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## **Scattering coefficients for a sphere in a visco-acoustic medium for arbitrary partial wave order**

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Pinfield, Valerie, Pierre Marechal, and M Mahbub Alam. 2020. "Scattering Coefficients for a Sphere in a Visco-acoustic Medium for Arbitrary Partial Wave Order". Loughborough University.  
<https://doi.org/10.17028/rd.lboro.12173667.v2>.

- > Analytical solution for the scattering coefficients for n'th order
- > using the nonthermal boundary equation matrix
- > to derive coefficients  $T_n$  for compressional and shear modes
- > for small compressional wavenumber  $kca$
- > by Valerie Pinfield, Loughborough University, April 2020
- > In Maple 2015.0 (Maplesoft)
- > Paper submitted to WaveMotion 2020.
- > "Scattering coefficients for a sphere in a visco-acoustic medium for arbitrary partial wave order"
- > by M. Mahbub Alam, Valerie J. Pinfield, Pierre Maréchal
- > Using an incident potential for both compressional or shear incident wave of partial wave type  $j_n(kr)P_n(\cos\theta)$
- > The incident shear wave is taken to be of the form  $u = \text{curl curl}(X \cdot \mathbf{r} \cdot \mathbf{e}_r)$  where  $\mathbf{e}_r$  is unit vector in radial direction.
- > The potential  $X$  is taken to be of plane wave type i.e.  $X = \sum_n i^n (2n+1) j_n(ksr) P_n(\cos\theta)$
- > Note that this is not a planar shear incident wave in displacement or velocity.

Using  $u = +\text{grad}(\phi)$  convention for compressional waves (so opposite to Pinfield's papers with Forrester)

All coefficients are scaled by the relevant bessel or hankel function to remove factors of  $x^n$  and the incident wave potential is also scaled.

Equations are in the order  $u_r, u_\theta, \sigma_{rr}, \sigma_{r\theta}$  ( $u$  is displacement,  $\sigma$  is stress)

Coefficients are in the order  $T_n^I C, T_n^I S, B_n^I C, B_n^I S$  where

$T$  are outside coefficients,  $B$  are inside coefficients, and  $I$  represents incident wave type

SCALE all wave potentials so that the terms in  $x^n$  are removed.

Scaling is as follows:

incident waves divided by  $j_n(kIa)$  where  $I$  indicates incident wave type  $C$  or  $S$

$T_n^I C = T_n^I C_{\text{scaled}} * j_n(kIa) / h_n(kca)$

$T_n^I S = T_n^I S_{\text{scaled}} * j_n(kIa) / h_n(ksa)$

so with coefficients matrix in order  $(T_n^I C, T_n^I S, B_n^I C, B_n^I S)$  the columns are scaled as:

$(/h_n(kca) /h_n(ksa) /j_n(kc'a) /j_n(ks'a))$

and incident matrix RHSC is  $/j_n(kca)$  and RHSS is  $/j_n(ksa)$

(primed denotes inside scatterer)

Initially make no assumptions about magnitude of  $y_s'$ , later expand solution as series in  $y_s'$

This is because the  $n=1$  results seem to come from the second term in series in  $y_s'$  ( $kspa$ ) in numerator and denominator

So obtain the leading order term in a series in  $kca$ , and then take a series in  $kspa$  of numerator and denominator

Solutions for the  $n=1$  coefficient were presented in a previous paper Pinfield and Forrester, Journal of the Acoustical Society of America, 141, 649 (2017); doi: 10.1121/1.4974142. These results have been used to identify simplifications for the formulae for general  $n$  here.

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Set up boundary equation in matrix form. Notation:  $kca, ksa, kcpa, kspa$  are wavenumbers  $\times$  radius for compressional, shear outside then inside

$jnp\_jnk^{**}a$  and similarly for  $hnp\_hn$  are ratios of  $j_n'(k^{**}a)/j_n(k^{**}a)$  etc.  $\rho_C$  and  $\rho_P$  are densities in embedding medium and particle respectively

> restart;

> with(LinearAlgebra) :

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[> LHS := Matrix(4, 4) :
[> RHSC := Matrix(4, 1) : # incident compressional wave
[> RHSS := Matrix(4, 1) : # incident shear wave
[> RHSS:
[> LHS(1, 1) := kca·hnp_hnkca :
[> LHS(2, 1) := 1 :
[> LHS(3, 1) :=  $\frac{\rho C}{(ksa \cdot ksa)} \cdot ((ksa \cdot ksa - 2 \cdot n \cdot (n + 1)) + 4 \cdot kca \cdot hnp\_hnkca) :$ 
[> LHS(4, 1) :=  $\frac{2 \cdot \rho C}{ksa \cdot ksa} \cdot (1 - kca \cdot hnp\_hnkca) :$ 
[> LHS(1, 2) := n·(n + 1) :
[> LHS(2, 2) := (1 + ksa·hnp_hnksa) :
[> LHS(3, 2) :=  $\frac{2 \cdot n \cdot (n + 1) \cdot \rho C}{ksa \cdot ksa} \cdot (1 - ksa \cdot hnp\_hnksa) :$ 
[> LHS(4, 2) :=  $\frac{\rho C}{ksa \cdot ksa} \cdot ((ksa \cdot ksa - 2 \cdot n \cdot (n + 1) + 2) + 2 \cdot ksa \cdot hnp\_hnksa) :$ 

[> LHS(1, 3) := -kcpa·jnp_jnkcpa :
[>
[> LHS(2, 3) := -1 :
[> LHS(3, 3) :=  $-\frac{\rho P}{kspa \cdot kspa} \cdot ((kspa \cdot kspa - 2 \cdot n \cdot (n + 1)) + 4 \cdot kcpa \cdot jnp\_jnkcpa) :$ 
[> LHS(4, 3) :=  $-\frac{2 \cdot \rho P}{kspa \cdot kspa} \cdot (1 - kcpa \cdot jnp\_jnkcpa) :$ 

[> LHS(1, 4) := -n·(n + 1) :
[> LHS(2, 4) := -(1 + kspa·jnp_jnksa) :
[> LHS(3, 4) :=  $-\frac{2 \cdot n \cdot (n + 1) \cdot \rho P}{kspa \cdot kspa} \cdot (1 - kspa \cdot jnp\_jnksa) :$ 
[> LHS(4, 4) :=  $-\frac{\rho P}{kspa \cdot kspa} \cdot ((kspa \cdot kspa - 2 \cdot n \cdot (n + 1) + 2) \cdot 1 + 2 \cdot kspa \cdot jnp\_jnksa) :$ 

[> RHSC(1, 1) := -kca·jnp_jnkca : # for incident compressional waves
[> RHSC(2, 1) := -1 :
[> RHSC(3, 1) :=  $-\left(\frac{\rho C}{ksa \cdot ksa}\right) \cdot ((ksa \cdot ksa - 2 \cdot n \cdot (n + 1)) + 4 \cdot kca \cdot jnp\_jnkca) :$ 
[> RHSC(4, 1) :=  $2 \frac{\rho C}{ksa \cdot ksa} \cdot (kca \cdot jnp\_jnkca - 1) :$ 

[> RHSS(1, 1) := -n·(n + 1) : #for incident shear wave
[> RHSS(2, 1) := -(1 + ksa·jnp_jnksa) :
[> RHSS(3, 1) :=  $2 \cdot \frac{\rho C}{ksa \cdot ksa} \cdot n \cdot (n + 1) \cdot (ksa \cdot jnp\_jnksa - 1) :$ 
[> RHSS(4, 1) :=  $-\frac{\rho C}{ksa \cdot ksa} \cdot ((ksa \cdot ksa - 2 \cdot n \cdot (n + 1) + 2) + 2 \cdot ksa \cdot jnp\_jnksa) :$ 

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Expand the Bessel and Hankel functions in the compressional wavenumbers for small arguments and find the ratios of  $j_{np}/j_n$  etc. using recurrence relations to work with scaled boundary equations

Check the series expansions for  $h_{nx}$

$$> xhn\_exp := i^{(-n-1)} \cdot \frac{(2 \cdot n)! \cdot i^{(n)}}{n! \cdot (2 \cdot x)^n} \cdot \left( 1 - i \cdot x - \frac{(n-1)}{(2 \cdot n-1)} \cdot x^2 \right) \# \text{this is } h_n(x) \cdot x \text{ for } n \geq 2 \text{ ONLY}$$

$$xhn\_exp := \frac{i^{-n-1} (2n)! i^n \left( 1 - ix - \frac{(n-1)x^2}{2n-1} \right)}{n! (2x)^n} \quad (1)$$

(it works for  $n=1$  but the  $x^2$  term in the bracket is identically zero in that case since  $(n-1)=0$ . Obtained from the Epstein and Carhart (1953) (E&C) paper expressions for  $h_n$  at small arguments

$$> xhn1\_exp := \text{subs}(n = n + 1, xhn\_exp) \# \frac{h_{n+1}(x) \cdot x}{\exp(i \cdot x)}$$

$$xhn1\_exp := \frac{i^{-n-2} (2n+2)! i^{n+1} \left( 1 - ix - \frac{nx^2}{2n+1} \right)}{(n+1)! (2x)^{n+1}} \quad (2)$$

$$> hn1\_hnRatio := \text{simplify} \left( \frac{xhn1\_exp}{xhn\_exp} \right) \# \text{the ratio of } h_{n+1} \text{ over } h_n \text{ for small arguments } x$$

$$hn1\_hnRatio := \frac{(2n-1) (2inx + nx^2 + ix - 2n - 1)}{x (2inx + nx^2 - ix - x^2 - 2n + 1)} \quad (3)$$

$$> hn1\_hnxx := \text{simplify}(\text{simplify}(\text{series}(hn1\_hnRatio, x, 2)));$$

#series expression for the ratio of  $h_{n+1}$  over  $h_n$  for small arguments  $x$

$$hn1\_hnxx := \frac{2n+1}{x} + O(x) \quad (4)$$

Now obtain the ratio of the derivative to the Hankel function i.e.  $h_n'/h_n$  using the recurrence relation  $R_n'/R_n = (n/x) - R_{n+1}/R_n$

$$> hnp\_hnxx := \frac{n}{x} \cdot \left( 1 - \frac{hn1\_hnxx \cdot x}{n} \right); \# \text{ratio of } \frac{h_n'}{h_n} \text{ using recurrence relation}$$

$$hnp\_hnxx := \frac{n \left( 1 - \frac{\left( \frac{2n+1}{x} + O(x) \right) x}{n} \right)}{x} \quad (5)$$

$$> hnp\_hnxx := \text{simplify}(\text{series}(hnp\_hnxx, x, 3))$$

$$hnp\_hnxx := \frac{-n-1}{x} + O(x) \quad (6)$$

Now do the same for the Bessel functions - this is easier because they can be expanded easily as series in  $x$  - see E&C

$$> jnxx := \frac{x^n}{\text{fac\_jn}} \cdot \left( 1 - \frac{x^2}{2 \cdot (2 \cdot n + 3)} + \frac{x^4}{2 \cdot 4 \cdot (2 \cdot n + 3) \cdot (2 \cdot n + 5)} + O(x^6) \right)$$

#extend this to the 4th order term in bracket for the assumption of general kspa - it is necessary in order to get even the second term in the series in kspa of the leading term in kca of the coefficients

$$jnxx := \frac{x^n \left( 1 - \frac{x^2}{4n+6} + \frac{1}{8} \frac{x^4}{(2n+3)(2n+5)} + O(x^6) \right)}{fac\_jn} \quad (7)$$

$$> jnlxx := \frac{subs(n=n+1, jnxx)}{(2n+3)}$$

# jn+1(x) - do a manual adjustment on the factor fac\_jn which is 1·3·...(2n+1)

$$jnlxx := \frac{x^{n+1} \left( 1 - \frac{x^2}{4n+10} + \frac{1}{8} \frac{x^4}{(2n+5)(2n+7)} + O(x^6) \right)}{fac\_jn (2n+3)} \quad (8)$$

$$> jnl\_jnxx := simplify\left(\left(\frac{jnlxx}{jnxx}\right)\right)$$

$$jnl\_jnxx := \left( (x^4 + 32 O(x^6) n^2 - 8 n x^2 + 192 O(x^6) n + 32 n^2 - 28 x^2 + 280 O(x^6) + 192 n + 280) x \right) / \left( (x^4 + 32 O(x^6) n^2 - 8 n x^2 + 128 O(x^6) n + 32 n^2 - 20 x^2 + 120 O(x^6) + 128 n + 120) (2n+7) \right) \quad (9)$$

$$> simplify(series(jnl\_jnxx, x, 4))$$

$$\frac{1}{2n+3} x + \frac{1}{(2n+3)(4n^2+16n+15)} x^3 + O(x^5) \quad (10)$$

>

$$> jnp\_jnxx := simplify\left(series\left(\frac{n}{x} \cdot \left(1 - \frac{jnl\_jnxx \cdot x}{n}\right), x, 5\right)\right)$$

$$jnp\_jnxx := \frac{n}{x} - \frac{1}{2n+3} x - \frac{1}{(2n+3)(4n^2+16n+15)} x^3 + O(x^5) \quad (11)$$

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Now substitute the wavenumber-radius products into the relevant Bessel/Hankel functions ONLY for compressional modes for which kca << 1 and kcpa << 1

Do not make any assumption about the magnitude of the shear wavenumber-radius products in the continuous phase

Also use kcpa=kca\*kcp\_kc since kcp and kc are the same order of magnitude.

First for compressional waves outside particles

$$> hnp\_hnkca := series(subs(x=kca, hnp_hnxx), kca, 3);$$

$$hnp\_hnkca := \frac{-n-1}{kca} + O(kca) \quad (12)$$

for compressional waves inside particles

$$> jnp\_jnkcpa := series(subs(x=kca \cdot kcp\_kc, jnp\_jnxx), kca, 3);$$

$$jnp\_jnkcpa := \frac{n}{kca} - \frac{kcp\_kc}{2n+3} kca + O(kca^3) \quad (13)$$

For compressional incident waves

$$> jnp\_jnkca := series(subs(x=kca, jnp\_jnxx), kca, 3)$$

$$jnp\_jnkca := \frac{n}{kca} - \frac{1}{2n+3} kca + O(kca^3) \quad (14)$$

```

> kcpa := kca·kcp_kc :
Check the matrix elements to see if all make sense and are expandable as series
> series(LHS[1, 1], kca, 3);#check matrix elements are series in kca
      -n - 1 + O(kca^2)
(15)

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Next, take the solution of the boundary matrix equation for the scattering coefficients and then expand them as series in kca

# Do the incident compressional wave first - the Underscore\_ indicates that these are scaled coefficients

```

> coeffsC := LinearSolve(LHS, RHSC) :
      #coefficients for incident compressional wave in order An,Cn,Anp,Cnp
> TnCC_ := coeffsC(1) : # incident compressional, scattered compressional wave=A1
> TnCS_ := coeffsC(2) : # incident compressional, scattered shear wave=C1

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Incident shear wave

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>
> coeffsS := LinearSolve(LHS, RHSS) :
      #coefficients for incident shear wave in order An,Cn,Anp,Cnp
> TnSC_ := coeffsS(1) : # incident shear wave, scattered compressional wave
> TnSS_ := coeffsS(2) : #incident shear wave, scattered shear wave

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Now that a general ks'a (kspa) has been assumed, take the **leading term of coefficients as series in kca, then express that leading term as a series in ks'a.**

The coefficient (leading term in kca) must be expressed as separate series in numerator and denominator i.e.

$TnCC_ = (A+B*ks'a^2)/(C+D*ks'a^2)$  where both A and C include factor of (n-1).

This means that the case n=1 comes from B/D and the cases n>=2 are A/C.

This cannot be obtained properly if take a series of  $TnCC=E+F*ks'a^2$  because of cancellation of terms in (n-1)

but the choice of what to include in numerator and denominator is not unique

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Work on TnCC coefficient first. Take leading term in kca

```

TnCCLeadkca_ := (series('leadterm'((TnCC_)), kca, 1))
      # take the complete leading term, not just its numerator and denominator as series in kca
TnCCLeadkca_ := -(n (4 ksa^5 kspa n^3 rhoP^2 hnp_hnksa_jnp_jnkspa
      - 8 ksa^3 kspa^3 n^3 rhoC rhoP hnp_hnksa_jnp_jnkspa
      + 4 ksa kspa^5 n^3 rhoC^2 hnp_hnksa_jnp_jnkspa + ksa^5 kspa^4 rhoC rhoP hnp_hnksa
      - ksa^5 kspa^4 rhoP^2 hnp_hnksa + 2 ksa^5 kspa^3 rhoC rhoP hnp_hnksa_jnp_jnkspa
      - 2 ksa^5 kspa^3 rhoP^2 hnp_hnksa_jnp_jnkspa + 4 ksa^5 kspa^2 n^2 rhoP^2 hnp_hnksa
      + 4 ksa^5 kspa n^2 rhoP^2 hnp_hnksa_jnp_jnkspa - 4 ksa^5 n^4 rhoP^2 hnp_hnksa

```

$$\begin{aligned}
& -ksa^4 kspa^5 rhoC^2 jnp\_jnksa + ksa^4 kspa^5 rhoC rhoP jnp\_jnksa \\
& -4 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa^4 rhoP^2 jnp\_jnksa \\
& -2 ksa^3 kspa^5 rhoC^2 hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& -4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 4 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& + 8 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa \\
& -4 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa - 4 kspa^5 n^4 rhoC^2 jnp\_jnksa \\
& -2 ksa^5 kspa^2 n rhoC rhoP hnp\_hnksa - 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa \\
& -4 ksa^5 n^3 rhoP^2 hnp\_hnksa + ksa^4 kspa^4 n rhoC^2 - 2 ksa^4 kspa^4 n rhoC rhoP \\
& + ksa^4 kspa^4 n rhoP^2 + 2 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^3 rhoC rhoP \\
& -4 ksa^4 kspa^2 n^3 rhoP^2 - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 4 ksa^4 n^5 rhoP^2 \\
& + 2 ksa^3 kspa^4 n rhoC^2 hnp\_hnksa + 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& -4 ksa^2 kspa^4 n^3 rhoC^2 + 4 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& -8 ksa^2 kspa^2 n^5 rhoC rhoP - 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& -4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 4 kspa^4 n^5 rhoC^2 \\
& -2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa + 8 ksa^5 n^2 rhoP^2 hnp\_hnksa \\
& + 2 ksa^4 kspa^3 rhoC rhoP jnp\_jnksa + 8 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& + 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa - 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& -2 ksa^2 kspa^5 rhoC^2 jnp\_jnksa - 16 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& -8 ksa^2 kspa^2 n^4 rhoC rhoP + 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 8 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 - 2 ksa^4 kspa^2 n rhoC rhoP \\
& + 2 ksa^4 kspa^2 n rhoP^2 - 8 ksa^4 n^3 rhoP^2 + 2 ksa^2 kspa^4 n rhoC^2 - 2 ksa^2 kspa^4 n rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^3 rhoC rhoP - 8 kspa^4 n^3 rhoC^2) ) / ( \\
& -4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& -4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& -8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& -ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& -4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& -2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& -2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa
\end{aligned}$$

$$\begin{aligned}
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 \\
& - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa \\
& - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP \\
& + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 24 ksa^2 kspa^2 n^5 rhoC rhoP \\
& + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP + ksa^4 kspa^4 rhoP^2 \\
& + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP - 2 ksa^4 kspa^2 n^2 rhoP^2 \\
& + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa \\
& - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP
\end{aligned}$$



$$+ 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2)$$

> TnCCLeadNum := numer(TnCCLeadkca\_):

#this is not giving me the numerator - instead, manually copy the output above to enter the numerator and denominator

> numTnCC\_ := -n·(4 ksa<sup>5</sup> kspa n<sup>3</sup> rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa  
 - 8 ksa<sup>3</sup> kspa<sup>3</sup> n<sup>3</sup> rhoC rhoP hnp\_hnksajnp\_jnkspa  
 + 4 ksa kspa<sup>5</sup> n<sup>3</sup> rhoC<sup>2</sup> hnp\_hnksajnp\_jnkspa + ksa<sup>5</sup> kspa<sup>4</sup> rhoC rhoP hnp\_hnksa  
 - ksa<sup>5</sup> kspa<sup>4</sup> rhoP<sup>2</sup> hnp\_hnksa + 2 ksa<sup>5</sup> kspa<sup>3</sup> rhoC rhoP hnp\_hnksajnp\_jnkspa  
 - 2 ksa<sup>5</sup> kspa<sup>3</sup> rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa + 4 ksa<sup>5</sup> kspa<sup>2</sup> n<sup>2</sup> rhoP<sup>2</sup> hnp\_hnksa  
 + 4 ksa<sup>5</sup> kspa n<sup>2</sup> rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa - 4 ksa<sup>5</sup> n<sup>4</sup> rhoP<sup>2</sup> hnp\_hnksa  
 - ksa<sup>4</sup> kspa<sup>5</sup> rhoC<sup>2</sup> jnp\_jnkspa + ksa<sup>4</sup> kspa<sup>5</sup> rhoC rhoP jnp\_jnkspa  
 - 4 ksa<sup>4</sup> kspa<sup>3</sup> n<sup>2</sup> rhoC rhoP jnp\_jnkspa - 4 ksa<sup>4</sup> kspa n<sup>4</sup> rhoP<sup>2</sup> jnp\_jnkspa  
 - 2 ksa<sup>3</sup> kspa<sup>5</sup> rhoC<sup>2</sup> hnp\_hnksajnp\_jnkspa + 2 ksa<sup>3</sup> kspa<sup>5</sup> rhoC rhoP hnp\_hnksajnp\_jnkspa  
 - 4 ksa<sup>3</sup> kspa<sup>4</sup> n<sup>2</sup> rhoC rhoP hnp\_hnksa - 8 ksa<sup>3</sup> kspa<sup>3</sup> n<sup>2</sup> rhoC rhoP hnp\_hnksajnp\_jnkspa  
 + 8 ksa<sup>3</sup> kspa<sup>2</sup> n<sup>4</sup> rhoC rhoP hnp\_hnksa + 4 ksa<sup>2</sup> kspa<sup>5</sup> n<sup>2</sup> rhoC<sup>2</sup> jnp\_jnkspa  
 + 8 ksa<sup>2</sup> kspa<sup>3</sup> n<sup>4</sup> rhoC rhoP jnp\_jnkspa + 4 ksa kspa<sup>5</sup> n<sup>2</sup> rhoC<sup>2</sup> hnp\_hnksajnp\_jnkspa  
 - 4 ksa kspa<sup>4</sup> n<sup>4</sup> rhoC<sup>2</sup> hnp\_hnksa - 4 kspa<sup>5</sup> n<sup>4</sup> rhoC<sup>2</sup> jnp\_jnkspa  
 - 2 ksa<sup>5</sup> kspa<sup>2</sup> n rhoC rhoP hnp\_hnksa - 8 ksa<sup>5</sup> kspa n rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa  
 - 4 ksa<sup>5</sup> n<sup>3</sup> rhoP<sup>2</sup> hnp\_hnksa + ksa<sup>4</sup> kspa<sup>4</sup> n rhoC<sup>2</sup> - 2 ksa<sup>4</sup> kspa<sup>4</sup> n rhoC rhoP  
 + ksa<sup>4</sup> kspa<sup>4</sup> n rhoP<sup>2</sup> + 2 ksa<sup>4</sup> kspa<sup>3</sup> n rhoP<sup>2</sup> jnp\_jnkspa + 4 ksa<sup>4</sup> kspa<sup>2</sup> n<sup>3</sup> rhoC rhoP  
 - 4 ksa<sup>4</sup> kspa<sup>2</sup> n<sup>3</sup> rhoP<sup>2</sup> - 4 ksa<sup>4</sup> kspa n<sup>3</sup> rhoP<sup>2</sup> jnp\_jnkspa + 4 ksa<sup>4</sup> n<sup>5</sup> rhoP<sup>2</sup>  
 + 2 ksa<sup>3</sup> kspa<sup>4</sup> n rhoC<sup>2</sup> hnp\_hnksa + 16 ksa<sup>3</sup> kspa<sup>3</sup> n rhoC rhoP hnp\_hnksajnp\_jnkspa  
 + 8 ksa<sup>3</sup> kspa<sup>2</sup> n<sup>3</sup> rhoC rhoP hnp\_hnksa - 2 ksa<sup>2</sup> kspa<sup>5</sup> n rhoC rhoP jnp\_jnkspa  
 - 4 ksa<sup>2</sup> kspa<sup>4</sup> n<sup>3</sup> rhoC<sup>2</sup> + 4 ksa<sup>2</sup> kspa<sup>4</sup> n<sup>3</sup> rhoC rhoP + 8 ksa<sup>2</sup> kspa<sup>3</sup> n<sup>3</sup> rhoC rhoP jnp\_jnkspa  
 - 8 ksa<sup>2</sup> kspa<sup>2</sup> n<sup>5</sup> rhoC rhoP - 8 ksa kspa<sup>5</sup> n rhoC<sup>2</sup> hnp\_hnksajnp\_jnkspa  
 - 4 ksa kspa<sup>4</sup> n<sup>3</sup> rhoC<sup>2</sup> hnp\_hnksa - 4 kspa<sup>5</sup> n<sup>3</sup> rhoC<sup>2</sup> jnp\_jnkspa + 4 kspa<sup>4</sup> n<sup>5</sup> rhoC<sup>2</sup>  
 - 2 ksa<sup>5</sup> kspa<sup>2</sup> rhoP<sup>2</sup> hnp\_hnksa + 8 ksa<sup>5</sup> n<sup>2</sup> rhoP<sup>2</sup> hnp\_hnksa  
 + 2 ksa<sup>4</sup> kspa<sup>3</sup> rhoC rhoP jnp\_jnkspa + 8 ksa<sup>4</sup> kspa n<sup>2</sup> rhoP<sup>2</sup> jnp\_jnkspa + 4 ksa<sup>4</sup> n<sup>4</sup> rhoP<sup>2</sup>  
 + 2 ksa<sup>3</sup> kspa<sup>4</sup> rhoC rhoP hnp\_hnksa - 16 ksa<sup>3</sup> kspa<sup>2</sup> n<sup>2</sup> rhoC rhoP hnp\_hnksa  
 - 2 ksa<sup>2</sup> kspa<sup>5</sup> rhoC<sup>2</sup> jnp\_jnkspa - 16 ksa<sup>2</sup> kspa<sup>3</sup> n<sup>2</sup> rhoC rhoP jnp\_jnkspa  
 - 8 ksa<sup>2</sup> kspa<sup>2</sup> n<sup>4</sup> rhoC rhoP + 8 ksa kspa<sup>4</sup> n<sup>2</sup> rhoC<sup>2</sup> hnp\_hnksa + 8 kspa<sup>5</sup> n<sup>2</sup> rhoC<sup>2</sup> jnp\_jnkspa  
 + 4 kspa<sup>4</sup> n<sup>4</sup> rhoC<sup>2</sup> - 2 ksa<sup>4</sup> kspa<sup>2</sup> n rhoC rhoP + 2 ksa<sup>4</sup> kspa<sup>2</sup> n rhoP<sup>2</sup> - 8 ksa<sup>4</sup> n<sup>3</sup> rhoP<sup>2</sup>  
 + 2 ksa<sup>2</sup> kspa<sup>4</sup> n rhoC<sup>2</sup> - 2 ksa<sup>2</sup> kspa<sup>4</sup> n rhoC rhoP + 16 ksa<sup>2</sup> kspa<sup>2</sup> n<sup>3</sup> rhoC rhoP  
 - 8 kspa<sup>4</sup> n<sup>3</sup> rhoC<sup>2</sup>) :

> denTnCC\_ := -4 ksa<sup>5</sup> kspa n<sup>4</sup> rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa

+ 8 ksa<sup>3</sup> kspa<sup>3</sup> n<sup>4</sup> rhoC rhoP hnp\_hnksajnp\_jnkspa  
 - 4 ksa kspa<sup>5</sup> n<sup>4</sup> rhoC<sup>2</sup> hnp\_hnksajnp\_jnkspa + ksa<sup>5</sup> kspa<sup>4</sup> n rhoC rhoP hnp\_hnksa  
 + ksa<sup>5</sup> kspa<sup>4</sup> n rhoP<sup>2</sup> hnp\_hnksa + 2 ksa<sup>5</sup> kspa<sup>3</sup> n rhoC rhoP hnp\_hnksajnp\_jnkspa  
 + 2 ksa<sup>5</sup> kspa<sup>3</sup> n rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa - 4 ksa<sup>5</sup> kspa<sup>2</sup> n<sup>3</sup> rhoP<sup>2</sup> hnp\_hnksa  
 - 8 ksa<sup>5</sup> kspa n<sup>3</sup> rhoP<sup>2</sup> hnp\_hnksajnp\_jnkspa + 4 ksa<sup>5</sup> n<sup>5</sup> rhoP<sup>2</sup> hnp\_hnksa  
 - ksa<sup>4</sup> kspa<sup>5</sup> n rhoC<sup>2</sup> jnp\_jnkspa - ksa<sup>4</sup> kspa<sup>5</sup> n rhoC rhoP jnp\_jnkspa  
 - 4 ksa<sup>4</sup> kspa<sup>3</sup> n<sup>3</sup> rhoC rhoP jnp\_jnkspa - 4 ksa<sup>4</sup> kspa n<sup>5</sup> rhoP<sup>2</sup> jnp\_jnkspa

$$\begin{aligned}
& -2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& -2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2 :
\end{aligned}$$

Need to get these leading terms (in kca) numerator and denominator for every coefficient BEFORE expanding as series in kspa for numerator and denominator separately.

> TnCSLeadkca\_ := (series('leadterm'((TnCS\_)), kca, 1))

#initially show output, then copy its numerator and denominator, then hide output

$$\begin{aligned}
 \text{TnCSLeadkca}_- := & (4 \text{ksa}^4 \text{kspa}^3 n^2 \text{rhoC rhoP jnp\_jnksa} + 8 \text{ksa}^4 \text{kspa} n^4 \text{rhoP}^2 \text{jnp\_jnksa} \\
 & - 4 \text{ksa}^2 \text{kspa}^5 n^2 \text{rhoC}^2 \text{jnp\_jnksa} - 16 \text{ksa}^2 \text{kspa}^3 n^4 \text{rhoC rhoP jnp\_jnksa} \\
 & + 8 \text{kspa}^5 n^4 \text{rhoC}^2 \text{jnp\_jnksa} + 2 \text{ksa}^4 \text{kspa}^4 n \text{rhoC rhoP} - 2 \text{ksa}^4 \text{kspa}^4 n \text{rhoP}^2 \\
 & + 2 \text{ksa}^4 \text{kspa}^3 n \text{rhoC rhoP jnp\_jnksa} - 4 \text{ksa}^4 \text{kspa}^3 n \text{rhoP}^2 \text{jnp\_jnksa} \\
 & - 4 \text{ksa}^4 \text{kspa}^2 n^3 \text{rhoC rhoP} + 8 \text{ksa}^4 \text{kspa}^2 n^3 \text{rhoP}^2 + 12 \text{ksa}^4 \text{kspa} n^3 \text{rhoP}^2 \text{jnp\_jnksa} \\
 & - 8 \text{ksa}^4 n^5 \text{rhoP}^2 - 2 \text{ksa}^2 \text{kspa}^5 n \text{rhoC}^2 \text{jnp\_jnksa} + 4 \text{ksa}^2 \text{kspa}^5 n \text{rhoC rhoP jnp\_jnksa} \\
 & + 4 \text{ksa}^2 \text{kspa}^4 n^3 \text{rhoC}^2 - 8 \text{ksa}^2 \text{kspa}^4 n^3 \text{rhoC rhoP} \\
 & - 24 \text{ksa}^2 \text{kspa}^3 n^3 \text{rhoC rhoP jnp\_jnksa} + 16 \text{ksa}^2 \text{kspa}^2 n^5 \text{rhoC rhoP} \\
 & + 12 \text{kspa}^5 n^3 \text{rhoC}^2 \text{jnp\_jnksa} - 8 \text{kspa}^4 n^5 \text{rhoC}^2 + \text{ksa}^4 \text{kspa}^4 \text{rhoC rhoP} \\
 & - \text{ksa}^4 \text{kspa}^4 \text{rhoP}^2 - 2 \text{ksa}^4 \text{kspa}^3 \text{rhoP}^2 \text{jnp\_jnksa} - 2 \text{ksa}^4 \text{kspa}^2 n^2 \text{rhoC rhoP} \\
 & + 4 \text{ksa}^4 \text{kspa}^2 n^2 \text{rhoP}^2 - 12 \text{ksa}^4 \text{kspa} n^2 \text{rhoP}^2 \text{jnp\_jnksa} - 12 \text{ksa}^4 n^4 \text{rhoP}^2 \\
 & + 2 \text{ksa}^2 \text{kspa}^5 \text{rhoC rhoP jnp\_jnksa} + 2 \text{ksa}^2 \text{kspa}^4 n^2 \text{rhoC}^2 - 4 \text{ksa}^2 \text{kspa}^4 n^2 \text{rhoC rhoP} \\
 & + 24 \text{ksa}^2 \text{kspa}^3 n^2 \text{rhoC rhoP jnp\_jnksa} + 24 \text{ksa}^2 \text{kspa}^2 n^4 \text{rhoC rhoP} \\
 & - 12 \text{kspa}^5 n^2 \text{rhoC}^2 \text{jnp\_jnksa} - 12 \text{kspa}^4 n^4 \text{rhoC}^2 - 4 \text{ksa}^4 \text{kspa}^2 n \text{rhoP}^2 \\
 & - 8 \text{ksa}^4 \text{kspa} n \text{rhoP}^2 \text{jnp\_jnksa} + 12 \text{ksa}^4 n^3 \text{rhoP}^2 + 4 \text{ksa}^2 \text{kspa}^4 n \text{rhoC rhoP} \\
 & + 16 \text{ksa}^2 \text{kspa}^3 n \text{rhoC rhoP jnp\_jnksa} - 24 \text{ksa}^2 \text{kspa}^2 n^3 \text{rhoC rhoP} \\
 & - 8 \text{kspa}^5 n \text{rhoC}^2 \text{jnp\_jnksa} + 12 \text{kspa}^4 n^3 \text{rhoC}^2 - 2 \text{ksa}^4 \text{kspa}^2 \text{rhoP}^2 + 8 \text{ksa}^4 n^2 \text{rhoP}^2 \\
 & + 2 \text{ksa}^2 \text{kspa}^4 \text{rhoC rhoP} - 16 \text{ksa}^2 \text{kspa}^2 n^2 \text{rhoC rhoP} + 8 \text{kspa}^4 n^2 \text{rhoC}^2) / ( \\
 & - 4 \text{ksa}^5 \text{kspa} n^4 \text{rhoP}^2 \text{hnp\_hnksa jnp\_jnksa} \\
 & + 8 \text{ksa}^3 \text{kspa}^3 n^4 \text{rhoC rhoP hnp\_hnksa jnp\_jnksa} \\
 & - 4 \text{ksa} \text{kspa}^5 n^4 \text{rhoC}^2 \text{hnp\_hnksa jnp\_jnksa} + \text{ksa}^5 \text{kspa}^4 n \text{rhoC rhoP hnp\_hnksa} \\
 & + \text{ksa}^5 \text{kspa}^4 n \text{rhoP}^2 \text{hnp\_hnksa} + 2 \text{ksa}^5 \text{kspa}^3 n \text{rhoC rhoP hnp\_hnksa jnp\_jnksa} \\
 & + 2 \text{ksa}^5 \text{kspa}^3 n \text{rhoP}^2 \text{hnp\_hnksa jnp\_jnksa} - 4 \text{ksa}^5 \text{kspa}^2 n^3 \text{rhoP}^2 \text{hnp\_hnksa} \\
 & - 8 \text{ksa}^5 \text{kspa} n^3 \text{rhoP}^2 \text{hnp\_hnksa jnp\_jnksa} + 4 \text{ksa}^5 n^5 \text{rhoP}^2 \text{hnp\_hnksa} \\
 & - \text{ksa}^4 \text{kspa}^5 n \text{rhoC}^2 \text{jnp\_jnksa} - \text{ksa}^4 \text{kspa}^5 n \text{rhoC rhoP jnp\_jnksa} \\
 & - 4 \text{ksa}^4 \text{kspa}^3 n^3 \text{rhoC rhoP jnp\_jnksa} - 4 \text{ksa}^4 \text{kspa} n^5 \text{rhoP}^2 \text{jnp\_jnksa} \\
 & - 2 \text{ksa}^3 \text{kspa}^5 n \text{rhoC}^2 \text{hnp\_hnksa jnp\_jnksa} \\
 & - 2 \text{ksa}^3 \text{kspa}^5 n \text{rhoC rhoP hnp\_hnksa jnp\_jnksa} + 4 \text{ksa}^3 \text{kspa}^4 n^3 \text{rhoC rhoP hnp\_hnksa} \\
 & + 16 \text{ksa}^3 \text{kspa}^3 n^3 \text{rhoC rhoP hnp\_hnksa jnp\_jnksa} - 8 \text{ksa}^3 \text{kspa}^2 n^5 \text{rhoC rhoP hnp\_hnksa} \\
 & + 4 \text{ksa}^2 \text{kspa}^5 n^3 \text{rhoC}^2 \text{jnp\_jnksa} + 8 \text{ksa}^2 \text{kspa}^3 n^5 \text{rhoC rhoP jnp\_jnksa} \\
 & - 8 \text{ksa} \text{kspa}^5 n^3 \text{rhoC}^2 \text{hnp\_hnksa jnp\_jnksa} + 4 \text{ksa} \text{kspa}^4 n^5 \text{rhoC}^2 \text{hnp\_hnksa} \\
 & - 4 \text{kspa}^5 n^5 \text{rhoC}^2 \text{jnp\_jnksa} + \text{ksa}^5 \text{kspa}^4 \text{rhoP}^2 \text{hnp\_hnksa} \\
 & + 2 \text{ksa}^5 \text{kspa}^3 \text{rhoP}^2 \text{hnp\_hnksa jnp\_jnksa} - 2 \text{ksa}^5 \text{kspa}^2 n^2 \text{rhoC rhoP hnp\_hnksa} \\
 & - 4 \text{ksa}^5 \text{kspa}^2 n^2 \text{rhoP}^2 \text{hnp\_hnksa} + 4 \text{ksa}^5 \text{kspa} n^2 \text{rhoP}^2 \text{hnp\_hnksa jnp\_jnksa}
 \end{aligned}
 \tag{17}$$

$$\begin{aligned}
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 \\
& - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa \\
& - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP \\
& + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 24 ksa^2 kspa^2 n^5 rhoC rhoP \\
& + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP + ksa^4 kspa^4 rhoP^2 \\
& + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP - 2 ksa^4 kspa^2 n^2 rhoP^2 \\
& + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa \\
& - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2)
\end{aligned}$$

$$\begin{aligned}
> \text{numTnCS}_- := & 4 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa + 8 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa \\
& - 4 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa - 16 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa \\
& + 8 kspa^5 n^4 rhoC^2 jnp\_jnksa + 2 ksa^4 kspa^4 n rhoC rhoP - 2 ksa^4 kspa^4 n rhoP^2
\end{aligned}$$

$$\begin{aligned}
& + 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa \\
& - 4 ksa^4 kspa^2 n^3 rhoC rhoP + 8 ksa^4 kspa^2 n^3 rhoP^2 + 12 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa \\
& - 8 ksa^4 n^5 rhoP^2 - 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa + 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^2 kspa^4 n^3 rhoC^2 - 8 ksa^2 kspa^4 n^3 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa + 16 ksa^2 kspa^2 n^5 rhoC rhoP \\
& + 12 kspa^5 n^3 rhoC^2 jnp\_jnksa - 8 kspa^4 n^5 rhoC^2 + ksa^4 kspa^4 rhoC rhoP \\
& - ksa^4 kspa^4 rhoP^2 - 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa - 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& + 4 ksa^4 kspa^2 n^2 rhoP^2 - 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa - 12 ksa^4 n^4 rhoP^2 \\
& + 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa + 2 ksa^2 kspa^4 n^2 rhoC^2 - 4 ksa^2 kspa^4 n^2 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa + 24 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 12 kspa^5 n^2 rhoC^2 jnp\_jnksa - 12 kspa^4 n^4 rhoC^2 - 4 ksa^4 kspa^2 n rhoP^2 \\
& - 8 ksa^4 kspa n rhoP^2 jnp\_jnksa + 12 ksa^4 n^3 rhoP^2 + 4 ksa^2 kspa^4 n rhoC rhoP \\
& + 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa - 24 ksa^2 kspa^2 n^3 rhoC rhoP \\
& - 8 kspa^5 n rhoC^2 jnp\_jnksa + 12 kspa^4 n^3 rhoC^2 - 2 ksa^4 kspa^2 rhoP^2 + 8 ksa^4 n^2 rhoP^2 \\
& + 2 ksa^2 kspa^4 rhoC rhoP - 16 ksa^2 kspa^2 n^2 rhoC rhoP + 8 kspa^4 n^2 rhoC^2 :
\end{aligned}$$

>  $denTnCS\_ := -4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa$

$$\begin{aligned}
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa
\end{aligned}$$

$$\begin{aligned}
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2 :
\end{aligned}$$

> denTnCS\_ - denTnCC\_  
*#check that the denominators are the same for the coefficients - this saves having to derive the simplified form of the denominator*

0

(18)

> TnSCLeadkca\_ := (series('leadterm'((TnSC\_)), kca, 1))

TnSCLeadkca\_ := -(n (n + 1) ksa (2 ksa^4 kspa^3 n rhoC rhoP hnp\\_hnksa jnp\\_jnksa

(19)

$$\begin{aligned}
& - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& + 4 ksa^4 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa \\
& - 8 ksa^2 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa jnp\_jnksa + 4 kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa jnp\_jnksa + ksa^4 kspa^4 rhoC rhoP hnp\_hnksa \\
& - ksa^4 kspa^4 rhoC rhoP jnp\_jnksa - ksa^4 kspa^4 rhoP^2 hnp\_hnksa \\
& + ksa^4 kspa^4 rhoP^2 jnp\_jnksa - 2 ksa^4 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa jnp\_jnksa - 2 ksa^4 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^4 kspa^2 n^2 rhoC rhoP jnp\_jnksa + 4 ksa^4 kspa^2 n^2 rhoP^2 hnp\_hnksa \\
& - 4 ksa^4 kspa^2 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^4 n^4 rhoP^2 hnp\_hnksa \\
& + 4 ksa^4 n^4 rhoP^2 jnp\_jnksa + 2 ksa^2 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa
\end{aligned}$$

$$\begin{aligned}
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa jnp\_jnksa + 2 ksa^2 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& - 2 ksa^2 kspa^4 n^2 rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^4 n^2 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^4 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^2 kspa^2 n^4 rhoC rhoP hnp\_hnksa \\
& - 8 ksa^2 kspa^2 n^4 rhoC rhoP jnp\_jnksa + 4 kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 kspa^5 n^2 rhoC^2 jnp\_jnksa jnp\_jnksa - 4 kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& + 4 kspa^4 n^4 rhoC^2 jnp\_jnksa - 8 ksa^4 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^4 n^3 rhoP^2 hnp\_hnksa \\
& + 4 ksa^4 n^3 rhoP^2 jnp\_jnksa + 16 ksa^2 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^2 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& - 8 ksa^2 kspa^2 n^3 rhoC rhoP jnp\_jnksa - 8 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& + 8 kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa - 4 kspa^4 n^3 rhoC^2 hnp\_hnksa \\
& + 4 kspa^4 n^3 rhoC^2 jnp\_jnksa - 2 ksa^4 kspa^2 rhoP^2 hnp\_hnksa \\
& + 2 ksa^4 kspa^2 rhoP^2 jnp\_jnksa + 8 ksa^4 n^2 rhoP^2 hnp\_hnksa - 8 ksa^4 n^2 rhoP^2 jnp\_jnksa \\
& + 2 ksa^2 kspa^4 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^4 rhoC rhoP jnp\_jnksa \\
& - 16 ksa^2 kspa^2 n^2 rhoC rhoP hnp\_hnksa + 16 ksa^2 kspa^2 n^2 rhoC rhoP jnp\_jnksa \\
& + 8 kspa^4 n^2 rhoC^2 hnp\_hnksa - 8 kspa^4 n^2 rhoC^2 jnp\_jnksa) / ( \\
& - 4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2
\end{aligned}$$



$$\begin{aligned}
& -12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& -2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& -16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& -2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& -12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& -2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& -4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 \\
& -2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa \\
& -4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP \\
& + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 24 ksa^2 kspa^2 n^5 rhoC rhoP \\
& + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa \\
& -4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa \\
& -8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP + ksa^4 kspa^4 rhoP^2 \\
& + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP - 2 ksa^4 kspa^2 n^2 rhoP^2 \\
& + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa \\
& -2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& -24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& -8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& -4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2)
\end{aligned}$$

$$\begin{aligned}
> \text{numTnSC}_- & := -n (n + 1) ksa (2 ksa^4 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& + 4 ksa^4 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa \\
& - 8 ksa^2 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa jnp\_jnksa + 4 kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa jnp\_jnksa + ksa^4 kspa^4 rhoC rhoP hnp\_hnksa
\end{aligned}$$



$$\begin{aligned}
& -ksa^4 kspa^4 rhoC rhoP jnp\_jnksa - ksa^4 kspa^4 rhoP^2 hnp\_hnksa \\
& + ksa^4 kspa^4 rhoP^2 jnp\_jnksa - 2 ksa^4 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa jnp\_jnksa - 2 ksa^4 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^4 kspa^2 n^2 rhoC rhoP jnp\_jnksa + 4 ksa^4 kspa^2 n^2 rhoP^2 hnp\_hnksa \\
& - 4 ksa^4 kspa^2 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^4 n^4 rhoP^2 hnp\_hnksa \\
& + 4 ksa^4 n^4 rhoP^2 jnp\_jnksa + 2 ksa^2 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa jnp\_jnksa + 2 ksa^2 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& - 2 ksa^2 kspa^4 n^2 rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^4 n^2 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^4 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^2 kspa^2 n^4 rhoC rhoP hnp\_hnksa \\
& - 8 ksa^2 kspa^2 n^4 rhoC rhoP jnp\_jnksa + 4 kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 kspa^5 n^2 rhoC^2 jnp\_jnksa jnp\_jnksa - 4 kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& + 4 kspa^4 n^4 rhoC^2 jnp\_jnksa - 8 ksa^4 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^4 n^3 rhoP^2 hnp\_hnksa \\
& + 4 ksa^4 n^3 rhoP^2 jnp\_jnksa + 16 ksa^2 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^2 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& - 8 ksa^2 kspa^2 n^3 rhoC rhoP jnp\_jnksa - 8 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& + 8 kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa - 4 kspa^4 n^3 rhoC^2 hnp\_hnksa \\
& + 4 kspa^4 n^3 rhoC^2 jnp\_jnksa - 2 ksa^4 kspa^2 rhoP^2 hnp\_hnksa + 2 ksa^4 kspa^2 rhoP^2 jnp\_jnksa \\
& + 8 ksa^4 n^2 rhoP^2 hnp\_hnksa - 8 ksa^4 n^2 rhoP^2 jnp\_jnksa \\
& + 2 ksa^2 kspa^4 rhoC rhoP hnp\_hnksa - 2 ksa^2 kspa^4 rhoC rhoP jnp\_jnksa \\
& - 16 ksa^2 kspa^2 n^2 rhoC rhoP hnp\_hnksa + 16 ksa^2 kspa^2 n^2 rhoC rhoP jnp\_jnksa \\
& + 8 kspa^4 n^2 rhoC^2 hnp\_hnksa - 8 kspa^4 n^2 rhoC^2 jnp\_jnksa) :
\end{aligned}$$

$$\begin{aligned}
> denTnSC_ := & -4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa
\end{aligned}$$

$$\begin{aligned}
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2 :
\end{aligned}$$

> denTnSC\_ - denTnCC\_ # confirm the denominator is identical to that of TnCC\_

0

(20)

> TnSSLeadkca\_ := (series('leadterm'((TnSS\_)), kca, 1))

#initially show output, then copy numerator and denominator and then hide output

TnSSLeadkca\_ := - ( - 4 ksa^5 kspa n^4 rhoP^2 jnp\\_jnksa jnp\\_jnksa

(21)

$$\begin{aligned}
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 jnp\_jnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP jnp\_jnksa \\
& + ksa^5 kspa^4 n rhoP^2 jnp\_jnksa + 2 ksa^5 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 jnp\_jnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 jnp\_jnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 jnp\_jnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa
\end{aligned}$$

$$\begin{aligned}
& -4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa^5 n^5 rhoP^2 jnp\_jnksa \\
& -2 ksa^3 kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa \\
& -2 ksa^3 kspa^5 n rhoC rhoP jnp\_jnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP jnp\_jnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP jnp\_jnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP jnp\_jnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 jnp\_jnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 jnp\_jnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 jnp\_jnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 jnp\_jnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP jnp\_jnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 jnp\_jnksa + 4 ksa^5 kspa n^2 rhoP^2 jnp\_jnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 jnp\_jnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP jnp\_jnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 jnp\_jnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP jnp\_jnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 jnp\_jnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 jnp\_jnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 jnp\_jnksa \\
& + 8 ksa^5 kspa n rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 jnp\_jnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP jnp\_jnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 jnp\_jnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 jnp\_jnksa - 8 ksa^5 n^2 rhoP^2 jnp\_jnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP jnp\_jnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP jnp\_jnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 jnp\_jnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2
\end{aligned}$$

$$\begin{aligned}
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2) / ( \\
& - 4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2
\end{aligned}$$

$$\begin{aligned}
& - 2 ksa^3 kspa^4 n \rho C \rho P hnp\_hnksa - 16 ksa^3 kspa^3 n \rho C \rho P hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^2 n^3 \rho C \rho P hnp\_hnksa + 2 ksa^2 kspa^5 n \rho C^2 jnp\_jnksa \\
& - 4 ksa^2 kspa^5 n \rho C \rho P jnp\_jnksa - 8 ksa^2 kspa^4 n^3 \rho C^2 + 8 ksa^2 kspa^4 n^3 \rho C \rho P \\
& + 8 ksa^2 kspa^3 n^3 \rho C \rho P jnp\_jnksa - 24 ksa^2 kspa^2 n^5 \rho C \rho P \\
& + 8 ksa kspa^5 n \rho C^2 hnp\_hnksa jnp\_jnksa - 4 ksa kspa^4 n^3 \rho C^2 hnp\_hnksa \\
& - 4 kspa^5 n^3 \rho C^2 jnp\_jnksa + 12 kspa^4 n^5 \rho C^2 + 2 ksa^5 kspa^2 \rho P^2 hnp\_hnksa \\
& - 8 ksa^5 n^2 \rho P^2 hnp\_hnksa - ksa^4 kspa^4 \rho C \rho P + ksa^4 kspa^4 \rho P^2 \\
& + 2 ksa^4 kspa^3 \rho P^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 \rho C \rho P - 2 ksa^4 kspa^2 n^2 \rho P^2 \\
& + 12 ksa^4 kspa n^2 \rho P^2 jnp\_jnksa + 4 ksa^4 n^4 \rho P^2 - 2 ksa^3 kspa^4 \rho C \rho P hnp\_hnksa \\
& + 16 ksa^3 kspa^2 n^2 \rho C \rho P hnp\_hnksa - 2 ksa^2 kspa^5 \rho C \rho P jnp\_jnksa \\
& - 2 ksa^2 kspa^4 n^2 \rho C^2 + 2 ksa^2 kspa^4 n^2 \rho C \rho P \\
& - 24 ksa^2 kspa^3 n^2 \rho C \rho P jnp\_jnksa - 8 ksa^2 kspa^2 n^4 \rho C \rho P \\
& - 8 ksa kspa^4 n^2 \rho C^2 hnp\_hnksa + 12 kspa^5 n^2 \rho C^2 jnp\_jnksa + 4 kspa^4 n^4 \rho C^2 \\
& + 4 ksa^4 kspa^2 n \rho P^2 + 8 ksa^4 kspa n \rho P^2 jnp\_jnksa - 12 ksa^4 n^3 \rho P^2 \\
& - 4 ksa^2 kspa^4 n \rho C \rho P - 16 ksa^2 kspa^3 n \rho C \rho P jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 \rho C \rho P + 8 kspa^5 n \rho C^2 jnp\_jnksa - 12 kspa^4 n^3 \rho C^2 \\
& + 2 ksa^4 kspa^2 \rho P^2 - 8 ksa^4 n^2 \rho P^2 - 2 ksa^2 kspa^4 \rho C \rho P \\
& + 16 ksa^2 kspa^2 n^2 \rho C \rho P - 8 kspa^4 n^2 \rho C^2)
\end{aligned}$$

$$\begin{aligned}
> \text{numTnSS\_} := & - (-4 ksa^5 kspa n^4 \rho P^2 jnp\_jnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 \rho C \rho P jnp\_jnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 \rho C^2 jnp\_jnksa jnp\_jnksa + ksa^5 kspa^4 n \rho C \rho P jnp\_jnksa \\
& + ksa^5 kspa^4 n \rho P^2 jnp\_jnksa + 2 ksa^5 kspa^3 n \rho C \rho P jnp\_jnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n \rho P^2 jnp\_jnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 \rho P^2 jnp\_jnksa \\
& - 8 ksa^5 kspa n^3 \rho P^2 jnp\_jnksa jnp\_jnksa + 4 ksa^5 n^5 \rho P^2 jnp\_jnksa \\
& - ksa^4 kspa^5 n \rho C^2 jnp\_jnksa - ksa^4 kspa^5 n \rho C \rho P jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 \rho C \rho P jnp\_jnksa - 4 ksa^4 kspa n^5 \rho P^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n \rho C^2 jnp\_jnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n \rho C \rho P jnp\_jnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 \rho C \rho P jnp\_jnksa \\
& + 16 ksa^3 kspa^3 n^3 \rho C \rho P jnp\_jnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 \rho C \rho P jnp\_jnksa \\
& + 4 ksa^2 kspa^5 n^3 \rho C^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 \rho C \rho P jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 \rho C^2 jnp\_jnksa jnp\_jnksa + 4 ksa kspa^4 n^5 \rho C^2 jnp\_jnksa \\
& - 4 kspa^5 n^5 \rho C^2 jnp\_jnksa + ksa^5 kspa^4 \rho P^2 jnp\_jnksa \\
& + 2 ksa^5 kspa^3 \rho P^2 jnp\_jnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 \rho C \rho P jnp\_jnksa \\
& - 4 ksa^5 kspa^2 n^2 \rho P^2 jnp\_jnksa + 4 ksa^5 kspa n^2 \rho P^2 jnp\_jnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 \rho P^2 jnp\_jnksa - ksa^4 kspa^5 \rho C \rho P jnp\_jnksa + ksa^4 kspa^4 n^2 \rho C^2 \\
& - 2 ksa^4 kspa^4 n^2 \rho C \rho P + ksa^4 kspa^4 n^2 \rho P^2 - 8 ksa^4 kspa^3 n^2 \rho C \rho P jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 \rho P^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 \rho C \rho P - 4 ksa^4 kspa^2 n^4 \rho P^2 \\
& - 12 ksa^4 kspa n^4 \rho P^2 jnp\_jnksa + 4 ksa^4 n^6 \rho P^2 \\
& - 2 ksa^3 kspa^5 \rho C \rho P jnp\_jnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 \rho C^2 jnp\_jnksa
\end{aligned}$$

$$\begin{aligned}
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP jnp\_jnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP jnp\_jnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 jnp\_jnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 jnp\_jnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 jnp\_jnksa \\
& + 8 ksa^5 kspa n rhoP^2 jnp\_jnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 jnp\_jnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP jnp\_jnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP jnp\_jnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 jnp\_jnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 jnp\_jnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 jnp\_jnksa - 8 ksa^5 n^2 rhoP^2 jnp\_jnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP jnp\_jnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP jnp\_jnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 jnp\_jnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2) :
\end{aligned}$$

$$\begin{aligned}
> denTnSS_ := ( & - 4 ksa^5 kspa n^4 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^3 kspa^3 n^4 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^5 n^4 rhoC^2 hnp\_hnksa jnp\_jnksa + ksa^5 kspa^4 n rhoC rhoP hnp\_hnksa \\
& + ksa^5 kspa^4 n rhoP^2 hnp\_hnksa + 2 ksa^5 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& + 2 ksa^5 kspa^3 n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 kspa^2 n^3 rhoP^2 hnp\_hnksa \\
& - 8 ksa^5 kspa n^3 rhoP^2 hnp\_hnksa jnp\_jnksa + 4 ksa^5 n^5 rhoP^2 hnp\_hnksa \\
& - ksa^4 kspa^5 n rhoC^2 jnp\_jnksa - ksa^4 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 4 ksa^4 kspa^3 n^3 rhoC rhoP jnp\_jnksa - 4 ksa^4 kspa n^5 rhoP^2 jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 2 ksa^3 kspa^5 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 4 ksa^3 kspa^4 n^3 rhoC rhoP hnp\_hnksa \\
& + 16 ksa^3 kspa^3 n^3 rhoC rhoP hnp\_hnksa jnp\_jnksa - 8 ksa^3 kspa^2 n^5 rhoC rhoP hnp\_hnksa \\
& + 4 ksa^2 kspa^5 n^3 rhoC^2 jnp\_jnksa + 8 ksa^2 kspa^3 n^5 rhoC rhoP jnp\_jnksa \\
& - 8 ksa kspa^5 n^3 rhoC^2 hnp\_hnksa jnp\_jnksa + 4 ksa kspa^4 n^5 rhoC^2 hnp\_hnksa \\
& - 4 kspa^5 n^5 rhoC^2 jnp\_jnksa + ksa^5 kspa^4 rhoP^2 hnp\_hnksa \\
& + 2 ksa^5 kspa^3 rhoP^2 hnp\_hnksa jnp\_jnksa - 2 ksa^5 kspa^2 n^2 rhoC rhoP hnp\_hnksa
\end{aligned}$$



$$\begin{aligned}
& -4 ksa^5 kspa^2 n^2 rhoP^2 hnp\_hnksa + 4 ksa^5 kspa n^2 rhoP^2 hnp\_hnksa jnp\_jnksa \\
& + 8 ksa^5 n^4 rhoP^2 hnp\_hnksa - ksa^4 kspa^5 rhoC rhoP jnp\_jnksa + ksa^4 kspa^4 n^2 rhoC^2 \\
& - 2 ksa^4 kspa^4 n^2 rhoC rhoP + ksa^4 kspa^4 n^2 rhoP^2 - 8 ksa^4 kspa^3 n^2 rhoC rhoP jnp\_jnksa \\
& + 2 ksa^4 kspa^3 n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 kspa^2 n^4 rhoC rhoP - 4 ksa^4 kspa^2 n^4 rhoP^2 \\
& - 12 ksa^4 kspa n^4 rhoP^2 jnp\_jnksa + 4 ksa^4 n^6 rhoP^2 \\
& - 2 ksa^3 kspa^5 rhoC rhoP hnp\_hnksa jnp\_jnksa + 2 ksa^3 kspa^4 n^2 rhoC^2 hnp\_hnksa \\
& + 4 ksa^3 kspa^4 n^2 rhoC rhoP hnp\_hnksa - 8 ksa^3 kspa^3 n^2 rhoC rhoP hnp\_hnksa jnp\_jnksa \\
& - 16 ksa^3 kspa^2 n^4 rhoC rhoP hnp\_hnksa + 8 ksa^2 kspa^5 n^2 rhoC^2 jnp\_jnksa \\
& - 2 ksa^2 kspa^5 n^2 rhoC rhoP jnp\_jnksa - 4 ksa^2 kspa^4 n^4 rhoC^2 + 4 ksa^2 kspa^4 n^4 rhoC rhoP \\
& + 24 ksa^2 kspa^3 n^4 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^6 rhoC rhoP \\
& + 4 ksa kspa^5 n^2 rhoC^2 hnp\_hnksa jnp\_jnksa + 8 ksa kspa^4 n^4 rhoC^2 hnp\_hnksa \\
& - 12 kspa^5 n^4 rhoC^2 jnp\_jnksa + 4 kspa^4 n^6 rhoC^2 + 2 ksa^5 kspa^2 n rhoP^2 hnp\_hnksa \\
& + 8 ksa^5 kspa n rhoP^2 hnp\_hnksa jnp\_jnksa - 4 ksa^5 n^3 rhoP^2 hnp\_hnksa \\
& - 2 ksa^4 kspa^4 n rhoC rhoP + 2 ksa^4 kspa^4 n rhoP^2 - 2 ksa^4 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 4 ksa^4 kspa^3 n rhoP^2 jnp\_jnksa + 8 ksa^4 kspa^2 n^3 rhoC rhoP - 8 ksa^4 kspa^2 n^3 rhoP^2 \\
& - 4 ksa^4 kspa n^3 rhoP^2 jnp\_jnksa + 12 ksa^4 n^5 rhoP^2 - 2 ksa^3 kspa^4 n rhoC rhoP hnp\_hnksa \\
& - 16 ksa^3 kspa^3 n rhoC rhoP hnp\_hnksa jnp\_jnksa + 8 ksa^3 kspa^2 n^3 rhoC rhoP hnp\_hnksa \\
& + 2 ksa^2 kspa^5 n rhoC^2 jnp\_jnksa - 4 ksa^2 kspa^5 n rhoC rhoP jnp\_jnksa \\
& - 8 ksa^2 kspa^4 n^3 rhoC^2 + 8 ksa^2 kspa^4 n^3 rhoC rhoP + 8 ksa^2 kspa^3 n^3 rhoC rhoP jnp\_jnksa \\
& - 24 ksa^2 kspa^2 n^5 rhoC rhoP + 8 ksa kspa^5 n rhoC^2 hnp\_hnksa jnp\_jnksa \\
& - 4 ksa kspa^4 n^3 rhoC^2 hnp\_hnksa - 4 kspa^5 n^3 rhoC^2 jnp\_jnksa + 12 kspa^4 n^5 rhoC^2 \\
& + 2 ksa^5 kspa^2 rhoP^2 hnp\_hnksa - 8 ksa^5 n^2 rhoP^2 hnp\_hnksa - ksa^4 kspa^4 rhoC rhoP \\
& + ksa^4 kspa^4 rhoP^2 + 2 ksa^4 kspa^3 rhoP^2 jnp\_jnksa + 2 ksa^4 kspa^2 n^2 rhoC rhoP \\
& - 2 ksa^4 kspa^2 n^2 rhoP^2 + 12 ksa^4 kspa n^2 rhoP^2 jnp\_jnksa + 4 ksa^4 n^4 rhoP^2 \\
& - 2 ksa^3 kspa^4 rhoC rhoP hnp\_hnksa + 16 ksa^3 kspa^2 n^2 rhoC rhoP hnp\_hnksa \\
& - 2 ksa^2 kspa^5 rhoC rhoP jnp\_jnksa - 2 ksa^2 kspa^4 n^2 rhoC^2 + 2 ksa^2 kspa^4 n^2 rhoC rhoP \\
& - 24 ksa^2 kspa^3 n^2 rhoC rhoP jnp\_jnksa - 8 ksa^2 kspa^2 n^4 rhoC rhoP \\
& - 8 ksa kspa^4 n^2 rhoC^2 hnp\_hnksa + 12 kspa^5 n^2 rhoC^2 jnp\_jnksa + 4 kspa^4 n^4 rhoC^2 \\
& + 4 ksa^4 kspa^2 n rhoP^2 + 8 ksa^4 kspa n rhoP^2 jnp\_jnksa - 12 ksa^4 n^3 rhoP^2 \\
& - 4 ksa^2 kspa^4 n rhoC rhoP - 16 ksa^2 kspa^3 n rhoC rhoP jnp\_jnksa \\
& + 24 ksa^2 kspa^2 n^3 rhoC rhoP + 8 kspa^5 n rhoC^2 jnp\_jnksa - 12 kspa^4 n^3 rhoC^2 \\
& + 2 ksa^4 kspa^2 rhoP^2 - 8 ksa^4 n^2 rhoP^2 - 2 ksa^2 kspa^4 rhoC rhoP \\
& + 16 ksa^2 kspa^2 n^2 rhoC rhoP - 8 kspa^4 n^2 rhoC^2) :
\end{aligned}$$

> denTnSS\_ - denTnCC\_

0

(22)

\*\*\*\*\*  
\*\*\*\*\*

So we now have numerator and denominator expressions for each coefficient (all denominators identical).

Now work on numerator and denominator separately, expressing as series in kspa (because kspa is small if kcpa is small and the particle is solid)

First do the first term of each series - these will be the terms that give the  $n \geq 2$  expressions

TnCC\_ = (A+B\*ks'a^2)/(C+D\*ks'a^2) where both A and C include factor of (n-1).

This means that the case  $n=1$  comes from B/D and the cases  $n \geq 2$  are A/C

and  
 $TnCS\_ = (E + Fks'a^2)/(C + D*ks'a^2)$  etc.

$$\begin{aligned} &> jnp\_jnkspa := subs(x = kspa, jnp\_jnxx) \\ jnp\_jnkspa &:= \frac{n}{kspa} - \frac{1}{2n+3} kspa - \frac{1}{(2n+3)(4n^2+16n+15)} kspa^3 + O(kspa^5) \end{aligned} \quad (23)$$

\*\*\*\*\*  
 \*\*\*\*\*

First work on TnCC\_

$$\begin{aligned} &> numTnCC\_Serkspa := simplify(series(numTnCC_, kspa, 8)) \\ numTnCC\_Serkspa &:= -\frac{1}{2n+3} (2n ksa^4 rhoP^2 (2ksa n^3 hnp\_hnksa + 2ksa n^2 hnp\_hnksa \\ &\quad - 2n^4 - ksa n hnp\_hnksa - 2n^3 - 3ksa hnp\_hnksa + n^2 + 3n) ) kspa^2 \\ &\quad - \frac{1}{(2n+3)(4n^2+16n+15)} (n ksa^2 rhoP (8ksa^3 n^3 rhoC hnp\_hnksa \\ &\quad - 12ksa^3 n^3 rhoP hnp\_hnksa + 36ksa^3 n^2 rhoC hnp\_hnksa - 40ksa^3 n^2 rhoP hnp\_hnksa \\ &\quad + 8ksa^2 n^4 rhoC + 12ksa^2 n^4 rhoP + 46ksa^3 n rhoC hnp\_hnksa \\ &\quad - 38ksa^3 n rhoP hnp\_hnksa + 20ksa^2 n^3 rhoC + 40ksa^2 n^3 rhoP \\ &\quad - 24ksa n^3 rhoC hnp\_hnksa + 15ksa^3 rhoC hnp\_hnksa - 15ksa^3 rhoP hnp\_hnksa \\ &\quad - 26ksa^2 n^2 rhoC + 38ksa^2 n^2 rhoP - 72ksa n^2 rhoC hnp\_hnksa + 24n^4 rhoC \\ &\quad - 77ksa^2 n rhoC + 15ksa^2 n rhoP + 6ksa n rhoC hnp\_hnksa + 72n^3 rhoC - 30ksa^2 rhoC \\ &\quad + 90ksa rhoC hnp\_hnksa - 6n^2 rhoC - 90n rhoC) ) kspa^4 + O(kspa^6) \end{aligned} \quad (24)$$

$$\begin{aligned} &> denTnCC\_Serkspa := simplify(series(denTnCC_, kspa, 8))\# \\ denTnCC\_Serkspa &:= -\frac{1}{2n+3} (2ksa^4 rhoP^2 (2ksa n^4 hnp\_hnksa + 4ksa n^3 hnp\_hnksa + 2n^5 \\ &\quad + ksa n^2 hnp\_hnksa + 6n^4 - 4ksa n hnp\_hnksa + 5n^3 - 3ksa hnp\_hnksa - 3n^2 - 7n \\ &\quad - 3) ) kspa^2 + \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2 rhoP (8ksa^3 n^4 rhoC hnp\_hnksa \\ &\quad + 12ksa^3 n^4 rhoP hnp\_hnksa + 36ksa^3 n^3 rhoC hnp\_hnksa + 52ksa^3 n^3 rhoP hnp\_hnksa \\ &\quad - 8ksa^2 n^5 rhoC + 12ksa^2 n^5 rhoP + 46ksa^3 n^2 rhoC hnp\_hnksa \\ &\quad + 78ksa^3 n^2 rhoP hnp\_hnksa - 60ksa^2 n^4 rhoC + 64ksa^2 n^4 rhoP \\ &\quad + 24ksa n^4 rhoC hnp\_hnksa + 15ksa^3 n rhoC hnp\_hnksa + 53ksa^3 n rhoP hnp\_hnksa \\ &\quad - 178ksa^2 n^3 rhoC + 130ksa^2 n^3 rhoP + 96ksa n^3 rhoC hnp\_hnksa + 24n^5 rhoC \\ &\quad + 15ksa^3 rhoP hnp\_hnksa - 261ksa^2 n^2 rhoC + 131ksa^2 n^2 rhoP \\ &\quad + 66ksa n^2 rhoC hnp\_hnksa + 120n^4 rhoC - 183ksa^2 n rhoC + 68ksa^2 n rhoP \\ &\quad - 96ksa n rhoC hnp\_hnksa + 162n^3 rhoC - 45ksa^2 rhoC + 15ksa^2 rhoP \\ &\quad - 90ksa rhoC hnp\_hnksa - 30n^2 rhoC - 186n rhoC - 90rhoC) ) kspa^4 + O(kspa^6) \end{aligned} \quad (25)$$

Check the case n=1 to make sure both numerator and denominator are defined - then comment this out to retain general n for the overall solution.

> #n:=1



> #numTnCC\_Serkspa # Confirms that only the second term in the series remains and the first term vanishes, this is numerator of the n=1 solution

> #denTnCC\_Serkspa # Similarly confirms only second term remains - this is the denom of the n=1 solution

Now simplify the series in numerator and denominator as far as possible (a factor of kspa^2 can be removed from all terms

so the kspa2 terms are the first terms (with A and C) in numerator and denominator

> numTnCC\_kspa2 := simplify(collect(coeff(numTnCC\_Serkspa, kspa, 2), hnp\_hnksa))  
# this is expression "A" as written in the text above

$$\text{numTnCC\_kspa2} := - \frac{2 \text{ksa}^4 n \text{rhoP}^2 (2 n^3 + 2 n^2 - n - 3) (\text{ksa hnp\_hnksa} - n)}{2 n + 3} \quad (26)$$

> numTnCC\_kspa2facs := factor(simplify( $\frac{\text{numTnCC\_kspa2} \cdot (2 \cdot n + 3)}{(\text{ksa} \cdot \text{hnp\_hnksa} - n)}$ )))

$$\text{numTnCC\_kspa2facs} := -2 \text{ksa}^4 n \text{rhoP}^2 (n - 1) (2 n^2 + 4 n + 3) \quad (27)$$

\*\*\*\*\*This (above) is the simplified version of the first term in the numerator (combined with the factors removed in the formula)

> denTnCC\_kspa2 := (collect((coeff(denTnCC\_Serkspa, kspa, 2)), hnp\_hnksa))  
#this is expression "C" as written in the text above

$$\text{denTnCC\_kspa2} := - \frac{2 \text{ksa}^4 \text{rhoP}^2 (2 \text{ksa} n^4 + 4 \text{ksa} n^3 + \text{ksa} n^2 - 4 \text{ksa} n - 3 \text{ksa}) \text{hnp\_hnksa}}{2 n + 3} \quad (28)$$

$$- \frac{2 \text{ksa}^4 \text{rhoP}^2 (2 n^5 + 6 n^4 + 5 n^3 - 3 n^2 - 7 n - 3)}{2 n + 3}$$

> denTnCC\_kspa2facs := factor(simplify( $\frac{\text{denTnCC\_kspa2} \cdot (2 \cdot n + 3)}{(\text{ksa} \cdot \text{hnp\_hnksa} + (n + 1)) \cdot \text{ksa}^4 \cdot \text{rhoP}^2}$ )))

$$\text{denTnCC\_kspa2facs} := -2 (n - 1) (n + 1) (2 n^2 + 4 n + 3) \quad (29)$$

\*\*\*\*\*This (above) is the expression for the first term in the denominator for all coefficients

Combining these gives the expression for n>=2

$$\text{TnCC\_nGE2} := \frac{n \cdot (\text{ksa} \cdot \text{hnp\_hnksa} - n)}{(n + 1) \cdot (\text{ksa} \cdot \text{hnp\_hnksa} + (n + 1))}$$

$$\text{TnCC\_nGE2} := \frac{n (\text{ksa hnp\_hnksa} - n)}{(n + 1) (\text{ksa hnp\_hnksa} + n + 1)} \quad (30)$$

Now the second term of each series - these are the terms that give the n=1 expression

> numTnCC\_kspa4 := simplify(coeff(numTnCC\_Serkspa, kspa, 4))  
#this is expression "B" as written in the text above

$$\text{numTnCC\_kspa4} := - \frac{1}{(2 n + 3) (4 n^2 + 16 n + 15)} (n \text{ksa}^2 \text{rhoP} (8 \text{ksa}^3 n^3 \text{rhoC hnp\_hnksa} \quad (31)$$

$$\begin{aligned} & - 12 \text{ksa}^3 n^3 \text{rhoP hnp\_hnksa} + 36 \text{ksa}^3 n^2 \text{rhoC hnp\_hnksa} - 40 \text{ksa}^3 n^2 \text{rhoP hnp\_hnksa} \\ & + 8 \text{ksa}^2 n^4 \text{rhoC} + 12 \text{ksa}^2 n^4 \text{rhoP} + 46 \text{ksa}^3 n \text{rhoC hnp\_hnksa} \\ & - 38 \text{ksa}^3 n \text{rhoP hnp\_hnksa} + 20 \text{ksa}^2 n^3 \text{rhoC} + 40 \text{ksa}^2 n^3 \text{rhoP} \\ & - 24 \text{ksa} n^3 \text{rhoC hnp\_hnksa} + 15 \text{ksa}^3 \text{rhoC hnp\_hnksa} - 15 \text{ksa}^3 \text{rhoP hnp\_hnksa} \\ & - 26 \text{ksa}^2 n^2 \text{rhoC} + 38 \text{ksa}^2 n^2 \text{rhoP} - 72 \text{ksa} n^2 \text{rhoC hnp\_hnksa} + 24 n^4 \text{rhoC} \end{aligned}$$

$$- 77 ksa^2 n rhoC + 15 ksa^2 n rhoP + 6 ksa n rhoC hnp\_hnksa + 72 n^3 rhoC - 30 ksa^2 rhoC + 90 ksa rhoC hnp\_hnksa - 6 n^2 rhoC - 90 n rhoC)$$

See what factors this expression - expecting (hnp\_hnksa-1) but it may only factorise for the case n=1

> numTnCC\_kspa4facs

$$:= collect \left( simplify \left( expand \left( \frac{numTnCC\_kspa4 \cdot (2 \cdot n + 3) \cdot (4 \cdot n^2 + 16 \cdot n + 15)}{n \cdot ksa^2 \cdot rhoP} \right) \right), hnp\_hnksa \right)$$

$$\begin{aligned} numTnCC\_kspa4facs := & (-8 ksa^3 n^3 rhoC + 12 ksa^3 n^3 rhoP - 36 ksa^3 n^2 rhoC \\ & + 40 ksa^3 n^2 rhoP - 46 ksa^3 n rhoC + 38 ksa^3 n rhoP + 24 ksa n^3 rhoC - 15 ksa^3 rhoC \\ & + 15 ksa^3 rhoP + 72 ksa n^2 rhoC - 6 ksa n rhoC - 90 ksa rhoC) hnp\_hnksa \\ & - 8 ksa^2 n^4 rhoC - 12 ksa^2 n^4 rhoP - 20 ksa^2 n^3 rhoC - 40 ksa^2 n^3 rhoP + 26 ksa^2 n^2 rhoC \\ & - 38 ksa^2 n^2 rhoP - 24 n^4 rhoC + 77 ksa^2 n rhoC - 15 ksa^2 n rhoP - 72 n^3 rhoC \\ & + 30 ksa^2 rhoC + 6 n^2 rhoC + 90 n rhoC \end{aligned} \quad (32)$$

Note that (4n^2+16n+15) factorises to (2n+3)(2n+5)

> denTnCC\_kspa4 := simplify(coeff(denTnCC\_Serkspa, kspa, 4))  
#this is the expression "D" as written in the text above

$$\begin{aligned} denTnCC\_kspa4 := & \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2 rhoP (8 ksa^3 n^4 rhoC hnp\_hnksa \\ & + 12 ksa^3 n^4 rhoP hnp\_hnksa + 36 ksa^3 n^3 rhoC hnp\_hnksa + 52 ksa^3 n^3 rhoP hnp\_hnksa \\ & - 8 ksa^2 n^5 rhoC + 12 ksa^2 n^5 rhoP + 46 ksa^3 n^2 rhoC hnp\_hnksa \\ & + 78 ksa^3 n^2 rhoP hnp\_hnksa - 60 ksa^2 n^4 rhoC + 64 ksa^2 n^4 rhoP \\ & + 24 ksa n^4 rhoC hnp\_hnksa + 15 ksa^3 n rhoC hnp\_hnksa + 53 ksa^3 n rhoP hnp\_hnksa \\ & - 178 ksa^2 n^3 rhoC + 130 ksa^2 n^3 rhoP + 96 ksa n^3 rhoC hnp\_hnksa + 24 n^5 rhoC \\ & + 15 ksa^3 rhoP hnp\_hnksa - 261 ksa^2 n^2 rhoC + 131 ksa^2 n^2 rhoP \\ & + 66 ksa n^2 rhoC hnp\_hnksa + 120 n^4 rhoC - 183 ksa^2 n rhoC + 68 ksa^2 n rhoP \\ & - 96 ksa n rhoC hnp\_hnksa + 162 n^3 rhoC - 45 ksa^2 rhoC + 15 ksa^2 rhoP \\ & - 90 ksa rhoC hnp\_hnksa - 30 n^2 rhoC - 186 n rhoC - 90 rhoC) \end{aligned} \quad (33)$$

> denTnCC\_kspa4facs

$$:= collect \left( simplify \left( expand \left( \frac{denTnCC\_kspa4 \cdot (2 \cdot n + 3) \cdot (4 \cdot n^2 + 16 \cdot n + 15)}{ksa^2 \cdot rhoP} \right) \right), hnp\_hnksa \right)$$

$$\begin{aligned} denTnCC\_kspa4facs := & (8 ksa^3 n^4 rhoC + 12 ksa^3 n^4 rhoP + 36 ksa^3 n^3 rhoC + 52 ksa^3 n^3 rhoP \\ & + 46 ksa^3 n^2 rhoC + 78 ksa^3 n^2 rhoP + 24 ksa n^4 rhoC + 15 ksa^3 n rhoC + 53 ksa^3 n rhoP \\ & + 96 ksa n^3 rhoC + 15 ksa^3 rhoP + 66 ksa n^2 rhoC - 96 ksa n rhoC - 90 ksa rhoC) \\ & hnp\_hnksa - 8 ksa^2 n^5 rhoC + 12 ksa^2 n^5 rhoP - 60 ksa^2 n^4 rhoC + 64 ksa^2 n^4 rhoP \\ & - 178 ksa^2 n^3 rhoC + 130 ksa^2 n^3 rhoP + 24 n^5 rhoC - 261 ksa^2 n^2 rhoC \end{aligned} \quad (34)$$

$$+ 131 ksa^2 n^2 rhoP + 120 n^4 rhoC - 183 ksa^2 n rhoC + 68 ksa^2 n rhoP + 162 n^3 rhoC \\ - 45 ksa^2 rhoC + 15 ksa^2 rhoP - 30 n^2 rhoC - 186 n rhoC - 90 rhoC$$

Try and simplify the parts of these expressions. First the second term of the numerator series, starting with the terms in hnp\_hnksa

We know that there is a factor of (ksa\*hnp\_hnksa-n) and a factor (rhoHat-1) in the n=1 solution

So collect terms in rhoC(ksa\*hnp\_hn) and then rhoP(ksa\*hnp\_hn) and those in n\*rhoC and n\*rhoP.

Use the matching terms in rhoP to collect (rhoP-rhoC)(ksa\*hnp\_hn-n) and leave remaining terms just in rhoC. These remaining terms turn out to have factors (n-1) which means they vanish for n=1. This simplification was completed by hand. Check it matches what was obtained above

$$\begin{aligned} &> numTnCC\_kspa4facsSimpl := (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15) \cdot (rhoP - rhoC) \cdot ksa^2 \cdot (ksa \\ &\quad \cdot hnp\_hnksa - n) + 4 \cdot n \cdot (n - 1) \cdot (n + 2) \cdot rhoC \cdot ksa^2 \cdot (ksa \cdot hnp\_hnksa - n) + 6 \cdot (n - 1) \cdot (2 \\ &\quad \cdot n + 3) \cdot (2 \cdot n + 5) \cdot rhoC \cdot (ksa \cdot hnp\_hnksa - n) - 2 \cdot (n - 1) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 3) \cdot (2 \cdot n \\ &\quad + 5) \cdot rhoC \cdot ksa^2 \\ numTnCC\_kspa4facsSimpl &:= (12 n^3 + 40 n^2 + 38 n + 15) (rhoP - rhoC) ksa^2 (ksa hnp\_hnksa - n) + 4 n (n - 1) (n + 2) rhoC ksa^2 (ksa hnp\_hnksa - n) + 6 (n - 1) (2 n + 3) (2 n \\ &\quad + 5) rhoC (ksa hnp\_hnksa - n) - 2 (n - 1) (2 n + 1) (2 n + 3) (2 n + 5) rhoC ksa^2 \end{aligned} \quad (35)$$

\*\*\*\*\*This (above) is the simplified result for the second term in numerator of TnCC (when combined with the factors in line 32)

$$\begin{aligned} &> expand(numTnCC\_kspa4facsSimpl - numTnCC\_kspa4facs) \\ &\quad \# confirm that this is zero, therefore the simplification is correct. \\ &\quad \quad \quad 0 \end{aligned} \quad (36)$$

Now try and simplify the second term in the series in kspa in the denominator denTnCC\_kspa4facs.

Based on some analytical simplifications, and by comparing with the n=1 known result, guessing that the remaining terms will be of the form

$$(2n+1)*rhoC*(ksa*hnp/hn-n)+(n+1)*(rhoP-rhoC)*(ksa*hnp/hn+(n+1))$$

$$\begin{aligned} &> denRhoC := simplify(denTnCC\_kspa4facs - (n + 1) \cdot (rhoP - rhoC) \cdot ksa^2 \cdot (ksa \cdot hnp\_hnksa \\ &\quad + (n + 1)) \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15)) \\ denRhoC &:= 20 ksa^3 n^4 rhoC hnp\_hnksa + 88 ksa^3 n^3 rhoC hnp\_hnksa + 4 ksa^2 n^5 rhoC \\ &\quad + 124 ksa^3 n^2 rhoC hnp\_hnksa + 4 ksa^2 n^4 rhoC + 24 ksa n^4 rhoC hnp\_hnksa \\ &\quad + 68 ksa^3 n rhoC hnp\_hnksa - 48 ksa^2 n^3 rhoC + 96 ksa n^3 rhoC hnp\_hnksa + 24 n^5 rhoC \\ &\quad + 15 ksa^3 rhoC hnp\_hnksa - 130 ksa^2 n^2 rhoC + 66 ksa n^2 rhoC hnp\_hnksa + 120 n^4 rhoC \\ &\quad - 115 ksa^2 n rhoC - 96 ksa n rhoC hnp\_hnksa + 162 n^3 rhoC - 30 ksa^2 rhoC \\ &\quad - 90 ksa rhoC hnp\_hnksa - 30 n^2 rhoC - 186 n rhoC - 90 rhoC \end{aligned} \quad (37)$$

$$\begin{aligned} &> denRhoCLeft := collect(simplify(expand((denRhoC - (2 \cdot n + 1) \cdot (ksa \cdot hnp\_hnksa - n) \cdot ksa^2 \\ &\quad \cdot rhoC \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15)))), hnp\_hnksa) \\ denRhoCLeft &:= (-4 ksa^3 n^4 rhoC - 4 ksa^3 n^3 rhoC + 8 ksa^3 n^2 rhoC + 24 ksa n^4 rhoC \\ &\quad + 96 ksa n^3 rhoC + 66 ksa n^2 rhoC - 96 ksa n rhoC - 90 ksa rhoC) hnp\_hnksa \\ &\quad + 28 ksa^2 n^5 rhoC + 96 ksa^2 n^4 rhoC + 68 ksa^2 n^3 rhoC + 24 n^5 rhoC - 62 ksa^2 n^2 rhoC \\ &\quad + 120 n^4 rhoC - 100 ksa^2 n rhoC + 162 n^3 rhoC - 30 ksa^2 rhoC - 30 n^2 rhoC \\ &\quad - 186 n rhoC - 90 rhoC \end{aligned} \quad (38)$$

$$> denRhoCLeft2 := simplify(denRhoCLeft - rhoC \cdot ksa^2 \cdot (-4 \cdot n^4 - 4 \cdot n^3 + 8 \cdot n^2) \cdot (ksa \cdot hnp\_hnksa$$

$$\begin{aligned}
& -n) - \rho C \cdot (24 \cdot n^4 + 96 \cdot n^3 + 66 \cdot n^2 - 96 \cdot n - 90) \cdot (ksa \cdot hnp\_hnksa - n)) \\
denRhoCLeft2 := & 24 \, ksa^2 \, n^5 \, \rho C + 92 \, ksa^2 \, n^4 \, \rho C + 76 \, ksa^2 \, n^3 \, \rho C + 48 \, n^5 \, \rho C \\
& - 62 \, ksa^2 \, n^2 \, \rho C + 216 \, n^4 \, \rho C - 100 \, ksa^2 \, n \, \rho C + 228 \, n^3 \, \rho C - 30 \, ksa^2 \, \rho C \\
& - 126 \, n^2 \, \rho C - 276 \, n \, \rho C - 90 \, \rho C
\end{aligned} \tag{39}$$

$$\begin{aligned}
> denRhoCLeft3 := & collect\left(simplify\left(expand\left(\frac{denRhoCLeft2}{(n-1)}\right)\right), ksa\right) \\
denRhoCLeft3 := & 2 \, (12 \, n^4 + 58 \, n^3 + 96 \, n^2 + 65 \, n + 15) \, \rho C \, ksa^2 + 2 \, (24 \, n^4 + 132 \, n^3 \\
& + 246 \, n^2 + 183 \, n + 45) \, \rho C
\end{aligned} \tag{40}$$

So we have obtained a simplified result for the second term of the series in the denominator. Because these remaining terms have a factor (n-1) and we have a reasonably short expression for them. Now reconstruct it (with factorisations done manually) and check that the simplified form is correct. Check by subtracting each part of the simplified expression from the full result (with factors removed) sequentially

$$\begin{aligned}
> denTnCC\_kspa4facsRed := & denTnCC\_kspa4facs - (n+1) \cdot (\rho P - \rho C) \cdot ksa^2 \cdot (ksa \\
& \cdot hnp\_hnksa + (n+1)) \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15) : \\
> denTnCC\_kspa4facsRed := & denTnCC\_kspa4facsRed - (2 \cdot n + 1) \cdot (ksa \cdot hnp\_hnksa - n) \cdot ksa^2 \\
& \cdot \rho C \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15) : \\
> denTnCC\_kspa4facsRed := & denTnCC\_kspa4facsRed - (-4 \cdot n^2) \cdot (n-1) \cdot (n+2) \cdot \rho C \cdot ksa^2 \\
& \cdot (ksa \cdot hnp\_hnksa - n) : \\
> denTnCC\_kspa4facsRed := & denTnCC\_kspa4facsRed - 6 \cdot (n-1) \cdot (n+1) \cdot (2 \cdot n + 3) \cdot (2 \cdot n \\
& + 5) \cdot \rho C \cdot (ksa \cdot hnp\_hnksa - n) : \\
> collect\left(simplify\left(expand\left(\frac{denTnCC\_kspa4facsRed}{(n-1)}\right)\right), ksa\right) \\
2 \, (12 \, n^4 + 58 \, n^3 + 96 \, n^2 + 65 \, n + 15) \, \rho C \, ksa^2 + & 2 \, (24 \, n^4 + 132 \, n^3 + 246 \, n^2 + 183 \, n \\
& + 45) \, \rho C
\end{aligned} \tag{41}$$

$$\begin{aligned}
> denTnCC\_kspa4facsRed := & simplify(denTnCC\_kspa4facsRed - 2 \cdot (n-1) \cdot (n+1) \cdot (2 \cdot n + 1) \\
& \cdot (6 \, n^2 + 20 \, n + 15) \cdot \rho C \cdot ksa^2) \\
denTnCC\_kspa4facsRed := & 48 \, n^5 \, \rho C + 216 \, n^4 \, \rho C + 228 \, n^3 \, \rho C - 126 \, n^2 \, \rho C \\
& - 276 \, n \, \rho C - 90 \, \rho C
\end{aligned} \tag{42}$$

$$\begin{aligned}
> denTnCC\_kspa4facsRed := & simplify(denTnCC\_kspa4facsRed - 6 \cdot (n-1) \cdot (n+1) \cdot (2 \cdot n + 1) \\
& \cdot (2 \cdot n + 3) \cdot (2 \cdot n + 5) \cdot \rho C) \\
denTnCC\_kspa4facsRed := & 0
\end{aligned} \tag{43}$$

Therefore these set of simplified expressions agree with the original second term of series in denominator. So the set of expressions just above allow the construction of the term

$$\begin{aligned}
> denTnCC\_kspa4facsSimpl := & (n+1) \cdot (\rho P - \rho C) \cdot ksa^2 \cdot (ksa \cdot hnp\_hnksa + (n+1)) \cdot (12 \cdot n^3 \\
& + 40 \cdot n^2 + 38 \cdot n + 15) : \\
> denTnCC\_kspa4facsSimpl := & denTnCC\_kspa4facsSimpl + (2 \cdot n + 1) \cdot (ksa \cdot hnp\_hnksa - n) \cdot ksa^2 \\
& \cdot \rho C \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15) : \\
> denTnCC\_kspa4facsSimpl := & denTnCC\_kspa4facsSimpl + (-4 \cdot n^2) \cdot (n-1) \cdot (n+2) \cdot \rho C \\
& \cdot ksa^2 \cdot (ksa \cdot hnp\_hnksa - n) : \\
> denTnCC\_kspa4facsSimpl := & denTnCC\_kspa4facsSimpl + 6 \cdot (n-1) \cdot (n+1) \cdot (2 \cdot n + 3) \cdot (2 \cdot n
\end{aligned}$$

$+ 5) \cdot \text{rhoC} \cdot (\text{ksa} \cdot \text{hnp\_hnksa} - n) :$

>  $\text{denTnCC\_kspa4facsSimpl} := \text{denTnCC\_kspa4facsSimpl} + 2 \cdot (n - 1) \cdot (n + 1) \cdot (2 \cdot n + 1) \cdot (6 n^2 + 20 n + 15) \cdot \text{rhoC} \cdot \text{ksa}^2 :$

>  $\text{denTnCC\_kspa4facsSimpl} := \text{denTnCC\_kspa4facsSimpl} + 6 \cdot (n - 1) \cdot (n + 1) \cdot (2 \cdot n + 1) \cdot (2 \cdot n + 3) \cdot (2 \cdot n + 5) \cdot \text{rhoC} :$

\*\*\*\*\*This (above) is the simplified expression for the second term in denominator, combined with factors removed in line 34.

>  $\text{simplify}(\text{denTnCC\_kspa4facs} - \text{denTnCC\_kspa4facsSimpl})$   
*# this is zero, so the simplification is correct.*  
 $0$  (44)

>  $\text{denTnCC\_kspa4facsSimpl}$  *# the expression after the factors were removed*  
 $(n + 1) (\text{rhoP} - \text{rhoC}) \text{ksa}^2 (\text{ksa} \cdot \text{hnp\_hnksa} + n + 1) (12 n^3 + 40 n^2 + 38 n + 15) + (2 n + 1) (\text{ksa} \cdot \text{hnp\_hnksa} - n) \text{ksa}^2 \text{rhoC} (12 n^3 + 40 n^2 + 38 n + 15) - 4 n^2 (n - 1) (n + 2) \text{rhoC} \text{ksa}^2 (\text{ksa} \cdot \text{hnp\_hnksa} - n) + 6 (n - 1) (n + 1) (2 n + 3) (2 n + 5) \text{rhoC} (\text{ksa} \cdot \text{hnp\_hnksa} - n) + 2 (n - 1) (n + 1) (2 n + 1) (6 n^2 + 20 n + 15) \text{rhoC} \text{ksa}^2 + 6 (n - 1) (n + 1) (2 n + 1) (2 n + 3) (2 n + 5) \text{rhoC}$  (45)

\*\*\*\*\*This (above) is the simplified expression for the second term in denominator, combined with factors removed in line 34.

We now have every expression needed to construct TnCC\_

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Now work on the other coefficients. First TnCS\_ (scaled) numerator only (since denominator is identical)

>  $\text{numTnCS\_Serkspa} := \text{simplify}(\text{series}(\text{numTnCS\_}, \text{kspa}, 8))$   
 $\text{numTnCS\_Serkspa} := \frac{2 \text{ksa}^4 \text{rhoP}^2 (4 n^4 + 6 n^3 - 7 n - 3)}{2 n + 3} \text{kspa}^2$  (46)

$-\frac{1}{(2 n + 3) (4 n^2 + 16 n + 15)} (\text{ksa}^2 \text{rhoP} (24 \text{ksa}^2 n^4 \text{rhoP} - 24 \text{ksa}^2 n^3 \text{rhoC} + 92 \text{ksa}^2 n^3 \text{rhoP} - 108 \text{ksa}^2 n^2 \text{rhoC} + 116 \text{ksa}^2 n^2 \text{rhoP} + 48 n^4 \text{rhoC} - 138 \text{ksa}^2 n \text{rhoC} + 68 \text{ksa}^2 n \text{rhoP} + 168 n^3 \text{rhoC} - 45 \text{ksa}^2 \text{rhoC} + 15 \text{ksa}^2 \text{rhoP} + 60 n^2 \text{rhoC} - 186 n \text{rhoC} - 90 \text{rhoC})) \text{kspa}^4 + O(\text{kspa}^6)$

>  $\text{numTnCS\_kspa2} := \text{simplify}((\text{coeff}(\text{numTnCS\_Serkspa}, \text{kspa}, 2)))$   
 $\text{numTnCS\_kspa2} := \frac{2 \text{ksa}^4 \text{rhoP}^2 (4 n^4 + 6 n^3 - 7 n - 3)}{2 n + 3}$  (47)

\*\*\*\*\*This (above) is the simplified result for the first term in the numerator of TnCS\_

>  $\text{numTnCS\_kspa4} := \text{simplify}(\text{coeff}(\text{numTnCS\_Serkspa}, \text{kspa}, 4))$   
*# this is expression "F" as written in the text above - second term in numerator*  
 $\text{numTnCS\_kspa4} := -\frac{1}{(2 n + 3) (4 n^2 + 16 n + 15)} (\text{ksa}^2 \text{rhoP} (24 \text{ksa}^2 n^4 \text{rhoP} - 24 \text{ksa}^2 n^3 \text{rhoC} + 92 \text{ksa}^2 n^3 \text{rhoP} - 108 \text{ksa}^2 n^2 \text{rhoC} + 116 \text{ksa}^2 n^2 \text{rhoP} + 48 n^4 \text{rhoC} - 138 \text{ksa}^2 n \text{rhoC} + 68 \text{ksa}^2 n \text{rhoP} + 168 n^3 \text{rhoC} - 45 \text{ksa}^2 \text{rhoC} + 15 \text{ksa}^2 \text{rhoP} + 60 n^2 \text{rhoC} - 186 n \text{rhoC} - 90 \text{rhoC}))$  (48)

$$- 138 ksa^2 n rhoC + 68 ksa^2 n rhoP + 168 n^3 rhoC - 45 ksa^2 rhoC + 15 ksa^2 rhoP \\ + 60 n^2 rhoC - 186 n rhoC - 90 rhoC)$$

remove factors

> numTnCS\_kspa4facs

$$:= collect\left(simplify\left(expand\left(\frac{numTnCS\_kspa4 \cdot (2 \cdot n + 3) \cdot (4 \cdot n^2 + 16 \cdot n + 15)}{ksa^2 \cdot rhoP}\right)\right), rhoP\right)$$

$$numTnCS\_kspa4facs := (-24 ksa^2 n^4 - 92 ksa^2 n^3 - 116 ksa^2 n^2 - 68 ksa^2 n - 15 ksa^2) rhoP \\ + 24 ksa^2 n^3 rhoC + 108 ksa^2 n^2 rhoC - 48 n^4 rhoC + 138 ksa^2 n rhoC - 168 n^3 rhoC \\ + 45 ksa^2 rhoC - 60 n^2 rhoC + 186 n rhoC + 90 rhoC \quad (49)$$

Collect terms to result in a (rhoP-rhoC) expression (which appears in the n=1 coefficient) and remaining terms with a factor of (n-1). Manual factorisation.

$$> numTnCS\_kspa4facsSimpl := -(rhoP - rhoC) \cdot ksa^2 \cdot (2 \cdot n + 1) \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n + 15) \\ - 2 \cdot rhoC \cdot ksa^2 \cdot (n - 1) \cdot (2 \cdot n + 1) \cdot (6 \cdot n^2 + 20 \cdot n + 15) - 6 \cdot rhoC \cdot (n - 1) \cdot (2 \cdot n + 1) \cdot (2 \\ \cdot n + 3) \cdot (2 \cdot n + 5)$$

$$numTnCS\_kspa4facsSimpl := -(rhoP - rhoC) ksa^2 (2 n + 1) (12 n^3 + 40 n^2 + 38 n + 15) \\ - 2 rhoC ksa^2 (n - 1) (2 n + 1) (6 n^2 + 20 n + 15) - 6 rhoC (n - 1) (2 n + 1) (2 n \\ + 3) (2 n + 5) \quad (50)$$

\*\*\*\*\*This is the simplified expression for the second term in the denominator of TnCS\_ (combined with factors in line 49)

$$> simplify(expand(numTnCS\_kspa4facsSimpl - numTnCS\_kspa4facs)) \\ \#check that the simplified version agrees with the original$$

0

(51)

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Now work on numerator of TnSC\_ (scaled). Denominator is identical to TnCC\_

$$> numTnSC\_Serksa := simplify(series(numTnSC_, kspa, 8))$$

$$numTnSC\_Serksa := -\frac{1}{2n+3} (2n(n+1) ksa^5 rhoP^2 (2n^3 hnp\_hnksa - 2n^3 jnp\_jnksa \\ + 2n^2 hnp\_hnksa - 2n^2 jnp\_jnksa - n hnp\_hnksa + n jnp\_jnksa - 3 hnp\_hnksa \\ + 3 jnp\_jnksa)) kspa^2 + \frac{1}{(2n+3)(4n^2+16n+15)} (n(n \\ + 1) ksa^3 rhoP (12 ksa^2 n^3 rhoP hnp\_hnksa - 12 ksa^2 n^3 rhoP jnp\_jnksa \\ - 12 ksa^2 n^2 rhoC hnp\_hnksa + 12 ksa^2 n^2 rhoC jnp\_jnksa + 40 ksa^2 n^2 rhoP hnp\_hnksa \\ - 40 ksa^2 n^2 rhoP jnp\_jnksa - 48 ksa^2 n rhoC hnp\_hnksa + 48 ksa^2 n rhoC jnp\_jnksa \\ + 38 ksa^2 n rhoP hnp\_hnksa - 38 ksa^2 n rhoP jnp\_jnksa + 24 n^3 rhoC hnp\_hnksa \\ - 24 n^3 rhoC jnp\_jnksa - 45 ksa^2 rhoC hnp\_hnksa + 45 ksa^2 rhoC jnp\_jnksa \\ + 15 ksa^2 rhoP hnp\_hnksa - 15 ksa^2 rhoP jnp\_jnksa + 72 n^2 rhoC hnp\_hnksa \\ - 72 n^2 rhoC jnp\_jnksa - 6 n rhoC hnp\_hnksa + 6 n rhoC jnp\_jnksa - 90 rhoC hnp\_hnksa \\ + 90 rhoC jnp\_jnksa)) kspa^4 + O(ksa^6) \quad (52)$$

$$> numTnSC\_ksa2 := simplify((coeff(numTnSC\_Serksa, kspa, 2)))$$

(53)

$$\text{numTnSC\_kspa2} := -\frac{1}{2n+3} (2n(n+1)ksa^5\rho P^2(2n^3\text{hnp\_hnksa} - 2n^3\text{jnp\_jnksa} + 2n^2\text{hnp\_hnksa} - 2n^2\text{jnp\_jnksa} - n\text{hnp\_hnksa} + n\text{jnp\_jnksa} - 3\text{hnp\_hnksa} + 3\text{jnp\_jnksa})) \quad (53)$$

$$\begin{aligned} &> \text{numTnSC\_kspa2Simpl1} := \text{simplify}\left(\text{expand}\left(\text{subs}\left(\text{jnp\_jnksa} = \text{hnp\_hnksa}\right.\right.\right. \\ &\quad \left.\left.\left.- \frac{I}{ksa^2 \cdot \text{jnksa} \cdot \text{hnksa}}, \text{numTnSC\_kspa2}\right)\right)\right) \# \text{use the Heine relation to simplify} \\ &\quad \text{numTnSC\_kspa2Simpl1} := -\frac{2Inksa^3\rho P^2(2n^4 + 4n^3 + n^2 - 4n - 3)}{(2n+3)\text{jnksa}\text{hnksa}} \quad (54) \end{aligned}$$

\*\*\*\*\*This (above) is the simplified expression for the first term in the numerator of TnSC\_

$$\begin{aligned} &> \text{numTnSC\_kspa4} := \text{simplify}((\text{coeff}(\text{numTnSC\_Serksa}, \text{kspa}, 4))) \\ &\quad \text{numTnSC\_kspa4} := \frac{1}{(2n+3)(4n^2+16n+15)} (n(n \\ &\quad + 1)ksa^3\rho P(12ksa^2n^3\rho P\text{hnp\_hnksa} - 12ksa^2n^3\rho P\text{jnp\_jnksa} \\ &\quad - 12ksa^2n^2\rho C\text{hnp\_hnksa} + 12ksa^2n^2\rho C\text{jnp\_jnksa} + 40ksa^2n^2\rho P\text{hnp\_hnksa} \\ &\quad - 40ksa^2n^2\rho P\text{jnp\_jnksa} - 48ksa^2n\rho C\text{hnp\_hnksa} + 48ksa^2n\rho C\text{jnp\_jnksa} \\ &\quad + 38ksa^2n\rho P\text{hnp\_hnksa} - 38ksa^2n\rho P\text{jnp\_jnksa} + 24n^3\rho C\text{hnp\_hnksa} \\ &\quad - 24n^3\rho C\text{jnp\_jnksa} - 45ksa^2\rho C\text{hnp\_hnksa} + 45ksa^2\rho C\text{jnp\_jnksa} \\ &\quad + 15ksa^2\rho P\text{hnp\_hnksa} - 15ksa^2\rho P\text{jnp\_jnksa} + 72n^2\rho C\text{hnp\_hnksa} \\ &\quad - 72n^2\rho C\text{jnp\_jnksa} - 6n\rho C\text{hnp\_hnksa} + 6n\rho C\text{jnp\_jnksa} - 90\rho C\text{hnp\_hnksa} \\ &\quad + 90\rho C\text{jnp\_jnksa})) \quad (55) \end{aligned}$$

$$\begin{aligned} &> \text{numTnSC\_kspa4fac} \\ &\quad := \left( \text{simplify}\left(\text{expand}\left(\frac{\text{numTnSC\_kspa4} \cdot (2 \cdot n + 3) \cdot (4 \cdot n^2 + 16 \cdot n + 15)}{ksa^2 \cdot \rho P}\right)\right) \right) \\ &\quad \text{numTnSC\_kspa4fac} := 12ksa^3n^5\rho P\text{hnp\_hnksa} - 12ksa^3n^5\rho P\text{jnp\_jnksa} \quad (56) \\ &\quad - 12ksa^3n^4\rho C\text{hnp\_hnksa} + 12ksa^3n^4\rho C\text{jnp\_jnksa} + 52ksa^3n^4\rho P\text{hnp\_hnksa} \\ &\quad - 52ksa^3n^4\rho P\text{jnp\_jnksa} - 60ksa^3n^3\rho C\text{hnp\_hnksa} + 60ksa^3n^3\rho C\text{jnp\_jnksa} \\ &\quad + 78ksa^3n^3\rho P\text{hnp\_hnksa} - 78ksa^3n^3\rho P\text{jnp\_jnksa} + 24ksa^5\rho C\text{hnp\_hnksa} \\ &\quad - 24ksa^5\rho C\text{jnp\_jnksa} - 93ksa^3n^2\rho C\text{hnp\_hnksa} + 93ksa^3n^2\rho C\text{jnp\_jnksa} \\ &\quad + 53ksa^3n^2\rho P\text{hnp\_hnksa} - 53ksa^3n^2\rho P\text{jnp\_jnksa} + 96ksa^4\rho C\text{hnp\_hnksa} \\ &\quad - 96ksa^4\rho C\text{jnp\_jnksa} - 45ksa^3n\rho C\text{hnp\_hnksa} + 45ksa^3n\rho C\text{jnp\_jnksa} \\ &\quad + 15ksa^3n\rho P\text{hnp\_hnksa} - 15ksa^3n\rho P\text{jnp\_jnksa} + 66ksa^3n^3\rho C\text{hnp\_hnksa} \\ &\quad - 66ksa^3n^3\rho C\text{jnp\_jnksa} - 96ksa^2n^2\rho C\text{hnp\_hnksa} + 96ksa^2n^2\rho C\text{jnp\_jnksa} \\ &\quad - 90ksa^2n\rho C\text{hnp\_hnksa} + 90ksa^2n\rho C\text{jnp\_jnksa} \end{aligned}$$

$$> \text{numTnSC\_kspa4facSimpl1} := \text{collect}\left(\text{simplify}\left(\text{expand}\left(\text{subs}\left(\text{jnp\_jnksa} = \text{hnp\_hnksa}\right.\right.\right.\right.$$



$$\begin{aligned} & - \frac{I}{ksa^2 \cdot jnksa \cdot hnksa}, numTnSC\_kspa4fac \Big) \Big) \Big) \Big) , rhoP \Big) \#use the Heine relation to simplify \\ numTnSC\_kspa4facSimpl1 := & \frac{In \left( 12 ksa^2 n^4 + 52 ksa^2 n^3 + 78 ksa^2 n^2 + 53 ksa^2 n + 15 ksa^2 \right) rhoP}{ksa jnksa hnksa} \end{aligned} \quad (57)$$

$$+ \frac{1}{ksa jnksa hnksa} ( \ln ( -12 ksa^2 n^3 rhoC - 60 ksa^2 n^2 rhoC + 24 n^4 rhoC - 93 ksa^2 n rhoC + 96 n^3 rhoC - 45 ksa^2 rhoC + 66 n^2 rhoC - 96 n rhoC - 90 rhoC ) )$$

$$\begin{aligned} & \textcolor{red}{> numTnSC\_kspa4facsSimpl} := \frac{1}{ksa \cdot jnk sa \cdot hnksa} (I \cdot n \cdot [(rhoP - rhoC) \cdot ksa^2 \cdot (n + 1) \cdot (12 \cdot n^3 \\ & \quad + 40 \cdot n^2 + 38 \cdot n + 15) + 2 \cdot rhoC \cdot (n - 1) \cdot (n + 1) \cdot ((6 \cdot n^2 + 20 \cdot n + 15) \cdot ksa^2 + 3 \cdot (2n \\ & \quad + 3) \cdot (2n + 5))]) \\ numTnSC\_kspa4facsSimpl &:= \frac{1}{ksa jnk sa hnksa} (In [(rhoP - rhoC) ksa^2 (n + 1) (12 n^3 \\ & \quad + 40 n^2 + 38 n + 15) + 2 rhoC (n - 1) (n + 1) ((6 n^2 + 20 n + 15) ksa^2 + 3 (2 n \\ & \quad + 3) (2 n + 5))]) \end{aligned} \tag{58}$$

\*\*\*\*\*This (above) is the simplified expression for the second term in the numerator of TnSC , combined with factors in line 56.

> expand(simplify( (collect( expand(ksa:jnksa·hnksa·numTnSC\_kspa4facsSimpl - ksa:jnksa·hnksa·numTnSC\_kspa4facsSimpl1), rhoP) ) ) )  
*#this is zero (but Maple won't simplify it to zero) so the simplification is correct*

$$[12 In^5 ksa^2 rhoP - 12 In^4 ksa^2 rhoC + 52 In^4 ksa^2 rhoP - 60 In^3 ksa^2 rhoC + 78 In^3 ksa^2 rhoP + 24 In^5 rhoC - 93 In^2 ksa^2 rhoC + 53 In^2 ksa^2 rhoP + 96 In^4 rhoC - 45 In ksa^2 rhoC + 15 In ksa^2 rhoP + 66 In^3 rhoC - 96 In^2 rhoC - 90 In rhoC] - 12 In^5 ksa^2 rhoP + 12 In^4 ksa^2 rhoC - 52 In^4 ksa^2 rhoP + 60 In^3 ksa^2 rhoC - 78 In^3 ksa^2 rhoP - 24 In^5 rhoC + 93 In^2 ksa^2 rhoC - 53 In^2 ksa^2 rhoP - 96 In^4 rhoC + 45 In ksa^2 rhoC - 15 In ksa^2 rhoP - 66 In^3 rhoC + 96 In^2 rhoC + 90 In rhoC \quad (59)$$

*#this is zero (but Maple won't simplify it to zero) so the simplification is correct*

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*****
*****
Now TnSS_ numerator (denominator is identical)
> numTnSS_Serkspa := simplify(series(numTnSS_, kspa, 8))
numTnSS_Serkspa := 
$$\frac{1}{2n+3} (2ksa^4\rho P^2 (2ksa n^4 jnp\_jnksa + 4ksa n^3 jnp\_jnksa + 2n^5$$


$$+ ksa n^2 jnp\_jnksa + 6n^4 - 4ksa n jnp\_jnksa + 5n^3 - 3ksa jnp\_jnksa - 3n^2 - 7n - 3))$$


$$kspa^2 - \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2\rho P (8ksa^3 n^4\rho C jnp\_jnksa$$


$$+ 12ksa^3 n^4\rho P jnp\_jnksa + 36ksa^3 n^3\rho C jnp\_jnksa + 52ksa^3 n^3\rho P jnp\_jnksa$$


$$- 8ksa^2 n^5\rho C + 12ksa^2 n^5\rho P + 46ksa^3 n^2\rho C jnp\_jnksa$$


$$+ 78ksa^3 n^2\rho P jnp\_jnksa - 60ksa^2 n^4\rho C + 64ksa^2 n^4\rho P$$

(60)

```

Now TnSS numerator (denominator is identical)

$$\begin{aligned} & \textcolor{blue}{> numTnSS\_Serksa := simplify(series(numTnSS_, ksa, 8))} \\ & \textcolor{blue}{numTnSS\_Serksa := \frac{1}{2n+3} (2ksa^4 rhoP^2 (2ksa n^4 jnp\_jnksa + 4ksa n^3 jnp\_jnksa + 2n^5} \\ & \textcolor{blue}{+ ksa n^2 jnp\_jnksa + 6n^4 - 4ksa n jnp\_jnksa + 5n^3 - 3ksa jnp\_jnksa - 3n^2 - 7n - 3))} \\ & \textcolor{blue}{ksa^2 - \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2 rhoP (8ksa^3 n^4 rhoC jnp\_jnksa} \\ & \textcolor{blue}{+ 12ksa^3 n^4 rhoP jnp\_jnksa + 36ksa^3 n^3 rhoC jnp\_jnksa + 52ksa^3 n^3 rhoP jnp\_jnksa} \\ & \textcolor{blue}{- 8ksa^2 n^5 rhoC + 12ksa^2 n^5 rhoP + 46ksa^3 n^2 rhoC jnp\_jnksa} \\ & \textcolor{blue}{+ 78ksa^3 n^2 rhoP jnp\_jnksa - 60ksa^2 n^4 rhoC + 64ksa^2 n^4 rhoP} \end{aligned} \quad (60)$$

$$\begin{aligned} numTnSS\_Serksa := & \frac{1}{2n+3} (2ksa^4rhoP^2 (2ksa^4jnp\_jnksa + 4ksa^3jnp\_jnksa + 2n^5 \\ & + ksa^2jnp\_jnksa + 6n^4 - 4ksanjnp\_jnksa + 5n^3 - 3ksajnp\_jnksa - 3n^2 - 7n - 3)) \\ & kspa^2 - \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2rhoP (8ksa^3n^4rhoCjnp\_jnksa \\ & + 12ksa^3n^4rhoPjnp\_jnksa + 36ksa^3n^3rhoCjnp\_jnksa + 52ksa^3n^3rhoPjnp\_jnksa \\ & - 8ksa^2n^5rhoC + 12ksa^2n^5rhoP + 46ksa^3n^2rhoCjnp\_jnksa \\ & + 78ksa^3n^2rhoPjnp\_jnksa - 60ksa^2n^4rhoC + 64ksa^2n^4rhoP \end{aligned} \quad (60)$$

$$\begin{aligned}
& kspa^2 - \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2 rhoP (8ksa^3 n^4 rhoC jnp\_jnksa \\
& + 12ksa^3 n^4 rhoP jnp\_jnksa + 36ksa^3 n^3 rhoC jnp\_jnksa + 52ksa^3 n^3 rhoP jnp\_jnksa \\
& - 8ksa^2 n^5 rhoC + 12ksa^2 n^5 rhoP + 46ksa^3 n^2 rhoC jnp\_jnksa \\
& + 78ksa^3 n^2 rhoP jnp\_jnksa - 60ksa^2 n^4 rhoC + 64ksa^2 n^4 rhoP
\end{aligned}$$

$$\begin{aligned}
& + 12 ksa^3 n^4 rhoP jnp\_jnksa + 36 ksa^3 n^3 rhoC jnp\_jnksa + 52 ksa^3 n^3 rhoP jnp\_jnksa \\
& - 8 ksa^2 n^5 rhoC + 12 ksa^2 n^5 rhoP + 46 ksa^3 n^2 rhoC jnp\_jnksa \\
& + 78 ksa^3 n^2 rhoP jnp\_jnksa - 60 ksa^2 n^4 rhoC + 64 ksa^2 n^4 rhoP
\end{aligned}$$

$$-8ksa^2n^5rhoC + 12ksa^2n^5rhoP + 46ksa^3n^2rhoCjnp\_jnksa$$

$$+ 78ksa^3n^2rhoPjnp\_jnksa - 60ksa^2n^4rhoC + 64ksa^2n^4rhoP$$

$$+ 78 ksa^3 n^2 rhoP jnp jnk sa - 60 ksa^2 n^4 rhoC + 64 ksa^2 n^4 rhoP$$



$$\begin{aligned}
& + 24 ksa n^4 rhoC jnp\_jnksa + 15 ksa^3 n rhoC jnp\_jnksa + 53 ksa^3 n rhoP jnp\_jnksa \\
& - 178 ksa^2 n^3 rhoC + 130 ksa^2 n^3 rhoP + 96 ksa n^3 rhoC jnp\_jnksa + 24 n^5 rhoC \\
& + 15 ksa^3 rhoP jnp\_jnksa - 261 ksa^2 n^2 rhoC + 131 ksa^2 n^2 rhoP \\
& + 66 ksa n^2 rhoC jnp\_jnksa + 120 n^4 rhoC - 183 ksa^2 n rhoC + 68 ksa^2 n rhoP \\
& - 96 ksa n rhoC jnp\_jnksa + 162 n^3 rhoC - 45 ksa^2 rhoC + 15 ksa^2 rhoP \\
& - 90 ksa rhoC jnp\_jnksa - 30 n^2 rhoC - 186 n rhoC - 90 rhoC ) ) kspa^4 + O(kspa^6)
\end{aligned}$$

> numTnSS\_kspa2 := simplify( (coeff(numTnSS\_Serkspa, kspa, 2)) )

$$\begin{aligned}
numTnSS\_kspa2 := & \frac{1}{2n+3} (2 ksa^4 rhoP^2 (2 ksa n^4 jnp\_jnksa + 4 ksa n^3 jnp\_jnksa + 2 n^5 \\
& + ksa n^2 jnp\_jnksa + 6 n^4 - 4 ksa n jnp\_jnksa + 5 n^3 - 3 ksa jnp\_jnksa - 3 n^2 - 7 n - 3) )
\end{aligned} \tag{61}$$

> numTnSS\_kspa2facs := simplify( expand( collect( ( numTnSS\_kspa2 \* (2\*n+3) / (2\*ksa^4\*rhoP^2), jnp\\_jnksa ) ) ) )

$$\begin{aligned}
numTnSS\_kspa2facs := & 2 ksa n^4 jnp\_jnksa + 4 ksa n^3 jnp\_jnksa + 2 n^5 + ksa n^2 jnp\_jnksa + 6 n^4 \\
& - 4 ksa n jnp\_jnksa + 5 n^3 - 3 ksa jnp\_jnksa - 3 n^2 - 7 n - 3
\end{aligned} \tag{62}$$

> numTnSS\_kspa2facsSimpl := (2\*n^4 + 4\*n^3 + n^2 - 4\*n - 3) \* (ksa\*jnp\\_jnksa + n + 1)

$$numTnSS\_kspa2facsSimpl := (2 n^4 + 4 n^3 + n^2 - 4 n - 3) (ksa jnp\_jnksa + n + 1) \tag{63}$$

> numTnSS\_kspa2facsSimpl := (n-1) \* (n+1) \* (2\*n^2 + 4\*n + 3) \* (ksa\*jnp\\_jnksa + n + 1)

$$numTnSS\_kspa2facsSimpl := (n-1) (n+1) (2 n^2 + 4 n + 3) (ksa jnp\_jnksa + n + 1) \tag{64}$$

\*\*\*\*\*This (above) is the simplified expression for the first term in the numerator of TnSS\_combined with factors in line 62.

> expand(numTnSS\_kspa2facs - numTnSS\_kspa2facsSimpl)  
# check that this agrees with the original.

0

(65)

> numTnSS\_kspa4 := simplify( (coeff(numTnSS\_Serkspa, kspa, 4)) )

$$\begin{aligned}
numTnSS\_kspa4 := & - \frac{1}{(2n+3)(4n^2+16n+15)} (ksa^2 rhoP (8 ksa^3 n^4 rhoC jnp\_jnksa \\
& + 12 ksa^3 n^4 rhoP jnp\_jnksa + 36 ksa^3 n^3 rhoC jnp\_jnksa + 52 ksa^3 n^3 rhoP jnp\_jnksa
\end{aligned} \tag{66}$$

$$\begin{aligned}
& - 8 ksa^2 n^5 rhoC + 12 ksa^2 n^5 rhoP + 46 ksa^3 n^2 rhoC jnp\_jnksa \\
& + 78 ksa^3 n^2 rhoP jnp\_jnksa - 60 ksa^2 n^4 rhoC + 64 ksa^2 n^4 rhoP \\
& + 24 ksa n^4 rhoC jnp\_jnksa + 15 ksa^3 n rhoC jnp\_jnksa + 53 ksa^3 n rhoP jnp\_jnksa \\
& - 178 ksa^2 n^3 rhoC + 130 ksa^2 n^3 rhoP + 96 ksa n^3 rhoC jnp\_jnksa + 24 n^5 rhoC \\
& + 15 ksa^3 rhoP jnp\_jnksa - 261 ksa^2 n^2 rhoC + 131 ksa^2 n^2 rhoP \\
& + 66 ksa n^2 rhoC jnp\_jnksa + 120 n^4 rhoC - 183 ksa^2 n rhoC + 68 ksa^2 n rhoP \\
& - 96 ksa n rhoC jnp\_jnksa + 162 n^3 rhoC - 45 ksa^2 rhoC + 15 ksa^2 rhoP \\
& - 90 ksa rhoC jnp\_jnksa - 30 n^2 rhoC - 186 n rhoC - 90 rhoC ) )
\end{aligned}$$

> numTnSS\_kspa4facs := collect( expand( ( numTnSS\_kspa4 \* (2\*n+3) \* (4\*n^2 + 16\*n + 15) / (ksa^2\*rhoP) ) ) ,

$$\left. \begin{array}{l} \text{rhoP} \end{array} \right)$$

$$\begin{aligned} \text{numTnSS\_kspa4facs} := & (-12 \text{ksa}^3 n^4 \text{jnp\_jnksa} - 52 \text{ksa}^3 n^3 \text{jnp\_jnksa} - 12 \text{ksa}^2 n^5 \\ & - 78 \text{ksa}^3 n^2 \text{jnp\_jnksa} - 64 \text{ksa}^2 n^4 - 53 \text{ksa}^3 n \text{jnp\_jnksa} - 130 \text{ksa}^2 n^3 - 15 \text{ksa}^3 \text{jnp\_jnksa} \\ & - 131 \text{ksa}^2 n^2 - 68 \text{ksa}^2 n - 15 \text{ksa}^2) \text{rhoP} - 8 \text{ksa}^3 n^4 \text{rhoCjnp\_jnksa} \\ & - 36 \text{ksa}^3 n^3 \text{rhoCjnp\_jnksa} + 8 \text{ksa}^2 n^5 \text{rhoC} - 46 \text{ksa}^3 n^2 \text{rhoCjnp\_jnksa} \\ & + 60 \text{ksa}^2 n^4 \text{rhoC} - 24 \text{ksa} n^4 \text{rhoCjnp\_jnksa} - 15 \text{ksa}^3 n \text{rhoCjnp\_jnksa} \\ & + 178 \text{ksa}^2 n^3 \text{rhoC} - 96 \text{ksa} n^3 \text{rhoCjnp\_jnksa} - 24 n^5 \text{rhoC} + 261 \text{ksa}^2 n^2 \text{rhoC} \\ & - 66 \text{ksa} n^2 \text{rhoCjnp\_jnksa} - 120 n^4 \text{rhoC} + 183 \text{ksa}^2 n \text{rhoC} + 96 \text{ksa} n \text{rhoCjnp\_jnksa} \\ & - 162 n^3 \text{rhoC} + 45 \text{ksa}^2 \text{rhoC} + 90 \text{ksa} \text{rhoCjnp\_jnksa} + 30 n^2 \text{rhoC} + 186 n \text{rhoC} \\ & + 90 \text{rhoC} \end{aligned} \quad (67)$$

Now attempt to extract similar terms to those appearing in the n=1 coefficient -see 2017 JASA paper. Work out the factors appearing in front of the Bessel function terms by hand by taking all the rhoP terms.

$$\begin{aligned} & \text{numTnSS\_kspa4facsRed} := \text{expand}(\text{numTnSS\_kspa4facs} - ((n+1) \cdot (12 \cdot n^3 + 40 \cdot n^2 + 38 \cdot n \\ & \quad + 15) \cdot \text{ksa}^2 \cdot \text{rhoP} \cdot (\text{ksa} \cdot \text{jnp\_jnksa} + (n+1)))) \\ \text{numTnSS\_kspa4facsRed} := & -8 \text{ksa}^3 n^4 \text{rhoCjnp\_jnksa} - 36 \text{ksa}^3 n^3 \text{rhoCjnp\_jnksa} \\ & + 8 \text{ksa}^2 n^5 \text{rhoC} - 46 \text{ksa}^3 n^2 \text{rhoCjnp\_jnksa} + 60 \text{ksa}^2 n^4 \text{rhoC} - 24 \text{ksa} n^4 \text{rhoCjnp\_jnksa} \\ & - 15 \text{ksa}^3 n \text{rhoCjnp\_jnksa} + 178 \text{ksa}^2 n^3 \text{rhoC} - 96 \text{ksa} n^3 \text{rhoCjnp\_jnksa} - 24 n^5 \text{rhoC} \\ & + 261 \text{ksa}^2 n^2 \text{rhoC} - 66 \text{ksa} n^2 \text{rhoCjnp\_jnksa} - 120 n^4 \text{rhoC} + 183 \text{ksa}^2 n \text{rhoC} \\ & + 96 \text{ksa} n \text{rhoCjnp\_jnksa} - 162 n^3 \text{rhoC} + 45 \text{ksa}^2 \text{rhoC} + 90 \text{ksa} \text{rhoCjnp\_jnksa} \\ & + 30 n^2 \text{rhoC} + 186 n \text{rhoC} + 90 \text{rhoC} \end{aligned} \quad (68)$$

There are no rhoP terms remaining. Now take a similar term in rhoC to get the (rhoP-rhoC) term as in the n=1 coefficient

$$\begin{aligned} & \text{numTnSS\_kspa4facsRed} := \text{simplify}(\text{expand}(\text{numTnSS\_kspa4facsRed} - ((n+1) \cdot (12 \cdot n^3 + 40 \\ & \quad \cdot n^2 + 38 \cdot n + 15) \cdot \text{ksa}^2 \cdot \text{rhoC} \cdot (\text{ksa} \cdot \text{jnp\_jnksa} + (n+1)))) \\ \text{numTnSS\_kspa4facsRed} := & -20 \text{ksa}^3 n^4 \text{rhoCjnp\_jnksa} - 88 \text{ksa}^3 n^3 \text{rhoCjnp\_jnksa} \\ & - 4 \text{ksa}^2 n^5 \text{rhoC} - 124 \text{ksa}^3 n^2 \text{rhoCjnp\_jnksa} - 4 \text{ksa}^2 n^4 \text{rhoC} - 24 \text{ksa} n^4 \text{rhoCjnp\_jnksa} \\ & - 68 \text{ksa}^3 n \text{rhoCjnp\_jnksa} + 48 \text{ksa}^2 n^3 \text{rhoC} - 96 \text{ksa} n^3 \text{rhoCjnp\_jnksa} - 24 n^5 \text{rhoC} \\ & - 15 \text{ksa}^3 \text{rhoCjnp\_jnksa} + 130 \text{ksa}^2 n^2 \text{rhoC} - 66 \text{ksa} n^2 \text{rhoCjnp\_jnksa} - 120 n^4 \text{rhoC} \\ & + 115 \text{ksa}^2 n \text{rhoC} + 96 \text{ksa} n \text{rhoCjnp\_jnksa} - 162 n^3 \text{rhoC} + 30 \text{ksa}^2 \text{rhoC} \\ & + 90 \text{ksa} \text{rhoCjnp\_jnksa} + 30 n^2 \text{rhoC} + 186 n \text{rhoC} + 90 \text{rhoC} \end{aligned} \quad (69)$$

Now take a similar term in rhoC by with the other combination of jnp\_jnksa to complete the betajS term (JASA2017 paper) in (2n+1)

$$\begin{aligned} & \text{numTnSS\_kspa4facsRed} := \text{simplify}(\text{expand}(\text{numTnSS\_kspa4facsRed} - ((2 \cdot n + 1) \cdot (12 \cdot n^3 \\ & \quad + 40 \cdot n^2 + 38 \cdot n + 15) \cdot \text{ksa}^2 \cdot \text{rhoC} \cdot (\text{ksa} \cdot \text{jnp\_jnksa} - n)))) \\ \text{numTnSS\_kspa4facsRed} := & 4 \text{ksa}^3 n^4 \text{rhoCjnp\_jnksa} + 4 \text{ksa}^3 n^3 \text{rhoCjnp\_jnksa} \\ & - 28 \text{ksa}^2 n^5 \text{rhoC} - 8 \text{ksa}^3 n^2 \text{rhoCjnp\_jnksa} - 96 \text{ksa}^2 n^4 \text{rhoC} - 24 \text{ksa} n^4 \text{rhoCjnp\_jnksa} \end{aligned} \quad (70)$$

$$\begin{aligned}
& - 68 ksa^2 n^3 rhoC - 96 ksa n^3 rhoC jnp\_jnksa - 24 n^5 rhoC + 62 ksa^2 n^2 rhoC \\
& - 66 ksa n^2 rhoC jnp\_jnksa - 120 n^4 rhoC + 100 ksa^2 n rhoC + 96 ksa n rhoC jnp\_jnksa \\
& - 162 n^3 rhoC + 30 ksa^2 rhoC + 90 ksa rhoC jnp\_jnksa + 30 n^2 rhoC + 186 n rhoC \\
& + 90 rhoC
\end{aligned}$$

> numTnSS\_kspa4facsRed := collect(numTnSS\_kspa4facsRed, jnp\_jnksa)

$$\begin{aligned}
numTnSS\_kspa4facsRed := & (4 ksa^3 n^4 rhoC + 4 ksa^3 n^3 rhoC - 8 ksa^3 n^2 rhoC - 24 ksa n^4 rhoC \\
& - 96 ksa n^3 rhoC - 66 ksa n^2 rhoC + 96 ksa n rhoC + 90 ksa rhoC) jnp\_jnksa \\
& - 28 ksa^2 n^5 rhoC - 96 ksa^2 n^4 rhoC - 68 ksa^2 n^3 rhoC - 24 n^5 rhoC + 62 ksa^2 n^2 rhoC \\
& - 120 n^4 rhoC + 100 ksa^2 n rhoC - 162 n^3 rhoC + 30 ksa^2 rhoC + 30 n^2 rhoC \\
& + 186 n rhoC + 90 rhoC
\end{aligned} \quad (71)$$

Construct a term in ksa\*jnp\_jnksa-n similarly to other terms in the coefficients.

> numTnSS\_kspa4facsRed := numTnSS\_kspa4facsRed - 2 \* (n - 1) \* rhoC \* (ksa\*jnp\_jnksa - n) \* (2 \* n^2 \* (n + 2) \* ksa^2 - 3 \* (n + 1) \* (2 \* n + 3) \* (2 \* n + 5))

$$\begin{aligned}
numTnSS\_kspa4facsRed := & (4 ksa^3 n^4 rhoC + 4 ksa^3 n^3 rhoC - 8 ksa^3 n^2 rhoC - 24 ksa n^4 rhoC \\
& - 96 ksa n^3 rhoC - 66 ksa n^2 rhoC + 96 ksa n rhoC + 90 ksa rhoC) jnp\_jnksa \\
& - 28 ksa^2 n^5 rhoC - 96 ksa^2 n^4 rhoC - 68 ksa^2 n^3 rhoC - 24 n^5 rhoC + 62 ksa^2 n^2 rhoC \\
& - 120 n^4 rhoC + 100 ksa^2 n rhoC - 162 n^3 rhoC + 30 ksa^2 rhoC + 30 n^2 rhoC \\
& + 186 n rhoC + 90 rhoC - 2 (n - 1) rhoC (ksa jnp\_jnksa - n) (2 n^2 (n + 2) ksa^2 - 3 (n \\
& + 1) (2 n + 3) (2 n + 5))
\end{aligned} \quad (72)$$

> collect(simplify(expand(numTnSS\_kspa4facsRed)), ksa^2)

#so we have eliminated all the terms in jnp\_jnksa

$$\begin{aligned}
& (-24 n^5 rhoC - 92 n^4 rhoC - 76 n^3 rhoC + 62 n^2 rhoC + 100 n rhoC + 30 rhoC) ksa^2 \\
& - 48 n^5 rhoC - 216 n^4 rhoC - 228 n^3 rhoC + 126 n^2 rhoC + 276 n rhoC + 90 rhoC
\end{aligned} \quad (73)$$

> numTnSS\_kspa4facsRed := simplify(expand(numTnSS\_kspa4facsRed - 2 \* (-1) \* (n - 1) \* (n + 1) \* (2 \* n + 1) \* rhoC \* ((6 \* n^2 + 20 \* n + 15) \* ksa^2 + 3 \* (2 \* n + 3) \* (2 \* n + 5))))

# this is zero, confirming that the simplification has removed all terms

$$numTnSS\_kspa4facsRed := 0 \quad (74)$$

Now construct the second term in the series in the numerator of TnSS\_ (scaled) i.e.

numTnSS\_kspa4facs

> numTnSS\_kspa4facsSimpl := - (12 \* n^3 + 40 \* n^2 + 38 \* n + 15) \* ksa^2 \* ((n + 1) \* (rhoP - rhoC) \* (ksa\*jnp\_jnksa + (n + 1)) + (2 \* n + 1) \* rhoC \* (ksa\*jnp\_jnksa - n)) :

> numTnSS\_kspa4facsSimpl := numTnSS\_kspa4facsSimpl + 2 \* (n - 1) \* rhoC \* (ksa\*jnp\_jnksa - n) \* (2 \* n^2 \* (n + 2) \* ksa^2 - 3 \* (n + 1) \* (2 \* n + 3) \* (2 \* n + 5)) :

> numTnSS\_kspa4facsSimpl := numTnSS\_kspa4facsSimpl - 2 \* (n - 1) \* (n + 1) \* (2 \* n + 1) \* rhoC \* ((6 \* n^2 + 20 \* n + 15) \* ksa^2 + 3 \* (2 \* n + 3) \* (2 \* n + 5))

$$\begin{aligned}
numTnSS\_kspa4facsSimpl := & - (12 n^3 + 40 n^2 + 38 n + 15) ksa^2 ((n + 1) (rhoP \\
& - rhoC) (ksa jnp\_jnksa + n + 1) + (2 n + 1) rhoC (ksa jnp\_jnksa - n)) + 2 (n \\
& - 1) rhoC (ksa jnp\_jnksa - n) (2 n^2 (n + 2) ksa^2 - 3 (n + 1) (2 n + 3) (2 n + 5)) \\
& - 2 (n - 1) (n + 1) (2 n + 1) rhoC ((6 n^2 + 20 n + 15) ksa^2 + 3 (2 n + 3) (2 n + 5))
\end{aligned} \quad (75)$$

\*\*\*\*\*This (above) is the second term in the numerator of TnSS\_ combined with the

factors in line 67.

```
> simplify(numTnSS_kspa4facSimpl - numTnSS_kspa4fac)
# check that it agrees with the original. Confirms that this simplified version is correct.
0 (76)
```

\*\*\*\*\*  
The derivation of the expressions for the scaled coefficients are now complete - see results above.

\*\*\*\*\*  
\*\*\*\*\*  
Full expressions for the unscaled coefficients can be obtained using the expansions for hn and jn for small arguments for kca terms, and not expanding those for ksa.  
For n>=2

```
> TnCC := (ksa·hnp_hnkca - n)·n / ((n + 1)·(ksa·hnp_hnkca + (n + 1))) · jnkca / hnkc
TnCC := (ksa hnp_hnkca - n) n jnkca / ((n + 1) (ksa hnp_hnkca + n + 1) hnkc) (77)
```

```
> TnCS := - (2·n + 1) / ((n + 1)·(ksa·hnp_hnkca + (n + 1))) · jnkca / hnkc
TnCS := - (2 n + 1) jnkca / ((n + 1) (ksa hnp_hnkca + n + 1) hnkc) (78)
```

```
> TnSC := i·n / (ksa·(ksa·hnp_hnkca + (n + 1))hnkc) · jnkca / hnkc
TnSC := i n / (ksa (ksa hnp_hnkca + n + 1) hnkc hnkc) (79)
```

```
> TnSS := - (ksa·jnp_jnkca + (n + 1)) / ((ksa·hnp_hnkca + (n + 1))) · jnkca / hnkc
TnSS := - (ksa jnp_jnkca + n + 1) jnkca / ((ksa hnp_hnkca + n + 1) hnkc) (80)
```

Now check out the orders of all these coefficients with kca - and possibly get the leading term of jn and hn from the expansions above (but need to get the first term of 1/hn)

```
> jnxx
x^n (1 - x^2 / (4 n + 6) + 1 / 8 x^4 / ((2 n + 3) (2 n + 5)) + O(x^6)) / fac_jn (81)
```

```
> jnkca_lead := 1 / fac_jn #times kca^n and facjn = 1·3·...·(2 n + 1)
jnkc_lead := 1 / fac_jn (82)
```

```
> jnkca := jnkca_lead
jnkc := 1 / fac_jn (83)
```

```
> xhn_exp
```

$$\frac{i^{-n-1} (2n)! i^n \left( 1 - ix - \frac{(n-1)x^2}{2n-1} \right)}{n! (2x)^n} \quad (84)$$

$$> hnkca := -\frac{i \cdot (2 \cdot n)!}{n! \cdot 2^n} \#times kca^{-(n+1)} \text{ leading term only}$$

$$hnkca := -\frac{i (2n)!}{n! 2^n} \quad (85)$$

Substituting into the coefficients to get the leading terms in kca gives (with fac\_jn=1\*3...(2n+1) which can be written in terms of double factorials but is retained here

Therefore the expressions for n>=2 are as follows at leading order (in kca and kspa)

$$> TnCC\_lead := simplify(TnCC) \#multiplied by kca^{(2n+1)}$$

$$TnCC\_lead := -\frac{(ksa hnp\_hnksa - n) n n! 2^n}{(n+1) (ksa hnp\_hnksa + n + 1) fac\_jn i (2n)!} \quad (86)$$

$$> TnCS\_lead := TnCS \#multiplied by kca^n$$

$$TnCS\_lead := -\frac{2n+1}{(n+1) (ksa hnp\_hnksa + n + 1) fac\_jn hnksa} \quad (87)$$

$$> TnSC\_lead := TnSC \#multiplied by kca^{(n+1)}$$

$$TnSC\_lead := -\frac{n n! 2^n}{ksa (ksa hnp\_hnksa + n + 1) hnksa (2n)!} \quad (88)$$

$$> TnSS\_lead := TnSS \#multiplied by kca^0$$

$$TnSS\_lead := -\frac{(ksa jnp\_jnksa + n + 1) jnksa}{(ksa hnp\_hnksa + n + 1) hnksa} \quad (89)$$

For n=1, scaled coefficients are given by the ksa4 terms of the series in numerator and denominator

$$> n := 1$$

$$n := 1 \quad (90)$$

$$> TnCC\_n1 := simplify\left(\frac{numTnCC\_kspa4facsSimpl}{denTnCC\_kspa4facsSimpl}\right)$$

$$TnCC\_n1 := -\frac{(-rhoP + rhoC) (ksa hnp\_hnksa - 1)}{ksa rhoC hnp\_hnksa + 2 ksa rhoP hnp\_hnksa - 7 rhoC + 4 rhoP} \quad (91)$$

$$> TnCS\_n1 := simplify\left(\frac{numTnCS\_kspa4facsSimpl}{denTnCC\_kspa4facsSimpl}\right)$$

$$TnCS\_n1 := \frac{3 (-rhoP + rhoC)}{ksa rhoC hnp\_hnksa + 2 ksa rhoP hnp\_hnksa - 7 rhoC + 4 rhoP} \quad (92)$$

$$> TnSC\_n1 := simplify\left(\frac{numTnSC\_kspa4facsSimpl}{denTnCC\_kspa4facsSimpl}\right)$$

$$TnSC\_n1 := \quad (93)$$

$$\frac{\frac{1}{105} I[-210 ksa^2 (-rhoP + rhoC)]}{ksa^3 jnksa hnksa (ksa rhoC hnp\_hnksa + 2 ksa rhoP hnp\_hnksa - 7 rhoC + 4 rhoP)}$$

$$> TnSS\_n1 := simplify\left(\frac{numTnSS\_kspa4facsSimpl}{denTnCC\_kspa4facsSimpl}\right)$$

$$T_{nSS\_n1} := - \frac{ksa\,rhoC\,jnp\_jnksa + 2\,ksa\,rhoP\,jnp\_jnksa - 7\,rhoC + 4\,rhoP}{ksa\,rhoC\,hnp\_hnksa + 2\,ksa\,rhoP\,hnp\_hnksa - 7\,rhoC + 4\,rhoP} \quad (94)$$

These are the scaled coefficients for n=1, these agree with Pinfield and Forrester, Journal of the Acoustical Society of America, 141, 649 (2017); doi: 10.1121/1.4974142