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Coupled Klein-Gordon equations and energy exchange in two-component systems

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Abstract. A system of coupled Klein-Gordon equations is proposed as a model for one-dimensional nonlinear wave processes in two-component media (e.g., long longitudinal waves in elastic bi-layers, where nonlinearity comes only from the bonding material). We discuss general properties of the model (Lie group classification, conservation laws, invariant solutions) and special solutions exhibiting an energy exchange between the two physical components of the system. To study the latter, we consider the dynamics of weakly nonlinear multi-phase wavetrains within the framework of two pairs of counter-propagating waves in a system of two coupled Sine-Gordon equations, and obtain a hierarchy of asymptotically exact coupled evolution equations describing the amplitudes of the waves. We then discuss modulational instability of these weakly nonlinear solutions and its effect on the energy exchange.

1 Introduction

During the past decades it has been understood that the dynamics of nonlinear waves is central to explaining a wide range of physical phenomena. Also, it has become clear that the same equations, such as, for example, the Korteweg – de Vries equation, the nonlinear Schrödinger (NLS) equation, the Sine-Gordon equation, etc., appear in many different physical situations (see, for example, [1,2] and the references therein). Since heterogeneity constitutes an essential feature of many physical problems, it makes sense to introduce some simple mathematical models which allow one to study peculiarities of nonlinear wave processes in heterogeneous media.

To derive such continuum models, one can consider the long-wave dynamics of discrete models. One of the most famous models of this type is the Frenkel-Kontorova (FK) model [3]. Proposed initially to describe dislocations in metals, it has found numerous applications, and has been generalised in order to describe different phenomena (see, for example, [4]). The original FK model deals with the situation when an additional semi-infinite plane of atoms is inserted into a perfect crystal lattice. The atoms of the “interface” layer are treated as a one-dimensional chain subjected to an external periodic potential produced by the surrounding atoms. For long waves, in dimensionless variables, one can obtain the integrable sine-Gordon equation whose kink and antikink solutions give the approximate description of the lattice behavior in the vicinity of the dislocation core.

In the FK model, one part of the crystal is treated as rigid and motionless. The natural generalisation of this model is that of coupled chains of particles [5], i.e., of two one-dimensional periodic chains with linear links between elements and non-linear interaction between the chains. The long wave dynamics of this system is described, in dimensionless variables, by the coupled

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Klein-Gordon equations

$$u_{tt} - u_{xx} = f_u(u, w), \quad w_{tt} - c^2 w_{xx} = f_w(u, w), \quad (1)$$

where the subscripts denote partial derivatives, c is the ratio of the acoustic velocities of non-interacting components, and $f(u, w)$ describes the interaction between the chains.

Although originating as a generalisation of the FK model, for appropriate choices of the function $f(u, w)$, the coupled Klein-Gordon equations (1) can also be considered as a long-wave limit of a simple lattice model for longitudinal nonlinear waves in elastic bi-layers, where the dominant nonlinearity comes only from the bonding material [6]. Many advanced structures, such as joints, coatings and electronic packages, use bi-layers, i.e., combinations of two materials with different properties. Bi-materials have been used for decades to obtain a structure with the properties better than those of the parent constituents. Their increasing use at present is mainly due to new developments in aerospace industry and microelectronics. When (1) is applied to the modelling of bi-layers, the potential function $f(u, w)$ and parameter c are determined by the type of interface and materials forming the bi-layer. It is also worth noting that similar equations describe some processes in the DNA double helix [7] (see also [8] and the references therein).

In such phenomenological models the function $f(u, w)$ should be found experimentally, and therefore, its analytic form is not unique. It is known that the existence of a sufficiently large group of symmetries allows a certain analytic investigation of properties of the equations (see, for example, [9, 10]). Thus, the problem of group classification of coupled Klein-Gordon equations (1) naturally arises in connection with the model introduced above and is discussed in Section 2. These results can be used to find conservation laws and to construct particular invariant solutions, which is discussed in Section 3.

In Section 4, we discuss solutions exhibiting periodic energy exchange between the two physical components of the system within the framework of two pairs of counter-propagating waves in a system of coupled Sine-Gordon equations, and derive various asymptotic models for weakly nonlinear solutions. The emphasis is on the case when the system is not integrable, and the group velocities of each pair of waves are arbitrary and usually different from each other. Thus, our methodology is generic and can be applied to other systems, and other multi-phase wavetrains. We then discuss modulational instability of these solutions and show that it can lead to the formation of localized structures, and to a modification of the linear energy exchange, which then continues for some time into the nonlinear regime as an energy exchange between these localized structures. When the system is close to being integrable, the time evolution is distinguished by a remarkable almost periodic sequence of energy exchange scenarios, with spatial patterns alternating between approximately uniform wavetrains and localized structures, similar to the famous Fermi-Pasta-Ulam recurrence [11].

We conclude in Section 5, where we discuss some possible applications and generalisations of our model.

2 Lie point symmetries

Equations of type (1) with $c = 1$ (and arbitrary functions of u and w in the right-hand side) were studied in [12], where the cases admitting Lie-Bäcklund symmetries were found, and completely or partially integrable examples were presented. If $f_{uw}(u, w) = 0$, the system (1) splits into two independent Klein-Gordon equations, whose group classification was given by S.Lie [13] (see also [14]). The results of Lie group classification of equations (1) for $c \neq 1$, $f_{uw}(u, w) \neq 0$ are briefly discussed below (see [5, 6]). In the next Section they are used to find conservation laws and to construct particular invariant solutions.

The classification is given up to the following equivalence transformations (which preserve the differential structure of equations and change only the arbitrary element, see, e.g., [9])

$$\tilde{t} = \alpha t + \beta, \quad \tilde{x} = \alpha x + \gamma, \quad \tilde{u} = \lambda u + \mu, \quad \tilde{w} = \lambda w + \nu, \quad \tilde{f} = \frac{\lambda^2}{\alpha^2} f + \rho, \quad (2)$$

where $\alpha, \beta, \gamma, \lambda, \mu, \nu$, and ρ are arbitrary constants and $\alpha\lambda \neq 0$;

$$\begin{aligned} t \rightarrow -t, \quad x \rightarrow x; \quad t \rightarrow t, \quad x \rightarrow -x; \\ u \rightarrow -u, \quad w \rightarrow w; \quad u \rightarrow u, \quad w \rightarrow -w; \end{aligned} \quad (3)$$

$$t \rightarrow t, \quad x \rightarrow \frac{x}{c}, \quad u \rightarrow w, \quad w \rightarrow u, \quad c \rightarrow \frac{1}{c}. \quad (4)$$

(Groups admitted by equations related via equivalence transformations are similar).

The generator of the admitted Lie point group is sought in the form

$$X = \xi^1(t, x, u, w) \frac{\partial}{\partial t} + \xi^2(t, x, u, w) \frac{\partial}{\partial x} + \eta^1(t, x, u, w) \frac{\partial}{\partial u} + \eta^2(t, x, u, w) \frac{\partial}{\partial w}. \quad (5)$$

Using the methods of the group analysis of differential equations (see, for example, [9], [10]), the coordinates of the generator can be found as follows:

$$\begin{aligned} \xi^1 &= C_1 t + C_2, \quad \xi^2 = C_1 x + C_3, \\ \eta^1 &= C_4 u + \phi(t, x), \quad \eta^2 = C_4 w + \chi(t, x), \end{aligned}$$

where functions $\phi(t, x)$ and $\chi(t, x)$ satisfy the equations

$$\begin{aligned} \phi_{tt} - \phi_{xx} &= (2C_1 - C_4)f_u + (C_4 u + \phi)f_{uu} + (C_4 w + \chi)f_{uw}, \\ \chi_{tt} - c^2 \chi_{xx} &= (2C_1 - C_4)f_w + (C_4 u + \phi)f_{uw} + (C_4 w + \chi)f_{ww}. \end{aligned} \quad (6)$$

Here C_i , $i = \overline{1, 4}$ are arbitrary constants.

It follows from (6) that when $f(u, w)$ is an arbitrary function, equations (1) admit 2-dimensional algebra with the basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}. \quad (7)$$

Extension of the admitted algebra (7) is possible in those cases when function $f(u, w)$ satisfies the following equations

$$\begin{aligned} (a_1 u + a_2)f_{uu} + (a_1 w + a_3)f_{uw} + (a_1 + a_4)f_u &= a_5, \\ (a_1 u + a_2)f_{uw} + (a_1 w + a_3)f_{ww} + (a_1 + a_4)f_w &= a_6 \end{aligned}$$

with constant coefficients a_i , $i = \overline{1, 6}$, or, consequently, the equation

$$(a_1 u + a_2)f_u + (a_1 w + a_3)f_w + a_4 f = a_5 u + a_6 w + a_7, \quad (8)$$

where a_7 is also a constant coefficient.

To analyse the classifying relation (8), we first find how its coefficients transform under the equivalence transformations. Then, we find all solutions of equation (8) up to the equivalence transformations. Finally, solving equations (6) for all cases we find the generators additional to (7). The results of group classification are presented in Table 1.

Table 1. Group classification of Eqs.(1) ($c^2 \neq 1$, $f_{uw} \neq 0$).

$f(u, w)$	Admitted generators
Arbitrary function	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}$
$u^\sigma F(\frac{u}{w}), \quad \sigma \neq 0,$ if $\sigma \neq 2, \quad F(z) \neq \varepsilon \left(\delta - \frac{1}{z} \right)^\sigma$ if $\sigma = 2, \quad F(z) \neq A + \frac{B}{z^2} + \frac{1}{z}$	$X_3 = (\sigma - 2)(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}) - 2(u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w})$
$\exp u F(\tilde{\delta}u - w),$ $F(z) \neq \varepsilon \exp z$	$X_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2(\frac{\partial}{\partial u} + \tilde{\delta} \frac{\partial}{\partial w})$
$F(\frac{u}{w}) + \tilde{\varepsilon} \ln u,$ $F(z) \neq \tilde{\varepsilon} \ln \left(\delta - \frac{1}{z} \right)$	$X_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w}$
$F(\delta u - w) + A u w + (\varepsilon - \delta A) \frac{u^2}{2},$ $F'''(z) \neq 0$ a) $\lambda^2 = \frac{\delta \varepsilon - A}{\delta(1 - c^2)} \neq 0,$ $\mu^2 = \frac{\delta \varepsilon c^2 - A}{\delta(1 - c^2)} \neq 0$ b) $\lambda = 0, \quad \mu^2 = -\frac{A}{\delta} = -\varepsilon$ c) $\lambda^2 = \frac{A}{\delta c^2} = \varepsilon, \quad \mu = 0$	$X_i = \phi_i(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right), \quad i = \overline{3, 6}$ $\phi_3 = \cos \lambda x \cos \mu t,$ $\phi_4 = \sin \lambda x \sin \mu t,$ $\phi_5 = \cos \lambda x \sin \mu t,$ $\phi_6 = \sin \lambda x \cos \mu t$ $\phi_3 = x \cos \mu t, \quad \phi_4 = \cos \mu t,$ $\phi_5 = x \sin \mu t, \quad \phi_6 = \sin \mu t$ $\phi_3 = t \cos \lambda x, \quad \phi_4 = \cos \lambda x,$ $\phi_5 = t \sin \lambda x, \quad \phi_6 = \sin \lambda x$
$F(\delta u - w) + \varepsilon u w - \delta \varepsilon \frac{u^2}{2}, \quad F'''(z) \neq 0$ $\lambda^2 = \frac{\varepsilon}{\delta(c^2 - 1)}$	$X_i = \phi_i(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right), \quad i = \overline{3, 6}$ $\phi_3 = \cos \lambda x \cos \lambda t, \quad \phi_4 = \sin \lambda x \sin \lambda t,$ $\phi_5 = \cos \lambda x \sin \lambda t, \quad \phi_6 = \sin \lambda x \cos \lambda t$

Table 1 (continued). Group classification of Eqs.(1) ($c^2 \neq 1$, $f_{uw} \neq 0$).

$F(\delta u - w) + Au, \quad F'''(z) \neq 0$	$X_i = \phi_i(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right), \quad i = \overline{3, 6}$ $\phi_3 = tx, \phi_4 = t, \phi_5 = x, \phi_6 = 1$
(i) $F(z) = \varepsilon z^\sigma + Bz, \quad \sigma \neq 0, 1, 2$	$X_7 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{2}{2-\sigma} \left(u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} \right)$ $+ \frac{1-\sigma}{2-\sigma} \psi(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right)$
(ii) $F(z) = \varepsilon \exp z + Bz$	$X_7 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial w} + \psi(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right)$
(iii) $F(z) = \varepsilon \ln z + Bz$	$X_7 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w}$ $+ \frac{1}{2} \psi(t, x) \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right)$
(iv) $F(z) = \varepsilon z \ln z$	$X_7 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2 \left(u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w} \right)$ $+ \frac{\varepsilon}{\delta(c^2 - 1)} [(1 + \delta^2 c^2) t^2$ $+ (1 + \delta^2) x^2] \left(\frac{\partial}{\partial u} + \delta \frac{\partial}{\partial w} \right)$
$\frac{A}{2} u^2 + \frac{B}{2} w^2 + uw + Cu + Dw$	$X_{\phi, \chi} = \phi(t, x) \frac{\partial}{\partial u} + \chi(t, x) \frac{\partial}{\partial w},$ where $\phi_{tt} - \phi_{xx} = A\phi + \chi,$ $\chi_{tt} - c^2 \chi_{xx} = \phi + B\chi$
a) $AB \neq 1, C = D = 0$	$X_3 = u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w}$
b) $AB = 1$	$X_3 = \left(u + \frac{D - BC}{2(A + B)} t^2 \right) \frac{\partial}{\partial u}$ $+ \left(w - \frac{A(D - BC)}{2(A + B)} t^2 + \frac{AC + D}{A + B} \right) \frac{\partial}{\partial w}$

Comments.

1. A, B, C, D and σ are arbitrary constants, δ is a positive real constant, $\tilde{\delta}$ is a nonnegative real constant; $\varepsilon = \pm 1$, $\tilde{\varepsilon} = 0, \pm 1$.
2. Constants λ and μ take real or imaginary values.
3. $\psi(t, x) = \frac{1}{\delta(c^2 - 1)} \{ [A\delta c^2 + B(1 + \delta^2 c^2)] t^2 + [A\delta + B(1 + \delta^2)] x^2 \}.$
4. In subcases the group of equivalence transformations may be larger than (2)–(4). For example, if $f(u, w) = F(\delta u - w) + Au$, where $F'''(z) \neq 0$, additional transformations have the form:

$$\tilde{u} = u + \theta tx + \omega t + \tau x, \quad \tilde{w} = w + \delta(\theta tx + \omega t + \tau x) \quad (9)$$

with arbitrary constants θ, ω and τ ;

$$\tilde{u} = u + \frac{c^2 t^2 + x^2}{2(c^2 - 1)}(\tilde{A} - A), \quad \tilde{w} = w + \frac{c^2 t^2 + x^2}{2(c^2 - 1)}\delta(\tilde{A} - A), \quad (10)$$

where $\tilde{A}$ satisfies the equation $\frac{d\tilde{A}}{da} = \Phi(\tilde{A}, c, \delta)$, $\tilde{A}|_{a=0} = A$;

$$\begin{aligned} \tilde{u} &= u + \frac{\kappa(A, c, \delta)}{2\delta(c^2 - 1)}[(1 + \delta^2 c^2)t^2 + (1 + \delta^2)x^2], \\ \tilde{w} &= w + \frac{\kappa(A, c, \delta)}{2(c^2 - 1)}[(1 + \delta^2 c^2)t^2 + (1 + \delta^2)x^2], \\ \tilde{f} &= f + \kappa(A, c, \delta)(\delta u - w). \end{aligned} \quad (11)$$

Here Φ and κ are arbitrary functions of their arguments. Up to these transformations, one may assume $A = B = 0$.

5. Additional generators for intersecting subcases are listed only once.

3 Conservation laws and invariant solutions

The results of classification can be used to find conservation laws and construct particular invariant solutions (self-similar, travelling waves, etc.)

The system (1) with an arbitrary function $f(u, w)$ is Lagrangian with the density

$$L = \frac{1}{2}(u_t^2 + w_t^2 - u_x^2 - c^2 w_x^2) + f(u, w).$$

Therefore, knowing infinitesimal operators (5) and using the Nöther theorem (see, e.g., [10]), one can find conservation laws:

$$\frac{\partial}{\partial t} A^1 + \frac{\partial}{\partial x} A^2 = 0,$$

where

$$A^i = L\xi_\alpha^i + (\eta^\alpha - \xi^k u_k^\alpha) \frac{\partial L}{\partial u_i^\alpha}, \quad \alpha = 1, 2, i = 1, 2; \quad (12)$$

and $u^1 = u, u^2 = w; u_1^\alpha = u_t^\alpha, u_2^\alpha = u_x^\alpha$.

The operators $X_1 = \frac{\partial}{\partial t}$ and $X_2 = \frac{\partial}{\partial x}$ are admitted for any $f(u, w)$. The corresponding conservation laws for the energy and momentum have the form

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{1}{2}(u_t^2 + w_t^2 + u_x^2 + c^2 w_x^2) - f(u, w) \right] - \frac{\partial}{\partial x} [u_t u_x + c^2 w_t w_x] &= 0, \\ \frac{\partial}{\partial t} [u_t u_x + w_t w_x] - \frac{\partial}{\partial x} [f(u, w) + \frac{1}{2}(u_t^2 + w_t^2 + u_x^2 + c^2 w_x^2)] &= 0. \end{aligned}$$

In those cases where the dimension of the admitted algebra is greater than 2, there are additional conservation laws, which can be easily written down explicitly, using (12).

Let us also find some particular invariant solutions of equations (1) for the case $f(u, w) = \cos(\delta u - w)$. Introducing $\tilde{u} = \delta u, \tilde{w} = w$ as new dependent variables we rewrite (1) as

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \quad w_{tt} - c^2 w_{xx} = \sin(u - w) \quad (13)$$

(the tildes are omitted). Although this case is not the most beneficial from the point of view of admitted symmetries (6-dimensional algebra), it is interesting as a possible generalisation of the FK model. Here, $\delta^2 = m_2/m_1$ has a meaning of the ratio of masses of particles in the “lower” and the “upper” chains. For $\delta^2 \rightarrow 0$, setting $u = 0$, we obtain for w the Sine-Gordon

equation, being the long-wave approximation of the FK model. Thus, the FK model appears as a natural limit of the coupled equations (13).

Equations (13) admit a generator $\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \alpha t (\frac{\partial}{\partial u} + \frac{\partial}{\partial w})$, where v is an arbitrary constant, and α may be taken as a nonnegative constant. The solution which is invariant with respect to this generator, for $v^2 \neq 1, c^2$, and $\frac{1 + \delta^2 c^2}{1 + \delta^2}$ has the form

$$\begin{aligned} u &= \frac{1}{\delta^2(v^2 - c^2) + v^2 - 1} \{ \delta^2(v^2 - c^2)p(x - vt) - \frac{\alpha}{2}[(1 + \delta^2 c^2)t^2 + \\ &\quad + (1 + \delta^2)x^2] + \alpha(1 + \delta^2)vtx \} + A(x - vt) + B, \\ w &= \frac{1}{\delta^2(v^2 - c^2) + v^2 - 1} \{ (1 - v^2)p(x - vt) - \frac{\alpha}{2}[(1 + \delta^2 c^2)t^2 + \\ &\quad + (1 + \delta^2)x^2] + \alpha(1 + \delta^2)vtx \} + A(x - vt) + B, \end{aligned} \quad (14)$$

$$\text{where } p'' = -\lambda \sin p - a, \quad (15)$$

$$\lambda = \frac{\delta^2(v^2 - c^2) + v^2 - 1}{(v^2 - 1)(v^2 - c^2)}, \quad a = \frac{\alpha(1 - c^2)}{(v^2 - 1)(v^2 - c^2)}.$$

Here, prime denotes derivative with respect to the argument of the function. As A and B are arbitrary constants, let us suppose $A = B = 0$.

If $\lambda < 0$, then, introducing $\tilde{p} = p + \pi$, (15) can be rewritten in the form $\tilde{p}'' = -|\lambda| \sin \tilde{p} - a$. Therefore, analysing (15) one can suppose $\lambda > 0$.

The energy integral of the equation (15) may be written in the form

$$(p')^2 + 2\lambda(1 - \cos p) + 2ap - 2E^* = 2E, \quad E^* = \lambda - \sqrt{\lambda^2 - a^2} - a \arcsin \frac{a}{\lambda}.$$

If the condition $|a| < \lambda$ is satisfied, then any value of E from the region

$$0 < E < 2\lambda - \pi|a| - 2E^*$$

defines a bounded periodic solution of (15) describing nonlinear oscillations near the point of equilibrium $p = -\arcsin \frac{a}{\lambda}$ in the region $p \in [p_1, p_2]$, where p_1 and p_2 are the smaller and the larger roots of equation

$$\lambda(1 - \cos p) + ap - E^* = E$$

on the interval $]-\pi + \arcsin \frac{a}{\lambda}, \pi + \arcsin \frac{a}{\lambda}[$. It is assumed here that $\arcsin \frac{a}{\lambda} \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

If $\alpha = 0$, the equation (15) turns out to be the pendulum equation which is integrable in elliptic functions (see, e.g, [15]). In this case solutions of the system (13) describing periodic steady waves have the form:

$$\begin{aligned} u &= \frac{2\delta^2(v^2 - c^2)}{\delta^2(v^2 - c^2) + v^2 - 1} \arcsin\{k \operatorname{sn}[\sqrt{\lambda}(x - vt), k]\} = \\ &= \frac{\delta^2(v^2 - c^2)}{1 - v^2} w, \quad 0 < k < 1, \end{aligned}$$

describing "fast" waves, propagating with velocities $v^2 \in]S, M[\cup]L, +\infty[$, where $S = \min\{1, c^2\}$, $M = \frac{1 + \delta^2 c^2}{1 + \delta^2}$, $L = \max\{1, c^2\}$,

$$\begin{aligned} u &= \frac{2\delta^2(v^2 - c^2)}{\delta^2(v^2 - c^2) + v^2 - 1} \arcsin\{\operatorname{dn}[\sqrt{|\lambda|}(x - vt), k]\} = \\ &= \frac{\delta^2(v^2 - c^2)}{1 - v^2} w, \quad 0 < k < 1, \end{aligned}$$

describing "slow" waves, propagating with velocities $v^2 \in [0, S[\cup]M, L[$.

In the $k = 1$ limiting case "slow" periodic waves become solitary waves (kinks):

$$u = \frac{4\delta^2(v^2 - c^2)}{\delta^2(v^2 - c^2) + v^2 - 1} \arctan\{\exp \sqrt{|\lambda|}(x - vt)\} = \frac{\delta^2(v^2 - c^2)}{1 - v^2} w.$$

Using reflections

$$t \rightarrow -t, \quad x \rightarrow x; \quad t \rightarrow t, \quad x \rightarrow -x; \quad u \rightarrow -u, \quad w \rightarrow -w;$$

admitted by the equations (13), one can obtain solutions with other combinations of signs.

If $\delta \rightarrow 0$ ($m_1 \gg m_2$), the solitary waves may propagate with the velocities $v^2 \in [0, c^2[$. In that case the displacement of particles does not depend on the velocity of wave propagation. If the masses m_1 and m_2 are comparable, the solitary waves may propagate with the velocities $v^2 \in [0, S[\cup]M, L[$. Therefore, if the acoustic velocities of non-interacting chains are different ($c^2 \neq 1$), a gap appears in the velocity spectrum of the solitary waves, i.e. the system acts as a filter of solitary waves. Here the relative displacement ("upper" particles relative to the "lower" ones) remains the same as in the FK model (per period of the chain), but the absolute displacement depends on the velocity of the wave.

Solutions in a presence of additional shear forces can be constructed using (14) and the equivalence transformations, listed in the Comments to Table 1. For example, using (14) and equivalence transformations (9) and (11), where one should change $u \rightarrow u/\delta, \tilde{u} \rightarrow \tilde{u}/\delta$ and substitute

$$\theta = -\frac{\alpha(1 + \delta^2)v}{\delta[\delta^2(v^2 - c^2) + v^2 - 1]}, \quad \omega = \tau = 0, \quad \kappa = \frac{\alpha(c^2 - 1)}{\delta^2(v^2 - c^2) + v^2 - 1},$$

we obtain a solution

$$u = \frac{\delta^2(v^2 - c^2)}{\delta^2(v^2 - c^2) + v^2 - 1} p(x - vt), \quad w = \frac{1 - v^2}{\delta^2(v^2 - c^2) + v^2 - 1} p(x - vt),$$

where the function $p(x - vt)$ satisfies equation (15). The solution describes travelling waves in u - and w -components subjected to additional shear forces of equal magnitude but opposite sign. Using equivalence transformations (9) and (10), where one should change $u \rightarrow u/\delta, \tilde{u} \rightarrow \tilde{u}/\delta$ and substitute the same values of θ, ω and τ as above, and using

$$A = 0, \quad \tilde{A} = \frac{\alpha(c^2 - 1)(1 + \delta^2 c^2)}{\delta c^2[\delta^2(v^2 - c^2) + v^2 - 1]},$$

we obtain a solution

$$u = \frac{1}{\delta^2(v^2 - c^2) + v^2 - 1} \left[\delta^2(v^2 - c^2) p(x - vt) - \frac{\alpha(c^2 - 1)}{2c^2} x^2 \right],$$

$$w = \frac{1}{\delta^2(v^2 - c^2) + v^2 - 1} \left[(1 - v^2) p(x - vt) - \frac{\alpha(c^2 - 1)}{2c^2} x^2 \right],$$

where the function $p(x - vt)$ satisfies equation (15). Here, only the u -component is subjected to additional shear forces.

4 Energy exchange

In this section, we again consider a system of coupled Sine-Gordon equations

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \quad w_{tt} - c^2 w_{xx} = \sin(u - w), \quad (16)$$

and discuss particular solutions, which can be viewed as a continuum generalisation of periodic energy exchange in a system of coupled pendulums [16–18].

The energy conservation law for the system (16) has the form

$$\frac{\partial}{\partial t} \left(\frac{1}{2} (u_t^2 + u_x^2 + \delta^2 w_t^2 + c^2 \delta^2 w_x^2) + \delta^2 (1 - \cos(u - w)) \right) - \frac{\partial}{\partial x} (u_t u_x + c^2 \delta^2 w_t w_x) = 0.$$

For periodic solutions in $0 < x < l$ considered below, we define domain integrated average energies

$$E_u = \frac{1}{l} \int_0^l \frac{1}{2} (u_t^2 + u_x^2) dx, \quad E_w = \frac{1}{l} \int_0^l \frac{\delta^2}{2} (w_t^2 + c^2 w_x^2) dx, \quad (17)$$

$$E_c = \frac{1}{l} \int_0^l \delta^2 (1 - \cos(u - w)) dx, \quad (18)$$

describing the energy in the u -component, the energy in the w -component, and the energy of coupling, respectively. Then the domain integrated average energy E is

$$E = E_u + E_w + E_c. \quad (19)$$

Although E is constant, we can identify solutions that have a time-periodic exchange of energy between E_u , E_w and E_c .

4.1 Linear solutions

When $|u - w| \ll 1$, the coupled Sine-Gordon equations (16) reduce to a linear system. Looking for a solution of the form $(u, w) = (1, \alpha) \exp i(kx - \omega t) + \text{c.c.}$ we obtain the dispersion relation

$$\omega_{1,2}^2 = \frac{1}{2} \left[\nu_1^2 + \nu_2^2 \mp \sqrt{(\nu_1^2 - \nu_2^2)^2 + 4\delta^2} \right], \quad (20)$$

where $\nu_1^2 = \delta^2 + k^2$ and $\nu_2^2 = 1 + c^2 k^2$. The corresponding values of α , giving the ratio w/u , are

$$\alpha_{1,2} = \frac{\nu_1^2 - \omega_{1,2}^2}{\delta^2} = \frac{1}{\nu_2^2 - \omega_{1,2}^2}.$$

We note that $\alpha_1 > 0$, and $\alpha_2 < 0$. Of course, for such solutions, conversion to normal mode coordinates uncouples the system. Thus, in this linearised system, energy exchange in the physical variables is due to the manner in which these normal modes are combined.

We first consider a superposition of two linear waves, taken to be right-propagating, with the same positive wavenumber k but different frequencies $\omega_1(k)$ and $\omega_2(k)$. The amplitudes of the waves are chosen in such a way that initially $u(x, 0) = U \cos kx$, $w(x, 0) = 0$. Then

$$\begin{pmatrix} u \\ w \end{pmatrix} = U \cos kx \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_t \\ w_t \end{pmatrix} = \frac{U \sin kx}{\alpha_1 + |\alpha_2|} \begin{pmatrix} \omega_1 |\alpha_2| + \omega_2 \alpha_1 \\ (\omega_1 - \omega_2) \alpha_1 |\alpha_2| \end{pmatrix}, \quad \text{at } t = 0. \quad (21)$$

The solution is then given by

$$u = \tilde{U}(t) \cos(kx - \gamma_+ t - \theta(t)), \quad w = -\frac{2\alpha_1 |\alpha_2|}{\alpha_1 + |\alpha_2|} U \sin \gamma_- t \sin(kx - \gamma_+ t), \quad \text{where} \quad (22)$$

$$\tilde{U}(t) = U \sqrt{1 - \frac{4\alpha_1 |\alpha_2| \sin^2 \gamma_- t}{(\alpha_1 + |\alpha_2|)^2}}, \quad \theta(t) = \arctan \left[\left(\frac{\alpha_1 - |\alpha_2|}{\alpha_1 + |\alpha_2|} \right) \tan \gamma_- t \right], \quad \gamma_{\pm} = \frac{\omega_2 \pm \omega_1}{2}. \quad (23)$$

This solution describes an exchange of energy in the travelling waves between the two components u and w . The amplitude of the u -component varies between its initial value U , and a smaller non-zero value $U_{min} = U|\alpha_1 - \alpha_2|/(\alpha_1 + |\alpha_2|)$. We shall say that there is “a full energy exchange” if $U_{min} = 0$, and that there is “a partial energy exchange” otherwise. In this sense, there is a full exchange when

$$(c^2 - 1)k^2 = \delta^2 - 1. \quad (24)$$

Then $\nu_1^2 = \nu_2^2 = \nu^2$, $\alpha_1 = -\alpha_2 = \delta^{-1}$, and the solution (22) reduces to

$$u = U \cos \gamma_- t \cos(kx - \gamma_+ t), \quad w = -U\delta^{-1} \sin \gamma_- t \sin(kx - \gamma_+ t). \quad (25)$$

This is illustrated in Figure 1, for the case $c = \delta = 3, k = 1$. Note that in the integrable case, when $c = 1$, (24) can be satisfied only for $\delta = 1$, while k is then arbitrary. Otherwise, when $c \neq 1$, (24) can be satisfied only when $(\delta^2 - 1)/(c^2 - 1) > 0$, for a single wavenumber $k = ((\delta^2 - 1)/(c^2 - 1))^{1/2}$.

Fig. 1. Full exchange in the linear two-wave solution (25).

For the solution (25), the energy is partitioned according to

$$\begin{pmatrix} E_u \\ E_w \\ E_c \end{pmatrix} = \frac{U^2 \cos^2 \gamma_- t}{4} \begin{pmatrix} \gamma_+^2 + k^2 \\ \gamma_-^2 \\ \delta^2 \end{pmatrix} + \frac{U^2 \sin^2 \gamma_- t}{4} \begin{pmatrix} \gamma_-^2 \\ \gamma_+^2 + c^2 k^2 \\ 1 \end{pmatrix}, \quad (26)$$

where, for these linear solutions, E_c is calculated using only the quadratic term of (18). Using $\gamma_-^2 + \gamma_+^2 = \delta^2 + k^2$, one may show from (19) and (26) that $E = U^2(\delta^2 + k^2)/2$. From (26) we see that there is a partial exchange of energy between u and w , with period $T = \pi/\gamma_-$. If γ_- is small enough, i.e. if the frequencies ω_1 and ω_2 are close enough for some wavenumber k , then at $t = \pi/2\gamma_-, 3\pi/2\gamma_-, \dots$ the u -component is almost in equilibrium and almost all of its energy is lost, whilst at $t = 0, \pi/\gamma_-, \dots$ the w -component is almost in equilibrium and almost all of its energy is lost. If $\nu^2 \gg \delta$, then the period T of the energy exchange tends to infinity.

There exists a corresponding solution to (22) consisting of a superposition of two left-propagating waves. We next consider a combination of these left-propagating waves and the right-propagating waves (22), choosing the amplitudes of the waves in a such a way that at $t = 0$,

$$u(x, 0) = U \cos kx, \quad u_t(x, 0) = 0, \quad w(x, 0) = 0, \quad w_t(x, 0) = 0. \quad (27)$$

We then obtain a standing wave involving two pairs of counter-propagating waves:

$$u = \tilde{U}(t) \cos(\gamma_+ t + \theta(t)) \cos kx, \quad w = \frac{2\alpha_1 |\alpha_2|}{\alpha_1 + |\alpha_2|} U \sin \gamma_- t \sin \gamma_+ t \cos kx, \quad (28)$$

with $\tilde{U}(t)$, $\theta(t)$ and $\gamma_{\pm}$ given by (23). The amplitude of the standing wave in the u component varies between U and U_{min} , so that once again there is only a partial exchange between the amplitude of the components, unless condition (24) is satisfied. Then, we have a full exchange, and (28) becomes

$$u = U \cos \gamma_- t \cos \gamma_+ t \cos kx, \quad w = U\delta^{-1} \sin \gamma_- t \sin \gamma_+ t \cos kx, \quad (29)$$

Fig. 2. Full exchange in the linear four-wave solution (29).

where we note that $\gamma_- < \gamma_+$, from (23). This standing wave solution is illustrated in Figure 2, for the case $c = \delta = 3$, $k = 1$.

The energy in solution (29) is $E = U^2(\delta^2 + k^2)/4$, although its partition is rather more complicated than (26). Nevertheless, the general pattern of a partial energy exchange on a timescale π/γ_- remains. For instance, for $\gamma_- \ll 1$ we can say that u loses almost all of its energy near $t = \pi/2\gamma_-$, $3\pi/2\gamma_-$, $\dots$ etc.

These processes represent a continuum generalisation of energy exchange in a system of coupled oscillators [19] (see also [20]). In the context of the possible applications of our model mentioned in the Introduction, one can infer an energy exchange between the two layers of a bi-layer with an imperfect interface, or, between the two strands of DNA, etc. However, as plane waves may be modulationally unstable when weakly nonlinear effects are taken into account, we are motivated to extend this linear analysis into the weakly nonlinear regime, and so determine how such energy exchanges are affected by modulational instability. Although our analysis is presented only for the specific system (16), our methodology is generic and can be applied to other systems.

Note that the role of t and x is interchangeable here; one can also consider a superposition of waves with the same frequency, but with different wavenumbers.

4.2 Weakly nonlinear solutions

If $c = 1$ then in the new variables $p = u - w$ and $q = u + \delta^2 w$ the system (16) uncouples into a Sine-Gordon equation for p and a linear wave equation for q . Thus, this case is completely integrable, and one can find an exact nonlinear two-wave solution describing energy exchange between the physical components u and w of the system (16). It is given in terms of Jacobi elliptic functions [16]:

$$\begin{pmatrix} u \\ w \end{pmatrix} = \frac{2}{1 + \delta^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \arcsin \phi_2 \begin{pmatrix} \delta^2 \\ -1 \end{pmatrix}, \quad \phi_{1,2} = \kappa \operatorname{sn}(kx - \omega_{1,2}(k)t + \theta_0, \kappa), \quad (30)$$

where $\omega_1 = k$, $\omega_2 = \sqrt{1 + \delta^2 + k^2}$, from (20). The form of the solution for small amplitudes can be calculated by taking the limit $\kappa \rightarrow 0$ in (30). Using the well-known approximation (e.g. [21])

$$\operatorname{sn}(x, \kappa) = \sin x - \frac{1}{4}\kappa^2(x - \sin x \cos x) \cos x + O(\kappa^4),$$

one can eliminate the secular terms by renormalizing the frequencies $\omega_{1,2}(k)$ in the solution (30), and thus find an asymptotic expansion which reduces to the linear solution (22) when $c = 1$ (see [16]). Note, that the exact nonlinear four-wave energy exchange solution (which for small amplitudes reduces to the four-wave linear solution (28)) is not available even for $c = 1$.

In the general case, we can study weakly nonlinear energy exchange solutions by considering the slow spatio-temporal evolution of the wave amplitudes. With our two- and four-wave energy exchange solutions in mind, we consider the case of two pairs of counter-propagating waves in a spatially periodic domain, which is a special (resonant) case of a general four-wave interaction. However, our approach is generic and can generally be applied to any number of waves [18].

Here, we consider the weakly nonlinear evolution of two pairs of counter-propagating plane waves within system (16). To do this we derive equations for the evolution of wave packets, each with the same dominant wavenumber k . We suppose that the wave amplitudes are characterised by a small parameter $\varepsilon \ll 1$, and thus introduce the asymptotic multiple-scales expansion

$$\begin{pmatrix} u \\ w \end{pmatrix} = \varepsilon \begin{pmatrix} u_1 \\ w_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} u_3 \\ w_3 \end{pmatrix} + O(\varepsilon^4),$$

where

$$\begin{pmatrix} u_1 \\ w_1 \end{pmatrix} = \left[A e^{i(kx - \omega_1 t)} + B e^{i(kx + \omega_1 t)} \right] \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} + \left[C e^{i(kx - \omega_2 t)} + D e^{i(kx + \omega_2 t)} \right] \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} + \text{c.c.} \quad (31)$$

The coupled Sine-Gordon equations (16) may then be reduced to coupled equations for the slow spatio-temporal evolution of the wave amplitudes A , B , C and D , which are functions of slow variables. We will consider modulations on a long lengthscale χ or on a super-long lengthscale X , with corresponding slow and super-slow timescales τ and T , defined by

$$\chi = \varepsilon x, \quad X = \varepsilon^2 x, \quad \tau = \varepsilon t, \quad T = \varepsilon^2 t. \quad (32)$$

We first derive an asymptotic model for modulations on the super-long lengthscale, leading to a system of four non-dispersive nonlinearly coupled envelope equations. We then do the same for modulations on the long lengthscale, leading to various coupled NLS type equations, where the type of coupling (local or non-local) between various modes depends crucially upon the group velocities of the waves.

4.2.1 Non-dispersive equations

To derive this model, one has to suppose that A , B , C and D are functions of $X = \varepsilon^2 x$, $T = \varepsilon^2 t$ (see, for example, [22, 23]). At leading order, i.e. $O(\varepsilon)$ in this expansion, we find a linear wave equation for u_1 and w_1 , which is taken to be the four-wave solution (31), with A , B , C and D as yet undetermined functions of X, T . At the next order, one obtains solvability conditions in the usual way from the equations for u_2 and w_2 , yielding

$$\begin{aligned} A_T + v_{g_1} A_X &= i\mu_1(|A|^2 + 2|B|^2)A + i\mu_2(|C|^2 + |D|^2)A + i\mu_2 B^* C D, \\ B_T - v_{g_1} B_X &= -i\mu_1(|B|^2 + 2|A|^2)B - i\mu_2(|C|^2 + |D|^2)B - i\mu_2 A^* C D, \\ C_T + v_{g_2} C_X &= i\mu_3(|C|^2 + 2|D|^2)C + i\mu_4(|A|^2 + |B|^2)C + i\mu_4 D^* A B, \\ D_T - v_{g_2} D_X &= -i\mu_3(|D|^2 + 2|C|^2)D - i\mu_4(|A|^2 + |B|^2)D - i\mu_4 C^* A B, \end{aligned} \quad (33)$$

where the group velocities v_{g_i} are given by

$$v_{g_i} = \omega'_i = \frac{k}{\omega_i} \frac{1 + \alpha_i^2 \delta^2 c^2}{1 + \alpha_i^2 \delta^2}, \quad (34)$$

and the nonlinear coefficients are

$$\begin{aligned} \mu_1 &= \frac{\delta^2(1 - \alpha_1)^4}{4\omega_1(1 + \alpha_1^2 \delta^2)}, \quad \mu_2 = \frac{\delta^2(1 - \alpha_1)^2(1 - \alpha_2)^2}{2\omega_1(1 + \alpha_1^2 \delta^2)}, \\ \mu_3 &= \frac{\delta^2(1 - \alpha_2)^4}{4\omega_2(1 + \alpha_2^2 \delta^2)}, \quad \mu_4 = \frac{\delta^2(1 - \alpha_1)^2(1 - \alpha_2)^2}{2\omega_2(1 + \alpha_2^2 \delta^2)}. \end{aligned} \quad (35)$$

The equations (33) are non-dispersive.

Note that for the case of two-wave interaction, with $B = D = 0$ for instance, (33) yields the system

$$A_T + v_{g_1} A_X = i(\mu_1 |A|^2 + \mu_2 |C|^2)A, \quad C_T + v_{g_2} C_X = i(\mu_3 |C|^2 + \mu_4 |A|^2)C,$$

which is integrable in quadratures [17] (see also [22]).

4.2.2 A hierarchy of coupled NLS type models

To derive various coupled NLS type models, we generalise the approach of [24–26] to the case of four waves. The generalisation is not straightforward, since the number of characteristic variables now differs from the number of independent variables, whereas they are the same in the two-wave case. However, these potential difficulties can be readily overcome, and the approach we describe here can generally be applied to any number of waves.

We suppose that A , B , C and D are functions of the long lengthscale $\chi = \varepsilon x$ and both slow timescales $\tau = \varepsilon t$ and $T = \varepsilon^2 t$. Once again, at $O(\varepsilon)$ (i.e. the leading order in this expansion), we find a linear wave equation for u_1 and w_1 , which is taken to be the four-wave solution (31), with A , B , C and D as yet undetermined functions of χ , τ and T . At $O(\varepsilon^2)$, there are solvability conditions $A_\tau + v_{g1}A_\chi = 0$, $B_\tau - v_{g1}B_\chi = 0$, $C_\tau + v_{g2}C_\chi = 0$, and $D_\tau - v_{g2}D_\chi = 0$. Introducing the variables

$$\eta_i = \chi - v_{gi}\tau, \quad \xi_i = \chi + v_{gi}\tau, \quad (36)$$

the solvability conditions give

$$A = A(\eta_1, T), \quad B = B(\xi_1, T), \quad C = C(\eta_2, T), \quad D = D(\xi_2, T), \quad (37)$$

i.e. there is a simple functional dependence for each wave amplitude in its own reference frame. If we suppose that our solutions have spatial period l in x , which must satisfy $l \sim \varepsilon^{-1}$ for modulations on the long lengthscale, then A , B , C and D are periodic in η_1 , ξ_1 , η_2 and ξ_2 respectively, with period $\lambda = \varepsilon l$.

The $O(\varepsilon^2)$ terms in the expansion may then be written as

$$\begin{aligned} \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} = & \left[A_2(\eta_1, \tau, T)e^{i(kx - \omega_1 t)} + B_2(\xi_1, \tau, T)e^{i(kx + \omega_1 t)} \right] \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} \\ & + \left[C_2(\eta_2, \tau, T)e^{i(kx - \omega_2 t)} + D_2(\xi_2, \tau, T)e^{i(kx + \omega_2 t)} \right] \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} + \text{c.c.} \end{aligned} \quad (38)$$

At $O(\varepsilon^3)$, one obtains solvability conditions in the usual way:

$$\partial_\tau A_2 + A_T = \frac{i}{2}\omega_1'' A_{\eta_1 \eta_1} + i\mu_1(|A|^2 + 2|B|^2)A + i\mu_2(|C|^2 + |D|^2)A + i\mu_2 B^* CD, \quad (39)$$

$$\partial_\tau B_2 + B_T = -\frac{i}{2}\omega_1'' B_{\xi_1 \xi_1} - i\mu_1(|B|^2 + 2|A|^2)B - i\mu_2(|C|^2 + |D|^2)B - i\mu_2 A^* CD, \quad (40)$$

$$\partial_\tau C_2 + C_T = \frac{i}{2}\omega_2'' C_{\eta_2 \eta_2} + i\mu_3(|C|^2 + 2|D|^2)C + i\mu_4(|A|^2 + |B|^2)C + i\mu_4 D^* AB, \quad (41)$$

$$\partial_\tau D_2 + D_T = -\frac{i}{2}\omega_2'' D_{\xi_2 \xi_2} - i\mu_3(|D|^2 + 2|C|^2)D - i\mu_4(|A|^2 + |B|^2)D - i\mu_4 C^* AB. \quad (42)$$

Here the partial derivatives of A_2 , B_2 , C_2 and D_2 with respect to τ are evaluated at constant η_1 , ξ_1 , η_2 and ξ_2 respectively, consistent with their functional form (37), the nonlinear coefficients are defined as before in (35), and

$$\omega_i'' = \frac{1 - v_{gi}^2 + \alpha_i^2 \delta^2 (c^2 - v_{gi}^2) + 4\omega_i^2 \alpha_i (v_{gi} - k/\omega_i)(v_{gi} - c^2 k/\omega_i)}{\omega_i (1 + \alpha_i^2 \delta^2)}. \quad (43)$$

We proceed further by integrating (39) with respect to τ at constant η_1 from 0 to τ_0 , and dividing by τ_0 . Letting $\tau_0 \rightarrow \infty$, and demanding that A_2 remains bounded in this limit, yields

$$\begin{aligned} A_T = & \frac{i}{2}\omega_1'' A_{\eta_1 \eta_1} + i\mu_1 |A|^2 A + iA \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} (2\mu_1 |B|^2 + \mu_2 |C|^2 + \mu_2 |D|^2) d\tau \right) \\ & + i\mu_2 \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} B^* CD d\tau \right), \end{aligned}$$

where we have recalled that $A = A(\eta_1, T)$, and where the integrals are taken at constant η_1 . Noting the functional forms (37), we may use (36) to convert the terms involving $|B|^2$, $|C|^2$ and $|D|^2$ into integrals with respect to ξ_1 , η_2 and ξ_2 respectively, still at fixed η_1 , eventually yielding

$$A_T = \frac{i}{2}\omega_1'' A_{\eta_1\eta_1} + i\mu_1 |A|^2 A + i\mu_2 \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} B^* C D \, d\tau \right) \\ + iA \lim_{\tau_0 \rightarrow \infty} \frac{1}{\tau_0} \left(2\mu_1 \int_{\eta_1}^{\eta_1 + \tau_0} |B|^2 \, d\xi_1 + \mu_2 \int_{\eta_1}^{\eta_1 + \tau_0} |C|^2 \, d\eta_2 + \mu_2 \int_{\eta_1}^{\eta_1 + \tau_0} |D|^2 \, d\xi_2 \right). \quad (44)$$

Here τ_0 has been rescaled in each of the final three integrals, and to perform this rescaling we have assumed that v_{g1} and v_{g2} are both $O(1)$, with $(v_{g1} - v_{g2}) = O(1)$. Finally, we use the periodicity of B , C and D to convert the final three integral terms of (44) to averages over one period. Performing similar operations for (40), (41) and (42) gives

$$A_T = \frac{i}{2}\omega_1'' A_{\eta_1\eta_1} + i\mu_1 \left(|A|^2 + 2\overline{|B|^2} \right) A + i\mu_2 \left[\left(\overline{|C|^2} + \overline{|D|^2} \right) A + \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} B^* C D \, d\tau \right) \right], \\ B_T = -\frac{i}{2}\omega_1'' B_{\xi_1\xi_1} - i\mu_1 \left(2\overline{|A|^2} + |B|^2 \right) B - i\mu_2 \left[\left(\overline{|C|^2} + \overline{|D|^2} \right) B + \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} A^* C D \, d\tau \right) \right], \\ C_T = \frac{i}{2}\omega_2'' C_{\eta_2\eta_2} + i\mu_3 \left(|C|^2 + 2\overline{|D|^2} \right) C + i\mu_4 \left[\left(\overline{|A|^2} + \overline{|B|^2} \right) C + \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} D^* A B \, d\tau \right) \right], \\ D_T = -\frac{i}{2}\omega_2'' D_{\xi_2\xi_2} - i\mu_3 \left(2\overline{|C|^2} + |D|^2 \right) D - i\mu_4 \left[\left(\overline{|A|^2} + \overline{|B|^2} \right) D + \lim_{\tau_0 \rightarrow \infty} \left(\frac{1}{\tau_0} \int_0^{\tau_0} C^* A B \, d\tau \right) \right],$$

where

$$\overline{|A|^2}(T) = \frac{1}{\lambda} \int_0^\lambda |A|^2 \, d\eta_1, \quad \overline{|B|^2}(T) = \frac{1}{\lambda} \int_0^\lambda |B|^2 \, d\xi_1, \\ \overline{|C|^2}(T) = \frac{1}{\lambda} \int_0^\lambda |C|^2 \, d\eta_2, \quad \overline{|D|^2}(T) = \frac{1}{\lambda} \int_0^\lambda |D|^2 \, d\xi_2. \quad (45)$$

These non-locally coupled NLS equations govern the weakly nonlinear evolution of the wave amplitudes in the general case, when v_{g1} , v_{g2} and $(v_{g1} - v_{g2})$ are all $O(1)$. Note, that the last “mixing” term appears due to the resonant nature of the case of two pairs of counter-propagating waves, and will be absent in the generic nonresonant case.

If v_{g1} , v_{g2} are $O(1)$ but $(v_{g1} - v_{g2}) = O(\varepsilon)$, then the averaging process described above fails, and we write

$$v_{g1} = \bar{v}_g - \varepsilon\Delta, \quad v_{g2} = \bar{v}_g + \varepsilon\Delta, \quad \text{where } \bar{v}_g = \frac{v_{g1} + v_{g2}}{2}, \quad \Delta = \frac{v_{g2} - v_{g1}}{2\varepsilon}. \quad (46)$$

Then, rather than (37), the solvability conditions at $O(\varepsilon)$ give $A = A(\bar{\eta}, T)$, $B = B(\bar{\xi}, T)$, $C = C(\bar{\eta}, T)$, $D = D(\bar{\xi}, T)$, where

$$\bar{\eta} = \chi - \bar{v}_g \tau, \quad \bar{\xi} = \chi + \bar{v}_g \tau.$$

At $O(\varepsilon^3)$ we obtain equations similar to (39)–(42), but with additional terms $-\Delta A_\chi$, ΔB_χ , ΔC_χ , $-\Delta D_\chi$ on the left-hand side. We then once again perform an averaging operation, yet this time A and C are now both functions of $\bar{\eta}$ and T , whilst B and D are both functions of $\bar{\xi}$ and T . Thus, the final integral terms can be considerably simplified, and rather than (45) we

obtain

$$\begin{aligned}
A_T - \Delta A_{\bar{\eta}} &= \frac{i}{2} \omega_1'' A_{\bar{\eta}\bar{\eta}} + i\mu_1 \left(|A|^2 + 2|\overline{B}|^2 \right) A + i\mu_2 \left(|C|^2 + |\overline{D}|^2 \right) A + i\mu_2 \overline{B^* D} C, \\
B_T + \Delta B_{\bar{\xi}} &= -\frac{i}{2} \omega_1'' B_{\bar{\xi}\bar{\xi}} - i\mu_1 \left(2|\overline{A}|^2 + |B|^2 \right) B - i\mu_2 \left(|\overline{C}|^2 + |D|^2 \right) B - i\mu_2 \overline{A^* C} D, \\
C_T + \Delta C_{\bar{\eta}} &= \frac{i}{2} \omega_2'' C_{\bar{\eta}\bar{\eta}} + i\mu_3 \left(|C|^2 + 2|\overline{D}|^2 \right) C + i\mu_4 \left(|A|^2 + |\overline{B}|^2 \right) C + i\mu_4 \overline{D^* B} A, \\
D_T - \Delta D_{\bar{\xi}} &= -\frac{i}{2} \omega_2'' D_{\bar{\xi}\bar{\xi}} - i\mu_3 \left(2|\overline{C}|^2 + |D|^2 \right) D - i\mu_4 \left(|\overline{A}|^2 + |B|^2 \right) D - i\mu_4 \overline{C^* A} B,
\end{aligned} \tag{47}$$

with the coefficients defined as before. There is now a local coupling between the A and C modes, and between the B and D modes.

If one permits v_{g1} and v_{g2} to be $O(\varepsilon)$, then three other scenarios arise in which the averaging process fails. If $v_{g1} = O(\varepsilon)$ but $v_{g2} = O(1)$, then using the obvious extension of the above analysis one obtains a system in which A and B are locally coupled, whilst C and D satisfy non-locally coupled NLS equations. If $v_{g2} = O(\varepsilon)$ but $v_{g1} = O(1)$, then C and D are locally coupled, whilst A and B satisfy non-locally coupled NLS equations. Finally, if v_{g1} and v_{g2} are both $O(\varepsilon)$, then the solvability conditions at $O(\varepsilon)$ give $A = A(\chi, T)$, $B = B(\chi, T)$, $C = C(\chi, T)$, $D = D(\chi, T)$. At $O(\varepsilon^3)$, the averaging operation immediately yields a system of coupled NLS equation for A , B , C and D

$$\begin{aligned}
A_T + \tilde{v}_{g1} A_\chi - \frac{i}{2} \omega_1'' A_{\chi\chi} &= i\mu_1 (|A|^2 + 2|B|^2) A + i\mu_2 (|C|^2 + |D|^2) A + i\mu_2 B^* C D, \\
B_T - \tilde{v}_{g1} B_\chi + \frac{i}{2} \omega_1'' B_{\chi\chi} &= -i\mu_1 (2|\overline{A}|^2 + |B|^2) B - i\mu_2 (|\overline{C}|^2 + |D|^2) B - i\mu_2 A^* C D, \\
C_T + \tilde{v}_{g2} C_\chi - \frac{i}{2} \omega_2'' C_{\chi\chi} &= i\mu_3 (|C|^2 + 2|D|^2) C + i\mu_4 (|A|^2 + |B|^2) C + i\mu_4 D^* A B, \\
D_T - \tilde{v}_{g2} D_\chi + \frac{i}{2} \omega_2'' D_{\chi\chi} &= -i\mu_3 (2|\overline{C}|^2 + |D|^2) D - i\mu_4 (|\overline{A}|^2 + |B|^2) D - i\mu_4 C^* A B.
\end{aligned} \tag{48}$$

where all four waves are coupled locally, and $\tilde{v}_{gi} = v_{gi}/\varepsilon = O(1)$ for $i = 1, 2$.

Usually, the model of locally coupled NLS equations is used universally, regardless of the values of group velocities of the waves. Strictly speaking, this model is often used outside the region of its formal asymptotic validity. Nevertheless, even then (48) is an appealing model on purely physical grounds. Indeed, since the effects of transport, dispersion, and leading order nonlinearity are captured, models of this type are widely used in studies of weakly nonlinear wave phenomena. There is an analogy here with the well-known Boussinesq equations often used to model the two-way propagation of water waves. Indeed, any of the asymptotic models (33), (45) and (47) already introduced, or those mentioned above, can be derived from (48) by an asymptotic multiple-scales reduction on the appropriate length and time-scales.

Note that all sets of envelope equations derived in this section collapse and partially decouple when $c = 1$. Indeed, in this case $\alpha_1 = 1$, so that three of the coupling coefficients (μ_1, μ_2 and μ_4) and the coefficient ω_1'' vanish. The equations for A and B are then just linear first-order nondispersive wave equations, which is a shadow of the uncoupling of the system (16) in this case. Further the equations for C and D collapse to two coupled NLS equations, in which we note that $\mu_3 > 0$ and $\omega_2'' > 0$, so that each NLS is of focussing type corresponding to modulational instability for plane waves in the Sine-Gordon equation (see, for instance, [27]). When c is close to 1, the coefficients ($\mu_1, \mu_2, \mu_4, \omega_1''$) are no longer zero, but are still very small, and so any coupling between the A, B modes and C, D modes will develop on a time-scale longer than in the generic case, when c is not close to 1. This feature manifests itself in the numerical simulations, discussed in subsection 4.3, where the the solutions become markedly less regular as c departs from 1.

4.3 The effect of modulational instability on energy exchange

The mechanism of the energy exchange discussed above is essentially linear, with energy being transferred between the physical components of the system, rather than between the wave components. It is well known that dispersive weakly nonlinear systems of the type being studied can support modulational instabilities. Therefore, a natural question to ask is, what happens to our linear multi-phase energy exchange solutions under the influence of nonlinearity if they are modulationally perturbed?

When there is just a single wave, say $A \exp i(kx - \omega(k)t) + c.c.$, the procedure is well-known. In a reference frame moving with the group velocity of the wave, the governing nonlinear partial differential equations can be reduced via the method of multiple scales to a nonlinear Schrödinger (NLS) equation for the wave amplitude (e.g. [28]). As is well known, if $\omega''(k)$ has the same sign as the coefficient of the cubic nonlinear term, then a plane wave is modulationally unstable [29] (sometimes called a Benjamin–Feir instability, or sideband instability [30]).

One can apply a similar methodology for the case of two waves, possibly counter-propagating, say $A \exp i(kx - \omega_1(k)t) + B \exp i(kx - \omega_2(k)t) + c.c.$. In general the governing nonlinear partial differential equations can then be reduced to non-locally coupled NLS equations for A and B [24–26]. For unbounded or periodic systems, the coupling terms either disappear or can be eliminated by a phase transformation [25], and so modulational instability is determined by the instability of the individual waves. However, if the group velocities ω'_1 and ω'_2 are equal, or differ only by an $O(\varepsilon)$ quantity, then one obtains locally coupled NLS equations, and there may be additional instabilities solely due to this coupling, even when both waves are individually stable [31, 32].

We can extend these studies to consider the modulational instability of two pairs of counter-propagating waves in a spatially periodic domain, which is a special, but important, case of a general four-wave interaction. One approach is to derive a set of coupled NLS equations (see, for example [33, 34]), and to consider modulational instability within this model. However, the resulting instability calculations are cumbersome. Further, as discussed in the previous subsection, such coupled NLS equations are not strictly asymptotically valid in the generic case when the group velocities are different. Thus, in the previous subsection we developed a more formal asymptotic expansion, yielding a four-wave analogue of the non-locally coupled NLS equations previously developed for two-wave solutions. These equations yield relatively simple predictions of modulational instability, so that the differences between two-wave and four-wave cases can easily be identified.

In this section we briefly summarize results of our studies of modulational instability of two- and four-wave solutions and discuss how modulational instability affects the processes of energy exchange between the physical components of the system [17, 18].

4.3.1 Modulational instability of two-wave solutions

We consider the modulational instability of spatially uniform coupled plane wave solutions of the form

$$A = A_0 e^{i\Omega_a T}, \quad C = C_0 e^{i\Omega_c T}, \quad B_0 = D_0 = 0. \quad (49)$$

This is a solution of the non-dispersive nonlinearly coupled envelope (33) model, the non-locally coupled NLS models (45) and (47), and the coupled NLS model (48), provided

$$\Omega_a = \mu_1 |A_0|^2 + \mu_2 |C_0|^2, \quad \Omega_c = \mu_3 |C_0|^2 + \mu_4 |A_0|^2.$$

Substituting (49) into (31) we obtain the leading order solution in the form of two counter-propagating waves:

$$\begin{pmatrix} u \\ w \end{pmatrix} = \varepsilon A_0 e^{i(kx - (\omega_1 - \varepsilon^2 \Omega_a)t)} \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} + \varepsilon C_0 e^{i(kx - (\omega_2 - \varepsilon^2 \Omega_c)t)} \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} + c.c. + O(\varepsilon^2). \quad (50)$$

This solution describes a weakly nonlinear interaction between the two components. For $\varepsilon \neq 0$, it generalises the linear solutions of Section 4.1 by nonlinearity-induced corrections of $O(\varepsilon^2)$ to

the frequencies $\omega_{1,2}(k)$ of the linear waves, given by (20). For $\varepsilon \rightarrow 0$, it reduces to the two-wave linear energy exchange solution (22) if

$$\varepsilon A_0 = \frac{|\alpha_2|U}{2(\alpha_1 + |\alpha_2|)}, \quad \varepsilon C_0 = \frac{\alpha_1 U}{2(\alpha_1 + |\alpha_2|)}. \quad (51)$$

In the special case $c = 1$, (50) matches the weakly nonlinear expansion of the exact two-wave energy-exchange solution (30). Since $\alpha_1 = 1$ and $\alpha_2 = -\delta^{-2}$ at $c = 1$, from (35) $\mu_1 = \mu_2 = \mu_4 = 0$, and $\mu_3 = (1 + \delta^2)^3/(4\omega_2\delta^4)$, so that taking

$$\varepsilon A_0 = \frac{\kappa}{1 + \delta^2}, \quad \varepsilon C_0 = \frac{\kappa\delta^2}{1 + \delta^2}, \quad B_0 = D_0 = 0,$$

we find $\Omega_a = 0$, $\varepsilon^2 \Omega_c = \mu_3 |\varepsilon C_0|^2 = \kappa^2(1 + \delta^2)/4\omega_2$, in agreement with the weakly nonlinear expansion of (30).

In [17] we considered the stability of the coupled plane wave solutions (49) in the context of two of the asymptotic models of Section 4.2. Firstly, we showed that the non-dispersive nonlinearly coupled envelope equations (33) with $B = D = 0$ do not support any modulational instabilities. Secondly, we studied instabilities with modulations on the long lengthscale $\chi = \varepsilon x$, with $v_{g1} - v_{g2}$ non-zero and $O(1)$, so that we used (45) with $B = D = 0$. However, the dynamics of that system are relatively simple, since as in [25] the equations may be transformed to two uncoupled NLS equations using the nonlocal transformation

$$\begin{aligned} A(\eta_1, T) &= \tilde{A}(\eta_1, T) \exp \left(i\mu_2 \int_0^T \overline{|C|^2}(T') dT' \right), \\ C(\eta_2, T) &= \tilde{C}(\eta_2, T) \exp \left(i\mu_4 \int_0^T \overline{|A|^2}(T') dT' \right). \end{aligned} \quad (52)$$

Alternatively, stability results can be established directly by looking for solutions of (45) of the form

$$\begin{aligned} A(\eta_1, T) &= A_0(T) \left(1 + a_1 e^{i(\hat{\kappa}\eta_1 - \hat{\Omega}_a T)} + a_2^* e^{-i(\hat{\kappa}\eta_1 - \hat{\Omega}_a^* T)} \right), \\ C(\eta_2, T) &= C_0(T) \left(1 + c_1 e^{i(\hat{\kappa}\eta_2 - \hat{\Omega}_c T)} + c_2^* e^{-i(\hat{\kappa}\eta_2 - \hat{\Omega}_c^* T)} \right), \end{aligned} \quad (53)$$

representing sinusoidal perturbations to the coupled-wave solution (49). Neglecting terms quadratic in disturbance amplitude, (45) leads to two uncoupled linear systems, as expected from (52), with corresponding dispersion relations

$$\hat{\Omega}_a^2 = -\mu_1 |A_0|^2 \omega_1'' \hat{\kappa}^2 + \frac{\omega_1''^2 \hat{\kappa}^4}{4}, \quad \hat{\Omega}_c^2 = -\mu_3 |C_0|^2 \omega_2'' \hat{\kappa}^2 + \frac{\omega_2''^2 \hat{\kappa}^4}{4}. \quad (54)$$

Using (32) and (36) we can rewrite these in terms of an unscaled disturbance wavenumber $\hat{k}$ and frequency $\hat{\omega}$, via

$$\varepsilon \hat{\kappa} = \hat{k}, \quad \varepsilon^2 \hat{\Omega}_a = \hat{\omega}_a - \hat{k} v_{g1}, \quad \varepsilon^2 \hat{\Omega}_c = \hat{\omega}_c - \hat{k} v_{g2}, \quad (55)$$

giving

$$\left(\hat{\omega}_a - \hat{k} v_{g1} \right)^2 = -\mu_1 |\varepsilon A_0|^2 \omega_1'' \hat{k}^2 + \frac{\omega_1''^2 \hat{k}^4}{4}, \quad \left(\hat{\omega}_c - \hat{k} v_{g2} \right)^2 = -\mu_3 |\varepsilon C_0|^2 \omega_2'' \hat{k}^2 + \frac{\omega_2''^2 \hat{k}^4}{4}, \quad (56)$$

where for formal validity we need $A_0, C_0 = O(1)$, $\hat{k} = O(\varepsilon)$, and $(v_{g1} - v_{g2}) = O(1)$.

From (35), we see that $\mu_1 > 0$ and $\mu_3 > 0$, so there will be instability only if $\omega_1'' > 0$ or $\omega_2'' > 0$. Using (43), we can show that ω_2'' is always positive. Thus for sufficiently small $\hat{k}$ there

will be a root for $\hat{\omega}_c$ with positive imaginary part, corresponding to an unstable mode. Using (20), we can show that $\omega_1'' < 0$ for small k , whilst $\omega_1'' > 0$ for large k , tending to zero at infinity. Thus $\hat{\omega}_a$ is real for small k , and hence does not correspond to an unstable mode, whilst for larger k , where $\omega_1'' > 0$, it will correspond to an unstable mode for sufficiently small $\hat{k}$.

In [18] we considered instability in the context of the two other asymptotic models of Section 4.2. The first model is for modulations on the long lengthscale $\chi = \varepsilon x$, but with $(v_{g1} - v_{g2}) = O(\varepsilon)$. Then we must use equations (47), with $B = D = 0$, giving a system that is locally coupled in A and C . The second model is the coupled NLS equations (48), with $B = D = 0$. However, with the appropriate rescaling, it is easy to see that these two systems of equations are equivalent.

We look for disturbances to the coupled-wave solution (49) in the form

$$\begin{aligned} A &= A_0(T) \left(1 + a_1 e^{i(\hat{k}x - \hat{\omega}_{ac}t)} + a_2^* e^{-i(\hat{k}x - \hat{\omega}_{ac}^*t)} \right), \\ C &= C_0(T) \left(1 + c_1 e^{i(\hat{k}x - \hat{\omega}_{ac}t)} + c_2^* e^{-i(\hat{k}x - \hat{\omega}_{ac}^*t)} \right), \end{aligned}$$

where we find it convenient to revert to the original unscaled variables x, t and work in terms of the unscaled disturbance frequency $\hat{\omega}$ and wavenumber $\hat{k}$. Substituting into (47) or (48) we obtain:

$$\begin{aligned} &\left(\hat{\omega}_{ac} - \hat{k}v_{g1} - \frac{1}{2}\omega_1''\hat{k}^2 + \mu_1|\varepsilon A_0|^2 \right) a_1 + \mu_1|\varepsilon A_0|^2 a_2 + \mu_2|\varepsilon C_0|^2 c_1 + \mu_2|\varepsilon C_0|^2 c_2 = 0, \\ &-\mu_1|\varepsilon A_0|^2 a_1 + \left(\hat{\omega}_{ac} - \hat{k}v_{g1} + \frac{1}{2}\omega_1''\hat{k}^2 - \mu_1|\varepsilon A_0|^2 \right) a_2 - \mu_2|\varepsilon C_0|^2 c_1 - \mu_2|\varepsilon C_0|^2 c_2 = 0, \\ &\mu_4|\varepsilon A_0|^2 a_1 + \mu_4|\varepsilon A_0|^2 a_2 + \left(\hat{\omega}_{ac} - \hat{k}v_{g2} - \frac{1}{2}\omega_2''\hat{k}^2 + \mu_3|\varepsilon C_0|^2 \right) c_1 + \mu_3|\varepsilon C_0|^2 c_2 = 0, \\ &-\mu_4|\varepsilon A_0|^2 a_1 - \mu_4|\varepsilon A_0|^2 a_2 - \mu_3|\varepsilon C_0|^2 c_1 + \left(\hat{\omega}_{ac} - \hat{k}v_{g2} + \frac{1}{2}\omega_2''\hat{k}^2 - \mu_3|\varepsilon C_0|^2 \right) c_2 = 0. \end{aligned}$$

This four-by-four linear homogeneous system of equations must have a vanishing determinant, from which we find that

$$Q_a(\hat{v}, \hat{k})Q_c(\hat{v}, \hat{k}) = R, \quad \text{where } R = \mu_2\mu_4\omega_1''\omega_2''|\varepsilon A_0|^2|\varepsilon C_0|^2, \quad (57)$$

$$Q_a(\hat{v}, \hat{k}) = (\hat{v} - v_{g1})^2 - \frac{\omega_1''\hat{k}^2}{4} + \mu_1\omega_1''|\varepsilon A_0|^2, \quad Q_c(\hat{v}, \hat{k}) = (\hat{v} - v_{g2})^2 - \frac{\omega_2''\hat{k}^2}{4} + \mu_3\omega_2''|\varepsilon C_0|^2. \quad (58)$$

Here $\hat{v} = \hat{\omega}_{ac}/\hat{k}$ is the phase speed of the disturbance, whilst R is proportional to the coupling constants in (47) and (48). Note that $Q_a = 0$ and $Q_c = 0$ are the dispersion relations for the uncoupled NLS equations for A and C respectively, cf. (56). Equation (57) is a fourth-order dispersion relation with real coefficients for $\hat{v}$. Complex solutions for $\hat{v}$ imply modulational instability.

Working within the context of (47), for formal validity of (57) we need $A_0, C_0 = O(1)$, $\hat{k} = O(\varepsilon)$, and $(v_{g1} - v_{g2}) = O(\varepsilon)$. However, within the context of (48), there are no specific scale restrictions, and we may investigate different scaling regimes.

We first probe the effects of the dispersive terms for modulations on the super-long lengthscale, by taking $\hat{k} = \varepsilon^2 \hat{K}$, with $\hat{K} = O(1)$ and $\varepsilon \ll 1$. Writing $\hat{v} = \hat{v}_1 + \varepsilon^2 \hat{v}_2 + \dots$, one may derive expressions for the four roots as

$$\begin{aligned} (\hat{v} - v_{g1})^2 &= -\varepsilon^2 \mu_1 \omega_1'' |A_0|^2 + \varepsilon^4 \left(\frac{1}{4} \omega_1''^2 \hat{K}^2 + \frac{\mu_2 \mu_4 \omega_1'' \omega_2'' |A_0|^2 |C_0|^2}{(v_{g1} - v_{g2})^2} \right) + O(\varepsilon^6), \\ (\hat{v} - v_{g2})^2 &= -\varepsilon^2 \mu_3 \omega_2'' |C_0|^2 + \varepsilon^4 \left(\frac{1}{4} \omega_2''^2 \hat{K}^2 + \frac{\mu_2 \mu_4 \omega_1'' \omega_2'' |A_0|^2 |C_0|^2}{(v_{g1} - v_{g2})^2} \right) + O(\varepsilon^6). \end{aligned}$$

Since $\mu_3 > 0$ and $\omega_2'' > 0$, at least one pair of roots for $\hat{v}$ is complex, and hence there is always an unstable mode. Since $\mu_1 > 0$, there will be an additional unstable mode if $\omega_1'' > 0$. Thus, our analysis suggests that the dispersive terms are destabilising for disturbances on the super-long lengthscale, leading to an instability with a growth rate of $O(\varepsilon^3)$.

We can recover results for modulations on the long lengthscale by taking $\hat{k} = \varepsilon \hat{\kappa}$, with $\hat{\kappa} = O(1)$ and $\varepsilon \ll 1$. If $(v_{g1} - v_{g2}) = O(1)$, then writing $\hat{v} = \hat{v}_1 + \varepsilon \hat{v}_2 + \dots$, one may derive expressions for the four roots as

$$\begin{aligned} (\hat{v} - v_{g1})^2 &= -\varepsilon^2 \left(\mu_1 \omega_1'' |A_0|^2 - \frac{1}{4} \omega_1''^2 \hat{\kappa}^2 \right) + \frac{\varepsilon^4 \mu_2 \mu_4 \omega_1'' \omega_2'' |A_0|^2 |C_0|^2}{(v_{g1} - v_{g2})^2} + O(\varepsilon^6), \\ (\hat{v} - v_{g2})^2 &= -\varepsilon^2 \left(\mu_3 \omega_2'' |C_0|^2 - \frac{1}{4} \omega_2''^2 \hat{\kappa}^2 \right) + \frac{\varepsilon^4 \mu_2 \mu_4 \omega_1'' \omega_2'' |A_0|^2 |C_0|^2}{(v_{g1} - v_{g2})^2} + O(\varepsilon^6). \end{aligned} \quad (59)$$

Since $\mu_3 > 0$ and $\omega_2'' > 0$, there will be an instability with a growth rate of $O(\varepsilon^2)$ if $\hat{\kappa}^2 < 4\mu_3 |C_0|^2 / \omega_2''$. Since $\mu_1 > 0$, there will be an additional unstable mode if $\omega_1'' > 0$ and $\hat{\kappa}^2 < 4\mu_1 |A_0|^2 / \omega_1''$. This is consistent with (54) and (56), derived formally from the non-locally coupled NLS equations. If $(v_{g1} - v_{g2}) = O(\varepsilon)$, then no further approximation of (57) is possible, since it is already the asymptotically consistent model for this regime.

Thus, we have considered how modulational instabilities of spatially uniform coupled two-wave solutions are predicted by the four weakly nonlinear models of Section 4.2. The non-dispersive nonlinearly coupled envelope equations (33) give an unambiguous prediction of stability for disturbances with $\hat{k}$ of $O(\varepsilon^2)$ on a timescale $\sim \varepsilon^{-2}$. Further, the non-locally coupled NLS equations (45) give an unambiguous prediction of instability for disturbances with $\hat{k}$ of $O(\varepsilon)$ on a timescale $\sim \varepsilon^{-2}$, whenever $v_{g1} - v_{g2}$ is $O(1)$, and either $\omega_1'' > 0$ or $\omega_2'' > 0$ (the latter is always true here). The coupled NLS equations (47) and (48) give a dispersion relation which is harder to analyse, but we showed that it reproduces the results for disturbances with $\hat{k}$ of $O(\varepsilon)$ or $O(\varepsilon^2)$ in their respective regimes of validity. However, in the context of (48), the coupled NLS dispersion relation also gives predictions which are strictly outside its asymptotic range of validity.

4.3.2 Modulational instability of four-wave solutions

We next examine how modulational instabilities of four-wave solutions are predicted by the asymptotic models derived in Section 4.2. We restrict our analysis to the non-locally coupled NLS equations, derived for modulations on the long lengthscale $\chi = \varepsilon x$ [18].

We start by examining the form of spatially uniform coupled plane waves. For such solutions, the non-locally coupled NLS models (45) and (47) reduce to the same system. Looking for solutions of the form

$$A = A_0 e^{i\Omega_a T}, \quad B = B_0 e^{-i\Omega_b T}, \quad C = C_0 e^{i\Omega_c T}, \quad D = D_0 e^{-i\Omega_d T}, \quad (60)$$

we see that a balance is only possible if

$$\Omega_a - \Omega_b = \Omega_c - \Omega_d. \quad (61)$$

Substitution of (60) into (45) or (47) gives

$$\begin{aligned} \Omega_a &= \mu_1 (|A_0|^2 + 2|B_0|^2) + \mu_2 (|C_0|^2 + |D_0|^2) + \mu_2 \phi^* / |A_0|^2, \\ \Omega_b &= \mu_1 (2|A_0|^2 + |B_0|^2) + \mu_2 (|C_0|^2 + |D_0|^2) + \mu_2 \phi^* / |B_0|^2, \\ \Omega_c &= \mu_3 (|C_0|^2 + 2|D_0|^2) + \mu_4 (|A_0|^2 + |B_0|^2) + \mu_4 \phi / |C_0|^2, \\ \Omega_d &= \mu_3 (2|C_0|^2 + |D_0|^2) + \mu_4 (|A_0|^2 + |B_0|^2) + \mu_4 \phi / |D_0|^2, \end{aligned}$$

where

$$\phi = A_0 B_0 C_0^* D_0^*. \quad (62)$$

Thus, for solutions of the form (60), the condition (61) for balance becomes

$$(|A_0|^2 - |B_0|^2) \left(\mu_1 + \frac{\mu_2 C_0 D_0}{A_0 B_0} \right) = (|C_0|^2 - |D_0|^2) \left(\mu_3 + \frac{\mu_4 A_0 B_0}{C_0 D_0} \right). \quad (63)$$

If (63) is satisfied, then we may substitute (60) into (31) to obtain a solution in the form of two pairs of counter-propagating waves,

$$\begin{aligned} \begin{pmatrix} u \\ w \end{pmatrix} = \varepsilon \left(A_0 e^{i(kx - (\omega_1 - \varepsilon^2 \Omega_a)t)} + B_0 e^{i(kx + (\omega_1 - \varepsilon^2 \Omega_b)t)} \right) \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} \\ + \varepsilon \left(C_0 e^{i(kx - (\omega_2 - \varepsilon^2 \Omega_c)t)} + D_0 e^{i(kx + (\omega_2 - \varepsilon^2 \Omega_d)t)} \right) \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix} + \text{c.c.} + O(\varepsilon^2). \end{aligned} \quad (64)$$

This solution generalises the linear plane waves of Section 4.1 by nonlinearly-induced corrections of $O(\varepsilon^2)$ to the frequencies $\omega_{1,2}(k)$ of the linear waves, given by (20). For $\varepsilon \rightarrow 0$, it reduces to the four-wave linear energy exchange solution (28) if

$$\varepsilon A_0 = \varepsilon B_0 = \frac{|\alpha_2|U}{4(\alpha_1 + |\alpha_2|)}, \quad \varepsilon C_0 = \varepsilon D_0 = \frac{\alpha_1 U}{4(\alpha_1 + |\alpha_2|)}, \quad (65)$$

which clearly satisfies the balance condition (63).

We study the instability of the plane-wave solution (60) in the case with v_{g1} and v_{g2} both $O(1)$, and $(v_{g1} - v_{g2})$ non-zero and $O(1)$. Accordingly we use equations (45), and we look for solutions in the form (53), with corresponding forms for B and D . Neglecting terms quadratic in disturbance amplitude, this yields four linear systems of the form

$$\begin{aligned} \left(\hat{\Omega}_a - \frac{1}{2} \omega_1'' \hat{k}^2 + \mu_1 |A_0|^2 - \frac{\mu_2 \phi^*}{|A_0|^2} \right) a_1 + \mu_1 |A_0|^2 a_2 = 0, \\ -\mu_1 |A_0|^2 a_1 + \left(\hat{\Omega}_a + \frac{1}{2} \omega_1'' \hat{k}^2 - \mu_1 |A_0|^2 + \frac{\mu_2 \phi}{|A_0|^2} \right) a_2 = 0. \end{aligned}$$

Note that although this system might appear only to involve $|A_0|$, in fact the terms involving ϕ give a coupling to the other basic state waves via B_0 , C_0 and D_0 , unlike the two-wave case. Each system has an associated dispersion relation of the form

$$\hat{\Omega}_a^2 + \frac{\mu_2}{|A_0|^2} (\phi - \phi^*) \hat{\Omega}_a - \left(\frac{1}{2} \omega_1'' \hat{k}^2 + \frac{\mu_2 \phi}{|A_0|^2} - \mu_1 |A_0|^2 \right) \left(\frac{1}{2} \omega_1'' \hat{k}^2 + \frac{\mu_2 \phi^*}{|A_0|^2} - \mu_1 |A_0|^2 \right) + \mu_1^2 |A_0|^4 = 0.$$

When A_0 , B_0 , C_0 and D_0 all have the same phase, as is the case for (65) for instance, then $\phi = |A_0 B_0 C_0 D_0|$, and the dispersion relation reduces to

$$\hat{\Omega}_a^2 = \left(\frac{1}{2} \omega_1'' \hat{k}^2 - \mu_1 |A_0|^2 + \frac{\mu_2 |B_0 C_0 D_0|}{|A_0|} \right)^2 - \mu_1^2 |A_0|^4. \quad (66)$$

Reverting to dimensional variables using (55) gives

$$\left(\hat{\omega}_a - \hat{k} v_{g1} \right)^2 = \left(\frac{1}{2} \omega_1'' \hat{k}^2 - (1 - r_a) \mu_1 |\varepsilon A_0|^2 \right)^2 - \mu_1^2 |\varepsilon A_0|^4, \quad r_a = \frac{\mu_2}{\mu_1} \left| \frac{B_0 C_0 D_0}{A_0^3} \right|, \quad (67)$$

with corresponding forms for $\hat{\omega}_b$, $\hat{\omega}_c$ and $\hat{\omega}_d$. Although equations (66) and (67) have a similar form to their two-wave counterparts (54) and (56), there are important differences. For instance, consider the instability of the A -mode, as given by (67).

When $\omega_1'' > 0$, (56) shows that for the two-wave solution a maximum growth rate of $s_* = \mu_1 |\varepsilon A_0|^2$ is achieved by the A -mode when $\hat{k}^2 = \hat{k}_*^2 = 2\mu_1 |\varepsilon A_0|^2 / |\omega_1''| > 0$. However, for the four-wave solution, instability of the A -mode is instead determined by the parameter r_a , and

only if $r_a = 0$ are the two-wave results recovered. If $0 < r_a \leq 1$, then (67) is minimised at $\hat{k}^2 = \hat{k}_*^2(1 - r_a)$, so that the disturbance wavenumber of the most unstable A -mode is reduced, although the growth rate remains s_* . If $r_a > 1$, (67) is instead minimised at $\hat{k} = 0$, with value $s_*^2(r_a^2 - 2r_a) > -s_*^2$. Therefore, if $1 < r_a < 2$ the most unstable A -mode occurs at $\hat{k} = 0$ with a reduced growth rate $s < s_*$, whilst if $r_a \geq 2$ the A -mode is stable. Thus, a two-wave solution can yield an instability in the A -mode whilst a corresponding four-wave solution does not.

When $\omega_1'' < 0$, (56) shows that the two-wave solution does not lead to an instability in the A -mode. However, when $r_a \geq 1$, (67) is minimised at $\hat{k}^2 = (r_a - 1)\hat{k}_*^2$, leading to an instability in the A -mode with growth rate s_* . If $0 < r_a < 1$, (67) is instead minimised at $\hat{k} = 0$, with value $s_*^2(r_a^2 - 2r_a) > -s_*^2$, leading to an instability in the A -mode with a reduced growth rate $s < s_*$. Thus, a four-wave solution can yield an instability in the A -mode whilst a corresponding two-wave solution does not.

There are corresponding conclusions for the B mode, and for the C and D -modes (for which $\omega_2'' > 0$), and the overall stability of (60) will be determined by the most unstable of these four modes. Although the possible stabilising and destabilising effects outlined above are of interest, we do not consider the details any further here.

Partial results for the case with v_{g1} and v_{g2} both $O(1)$, but $(v_{g1} - v_{g2}) = O(\varepsilon)$ can be found in [18].

4.3.3 Modification of the energy exchange

Numerical simulations of the original system (16) of coupled Sine-Gordon equations were performed both to examine the accuracy and validity of the theoretical predictions of instability and to visualize the long-time evolution of our energy exchange solutions [17,18]. Simulations were performed in a periodic domain, using plane-wave initial conditions of the energy exchange solutions. A major issue was to assess the performance of the stability predictions obtained from the different weakly nonlinear models. For cases with $(v_{g1} - v_{g2}) = O(1)$, the quadratic dispersion relations of the non-locally coupled NLS equations give good predictions: relative errors are about 2% when the amplitude parameter $U = 0.5$, and these decrease with U . For two-wave cases, the quartic dispersion relation of the coupled NLS equations gives no more accurate predictions, even at large amplitudes. As $|v_{g1} - v_{g2}|$ decreases, the quadratic dispersion relations become less accurate, and the quartic dispersion relations of the non-locally coupled NLS regime with $(v_{g1} - v_{g2}) = O(\varepsilon)$, or equivalently of the coupled NLS equations in the two-wave case, become more useful. For cases where the group velocities are very close or equal, the quadratic dispersion relations of the non-locally coupled NLS equations no longer even predict qualitatively the correct form of the dispersion curve, and a quartic dispersion relation must be used. The region of maximum growth is then well described, with relative errors decreasing with U , although at larger disturbance wavenumbers convergence appears to be poor and relative errors are about 10%. In some four-wave cases there was an additional dominant short-wavelength instability, which did not correspond to a long spatial modulation of the type considered theoretically, and was not predicted by any of our theories.

The numerical simulations also enabled us to examine the long-time evolution of the energy exchange solutions. For the energy exchange parameters of interest, the initial solutions are spatially uniform plane-waves with an almost complete energy exchange between the components, whilst later in the evolution the solutions can become spatially localized with an incomplete energy exchange. In order to make clear conclusions, we considered a restricted class of scenarios. First, we restrict the plane-wave parameters to $c = \delta$ and $k = 1$, all of which satisfy the full exchange condition (24). Then, for c and δ of $O(1)$, $v_{g1} - v_{g2} \approx 0.5$. Second, we restrict the computational geometry so that just one of the numerically resolved sidebands is unstable. We achieve this by taking $l \approx 2\pi/\hat{k}_{c*}$, where $\hat{k}_{c*}$ is the most unstable wavenumber associated with the prediction $\hat{s}_c$, as determined by (56) or by the C-mode version of (67). We then obtain an evolution in which the initial plane-wave structure becomes modulated on the lengthscale of the computational domain, with no possibility of an interaction between modulational instabilities of two different wavelengths.

Fig. 3. Time evolution of a two-wave solution at $c = \delta = 1$, $k = 1$, $U = 0.4$, with $l = 8\pi$.

Fig. 4. Time evolution of a four-wave solution at $c = \delta = 1$, $k = 1$, $U = 0.4$, with $l = 16\pi$.

Although the nonintegrable case is the focus of our study, it is useful to begin our discussion here with a brief display of the results for the integrable case $c = 1$. We thus set $c = \delta = 1$, so that the system (16) uncouples into a Sine-Gordon equation and a linear wave equation, and consequently is integrable. Thus, the modulational instability of plane-wave solutions shows a recurrence, similar to the famous Fermi–Pasta–Ulam recurrence [11]. We view this behaviour by considering the time evolution of three quantities: E_{plane} , representing the strength of the plane-wave and its nonlinear harmonics; $E_{side|1}$, representing the strength of the modulational instability in the first sideband pair, with disturbance wavenumber $\hat{k} = 2\pi/l$; and $E - E_{plane} - E_{side|1}$, representing the contribution from other wavenumbers.

These quantities are shown in the top row of Figure 3, during the instability of a two-wave solution at $c = \delta = 1$, with $U = 0.4$ and $l = 8\pi$. One can clearly see an almost periodic recurrence, with most of the energy either in E_{plane} , or $E_{side|1}$. In the second row, one can see how E is partitioned between u , w and the coupling, as defined in (19), whilst in the third and fourth rows one can see $u(x, t)$ and $w(x, t)$ respectively (the complete computational domain is shown). At first the plane-wave solution has a periodic energy exchange, which is

almost complete between E_u and E_w , as approximately described by (25) and (26). At the first minimum in E_{plane} , near $t = 1300$, there is still a periodic energy exchange between u and w , although the exchange is no longer almost complete, i.e. E_u and E_w no longer almost vanish periodically. Further, at this time $u(x, t)$ and $w(x, t)$ are strongly modulated on the largest resolved scale, and the large amplitude behaviour pulses between u and w . Thus, the almost complete energy exchange with spatially uniform structures initially observed is modified in the nonlinear regime to an incomplete energy exchange with structures localized on the lengthscale of the computational domain. Note that these same two scenarios alternate, in line with the recurrence of E_{plane} and $E_{side|1}$. One can see how $u(x, t)$ and $w(x, t)$ return almost to their initial plane-wave form when $E_{side|1}$ is minimised, near $t = 13,170$ for instance, whilst the pulsing modulations return when $E_{side|1}$ is maximised, near $t = 17,150$ for instance.

The four-wave case at $c = \delta = 1$ with $U = 0.4$ is similar, as shown in Figure 4. Once again there is an almost periodic recurrence in E_{plane} and $E_{side|1}$. However, the partitioning of E between the components E_u , E_w and E_c is more complicated than in the two-wave case, with rapid oscillations in each component superimposed on a slower exchange between E_u and E_w . Once again, when E_{plane} is maximised the slow exchange between E_u and E_w is almost complete, whilst when $E_{side|1}$ is maximised the slow exchange becomes incomplete. The general pattern of an almost periodic sequence of alternating exchange patterns remains, from approximately uniform wavetrains to localized structures and back, and so on.

Fig. 5. Time evolution of solutions with $k = 1$ and $U = 0.4$: (a) a two-wave solution at $c = \delta = 1.01$, with $l = 8\pi$; (b) a four-wave solution at $c = \delta = 1.01$, with $l = 16\pi$; (c) a two-wave solution at $c = \delta = 1.1$ with $l = 8\pi$; (d) a four-wave solution at $c = \delta = 1.1$ with $l = 16\pi$.

A natural question to ask is to what extent similar behaviour occurs when $c \neq 1$. Shown in Figure 5 are results of corresponding simulations at $c = \delta = 1.01$ and $c = \delta = 1.1$, with $U = 0.4$ in both cases. For the two-wave cases, shown in panels (a) and (c), the period of the recurrence now varies, although the character of each recurrence event appears to remain almost the same. For the four-wave cases, shown in panels (b) and (d), the character of the recurrence events now varies, but the evolution is still predominantly characterized by an exchange between E_{plane} and $E_{side|1}$. Thus both two-wave and four-wave cases exhibit an imperfect recurrence, which we might expect since, for c close to 1, the equations are close to being integrable. Interestingly, the system (16) with c close to 1 has been used to model some dynamical processes in the DNA [7, 8], where such energy exchange processes could, potentially, play an important role. Thus, this special case invites further theoretical studies.

Although the recurrence becomes imperfect when $c \neq 1$, the alternating pattern of approximately uniform wavetrains and localized structures remains. Shown in Figures 6 and 7 are the

Fig. 6. Time evolution of a two-wave solution at $c = \delta = 1.2$, $k = 1$ and $U = 0.4$, with $l = 10\pi$.

Fig. 7. Time evolution of a four-wave solution at $c = \delta = 1.2$, $k = 1$ and $U = 0.4$, with $l = 20\pi$.

results of two-wave and four-wave simulations at $c = \delta = 1.2$, with $U = 0.4$. The two-wave case of Figure 6 still looks remarkably similar to that of Figure 3, taken at $c = \delta = 1$. However, in the four-wave case of Figure 7, the recurrence breaks down around $t = 13,000$, only to reappear later in the evolution. Further, during the second sequence of recurrence events, around $t = 17,000$, the alternating pattern of approximately uniform wavetrains and localized structures reappears. For instance, around $t = 17,100$ one can see that the solution only has a small spatial modulation, whilst around $t = 17,730$ the solution has a spatial character almost identical to that of the first recurrence event around $t = 5680$. More remarkably, note from the second row of Figure 7 how even the details of the exchange between E_u , E_w and E_c are almost identical around $t = 5680$ and $t = 17,730$.

However, for larger values of $c = \delta$, evolutions which might be described as recurrent become rare, and typically the solutions display a more irregular behaviour. To illustrate this, we show results for the case $c = \delta = 1.5$. Shown in Figure 8 is the evolution of a two-wave solution, with $U = 0.4$ and $l = 12\pi$, whilst shown in Figure 9 is the evolution a four-wave solution, with $U = 0.4$ and $l = 24\pi$. In the two-wave case, at large times the solution loses much

Fig. 8. Time evolution of a two-wave solution at $c = \delta = 1.5$, $k = 1$ and $U = 0.4$, with $l = 12\pi$.

Fig. 9. Time evolution of a four-wave solution at $c = \delta = 1.5$, $k = 1$ and $U = 0.4$, with $l = 24\pi$.

of the character of the earlier recurrence. The four-wave case displays an irregular, imperfect recurrence, the character of which does seem to be similar to the earlier events.

5 Conclusion

The system (1) discussed in this paper is a natural two-component generalisation of the Klein-Gordon equation. It originates as a model describing the long-wave dynamics of two coupled periodic chains of particles, the elements of each chain being linked by linear coupling, and the chains themselves being linked by nonlinear coupling [5]. Known applications of this model include long longitudinal nonlinear waves in elastic by-layers with imperfect interfaces, where nonlinearity comes only from the bonding material [6], and some dynamical processes in DNA [7, 8]. A particular case of coupled Sine-Gordon equations can be viewed as a generalisation of the Frenkel - Kontorova dislocation model [3] (see also [4]).

The Lie group classification of coupled Klein-Gordon equations given in Section 2 provides the complete description of cases when equations admit additional point symmetries (for

$c \neq 1, f_{uw}(u, w) \neq 0$). These symmetries can be used to find conservation laws and invariant solutions. An example is shown in Section 3.

Energy exchange processes of the type considered in Section 4 are an important feature of two-component physical systems to which the model applies. The modulational instability results in a significant change in the dynamics, leading to the appearance of localized structures. In many cases we could observe that the initial linear energy exchange then proceeds for some time as energy exchange between these localized structures. In that sense, one can say that energy exchange has continued into the nonlinear regime. When the system is close to the integrable case at $c = 1$, the time evolution is distinguished by a remarkable almost periodic sequence of energy exchange scenarios, with spatial patterns alternating between approximately uniform wavetrains and localized structures. As c moves away from 1, the time evolution becomes more disordered. Thus, the case when c is close to 1 is of special interest and requires further theoretical study. Possibly, one could try to exploit the integrability of coupled Sine-Gordon equations at $c = 1$ in the spirit of work on the perturbed Sine-Gordon equation (see for instance [35–37]). Although our study was only performed for a two-component set of coupled Sine-Gordon equations, we would expect similar phenomena to arise in a wide class of nonlinear wave systems, which can support several wave modes. This is typical for multi-component systems. The methodology developed for weakly nonlinear multi-phase wavetrains is generic, and can be used for any number of waves, and for other systems.

The simple lattice model leading to the system (1) has been used to study some features of propagation of nonlinear waves in bi-layers with delamination [38]. This is in the spirit of lattice models proposed for the modelling of crack propagation in composites (see, for example, [39]). Our model can be naturally modified to take into account nonlinear interactions in the layers and other degrees of freedom (for example, by considering chains of interacting mechanical dipoles [40] instead of chains of point masses). It can also be easily two-dimensionalised. We expect such models to be useful for the modelling of nonlinear waves in a large class of bi-layers with delamination. Results in this direction will be reported elsewhere.

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