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On the statistical stability of Lorenz attractors with a $C^{1+\alpha}$ stable foliation

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ON THE STATISTICAL STABILITY OF LORENZ ATTRACTORS WITH A $C^{1+\alpha}$ STABLE FOLIATION

WAEEL BAHSOUN AND MARKS RUZIBOEV

ABSTRACT. We prove statistical stability for a family of Lorenz attractors with a $C^{1+\alpha}$ stable foliation.

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1. INTRODUCTION

In his seminal work [21] Lorenz introduced the following system of equations

$$\begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -\frac{8}{3}z + xy \end{cases} \quad (1)$$

as a simplified model for atmospheric convection. Numerical analysis performed by Lorenz showed that the above system exhibits sensitive dependence on initial conditions and has a non-periodic “strange” attractor. A rigorous mathematical framework of similar flows was initiated with the introduction of the so called geometric Lorenz flow in [1, 14]. Nowadays it is well known that the geometric Lorenz attractor, whose vector field will be denoted by X_0 , is robust in the C^1 topology [8]. This means that vector fields X_ε that are sufficiently close in the C^1 topology to X_0 admit invariant contracting foliations $\mathcal{F}_\varepsilon$ on the Poincaré section Σ and they admit strange

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attractors. More precisely, there exists an open neighbourhood U in $\mathbb{R}^3$ containing Λ , the attractor of X_0 , and an open neighbourhood $\mathcal{U}$ of X_0 in the C^1 topology such that for all vector fields $X_\varepsilon \in \mathcal{U}$, the maximal invariant set $\Lambda_{X_\varepsilon} = \cap_{t \geq 0} X_\varepsilon^t(U)$ is a transitive set which is invariant under the flow of X_ε [24]. The papers [26, 27] provided a computer-assisted proof that the classical Lorenz flow; i.e., the flow defined in (1) has a robustly transitive invariant set containing an equilibrium point. In [22] it was proved that the Lorenz flow is mixing. Statistical limit laws were first obtained in [16]. Then rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps was obtained in [7]. Recently, Araújo and Melbourne proved in [5] that the stable foliation of the Lorenz flow is $C^{1+\alpha}$. Moreover, their methods also imply that C^1 perturbations X_ε of the Lorenz flow admit a $C^{1+\alpha}$ stable foliation and the stable foliation of X_ε is C^1 in ε (see Theorem 2.2 in [11]). Further, in another paper Araújo and Melbourne [6] showed that the Lorenz system is exponentially mixing and that this property is robust in the C^1 topology.

In this paper we study a family of perturbations X_ε which are consistent with the results of [5, 6, 8, 27] and prove statistical stability of Lorenz attractors with a $C^{1+\alpha}$ stable foliation. For a precise statement see Theorem 2.2 in section 2. Previous results on the statistical stability of Lorenz attractors was announced in [3] but only for flows with C^2 stable foliations. In [13] among other things, statistical stability of Poincaré maps for ‘BV-like’ Lorenz attractors is studied. In our work, we only assume $C^{1+\alpha}$ stable foliation for the family of flows and we prove that the corresponding 1-d maps are strongly statistically stable. We then obtain statistical stability for the family of flows. Our proofs allow the discontinuities in the base of the corresponding Poincaré maps to change with the perturbation (See Figure 2 for an illustration). In fact, this is the main issue with perturbations of Lorenz systems since the derivative blows up at the discontinuity point.

The rest of the paper is organised as follows. In section 2 we introduce the family of flows that we study in this paper. The statement of our main result (Theorem 2.2) is also in this section. In section 3 we provide proofs of our result in a series of lemmas and propositions.

2. SETUP AND STATEMENT OF MAIN RESULT

2.1. A geometric Lorenz flow with a $C^{1+\alpha}$ stable foliation. Let $X_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field associated with a flow that has an equilibrium point at 0. We assume that X_0 satisfies the following assumptions:

- The differential $DX_0(0)$ has three real eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$, $\lambda_1 + \lambda_3 > 0$ (Lorenz-like singularity) and $\lambda_1 + \lambda_2 < \lambda_3^1$.
- Let

$$\Sigma := \left\{ (x, y, 1) \mid -\frac{1}{2} \leq x, y \leq \frac{1}{2} \right\}$$

¹Note that this condition is required to get the $C^{1+\alpha}$ regularity of the stable foliation for the flow (see [5] section 5).

and let $\Gamma = \{(0, y, 1) \mid -\frac{1}{2} \leq y \leq \frac{1}{2}\}$ be the intersection of Σ with the local stable manifold of the fixed point 0. The segment Γ divides Σ into two parts

$$\Sigma^+ = \{(x, y, 1) \in \Sigma : x > 0\} \text{ and } \Sigma^- = \{(x, y, 1) \in \Sigma : x < 0\}.$$

There exists a well defined Poincaré map $F : \Sigma \rightarrow \Sigma$ such that the images of $\Sigma^\pm$ by this map are curvilinear triangles $S^\pm$ as in Figure 1, without the vertexes $(\pm 1, 0, 1)$ and every line segment in $\mathcal{F} = \{(x, y, 1) \in \Sigma \mid x = \text{const}\}$ except Γ is mapped into a segment $\{(x, y, 1) \in S^\pm \mid x = \text{const}\}$. The return time $\tau : \Sigma \setminus \Gamma \rightarrow \mathbb{R}$ to $S^\pm$ is given by $\tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x|$.

• The flow maps Σ into $S^\pm$ in a smooth way so that the Poincaré map $F(x, y) = X_0^{\tau(x, y, 1)}((x, y, 1), t)$ has the form

$$F(x, y) = (T(x), g(x, y)),$$

- $T : I \rightarrow I$, where $I := [-\frac{1}{2}, \frac{1}{2}]$, has discontinuity at $x = 0$ with side limits $T(0^+) = -\frac{1}{2}$ and $T(0^-) = \frac{1}{2}$;
- T is piecewise monotone increasing and T is C^1 on $I \setminus \{0\}$;
- $\lim_{x \rightarrow 0^\pm} T'(x) = +\infty$;
- $\frac{1}{T'}$ is α -Hölder on I_i , $i = 1, 2$; $I_1 := [-\frac{1}{2}, 0]$ and $I_2 := [0, \frac{1}{2}]$ with $0 < \alpha \leq 1$;
- There are $C > 0$ and $\theta > 1$ such that $(T^n)'(x) \geq C\theta^n$ for any $n \in \mathbb{N}$;
- T is transitive;
- g preserves $\mathcal{F}$ and it is uniformly contracting; i.e., there exists $K > 0$ and $0 < \rho < 1$ such that for any given leaf γ of the foliation and $\xi_1, \xi_2 \in \gamma$ and $n \geq 1$,

$$\text{dist}(F^n(\xi_1), F^n(\xi_2)) \leq K\rho^n \text{dist}(\xi_1, \xi_2).$$

FIGURE 1. The images $S^\pm$ of $\Sigma^\pm$ under the Poincaré map.

2.2. Universally bounded p -variation. We now define a space that captures the regularity of $\frac{1}{T'}$. For $p \geq 1$, we say $f : I \rightarrow \mathbb{R}$ is a function of universally bounded p -variation if

$$V_p(f) := \sup_{-1/2 \leq x_0 < \dots < x_n \leq 1/2} \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{1/p} < +\infty.$$

The space of universally bounded p -variation functions is denoted by $UBV_p(I)$ and it will play a key role in studying perturbations of X_0 .

2.3. Perturbations: a family $\mathcal{X}$ of flows with $C^{1+\alpha}$ stable foliations.

We now consider perturbations of X_0 which are consistent with the results of [5, 8]. From now on we are going to set $p := \frac{1}{\alpha}$. Let $\mathcal{X}$ be the family of C^1 perturbations of X_0 ; i.e., there exists an open neighbourhood U in $\mathbb{R}^3$ containing Λ , the attractor of X_0 , and an open neighbourhood $\mathcal{U}$ of X_0 containing $\mathcal{X}$ such that

- a) for each $X_\varepsilon \in \mathcal{X}$, the maximal forward invariant set Λ_{X_ε} is contained in U and is an attractor containing a hyperbolic singularity;
- b) for each $X_\varepsilon \in \mathcal{X}$, Σ is a cross-section for the flow with a return time τ_ε and a Poincaré map F_ε ;
- c) for each $X_\varepsilon \in \mathcal{X}$, the map F_ε admits a $C^{1+\alpha}$ uniformly contracting invariant foliation $\mathcal{F}_\varepsilon$ on Σ ;
- d) $F_\varepsilon : \Sigma \rightarrow \Sigma$ is given by²

$$F_\varepsilon(x, y) = (T_\varepsilon(x), g_\varepsilon(x, y));$$

- e) the map $T_\varepsilon : I \rightarrow I$ is transitive piecewise C^1 expanding with two branches and a discontinuity point O_ε such that $T(O_\varepsilon^+) = -\frac{1}{2}$, $T(O_\varepsilon^-) = \frac{1}{2}$ and $\lim_{x \rightarrow O_\varepsilon^\pm} T'_\varepsilon(x) = +\infty$ (see Figure 2 for an illustration);
- f) for any $\eta > 0$ there exists ε_0 and an interval $H(\varepsilon_0) \subset I$ such that for all $0 \leq \varepsilon < \varepsilon_0$, $\{O_\varepsilon, 0\} \in H$, $|H| = 2|O_{\varepsilon_0}| < \eta$; and

$$d(T, T_\varepsilon) := \sup_{x \in H^c} \{|T_\varepsilon(x) - T(x)| + |T'_\varepsilon(x) - T'(x)|\} \leq C\eta,$$

where $H^c := I \setminus H$;

- g) there are uniform (in ε and x) constants $C > 0$ and $\theta > 1$ such that $(T_\varepsilon^n)'(x) \geq C\theta^n$ except at the discontinuity point O_ε ;
- h) there is a uniform (in ε) constant $W > 0$ such that

$$\max_{i \in 1, 2} V_{p|I_{i, \varepsilon}} \left(\frac{1}{T'_\varepsilon} \right) \leq W,$$

where $I_{i, \varepsilon}$ is a monotonicity interval of T_ε , and $V_p(\cdot)$ is the p -variation;

- i) for any $n > 0$ let $\mathcal{P}_\varepsilon^{(n)} := \bigvee_{j=0}^{n-1} T_\varepsilon^{-j}(\mathcal{P}_\varepsilon)$, where $\mathcal{P}_\varepsilon = \{I_{1, \varepsilon}, I_{2, \varepsilon}\}$. There exists $\delta_n > 0$ independent of ε such that $\min_{J \in \mathcal{P}_\varepsilon^{(n)}} |J| \geq \delta_n$;
- j) the return time $\tau_\varepsilon : \Sigma \setminus \Gamma_\varepsilon \rightarrow \mathbb{R}$ satisfies the following: there is a constant $C > 0$ such that for each $X_\varepsilon \in \mathcal{X}$

$$\tau_\varepsilon(\xi) \leq -C \log |\pi_\varepsilon(\xi) - O_\varepsilon|,$$

where π_ε is the projection along the leaves of $\mathcal{F}_\varepsilon$ onto I .

Remark 2.1. In [3] the authors impose more regularity conditions on the stable foliations and consequently on T_ε . In particular, they assume that T_ε is piecewise C^2 . They also rely on the result of [17] which assumes that T_0 and T_ε are close in the Skorohod distance and that the transfer operators admit a uniform, in ε , Lasota-Yorke inequality. See [17] § 3); in particular the cautionary Remark 15, items (ii) and (iii). In our work, we only assume that the map T_ε is piecewise C^1 (see condition e)) and $\max_{i \in 1, 2} V_{p|I_{i, \varepsilon}} \left(\frac{1}{T'_\varepsilon} \right) \leq W$, for some $W > 0$ (see condition h)). We also assume that the maps are close in sense of assumption f). We would also like to stress that in our setting

²By c) g_ε is $C^{1+\alpha}$. Moreover, it is a uniform contraction on stable leaves.

FIGURE 2. A graph of T_ε with discontinuity O_ε versus the graph of T with discontinuity at 0.

$\sup_{x \in [-1, O_\varepsilon)} |T'_\varepsilon(x)| = \sup_{x \in (O_\varepsilon, 1]} |T'_\varepsilon(x)| = \infty$ and hence T'_ε does not admit a uniform Hölder constant on its domain.

Before stating our main result, we define an appropriate Banach space, which was first introduced by Keller [18], that will play a key role in our analysis.

2.4. A Banach space. Let $S_\rho(x) := \{y \in I : |x - y| < \rho\}$ and $f : I \rightarrow \mathbb{R}$ be any function defined on I . Let

$$\text{osc}(f, \rho, x) := \text{esssup}\{|f(y_1) - f(y_2)| : y_1, y_2 \in S_\rho(x)\},$$

and

$$\text{osc}_1(f, \rho) = \|\text{osc}(f, \rho, x)\|_1,$$

where the essential supremum is taken with respect to the two dimensional Lebesgue measure on $I \times I$ and $\|\cdot\|_1$ is the L^1 - norm with respect to Lebesgue measure on I . Fix $\rho_0 > 0$ and let $BV_{1,1/p} \subset L^1$ be the Banach space equipped with the norm

$$\|f\|_{1,1/p} = V_{1,1/p}(f) + \|f\|_1,$$

where

$$V_{1,1/p}(f) = \sup_{0 < \rho \leq \rho_0} \frac{\text{osc}_1(f, \rho)}{\rho^{1/p}}.$$

Notice that $V_{1,1/p}(\cdot)$ depends on ρ_0 . The fact that $BV_{1,1/p}$ is a Banach space is proved in [18]. Moreover, it is proved in [18] that the unit ball of $BV_{1,1/p}$ is compact in L^1 . We now list several inequalities, involving functions in $BV_{1,1/p}$, that were proved in [18]. For any $p \geq 1$ and $f \in BV_p(I)$ we have

$$V_{1,1/p}(f) \leq 2^{1/p} V_p(f). \quad (2)$$

Moreover, if $f \in BV_{1,1/p}$ and $Y \subset I$ is an interval with $|Y| \geq 4\rho_0$, then for each $0 < \rho \leq \rho_0$ we have

$$\text{osc}_1(f \cdot \mathbf{1}_Y, \rho) \leq \left(2 + \frac{8\rho_0}{|Y| - 2\rho_0}\right) \int_Y \text{osc}(f|_Y, \rho, x) dx + \frac{4\rho}{|Y|} \int_Y |f(x)| dx. \quad (3)$$

Further, if $Y, Z \subset I$ are intervals such that $T : Y \rightarrow Z$ is differentiable then

$$\begin{aligned} \int_Z \text{osc}\left(\frac{f}{T'} \circ T^{-1}|_Z, \rho, z\right) dx &\leq \int_Y \text{osc}(f|_Y, (C\theta)^{-1}\rho, y) dy \\ &+ 5 \int_Z \text{osc}\left(\frac{1}{T'} \circ T^{-1}|_Z, \rho, z\right) dx \left(\frac{1}{|Y|} \int_Y |f| dx + \frac{1}{\rho_0} \int_Y \text{osc}(f|_Y, \rho_0, y) dy \right). \end{aligned} \quad (4)$$

2.5. Statement of the main results. We first recall the definition of statistical stability for continuous-time dynamical systems: Let V be a neighbourhood of 0. Let $(X_\varepsilon)_{\varepsilon \in V}$ be a family of flows which is endowed with some topology $\mathfrak{T}$. Assume that every X_ε admits a unique SRB measure³ μ_ε . The family $(X_\varepsilon)_{\varepsilon \in V}$ is called statistically stable if $\varepsilon \mapsto X_\varepsilon$ is continuous at $\varepsilon = 0$ in the weak *-topology, i.e.

$$\lim_{\varepsilon \rightarrow 0} \int f d\mu_\varepsilon = \int f d\mu_0,$$

for any continuous function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. Statistical stability is defined analogously for discrete-time dynamical systems. We refer the reader to the articles [2, 4] for more information.

Theorem 2.2. *Let $X_\varepsilon \in \mathcal{X}$. Then*

- 1) X_ε admits a unique invariant probability SRB measure μ_ε .
- 2) For any continuous $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ we have

$$\lim_{\varepsilon \rightarrow 0} \int \varphi d\mu_\varepsilon = \int \varphi d\mu;$$

where μ is the SRB measure associated with X_0 ; i.e. the family $\mathcal{X}$ is statistically stable.

Remark 2.3. 1) in Theorem 2.2 is well known. See for instance [9]. We prove 2) in the following section.

3. PROOFS

3.1. Statistical stability of the family $\mathcal{X}$. Let $\bar{\mu}_\varepsilon$ and $\bar{\mu}_0$ denote the unique absolutely continuous invariant measures of the one dimensional maps T_ε, T_0 respectively. Let h_ε, h_0 denote the densities corresponding to $\bar{\mu}_\varepsilon, \bar{\mu}_0$ respectively. For $\varepsilon \geq 0$, let

$$P_\varepsilon : L^1(I) \rightarrow L^1(I)$$

denote the transfer operator (Perron-Frobenius) associated with T_ε [10, 12]; i.e., for any $f \in L^1(I)$

$$P_\varepsilon f(x) = \sum_{i=1}^2 \left(f \cdot \frac{1}{T'_\varepsilon} \right) (T_\varepsilon|_{I_{i,\varepsilon}})^{-1}(x) \cdot \mathbf{1}_{T_\varepsilon(I_{i,\varepsilon})}(x).$$

Our first goal is to prove that $\lim_{\varepsilon \rightarrow 0} \|h_\varepsilon - h\|_1 = 0$. This will be achieved by showing that P_ε , when acting on $BV_{1,1/p}$, satisfies a uniform (in ε) Lasota-Yorke inequality and that $\varepsilon \mapsto P_\varepsilon$ is continuous at $\varepsilon = 0$ in an appropriate topology. We will be then in a setting where we can apply the spectral stability result of [19], and hence achieve our first goal.

³For more information about SRB measures, we refer to [28].

3.1.1. A uniform Lasota-Yorke inequality. In this subsection, we show that P_ε admits a uniform (in ε) Lasota-Yorke inequality when acting on $BV_{1,1/p}$. We first start with two lemmas to control, uniformly in ε , the p -variation of $\frac{1}{(T_\varepsilon^n)'}.$

Lemma 3.1. *For any $\ell \in \mathbb{N}$*

$$\max_{J \in \mathcal{P}_\varepsilon^{(\ell+1)}} V_{p|J} \left(\frac{1}{|(T_\varepsilon^{\ell+1})'|} \right) < \frac{W(\ell)}{C\theta^\ell},$$

where $W(\ell) := \frac{C^\ell + C - 2}{C^{\ell-1}(C-1)}W$ and W is as in assumption h).

Proof. The proof is by induction on ℓ . Conclusion holds for $\ell = 0$ by h). Suppose it holds for ℓ . Since $J \in \mathcal{P}_\varepsilon^{(\ell+1)}$ is a subset of some $I_{i,\varepsilon} \in \mathcal{P}_\varepsilon$ and $T_\varepsilon(J) \in \mathcal{P}_\varepsilon^\ell$, using the standard properties of variation we get:

$$\begin{aligned} V_p \left(\frac{1}{|(T_\varepsilon^{\ell+1})'|} \right) &\leq \sup_{x \in I} \frac{1}{|T'_\varepsilon|} V_p \left(\frac{1}{|(T_\varepsilon^\ell)' \circ T_\varepsilon|} \right) + \sup_{x \in I} \left(\frac{1}{|(T_\varepsilon^\ell)' \circ T_\varepsilon|} \right) V_p \left(\frac{1}{|T'_\varepsilon|} \right) \\ &\leq \frac{1}{C\theta} V_p \left(\frac{1}{|(T_\varepsilon^\ell)' \circ T_\varepsilon|} \right) + \frac{1}{C\theta^\ell} W \\ &\leq \frac{1}{C\theta} \cdot \frac{1}{C\theta^{\ell-1}} \cdot \frac{C^{\ell-1} + C - 2}{C^{\ell-2}(C-1)} W + \frac{1}{C\theta^\ell} W \\ &\leq \frac{1}{C\theta^\ell} \frac{C^\ell + C - 2}{C^{\ell-1}(C-1)} W. \end{aligned}$$

□

Lemma 3.2. *Fix ℓ such that $\bar{\theta} := C\theta^{\ell-1} > 1$. Then for any ε and n the following holds*

$$\max_{J \in \mathcal{P}_\varepsilon^{(n\ell)}} V_{p|J} \left(\frac{1}{|(T_\varepsilon^{n\ell})'|} \right) \leq \frac{n}{\bar{\theta}^n} W(\ell-1),$$

where $W(\ell)$ is as in Lemma 3.1.

Proof. The proof is again by induction on n . For $n = 1$, by Lemma 3.1, we have

$$\max_{J \in \mathcal{P}_\varepsilon^{(\ell)}} V_{p|J} \left(\frac{1}{|(T_\varepsilon^\ell)'|} \right) \leq W(\ell-1)/(C\theta^{\ell-1}) = W(\ell-1)/\bar{\theta}.$$

Suppose that

$$\max_{J \in \mathcal{P}_\varepsilon^{((n-1)\ell)}} V_{p|J} \left(\frac{1}{|(T_\varepsilon^{(n-1)\ell})'|} \right) \leq \frac{n-1}{\bar{\theta}^{n-1}} W(\ell-1).$$

Let J be any interval in $\mathcal{P}_\varepsilon^{(n\ell)}$ then $T_\varepsilon^\ell J \in \mathcal{P}_\varepsilon^{(n-1)\ell}$. Hence, we get:

$$\begin{aligned} V_{p,J} \left(\frac{1}{|(T_\varepsilon^{n\ell})'|} \right) &\leq \sup_{x \in J} \frac{1}{|(T_\varepsilon^\ell)'|} V_{p,J} \left(\frac{1}{|(T_\varepsilon^{(n-1)\ell})' \circ T_\varepsilon^\ell|} \right) \\ &\quad + \sup_{x \in J} \left(\frac{1}{|(T_\varepsilon^{(n-1)\ell})' \circ T_\varepsilon|} \right) V_{p,J} \left(\frac{1}{|(T_\varepsilon^\ell)'|} \right) \\ &\leq \frac{1}{C\theta^\ell} \frac{n-1}{\theta^{n-1}} W(\ell-1) + \frac{1}{(C\theta^\ell)^{n-1}} \frac{1}{\theta} W(\ell-1) \\ &\leq \frac{n-1}{\theta^n} W(\ell-1) + \frac{1}{\theta^n} W(\ell-1) = \frac{n}{\theta^n} W(\ell-1). \end{aligned}$$

□

Lemma 3.3. *There exists⁴ ρ_0 , $0 < A_1, A_2 < \infty$, $0 < \kappa < 1$, such that for any $0 < \rho < \rho_0$, $n \in \mathbb{N}$ and for any $f \in BV_{1,1/p}$ the following holds*

$$\|P_\varepsilon^n f\|_{1,1/p} \leq A_1 \kappa^n \|f\|_{1,1/p} + A_2 \|f\|_1.$$

Proof. We first obtain an inequality for $n = 1$. Let

$$\underline{K} := \inf_{0 \leq \varepsilon < \varepsilon_0} \{|T_\varepsilon I_{1,\varepsilon}|, |T_\varepsilon I_{2,\varepsilon}|\}.$$

Fix ρ_0 such that

$$\rho_0 < \underline{K}/10. \quad (5)$$

Since $\text{esssup}(f+g) \leq \text{esssup}(f) + \text{esssup}(g)$ we have

$$\begin{aligned} \text{osc}_1(P_\varepsilon f, \rho) &\leq \sum_{i=1}^2 \text{osc}_1 \left(f \cdot \frac{1}{T'_\varepsilon} \circ (T_\varepsilon|I_{i,\varepsilon})^{-1} \cdot \mathbf{1}_{T_\varepsilon(I_{i,\varepsilon})}, \rho \right) \\ &\leq \sum_{i=1}^2 \left(\left(2 + \frac{8\rho_0}{|T(I_{i,\varepsilon})| - 2\rho_0} \right) \int_{T_\varepsilon(I_{i,\varepsilon})} \text{osc} \left(\frac{f}{T'_\varepsilon} \circ (T_\varepsilon|I_{i,\varepsilon})^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho, x \right) dx \right. \\ &\quad \left. + \frac{4\rho}{|T_\varepsilon(I_{i,\varepsilon})|} \int_{T_\varepsilon(I_{i,\varepsilon})} |f \circ (T|I_{i,\varepsilon})^{-1}| dx \right). \end{aligned} \quad (6)$$

The last inequality follows from inequality (3). Now, using the notation $T_\varepsilon|I_{i,\varepsilon} = T_{i,\varepsilon}$ and change of variable formula we have

$$\int_{T(I_{i,\varepsilon})} \left(|f| \circ T_{i,\varepsilon}^{-1} \right) \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} \right) dx = \int_{I_{i,\varepsilon}} |f| dx. \quad (7)$$

Inequality (5) implies that

$$2 + \frac{8\rho_0}{|T_\varepsilon(I_{i,\varepsilon})| - 2\rho_0} < 3. \quad (8)$$

⁴All the constants in this lemma are independent of ε .

On the other hand from (4) it follows that

$$\begin{aligned} \int_{T_\varepsilon(I_{i,\varepsilon})} \text{osc} \left(\frac{f}{T'_\varepsilon} \circ T_{i,\varepsilon}^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho, x \right) dx &\leq \int_{I_{i,\varepsilon}} \text{osc}(f|_{I_{i,\varepsilon}}, (C\theta)^{-1}\rho, x) dx \\ &\quad + 5 \int_{T_\varepsilon(I_{i,\varepsilon})} \text{osc} \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho, x \right) dx \quad (9) \\ &\quad \cdot \left(\frac{1}{|I_{i,\varepsilon}|} \int_{I_{i,\varepsilon}} |f| dx + \frac{1}{\rho_0} \int_{I_{i,\varepsilon}} \text{osc}(f|_{I_{i,\varepsilon}}, \rho_0, y) dy \right). \end{aligned}$$

Using the relation between L^p norms⁵, the definition of $V_{1,1/p}(\cdot)$ and (2) lead to

$$\begin{aligned} &\int_{T_\varepsilon(I_{i,\varepsilon})} \text{osc} \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho, x \right) dx \\ &\leq \left(\int_{T_\varepsilon(I_{i,\varepsilon})} \text{osc} \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho, x \right) dx \right)^{1/p} |T_\varepsilon(I_{i,\varepsilon})|^{1-1/p} \\ &\leq \rho^{1/p} V_{1,1/p} \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T(I_{i,\varepsilon})} \right) \leq (2\rho)^{1/p} V_p \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T(I_{i,\varepsilon})} \right) \\ &\leq (2\rho)^{1/p} V_p \left((T'_{i,\varepsilon})^{-1} |_{I_{i,\varepsilon}} \right). \end{aligned}$$

Therefore,

$$\text{osc}_1 \left(\frac{1}{T'_{i,\varepsilon}} \circ T_{i,\varepsilon}^{-1} |_{T_\varepsilon(I_{i,\varepsilon})}, \rho \right) \leq 2\rho^{1/p} W. \quad (10)$$

Substituting equation (10) first into (9) and then substituting (7), (8) and (9) into (6) and using property h) gives

$$\begin{aligned} \text{osc}_1(P_\varepsilon f, \rho) &\leq 3 \sum_{i=1}^2 \int_{I_i} \text{osc}(f|_{I_{i,\varepsilon}}, (C\theta)^{-1}\rho, y) dy \\ &\quad + 30\rho^{1/p} W \sum_{i=1}^2 \left(\frac{1}{|I_{i,\varepsilon}|} \int_{I_{i,\varepsilon}} |f| dm + \frac{1}{\rho_0} \int_{I_{i,\varepsilon}} \text{osc}(f|_{I_{i,\varepsilon}}, \rho_0, y) dy \right) \\ &\quad + \sum_{i=1}^2 \frac{4\rho}{|T_\varepsilon(I_{i,\varepsilon})|} \int_{I_{i,\varepsilon}} |f| dx \\ &\leq 3 \int_I \text{osc}(f, \frac{\rho}{C\theta}, x) dx \\ &\quad + 30W\rho^{1/p} \left(\frac{1}{\delta_1} \int_I |f| dx + \frac{1}{\rho_0} \int_I \text{osc}(f, \rho_0, x) dx \right) + \frac{4\rho}{\underline{K}} \int_I |f| dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\text{osc}_1(P_\varepsilon f, \rho)}{\rho^{1/p}} &\leq \left(\frac{3}{C\theta} + 30W\rho^{1/p-1} \right) V_{1,1/p}(f) \\ &\quad + \left(30W\frac{1}{\delta_1} + \frac{4\rho^{1-1/p}}{\underline{K}} \right) \|f\|_1. \end{aligned} \quad (11)$$

⁵ $\|f\|_1 \leq \mu(X)^{1-1/p} \|f\|_p$, where X is the space and μ is a measure on it.

Consequently, we have

$$\begin{aligned} V_{1,1/p}(P_\varepsilon f) &\leq \left(\frac{3}{C\theta} + 30W\rho^{1/p-1} \right) V_{1,1/p}(f) \\ &\quad + \left(30W\frac{1}{\delta_1} + \frac{4\rho^{1-1/p}}{\underline{K}} \right) \|f\|_1. \end{aligned} \quad (12)$$

We now prove an inequality for all n as stated in the lemma. Fix $\ell \in \mathbb{N}$ such that $\bar{\theta} := C\theta^{\ell-1} > 1$. By Lemma 3.2 and (12) applied to $(P_\varepsilon^\ell)^k$ we get

$$\begin{aligned} V_{1,1/p}((P_\varepsilon^\ell)^k f) &\leq \left(\frac{3}{\bar{\theta}^k} + \frac{30}{\bar{\theta}^k} kW(\ell-1)\rho^{1/p-1} \right) V_{1,1/p}(f) + \\ &\quad + \left(30W(\ell-1)\frac{1}{\bar{\theta}^k \delta_{k\ell}} + \frac{4\rho^{1-1/p}}{\underline{K}_{k\ell}} \right) \|f\|_1, \end{aligned} \quad (13)$$

where $\underline{K}_{k\ell} := \sup_{0 < \varepsilon < \varepsilon_0} \min_{J \in \mathcal{P}_\varepsilon^{(k\ell)}} \{|T_\varepsilon^{k\ell} J|\}$ and $\delta_{k\ell} > 0$ by assumption i). Since ℓ is fixed we can choose k large enough so that

$$\beta := \left(\frac{3}{\bar{\theta}^k} + \frac{30}{\bar{\theta}^k} kW(\ell-1)\rho^{1/p-1} \right) < 1$$

and let

$$K := \left(30W(\ell-1)\frac{1}{\bar{\theta}^k \delta_{k\ell}} + \frac{4\rho^{1-1/p}}{\underline{K}_{k\ell}} \right).$$

Thus, we have

$$V_{1,1/p}((P_\varepsilon^\ell)^k f) \leq \beta V_{1,1/p}(f) + K\|f\|_1. \quad (14)$$

Similar to (13), by using (12) and Lemma 3.2, for any $j \in \mathbb{N}$ we have

$$\begin{aligned} V_{1,1/p}(P_\varepsilon^j f) &\leq \left(\frac{3}{(C\theta)^j} + 30\frac{W(j-1)}{C\theta^{j-1}}\rho^{1/p-1} \right) V_{1,1/p}(f) \\ &\quad + \left(30\frac{W(j-1)}{C\theta^{j-1}}\frac{1}{\delta_j} + \frac{4\rho^{1-1/p}}{\underline{K}_j} \right) \|f\|_1. \end{aligned} \quad (15)$$

Set $k_0 = \ell k$, where k and ℓ are chosen so that $P_\varepsilon^{k_0}$ satisfies (13). Then for any $n \in \mathbb{N}$ we can write $n = k_0 m + j$ for some $j = 1, \dots, k_0 - 1$. Applying (14) consecutively implies

$$\begin{aligned} V_{1,1/p}(P_\varepsilon^n f) &= V_{1,1/p}((P_\varepsilon^{k_0})^m \circ P_\varepsilon^j f) \leq \beta V_{1,1/p}((P_\varepsilon^{k_0})^{m-1} \circ P_\varepsilon^j f) + K\|f\|_1 \\ &\leq \dots \leq \beta^m V_{1,1/p}(P_\varepsilon^j f) + K\frac{\beta}{1-\beta}\|f\|_1. \end{aligned}$$

Using (15) and setting

$$A_1 := \max_{1 \leq j \leq k_0} \left\{ \frac{3}{(C\theta)^j} + 30\frac{W(j-1)}{C\theta^{j-1}}\rho^{1/p-1} \right\} \cdot \beta^{-j/k_0},$$

$$A_2 := \max_{1 \leq j \leq k_0} \left\{ 30\frac{W(j-1)}{C\theta^{j-1}}\frac{1}{\delta_j} + \frac{4\rho^{1-1/p}}{\underline{K}_j} \right\} + K\frac{\beta}{1-\beta} + 1, \text{ and } \kappa := \beta^{1/k_0},$$

we obtain

$$\|P_\varepsilon^n f\|_{1,1/p} \leq A_1 \kappa^n \|f\|_{1,1/p} + A_2 \|f\|_1.$$

□

3.1.2. *Estimating the difference of the transfer operators in the mixed norm.* Define the following operator ‘mixed’ norm:

$$|||P_\varepsilon||| := \sup_{\|f\|_{1,1/p} \leq 1} \|P_\varepsilon f\|_1.$$

To apply the spectral stability result of [19], we still need to prove that $\lim_{\varepsilon \rightarrow 0} |||P_\varepsilon - P||| = 0$. Firstly, we start with a simple lemma that is similar⁶ to Lemma 11 in [17].

Lemma 3.4. *For any $f \in BV_{1,1/p}(I)$ and $u \in L^1(I)$*

$$\left| \int_I f \cdot u dx \right| \leq (1 + \rho_0^{1/p}) \|f\|_{1,1/p} \sup_{z \in I} \int_{x \leq z} u(x) dx.$$

Proof. We prove the lemma when u is a simple function. Then general case follows, since $BV_{1,1/p} \subset L^1$ and any L^1 function can be approximated by a sequence of simple functions. Let $-\frac{1}{2} = a_0 < a_2 < \dots < a_n = \frac{1}{2}$ be a partition of I and suppose that u is constant on each $J_i = (a_{i-1}, a_i)$, $i = 1, 2, \dots, n$. Let $\overline{\text{conv}(f(J_i))}$ denote the closure of the convex hull of $f(J_i)$ and set $G(x) := \int_{-\frac{1}{2}}^x u dx$. Then

$$\begin{aligned} \left| \int f \cdot u dx \right| &= \left| \sum_{i=1}^n \int_{J_i} f \cdot u dx \right| = \left| \sum_{i=1}^n u|_{J_i} \int_{J_i} f dx \right| \\ &= \left| \sum_{i=1}^n u|_{J_i} y_i \right| \quad \text{where } y_i \in \overline{\text{conv}(f(J_i))} \\ &= \left| \sum_{i=1}^n y_i \int_{J_i} u dx \right| = \left| \sum_{i=1}^n y_i (G(a_i) - G(a_{i-1})) \right| \\ &\leq \sum_{i=2}^n |y_i - y_{i+1}| |G(a_i)| + |G(-1/2)y_1| + |G(1/2)y_n| \\ &\leq \sup_{z \in I} |G(z)| \sum_{i=1}^n |y_i - y_{i+1}| + \|f\|_\infty \sup_{z \in I} |G(z)|. \end{aligned}$$

In the last inequality we used the facts $G(-1/2) = 0$ and $G(1/2) \leq \sup_{z \in I} |G(z)|$. By definition of $\text{osc}(f, \rho, x)$ we have

$$|y_i - y_{i+1}| \leq \text{osc}(f|_{J_i}, 1, z)$$

which implies

$$\int \sum_{i=1}^n |y_i - y_{i+1}| dx \leq \text{osc}_1(f, 1) \leq \rho_0^{1/p} V_{1,1/p}(f).$$

Substituting the latter into above equation finishes the proof. $\square$

Lemma 3.5.

$$\lim_{\varepsilon \rightarrow 0} |||P_\varepsilon - P_0||| = 0.$$

⁶Lemma 11 in [17] was proved for $f \in BV$, the space of one dimensional functions of bounded variation. Here we deal with functions in $BV_{1,1/p}$. To keep the paper self contained, we include a proof.

Proof. For any $f \in BV_{1,1/p}$ we have

$$\begin{aligned} \|P_\varepsilon f - Pf\|_1 &\leq \|P_\varepsilon f - P_\varepsilon(\mathbf{1}_{H^c} f)\|_1 + \|P_\varepsilon(\mathbf{1}_{H^c} f) - P(\mathbf{1}_{H^c} f)\|_1 \\ &\quad + \|P(\mathbf{1}_{H^c} f) - Pf\|_1. \end{aligned} \quad (16)$$

We first estimate the first and the last term in (16). By linearity of P we have

$$\begin{aligned} \|P(\mathbf{1}_{H^c} f) - Pf\|_1 &= \|P(\mathbf{1}_{H^c} f - f)\|_1 \leq \|\mathbf{1}_H f\|_1 \\ &\leq \|f\|_{1,1/p} |H| \leq 2|O_{\varepsilon_0}| \|f\|_{1,1/p} \leq \eta \|f\|_{1,1/p}. \end{aligned} \quad (17)$$

Similarly, for the first term we have

$$\|P_\varepsilon f - P_\varepsilon(\mathbf{1}_{H^c} f)\|_1 \leq \eta \|f\|_{1,1/p}. \quad (18)$$

It remains to prove that the second term in equation (16) goes to zero as $\varepsilon \rightarrow 0$. Let $u = \text{sgn}(P_T(\mathbf{1}_{H^c} f) - P_{T_\varepsilon}(\mathbf{1}_{H^c} f))$. Using the dual operators of P , P_ε and Lemma 3.4, we have:

$$\begin{aligned} \|P(\mathbf{1}_{H^c} f) - P_\varepsilon(\mathbf{1}_{H^c} f)\|_1 &= \left| \int u(P(\mathbf{1}_{H^c} f) - P_\varepsilon(\mathbf{1}_{H^c} f)) dx \right| \\ &= \left| \int \mathbf{1}_{H^c} u \circ T f dx - \int \mathbf{1}_{H^c} u \circ T_\varepsilon f dx \right| = \left| \int \mathbf{1}_{H^c} f (u \circ T - u \circ T_\varepsilon) dx \right| \\ &\leq (1 + \rho_0^{1/p}) \|f\|_{1,1/p} \sup_z \int_{-1/2}^z \mathbf{1}_{H^c} (u \circ T - u \circ T_\varepsilon) dx \\ &= (1 + \rho_0^{1/p}) \|f\|_{1,1/p} \sup_z \int_{[-1/2, z] \cap H^c} (u \circ T - u \circ T_\varepsilon) dx \\ &= (1 + \rho_0^{1/p}) \|f\|_{1,1/p} \sup_z \sum_{i=1}^2 \left(\int_{T_i(H_z^c)} \frac{u(y)}{T'(T_i^{-1}(y))} dy - \int_{T_{i,\varepsilon}(H_z^c)} \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} dy \right) \end{aligned} \quad (19)$$

where $H_z^c = [-1/2, z] \cap H^c$ and we used change of variables $y = T(x)$ and $y = T_\varepsilon(x)$ for the first and second summands respectively. For $i = 1, 2$ we have

$$\begin{aligned} &\int_{T_i(H_z^c)} \frac{u(y)}{T'(T_i^{-1}(y))} dy - \int_{T_{i,\varepsilon}(H_z^c)} \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} dy \\ &= \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} \left(\frac{u(y)}{T'(T_i^{-1}(y))} - \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} \right) dy \\ &\quad + \int_{T_i(H_z^c) \setminus T_{i,\varepsilon}(H_z^c)} \frac{u(y)}{T'(T_i^{-1}(y))} dy - \int_{T_{i,\varepsilon}(H_z^c) \setminus T_i(H_z^c)} \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} dy \\ &:= E_1 + E_2 + E_3 \end{aligned} \quad (20)$$

Now we estimate each of the terms in the right hand side separately. Using $T'_\varepsilon \geq C\theta$ and the fact that $\|u\|_\infty \leq 1$ we have

$$\begin{aligned}
E_1 &:= \left| \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} \left(\frac{u(y)}{T'(T_i^{-1}(y))} - \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} \right) dy \right| \\
&\leq \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} \left| \frac{1}{T'(T_i^{-1}(y))} - \frac{1}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} \right| dy \\
&\quad + \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} \left| \frac{1}{T'(T_i^{-1}(y))} - \frac{1}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} \right| dy \\
&\leq \left\| \frac{1}{T'} \right\|_\alpha \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} |T_{i,\varepsilon}^{-1}(y) - T_i^{-1}(y)|^\alpha dy \\
&\quad + \frac{1}{(C\theta)^2} \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} |T'_\varepsilon(T_{i,\varepsilon}^{-1}(y)) - T'(T_{i,\varepsilon}^{-1}(y))| dy \\
&\leq \left\| \frac{1}{T'} \right\|_\alpha \int_{T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)} |T_{i,\varepsilon}^{-1}(y) - T_i^{-1}(y)|^\alpha dy \\
&\quad + \frac{1}{(C\theta)^2} |T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)| d(T, T_\varepsilon).
\end{aligned}$$

Notice that for $y \in T(H_z^c) \cap T_\varepsilon(H_z^c)$ there exists $x \in H_z^c$ such that $T_\varepsilon(x) = y$ we have

$$|T_{i,\varepsilon}^{-1}(y) - T_i^{-1}(y)| = |x - T_i^{-1}(T_\varepsilon(x))| \leq \text{Lip}(T_i^{-1}) d(T, T_\varepsilon) \leq \frac{1}{C\theta} d(T, T_\varepsilon). \quad (21)$$

Taking into account the fact that $|T_i(H_z^c) \cap T_{i,\varepsilon}(H_z^c)| \leq 1$ the relation (21) implies that

$$E_1 \leq C_2 \eta^\alpha.$$

Now note that by assumption f) for any $x \in H^c$ we have $|T_\varepsilon(x) - T(x)| < C\eta$. This implies $|T_\varepsilon(H^c) \triangle T(H^c)| \leq 2C\eta$. Hence, for all sufficiently small ε we have

$$E_2 := \left| \int_{T_i(H_z^c) \setminus T_{i,\varepsilon}(H_z^c)} \frac{u(y)}{T'(T_i^{-1}(y))} dy \right| \leq \frac{1}{C\theta} |T_i(H_z^c) \setminus T_{i,\varepsilon}(H_z^c)| = C_3 \eta.$$

Similarly,

$$E_3 := \left| \int_{T_{i,\varepsilon}(H_z^c) \setminus T_i(H_z^c)} \frac{u(y)}{T'_\varepsilon(T_{i,\varepsilon}^{-1}(y))} dy \right| \leq C_3 \eta.$$

Substituting estimates for E_1 , E_2 and E_3 first into equation (20) and then substituting the result into (19) gives

$$\|P(\mathbf{1}_{H^c} f) - P_\varepsilon(\mathbf{1}_{H^c} f)\|_1 \leq (1 + \rho_0^{1/p})(2C_2 + 4C_3) \|f\|_{1,1/p} \eta^\alpha.$$

Substituting this and equations (17) and (18) into (16) implies

$$\|P_\varepsilon f - Pf\|_1 \leq C\eta^\alpha \|f\|_{1,1/p}$$

which finishes the proof. $\square$

We are now ready to prove that the 1-d family T_ε is strongly statistically stable. Firstly, we set some notation. Consider the set

$$V_{\delta,r}(P) = \{z \in \mathbb{C} : |z| \leq r \text{ or } \text{dist}(z, \sigma(P)) \leq \delta\},$$

where $\sigma(P)$ is the spectrum of P when acting on $BV_{1,1/p}$.

Proposition 3.6.

$$\lim_{\varepsilon \rightarrow 0} \|h_\varepsilon - h\|_1 = 0.$$

Proof. By Lemma 3.3 and Lemma 3.5, for any $z \in V_{\delta,r}(P)$ the Keller-Liverani [19] stability result implies

$$\lim_{\varepsilon \rightarrow 0} \|(zI - P_\varepsilon)^{-1} - (zI - P)^{-1}\| = 0.$$

Consequently,

$$\lim_{\varepsilon \rightarrow 0} \|\Pi_{1,\varepsilon} - \Pi_1\| = 0,$$

where $\Pi_{1,\varepsilon}$ and Π_1 are the spectral projections of P_ε and P associated with the eigenvalue 1. This completes the proof since both $\Pi_{1,\varepsilon}$ and Π_1 have rank 1. $\square$

3.1.3. Statistical stability: from the 1-d family, to the Poincaré maps, to the family of flows. We now discuss how to obtain continuity of the SRB measures (3) of Theorem 2.2) from Proposition 3.6. We first show how the absolutely continuous invariant measures of the family of 1-d maps are related to the SRB measures of the family of the flows via the Poincaré maps. This construction is well known (see for instance [9]). Let $\tilde{\psi} : \Sigma \rightarrow \mathbb{R}$ be any bounded function. Notice that Σ is foliated by stable manifolds, and any $x \in I$ defines unique stable manifold $\pi^{-1}(x)$. Therefore $\psi_\varepsilon^+ : I \rightarrow \mathbb{R}$ and $\psi_\varepsilon^- : I \rightarrow \mathbb{R}$ are well defined by

$$\psi_\varepsilon^+(x) := \sup_{\xi \in \pi_\varepsilon^{-1}(x)} \tilde{\psi}(\xi) \quad \text{and} \quad \psi_\varepsilon^-(x) := \inf_{\xi \in \pi_\varepsilon^{-1}(x)} \tilde{\psi}(\xi).$$

There exists a unique F_ε -invariant probability measure μ_{F_ε} on Σ such that for every continuous function $\tilde{\psi} : \Sigma \rightarrow \mathbb{R}$

$$\int \tilde{\psi} d\mu_{F_\varepsilon} = \lim_{n \rightarrow \infty} \int_I (\tilde{\psi} \circ F_\varepsilon^n)_\varepsilon^+ d\bar{\mu}_\varepsilon = \lim_{n \rightarrow \infty} \int_I (\tilde{\psi} \circ F_\varepsilon^n)_\varepsilon^- d\bar{\mu}_\varepsilon,$$

where $\bar{\mu}_\varepsilon$ is the T_ε -invariant absolutely continuous measure (see for instance, Lemma 6.1, [9]). To pass from the Poincaré map to the flow we use standard procedure: first consider suspension flow from the Poincaré map and then embed the suspension flow into the original flow. To apply the construction we first need to prove the following

Lemma 3.7. *For every $X_\varepsilon \in \mathcal{X}$ let $F_\varepsilon : \Sigma \rightarrow \Sigma$ be its Poincaré map and define μ_{F_ε} as above. Then τ_ε is μ_{F_ε} -integrable.*

Proof. Let $\tau_{N,\varepsilon} = \min\{N, \tau_\varepsilon\}$. Then $\tau_{N,\varepsilon}$ is monotone increasing in N and it converges to τ_ε almost everywhere. Since $\tau_{N,\varepsilon}$ is continuous and $d\bar{\mu}_\varepsilon/dm$

is uniformly bounded, in ε , we have

$$\begin{aligned} \int \tau_{N,\varepsilon} d\mu_{F_\varepsilon} &= \lim_{n \rightarrow \infty} \int (\tau_{N,\varepsilon} \circ F_\varepsilon^n)_\varepsilon^+ d\bar{\mu}_\varepsilon = \lim_{n \rightarrow \infty} \int (\tau_{N,\varepsilon} \circ F^n)_\varepsilon^+ (d\bar{\mu}_\varepsilon/dm) dm \\ &\leq C \|d\bar{\mu}_\varepsilon/dm\|_\infty \int_I \log |x - O_\varepsilon| dx < +\infty, \end{aligned}$$

which implies that $\lim_{N \rightarrow \infty} \int \tau_{N,\varepsilon} d\mu_F$ exists and finite. Hence by monotone convergence theorem $\int \tau_\varepsilon d\mu_{F_\varepsilon} < \infty$. $\square$

Let

$$\Sigma_{\tau_\varepsilon} = \Sigma \times [0, +\infty) / \sim,$$

where $\sim$ is the equivalence relation on $\Sigma \times [0, +\infty)$ generated by $(\xi, \tau_\varepsilon(\xi)) \sim (F_\varepsilon(\xi), 0)$. Then there is a natural projection $\pi_{\tau_\varepsilon} : \Sigma \times [0, \infty) \rightarrow \Sigma_{\tau_\varepsilon}$ which induces a topology and a Borel σ -algebra on $\Sigma_{\tau_\varepsilon}$. The suspension flow of F_ε with return time τ_ε is the semi-flow $(X_\varepsilon^t)_{t \geq 0}$ defined on $\Sigma_{\tau_\varepsilon}$ as

$$X_\varepsilon^t(\pi_{\tau_\varepsilon}(\xi, s)) = \pi_{\tau_\varepsilon}(\xi, s + t) \quad \text{for any } (\xi, s) \in \Sigma \times [0, +\infty).$$

Lemma 3.8. ([9], Lemma 6.7.) *The suspension flow X_ε^t admits a unique invariant probability measure $\bar{\mu}_{X_\varepsilon}$. Moreover, for every bounded measurable $\varphi : \Sigma_{\tau_\varepsilon} \rightarrow \mathbb{R}$, we have*

$$\int \varphi d\bar{\mu}_{X_\varepsilon} = \frac{1}{\mu_{F_\varepsilon}(\tau_\varepsilon)} \int \int_0^{\tau_\varepsilon(\xi)} \varphi(\pi_{\tau_\varepsilon}(\xi, t)) dt d\mu_{F_\varepsilon}(\xi),$$

where $\mu_{F_\varepsilon}(\tau) = \int \tau_\varepsilon d\mu_{F_\varepsilon}$.

Now, we can define the unique SRB measure of the original flow $X_\varepsilon(\xi, t)$. Define

$$\Phi_\varepsilon : \Sigma \times [0, +\infty) \rightarrow U \quad \text{by letting } \Phi_\varepsilon(\xi, t) = X_\varepsilon(\xi, t).$$

Since $\Phi_\varepsilon(\xi, \tau_\varepsilon(\xi)) = (F_\varepsilon(\xi), 0) \in \Sigma \times \{0\}$, map Φ_ε induces a map

$$\phi_\varepsilon : \Sigma_{\tau_\varepsilon} \rightarrow U, \quad \text{such that } \phi_\varepsilon \circ X_\varepsilon^t = X_\varepsilon(\cdot, t) \circ \phi_\varepsilon, \quad \text{for } t \geq 0. \quad (22)$$

via the identification $\sim$. Now the invariant measure of X_ε^t is naturally transferred to an invariant measure for $X_\varepsilon(\cdot, t)$ via pushing it forward $\mu_\varepsilon = \phi_{\varepsilon*} \bar{\mu}_{X_\varepsilon}$ (see [9], Section 7). Hence, we can define the SRB measure of the flow as follows:

Lemma 3.9. *The flow of each $X_\varepsilon \in \mathcal{X}$ has a unique SRB measure μ_ε . In particular, for any continuous function $\varphi : U \rightarrow \mathbb{R}$*

$$\int \varphi d\mu_\varepsilon = \frac{1}{\mu_{F_\varepsilon}(\tau_\varepsilon)} \int \int_0^{\tau_\varepsilon(\xi)} \varphi \circ \phi_\varepsilon \circ \pi_\varepsilon(\xi, t) dt d\mu_{F_\varepsilon}(\xi)$$

where $\mu_{F_\varepsilon}(\tau_\varepsilon) = \int \tau_\varepsilon d\mu_{F_\varepsilon}$.

The proof of 2) of Theorem 2.2, then proceeds as follows. We first note that Lemma 3.3 implies that the densities h_ε are in $BV_{1,1/p}$ and hence in L^∞ . Then by our Proposition 3.6 above and Proposition 3.3 of [3] we obtain that the Poincaré map is statistically stable. Then statistical stability of the Poincaré map is first lifted to the suspension flow and finally to the original flow. Notice that the key ingredients in the proof of Propositions 3.3 and Lemma 4.2. in [3] are that the densities h_ε are in L^∞ , the compactness

of the Poincaré section and the fact that the Lebesgue measure of the set where $\tau_\varepsilon > n$ decays sufficiently fast, which is the case of our setting.

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