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ROSSBY WAVES ON A SHEAR FLOW WITH RECIRCULATION CORES

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Abstract

Large amplitude Rossby waves riding on a background flow with a weak shear can be calculated up to a critical amplitude for which the meridional velocity, in a frame travelling with the wave, approaches zero at some point. Here we consider waves with an amplitude slightly greater than the critical amplitude by incorporating a region of recirculating fluid (vortex core) near this critical point. The effect of the vortex core is to introduce an extra nonlinear term into the equation for the wave amplitude proportional to the $3/2$ power of the difference between the wave amplitude and the critical amplitude. The main effect due to the vortex core is a broadening of the wave profile. Further we show that the vortex core family has a limiting amplitude, with the limiting amplitude corresponding to a semi-infinite bore.

1 Introduction

Rossby waves are a fundamental component of planetary scale atmospheric and oceanic flows. In the presence of background flows, they may develop critical layers where strong wave-mean flow interactions can develop, possibly leading to wave-breaking and eventual saturation (see McIntyre and

Killworth [1] and Haynes [2] for instance). One possible scenario for the outcome of a critical layer interaction is the emergence of a steady state, consisting of a wave-like structure containing a vortex core, that is, a zone of recirculating fluid. In this paper, we use the simplest model of nonlinear Rossby waves, namely the barotropic quasi-vorticity equation, and show that in certain circumstances it is possible to construct explicitly such steady states.

Our work is based on the pioneering paper of Benney [3], who showed that finite amplitude steady long Rossby waves, riding on a background flow with a weak shear, are governed by a generalized Korteweg-de Vries (KdV) equation; this contains the typical KdV dispersion term, but the nonlinearity is not necessarily quadratic, but instead depends on the specific shear profile. Warn [4] included weak transient effects into this analysis, and showed that the temporal term becomes non-local in this case, reducing to the conventional time derivative only in the small amplitude limit. However, the above-mentioned studies assume that there is no stagnation point or closed streamlines in the flow. that is, there is an implicit limitation on the wave amplitude. In our earlier paper on large amplitude internal waves [5] we proposed a technique capable to deal with waves of sufficiently large amplitude which may contain regions of recirculating fluid. In this paper we apply the same technique for long large amplitude Rossby waves, thus generalizing the earlier paper by Benney [3] to the case of waves containing stagnation points in their interior.

2 Formulation

We consider the usual barotropic quasi-geostrophic vorticity equation on a mid-latitude β -plane.

$$(\nabla^2 \psi)_t + J(\nabla^2 \psi, \psi) + \beta \psi_x = 0, \quad (1)$$

where $J(a, b) = a_x b_y - a_y b_x$ and ∇^2 is the usual Laplacian operator. Here ψ is the streamfunction such that the respective velocity fields are given by

$$u = \psi_y, \quad v = -\psi_x. \quad (2)$$

We suppose that the flow is in a meridional channel, bounded by rigid latitudinal boundaries placed at $y = y_0$ and $y = y_0 + H$, while as $x \rightarrow \pm\infty$ the flow becomes a weakly sheared mean flow, given by

$$U(y) = U_0(1 + \bar{\alpha}g(y)), \quad \bar{\alpha} \ll 1 \quad (3)$$

Here $\bar{\alpha}$ is the determining small parameter of the problem. We will be seeking steady travelling wave solutions of (1), so that in the frame of reference of the wave, and after one integration, (1) becomes

$$\nabla^2\psi - \beta(y - y_0) = \bar{F}(\psi - cy). \quad (4)$$

Here $\bar{F}$ is an arbitrary function; for those streamlines which pass upstream it can be determined from the upstream flow, but has to be determined otherwise on any closed streamlines.

Next we can introduce the dimensionless variables

$$\psi' = \frac{\psi - cy}{(U_0 - c)H}, \quad y' = \frac{(y - y_0)}{H}, \quad x' = \frac{x}{H}, \quad c' = \frac{c}{U_0}. \quad (5)$$

Here H is a typical length scale in the problem, which in our case is the width of the channel. Then equation (4) becomes, on omitting the prime superscripts, and redefining the arbitrary function $\bar{F}$ appropriately,

$$\nabla^2\psi - \frac{\beta H^2}{U_0} \frac{y}{1 - c} = F(\psi) \quad (6)$$

The boundary conditions are that

$$\psi_x = 0 \quad \text{at} \quad y = 0, 1, \quad (7)$$

$$\text{and} \quad \psi = y + \alpha \int_0^y g(\xi) d\xi, \quad \text{as} \quad x \rightarrow \pm\infty, \quad \text{where} \quad \alpha = \bar{\alpha}/(1 - c). \quad (8)$$

Here we have redefined the function $g(y)$ in (3) appropriately.

In this paper we will consider the case of Rossby waves in a narrow channel, that is we assume that the wavelength is much greater than the channel width, so that we put

$$\psi = \psi(y, X) \quad \text{where} \quad X = \epsilon x, \quad \epsilon \ll 1. \quad (9)$$

The relationship between the small parameters ϵ and α will be determined later by requiring an optimum balance between nonlinearity and dispersion in the asymptotic expansion.

For streamlines originating upstream, we can determine F from the boundary condition (8) as follows,

$$F(\psi) = -\frac{\beta H^2}{U_0} \frac{\psi}{1-c} + \frac{\beta H^2}{U_0} \frac{\alpha}{1-c} \int_0^\psi g(\chi) d\chi + \alpha \frac{dg(\psi)}{d\psi} + O(\alpha^2). \quad (10)$$

Equation (6) then becomes

$$\psi_{yy} + \epsilon^2 \psi_{xx} + \lambda(\psi - y) = \lambda \alpha \int_0^\psi g(\chi) d\chi + \alpha \frac{dg(\psi)}{d\psi} + O(\alpha^2), \quad (11)$$

$$\text{where } \lambda = \frac{\beta H^2}{U_0(1-c)} \quad (12)$$

We will show that equation (11) can be solved asymptotically with the required boundary conditions (7, 8) for a wave whose amplitude lies below a certain critical value. Above this value, a recirculation zone region with closed streamlines appears inside the wave.

In the absence of a recirculation zone (11, 7, 8) provide a complete formulation of the problem. The crucial assumption is that all streamlines must originate upstream. This condition is satisfied when

$$\psi_y > 0, \quad (13)$$

everywhere. That is, the X -component of the velocity in the frame of reference moving with the wave speed is everywhere positive. In this paper we will examine the case when (13) is violated, which we shall refer to as wave breaking.

The wave breaking condition is

$$\psi_y = 0, \quad (14)$$

which defines a critical wave amplitude. We shall show that for waves with amplitudes greater than the critical value, there is a core containing recirculating fluid. The recirculation zone can be attached to either the boundary $y = 0$ or $y = 1$. We shall develop the theory for the case when the recirculation zone is attached to $y = 0$, as the other case is completely analogous. Thus we suppose that the recirculation zone is bounded by $y = 0$ and

$$y = \eta(x), \quad \text{for } |x| < x_0, \quad (15)$$

while $\eta(x) = 0$ for $|x| > x_0$, where x_0 denotes the half-width of the recirculation zone.

Outside the recirculation zone, the flow is determined by (11) with the boundary conditions

$$\psi = 1 + \alpha \int_0^1 g(\chi) d\chi \quad \text{at} \quad y = 1, \quad (16)$$

$$\text{and} \quad \psi = 0 \quad \text{at} \quad y = \eta(x), \quad (17)$$

as well as condition (8) at infinity. There is no wave breaking outside the recirculation zone as (13) is then always satisfied.

Inside the recirculation zone, equation (6) is still valid but the exact form of $F(\psi)$ cannot be deduced from the condition at infinity, so we must leave it as undetermined at present. It will emerge that its precise form is not needed in our asymptotic construction. Finally, since $\eta(x)$ is to be determined as part of the full solution, we need to impose the dynamic condition of continuity of pressure across the recirculation zone boundary. It can be shown that this leads to continuity of the velocity field so that,

$$[\psi_y] = 0 \quad \text{at} \quad y = \eta(x). \quad (18)$$

We shall refer the region where $1 > y > \eta(x) > 0$ as the *inner* zone, i.e. $|x| < x_0$. Then $0 < y < 1$, $|x| > x_0$ is the *outer* zone. The *recirculation* zone is $0 < y < \eta(x)$, $|x| < x_0$.

3 Asymptotic solution

3.1 Outer zone

In (9) we have already introduced the small parameter ϵ which ensures that the typical scale of a wave in the x -direction (i.e. along a latitude) is much greater than the channel width; we recall that $X = \epsilon x$ is the new variable. It is then reasonable to assume that the dispersive term $\sim \epsilon^2$ should be balanced by nonlinear terms due to a weak variation in the zonal shear, that is, terms $\sim \alpha$ in (11), and so we put

$$\alpha \sim \epsilon^2. \quad (19)$$

We then seek a solution of (11, 16, 17, 8) in the form

$$\psi(y, X) = \psi^{(0)}(y, X) + \epsilon^2 \psi^{(1)}(y, X) + \dots, \quad (20)$$

$$\lambda = \lambda^{(0)} + \epsilon^2 \lambda^{(1)} + \dots. \quad (21)$$

It is useful to note that the expansion (21) in fact determines the asymptotic expansion for the wave velocity in the laboratory frame of reference, and from (12) we have

$$c = 1 - \frac{\beta H^2}{U_0 \lambda^{(0)}} + \epsilon^2 \frac{\lambda^{(1)}}{\lambda^{(0)}} \frac{\beta H^2}{U_0 \lambda^{(0)}} + \dots. \quad (22)$$

At the zeroth order, we obtain that

$$\psi^{(0)}(y, X) = y + A(X) W(y), \quad (23)$$

$$\text{where } W(y) = \sin(\pi y), \quad (24)$$

$$\text{and } \lambda^{(0)} = \pi^2. \quad (25)$$

The amplitude function $A(X)$ is to be determined from the compatibility condition at the next order in the expansion. We note that (24) represents the the lowest possible mode ($n = 1$), which is also the fastest as $1 - c$ decreases as $\lambda_n^{(0)} = (\pi n)^2$ increases for higher modes with $n \geq 2$.

Next it is convenient to introduce

$$\Psi^{(1)}(y, X) = \psi^{(1)}(y, X) - \int_0^y g(\hat{y}) d\hat{y}. \quad (26)$$

After some straightforward algebra we will arrive at the equation for $\Psi^{(1)}(y, X)$,

$$\Psi_{yy}^{(1)} + \lambda^{(0)} \Psi^{(1)} = S^{(1)}, \quad (27)$$

$$\text{where } S^{(1)} = \frac{\alpha}{\epsilon^2} \left[\frac{dg}{dy} \right]_y^{\psi^{(0)}} + \frac{\alpha}{\epsilon^2} \lambda^{(0)} \int_y^{\psi^{(0)}} g(\hat{y}) d\hat{y} - \psi_{XX}^{(0)} - \lambda^{(1)} (\psi^{(0)} - y). \quad (28)$$

This equation is to be solved with the boundary conditions

$$\Psi^{(1)} = 0 \quad \text{at } y = 0, 1. \quad (29)$$

The solvability condition for the problem (27, 29) is

$$\int_0^1 S^{(1)} W(y) dy = 0. \quad (30)$$

Then substitution of (28) into (30) finally yields the amplitude equation

$$A_{XX} + \lambda^{(1)} A + M(A) = 0, \quad (31)$$

$$\text{where } M(A) = -\frac{2\alpha}{\epsilon^2} \int_0^1 W(y) dy \left\{ \int_y^{y+AW(y)} G(\hat{y}) d\hat{y} \right\}, \quad (32)$$

$$\text{and } G(y) = \lambda^{(0)} g(y) + \frac{d^2 g(y)}{dy^2}. \quad (33)$$

The wave-breaking condition (14), when evaluated at leading order yields

$$1 + \pi A \cos(\pi y) = 0.$$

This is achieved at the boundary $y = 0, 1$ according as $A < 0, A > 0$ and we see that correspondingly $A^* = \mp 1/\pi$. Thus the results described above are valid whenever the amplitude $|A| \leq |A_*| = 1/\pi$. Note that $A < 0$ for the case when the recirculation zone is attached to the southern boundary of the channel ($y = 0$), while $A > 0$ for the case when the recirculation zone is attached to the northern boundary ($y = 1$). As we have already mentioned, only the former case $A < 0$ shall be considered here.

Note that if the solution of (31) is such that $|A| < 1/\pi$ for *all* X , then there is no inner zone or recirculation zone, and the solution is completed at this stage. Equations similar to (32) are now quite common in the studies of finite non-breaking waves in a "shallow" environment, and have been discussed by Benney and Ko [6], Clarke and Grimshaw [7] for the case of internal waves in stratified flows, and by Benney [3] and Warn [4] for the case of Rossby waves.

On the other hand, the case of interest here is when A achieves the critical value $-A_*$ for some value of X . We shall suppose that this occurs at $x = \pm x_0$ or $X = \pm X_0$ where $X_0 = \epsilon x_0$, and that for all $|X| > X_0, |A| < A_*$. In this case equation (32) holds only in the outer zone $|X| > X_0$ outside the inner zone and recirculation zone, and so we must obtain a solution in the inner zone, and then match it to the solution in the outer zone. Moreover, in the inner zone a solution outside the recirculation core must be matched with an appropriate solution inside this core.

3.2 Inner zone

In the region $|x| < x_0$, which we refer to as the inner zone, we shall suppose that a long length-scale assumption (in the latitudinal direction) is still valid, but the actual aspect ratio is different from that in the outer zone. Thus (20, 21) still hold but the slow variable X is replaced by another slow variable ξ , so that

$$\xi = \gamma x, \quad \gamma \ll 1 \quad (34)$$

$$\psi(y, \xi) = \psi^{(0)}(y, \xi) + \epsilon^2 \psi^{(1)}(y, \xi) + \dots \quad (35)$$

$$\text{and } \psi^{(0)}(y, \xi) = y + A(\xi) W(y). \quad (36)$$

Next we suppose that in the inner zone, the amplitude $A(\xi)$ is close to the critical amplitude, which for a wave with a recirculation zone attached to southern wall ($y = 0$), is $A_* = -1/\pi$ as $A < 0$ in this case. Thus we put

$$A(\xi) = A_* - \mu B(\xi), \quad \text{where } \mu \ll 1 \quad \text{and } B \geq 0. \quad (37)$$

The relationship of μ to the other small parameters will be determined later. The recirculation zone itself occupies the region $0 < y < \eta(\xi)$ where we now put

$$\eta(\xi) = \delta f(\xi), \quad \text{where } \delta \ll 1. \quad (38)$$

The condition $\delta \ll 1$ implies that the height (that is, the extent in the latitudinal direction) of the recirculation zone is small compared with the width of the channel.

On substituting (35, 37) into (11), and using (26) again, we get for the first order terms,

$$\Psi_{yy}^{(1)} + \lambda^{(0)} \Psi^{(1)} = R^{(1)}, \quad (39)$$

$$\text{where } R^{(1)} = \left[\frac{dg}{dy} \right]_y^{y+A_*W} + \lambda^{(0)} \int_y^{y+A_*W} g(\hat{y}) d\hat{y} + B_{\xi\xi} W - \lambda^{(1)} A_* W. \quad (40)$$

Here, in order to keep all terms in (39) of the same order, we put

$$\mu \gamma^2 = \epsilon^2 \quad \text{and} \quad \alpha = \epsilon^2. \quad (41)$$

The problem (39) is to be solved with the boundary conditions

$$\Psi^{(1)} = 0 \quad \text{at} \quad y = 1, \quad (42)$$

and the conditions (17. 18) at the vortex boundary. The solvability condition for this problem then yields

$$B_{\xi\xi} - \lambda^{(1)} A_* - M(A_*) = 2\pi \Psi^{(1)}(0, \xi). \quad (43)$$

Here we have used the limit $\delta \ll 1$ to replace the boundary condition at $y = \eta$ with an approximate boundary condition at $y = 0$.

Next, to determine $\Psi^{(1)}(0, \xi)$ we use the kinematic condition at the boundary of the recirculation zone. Thus from (17) we get that

$$-\frac{\pi^2 \delta^3 f^3}{6} - \mu \pi \delta B f + \epsilon^2 \Psi^{(1)}(0, \xi) = 0. \quad (44)$$

To ensure that all terms in (44) are of the same order, we must now put

$$\delta^3 = \mu \delta = \epsilon^2. \quad (45)$$

Thus, using these relations and (41) we finally determine all the small parameters in terms of ϵ where we recall that $\alpha = \epsilon^2$,

$$\gamma = \epsilon^{1/3} \quad \delta = \epsilon^{2/3}, \quad \mu = \epsilon^{4/3}, \quad (46)$$

Finally, in order to apply the dynamic boundary condition (18) we note that

$$\lim_{y \rightarrow \eta+0} \psi_y = \frac{\pi^2 \delta^2 f^2}{2} - \mu \pi B(\xi) + \dots, \quad (47)$$

In the next section we will construct the solution inside the recirculation zone. It will be shown that

$$\lim_{y \rightarrow \eta-0} \psi_y = \frac{\pi^2 \delta^2 f^2}{3} + \frac{\delta^2 f H_0}{2}, \quad (48)$$

where H_0 is the vorticity on the recirculation zone boundary (that is, on $y = 0$ and $y = \eta(x)$). This constant is not yet determined in our procedure. From (47, 48) we finally obtain an expression for $B(\xi)$

$$\frac{\pi^2 f^2}{3} - f H_0 = \frac{2 \mu \pi B(\xi)}{\delta^2} \quad (49)$$

$$\text{or } f = \frac{3H_0}{2\pi^2} \pm \sqrt{\left(\frac{3H_0}{2\pi^2}\right)^2 + \frac{6 \mu B(\xi)}{\delta^2 \pi}} \quad (50)$$

But if $H_0 > 0$ both solutions of (50) are not allowed, since we require that $f \geq 0$ and $f(B=0) = 0$ (recall from (37) that $B \geq 0$). On the other hand, if $H_0 < 0$, then the requirement that $\psi_y \geq 0$ in (48) for all $B \geq 0$ implies that $f > -3H_0/2\pi^2$ for all $B > 0$. This is impossible. Hence we conclude that we must put $H_0 = 0$ and (50) becomes

$$f = \sqrt{\frac{6 \mu B(\xi)}{\delta^2 \pi}}. \quad (51)$$

Thus, on using (44) and (51) we finally get the required equation for B from (43), that is,

$$B_{\xi\xi} - \lambda^{(1)} A_* - M(A_*) = \frac{2}{3} \nu B^{3/2}. \quad (52)$$

$$\nu = (6\pi)^{3/2} \frac{\mu^{3/2}}{\epsilon^2}, \quad (53)$$

where ν is the supercriticality parameter.

Thus we have now determined the equations governing the wave amplitude, both in the outer zone, $|X| > X_0$, (31) and in the inner zone, $|X| < X_0$ (52). To complete this formulation, we also need the matching conditions,

$$A = A_*, \quad B = 0, \quad \text{at} \quad X = \pm X_0 \quad (54)$$

$$A_X = -\mu^{1/2} B_\xi, \quad \text{at} \quad |X| = X_0 \quad (55)$$

Here we recall that X_0 is defined as the place where A reaches A_* in the outer solution.

3.3 Recirculation zone

Finally, we need to consider the solution inside the recirculation zone. We argue that the vorticity inside the recirculation zone is of the same order as the vorticity just outside the recirculation zone, i.e. $O(\delta)$, and hence replace (6) with

$$\psi_{yy} + \gamma^2 \psi_{xx} - \lambda y = \delta H(\psi). \quad (56)$$

Here the scaled vorticity $H(\psi)$ is undetermined at this stage. The boundary conditions are

$$\psi = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = \delta f(x). \quad (57)$$

It is now useful to rescale as follows,

$$\psi = \delta \Psi, \quad y = \delta Y. \quad (58)$$

Thus, recalling from (46) that $\gamma^2 = \delta$, and using the transformation (34), we obtain the following equation and boundary conditions for the streamfunction inside the recirculation zone,

$$\Psi_{YY} + \delta^3 \Psi_{\xi\xi} = \delta^2 H_0 + \delta^3 H_1 \Psi + \delta^2 \lambda Y + O(\delta^4), \quad (59)$$

$$\Psi = 0 \quad \text{at} \quad Y = 0 \quad \text{and} \quad Y = f(\xi). \quad (60)$$

where $H_0 = H(0)$, and $H_1 = H_\psi(0)$.

We search for a solution of (59–60) in the form

$$\Psi(Y, \xi) = \Psi^{(0)}(Y, \xi) + \delta^2 \Psi^{(1)}(Y, \xi) + \dots, \quad (61)$$

The solution is $\Psi^{(0)}(Y, \xi) = 0$ and

$$\Psi^{(1)}(Y, \xi) = \lambda \frac{Y}{6} (Y^2 - f^2) + H_0 \frac{Y}{2} (Y - f). \quad (62)$$

$$\text{and so} \quad \Psi_Y^{(1)}(Y, \xi) = \frac{\lambda}{2} (Y^2 - \frac{f^2}{3}) + H_0 (Y - \frac{f}{2}). \quad (63)$$

Thus, on using (58) we obtain (48), that is,

$$\lim_{y \rightarrow \delta f - 0} \psi_y^{(1)} = \delta^2 \lim_{Y \rightarrow f - 0} \Psi_Y^{(1)} = \frac{\pi^2 \delta^2 f^2}{3} + \frac{\delta^2 f H_0}{2}. \quad (64)$$

But, as we have shown earlier that $H_0 = 0$, we see that

$$\Psi^{(1)}(Y, \xi) = \pi^2 \frac{Y}{6} (Y^2 - f^2(\xi)). \quad (65)$$

Note that the leading order expression (25) for λ has been used in (64, 65).

4 Solutions of the amplitude equation

The amplitude equations (31, 52) have the same structure as those found by Derzho and Grimshaw [5] in an analogous problem for internal waves. Hence

it is sufficient here to just give a brief summary of the solution procedure. First, equation (31) can be integrated once to yield

$$A_X^2 + \lambda^{(1)} A^2 + 2 \int_0^A M(\hat{A}) d\hat{A} = 0, \quad (66)$$

where the conditions $A_X \rightarrow 0$ and $A \rightarrow 0$ as $X \rightarrow \pm\infty$ have been used. At the end points of the inner zone, the matching conditions (54, 55) give

$$\mu B_\xi^2 + \lambda^{(1)} A_*^2 + 2 \int_0^{A_*} M(A) dA = 0. \quad (67)$$

Next, on using the assumed symmetry of the problem, we can set

$$B_\xi = 0 \quad B = 1 \quad \text{at} \quad \xi = 0. \quad (68)$$

Here, we have normalized B to have a maximum value of 1, which in effect determines the value of μ (see (37)). Now, equation (52) also can be integrated once to yield

$$B_\xi^2 - 2 (\lambda^{(1)} A_* + M(A_*)) B = \frac{8\nu}{15} B^{5/2} + \text{constant}. \quad (69)$$

The constant of integration in (69) can be determined using (68), and so we arrive at

$$B_\xi^2 = 2 (\lambda^{(1)} A_* + M(A_*)) (B - 1) + \frac{8\nu}{15} (B^{5/2} - 1) \quad (70)$$

We now can substitute for $\lambda^{(1)}$ from (67), and finally obtain

$$B_\xi^2 = (1 - B) R(A_*) - \frac{8\nu}{15} (1 - B^{5/2}), \quad (71)$$

$$\text{where} \quad R(A_*) = \frac{4}{A_*} \int_0^{A_*} M(A) dA - 2 M(A_*). \quad (72)$$

The explicit solution of (71) is

$$|\xi| = \int_B^1 \left\{ (1 - u) R(A_*) - \frac{8\nu}{15} (1 - u^{5/2}) \right\}^{-1/2} du, \quad (73)$$

and the half-width of the recirculation zone is then given by the following expression

$$|\xi_0| = \int_0^1 \left\{ (1-u) R(A_*) - \frac{8\nu}{15}(1-u^{5/2}) \right\}^{-1/2} du. \quad (74)$$

Note that ξ_0 is a function of ν , and that ξ_0 increases as ν increases. Also, note that since the left-hand side of (71) must be positive for all $0 < B < 1$ it follows that

$$R(A_*) > \frac{4\nu}{3}. \quad (75)$$

Here equality holds in the limit $\xi_0 \rightarrow \infty$. Thus as the amplitude of the recirculation zone, measured here by ν , increases, so does the half-width. The maximum amplitude ν_* for a recirculation zone of infinite width is determined by

$$R(A_*) = \frac{4\nu_*}{3}. \quad (76)$$

As an illustration, consider the special case of a linear shear $g(y) = y$, so using (32) we can calculate $M(A)$

$$M(A) = \alpha_1 A + \alpha_2 A^2, \quad (77)$$

$$\text{where } \alpha_1 = -\frac{\alpha \pi^2}{\epsilon^2 2}, \alpha_2 = -\frac{\alpha 4 \pi}{\epsilon^2 3}. \quad (78)$$

The solution in the outer zone is found from (66)

$$A(X) = A_0 \text{sech}^2(\gamma_*(|X| - X_*)), \quad (79)$$

$$\text{where } A_0 = A_* - \mu - \frac{4 \mu \nu}{5 \alpha_2 A_*^2}, \quad (80)$$

$$\gamma_*^2 = \frac{1}{6} \alpha_2 (A_* - \mu) - \frac{2}{15} \frac{\mu \nu}{A_*^2}, \quad (81)$$

$$\text{and } \lambda^{(1)} = -\alpha_1 - \frac{2}{3} \alpha_2 (A_* - \mu) + \frac{8}{15} \frac{\mu \nu}{A_*^2}. \quad (82)$$

Now we can determine X_* using the requirement $A(X_0) = A_*$, so that

$$\mu \left(\frac{9 \pi}{2} - \frac{27 \pi^2 \nu}{10} \right) = (X_0 - X_*)^2. \quad (83)$$

Thus the required solution is completely defined. In particular, (72) is

$$R(A_*) = \frac{8}{9\pi^2} \frac{\alpha}{\epsilon^2}, \quad \nu_* = \frac{2}{3\pi^2} \frac{\alpha}{\epsilon^2}. \quad (84)$$

These expressions determines the maximum amplitude of the wave with recirculation zone as

$$\mu_* = \left(\frac{2}{3\pi^2} \frac{\alpha}{\epsilon^2} \right)^{2/3} \frac{1}{6\pi} \quad (85)$$

A typical wave profiles is shown in Fig.2 and 3, while Fig. 1 shows how the recirculation zone expands as the wave amplitude increases.

Also (67) yields

$$\lambda^{(1)} = -\frac{\alpha}{\epsilon^2} \frac{7\pi^2}{18} - \mu \left[2\nu^* - \frac{8\nu}{15\pi^2} \right], \quad (86)$$

so that the wave is supercritical as expected, that is, the magnitude of the speed is increased compared to the linear wave speed (see (22)). Here the first term, which is negative, is the value predicted by a KdV theory which ignores the recirculation zone, and the second term in brackets, proportional to μ , is the correction due to the presence of the recirculation zone. We see that this correction is also negative, and so increases the speed magnitude slightly.

Figure 1(a): The shape of the recirculation zone for several wave amplitudes; the recirculation zone becomes larger and wider as the modulus of the amplitude increases. The particular cases shown correspond to $A_* = 0.5\mu_*$; $A_* = 0.9\mu_*$; $A_* = 0.999\mu_*$.

Figure 1(b): The shape of the recirculation core, and flow inside the core for the case $A_* = 0.999\mu_*$. In both cases, (a) and (b), the ambient shear is linear, with $\alpha = 0.2$. The limiting wave amplitude is $A_{\max} = A_* - \mu_*$, $\mu_* = 0.003006$.

5 Higher modes and interior recirculation zones

To this point we have considered just the lowest mode ($n = 1$). Analogous constructions can be made for the higher modes ($n \geq 2$). For instance, let us briefly consider the scenario for the second mode, $n = 2$, for which the modal function $W(y) = \sin(\pi y)$ and eigenvalue $\lambda^{(0)} = \pi^2$ (23, 24, 25) are replaced by

$$W_2(y) = -\sin(2\pi y) \quad \text{and} \quad \lambda_2^{(0)} = (2\pi)^2. \quad (87)$$

The negative sign in (36) is purely for convenience. The outer zone expansion now proceeds just as in Section 3.1, and the outcome is again the amplitude equation (31), where $W(y)$ and $\lambda^{(0)}$ are replaced by $W_2(y)$ and $\lambda_2^{(0)}$ respectively in the expressions (32, 33).

However, the wave-breaking condition (14) now yields

$$1 - 2\pi A \cos(2\pi y) = 0.$$

This can be achieved either at the boundaries $y = 0, 1$ for $A > 0$ with $A_* = 1/2\pi$, or in the interior at $y = 1/2$ for $A < 0$ with $A_* = -1/2\pi$. The former case leads to boundary recirculation zones similar to those described in Section 3, and we shall not give any further details here. However, the latter case leads to a recirculation zone in the interior of the channel, and we shall now give a brief summary of its structure.

Figure 2: The streamline pattern for the case of a wave amplitude $A_* = 0.5\mu_*$.

Figure 3: The streamline pattern for the case of a wave amplitude $A_* = 0.999\mu_*$.

First, let us note that for this second mode there is an antisymmetry about the midline $y = 1/2$; in effect, there is a rigid boundary at $y = 1/2$. This antisymmetry is present also in the full equations (1); hence if the upstream shear $g(y)$ also has this antisymmetry, then we can analyze the interior recirculation zone just as in Section 3 by placing a rigid boundary at $y = 1/2$ and then considering only the region $1/2 < y < 1$. The outcome is a small recirculation zone just above $y = 1/2$. When we then use the antisymmetry to include the region $y < 1/2$, we see that the full interior recirculation zone is a dipole, with two oppositely directed circulations. There is an obvious analogy here with the well-known modon constructions.

Next, we give a brief account of the construction of this interior recircula-

tion zone where we do not invoke antisymmetry. The outer zone construction proceeds as indicated above in close analogy with that described in Section 3.1. The recirculation zone is now described by

$$-\eta_-(\xi) < y - \frac{1}{2} < \eta_+(\xi), \quad \text{where} \quad \eta_{\pm} = \delta f_{\pm}(\xi). \quad (88)$$

The inner expansion now proceeds just as in Section 3.2, except that it is now valid in the regions

$$0 < y < \frac{1}{2} - \eta_-(\xi), \quad \text{and} \quad \frac{1}{2} + \eta_+(\xi) < y < 1.$$

We then obtain the counterpart of (27) and the same balance of parameters as in (19). The solvability condition then again leads to (43) where now the right-hand side is replaced with

$$8\pi\Psi^{(1)}(\frac{1}{2}+) = -8\pi\Psi^{(1)}(\frac{1}{2}-). \quad (89)$$

Thus, antisymmetry is found to hold at this leading order, and so, again to leading order,

$$f_-(\xi) = f_+(\xi) = f(\xi).$$

We readily find that the counterpart of (65) is

$$\Psi^{(1)}(Y, \xi) = 4\pi^2 \frac{Z}{6} (Z^2 - f^2(\xi)) \quad \text{where} \quad Z = Y - \frac{1}{2}. \quad (90)$$

The dipole structure is immediately evident. Further, with $\lambda^{(0)}$ replaced by $\lambda_2^{(0)}$ where appropriate, the counterparts of (51) and (52) for the determination of f and B hold respectively; essentially, the same expressions are valid, with π replaced with 2π .

Similar constructions can be made for the higher modes, $n \geq 3$. For instance the third mode with $W(y) = \sin(3\pi y)$ has potential recirculation zones at $y = 0$ and $y = 2/3$ when $A > 0$, or at $y = 1$ and $y = 1/3$ when $A < 0$. But note that now a boundary and an interior recirculation zone appear simultaneously.

6 Conclusion

In this paper we have presented an analytical model for long Rossby waves in a meridional channel with weak shear. The main result is that we can describe waves with a small zone of recirculating fluid whenever the wave amplitude slightly exceeds the critical value for which overturning occurs. Most of our discussion has been for the lowest mode, but we have also briefly indicated the extension to higher modes.

We have derived a new nonlinear wave equation (Eqs. (52) or (71)) which governs the amplitude of large Rossby waves; this equation is valid in the inner zone of the flow (i.e. the region where fluid flows over the recirculation core). Flow inside the recirculation (vortex) core has also been obtained (see (65)) as a result of a rigorous asymptotic procedure. The solution of (71) is matched with (31), which holds in the outer zone. Equation (31) was originally derived by Benney [3] for waves with amplitudes below the critical amplitude, for which there are no closed streamlines. The governing equations for the inner and outer zones are both of the KdV type where weak linear dispersion is balanced by weak nonlinearity. Despite the large amplitude of the disturbances, the resulting nonlinearity in the outer zone is weak because of the weak shear in the background flow, and the fact that the governing equation is exactly linear when the shear is zero; thus the specific form of the nonlinear term in the outer zone depends on the specific shear profile. The inner zone nonlinearity is also weak since the size of the recirculation zone is small; however the inner zone nonlinearity does not depend on the details of the upstream shear. A novel nonlinear term proportional to the $3/2$ power of the difference between the wave amplitude and the critical amplitude then arises to account for a nonlinearity due to the flow over the vortex core.

We have found that the amplitude of the stationary solution with a vortex core is bounded (see Eq. (85)), and this limiting amplitude is generally quite close to the critical amplitude (at which overturning first occurs); thus our initial assumption is confirmed. Our analysis also reveals that for waves with vortex cores the width of the wave at the crest increases as the amplitude approaches the maximum value. The wave of maximum amplitude is a (semi-infinite) bore. Concerning the wave speed, we find that a wave with a vortex core is a little faster than a KdV theory without a vortex core would predict

for the same amplitude.

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Figure captions

Fig.1. (a) The shape of the recirculation zone for several wave amplitudes; the recirculation zone becomes larger and wider as the modulus of the amplitude increases. The particular cases shown correspond to $A_* - 0.5\mu_*$; $A_* - 0.9\mu_*$; $A_* - 0.999\mu_*$; (b) The shape of the recirculation core, and flow inside the core for the case $A_* - 0.999\mu_*$. In both cases, (a) and (b), the ambient shear is linear, with $\alpha = 0.2$. The limiting wave amplitude is $A_{\max} = A_* - \mu_*$, $\mu_* = 0.003006$

Fig.2. The streamline pattern for the case of a wave amplitude $A_* - 0.5\mu_*$.

Fig.3. The streamline pattern for the case of a wave amplitude $A_* - 0.999\mu_*$.