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Long wave instability in a three-layer stratified shear flow

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Abstract

In inviscid fluid flows instability can occur due to a resonance between two wave modes. For the case when the modes remain distinct at the critical point where the wave phase speeds coincide, then in the weakly nonlinear, long wave limit there is an expectation that the generic outcome is a model consisting of two coupled Korteweg-deVries equations. This situation is examined for a certain three-layer stratified shear flow.

1 Introduction

In inviscid fluid flows it is well-known that instability generally arises due to a resonance between two waves. Thus, as an appropriate external parameter is varied, the phase speeds of two waves coincide for some critical parameter value. The generic unfolding of this resonance yields either a stable “kissing” configuration, or a “bubble” of instability in the space of the external parameter (see Figure 1). There are many examples of this situation (see, for instance, the monograph by Craik (1985) for shear flows, or Baines and Mitsudera (1994) for a discussion of the physical processes involved).

Our concern here is with the unfolding of this resonance in the limit of long waves. Recent work has established that there are two generic canonical models (see, for instance, Grimshaw (2000)). For the case when the wave modes coincide at criticality, the canonical model is the Boussinesq equation

$$A_{tt} - \Delta^2 A_{xx} + \frac{1}{2}\mu(A^2)_{xx} + \lambda A_{xxxx} = \delta A_{xx}, \quad (1.1)$$

Here Δ is the aforementioned external parameter, δ is an unfolding parameter, while μ and λ are the nonlinear and dispersive coefficients respectively. In the linear, long-wave limit the dispersion relation for waves of speed c is just $c^2 = \Delta^2 + \delta$ which is just that sketched in Figure 1. Resonance occurs for $\Delta = \delta = 0$, and the flow is linearly stable, or unstable, according as $\delta > 0$, or < 0 . Equations of this form have been derived by Hickernell (1938a, b) for Kelvin-Helmholtz instability, and by Pedlosky and Helfrich (1993) and Mitsudera (1994) for certain geophysical flows.

However, our main interest here is with the alternative scenario, when the wave modes remain distinct at criticality. In this case a suitable canonical model consists of the coupled Korteweg-de-Vries (KdV) equations

$$\begin{aligned} A_t + \Delta_1 A_x + \mu_1 A A_x + \lambda_1 A_{xxx} + \kappa_1 B_x &= 0, \\ B_t + \Delta_1 B_x + \mu_2 B B_x + \lambda_2 B_{xxx} + \kappa_2 A_x &= 0, \end{aligned} \quad (1.2)$$

Here $\Delta_1 - \Delta_2$ is the detuning parameter, κ_1, κ_2 are unfolding parameters, while μ_1, μ_2 and λ_1, λ_2 are nonlinear and dispersive coefficients respectively. In the linear long-wave limit the dispersion relation for waves of speed c is

$$(c - \Delta_1)(c - \Delta_2) = \kappa_1 \kappa_2. \quad (1.3)$$

This is equivalent to that for the Boussinesq equation (1.1) if we put $\Delta = \frac{1}{2}(\Delta_1 - \Delta_2)$ and $\delta = \kappa_1 \kappa_2$. Hence there is instability if $\kappa_1 \kappa_2 < 0$, and stability if $\kappa_1 \kappa_2 > 0$. Equations of the form (1.2) have been derived by Mitsudera (1994) and Gottwald and Grimshaw (1999) for certain geophysical flows, and by Grimshaw (2000) for a certain three-layer stratified shear flow.

It is noteworthy that in the coupled KdV system (1.2) the only coupling terms are those with coefficients κ_1, κ_2 respectively. In general, we might also expect cross-coupling linear dispersive terms and nonlinear terms. That is, we can anticipate the possibility of terms B_{xxx} and A_{xxx} in the A - and B - equations respectively, and a full suite of nonlinear terms $AA_x, (AB)_x$ and BB_x in both equations. Our primary purpose in this paper is to demonstrate

that is indeed the case, and in the next Section we present a schematic argument to show this. Then, in the third Section, we consider a certain three-layer stratified shear flow, for which a more general coupled KdV system than (1.2) can be derived for some special parameter values. It is also pertinent to note here that Grimshaw (2000) also considered a three-layer stratified shear flow, and from that made a reduction to the coupled KdV system (1.2). However, in that case, the basic shear flow profile contained velocity jumps across the interfaces, thus possibly rendering the flow susceptible to a Kelvin-Helmholtz instability at short wavenumbers. The flow studied here in Section 3 has a continuous basic velocity profile, with constant shear in each layer, and in this respect is a more comfortable basic profile to work with. Further, the assumption of constant shear in each layer allows the analysis to proceed smoothly without the necessity to consider the possible role of critical layers. Nevertheless, we note that to develop the present theory further will almost certainly necessitate a detailed study of critical layer dynamics coupled to the instability mechanism.

2 Schematic derivation of the amplitude equations

In many physical systems the linearized stability problem, for which solutions are sought proportional to $\exp\{ik(x - ct)\}$, can be reduced to the following algebraic problem for the wave amplitudes A, B

$$\begin{aligned} D_1 A + E B &= 0, \\ E A + D_2 B &= 0. \end{aligned} \tag{2.1}$$

Here $D_{1,2}$ depend on k, c and a set of parameters $\Delta_1, \Delta_2, \dots$ say, which we shall represent by the single vector parameter $\Delta = (\Delta_1, \Delta_2, \dots)$. The dispersion relation is then

$$D \equiv D_1 D_2 - E^2 = 0, \tag{2.2}$$

which defines the wave speed c as a multi-valued function of k and Δ . In a non-dissipative, i.e. Hamiltonian, system, instability arises whenever $\text{Im } c \neq 0$. It is readily established that the onset of an instability occurs when the criterion

$$D_c = 0 \tag{2.3}$$

is adjoined to (2.2), where the subscript denotes the derivative of D with respect to c . At this point we shall take the limit $k \rightarrow 0$ and hence suppress the k -dependence for most of this section. In this case the conditions (2.2) and (2.3) define critical values of c and Δ for the onset of instability. Without loss of generality we can suppose that these are $c = 0$ and $\Delta = 0$. Our strategy then is to expand D_1, D_2 and E in powers of c and Δ , and explore the consequences. In the sequel all quantities are evaluated at criticality, $c = \Delta = 0$.

First, let us suppose that $E \neq 0$ at criticality. Then from (2.2) it follows that $D_{1,2} \neq 0$ also, and the system (2.1) is readily reduced to a single equation, for example, by the substitution $B = -EA/D_2$ which leads to

$$D(c, \Delta, k)A = 0 \tag{2.4}$$

Expansion of D as an operator in c , Δ and k , followed by incorporation of a weakly nonlinear term, then readily leads to the Boussinesq equation (1.1). We shall not explore this case any further here. Next, if $E = 0$ at criticality, then so does $D_1 D_2 = 0$. However, if either of $D_{1,2} \neq 0$, say $D_2 \neq 0$, then the system (2.1) can be again reduced to (2.4), and the Boussinesq equation (2.1) is again the outcome.

The case of most interest to us here is thus when $E = D_1 = D_2 = 0$ at criticality, from which we see immediately that A and B are independent at criticality. It is this case which will lead to a coupled KdV system of the form (1.1). Further, we shall suppose that while $D_c = 0$ (2.3), $D_{1c}, D_{2c} \neq 0$ to avoid degeneracy. It will also be assumed that $K = \frac{1}{2}D_{cc} = D_{1c}D_{2c} - E_c^2 \neq 0$. The next step is to expand D_1, D_2 and E in (2.1) in powers of c and Δ to obtain

$$\begin{aligned} (cD_{1c} + \Delta D_{1\Delta})A + (cE_c + \Delta E_\Delta)B &= 0, \\ (cE_c + \Delta E_\Delta)A + (cD_{2c} + \Delta D_{2\Delta})B &= 0. \end{aligned} \quad (2.5)$$

We can now identify $-ikc$ with $\partial/\partial t$ and ik with $\partial/\partial x$, and so (2.5) can be converted to a pair of coupled amplitude equations

$$\begin{aligned} D_{1c}A_t - \Delta D_{1\Delta}A_x + E_cB_t - \Delta E_\Delta B_x &= 0, \\ D_{2c}B_t - \Delta D_{2\Delta}B_x + E_cA_t - \Delta E_\Delta A_x &= 0. \end{aligned} \quad (2.6)$$

If $E_c = 0$, the amplitude equations reduce to the standard form

$$\begin{aligned} A_t + \Delta_1 A_x + \kappa_1 B_x &= 0, \\ B_t + \Delta_2 B_x + \kappa_2 A_x &= 0. \end{aligned} \quad (2.7)$$

Here we have made the identifications,

$$\begin{aligned} D_{1c}\Delta_1 &= -\Delta D_{1\Delta} \quad , \quad D_{2c}\Delta_2 = -\Delta D_{2\Delta}, \\ D_{1c}\kappa_1 &= -\Delta E_\Delta \quad , \quad D_{2c}\kappa_2 = -\Delta E_\Delta, \end{aligned} \quad (2.8)$$

First, we note that linear long wave instability requires that $\kappa_1 \kappa_2 < 0$, or equivalently, that $D_{1c}D_{2c} < 0$ (i.e. $K < 0$). Further, because Δ is a vector parameter, we can assume that there are sufficient degrees of freedom to allow $\Delta_1 - \Delta_2$ and $\kappa_1 \kappa_2$ to be independent parameters. Next, we consider the weakly nonlinear, weakly dispersive unfolding of the system (2.7). Let ϵ be a small parameter characterising weak dispersion so that $\partial/\partial x$ is $0(\epsilon)$. Then the appropriate balance for the onset of instability is that $\Delta_{1,2}$ and $\kappa_{1,2}$ are $0(\epsilon^2)$ (i.e. Δ is $0(\epsilon^2)$), while $\partial/\partial t$ is $0(\epsilon^3)$. Since here we also have $E_c = 0$ at criticality, this scaling implicitly assumes that $E(c, \Delta, k)$ is $0(\epsilon^2)$ so that in (2.1) A and B are only weakly coupled. Anticipating that the nonlinearity will be quadratic, so that A, B are $0(\epsilon^2)$, the natural weakly dispersive, weakly nonlinear unfolding of the system (2.7) is the coupled KdV system (1.2), where we can now see that $D_{1c}\lambda_1 = \frac{1}{2}D_{1kk}$ and $D_{2c}\lambda_2 = \frac{1}{2}D_{2kk}$, and all coefficients are evaluated at $c = \Delta = k = 0$. Note that the underlying reality of the physical system induces the symmetry that $k \rightarrow -k$ leaves the system unchanged, so that $D_{1k} = D_{2k} = 0$ for instance. It is important to note that there is no cross-coupling in the linear dispersive terms here, that is, there is no term $B_{xxx}(A_{xxx})$ in the $A - (B -)$ equation. The reason is that $E(c, \Delta, k)$ is $0(\epsilon^2)$ here. In principle, there could be cross-coupling in the nonlinear terms, but we have assumed that if the system is weakly coupled in the linear régime, it will also be weakly coupled in the nonlinear régime.

Let us now turn to the case of main interest here when $E_c \neq 0$. In this case A and B are strongly coupled, and it is useful to change variables as follows

$$\begin{aligned}\hat{A} &= D_{1c}A + E_cB, \\ \hat{B} &= D_{2c}B + E_cA,\end{aligned}\tag{2.9}$$

Note that since $K = D_{1c}D_{2c} - E_c^2 \neq 0$, this is a nonsingular transformation. The equation system (2.6) now adopts the standard form (2.7), that is,

$$\begin{aligned}\hat{A}_t + \Delta_1 \hat{A}_x + \kappa_1 \hat{B}_x &= 0, \\ \hat{B}_t + \Delta_2 \hat{B}_x + \kappa_2 \hat{A}_x &= 0\end{aligned}\tag{2.10}$$

Here, in place of (2.8), we now have

$$\begin{aligned}K\Delta_1 &= \Delta(-D_{2c}D_{1\Delta} + E_cE_\Delta) \\ K\Delta_2 &= \Delta(-D_{1c}D_{2\Delta} + E_cE_\Delta) \\ K\kappa_1 &= \Delta(-D_{1c}E_{1\Delta} + E_cD_{1\Delta}) \\ K\kappa_2 &= \Delta(-D_{2c}E_\Delta + E_cD_{2\Delta})\end{aligned}\tag{2.11}$$

Of course, these expression reduce to (2.8) when $E_c = 0$, except that κ_1, κ_2 are interchanged due to the transformation $\hat{A} = D_{1c}A, \hat{B} = D_{2c}B$ in this case. Linear long wave instability again requires that $\kappa_1\kappa_2 < 0$. More precisely, instability occurs whenever.

$$(\Delta_1 - \Delta_2)^2 + 4\kappa_1\kappa_2 < 0\tag{2.12}$$

Using the definitions (2.11) it can then be shown that (2.12) is equivalent to

$$D_{1c}D_{2c}\delta^2 + \frac{K}{D_{1c}D_{2c}}(\Delta_1 - \Delta_2)^2 < 0,\tag{2.13}$$

where

$$K(\Delta_1 - \Delta_2) = -D_{2c}\Delta D_{1\Delta} + D_{1c}\Delta D_{2\Delta},\tag{2.14}$$

$$\text{and} \quad \delta = \frac{\kappa_1}{D_{1c}} + \frac{\kappa_2}{D_{2c}} = \frac{1}{K} \left\{ -2\Delta E_\Delta + E_c \left(\frac{\Delta D_{1\Delta}}{D_{1c}} + \frac{\Delta D_{2\Delta}}{D_{2c}} \right) \right\}\tag{2.15}$$

Thus, instability occurs either when $D_{1c}D_{2c} < 0$ (when also $K < 0$), or when $0 < D_{1c}D_{2c} < E_c^2$ (and so also $K < 0$). In the former case it follows that $V = 1/2(\Delta_1 - \Delta_2)$ is the natural detuning parameter, while δ is the natural unfolding parameter, but these roles are reversed in the latter case. Note that since Δ is a vector parameter, we can always assume that it has at least two degrees of freedom, and hence we can vary V and δ independently.

We can now turn to the weakly nonlinear, weakly dispersive unfolding of the system (2.6), or equivalently (2.10). As before, let ϵ be a small parameter characterising weak dispersion so that $\partial/\partial x$ is $0(\epsilon)$. Then, as before, the appropriate balance requires Δ to be $0(\epsilon^2)$ (so that $\Delta_{1,2}$ and κ_1, κ_2 are $0(\epsilon^2)$), $\partial/\partial t$ is $0(\epsilon^3)$ and A, B are $0(\epsilon^2)$. The outcome is the coupled KdV system.

$$\begin{aligned}
D_{1c}A_t - \triangle D_{1\triangle}A_x + E_cB_t - \triangle E_{\triangle}B_x + \frac{1}{2}D_{1kk}A_{xxx} \\
+ \frac{1}{2}E_{kk}B_{xxx} + \mu_1AA_x + \nu(AB)_x + \sigma BB_x &= 0, \\
D_{2c}B_t - \triangle D_{2\triangle}B_x + E_cA_t - \triangle E_{\triangle}A_x + \frac{1}{2}D_{2kk}B_{xxx} \\
+ \frac{1}{2}E_{kk}A_{xxx} + \mu_2BB_x + \sigma(AB)_x + \nu AA_x &= 0.
\end{aligned} \tag{2.16}$$

The symmetry in the nonlinear coefficients is to ensure that the system is Hamiltonian. Indeed the Hamiltonian form is

$$J \frac{\partial \mathbf{A}}{\partial t} = - \frac{\partial}{\partial x} \left(\frac{\delta \mathcal{H}}{\delta \mathbf{A}} \right), \tag{2.17}$$

where $\mathbf{A}$ is the vector (A, B) , J is the symmetric matrix

$$J = \begin{bmatrix} D_{1c} & E_c \\ E_c & D_{2c} \end{bmatrix} \tag{2.18}$$

and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \int_{-\infty}^{\infty} H dx, \tag{2.19}$$

$$\begin{aligned}
\text{where } H = & -\frac{1}{2}\triangle D_{1\triangle}A^2 - \triangle E_{\triangle}AB - \frac{1}{2}\triangle D_{2\triangle}B^2 \\
& -\frac{1}{4}D_{1kk}A_x^2 - \frac{1}{2}E_{kk}A_xB_x - \frac{1}{4}D_{2kk}B_x^2 \\
& + \frac{1}{6}\mu_1A^3 + \frac{1}{2}\nu A^2B + \frac{1}{2}\sigma AB^2 + \frac{1}{6}\mu_2B^3.
\end{aligned} \tag{2.20}$$

The Hamiltonian is an invariant, that is $\mathcal{H}$ is conserved, and can be interpreted as an energy. Another invariant is

$$\mathcal{P} = \int_{-\infty}^{\infty} P dx, \tag{2.21}$$

where $P = \frac{1}{2}D_{1c}A^2 + E_cAB + \frac{1}{2}D_{2c}B^2,$

which can be interpreted as a momentum invariant. Two other conserved quantities are the Casimirs.

$$\mathcal{A} = \int_{-\infty}^{\infty} A dx, \quad \mathcal{B} = \int_{-\infty}^{\infty} B dx. \quad (2.22)$$

Note here, that if $D_{1c}D_{2c} < E_c^2$ then P is sign-indefinite, and this is precisely the criterion for linear instability (see (2.13)). On the other hand if $D_{1c}D_{2c} > E_c^2$ then P is definite, and its conservation ensures a form of stability; thus the criterion for linear stability and nonlinear stability coincide. An appropriate Poisson bracket for the Hamiltonian system (2.17) is

$$[F, G] = \int_{-\infty}^{\infty} \frac{\delta F}{\delta \mathbf{A}} J^{-1} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta \mathbf{A}} \right) dx, \quad (2.23)$$

for which all the usual properties, including the Jacobi identity, can now be readily verified.

Next we note that the substitution (2.9) will transform the coupled KdV system (2.16) into a similar system, but one in which each equation contains only a single time derivative (i.e. $\hat{A}_t$ or $\hat{B}_t$). Indeed, this transformation is just $\hat{\mathbf{A}} = J\mathbf{A}$. The Hamiltonian structure (2.17) is of course preserved and, in particular, $P = \frac{1}{2}\mathbf{A}^T \cdot \hat{\mathbf{A}} = \frac{1}{2}\hat{\mathbf{A}}^T J^{-1} \hat{\mathbf{A}}$ is the conserved density in (2.21). However the corresponding invariant (2.20) becomes rather cumbersome, so we shall not pursue this further here.

3 Three-layer shear flow

In this section we shall illustrate the schematic presentation of the previous section with a certain three-layer shear flow. The fluid is confined between two rigid boundaries located at $z = -H_1$ and $z = H_2 + H_3$, where z is the vertical coordinate. In the undisturbed state there are two free interfaces located at $z = 0$ and $z = H_2$, such that the lower layer of depth H_1 has a constant density ρ_1 , the middle layer of depth H_2 has a constant density ρ_2 , and the upper layer of depth H_3 has a constant density ρ_3 . In each layer there is a basic flow in the horizontal x -direction given by

$$\begin{aligned} U_1 + \Omega_1 z & \quad , \quad -H_1 \leq z \leq 0, \\ U_1 + \Omega_2 z & \quad , \quad -0 \leq z \leq H_2, \\ U_3 + \Omega_3(z - H_2) & \quad , \quad H_2 \leq z \leq H_2 + H_3, \end{aligned} \quad (3.1)$$

where $U_3 = U_1 + \Omega_2 H_2$. The basic flow is thus continuous and has constant shear in each layer. The continuity of the basic flow is put in place to prevent Kelvin-Helmholtz instability, and hence can be regarded as an improvement over the model discussed by Grimshaw (2000). However, note that there are implicit points of inflexion at the interfaces where the basic vorticity is discontinuous, and hence there is an expectation of an instability.

Confining our attention to two-dimensional flow, we can assume that the perturbed flow in each layer is irrotational. At each interface, there are two kinematic boundary conditions to ensure that there is no flow through the interface, and one dynamic boundary condition to enforce continuity of pressure. We shall represent the perturbed interfaces by $z = \eta(x, t)$

and $z = H_2 + \zeta(x, t)$ respectively. Next, let us linearise these boundary conditions, and then seek solutions of the form

$$\begin{aligned}\eta &= A \exp(ik(x - ct)) + c.c., \\ \zeta &= B \exp(ik(x - ct)) + c.c.,\end{aligned}\tag{3.2}$$

where c.c. denotes the complex conjugate. A straightforward analysis then shows that the linearised equations of motion can be reduced to an algebraic system of the form (2.1) where

$$\begin{aligned}D_1 &= g(\rho_1 - \rho_2) + (U_1 - c)(\rho_1\Omega_1 - \rho_2\Omega_2) - k(U_1 - c)^2 \left\{ \frac{\rho_1}{\tanh kH_1} + \frac{\rho_2}{\tanh kH_2} \right\}, \\ D_2 &= g(\rho_2 - \rho_3) + (U_3 - c)(\rho_2\Omega_2 - \rho_3\Omega_3) - k(U_3 - c)^2 \left\{ \frac{\rho_2}{\tanh kH_2} + \frac{\rho_3}{\tanh kH_3} \right\}, \\ E &= \frac{\rho_2 k (U_1 - c)(U_3 - c)}{\sinh kH_2}.\end{aligned}\tag{3.3}$$

In the limit $k \rightarrow 0$, these reduce to

$$\begin{aligned}D_1 &\rightarrow g(\rho_1 - \rho_2) + (U_1 - c)(\rho_1\Omega_1 - \rho_2\Omega_2) - (U_1 - c)^2 \left\{ \frac{\rho_1}{H_1} + \frac{\rho_2}{H_2} \right\}, \\ D_2 &\rightarrow g(\rho_2 - \rho_3) + (U_3 - c)(\rho_2\Omega_2 - \rho_3\Omega_3) - (U_3 - c)^2 \left\{ \frac{\rho_2}{H_2} + \frac{\rho_3}{H_3} \right\}, \\ E &\rightarrow \frac{\rho_2 (U_1 - c)(U_3 - c)}{H_2}.\end{aligned}\tag{3.4}$$

The case of interest to us here requires that $D_1 = D_2 = E = 0$ simultaneously in (3.4). Thus either $c = U_1$ or $c = U_3$, and without loss of generality, we choose the case $c = U_1$. It then that also $\rho_1 = \rho_2$ and

$$\Omega_2 H_2 (\Omega_2 H_2 + \Omega_3 H_3) = g H_3 \frac{(\rho_2 - \rho_3)}{\rho_3}\tag{3.5}$$

This represents a resonance between a vorticity wave on the lower interface with a mixed gravity-vorticity wave on the upper interface. At criticality we can now use (3.4) to obtain

$$\begin{aligned}D_{1c} &= \rho_2 (\Omega_2 - \Omega_1), \\ D_{2c} &= \rho_3 \Omega_3 + \rho_2 \Omega_2 \left(1 + \frac{2\rho_3 H_2}{\rho_2 H_3} \right), \\ \text{and } E_c &= -\rho_2 \Omega_2.\end{aligned}\tag{3.6}$$

The condition for linear instability that $D_{1c} D_{2c} < E_c^2$ (i.e. $K < 0$) can now be satisfied with a suitable choice of Ω_1 and Ω_3 , noting that then Ω_2 is constrained by the condition (3.5). For instance, if $\rho_1 = \rho_2 = \rho_3$ so that the flow is not stratified, the resonance condition (3.5) becomes $\Omega_2 H_2 = -\Omega_3 H_3$, and the instability condition reduces to $\Omega_3^2 H_3 + \Omega_1 \Omega_3 (H_2 + H_3) < 0$; alternatively one can satisfy (3.5) with $\Omega_2 H_2 = -\Omega_3 H_2$, and then the instability condition reduces to $\Omega_1 \Omega_3 > 0$. The former case represents a jet-like flow, and the latter represents a sheared flow. When $\rho_2 > \rho_3$, the resonance condition (3.5) implies that $\Omega_2 \neq 0$, and indeed, either Ω_2 and Ω_3 should have the same sign, or $|\Omega_2| H_2 > |\Omega_3| H_3$ if the signs are opposite. In both cases, the instability condition implies that $\Omega_1 \Omega_2 > 0$ and $|\Omega_1|$ should be sufficiently large. Again, both jet-like flows and sheared flows may be unstable.

Next we need to determine suitable expressions for $\Delta_{1,2}$ and $\kappa_{1,2}$ (2.11). The available parameters are $\rho_1 - \rho_2$, $\rho_2 - \rho_3$, $\Omega_1, \Omega_2, \Omega_3$ and $H_1/H_2, H_2/H_3$. The resonance conditions $\rho_1 = \rho_2$ and (3.5) impose two constraints on the parameter set, but this still leaves five degrees of freedom. Since we only need two degrees of freedom, we can let the vector parameter Δ be given by $\Delta = (g(\rho_1 - \rho_2), \Omega_2 - \Omega_{2c})$, where Ω_{2c} is the critical value of Ω_2 determined from (3.5) for fixed values of $\Omega_3, H_2/H_3$ and $\rho_2 - \rho_3$. Further without any loss of generality, we can set $U_1 = 0$. Thus, at criticality, $c = 0$ and $\Delta = 0$ and hence we can use the results of Section 2 directly. We find that

$$\begin{aligned}\Delta D_{1\Delta} &= g(\rho_1 - \rho_2), \\ \Delta D_{2\Delta} &= -\rho_3 H_2 \Omega \left(\Omega_3 + 2\Omega_{2c} \frac{H_2}{H_3} \right), \\ \Delta E_{\Delta} &= 0,\end{aligned}\tag{3.7}$$

where $\Omega = \Omega_2 - \Omega_{2c}$. We are now in a position to identify the detuning and unfolding parameters $V = \frac{1}{2}(\Delta_1 - \Delta_2)$ and δ from (2.14) and (2.15). The instability criterion (2.12) (or (2.13)) becomes

$$(-D_{2c}\Delta D_{1\Delta} + D_{1c}\Delta D_{2\Delta})^2 + 4E_c^2(\Delta D_{1\Delta})(\Delta D_{2\Delta}) < 0\tag{3.8}$$

and hence can only be realised if the product $(\Delta D_{1\Delta})(\Delta D_{2\Delta}) < 0$. Since $(\Delta D_{1\Delta}) > 0$ for a stable stratification, this implies that $\Omega(\Omega_3 + 2\Omega_{2c}H_2/H_3) > 0$ to achieve an instability; that is, the shear must be increased. Further, the expression (3.8) can be rearranged to give

$$\left(\Delta D_{2\Delta} - \left[1 - \frac{2E_c^2}{D_{1c}D_{2c}} \right] \frac{D_{2c}\Delta D_{1\Delta}}{D_{1c}} \right)^2 + \frac{4E_c^2 K}{D_{1c}^2 D_{2c}^2} \left(\frac{D_{2c}\Delta D_{1\Delta}}{D_{1c}} \right)^2 < 0,\tag{3.9}$$

which confirms that $K < 0$ is a necessary condition for instability as shown in Section 2, and also indicates how large $\Omega\Omega_{2c}$ must be to overcome the stabilizing stratification.

The coefficients of the linear dispersive terms in (2.16) are readily calculated from (3.3) and we find that

$$\begin{aligned}\frac{1}{2}D_{1kk} &= 0, & \frac{1}{2}E_{kk} &= 0, \\ \text{and} \quad \frac{1}{2}D_{2kk} &= -\frac{1}{3}\Omega_2^2 H_2^2 (\rho_2 H_2 + \rho_3 H_3).\end{aligned}\tag{3.10}$$

Thus there are no such terms for this present case in the ‘‘A-equation’’, and the only such term is in the ‘‘B-equation’’. It follows that for this particular three-layer shear flow the coupled KdV system (2.16) is degenerate. The possible implications of this are presently being explored.

Finally, we need to calculate the coefficients of the nonlinear terms in (2.16). This can only be achieved through a multi-scale perturbation expansion of the full equations of the motion, with $\partial/\partial x$ of $0(\epsilon)$, $\partial/\partial t$ of $0(\epsilon^3)$, Δ of $0(\epsilon^2)$ and η, ζ of $0(\epsilon^2)$. We shall omit the details and just record here the outcome,

$$\begin{aligned}
\mu_1 &= \rho_2(\Omega_{2c}^2 - \Omega_1^2), \\
\sigma &= -\nu = \rho_2 \Omega_{2c}^2, \\
\mu_2 &= \rho_3 \Omega_3^2 - \rho_2 \Omega_{2c}^2 + 3\rho_3 \Omega_{2c} \frac{H_2}{H_3^2} (\Omega_{2c} H_2 + \Omega_3 H_3).
\end{aligned} \tag{3.11}$$

This then completes the description of the coupled KdV system (2.16) for this three-layer shear flow. The general features expected are confirmed, although it is interesting to find that the “A-equation” contains no linear dispersive terms, but does contain the full set of nonlinear terms. It is not our intention here to explore the solution set of this coupled KdV system, which is presently under study and will be reported elsewhere.

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Figure 2: A schematic sketch of the dispersion relation for mode resonance, in which the phase speed c is plotted as a function of an external parameter Δ and an unfolding parameter δ . (a) the uncoupled case, $\delta = 0$, (b) the stable case $\delta > 0$, (c) the unstable case $\delta < 0$ (only $\text{Real}(c)$ is shown).