable $\tilde{\mu}_{ILI}^{Tra}$ are

$$\begin{aligned} E(\tilde{\mu}_{ILI}^{Tra}) &= \frac{m_- + m_+}{2} + \frac{\sigma_+ - \sigma_-}{6} \\ &= \frac{1.285 + 7.18}{2} + \frac{0 - 5.895}{6} \\ &= 3.25 \end{aligned}$$

and

$$\begin{aligned} Var(\tilde{A}^{Tra}) &= \left[\frac{m_+ - m_-}{2} + \frac{\sigma_- + \sigma_+}{6}\right]^2 + \frac{(\sigma_- + \sigma_+)^2}{72} \\ &= \left[\frac{7.18 - 1.285}{2} + \frac{5.895 + 0}{6}\right]^2 + \frac{(5.895 + 0)^2}{72} \\ &= 15.93 \end{aligned}$$

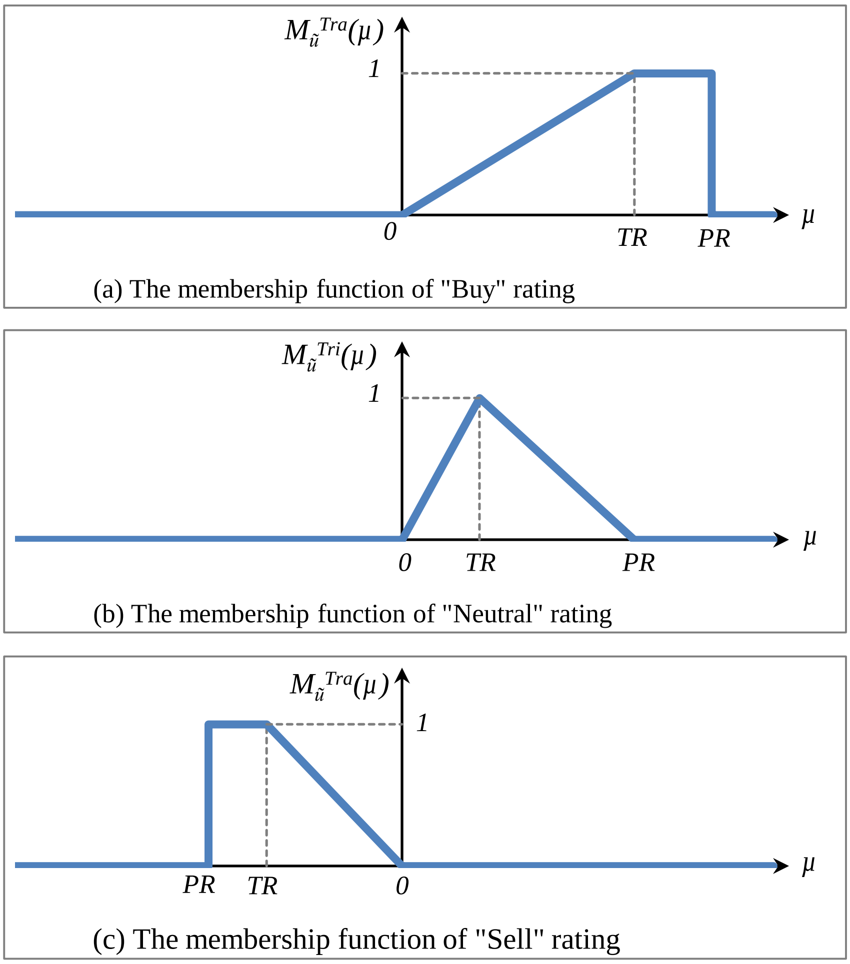
2) The second typical expression of the stock forecasts is the detailed analysis for a particular stock with clear investment action provided and, in most cases, this particular form of recommendation comes with a target of the share price and a potential rate PR. Similarly, the matching target return TR of the stock is obtained from the suggested target of the stock price. Then, by utilising the suggested investment action as one of the vertices for the corresponding trapezoidal fuzzy variable $\tilde{\mu}^{Tra} = (m_{-}, m_{+}, \sigma_{-}, \sigma_{+})$, the stock recommendations can be expressed as

$$\tilde{\mu}^{Tra}_{Buy} = (m_{-}, m_{+}, \sigma_{-}, \sigma_{+}) = (\mathrm{TR}, \mathrm{PR}, \mathrm{TR}, 0),$$

$$\tilde{\mu}^{Tri}_{\mathrm{Neutral}} = (m, \sigma_{-}, \sigma_{+}) = (\mathrm{TR}, \mathrm{TR}, \mathrm{PR} - \mathrm{TR}),$$

$$\tilde{\mu}^{Tra}_{Sell} = (m_{-}, m_{+}, \sigma_{-}, \sigma_{+}) = (\mathrm{PR}, \mathrm{TR}, 0{,}0 - \mathrm{TR}).$$

Figure 5.3 Fuzzy Expressions of the Second Type Forecast



(a) The membership function of "Buy" rating

(b) The membership function of "Neutral" rating

(c) The membership function of "Sell" rating

A sample of a stock recommendation with "neutral" rating issued by an analyst is presented in the following table.

Table 5.2 Sample of the Second Basic Format for Stock Recommendations

<table>
<tr><td colspan="3">HUA NAN FINANCIAL HOLDINGS</td><td><2880></td><td>Neutral</td><td>Analyst A</td></tr>
<tr><td>Closing Price</td><td>17.5</td><td colspan="4" rowspan="3">Remain "Neutral" rating for Hua Nan Financial Holdings with a NT$ 17.8 price target.
We keep the same recommended investment strategy for the stock as "Neutral" based on the following considerations: 1) Hua Nan Bank is the third largest domestic bank in Taiwan in terms of enterprise size, and it has a relatively decent market share in the Taiwanese financial service sector. Although…"</td></tr>
<tr><td>Targeted Price</td><td>17.8</td></tr>
<tr><td>Potential %</td><td>2%</td></tr>
</table>

Source: Based on the newsletter provided by the analyst 1 considered for the empirical investigation in Chapter 6, Feb 2013.

The fuzzy interpretation is $\tilde{\mu}^{Tri}_{\text{HuaNan}} = (1.71\%, 1.71\%, 0.29\%)$. According to Carlsson and Fuller (2001), the crisp possibilistic mean value and variance of the triangular fuzzy variable $\tilde{\mu}^{Tri}_{\text{HuaNan}}$ are:

$$E\left(\tilde{\mu}^{Tri}_{\text{HuaNan}}\right) = m + \frac{\sigma_+ - \sigma_-}{6}$$
$$= 1.71 + \frac{0.29 - 1.71}{6}$$
$$= 1.47$$

and

$$Var\left(\tilde{\mu}^{Tri}_{\text{HuaNan}}\right) = \frac{(\sigma_- + \sigma_+)^2}{24}$$
$$= \frac{(1.71 + 0.29)^2}{24}$$
$$= 0.17$$

3) The third typical expression of the stock forecasts is a collection of stocks which are advised as having a poorer performance as compared with other stocks in the short term. In addition to the basic information and the explanation of why the chosen stock is likely to perform poorly, a sequence of four price boundaries PB is given by the analyst for every advised stock, i.e.,

$$PB_1 < PB_2 < Closing\ Price < PB_3 < PB_4.$$

By using the similar setting, those price boundaries are converted into return boundaries RB, i.e., $RB_n = \frac{PB_n - Closing\ Price}{Closing\ Price}$ with $n = 1,2,3,4$ and

$$RB_1 < RB_2 < RB_3 < RB_4.$$

As the trapezoidal fuzzy variable can only take up to four vertices, the fuzzy variable for representing the third type of stock recommendation is based only on the return boundaries, i.e.,

$$\tilde{\mu}^{Tra} = (RB_2, RB_3, RB_2 - RB_1, RB_4 - RB_3).$$

A sample of the third typical expression of stock forecasts is displayed in Table 5.3, and the corresponding trapezoidal fuzzy expressions are given as

$$\tilde{\mu}^{Tra}_{RunLong} = (-2.04\%, 3.50\%, 4.08\%, 2.48\%),$$

$$\tilde{\mu}^{Tra}_{Macronix} = (-1.92\%, 1.28\%, 8.33\%, 2.31\%).$$

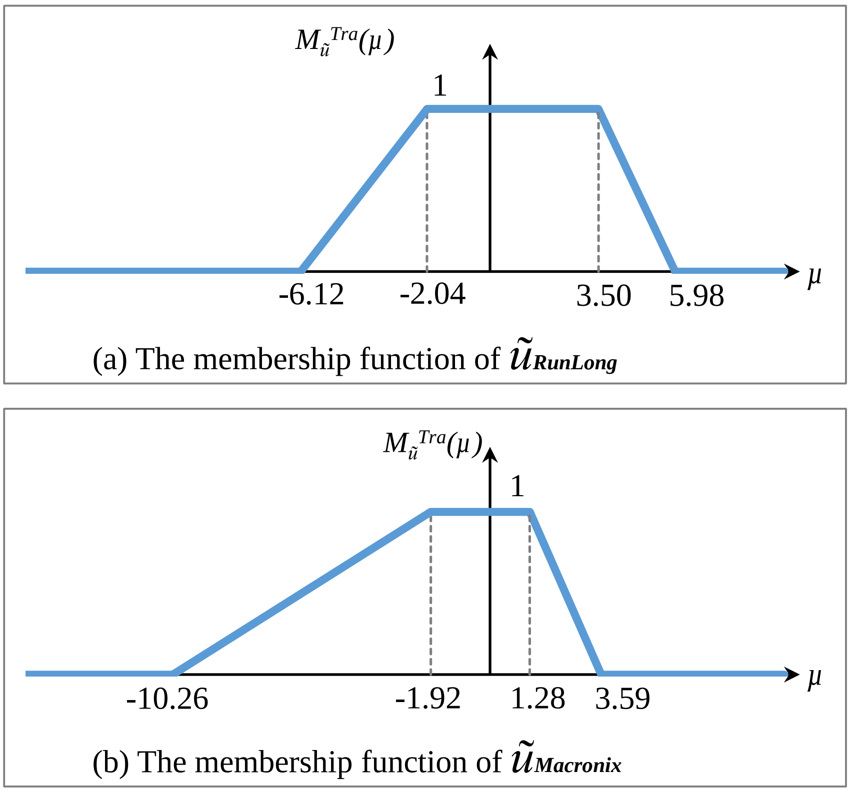
The crisp possibilistic interpretations for the trapezoidal fuzzy variables, $\tilde{\mu}^{Tra}_{RunLong}$ and $\tilde{\mu}^{Tra}_{Macronix}$, can be obtained by following the same equations in the first sample, where the crisp possibilistic mean values of $\tilde{\mu}^{Tra}_{RunLong}$ and $\tilde{\mu}^{Tra}_{Macronix}$ are 0.46 and -1.32, respectively. The crisp possibilistic variances are 15.52 and 12.95 for $\tilde{\mu}^{Tra}_{RunLong}$ and $\tilde{\mu}^{Tra}_{Macronix}$.

Table 5.3 Sample of the Third Basic Format for Stock Recommendations

Company	1st Price Boundary	2nd Price Boundary	Closing Price	3rd Price Boundary	4th Price Boundary
RUN LONG CONSTRUCTION <1808>	32.2	33.6	34.3	35.5	36.35
The share price is falling due to the disappointing quarter revenue and the lag effect after the ex-dividend date.					
MACRONIX INTERNATIONAL <2337>	7.0	7.65	7.8	7.9	8.08
Constantly reports quarterly net losses, thus, it is possible that the share price may break under the last trend line bottom.					

Source: Based on the newsletter provided by the analyst 1 considered for the empirical investigation in Chapter 6, July 2012.

Figure 5.4 The Membership Functions of $\tilde{\mu}^{Tra}_{RunLong}$ and $\tilde{\mu}^{Tra}_{Macronix}$



(a) The membership function of $\tilde{u}_{RunLong}$

(b) The membership function of $\tilde{u}_{Macronix}$

5.4 Summary

Following the developments of the multi-analyst approaches for the portfolio selection problem in the previous chapters, we have looked into the practical analysts' data which is considered as the important component of the models proposed in this research.

In Section 5.1, we have detailed the actual data collection with information about the background and the source of the investment forecasts. In Sections 5.2 and 5.3 we have described and discussed the formats of the newsletters collected from the securities brokerage firms, and then reviewed and discussed several alternatives that have been proposed for interpreting market forecasts in the existing literature. With regard to the limitations of the practical analysts' data collected for this research, we have used fuzzy set theory to deal with market forecasts and to express stock recommendations in terms of either triangular or trapezoidal fuzzy variables. A couple of samples of the common stock recommendation formats are provided to illustrate the interpretation of the data.

In the next chapter we will present practical implementations of the multi-analyst portfolio selection approaches in the Taiwanese stock market and report the empirical results obtained from employing the practical analysts' data discussed in this chapter.

Chapter 6
Empirical Application to Portfolio Management with Multiple Analysts

The stock market is one of the most active and important markets in finance. Many financial decision making methods for selecting appropriate asset allocations have been developed based on the return-risk portfolio selection theory proposed by Markowitz in 1952 to fulfil different investment requirements of the user. The framework of asset allocation models mostly depends on the expected returns and the standard deviations of the assets, and the correlations between assets are used to quantify the relationships between assets. Essentially, only two types of input parameter are required for most of the portfolio selection problems, i.e., a vector of expected returns μ and covariance matrix of the returns, Σ, which contains the information on the volatilities of the individual assets and the correlations of assets. The values of these input parameters are usually estimated via a chosen sample of asset performances from the past.

As discussed previously, obtaining an optimal portfolio based on point estimates of input parameters could be unreliable, because small fluctuations in the estimation values of the input parameters may lead to a quite different asset allocation. Hence, in order to overcome the undesirable sensitivity feature of the classical return-risk portfolio model, in Chapters 3 and 4 we proposed to incorporate additional investment information for obtaining more reliable asset allocation. More specifically, we have developed the multi-analyst portfolio selection approach $(\boldsymbol{F_{MV}})$ and the corresponding robust counterpart approach $(\boldsymbol{RE_{HP}})$. In order to provide a better understanding and to evaluate the portfolio performances of the two multi-analyst approaches, this chapter focuses on practical application of the proposed portfolio allocation models, using data from the Taiwanese stock market.

Section 6.1 outlines the necessary background for understanding the Taiwanese stock market. Section 6.2 explains the input data selected for the empirical investigation. Section 6.3 specifies the portfolio selection models considered in this chapter and defines the measures of portfolio performance for evaluating resulting portfolios. Sections 6.4 and 6.5 report and discuss the main findings. Finally, Section 6.6 concludes this chapter.

6.1 Introduction of the Taiwan Stock Market

The Taiwan Stock Exchange (TWSE) was founded in 1961 and operated as a stock exchange from 1962. The most widely quoted index for representing the market movement and the economy of Taiwan is the Taiwan Capitalisation Weighted Stock Index (TAIEX) which was created in 1966 by TWSE. All of the listed common stocks traded on TWSE are included in TAIEX, except for the preferred stocks, full delivery stocks and newly listed stocks, which are listed within the most recent calendar month. At the end of 2013, there were 809 stocks listed on the Taiwan Stock Exchange, with a total market capitalisation of NT$ 24519.6 billion[8]. The TWSE categorises the companies into 28 industrial sectors. The semiconductor sector, finance and insurance sector, and communications and internet sector are the major industries in Taiwan, and their total market share by market values captured is approximately 40% of the entire Taiwanese stock market.

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[8] NT$ is the common abbreviation of the official currency of Taiwan, which is also indicated by the currency code TWD.

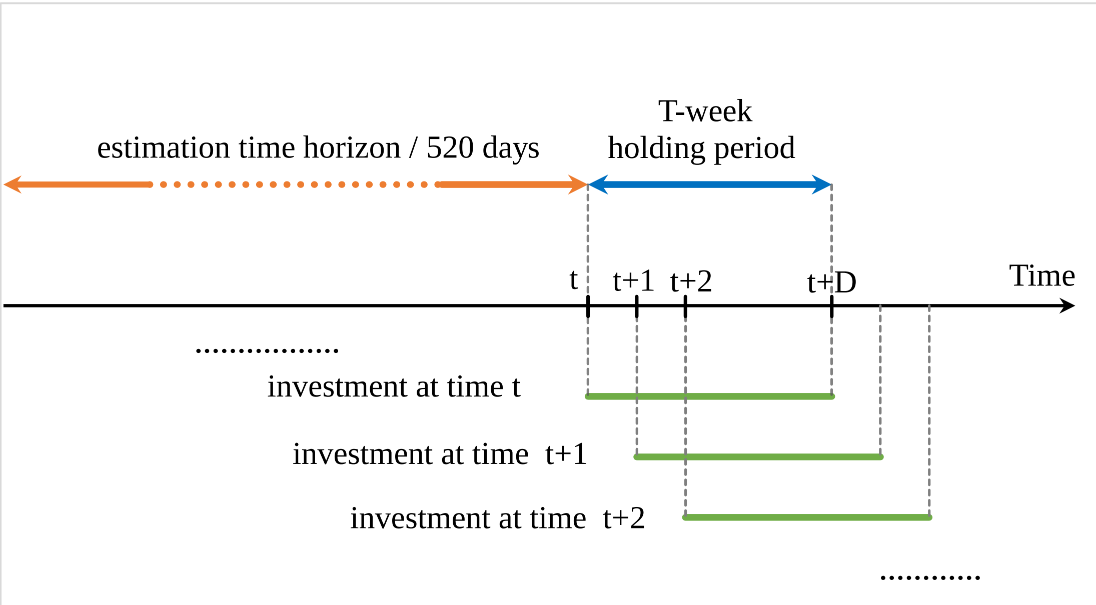
The daily trading session of the TWSE begins at 9:00 a.m. and finishes at 13:30 p.m., Monday to Friday. During the trading session the stock price movements are restricted by the price limit system, which has been imposed by TWSE in order to avoid irrational and dramatic market fluctuations and to enhance stock market stability. The price limit system has been implemented since TWSE was established. In response to various issues, such as major domestic and international political events, the price limit system has been adjusted a few times over the last three decades. Nowadays, TWSE sets the limit of the daily price movement at 7% for all listed stocks. In other words, the maximum and minimum daily returns of any stock in Taiwan's stock market are $+7\%$ and -7%, respectively.

6.2 The Historical Data and the Practical Analysts' Data

In order to evaluate the performances of the multi-analyst approach and the corresponding robust counterpart approach, we use historical as well as practical analysts' data in the modelling, for obtaining optimal portfolio outcomes. The historical data is the Taiwanese stock market data quoted from DataStream for the period from April 2010 to April 2014 (the observed sample period is based on 492 trading days that spans from April 2012 to April 2014) and the basis of the practical analysts' data is a collection of 2,133 newsletters obtained from four securities brokerage firms, as described previously in Chapter 5. The daily analysts' predictions of the stock returns are usually much higher than 7%, the legal daily return limits for any stock in Taiwan, and it is very unlikely that the performance of any stock can achieve the maximum return limit for a few days in a row. Therefore, the recommendations provided by analysts are assumed to have a time window before the predicted price is reached. However, as mentioned in Chapter 5, there is no clear information for determining the time frame needed before the predicted price is reached. Hence, the investor has to make a rational and reasonable prior guess regarding the time frame of the analysts' recommendations.

According to the newsletters collected for this research, the return prediction of stocks provided by the analysts varies substantially, i.e., ranging from -36% to 61%. It is difficult to decide on an appropriate time frame for analysts' recommendations, as no such indications are available in the newsletters. Therefore, we consider four different time frames, D days for $D = 5,10,15,20$, to assess the effectiveness of the multi-analyst approaches. The optimal portfolios are constructed based on different investment strategies for every trading day, with a holding period of D days. The estimates of the input parameters for the historical dataset are obtained from a rolling-horizon procedure, with the underlying historical market data assumed to be normally distributed, and the estimation time horizon considered for this study is 520 days. The following figure provides a schematic view of the portfolio selection process considered for this empirical application.

Figure 6.1 The Time Schedule for the Portfolio Selection Process

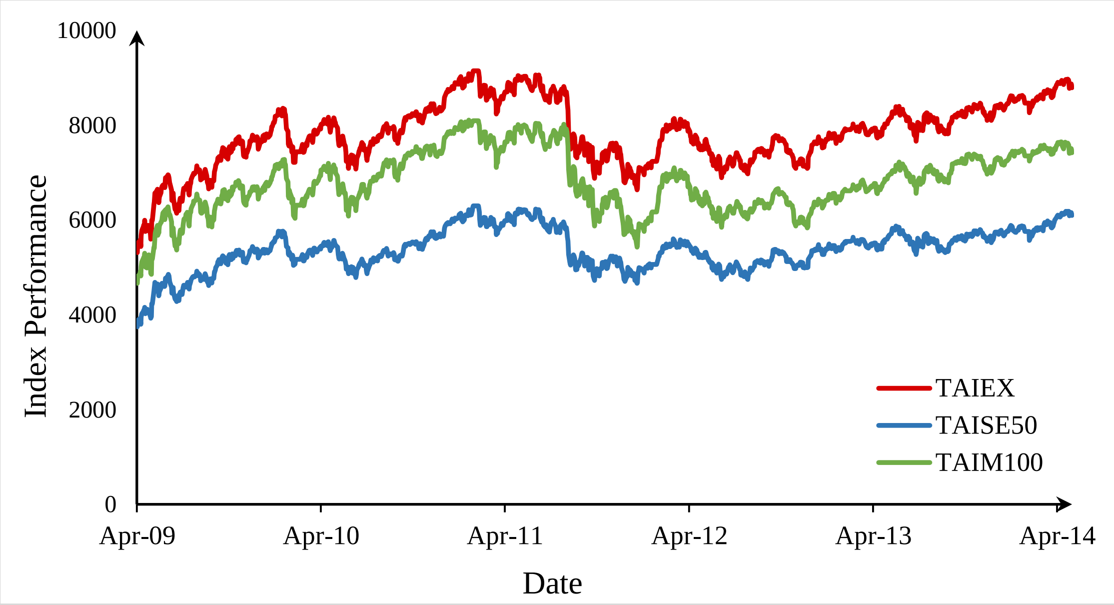


During the selection of the optimal portfolio at time t, an investor uses the historical data from a rolling-window of 520 days, i.e., 520 days is the estimation period beginning from $t-520$ to $t-1$ for the investment at time t, and the analysts' recommendations published at time t to generate parameter estimates for solving portfolio selection problems. The historical data can be used directly for generating parameter estimates via a choice of statistical estimation, and on the other hand, the analysts' data need to be first converted into trapezoidal fuzzy variables and then transformed into crisp possibilistic parameter estimates (see Section 5.3 for further information). Afterwards, the optimal asset allocation is obtained and used for calculating the corresponding in-sample portfolio performances and portfolio behaviours at time t. Then the investor holds the investment for D days. At time $t+D$, the investor sells all stocks which were bought at time t and calculates the out-of-sample portfolio performances.

The market sample considered in this study is given by the stocks listed in the FTSE TWSE Taiwan 50 Index (TAISE50) and FTSE TWSE Taiwan Mid-Cap 100 Index (TAIM100), rather than the entire Taiwanese stock market, which consists of 809 listed stocks. The TAISE50 Index is comprised of the top 50 capitalised blue chip stocks, which represent nearly 70% of the Taiwanese stock market and the TAIM100 Index is composed of the next 100 eligible Taiwanese stocks after the TAISE50 Index, which represent nearly 20% of the market. There are two stocks from the TAIM100 Index, TW:CSL and TW:GSE, that are removed from the market sample, as they are newly listed on TWSE, with insufficient historical data for obtaining parameter estimates. Therefore, the final sample considered for this research consists of 148 stocks and represents an investment universe that captures almost 90% of the Taiwanese stock market.

To get a clearer idea of the chosen sample, Figure 6.2 illustrates the index performance of the selected indices and the TAIEX Index over the time period from April 2009 to April 2014.

Figure 6.2 Historical Performances of the Selected Indices of Taiwanese Markets

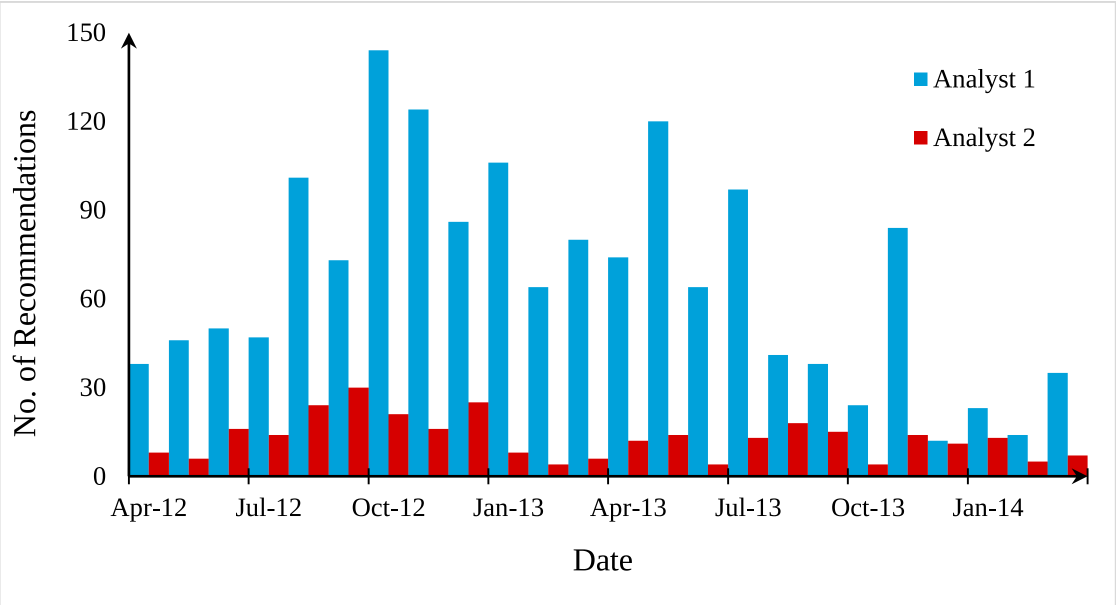


Source : TWSE Website (2005); DataStream.

As noted in Chapter 5, the qualities and the contents of the newsletters vary across the four securities brokerage firms. Therefore, the newsletters with incomplete or imprecise recommendations during the observation period are excluded. In addition, recommendations for stocks which are newly listed in the indices or de-listed from the indices over the sample period are omitted from the practical analysts' sample. Considering that the detailed recommendations provide more sufficient information for estimating parameter values, the securities brokerage firms which don't provide sufficient details are therefore ignored. After eliminating newsletters and recommendations that are not suitable for this study, the remaining sample for the practical analysts' data comprises 984 newsletters provided by two securities brokerage firms, made up of a total of 1,893 stock recommendations, published in the period from

April 2012 to April 2014. To be exact, Analyst 1 and Analyst 2 have made 1,585 and 308 stock recommendations during the sample period, respectively. In other words, 3.22 forecasts are provided by analyst 1 and 0.63 forecasts are given by analyst 2 for every trading day. In addition to the fact that one of the analysts, who has a greater trading volume as a security brokerage firm, always provides more detailed stock recommendations than the others, the amount of daily available stock recommendations varies from time to time and, on some occasions, both analysts have provided forecasts for the same stock simultaneously. Overall, approximately 3.82 stocks are recommended and 3.85 forecasts are provided for every trading day. The monthly volume of stock recommendations is presented in Figure 6.3 for explicitly comparing the differences in the volume of the recommendations between the analysts considered for the empirical investigation.

Figure 6.3 Monthly Volume of Stock Recommendations



Source: Based on the newsletters provided by the securities brokerage firms from April 2012 to April 2014.

6.3 Specification for Portfolio Models and Performance Measures

In the previous sections, we provided a general picture of the investment environment in the Taiwanese stock market and explained the data used in the empirical investigation. In the following part we specify the investment strategy models employed in the investigation and state the measures for evaluating performance of resulting portfolios.

Recall from Chapter 3 that the multi-analyst approach $(\boldsymbol{F_{MV}})$ developed for solving portfolio selection problems of n stocks with recommendations of Z analysts is given by

$$(\boldsymbol{F_{MV}}) \qquad \begin{aligned} &\max_{x\in\mathbb{R}^n} \min_{z\in Z} \quad \frac{1}{1+\exp\left(-\theta_z\left(\check{\hat{\mu}}_z{}^T x - \frac{\lambda}{2}x^T\check{\hat{\Sigma}}_z x - R^{Target}\right)\right)} \\ &s.t. \qquad x^T\,\mathbf{1} = 1 \\ &\qquad\qquad x_i \geq 0 \quad i = 1,2,\dots n \end{aligned} \qquad (6.1)$$

where θ_z is the credibility level of the analyst z given by the investor, $\check{\hat{\mu}}_z = \begin{pmatrix} \hat{\mu}_{H_Z} \\ \check{\mu}_{P_Z} \end{pmatrix}$ is the vector denoting the expected returns with $\hat{\mu}_{H_Z}$ representing the maximum likelihood estimate of the expected return vector and $\check{\mu}_{P_Z}$ representing the crisp possibilistic expected returns according to analyst z. $x \in \mathbb{R}^n$ is the decision vector denoting the weightings of the portfolio with x_i representing the proportion of capital invested in the i^{th} stock, λ is the risk aversion coefficient prescribed by the investor. $\check{\hat{\Sigma}}_z = \begin{pmatrix} \hat{\Sigma}_{HH_Z} & \check{\hat{\Sigma}}_{HP_Z} \\ \check{\hat{\Sigma}}_{PH_Z} & \check{\Sigma}_{PP_Z} \end{pmatrix}$ is the covariance matrix obtained from the maximum likelihood estimate of the variance $\hat{\sigma}_i{}^2$ of the historical dataset and the crisp possibilistic variance $\check{\sigma}_{z_i}^2$ of the practical analysts' dataset, based on the historical correlation coefficient matrix $Corr(\mu)$. R^{Target} is the middle aspiration level of the investment performance prescribed by the investor. See Section 3.2 for the details of the multi-analyst portfolio selection approach $(\boldsymbol{F_{MV}})$.

The corresponding robust multi-analyst approach $(\boldsymbol{RE_{HP}})$, with the ellipsoid uncertainty set describing estimation errors and uncertainties in the expected return estimates, is formulated as

$$(\boldsymbol{RE_{HP}}) \quad \begin{array}{ll} \max\limits_{x\in\mathbb{R}^n} \min\limits_{z\in Z} \min\limits_{\mu_z} & \dfrac{1}{1+exp\left(-\theta_z\left({\mu_z}^T x - \dfrac{\lambda}{2}x^T\check{\hat{\Sigma}}_z x - R^{Target}\right)\right)} \\ s.t. & x^T\,\mathbf{1} = 1 \\ & x_i \geq 0 \quad i = 1,2,\dots n \\ & \left(\mu_{H_z} - \hat{\mu}_{H_z}\right)^T {\hat{\Sigma}_{HH_z}}^{-1}\left(\mu_{H_z} - \hat{\mu}_{H_z}\right) \leq \delta_{H_z}^2 \\ & \mu_{P_z} = \check{\mu}_{P_z} \end{array} \tag{6.2}$$

where $\mu_z = \begin{pmatrix}\mu_{H_z} \\ \mu_{P_z}\end{pmatrix}$ denotes the vector of returns with μ_{H_z} and μ_{P_z} representing the returns of the historical dataset and the professional dataset, respectively. $\hat{\mu}_{H_z}$ is the maximum likelihood estimate of the return vector μ_{H_z} and $\check{\mu}_{P_z}$ is the crisp possibilistic expected returns according to analyst z's suggestions. $\check{\hat{\Sigma}}_z = \begin{pmatrix}\hat{\Sigma}_{HH_z} & \check{\hat{\Sigma}}_{HP_z} \\ \check{\hat{\Sigma}}_{PH_z} & \check{\Sigma}_{PP_z}\end{pmatrix}$ is the covariance matrix. $\hat{\mu}_{H_z}$ and $\hat{\Sigma}_{HH_z}$ denote the maximum likelihood estimate of the expected return vector and covariance matrix based on the historical performance. δ_{H_z} describes the desired robustness level for the return estimate $\hat{\mu}_{H_z}$. $\check{\mu}_{P_z}$ denotes the crisp possibilistic expected returns according to analyst z. See Section 4.2.2 for details of the robust multi-analyst portfolio selection approach $(\boldsymbol{RE_{HP}})$.

As the focus of the empirical study is to investigate whether the proposed multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, overcome the drawback of the conventional robust portfolio optimisation model (Ben-Tal & Nemirovski, 1998) and to evaluate the advantages and disadvantages of adopting analysts' suggestions as one of the inputs for investment decision making, we consider the classical mean-variance portfolio selection model, the robust counterpart approach, and the equally-weighted $(\mathbf{1/N})$ asset allocation as the benchmarks for comparison purposes. Note that the parameter estimates of the following standard investment strategies are obtained from the historical stock performances.

The first benchmark is the risk aversion formulation of the mean-variance portfolio optimisation problem (Markowitz, 1952), which is also considered as the fundamental framework for determining optimal portfolios

$$
(\boldsymbol{P_{MV}}) \qquad \begin{array}{ll} \max\limits_{x\in\mathbb{R}^n} & \hat{\mu}^T x - \dfrac{\lambda}{2} x^T \hat{\Sigma} x - R^{Target} \\ s.t. & x^T \mathbf{1} = 1 \\ & x_i \geq 0 \quad i = 1,2,\dots n \end{array} \tag{6.3}
$$

where $\hat{\mu} \in \mathbb{R}^n$ is the vector denoting the maximum likelihood estimate of the expected returns, $\hat{\Sigma} = [\hat{\sigma}_{ij}] \in \mathbb{R}^n \times \mathbb{R}^n$ is the maximum likelihood estimate of the covariance matrix with variance $\hat{\sigma}_{ii} = \hat{\sigma}_i^2$ for $i = j$ and covariance $\hat{\sigma}_{ij}$ for $i \neq j$.

The second benchmark is the robust counterpart approach (Ben-Tal & Nemirovski, 1998) to the mean-variance portfolio optimisation problem, with ellipsoid uncertainty set describing the uncertainties in the expected return estimates as follows

$$
(\boldsymbol{R_{MV}}) \qquad \begin{array}{ll} \max\limits_{x\in\mathbb{R}^n} \min\limits_{\mu} & \mu^T x - \dfrac{\lambda}{2} x^T \hat{\Sigma} x - R^{Target} \\ s.t. & x^T \mathbf{1} = 1 \\ & x_i \geq 0 \quad i = 1,2,\dots n \\ & (\mu - \hat{\mu})^T \hat{\Sigma}^{-1} (\mu - \hat{\mu}) \leq \delta^2 \end{array} \tag{6.4}
$$

where $\hat{\mu}$ and $\hat{\Sigma}$ represent the maximum likelihood estimates of the expected returns and the covariance matrix, respectively. δ denotes the desired robustness level for the ellipsoid uncertainty set.

The third benchmark is the equally-weighted $(\mathbf{1/N})$ asset allocation (DeMiguel et al., 2009). The reason for including the equally-weighted $(\mathbf{1/N})$ asset allocation as one of the benchmarks for the investigation is that the equally-weighted ($\mathbf{1/N}$) approach does not require any input parameter estimates or optimisation models for asset allocation. Hence the investor can arrange the investment without expending any additional efforts or costs. Table 6.1 summaries the asset allocation models considered for the empirical study.

Table 6.1 List of Asset Allocation Models Considered in the Empirical Study

No.	Abbreviation	Model
Asset allocation models developed in this research		
1	$\boldsymbol{F_{MV}}$	Multi-analyst portfolio selection with fuzzy aspiration
2	$\boldsymbol{RE_{HP}}$	Robust multi-analyst portfolio selection with fuzzy aspiration
Asset allocation models from existing literature		
3	$\boldsymbol{P_{MV}}$	Classical mean-variance portfolio selection (Markowitz, 1952)
4	$\boldsymbol{R_{MV}}$	Robust portfolio selection (Ben-Tal & Nemirovski, 1998)
5	$\mathbf{1/N}$	Equally-weighted asset allocation (DeMiguel et al., 2009)

Analysing the performance of the optimal portfolio is the essential step after constructing the portfolio selection model. Generally speaking, the primary criteria for justifying whether a portfolio outperforms another are the return, volatility, Sharpe ratio, utility value, and efficient frontier (e.g. Carlsson et al., 2002; Delage and Ye, 2010; Garlappi et al., 2007; Schöttle, 2007). On top of these basic measures for portfolio performance, Lutgens and Schotman (2010) utilise "investor's loss" and "investor's disappointment" in further evaluating ex-post realised portfolio performance. The loss function calculates the loss obtained from selecting one resulting portfolio x^* of a specific portfolio selection model, which is different with the true portfolio x^0 that is generated from the realised parameter values, μ^0 and Σ^0. In other words, the value of the loss function increases as the resulting portfolio x^* moves further from the true portfolio x^0. That is,

$$Loss(\mu^0, \Sigma^0 | x^*) = {\mu^0}^T x^0 - \acute{\mu}^T x^* \tag{6.5}$$

with $\acute{\mu} = (\acute{\mu}_1, \acute{\mu}_2, \dots, \acute{\mu}_n)$ denoting a choice of parameter estimator of returns μ adopted for a specific portfolio selection model. The realised parameter values, μ^0 and Σ^0, are also referred to as the true parameter values. The disappointment function is the difference between the expected and the realised profit by choosing the portfolio x^* of a specific portfolio selection model.

$$\begin{aligned} Disappointment(x^*) &= \acute{\mu}^T x^* - {\mu^0}^T x^* \\ &= (\acute{\mu} - \mu^0)^T x^* . \end{aligned} \tag{6.6}$$

This disappointment indicator characterises the predictive power of the chosen parameter estimate and the value of the disappointment can be either positive or negative. The higher the value of the disappointment, the more inferior the quality of a particular choice of parameter estimates.

On the other hand, Gregory et al. (2011) examined the portfolio optimisation models from a different perspective. In their study, they have investigated the robust effects on the performance of the robust portfolio optimisation model by adopting different levels of robustification and comparing the portfolio performance using the portfolio behaviour, portfolio robustness and the cost of the application.

In this study the portfolio performances are evaluated in terms of: (i) the ex-ante expected performance, (ii) the ex-post realised performance, and (iii) the portfolio behaviours. To be more specific in respect of the criteria of the portfolio performance for the investigation, the ex-ante expected performance includes only the expected portfolio return, expected standard deviation and the expected risk adjusted return. On the other hand, the ex-post realised performance includes three measures, i.e., the realised portfolio return, the loss, and the disappointment. The last investigation focuses on the characteristics of the portfolio, such as the number of stocks in the portfolio and the percentage of investment in the recommended stocks.

Furthermore, in a similar spirit to Gregory et al. (2011), four hypothetical situations are considered in this empirical investigation to represent different behaviours of the investor. More specifically, instead of examining the multi-analyst portfolio selection approach $(\boldsymbol{F_{MV}})$ and the related robust counterpart approach $(\boldsymbol{RE_{HP}})$ under each particular condition, different types and levels of confidence are considered. From a broad perspective, the behaviours of an investor are categorised according to the investor's belief in the parameter estimates of different datasets.

By adopting the level of credibility θ to express the confidence in the analysts' recommendations and the desired robustness level δ for describing the investor's aversion to estimation errors and uncertainties in the historical dataset, four types of investors are considered in this empirical study. Table 6.2 provides a summary of different types of investor.

Table 6.2 Description of Investment Behaviours

	Stronger Belief in Historical Performance	**Less Belief in Historical Performance**
Equal Preference for the Analysts	Type A Analyst Level $\theta_1 = \theta_2$ Robust Level $\delta \cong 0.23$	Type B Analyst Level $\theta_1 = \theta_2$ Robust Level $\delta = 1$
Unequal Preference for the Analysts	Type C Analyst Level $\theta_1 > \theta_2$ Robust Level $\delta \cong 0.23$	Type D Analyst Level $\theta_1 > \theta_2$ Robust Level $\delta = 1$

In the following sections we report the results of the empirical investigation for the multi-analyst approach $(\boldsymbol{F_{MV}})$ and the related robust counterpart approach $(\boldsymbol{RE_{HP}})$ proposed in this study. The objective of this investigation is to compare the performance of the classical portfolio selection methods with that of the proposed investment methods on real market data. Two empirical cases are considered from different investment perspectives.

The first empirical case examines the portfolio performances of the proposed portfolio selection methods under the assumption that the investor does not have sufficient knowledge regarding the credibility of the analysts and therefore has an equal preference for both analysts, i.e., $\theta_1 = \theta_2 = \frac{1}{2}$. The second empirical case examines the proposed models under the assumption that the investor has a stronger preference for one of the analysts. In the subsequent analysis, the level of credibility θ_z to each individual analyst z is taken as the market share of the financial institution. That is, the greater the market share of the securities brokerage firm, the more confident the investor is with its investment forecasts. In the second empirical case, the levels of credibility of the analyst are $\theta_1 = 0.7635$ and $\theta_2 = 0.2365$.

The software package MATLAB was used for solving the portfolio selection problems and generating the test results.

6.4 Empirical Analysis: Portfolio Management with Equal Preference for the Analysts

Of the four types of investors given in Table 6.2, the Type A and Type B investors are assumed to have no knowledge regarding the credibility of the analysts. Hence they treat the additional investment recommendations as equally important, i.e., both types of investor have the same value of θ, as $\theta_1 = \theta_2 = \frac{1}{2}$. However, they have differing opinions regarding the parameter estimates generated from the historical performances of stocks and therefore have differing desired robustness levels for the ellipsoidal uncertainty set. The Type A investor is assumed to have strong belief that historical stock performances are good reflections of their future performance and therefore assigns a tighter uncertainty set for the return estimates with $\delta \cong 0.23$ for the confidence ellipsoid, where the true values of the stock returns are expected to fall in the ellipsoidal uncertainty set $U^{Ellipsoid}$ with probability at least 95%. By contrast, the Type B investor is more hesitant about employing historical performances to estimate future stock performances. Hence, the Type B investor assigns $\delta = 1$ for the loose ellipsoidal uncertainty set $U^{Ellipsoid}$, where the true values of the stock returns are expected to be in the ellipsoidal uncertainty set $U^{Ellipsoid}$ at a 50% confidence level[9]. In the following analysis $(\boldsymbol{F_{MV}})$, $(\boldsymbol{RE_{HP-A}})$ and $(\boldsymbol{RE_{HP-B}})$ denote the multi-analyst approach, the robust multi-analyst approach of the Type A investor and the Type B investor, respectively.

6.4.1 The Ex-Ante Expected Performances of Various Investment Strategies

As discussed earlier in Chapter 2, the robust counterpart approach of the portfolio selection model is the worst-case scenario approach, which always considers the most pessimistic outcome and assumes that the parameter uncertainty will have a negative impact on stock returns; therefore, the resulting robust portfolio is expected to be more conservative, especially in terms of expected return, compared to the asset allocation of the original portfolio selection model without robustification.

Table 6.3 reports the average expected portfolio returns and the standard deviations of the investment strategies at different risk aversion levels with D days investment holding period and Table 6.4 reports the average expected risk adjusted returns of the

[9] See Section 2.2.2 for further information about the size of the uncertainty set.

Table 6.3 Expected Returns and Standard Deviations of the Optimal Portfolios

Holding Period	Risk Aversion	Mean (%)						Standard Deviation (%)					
		F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}	F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}
5 Days	$\lambda = 0$	2.2011	1.9270	1.0454	2.0888	1.7145	0.4443	5.6543	4.0691	1.5842	5.8578	4.0317	1.0768
	$\lambda = 0.5$	0.9148	0.7981	0.5836	0.7736	0.6177	0.3163	1.4335	1.2914	1.0868	1.3821	1.2031	0.9577
	$\lambda = 5$	0.2895	0.2732	0.2194	0.2299	0.2131	0.1566	0.8879	0.8872	0.8860	0.8857	0.8848	0.8828
	$\lambda \to \infty$	0.1958	0.1539	0.0053	0.1743	0.1610	0.1138	2.2208	2.2208	2.2208	0.8785	0.8785	0.8785
10 Days	$\lambda = 0$	4.2470	3.8198	1.8993	4.2119	3.7592	1.5164	8.1407	6.3396	2.1428	8.1934	6.3965	1.8221
	$\lambda = 0.5$	1.8308	1.6297	1.1564	1.7264	1.5112	0.9734	1.9034	1.7325	1.4070	2.7226	2.2888	1.3389
	$\lambda = 5$	0.6428	0.6329	0.5373	0.5909	0.5668	0.4850	1.1456	1.1629	1.1864	1.1475	1.1456	1.1404
	$\lambda \to \infty$	0.3285	0.2654	0.0417	0.4326	0.4162	0.3588	3.3021	3.3021	3.3021	1.1282	1.1282	1.1281
15 Days	$\lambda = 0$	6.1615	5.6654	2.9494	6.1341	5.6509	2.7193	9.7356	7.7637	2.6263	9.7851	7.8181	2.4533
	$\lambda = 0.5$	2.7236	2.4692	1.8285	2.6333	2.3758	1.7049	2.1777	1.9863	1.6025	2.1797	1.9828	1.5687
	$\lambda = 5$	0.9510	0.8924	0.7336	0.9893	0.9597	0.8603	1.5945	1.6923	1.8768	1.2667	1.2638	1.2555
	$\lambda \to \infty$	0.4636	0.3824	0.0945	0.7068	0.6895	0.6307	4.2190	4.2190	4.2190	1.2340	1.2339	1.2338
20 Days	$\lambda = 0$	8.0995	7.5797	4.1391	8.1109	7.5741	3.9998	11.8086	9.2197	3.0965	11.8270	9.2343	3.0143
	$\lambda = 0.5$	3.6472	3.3501	2.5926	3.5757	3.2820	2.5337	2.3969	2.1925	1.7726	2.4050	2.1984	1.7679
	$\lambda = 5$	0.8891	0.7965	0.3662	1.4168	1.3814	1.2622	3.2437	3.3475	4.0370	1.3541	1.3504	1.3391
	$\lambda \to \infty$	0.6072	0.5077	0.1551	0.9580	0.9411	0.8814	5.1524	5.1524	5.1524	1.3024	1.3023	1.3024

Notes: This table displays the average expected portfolio returns and the standard deviations of 492 observations for the multi-analyst portfolios, i.e., the multi-analyst approach (F_{MV}), the robust multi-analyst approach of the Type A investor (RE_{HP-A}) and the Type B investor (RE_{HP-B}). The alternative approaches are the mean-variance approach (P_{MV}), and the conventional robust approach of the Type A investor (R_{MV-A}) and the Type B investor (R_{MV-B}).

Table 6.4   Expected Risk Adjusted Returns of the Optimal Portfolios

Holding Period	Risk Aversion	Expected Risk Adjusted Return (%)						
		F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}	$1/N$
5 Days	$\lambda = 0$	0.3893	0.4736	0.6599	0.3566	0.4253	0.4126	0.0631
	$\lambda = 0.5$	0.6382	0.6180	0.5370	0.5597	0.5134	0.3303	
	$\lambda = 5$	0.3261	0.3079	0.2476	0.2595	0.2408	0.1773	
	$\lambda \to \infty$	0.0882	0.0693	0.0024	0.1985	0.1833	0.1295	
10 Days	$\lambda = 0$	0.5217	0.6025	0.8864	0.5141	0.5877	0.8322	0.0828
	$\lambda = 0.5$	0.9619	0.9407	0.8219	0.9140	0.8870	0.7271	
	$\lambda = 5$	0.5611	0.5442	0.4529	0.5150	0.4947	0.4253	
	$\lambda \to \infty$	0.0995	0.0804	0.0126	0.3835	0.3689	0.3180	
15 Days	$\lambda = 0$	0.6329	0.7297	1.1230	0.6269	0.7228	1.1084	0.0969
	$\lambda = 0.5$	1.2507	1.2431	1.1410	1.2081	1.1982	1.0868	
	$\lambda = 5$	0.5964	0.5273	0.3909	0.7810	0.7593	0.6852	
	$\lambda \to \infty$	0.1099	0.0906	0.0224	0.5728	0.5588	0.5112	
20 Days	$\lambda = 0$	0.6859	0.8221	1.3367	0.6858	0.8202	1.3269	0.1072
	$\lambda = 0.5$	1.5216	1.5280	1.4626	1.4868	1.4929	1.4332	
	$\lambda = 5$	0.2741	0.2379	0.0907	1.0463	1.0229	0.9425	
	$\lambda \to \infty$	0.1178	0.0985	0.0301	0.7356	0.7226	0.6768	

Note: This table displays the expected risk adjusted returns (Mean/SD) of portfolio for various investment strategies. The equally weighted portfolio is denoted by $(1/N)$. All the figures in this table are the average performances of 492 observations in percentage.

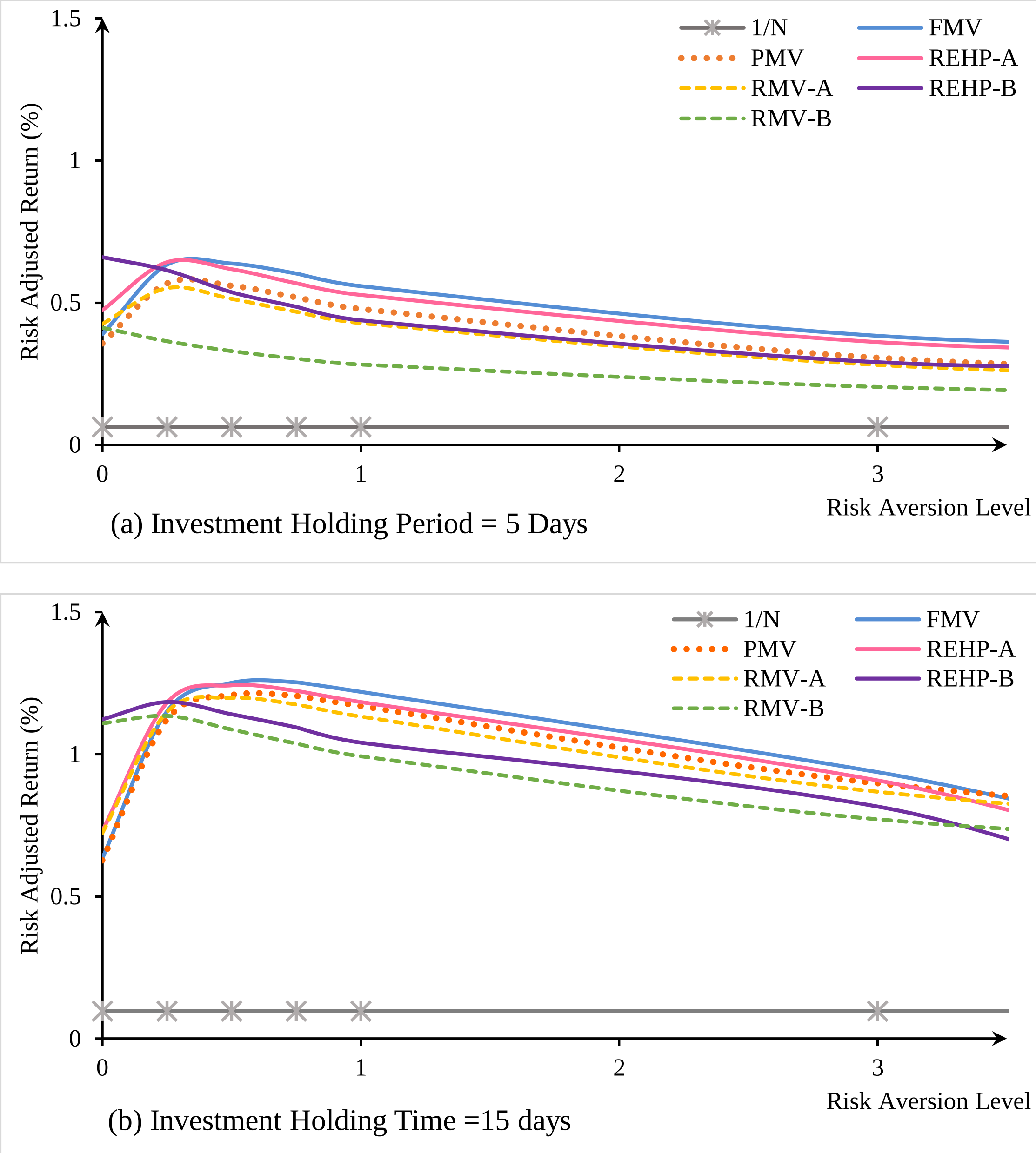
investment strategies considered in this study. Note that the results of $\lambda = 0$ correspond to the performances of the maximum return portfolios, whereas the results of $\lambda \to \infty$ correspond to the performances of the minimum variance portfolio.

From Table 6.3, the classical mean-variance portfolio allocation $(\boldsymbol{P_{MV}})$ based on the point estimates of the historical data has a higher expected return and higher variance than the conventional robust portfolio allocation $(\boldsymbol{R_{MV}})$ under different risk aversion levels. The difference between the mean-variance portfolio and the robust portfolio widens as the desired robustness level of the uncertainty set increases. In other words, the mean-variance and the robust portfolios have similar performance if the desired robustness level δ of the uncertainty set is low. A similar finding has also been reported by Goldfarb and Iyengar (2003). Furthermore, it is noted that the difference between the mean-variance and the robust portfolios decreases as the risk aversion increases. This is because we only consider estimation errors and parameter uncertainties in the expected returns for the robust portfolio selection problems, so that the robustification has stronger effect as risk aversion decreases (i.e., towards the maximum return allocation) and has less impact on the portfolio performance as risk aversion increases.

The same patterns of expected portfolio returns can be found between the multi-analyst portfolios $(\boldsymbol{F_{MV}})$ and the robust multi-analyst portfolios $(\boldsymbol{RE_{HP}})$, i.e., the expected portfolio return decreases as the risk aversion increases and the difference in the expected returns between investment strategies widens as the desired robustness level δ of the uncertainty set increases. The potential benefits from incorporating additional professional investment information are indicated in Table 6.3 and Table 6.4.

It is not surprising that the multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{P_{MV}})$, achieve greater expected returns and expected risk adjusted returns than the conventional investment strategies, $(\boldsymbol{P_{MV}})$ and $(\boldsymbol{R_{MV}})$, under a more risk-loving setting because the stock recommendations provided by the analysts are usually buy-side suggestions with relatively higher return estimates than the past stock performances. Furthermore, as in Figure 6.4 below, the advantage of employing robustification can be observed clearly under the case of $\lambda = 0$. Unlike the mean-variance portfolio $(\boldsymbol{P_{MV}})$ and the multi-analyst portfolio $(\boldsymbol{F_{MV}})$ that have very extreme weightings when the risk aversion coefficient is $\lambda = 0$, the robust counterparts to these two models, $(\boldsymbol{R_{MV}})$ and $(\boldsymbol{RE_{HP}})$, generated more diversified portfolios and therefore achieved greater expected risk adjusted returns (see Table 6.7 for further information regarding the portfolio weightings).

Figure 6.4 Expected Risk Adjusted Returns under Different Risk Aversion Levels



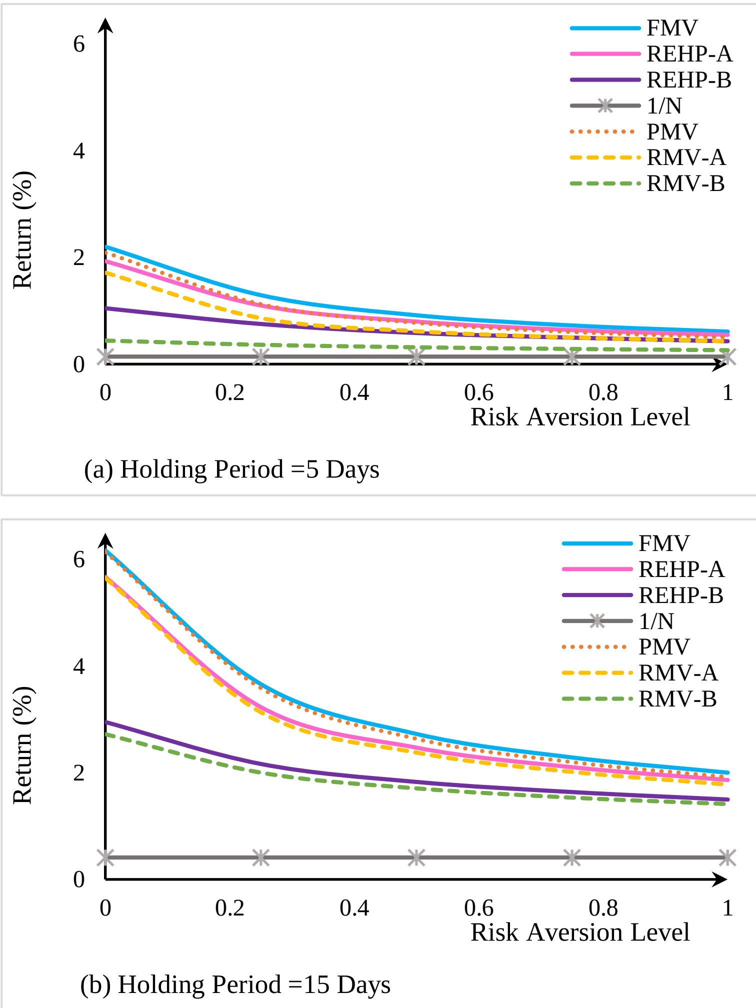
(a) Investment Holding Period = 5 Days

(b) Investment Holding Time =15 days

Note: This figure reports the expected risk adjusted returns under different risk aversion levels. Panel A and B show the results of various investment strategies with holding time frame of 5 days and 15 days, respectively. The figures for 10 days and 20 days investment holding time frame can be found in Appendix B.1.

One may notice that as the investment holding time frame increases, the multi-analyst portfolios and the corresponding conventional investment strategies tend to have similar expected performance in portfolio returns. This can be explained by the fact that the return estimates generated from the analysts' data don't have any information regarding that particular time frame and, therefore, examining investment strategies with a longer investment time frame, i.e., 20 days, may lead to the situation that the analysts' return estimates become more pessimistic, i.e., anticipating lower expected returns, than the historical return estimates and they are therefore unlikely to be chosen by the optimal portfolio. More specifically, the benefit of adopting professional investment recommendations is more significant for short-term investment. The following figure graphically compares the expected portfolio returns between various investment strategies over two different holding time frames.

Figure 6.5 Expected Portfolio Returns over Sample Period



Note: The figures for 10 days and 20 days investment holding time frame can be found in Appendix B.1.

From the above results, it can be seen that the advantages of the multi-analyst approaches $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$ diminish as the risk aversion parameter increases. To explain this, we recall equation (6.1) of the multi-analyst approach with fuzzy aspiration level $(\boldsymbol{F_{MV}})$:

$$(\boldsymbol{F_{MV}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \quad \frac{1}{1+\exp\left(-\theta_z\left(\check{\hat{\mu}}_z{}^T x - \frac{\lambda}{2} x^T \check{\hat{\Sigma}}_z x - R^{Target}\right)\right)}$$

In the above equation, the parameter θ_z denotes the credibility level for each individual analyst z and determines the curve of the membership function for the investment goal[10]. As the risk aversion coefficient λ increases, the resulting asset allocation is more heavily penalised by the expected risk measures. That is, as λ increases, the portfolio selection model varies from a trade-off between the return and risk to the scenario that focuses only on the risk. Since the stock recommendations of the analysts are usually predicted with greater returns but higher variations, therefore, the analysts' recommendations that aim to enhance the portfolio return do not affect much of the asset allocation when λ increases.

6.4.2 The Ex-Post Realised Performances of Various Investment Strategies

Next, we turn to consider the realised portfolio performances of various investment strategies based on the simple average of 492 observations. Table 6.5 reports the realised portfolio returns and Table 6.6 provides values of the portfolio loss and disappointment. By definition of the loss function (6.5) and the disappointment function (6.6), the values of the functions are correlated with the accuracy of the parameter estimates of the expected returns and the covariance matrix. Both loss and disappointment values can be positive and negative and the smaller the values are, the better the portfolio has performed.

[10] Reader should refer to Section 3.2 for more details.

Table 6.5 Realised Returns of the Optimal Portfolios

Holding Period	Risk Aversion	Realised Return (%)						
		F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}	$1/N$
5 Days	$\lambda = 0$	1.2719	1.2753	0.5556	1.1770	1.2394	0.4825	0.2382
	$\lambda = 0.5$	0.7022	0.6034	0.4384	0.6535	0.5590	0.3893	
	$\lambda = 5$	0.3074	0.3053	0.3005	0.2957	0.2940	0.2891	
	$\lambda \to \infty$	0.2382	0.2382	0.2382	0.2613	0.2610	0.2616	
10 Days	$\lambda = 0$	2.2190	2.3045	1.4857	2.2526	2.2895	1.3742	0.4851
	$\lambda = 0.5$	1.4044	1.2969	1.0443	1.3810	1.2703	0.9739	
	$\lambda = 5$	0.7078	0.6938	0.6804	0.6749	0.6692	0.6535	
	$\lambda \to \infty$	0.4851	0.4851	0.4851	0.5856	0.5851	0.5846	
15 Days	$\lambda = 0$	3.8485	3.4289	2.3381	3.9046	3.4262	2.2184	0.7456
	$\lambda = 0.5$	2.0217	1.9033	1.5770	1.9952	1.8731	1.5000	
	$\lambda = 5$	0.9504	0.9009	0.8850	1.0183	1.0122	0.9960	
	$\lambda \to \infty$	0.7456	0.7456	0.7456	0.9185	0.9195	0.9180	
20 Days	$\lambda = 0$	4.9979	4.3966	3.3945	5.1346	4.4680	3.3458	1.0252
	$\lambda = 0.5$	2.6823	2.5023	2.1584	2.6661	2.4758	2.1003	
	$\lambda = 5$	1.2532	1.1966	1.0675	1.5138	1.5053	1.4791	
	$\lambda \to \infty$	1.0252	1.0252	1.0252	1.2691	1.2689	1.2699	

Notes: This table displays the realised portfolio returns for various investment strategies. All the figures in this table are the average performances of 492 observations, expressed as percentages.

Table 6.6 Loss and Disappointment Rates of the Optimal Portfolios

Holding Period	Risk Aversion	Loss (%)						Disappointment (%)					
		F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}	F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}
5 Days	$\lambda = 0$	12.5417	12.5383	13.2580	12.6366	12.5742	13.3311	0.9292	0.6517	0.4898	0.9117	0.4751	-0.0382
	$\lambda = 0.5$	13.1114	13.2102	13.3752	13.1601	13.2546	13.4243	0.2011	0.1510	0.0140	0.1201	0.0586	-0.0730
	$\lambda = 5$	13.5062	13.5083	13.5131	13.5179	13.5196	13.5245	-0.0179	-0.0321	-0.0811	-0.0658	-0.0809	-0.1325
	$\lambda \to \infty$	13.5754	13.5754	13.5754	13.5523	13.5526	13.5520	-0.0424	-0.0843	-0.2329	-0.0869	-0.1000	-0.1478
10 Days	$\lambda = 0$	18.0110	17.9255	18.7443	17.9774	17.9405	18.8558	2.0280	1.5153	0.4136	1.9593	1.4697	0.1422
	$\lambda = 0.5$	18.8256	18.9331	19.1857	18.8490	18.9597	19.2561	0.4264	0.3328	0.1121	0.3454	0.2409	-0.0005
	$\lambda = 5$	19.5222	19.5362	19.5496	19.5551	19.5608	19.5765	-0.0650	-0.0609	-0.1431	-0.0840	-0.1024	-0.1685
	$\lambda \to \infty$	19.7449	19.7449	19.7449	19.6444	19.6449	19.6454	-0.1566	-0.2197	-0.4434	-0.1530	-0.1689	-0.2258
15 Days	$\lambda = 0$	22.1830	22.6026	23.6934	22.1269	22.6053	23.8131	2.3130	2.2365	0.6113	2.2294	2.2246	0.5009
	$\lambda = 0.5$	24.0098	24.1282	24.4545	24.0363	24.1584	24.5315	0.7019	0.5659	0.2515	0.6381	0.5027	0.2049
	$\lambda = 5$	25.0811	25.1306	25.1465	25.0132	25.0193	25.0355	0.0006	-0.0085	-0.1514	-0.0289	-0.0525	-0.1357
	$\lambda \to \infty$	25.2859	25.2859	25.2859	25.1130	25.1120	25.1135	-0.2820	-0.3632	-0.6511	-0.2118	-0.2301	-0.2873
20 Days	$\lambda = 0$	26.0393	26.6406	27.6427	25.9026	26.5692	27.6914	3.0991	3.1831	0.7446	2.9763	3.1061	0.6540
	$\lambda = 0.5$	28.3549	28.5349	28.8788	28.3711	28.5614	28.9369	0.9649	0.8478	0.4342	0.9096	0.8062	0.4334
	$\lambda = 5$	29.7840	29.8406	29.9697	29.5234	29.5319	29.5581	-0.3641	-0.4001	-0.7013	-0.0970	-0.1240	-0.2169
	$\lambda \to \infty$	30.0120	30.0120	30.0120	29.7681	29.7683	29.7673	-0.4180	-0.5175	-0.8701	-0.3111	-0.3278	-0.3885

Note: This table displays the average investor's losses and investor's disappointments of 492 observations for various investment strategies. See Section 6.3 for further details of the loss and the disappointment functions.

First, we note that the realised returns and the losses of the multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, are consistent with the realised performance of the equally weighted portfolio $(\mathbf{1}/\boldsymbol{N})$ when risk aversion level $\lambda \to \infty$. This is because the required investment benchmark R^{Target} is fixed at the expected return of the equally weighted portfolio $(\mathbf{1}/\boldsymbol{N})$, and therefore, the resulting asset allocations of the multi-analyst approaches become equally weighted as $\lambda \to \infty$. On the other hand, the respective portfolio disappointments of the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, differ under $\lambda \to \infty$, which is caused by the expected return estimates of the practical analysts' data and the robustification of the uncertainty sets.

Next, we consider the cases where $\lambda < +\infty$. It can be seen from Table 6.5 that the multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, usually have superior realised returns to the corresponding conventional asset allocations when holding the investment for 5 days and 10 days. As the investment holding period increases, the benefit of applying multi-analyst approaches gradually disappears. However, the multi-analyst approaches generally achieve greater realised portfolio returns for all chosen holding time frames under a more risk-loving setting, i.e., $\lambda = 0$ and $\lambda = 0.5$. It is very interesting to observe that the multi-analyst approach $(\boldsymbol{F_{MV}})$ obtained lower portfolio returns than the classical mean-variance portfolio selection at $\lambda = 0$ for $D = 10{,}15{,}20$. In addition to the fact that the stock recommendations have less impact on the multi-analyst portfolio selection process as the investment holding time frame D increases, i.e., the portion of wealth invested in the recommended stocks is very low for the multi-analyst portfolio $(\boldsymbol{F_{MV}})$ at $D = 10{,}15{,}20$ (see Table 6.7), this also indicates the need of employing robustification for dealing with the estimation errors and parameter uncertainties of the historical dataset in the multi-analyst portfolio selection framework.

The results of portfolio loss have similar patterns to the realised portfolio returns between various investment strategies. That is, the multi-analyst approaches outperform the corresponding conventional investment strategies in either a short-term investment or a more risk-loving investment. This is because the loss function calculates the difference of the realised return between the true portfolio and the optimal portfolio. Therefore the higher the realised return of the chosen portfolio, the lower the portfolio loss derived from this particular choice.

Although the multi-analyst approaches have some exciting performances under certain circumstances, they mostly have poorer performances in terms of portfolio disappointment compared with conventional investment strategies. This is not entirely surprising: the analysts' recommendations are usually more optimistic with higher expected returns than the historical data, leading to a higher hope and hence potentially greater disappointment. Therefore, the poorer 'disappointment' performance is mainly due to the predictive power of the analysts.

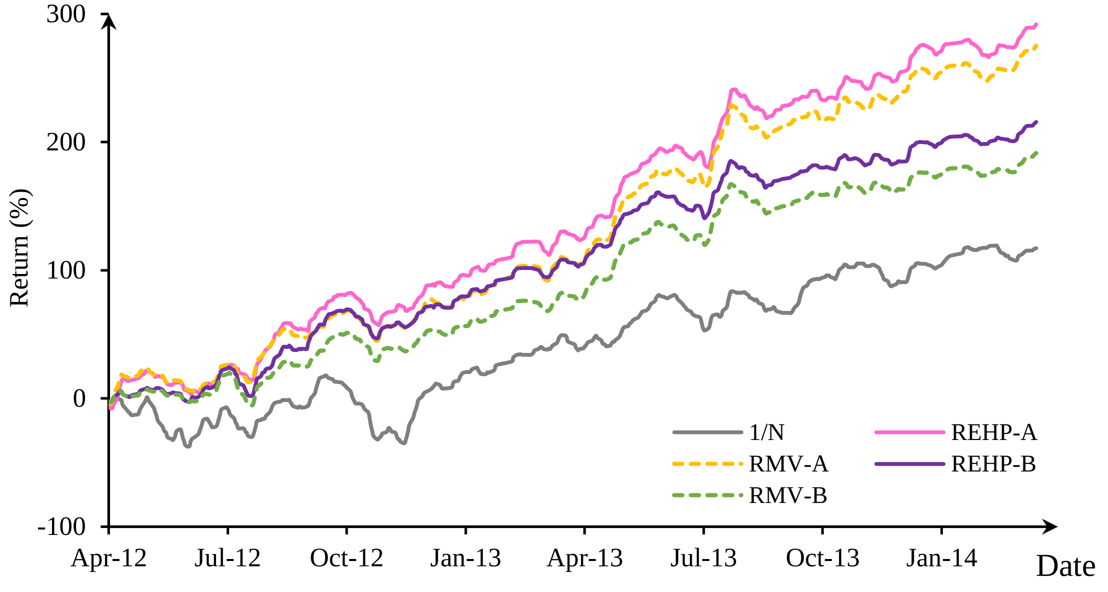
This also applies to the performances of the robust methods. Because of the impact of the uncertainty set, the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ generates more conservative asset allocations compared to the multi-analyst approach $(\boldsymbol{F_{MV}})$. As a result, the difference between the predicted and realised portfolio returns is reduced, which then leads to lower portfolio disappointments for the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$. The same effect can be found between the mean-variance portfolios $(\boldsymbol{P_{MV}})$ and the robust portfolios $(\boldsymbol{R_{MV}})$, where the mean-variance portfolios $(\boldsymbol{P_{MV}})$ generally have higher expected returns, higher expected risk levels, and higher disappointments than the related robust portfolios $(\boldsymbol{R_{MV}})$.

Overall, the conventional and the proposed robust counterpart approaches, $(\boldsymbol{R_{MV}})$ and $(\boldsymbol{RE_{HP}})$, are more conservative than their original frameworks. This feature of the robust models brings less profitable investment and can be understood as the cost of accounting for estimation errors and uncertainties in the portfolio allocation problems. Nevertheless, having incorporated the professional investment recommendations for asset allocation, the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ has improved the conventional robust counterpart approach $(\boldsymbol{R_{MV}})$ in terms of returns.

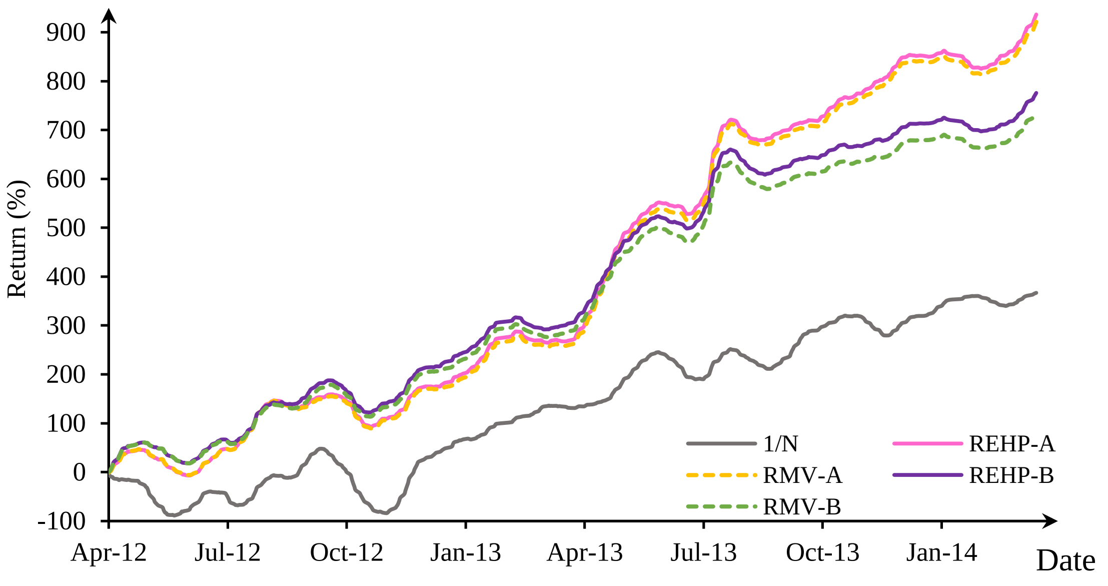
The following figure illustrates the realised cumulative returns of the robust portfolios, $(\boldsymbol{RE_{HP}})$ and $(\boldsymbol{R_{MV}})$, at $\lambda = 0.5$ over the sample period. As shown in Figure 6.6 (a), for both types of investor, the robust multi-analyst approaches $(\boldsymbol{RE_{HP}})$ exceed the equally weighted allocation and the conventional robust approach $(\boldsymbol{R_{MV}})$ for $D = 5$ days. In addition, with the more conservative robust multi-analyst approach, the type B investor $(\boldsymbol{RE_{HP-B}})$, has an outstanding performance during the period from September 2012 to May 2013. This result can be explained by the fact that the robust multi-analyst portfolios $\boldsymbol{RE_{HP-B}}$ have allocated more wealth in the advised stocks (see Figure 6.7) during this period and this therefore indicates the contribution of the professional investment recommendations. It is expected that imposing an uncertainty set on the multi-analyst approach for accounting for estimation errors and uncertainties of the historical dataset increases the portion of wealth allocated to the recommended stocks. This is because the stocks whose expected return estimates are based on historical data are penalised by robustification, hence, will have less weightings in the optimal portfolio[11]. In contrast, the recommended stocks are not penalised in the robust multi-analyst model, and furthermore, the more robust the model, the greater the weighting assigned to the recommended stocks. Figure 6.6 (b) shows the observation that the conservative robust multi-analyst approach $(\boldsymbol{RE_{HP-B}})$ also has an outstanding performance during the period from September 2012 to May 2013 and explicitly illustrates the impact of different robustness levels on portfolio performance.

[11] The purpose of applying robustification, i.e., adopting an uncertainty set for the expected returns, is to account for the estimation errors and uncertainties. By incorporating an uncertainty set into the portfolio selection framework, the optimal portfolio is based on the worst case scenario of the expected returns. In the proposed robust multi-analyst approach, the uncertainty set is only applied for the historical dataset. Hence the effect of robustification will only penalise stocks from the historical dataset.

Figure 6.6 Realised Cumulative Returns of Robust Portfolios



(a) Cumulative Realised Returns for *D*=5 days



(b) Cumulative Realised Returns for *D*=15 days

Note: This figure compares the realised returns of robust portfolios, (RE_{HP}) and (R_{MV}), with different robustness levels over the sample period. Panels (a) and (b) show the results with investment holding period of 5 days and 15 days, respectively. In Panel (a) the differences in total return between the robust and the robust multi-analyst portfolio are 16.88% for the Type A investor and 24.13% for the Type B investor. In Panel (b) the differences in total return between the robust and the robust multi-analyst portfolio are 14.87% for the Type A investor and 37.84% for the Type B investor. The figures for 10 days and 20 days investment holding period can be found in Appendix B.2.

As discussed earlier, the longer the investment holding period, the less effective the professional investment recommendations in the multi-analyst approaches. This is confirmed by the trend movements of the robust portfolios in Figure 6.6 (b), where the conventional robust and the robust multi-analyst portfolios of the Type A investor have similar patterns during the sample period. On the other hand, for the more conservative Type B investor, there is a larger gap between the trend movements of the robust and the robust multi-analyst portfolios, which indicates the connection between the size of the prescribed robustness level and the resulting portfolio weightings of the robust multi-analyst approach.

6.4.3 The Characteristics of the Portfolios

The following table provides an insight into the portfolio characteristics, where the number of selected stocks in the portfolio and the ratio of wealth invested in the recommended stocks help to understand the impact of the robustification and the risk aversion level on the multi-analyst approaches.

In general, we expect the mean-variance portfolios $(\boldsymbol{P_{MV}})$ and the conventional robust portfolios $(\boldsymbol{R_{MV}})$ to be more diversified than their counterparts of the multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HE}})$. This is because the analysts' investment recommendations are usually more optimistic, with higher expected returns than the historical performances. Hence the portfolios obtained via the multi-analyst approaches are more likely to focus on the recommended stocks with relatively higher weighting. Furthermore, as the size of the uncertainty set for the robust multi-analyst approach increases, a larger portion of wealth will be invested according to the analysts' recommendations. The reason for this is that the framework of the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ has distinguished the stocks into two sets, those with recommendations and those without them, and handles the estimation errors and

Table 6.7 The Characteristics of Optimal Portfolios

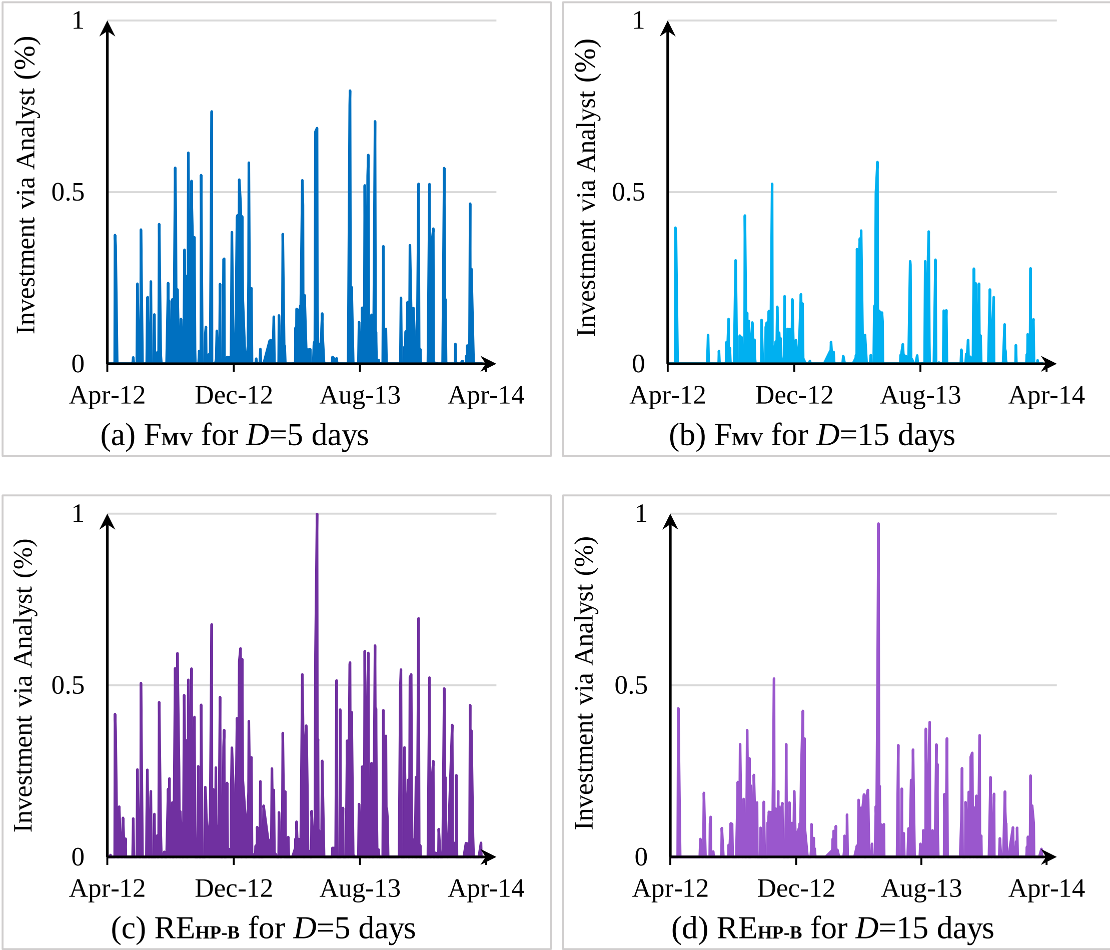
Holding Period	Risk Aversion	No. of Stocks						Investment via Analysts (%)					
		F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}	F_{MV}	RE_{HP-A}	RE_{HP-B}	P_{MV}	R_{MV-A}	R_{MV-B}
5 Days	$\lambda = 0$	1.10	3.75	16.65	1.00	3.90	19.15	9.55	15.99	22.40	2.24	2.00	1.19
	$\lambda = 0.5$	15.98	17.71	20.24	16.36	18.19	20.93	6.84	7.58	9.88	1.36	1.24	1.17
	$\lambda = 5$	20.58	20.63	20.83	20.17	20.22	20.27	3.47	3.52	3.57	1.21	1.21	1.22
	$\lambda \to \infty$	148.00	148.00	148.00	19.80	19.78	19.79	2.58	2.58	2.58	1.23	1.23	1.23
10 Days	$\lambda = 0$	1.02	2.53	13.31	1.00	2.47	13.99	2.44	5.53	14.69	2.24	1.84	1.71
	$\lambda = 0.5$	13.56	14.45	16.38	13.49	14.40	16.28	4.41	4.93	6.84	1.75	1.64	1.53
	$\lambda = 5$	22.81	22.90	24.17	18.67	18.70	18.83	2.66	2.76	3.02	1.55	1.55	1.56
	$\lambda \to \infty$	148.00	148.00	148.00	18.48	18.49	18.46	2.58	2.58	2.58	1.56	1.56	1.56
15 Days	$\lambda = 0$	1.03	2.07	11.68	1.00	2.03	11.88	1.76	2.60	9.29	2.44	2.03	2.45
	$\lambda = 0.5$	12.55	13.03	14.91	12.43	12.94	14.69	3.61	3.82	4.97	2.24	2.05	1.17
	$\lambda = 5$	29.49	32.48	38.65	16.91	16.96	17.09	2.53	2.59	2.84	1.67	1.68	1.69
	$\lambda \to \infty$	148.00	148.00	148.00	17.16	17.20	17.15	2.58	2.58	2.58	1.72	1.72	1.72
20 Days	$\lambda = 0$	1.03	1.90	9.94	1.00	1.89	9.96	0.81	1.72	6.22	2.44	2.08	2.73
	$\lambda = 0.5$	11.87	12.39	13.72	11.83	12.29	13.50	2.83	2.97	3.63	2.36	2.17	1.76
	$\lambda = 5$	79.00	81.76	115.06	16.65	16.65	16.69	2.07	2.13	2.40	1.70	1.71	1.72
	$\lambda \to \infty$	148.00	148.00	148.00	16.01	16.00	16.00	2.58	2.58	2.58	1.76	1.76	1.76

Notes: This table displays the portfolio characteristics including the number of stocks in the resulting portfolio (No. of Stocks) and the proportion of investment via analysts' recommendations (Investment via Analysts). The number of stocks is the sum of the stocks in the resulting portfolio and the investment via analysts represents the total fraction of wealth invested in the recommended stocks. The results presented in this table are the average of 492 observations.

uncertainties of these two sets differently. In the proposed robust multi-analyst approach $(\boldsymbol{RE_{HP}})$, the protection against the possible inaccurate parameter estimates is given by the uncertainty set. This uncertainty set only handles the estimation errors and uncertainties for the stocks without recommendations from the analysts, and, as the size of the uncertainty set increases, the worse the scenario is considered for these stocks. Consequently, this exaggerates the already optimistic prediction of the recommended stocks and leads to higher weightings for the recommended stocks.

Before further discussion, it is important to note that the analysts have made suggestions on approximately 3.82 stocks for every trading day. Therefore, on average, the equally weighted portfolio $(\boldsymbol{1/N})$ based on our sample of 148 stocks allocates a total of $\frac{3.82}{148} = 2.58\%$ capital to the recommended stocks. As shown in Table 6.7, the multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, have the same ratio of wealth invested in the recommended stocks as the equally weighted portfolio $(\boldsymbol{1/N})$ at $\lambda \to \infty$; this is because the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, turn into the equally weighted framework as $\lambda \to \infty$. Furthermore, the rate at which the multi-analyst approaches transform into the equally weighted framework as λ increases is positively correlated with the duration of the investment holding period. Apart from these factors, Table 6.7 confirms that the total ratio of wealth invested in the recommended stocks increases as the level of robustification increases for the multi-analyst approaches and, also, the potential advantage of adopting professional investment recommendations is less significant for long-term investment because the total ratio of wealth invested in the recommended stocks drops as the investment holding period increases. Figure 6.7 graphically illustrates the impact of the robustness level and the duration of the investment holding period on portfolio weightings for the multi-analyst approaches.

Figure 6.7 The Ratio of Wealth Invested in the Recommended Stocks



Note: This figure displays the total investment in the recommended stocks of the multi-analyst portfolios, (F_{MV}) and (RE_{HP-B}), at risk aversion coefficient $\lambda = 0.5$. The figures for $D = 10{,}20$ can be found in Appendix B.3.

In summary, we have investigated the changes in the portfolio performances along with different risk aversion coefficients λ and investment holding periods D. Based on the analysts' investment recommendations considered for this study, the proposed multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, are more suitable for short-term investment and normally outperform the corresponding conventional investment strategies in the more risk-loving setting. Although the multi-analyst approaches have inferior outcomes as $\lambda \to \infty$ and generate greater portfolio disappointments than other investment strategies, the more optimistic expected returns and also the more profitable ex-post outcomes, with relatively less regrets in terms of losses, may compensate for accepting these disadvantages of the multi-analyst approaches and, thus, encourage its user.

In the results reported above, the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, are examined under the circumstances that the investor has no knowledge regarding the credibility of the analysts and hence treats the investment recommendations provided by different analysts as equally important. Therefore, it is of interest to investigate the performances of the multi-analyst portfolios under the condition that the investor is more dependent on a particular analyst.

6.5 Empirical Analysis: Portfolio Management with Unequal Preference for the Analysts

On the basis of the previous empirical study, the following empirical investigation extends the earlier setting by assigning different levels of credibility to the analysts. As reported in Section 6.4, the analysts' recommendations have stronger effects on the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, for short-term investments. Therefore the second empirical investigation focuses on the results and findings of the weekly portfolio performances of the multi-analyst approaches, i.e., $D = 5$ days.

The portfolio selection framework of the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, considered in this section are the same as those in the first case but with an additional assumption that the investor has a stronger preference for one of the analysts. Furthermore, analogous to the previous empirical investigation, the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ will be examined with two levels of robustification. Recalling the types of investors stated in Table 6.2, we thus pay attention to the comparison between portfolios, which is generated according to different investors' requirements. In contrast to the Type A and Type B investors, suppose the Type C and Type D investors agree that analyst 1, who has produced more investment recommendations and comes from the securities brokerage firm with the greater market share, is more reliable than the other, and assign the respective levels of credibility of the analysts as $\theta_1 = 0.7635$ and $\theta_2 = 0.2365$. According to the investment newsletters collected for this study, analyst 2 is more conservative than analyst 1 in the sense that the stock recommendations are usually given with smaller price ranges. In other words, the stock forecasts provided by analyst 1 have greater predicted returns but come with higher variations. Although the Type C and Type D investors agree about the levels of credibility of the analysts, they have different opinions regarding the desired robustness level of the parameter estimates for the historical dataset, and adopt the confidence ellipsoids as $\delta \cong 0.23$ and $\delta = 1$, respectively. In the following analysis, $(\boldsymbol{F_{MV}}\,*)$, $(\boldsymbol{RE_{HP-C}})$, and $(\boldsymbol{RE_{HP-D}})$ represent the multi-analyst approach, and the robust multi-analyst approach of the Type C investor and the Type D investor respectively, with unequal credibility of analysts, i.e., $\theta_1 > \theta_2$.

6.5.1 The Ex-Ante Expected Performances of the Multi-Analyst Portfolios

Table 6.8 summarises the expected returns and the standard deviations of the multi-analyst portfolios for all types of investors. The first thing to note is that the expected performances of the portfolios with equal preference, $\theta_1 = \theta_2$, and the portfolios with unequal preference, $\theta_1 > \theta_2$, react in a similar way to changes in the desired robustification of the uncertainty set. More specifically, the expected portfolio performance decreases as the desired robustness of the uncertainty set increases for both situations, $\theta_1 = \theta_2$ and $\theta_1 > \theta_2$, and also, the difference in the expected returns between multi-analyst portfolios widens as the level of robustification increases[12]. On the other hand, the expected performances react differently to changes in the risk aversion coefficient λ among portfolios with different settings for the credibility level,

[12] The multi-analyst portfolios $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{F_{MV}}^*)$, are equivalent to their robust counterparts with the desired robustness level for the uncertainty sets equals to zero, i.e., $\delta = 0$.

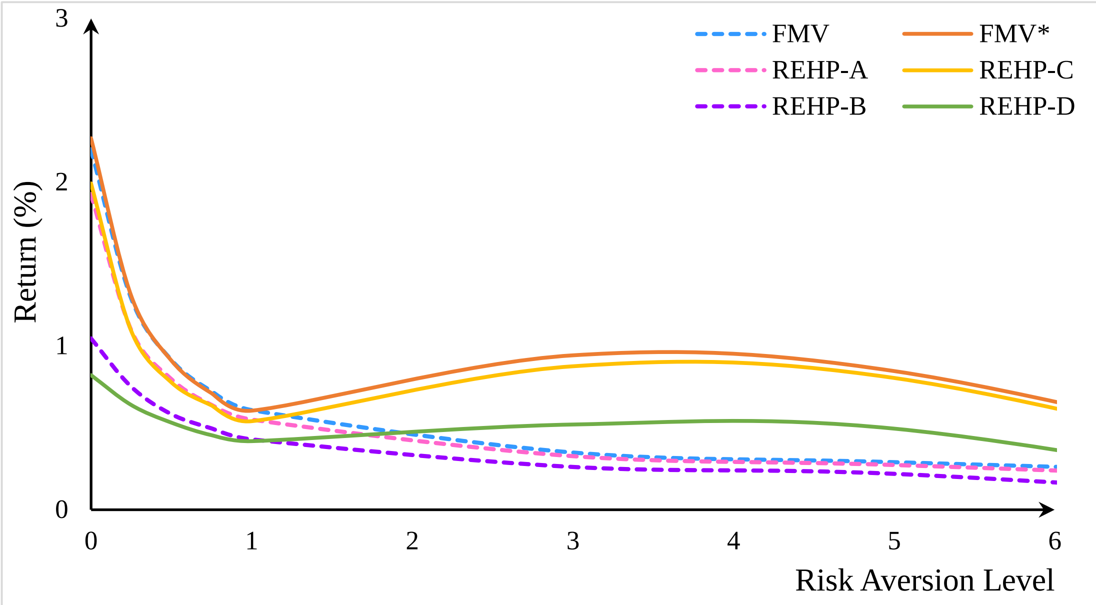
i.e., $\theta_1 = \theta_2$ or $\theta_1 > \theta_2$. Unlike the expected performances of the portfolio with $\theta_1 = \theta_2$ that mostly decrease as the risk aversion coefficient λ increases, the expected performances of the portfolio with $\theta_1 > \theta_2$ do not necessarily decrease as the risk aversion coefficient λ increases; only the expected standard deviations behave similarly as the risk aversion coefficient λ changes. The expected returns of the multi-analyst portfolios with unequal credibility $\theta_1 > \theta_2$ have wavy patterns as the risk aversion coefficient λ increases. Figure 6.8 graphically illustrates the expected returns of the multi-analyst portfolios.

Table 6.8 Expected Returns and Standard Deviations of the Optimal Portfolios

	Risk Aversion	F_{MV}	RE_{HP-A}	RE_{HP-B}	$F_{MV}*$	RE_{HP-C}	RE_{HP-D}
Mean (%)	$\lambda = 0$	2.2011	1.9270	1.0454	2.2683	1.9925	0.8215
	$\lambda = 0.5$	0.9148	0.7981	0.5836	0.9108	0.7774	0.5314
	$\lambda = 1$	0.6078	0.5502	0.4296	0.6044	0.5402	0.4182
	$\lambda = 3$	0.3484	0.3271	0.2613	0.9424	0.8755	0.5198
	$\lambda = 5$	0.2895	0.2732	0.2194	0.8463	0.8043	0.4948
	$\lambda \to \infty$	0.1958	0.1539	0.0053	0.1989	0.1620	0.0473
Standard Deviation (%)	$\lambda = 0$	5.6543	4.0691	1.5842	5.6231	4.0465	1.4501
	$\lambda = 0.5$	1.4335	1.2914	1.0868	1.4188	1.2332	0.9705
	$\lambda = 1$	1.0878	1.0430	0.9804	1.0798	0.9715	0.9367
	$\lambda = 3$	0.9073	0.9039	0.8977	0.9815	0.9288	0.8996
	$\lambda = 5$	0.8879	0.8872	0.8860	0.8956	0.8896	0.8890
	$\lambda \to \infty$	2.2208	2.2208	2.2208	2.2260	2.2260	2.2260

Notes: This table displays the average expected returns and the standard deviations of 492 observations for multi-analyst portfolios. (F_{MV}), (RE_{HP-A}), and (RE_{HP-B}) represent the portfolios with equal credibility of analysts, i.e., $\theta_1 = \theta_2$. $(F_{MV}*)$, (RE_{HP-C}), and (RE_{HP-D}) represent portfolios with unequal credibility of analysts, i.e., $\theta_1 > \theta_2$. See Table 6.2 for further descriptions of different robust multi-analyst portfolios (RE_{HP}).

Figure 6.8 Expected Weekly Returns of Multi-Analyst Portfolios



Notes: This figure shows the expected returns of the multi-analyst portfolios. The dashed lines represent the portfolios with equal credibility of analysts, i.e., $\theta_1 = \theta_2$, and the solid lines represent portfolios with unequal credibility of analysts, i.e., $\theta_1 > \theta_2$. Details of the results are contained in Table 6.8.

From Table 6.8 and Figure 6.8, we observe that the expected return of portfolio with either $\theta_1 = \theta_2$ or $\theta_1 > \theta_2$ drops as the risk aversion coefficient changes from $\lambda = 0$ to $\lambda = 1$. Beyond $\lambda = 1$, the expected returns of the portfolios with equal credibility level continue with the downward trend, whereas the expected returns of the portfolios with unequal credibility level increase and, after a turning point, decrease as the risk aversion coefficient λ increases. This is because the multi-analyst portfolio selection problems, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HE}})$, are no longer handling the analysts' recommendations equally after assigning a stronger preference for analyst 1. Hence the portfolio weightings obtained for calculating portfolio performances are based on the recommendations from a particular analyst rather than both analysts, which then leads to an interesting movement of the expected returns for the multi-analyst portfolios with $\theta_1 > \theta_2$.

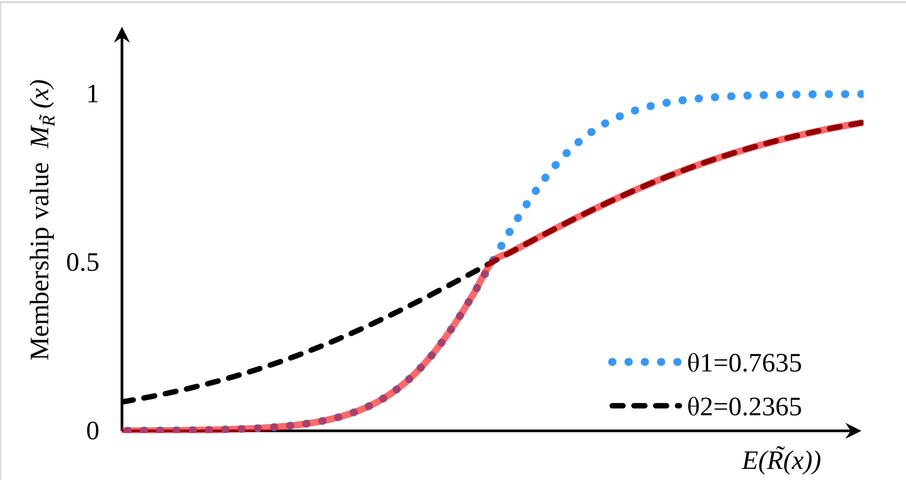
To explain this outcome in detail, recall equation (6.1) of the multi-analyst approach for solving portfolio selection problems.

$$(\boldsymbol{F_{MV}}) \qquad \max_{x\in\mathbb{R}^n} \min_{z\in Z} \quad \frac{1}{1+\exp\left(-\theta_z\left(\check{\mu}_z{}^T x - \frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}\right)\right)} \ .$$

Under a risk-loving setting with $\lambda \to 0$, the inner optimisation will assign a greater ratio to the analyst who provides relatively prudent recommendations that are more pessimistic than the other analyst; on the other hand, the outer optimisation will solve the portfolio selection problem by maximising the portfolio return based on the forecasts provided by the chosen analyst. In contrast, under a risk-averse setting with $\lambda \to \infty$, the inner optimisation will assign a greater ratio to the analyst who provides more risky recommendations that come with higher stock return variations, because the inner optimisation becomes to minimise $-\frac{\lambda}{2}x^T\check{\Sigma}_z x - R^{Target}$ as λ increases; then the outer optimisation will solve the portfolio selection problem by minimising the portfolio risk based on the forecasts provided by the chosen analysts. In other words, the multi-analyst approaches convert into the risk minimum portfolio selection framework based on the more risky forecasts as λ increases.

By choosing $\theta_1 = 0.7635$ and $\theta_2 = 0.2365$, the substantial difference between the credibility level of analyst 1 and that of analyst 2 has influenced the portfolio allocations significantly. The optimal portfolio weightings and the related expected portfolio performances are obtained mostly based on the relatively more careful analyst 2's predictions under a risk-loving setting; as a result, the portfolios with $\theta_1 > \theta_2$ have relatively conservative expected performances compared to the portfolios with $\theta_1 = \theta_2$ for $\lambda \to 0$. Conversely, the optimal portfolio weightings and the expected portfolio performances are obtained mostly based on the more optimistic analyst 1's predictions under a risk-averse setting. As a result, the portfolios with $\theta_1 > \theta_2$ have relatively higher expected returns but greater standard deviations compared to the portfolios with $\theta_1 = \theta_2$ for $\lambda \to \infty$. Figure 6.9 graphically illustrates the impact of defining different values of credibility levels for the multi-analyst approaches on the asset allocation.

Figure 6.9 The Effect of Credibility Level on Multi-Analyst Asset Allocation



Notes: The figure shows the membership functions (vertical axis) of the expected utility of the portfolio (horizontal axis). The dotted and the dashed lines represent the membership functions according to the advice of analysts 1 and 2, respectively. The solid line is the minimum of the two membership functions that represents the robust objective function.

Table 6.9 Expected Risk Adjusted Returns of the Optimal Portfolios

Expected Risk Adjusted Return (%)						
Risk Aversion	F_{MV}	RE_{HP-A}	RE_{HP-B}	$F_{MV}*$	RE_{HP-C}	RE_{HP-D}
$\lambda = 0$	0.3893	0.4736	0.6599	0.4034	0.4924	0.5665
$\lambda = 0.5$	0.6382	0.6180	0.5370	0.6420	0.6304	0.5476
$\lambda = 1$	0.5588	0.5275	0.4382	0.5597	0.5560	0.4465
$\lambda = 3$	0.3841	0.3619	0.2910	0.9602	0.9426	0.5778
$\lambda = 5$	0.3261	0.3079	0.2476	0.9450	0.9041	0.5566
$\lambda \to \infty$	0.0882	0.0693	0.0024	0.0894	0.0728	0.0212

Notes: This table displays the average expected risk adjusted returns (Mean/SD) of 492 observations for multi-analyst portfolios. (F_{MV}), (RE_{HP-A}), and (RE_{HP-B}) represent the portfolios $\theta_1 = \theta_2$. ($F_{MV}*$), (RE_{HP-C}), and (RE_{HP-D}) represent portfolios with $\theta_1 > \theta_2$.

Similar wavy patterns occurred in the expected risk adjusted returns for the multi-analyst portfolios with unequal credibility levels. In Table 6.9 the multi-analyst portfolios with $\theta_1 > \theta_2$ have only slightly better expected risk adjusted returns up to $\lambda = 1$; after that, the portfolios with unequal credibility levels have much higher expected risk adjusted returns than the portfolios with $\theta_1 = \theta_2$.

6.5.2 The Ex-Post Realised Performances of the Multi-Analyst Portfolios

Next we consider the ex-post realised performances of the multi-analyst portfolios for all types of investors. Table 6.10 reports the realised portfolio returns and Table 6.11 reports the investor's loss and disappointment.

Table 6.10 Realised Returns of the Optimal Portfolios

	Realised Return (%)					
Risk Aversion	F_{MV}	RE_{HP-A}	RE_{HP-B}	$F_{MV}*$	RE_{HP-C}	RE_{HP-D}
$\lambda = 0$	1.2687	1.2753	0.5556	1.2803	1.2845	0.5742
$\lambda = 0.5$	0.7022	0.6034	0.4384	0.7089	0.6104	0.4520
$\lambda = 1$	0.4932	0.4580	0.3883	0.4961	0.4648	0.3994
$\lambda = 3$	0.3379	0.3330	0.3210	0.3508	0.3506	0.3488
$\lambda = 5$	0.3074	0.3053	0.3005	0.3412	0.3440	0.3419
$\lambda \to \infty$	0.2382	0.2382	0.2382	0.2382	0.2382	0.2382

Notes: This table displays the average realised returns of 492 observations for multi-analyst portfolios. (F_{MV}), (RE_{HP-A}), and (RE_{HP-B}) represent the portfolios $\theta_1 = \theta_2$. ($F_{MV}*$), (RE_{HP-C}), and (RE_{HP-D}) represent portfolios with $\theta_1 > \theta_2$.

First of all, the portfolios with unequal credibility levels $\theta_1 > \theta_2$ have outperformed the portfolios with equal credibility levels in most circumstances. The only exception is the outcome under the minimum variance setting, where the multi-analyst portfolios for every type of investor have the same realised returns. This is simply because the multi-analyst approaches transfer into the equally weighted approach $(\mathbf{1/N})$ as the risk aversion coefficient λ increases.

By comparing the outcomes under different conditions, we have noticed that the effects of the robustification and the changes in the risk aversion coefficient λ have led to similar movements in the realised returns for portfolios of all types of investor. To be exact, the realised return decreases as the multi-analyst portfolio selection framework becomes more robust or risk averse, no matter whether the credibility levels of analysts are equal or unequal. Moreover, the same effect of robustification on asset allocation as in Section 6.4 can be found in Table 6.10, where the changes in the desired robustness levels δ have less influence on the realised returns as the risk aversion coefficient increases. Note that there are significant increases in the realised returns for the portfolios with $\theta_1 > \theta_2$ after $\lambda = 1$. This may be due to the impact of higher weightings being allocated to the recommended stocks (see Table 6.12), as the multi-analyst approaches with unequal credibility levels follow mostly the recommendations of analyst 1, who usually provides predictions with higher returns, in a risk averse scenario.

Table 6.11 compares the investor's losses and disappointments of the multi-analyst portfolios under various conditions. As explained earlier, the investor's loss is calculated based on the realised portfolio return. Therefore the pattern of the loss measured also corresponds to the changes in the desired robustness level δ and the risk aversion coefficient λ. That is, the more conservative the portfolio allocation, the greater the investor's loss. Furthermore, the investor who determines the credibility levels as $\theta_1 > \theta_2$ has lower portfolio loss than the investor who considers the analysts as equally reliable; this is simply because the portfolios with $\theta_1 > \theta_2$ have superior realised returns than the portfolios with $\theta_1 = \theta_2$.

On the other hand, the investor's disappointment is the difference between the expected and the realised returns of a specific investment strategy. Normally, the pattern of the disappointment measure is a downwards trend as the portfolio selection framework becomes more conservative. Nevertheless, the investor's disappointments for the portfolios with unequal credibility levels form a wavy pattern as the risk aversion coefficient λ increases. This is not surprising because the investor's disappointment is related to the expected portfolio return, so that the expected return and the investor's disappointment are supposed to have similar patterns as the portfolio selection framework becomes more conservative. In addition, the investor is more disappointed in the portfolios with $\theta_1 > \theta_2$, especially for the risk averse portfolios.

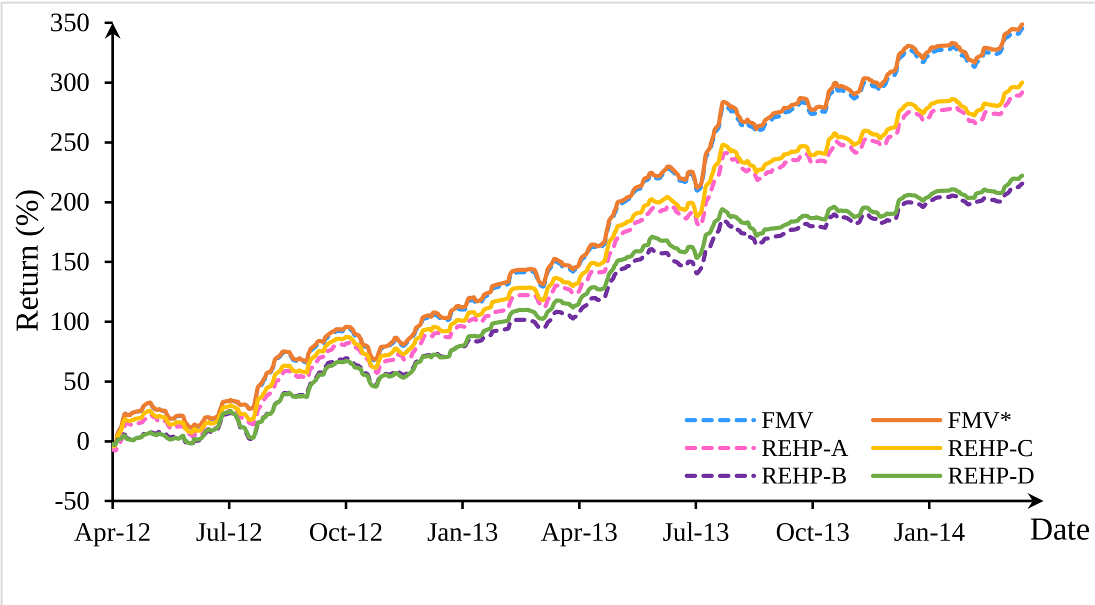
Table 6.11 Loss and Disappointment Rates of the Optimal Portfolios

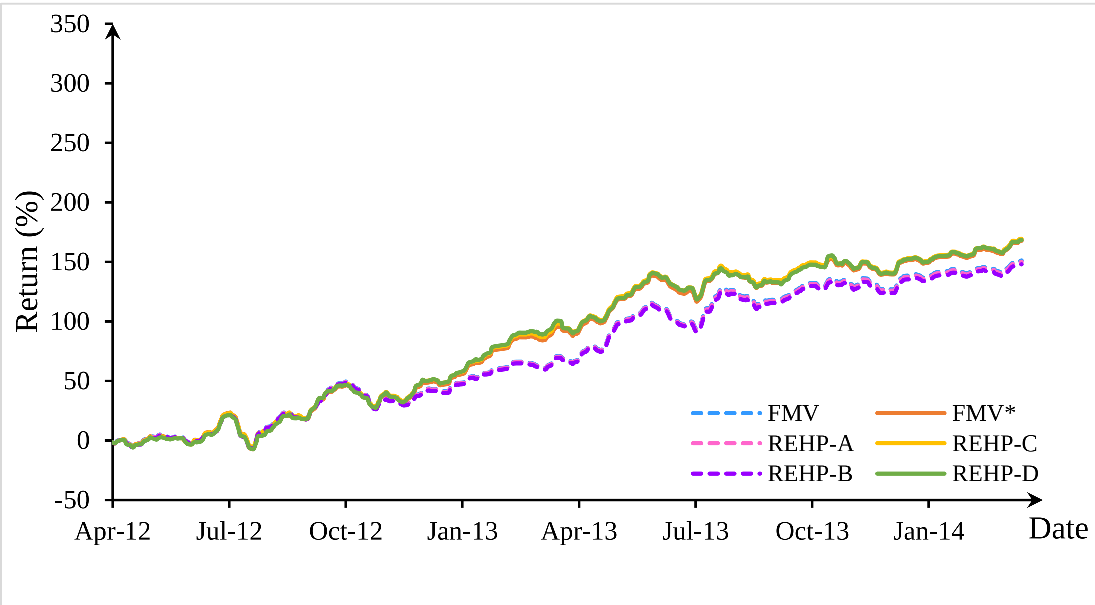
	Risk Aversion	F_{MV}	RE_{HP-A}	RE_{HP-B}	$F_{MV}*$	RE_{HP-C}	RE_{HP-D}
Loss (%)	$\lambda = 0$	12.5449	12.5383	13.2580	12.5333	12.5291	13.2394
	$\lambda = 0.5$	13.1114	13.2102	13.3752	13.1047	13.2032	13.3619
	$\lambda = 1$	13.3203	13.3555	13.4252	13.3175	13.3488	13.4142
	$\lambda = 3$	13.4756	13.4805	13.4926	13.4628	13.4630	13.4648
	$\lambda = 5$	13.5062	13.5083	13.5131	13.4724	13.4696	13.4717
	$\lambda \to \infty$	13.5754	13.5754	13.5754	13.5754	13.5754	13.5754
Disappointment (%)	$\lambda = 0$	0.9324	0.6517	0.4898	0.9880	0.7080	0.2473
	$\lambda = 0.5$	0.2011	0.1510	0.0140	0.2019	0.1670	0.0797
	$\lambda = 1$	0.1146	0.0922	0.0412	0.1083	0.0754	0.0188
	$\lambda = 3$	0.0105	-0.0059	-0.0597	0.5916	0.5249	0.1710
	$\lambda = 5$	-0.0179	-0.0321	-0.0811	0.5051	0.4603	0.1529
	$\lambda \to \infty$	-0.0424	-0.0843	-0.2329	-0.0393	-0.0762	-0.1909

Notes: This table displays the average investor's losses and investor's disappointments of 492 observations for multi-analyst portfolios. (F_{MV}), (RE_{HP-A}), and (RE_{HP-B}) represent the portfolios $\theta_1 = \theta_2$. $(F_{MV}*)$, (RE_{HP-C}), and (RE_{HP-D}) represent portfolios with $\theta_1 > \theta_2$. See Section 6.3 for further details of the loss and the disappointment functions.

At the end of this section, a figure is provided to illustrate the impact of assigning unequal credibility levels on multi-analyst approaches by comparing the realised cumulative returns.

Figure 6.10 Realised Cumulative Returns of Multi-Analyst Portfolios





Note: This figure shows the total realised returns of multi-analyst portfolios, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{HE_{HP}})$, from April 2012 to April 2014 at different risk aversion levels. The upper and lower panels show the results for $\lambda = 0.5$ and $\lambda = 5$, respectively. From the upper panel, there is not much difference in the returns between portfolios with $\theta_1 = \theta_2$ and $\theta_1 > \theta_2$. This is because the asset allocation of portfolios with $\theta_1 > \theta_2$ mainly follow the relatively more conservative recommendations of analyst 2, and therefore, the portfolios with $\theta_1 > \theta_2$ have very similar performance to the portfolios with $\theta_1 = \theta_2$. In contrast, for the lower panel where $\lambda = 5$, the asset allocation for the portfolios with $\theta_1 > \theta_2$ are based on the more optimistic recommendations of analyst 1 and obtained higher realised returns than the portfolios with $\theta_1 = \theta_2$.

6.5.3 The Characteristics of the Portfolios

Table 6.12 shows some key information in addition to the portfolio performances of the multi-analyst portfolios, where the number of selected stocks in the portfolio and the ratio of wealth invested in the recommended stocks provide further understanding about the impact of assigning unequal credibility levels of analysts on portfolio allocations.

Table 6.12 The Characteristics of Optimal Portfolios

	Risk Aversion	F_{MV}	RE_{HP-A}	RE_{HP-B}	$F_{MV}*$	RE_{HP-C}	RE_{HP-D}
No. of Stocks	$\lambda = 0$	1.10	3.75	16.65	1.14	3.78	16.70
	$\lambda = 0.5$	15.98	17.71	20.24	15.98	17.95	19.81
	$\lambda = 1$	19.29	20.10	20.26	19.50	19.83	20.07
	$\lambda = 3$	20.26	20.29	20.36	18.78	19.09	19.30
	$\lambda = 5$	20.58	20.63	20.83	18.64	18.80	18.85
	$\lambda \to \infty$	148.00	148.00	148.00	148.00	148.00	148.00
Investment via Analysts (%)	$\lambda = 0$	9.55	15.99	22.40	12.51	18.03	25.62
	$\lambda = 0.5$	6.85	7.58	9.88	6.96	8.36	14.72
	$\lambda = 1$	4.61	5.03	6.51	5.91	7.77	12.97
	$\lambda = 3$	3.93	4.17	4.89	11.36	12.69	16.69
	$\lambda = 5$	3.47	3.52	3.57	16.30	17.15	20.04
	$\lambda \to \infty$	2.58	2.58	2.58	2.58	2.58	2.58

Notes: This table displays the portfolio characteristics of multi-analyst approaches, including the number of stocks in the resulting portfolio (No. of Stocks) and the proportion of investment via analysts' recommendations (Investment via Analysts). The results presented in this table are the average of 492 observations.

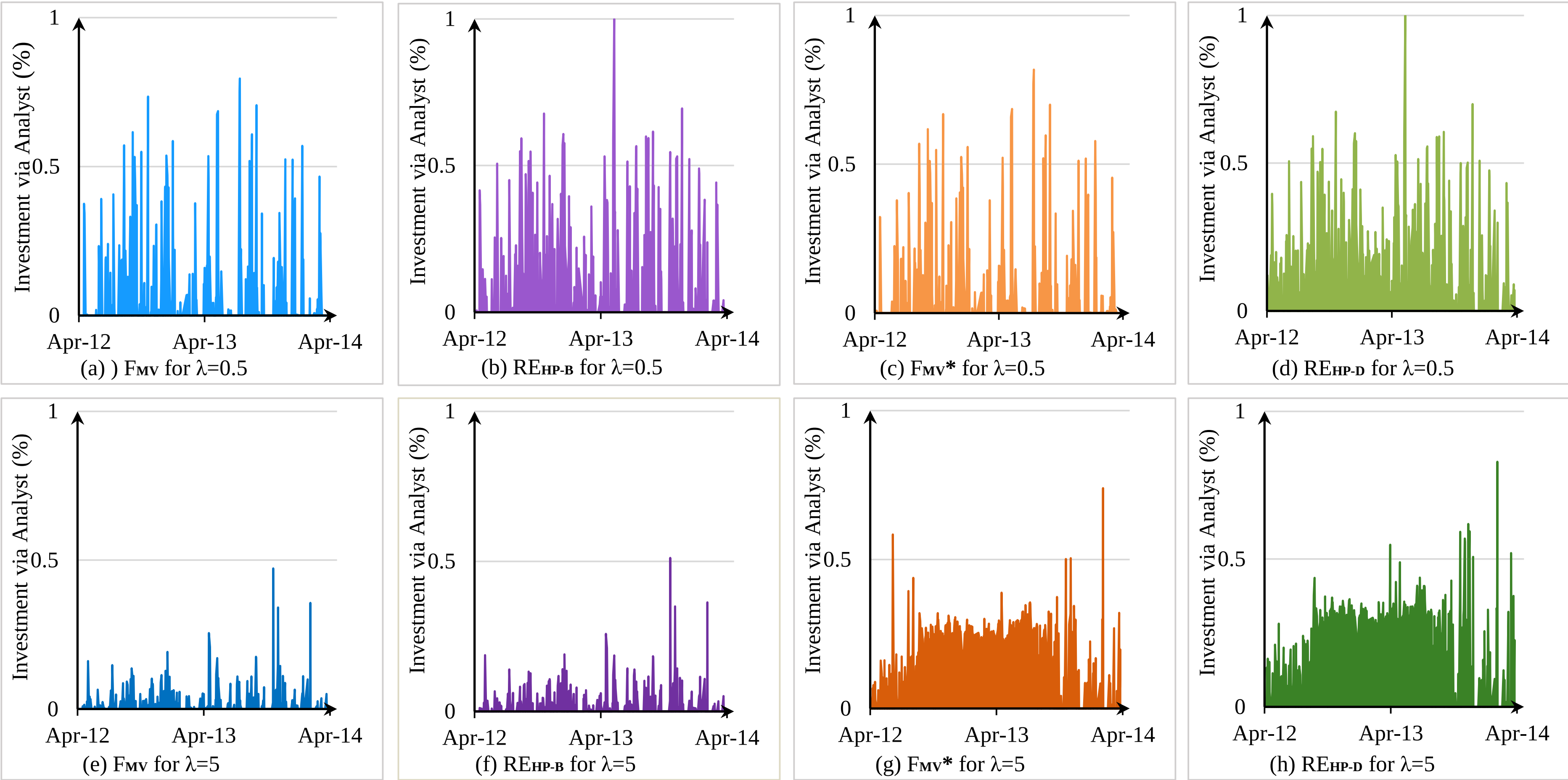
As explained in Section 6.4.3, the number of stocks in the portfolio and the ratio of capital invested in the recommended stocks are positively correlated with the desired robustness level of the multi-analyst approaches[13] if the credibility levels are equal, i.e., $\theta_1 = \theta_2$. That is, the number of stocks and the proportion of capital invested in the recommended stocks increase as the uncertainty set becomes more robust for the multi-analyst portfolio selections. By assigning unequal credibility levels $\theta_1 > \theta_2$ in the multi-analyst frameworks, the same relationship between the desired robustness level δ and the characteristics of the portfolio can be found. Nevertheless, the changes in the desired robustness level have slightly stronger effects on the asset allocation for the portfolios with unequal credibility levels (see Figure 6.11 for graphical comparison).

Unlike the correlation between the portfolio characteristics and the desired robustness level δ, which has a pattern similar to the case of equal credibility levels, the characteristics of the portfolios with unequal credibility levels have wavy patterns as the risk aversion level λ increases. As shown in Table 6.12, for every type of investor, the characteristics of the portfolios behave similarly for $\lambda \leq 1$. Beyond $\lambda = 1$, the portfolio with equal credibility levels $\theta_1 = \theta_2$ has gradually changed into the equally weighted portfolio as the risk aversion level increases. On the other hand, the portfolios with unequal credibility levels $\theta_1 > \theta_2$ have performed rather differently before eventually turning into the equally weighted portfolio. That is, for the cases where $1 \leq \lambda < +\infty$, the number of stocks has initially decreased and the ratio of capital invested in the recommended stocks has increased for the portfolio with unequal credibility levels, which is in fact the opposite movement to converging towards to the equally weighted portfolio.

[13] The multi-analyst approach $(\boldsymbol{F_{MV}})$ is equivalent to the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ with the desired robustness level $\delta = 0$.

The reason for this is that the multi-analyst approaches with unequal credibility levels have changed from the framework of solving portfolio problems according to the forecasts from both analysts to the framework that focuses on one particular analyst (see Figure 6.9). Hence, by assigning unequal credibility levels to analysts, the portfolio allocation and the corresponding ex-ante expected portfolio performances will mostly coincide with the forecasts of one specific analyst and the ex-post realised portfolio performances will more or less reflect the quality of the forecasts. According to our data, analyst 1 gives much more stock recommendations than analyst 2 (see Figure 6.3). In addition, the stock recommendations of analyst 1 are usually more optimistic and have greater predicted returns but higher variations and the stock recommendations of analyst 2 are usually more conservative. The multi-analyst portfolio with equal credibility levels is the robust portfolio that considers the worst possible investment scenarios provided by the two analysts. Therefore for $\lambda \leq 1$, the portfolio with unequal credibility levels mostly follows analyst 2's recommendations and, thus, is expected to have very similar performance to the portfolio with equal credibility levels for $\lambda \leq 1$, because analyst 2 is more prudent and only provides recommendations occasionally. On the other hand, for $1 \leq \lambda < +\infty$, the portfolio with unequal credibility levels mostly follows analyst 1's recommendations and, therefore, it is supposed to allocate relatively higher weightings to the recommended stocks and obtain portfolios with greater expected returns and standard deviations, because analyst 1 is more optimistic and provides many recommendations for every trading day. The following figure graphically illustrates the impact of assigning unequal credibility levels on portfolio weightings for the multi-analyst portfolio.

Figure 6.11 The Ratio of Wealth Invested in the Recommended Stocks



Note: This figure shows the ratio of capital invested in the recommended stocks over the sample period. The upper panels display results at λ=0.5 and the lower panels display results at λ=5. F_{MV} and RE_{HP-B} represent portfolios with $\theta_1 = \theta_2$, and $F_{MV}*$ and RE_{HP-D} represent portfolios with $\theta_1 > \theta_2$.

Overall, the investor allocates more wealth to the recommended stocks after assigning unequal credibility levels to analysts. Unlike the multi-analyst approaches with equal credibility levels that consider recommendations equally and obtain portfolios based on the combinations of both analysts' recommendations, the multi-analyst approaches with unequal credibility levels mostly allocate investment according to one particular analyst.

For this particular sample, our results illustrate that the portfolios with $\theta_1 > \theta_2$ generally have superior expected performances than the portfolios with $\theta_1 = \theta_2$. By assigning unequal credibility levels to analysts in the multi-analyst approaches, a higher ratio of wealth is invested in the recommended stocks. In addition, the portfolios with unequal credibility levels obtained higher realised returns than the portfolios with equal credibility levels under most scenarios. Nevertheless, the realised portfolio returns for the case in which $\theta_1 > \theta_2$ are more dependent on the predictive power, the accuracy of the forecasts, of analysts than is the case when $\theta_1 = \theta_2$. In other words, the multi-analyst portfolios with $\theta_1 > \theta_2$ may have lower realised returns than the others when different data is applied with the higher credibility level is assigned to an analyst whose forecasts are poorer.

This empirical study reveals the important role played by the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, among other conventional portfolio selection models, as the aim of the multi-analyst approaches is to account for additional investment information so that the resulting portfolio can be robust but also profitable. Nevertheless, by assigning unequal credibility levels to analysts, the focus of the multi-analyst approaches has shifted away from considering all the investment possibilities equally to emphasising an individual analyst. In this regards, the investment turns out to be less robust with more exposure to risk.

6.6 Summary

This chapter investigates the portfolio performances of the proposed multi-analyst approaches by implementing and solving the portfolio selection problems in the Taiwanese stock market over the period from April 2012 to April 2014. To provide a comprehensive empirical investigation, the proposed multi-analyst approaches are compared with other conventional investment strategies and examined under different scenarios, such as the duration of investment, robustness preference and risk preference. In addition, the empirical investigation is divided into two cases in order to explore the impact of the investor's preference for analysts on the multi-analyst approaches.

Based on the sample of 148 stocks with 492 observations, our results show that when the investor has an equal preference for analysts, i.e., $\theta_1 = \theta_2$, the proposed multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, generally outperform the corresponding conventional investment strategies and the equally weighted allocation method in terms of both expected and realised returns for shorter investment holding periods of 5 days and 10 days. In contrast, the benefit of incorporating additional investment information on the portfolio selection process disappears as the investment holding period D increases. This indicates that the duration of the investment holding period D has a substantial impact on the effectiveness of employing stock recommendations provided by the analysts. More specifically, as the investment holding period increases, the stock recommendations become less notable compared to the historical stock performances and therefore have less impact on asset allocation.

Apart from the already known effect of robustification on portfolio selections, that the difference between the portfolio performances of the 'original' approach and the corresponding robust counterpart approach widens as the robustness of the uncertainty set increases (Goldfarb and Iyengar, 2003), our result further shows that the impact of robustification on portfolio performance is weaker as the risk aversion level increases and the ratio of wealth invested in the recommended stocks is positively correlated with the desired robustness level of the multi-analyst approach. On the other hand, the changes in the risk aversion coefficients have influenced the multi-analyst approaches differently when the investor has assigned different credibility levels to the analysts.

Generally speaking, as the risk aversion coefficient increases, the expected portfolio performances become more conservative and, consequently, the realised portfolio return is lower. The same effect of the changes in the risk aversion coefficients has been found on the multi-analyst portfolios with equal credibility levels of analysts and, furthermore, less capital is invested according to the analysts' recommendations, as the analysts' recommendations considered for the empirical investigation are mostly more optimistic than the historical stock performances. In short, for the multi-analyst approaches with equal credibility levels, the advantage of incorporating additional investment information is more significant for the risk-loving investor.

Finally, this chapter has investigated the impact of the investor's preference for analysts on the multi-analyst approaches. By assigning unequal credibility levels to analysts in the multi-analyst approaches, a higher ratio of capital is invested according to the recommendations provided by the analysts. Unlike the multi-analyst approaches with equal credibility levels that allocate portfolios according to the combination of both analysts' recommendations, the multi-analyst approaches with unequal credibility levels allocate investment according to one particular analyst. Although the multi-analyst portfolios with unequal credibility levels have more profitable realised returns, the superior performances are highly dependent on the predictive power of the analyst. In other words, the multi-analyst approaches with equal credibility levels generate a more pessimistic portfolio allocation as the worst case scenarios are considered for asset allocation, but at the same time, the portfolio allocation is more robust against estimation errors and parameter uncertainties.

To conclude, the multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$, are more robust when equal credibility levels are adopted for the framework, and are more beneficial to risk loving investors for short-term investment.

Chapter 7
Conclusions and Discussion

The more information we obtain, the better decisions we can make. This also applies to investors when they are making decisions to allocate financial assets in their portfolios. However, in reality, information collection comes at a cost. Apart from the difficulty of obtaining useful and efficient information from the massive amount of investment newsletters, it is also hard to verify the reliability of the professional analysts' forecasts. In order to incorporate multiple analysts' opinions, which are made available from the investment newsletters, into the decision making process of asset allocation, we have developed a multi-analyst approach and the corresponding robust counterpart approach for portfolio selection problems, and empirically implemented both portfolio selection approaches to analyse the Taiwanese stock market.

In this final chapter, we summarise this thesis and discuss future research. It is organised as follows. Section 7.1 reviews the developments and results of the research. Section 7.2 outlines the key contributions of this research study. Finally, Section 7.3 draws attention to the limitations of the research and suggests potential directions for future research.

7.1 Summary

This thesis is organised in two parts. The first part of the thesis, which consists of Chapters 2, 3 and 4, reviews the literature relevant to this research and develops two new approaches to portfolio selection when market analysts' recommendations are available. The second part of the thesis comprises Chapters 5 and 6, which detail the analysts' recommendations collected for this research and illustrate the application and benefits of the proposed multi-analyst approaches, $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$.

In Chapter 2, we first introduced the well-known portfolio selection theory of Harry Markowitz (1952) and drew attention to the weaknesses of the mean-variance portfolio optimisation model. Among various suggestions for improving on the performance of Markowitz's classical portfolio selection model, the robust counterpart approach of Ben-Tal and Nemirovski (1998) is one of the most highly regarded methods for addressing the issues caused by estimation errors and parameter uncertainties. The popularity of the robust counterpart approach comes from its intuitive conceptual character and computational efficiency. However, as stated in Chapter 2 and observed in Chapter 6, the robust counterpart approach has its own weakness, as the robust portfolio is generally less profitable than the others. This undesirable outcome is caused by the excessively pessimistic character of the robust asset allocation, which always assumes the uncertainties of the portfolio selection problem will appear to be against the investors' benefits. In order to overcome the drawback of the existing robust portfolio optimisation model by providing a potentially profitable robust asset allocation, we proposed including additional investment information sources into the process of asset allocation, as they provide an opportunity to obtain better quality investments and, at the same time, help with better decision making when facing the underlying parameter uncertainty.

In the literature, not much has been done to adopt multiple information sources and pool those sources together to generate a final portfolio selection. The Bayesian approach is one of the most widely recognised methods in such cases, which can deal with uncertainties in decision making very well. However, it is difficult, if not impossible, for the Bayesian approach to address the issue of ambiguities associated with analysts' verbal recommendations. In addition, it requires an investor to assign prior probabilities to each individual expert. On the other hand, the multiple experts approach of Lutgens and Schotman (2010) doesn't require the user to have any knowledge regarding the reliability of the experts. In this regard, Lutgens and Schotman's multiple experts approach has fewer restrictions. In this thesis, we followed Lutgens and Schotman's approach when dealing with the additional investment information.

In Chapter 3, we improved the existing multi-expert approach of Lutgens and Schotman by using the concept of fuzzy set theory to deal with the verbal and imprecise investment recommendations provided by analysts. However, solving portfolio optimisation problems with fuzzy variables is a challenging task, as the original objective functions are turned into fuzzy functions with varying degrees of membership. To address this issue, the crisp possibilistic interpretation method of Carlsson and Fuller (2001) is incorporated for defuzzifying purposes, so that the fuzzy variables of the analysts' recommendations can be transformed into ordinary numbers. In addition, we adopted the work of Gupta et al. (2008) to define the investor's ambiguous aspiration level toward the investment. The developed multi-analyst approach $(\boldsymbol{F_{MV}})$ allows more flexibility than the original approach. Apart from the choice of using either the fuzzy set theory or probability theory for expressing analysts' recommendations, the user can also apply the developed multi-analyst approach $(\boldsymbol{F_{MV}})$ with or without assigning vague credibility level θ to each individual analyst.

The multi-analyst approach $(\boldsymbol{F_{MV}})$ developed in Chapter 3 focuses on how to incorporate the various investment information sources into the portfolio selection model without paying attention to parameter uncertainties. In Chapter 4, to handle the issues arising from estimation errors and parameter uncertainties, we extended the multi-analyst approach $(\boldsymbol{F_{MV}})$ by incorporating the concept of the robust counterpart approach. We first formulated the standard robust counterpart to the multi-analyst approach $(\boldsymbol{F_{MV}})$ by following the work of Ben-Tal and Nemirovski (1998). However, this standard robust counterpart of the multi-analyst approach is imperfect, as the parameter uncertainty levels of different assets are assumed to be identical and are treated equally via a joint uncertainty set. This is clearly not the case for the problem considered here because the levels of uncertainty differ for the assets with and without the analysts' recommendations. To overcome this problem, we introduced the concept of non-overlapping uncertainty set of Garlappi et al. (2007) to provide robustification for different subclasses of assets. Therefore, compared with the existing robust counterpart approach of the famous mean-variance portfolio optimisation problem of Markowitz, the robust multi-analyst approach $(\boldsymbol{RE_{HP}})$ is theoretically capable of generating a robust optimal portfolio with better portfolio return.

In Chapter 5, to examine the performance of the multi-analyst approaches $(\boldsymbol{F_{MV}})$ and $(\boldsymbol{RE_{HP}})$ developed in the first part of the thesis, we conducted a comprehensive investigation, using the developed portfolio selection methods. In order to present a more realistic picture of the model implementations and to have a better understanding of the impact of adopting analysts' recommendations on portfolio performances, the multi-analyst approaches are tested with real world data instead of using simulated expert data or market return models, as most of the existing studies did. We used stock market forecasts from various institutional analysts. We discussed the investment newsletters collected from different Taiwanese securities brokerage institutions and explained the procedure for expressing stock forecasts in terms of either triangular or trapezoidal fuzzy variables.

In Chapter 6, we applied the multi-analyst approaches to portfolio selection problems to analyse the Taiwanese stock market. Apart from the analysts' recommendations mentioned previously, historical performances of stocks were also adopted for calculating estimates of input parameters. In order to evaluate the performances of the proposed portfolio selection models, we conducted two tests with different descriptions of credibility levels θ. We first examined the multi-analyst approaches under the assumption that the investor has no preference of analyst. The resulting portfolio performances are compared to the results generated by the equally weighted method, mean-variance method of Markowitz, and the robust method of Ben-Tal and Nemirovski. As shown in Section 6.4, due to the analysts' recommendations, the multi-analyst approaches turned out to be less diversified, with greater expected volatilities, but the expected return and risk adjusted return of the proposed approaches mostly exceeded the corresponding investment strategies. Moreover, the multi-analyst approaches outperformed the corresponding conventional investment strategies in terms of realised returns for more risk-loving investment. The research has also shown that the benefit of employing multi-analyst approaches is more significant for shorter investment holding periods, of 5 days and 10 days. Therefore, the multi-analyst approaches with equal credibility levels seem promising for risk-loving investors to apply to short term investment. In contrast, the second empirical test assumed that the investor has unequal preference for the analysts. The multi-analyst approaches with unequal credibility levels allocated more wealth in the recommended stocks and generated relatively more optimistic portfolios. In view of obtaining a robust and also potentially profitable portfolio, it may not be a good idea to assign unequal credibility levels to the analysts, as the superior realised returns of the portfolios with unequal credibility levels are highly dependent on the predictive power of the particular analyst. Duration

Generally speaking, employing the model robust approach and estimation robust approach together for solving portfolio selection problems is supposed to come up with a rather pessimistic asset allocation. Nevertheless, this undesirable outcome can be improved by incorporating additional investment information sources. The theoretical developments and empirical studies of this thesis have introduced a different investment strategy for investors to optimise their investments.

7.2 Contributions

We summarise the main contributions of this thesis as follows:

1) Market analysts' recommendations in the real world are usually expressed verbally with a great deal of ambiguity. We have developed a new approach via fuzzy set theory and the multi-expert approach, termed the multi-analyst portfolio selection approach, to improve on the existing multi-prior approaches in the literature by taking into account the nature of the analysts' suggestions and the preferences of the investor.
2) In reality, market analysts usually only select a small proportion of assets and make a comment/recommendation on each asset. We have developed a robust counterpart approach of our multi-analyst approach to address the issue that the uncertainty levels differ for the assets with and without the analysts' recommendations. In this regard, the proposed robust multi-analyst approach possesses the benefits of both model and estimation robust approaches. To the best of our knowledge, this robust counterpart to the multi-analyst approach has not been considered in the literature so far.
3) We have also carried out an empirical study to investigate how the proposed investment strategies work with the real world application, as most of the existing studies in this field focus more on the theoretical aspects of the robust optimisation frameworks. Unlike the other portfolio optimisation studies, which use simulated expert data (Garlappi et al., 2007; Huang et al., 2010; Lutgens and Schotman, 2010), our empirical study is conducted with real analysts'

recommendations that are unique and collected from various financial institutions.

7.3 Limitations of the Study and Future Research

Our research is confined by the scope of this thesis and, in common with all research, there are several limitations embedded in this thesis that provide potential directions for future research.

In addition to the weakness, addressed in Section 6.4, that the multi-analyst approaches are only compatible with the return maximisation or the risk aversion portfolio selection models, it is worth noting that the risk measure adopted for portfolio selection problems in this research is the variance of the returns. Although variance is one of the more common and basic risk measurements, there are several voices that criticise the suitability of using variance as the measure of the investment risk. Variance is a measure used to describe the dispersion of a random variable or of a sample, and hence, by choosing variance as the measure for the investment risk, the overperformance and underperformance of the investment are treated as equally important. Nevertheless, investors never consider both situations in the same way. Therefore, the downside risk measures, such as semivariance, VaR, and CVaR, may be more appropriate for describing the investment risk as these measures only take the unfavourable outcomes into consideration.

On the other hand, although there is no explicit formulation for constructing the uncertainty set of the robust portfolio optimisation problems, researchers and practitioners usually follow the basic guideline to define the uncertainty set as centered on a point estimate with the level of robust δ denoting the robustness imposed on the portfolio optimisation problem. A natural question is whether we have chosen the most suitable statistical estimate for this centre point of the uncertainty set. In our empirical study, we have only considered the maximum likelihood estimator for approximating the true value of the input parameter. However, it is well known that the maximum likelihood estimator may perform poorly in some circumstances (for example with a substantial proportion of outliers). Therefore, it is of interest to consider an uncertainty set based on various other statistical estimators and examine the robustness imposed on the portfolio optimisation problem.

A further aspect that could be worth investigating more closely is the membership functions for translating the analysts' recommendations. It is known that there is no unique formulation or approach to express properly the vague recommendations and one could only convert the investment forecasts based on one's own judgement and perception regarding the recommendations. The procedure and the outcome of the translation have a great impact on the resulting optimal portfolios of the multi-analyst approach. Hence it is worth exploring further to figure out better alternatives for interpreting the analysts' recommendations.

Finally, the empirical study presented in this thesis applies the proposed multi-analyst approaches in the Taiwanese stock market. As mentioned in Chapter 6, the stock market in Taiwan is controlled by the price limits system for avoiding extreme price movements, hence, preventing dramatic losses for investors. The Taiwanese stock market is recognised as one of the most restricted stock markets, due to these comparatively tight boundaries on the daily price movements. Given that there is a lack of empirical studies focusing on the applications of robust portfolio selection problems with advice from multiple analysts, the existing studies on robust optimisation problem rarely discuss the impact of external market systems on the performance of the robust optimisation model. Hence, in further empirical research it is worth investigating the robust multi-analyst portfolio selection approach both with and without the price limits system.

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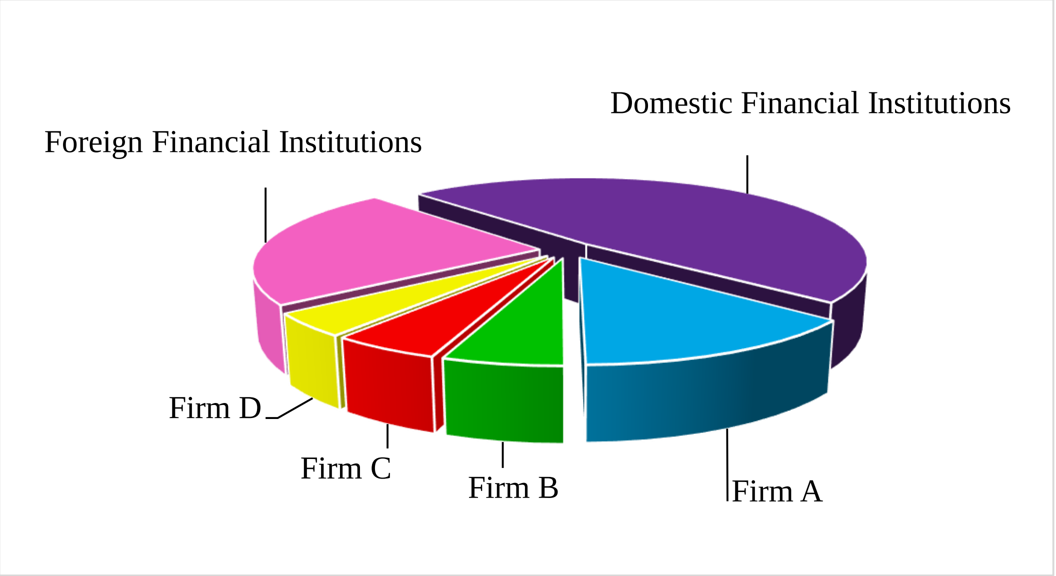
Appendix A

Details of the Securities Brokerage Firms

In this Appendix we provide details of the financial institutions selected for this research. As we discussed in Chapter 5, the stock market newsletters are collected from domestic securities brokerage firms, with their market share by total trading volume captured approximately at 30% in 2013 (see Figure A.1 for further details). The information regarding the securities brokerage firms considered for this research is detailed below.

Figure A.1 Market Share of Taiwanese Securities Brokerage Firms



Source: The TWSE Website.

Note: This figure graphically shows the market share by total trading volume of the securities brokerage firms considered for this research. According to the Taiwan Stock Exchange Corporation, at the end of 2013, Taiwan's financial sector consisted of 87 financial institutions which provide brokerage services. There are 68 domestic financial institutions and 19 foreign bank subsidiaries.

1) The first securities brokerage firm, A, is the subsidiary of a financial institution that has held the dominant position in the Taiwanese investment and banking industry by providing a full range of financial services, such as asset management, banking, futures, insurance, investment advisory services, investment trust, securities brokerage, and venture capital management. The leadership of this financial institution has been recognised by both local and foreign investors for more than a decade. Furthermore, instead of focusing only on Taiwan's market, this financial group also sets up overseas subsidiaries and representative offices for providing cross–border financial services.
2) The second securities brokerage firm, B, is a member of the second largest listed financial holding company in Taiwan. The key subsidiaries of this financial institution include asset management, national and international banks, futures, insurance and life insurance, securities brokerage firms, and venture capital management. Although this financial institution has a lower market share in the finance sector compared to the previous financial institution, it has been recognised as a profitable financial institution in Taiwan for the last five consecutive years.
3) The third securities brokerage firm, C, belongs to a financial institution that offers services in some major areas of the Taiwanese finance sector. For instance, corporate finance management, domestic and foreign stock markets listing services, futures and securities brokerage, insurance planning and consulting, mergers and acquisition, and investment advisory and wealth management services to institutions and individual investors. Unlike the other two securities brokerage firms mentioned earlier, which belong to a financial group with banking and insurance services, this financial institution focuses more on providing management and brokerage services.

4) The fourth securities brokerage firm, D, is a financial services firm. In addition to the securities brokerage and financing services, it also offers a wide range of additional complementary services through its various subsidiaries, such as futures brokerage and futures-related businesses, insurance brokerage and consulting services, investment advisory and wealth management services, and venture capital management. Similar to firm C, securities brokerage firm D focuses more on the management and brokerage services.

Appendix B

Test Results of the Portfolio Management with Equal Preference for the Analysts

This Appendix graphically illustrates full set of test results for the multi-analyst approaches with equal credibility levels to analysts. Table B.1 summaries the investment strategies considered for the empirical examination of the multi-analyst approaches with equal credibility levels to analysts.

Table B.1 List of Investment Strategies

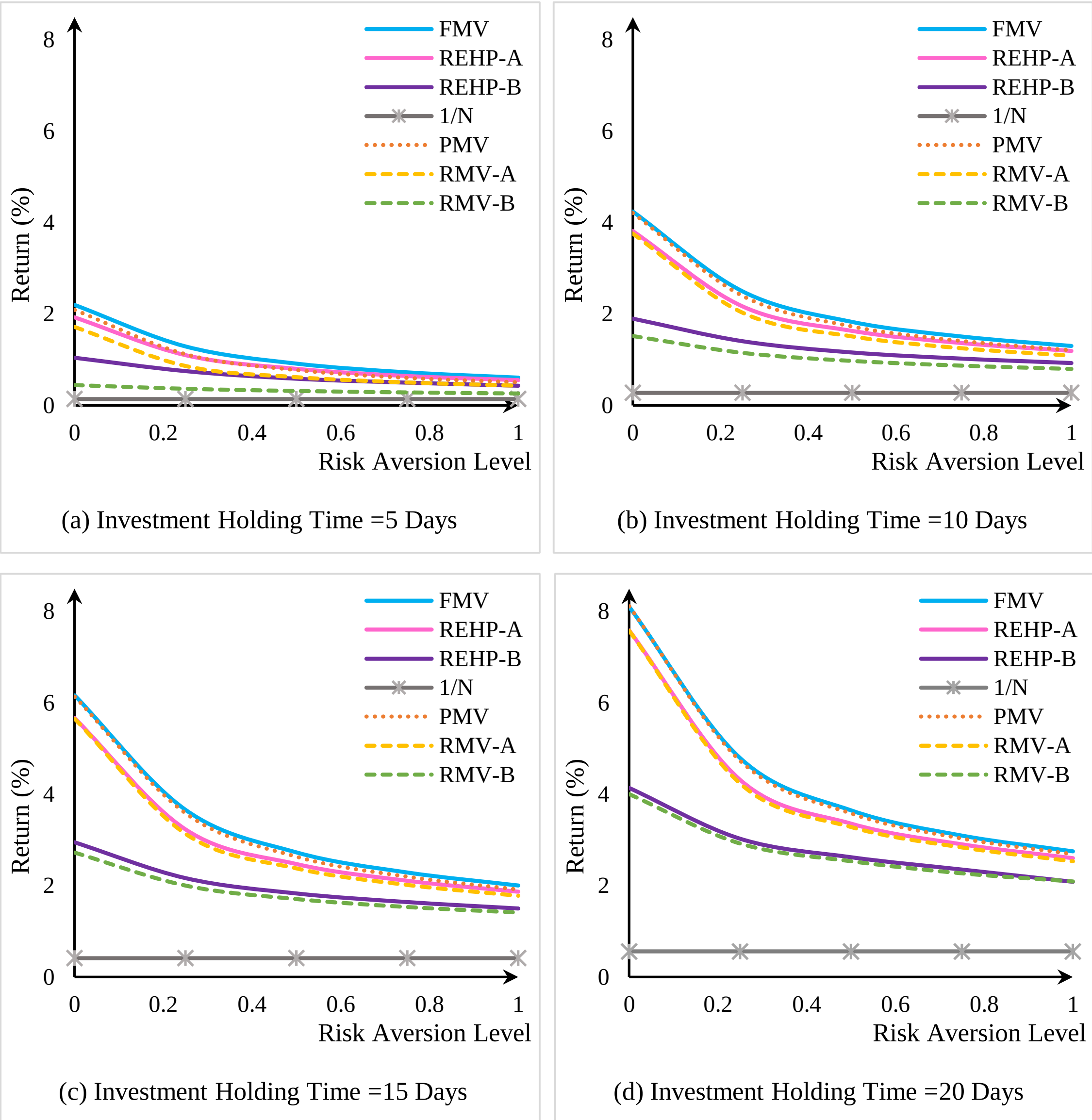
No.	Abbreviation	Model
Asset allocation models developed in this research		
1	$\boldsymbol{F_{MV}}$	Multi-analyst portfolio selection
2	$\boldsymbol{RE_{HP-A}}$	Robust multi-analyst portfolio selection of Type A investor
3	$\boldsymbol{RE_{HP-B}}$	Robust multi-analyst portfolio selection of Type B investor
Asset allocation models from existing literature		
4	$\boldsymbol{P_{MV}}$	Mean-variance portfolio selection
5	$\boldsymbol{R_{MV-A}}$	Robust portfolio selection of Type A investor
6	$\boldsymbol{R_{MV-B}}$	Robust portfolio selection of Type B investor
7	$\boldsymbol{1/N}$	Equally-weighted asset allocation

Note: See Section 6.3 for further details of the chosen investment strategies.

B.1 Ex-Ante Expected Portfolio Performances

The following figures illustrate the ex-ante expected portfolio performances of various investment strategies. Figure B.1 graphically compares the expected portfolio returns between various investment strategies for investment holding time frame $D = 5,10,15,20$. Figure B.2 displays the expected risk adjusted returns under different risk aversion coefficients over different investment holding time frames.

Figure B.1 Expected Portfolio Returns over Sample Period



(a) Investment Holding Time =5 Days

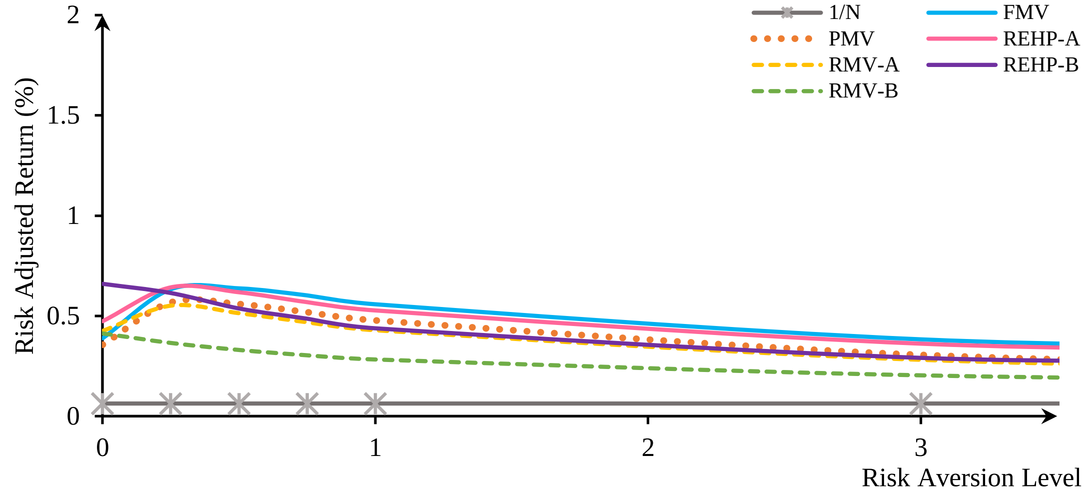
(b) Investment Holding Time =10 Days

(c) Investment Holding Time =15 Days

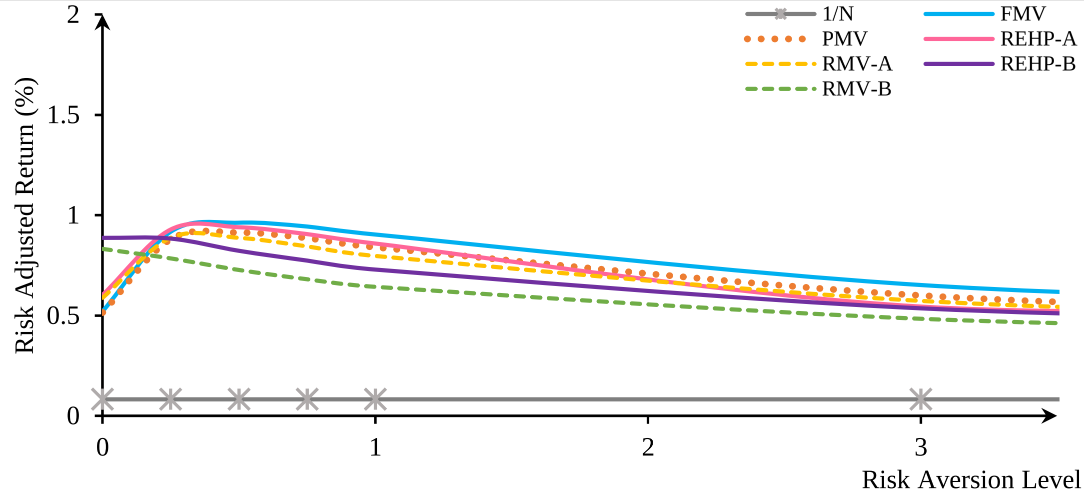
(d) Investment Holding Time =20 Days

Note: This figure shows the expected portfolio returns of various investment strategies. Details of the results are contained in Table 6.3.

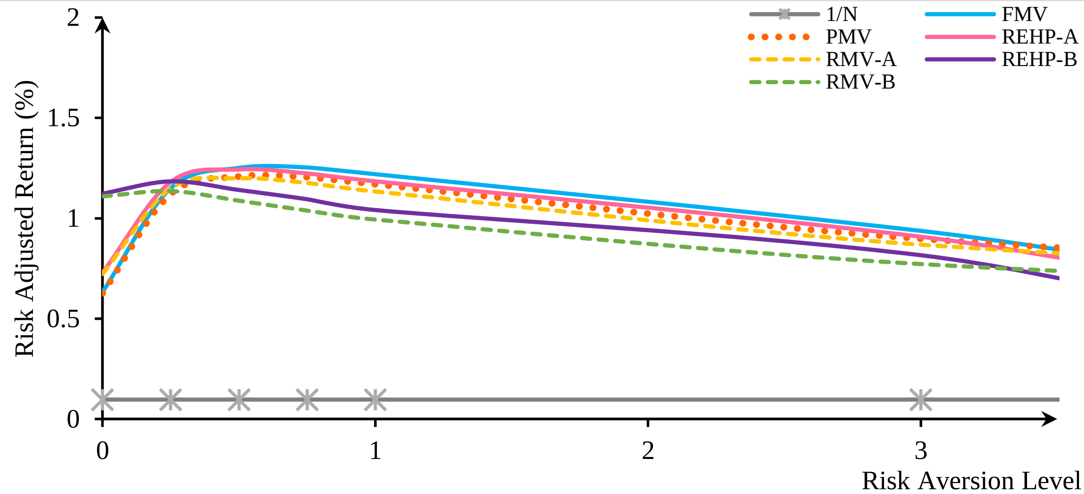
Figure B.2 Expected Risk Adjusted Returns under Different Risk Aversion Levels



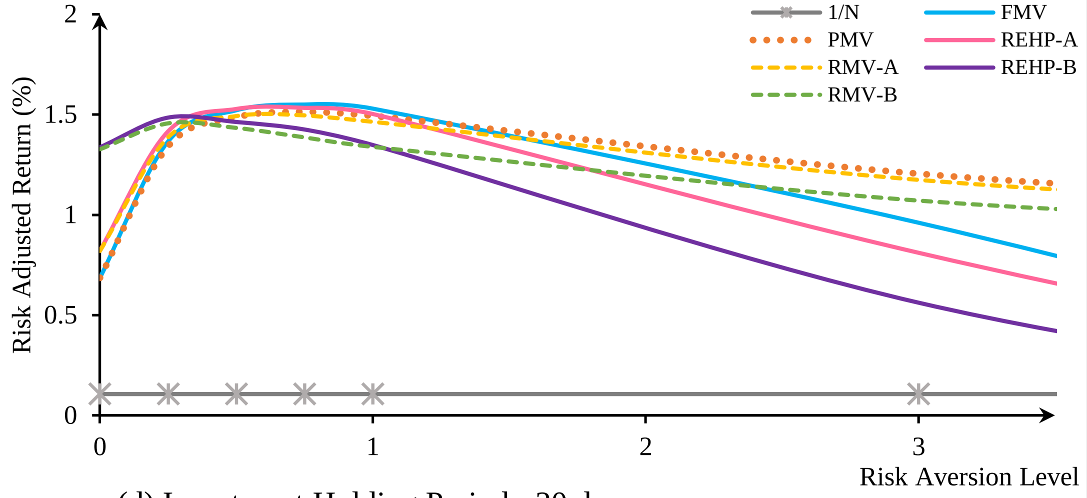
(a) Investment Holding Period = 5 Days



(b) Investment Holding Period =10 days



(c) Investment Holding Period =15 days



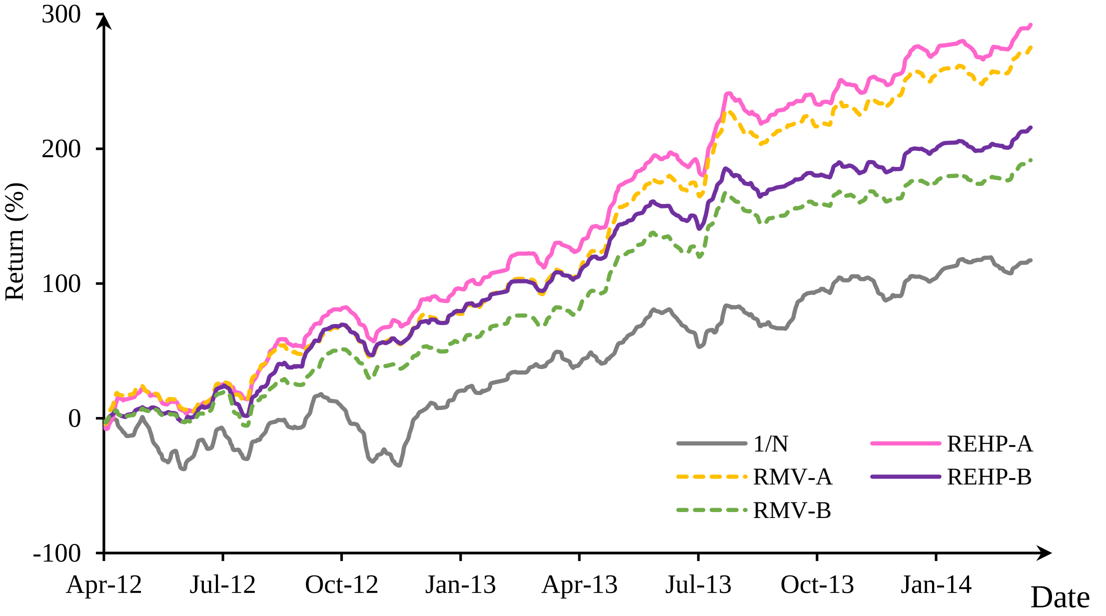
(d) Investment Holding Period =20 days

Note: This figure shows the expected risk adjusted returns of various investment strategies. Details of the results are contained in Table 6.4.

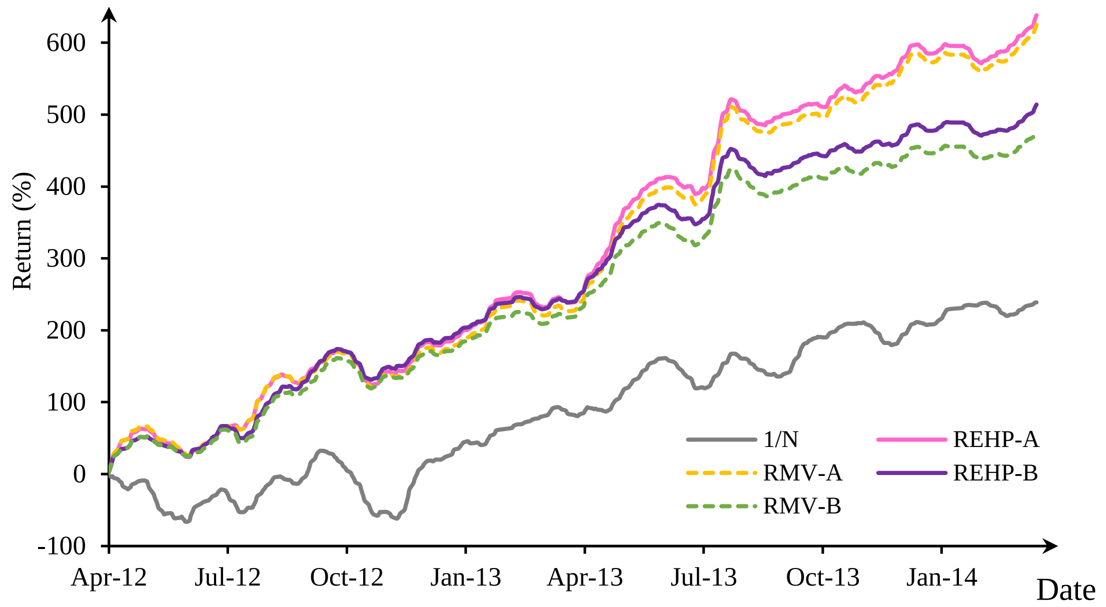
B.2 Ex-Post Realised Portfolio Performances

Figure B.3 illustrates the total realised returns of robust portfolios, (RE_{HP}) and (R_{MV}), at risk aversion level $\lambda = 0.5$ for different levels of robustness over the sample period.

Figure B.3 Realised Cumulative Returns of Robust Portfolios

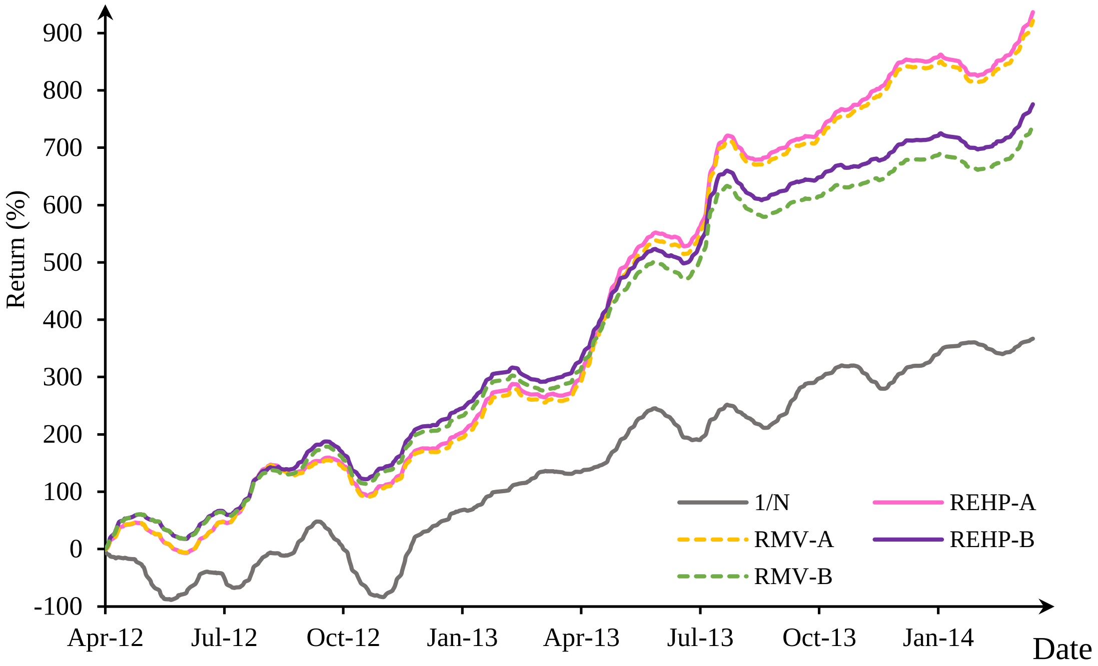


(a) Cumulative Realised Returns for D =5 days

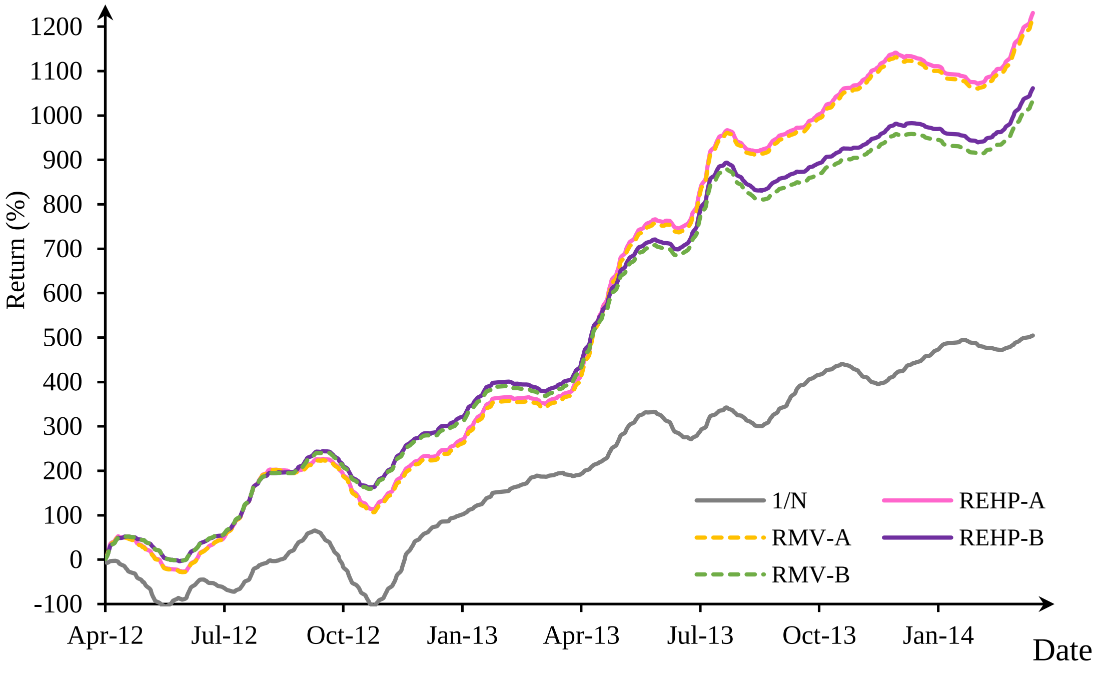


(b) Cumulative Realised Returns for *D* =10 Days

Note: Panel (a) shows the case when holding investment for 5 days, where the differences in cumulative returns between the robust and the robust multi-analyst portfolio are 16.88% for Type A investor and 24.13% for Type B investor. Panel (b) shows the case when holding investment for 10 days, where the differences in cumulative returns between the robust and the robust multi-analyst portfolio are 13.10% for Type A investor and 34.68% for Type B investor. Details of the results are contained in Table 6.5.



(c) Cumulative Realised Returns for *D*=15 days



(d) Cumulative Realised Returns for *D*=20 days

Note: Panel (c) shows the case when holding investment for 15 days, where the differences in cumulative returns between the robust and the robust multi-analyst portfolio are 14.87% for Type A investor and 37.85% for Type B investor. Panel (d) shows the case when holding investment for 10 days, where the differences in cumulative returns between the robust and the robust multi-analyst portfolio are 12.75% for Type A investor and 28.40% for Type B investor. Details of the results are contained in Table 6.5.

B.3 Characteristics of the Portfolios

The following figure graphically illustrates the impact of the desired robustness level and the duration of the investment holding period on portfolio weights for the multi-analyst approaches at $\lambda = 0.5$, where the multi-analyst portfolio (F_{MV}) is the robust multi-analyst portfolio with the desired robustness level $\delta = 0$ and (RE_{HP-B}) is the robust multi-analyst portfolio with the desired robustness level $\delta = 1$. Details of the results are contained in Table 6.7.

Figure B.4 The Ratio of Wealth Invested in the Recommended Stocks