I would say that in its abstract form a vector space is a concept that generalises physical entities we know as lines ($\mathbb{R}^1$), planes ($\mathbb{R}^2$), and 3-d space ($\mathbb{R}^3$). Considering these as sets of vectors, with the usual addition and multiplication by a real constant, we can confirm that they satisfy all ten properties of a vector space. There are a variety of spaces which are not geometrically recognisable as vector spaces, but which have an analogous structure. For example: Consider V—the set of all 2×2 matrices with real entries, with the standard addition and scalar multiplication.

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in V$$

Hence, V is closed under addition (satisfies the first property of a vector space). Also, V is closed under scalar multiplication (check), and all other eight properties. Hence, V is a vector space. Now, consider W—the set of all 2×2 matrices with integer entries, with the standard addition and scalar multiplication. Is W a vector space? Let's check: closed under addition—yes. There is a zero vector:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Is it closed under scalar multiplication? No, it is not. Here's a counterexample:

$$\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & \frac{1}{2} \end{bmatrix} \notin W.$$

Hence, W is not a vector space. Working with a vector space ensures that you have a 'nice' structure and allows for efficient ways of reaching conclusions. For example, you can conceptualise the set of solutions of a homogeneous linear ordinary differential equation as a vector space of functions.