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HILBERT $\widetilde{\mathbb{C}}$-MODULES: STRUCTURAL PROPERTIES AND APPLICATIONS TO VARIATIONAL PROBLEMS

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Abstract. We develop a theory of Hilbert $\widetilde{\mathbb{C}}$-modules which forms the core of a new functional analytic approach to algebras of generalized functions. Particular attention is given to finitely generated submodules, projection operators, representation theorems for $\widetilde{\mathbb{C}}$-linear functionals and $\widetilde{\mathbb{C}}$-sesquilinear forms. We establish a generalized Lax-Milgram theorem and use it to prove existence and uniqueness theorems for variational problems involving a generalized bilinear or sesquilinear form.

0. Introduction

Over the past thirty years, nonlinear theories of generalized functions have been developed by many authors [1, 6, 12, 19], mainly inspired by the work of J. F. Colombeau [3, 4]. They have proved to be a valuable tool for treating partial differential equations with singular data or coefficients [11, 14, 15, 16, 20]. Also, they have found a wealth of applications to differential geometry [13, 18] and relativity theory (cf. [12] and the references therein). As a consequence of intense research in the field, increasing importance is attached to an understanding of algebraic [2, 21] and topological structures [5, 7] in spaces of generalized functions.

This paper is part of a wider project which aims to introduce functional analytic methods into Colombeau algebras of generalized functions. Our intent is to deal with the general problem of existence and qualitative properties of solutions of partial differential equations in the Colombeau setting by means of functional analytic tools adapted and generalized to the range of topological modules over the ring $\widetilde{\mathbb{C}}$ of generalized numbers. Starting from the topological background given in [7, 8], in this paper we develop a theory of Hilbert $\widetilde{\mathbb{C}}$-modules. This will be the framework used to investigate variational equalities and inequalities generated by highly singular problems in partial differential equations.

A first example of a Hilbert $\widetilde{\mathbb{C}}$-module is the Colombeau space $\mathcal{G}_H$ of generalized functions based on a Hilbert space $(H, (\cdot|\cdot))$ [7, Definition 3.1], where the scalar product is obtained by letting $(\cdot|\cdot)$ act componentwise at the representatives level. A number of theorems, such as projection theorem, Riesz-representation and the Lax-Milgram theorem, can be obtained in a direct way when we work on $\mathcal{G}_H$ by applying the corresponding classical results at the level of representatives at first and then by checking the necessary moderateness conditions. This is a sort of

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transfer method, which has been exclusively employed so far, for producing results in a Colombeau context deeply related to a classical one. The novelty of our work is the introduction of a general notion of a Hilbert $\widetilde{\mathbb{C}}$-module which no longer has the internal structure of $\mathcal{G}_H$, and the completely intrinsic way of developing a topological and functional analytic theory within this abstract setting. As we will see in the course of the paper, the wide generality of our approach on the one hand entails some technicalities in the proofs and on the other hand leads to the introduction of a number of new concepts, such as edged subsets of a $\widetilde{\mathbb{C}}$-module, normalization property, etc.

We now describe the contents of the sections in more detail.

The first section serves to collect some basic notions necessary for the comprehension of the paper. We begin in Subsection 1.1 by recalling the definition of the Colombeau space $\mathcal{G}_E$ of generalized functions based on a locally convex topological vector space E. In order to view $\mathcal{G}_E$ as a particular example of a locally convex topological $\widetilde{\mathbb{C}}$-module, where $\widetilde{\mathbb{C}}$ is the ring $\mathcal{G}_{\mathbb{C}}$ of generalized constants, we make use of concepts such as valuation and ultra-pseudo-seminorm and of some fundamental ingredients of the theory of topological $\widetilde{\mathbb{C}}$-modules elaborated in [7, 8]. Particular attention is given to $\widetilde{\mathbb{C}}$-linear maps and $\widetilde{\mathbb{C}}$-sesquilinear forms acting on locally convex topological $\widetilde{\mathbb{C}}$-modules and to their basic structure when we work on spaces of $\mathcal{G}_E$-type [10, Definition 1.1]. We introduce the property of being internal for subsets of $\mathcal{G}_E$ as the analogue of the basic structure for maps. Internal subsets will play a main role in the paper, in the existence and uniqueness theorems for variational equalities and inequalities of Sections 7 and 8. We conclude Subsection 1.1 by discussing some issues concerning the ring $\widetilde{\mathbb{R}}$ of real generalized numbers: definition and properties of the order relation $\geq$, invertibility and negligibility with respect to a subset S of $(0, 1]$, characterization of zero divisors and idempotent elements, and infimum and close infimum in $\widetilde{\mathbb{R}}$.

The second part of Section 1 deals with the class of $\widetilde{\mathbb{C}}$-modules with $\widetilde{\mathbb{R}}$-seminorms. Making use of the order relation $\geq$ in $\widetilde{\mathbb{R}}$ and of the classical notion of seminorm as a blueprint, we introduce the concept of $\widetilde{\mathbb{R}}$-seminorm on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. This induces a topology on $\mathcal{G}$ which turns out to be $\widetilde{\mathbb{C}}$-locally convex. In other words we find a special class of locally convex topological $\widetilde{\mathbb{C}}$-modules which contains the spaces of generalized functions based on a locally convex topological vector space as a particular case.

Section 2 is devoted to the definition and the first properties of the family of topological $\widetilde{\mathbb{C}}$-modules which are the mathematical core of the paper: the Hilbert $\widetilde{\mathbb{C}}$-modules. They are defined by means of a generalized scalar product $(\cdot|\cdot)$ with values in $\widetilde{\mathbb{C}}$ which determines the $\widetilde{\mathbb{R}}$-norm $\|u\| = (u|u)^{\frac{1}{2}}$. This means that they are particular $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{C}}$-modules. As first examples of Hilbert $\widetilde{\mathbb{C}}$-modules we consider the space $\mathcal{G}_H$ based on a vector space H with scalar product and, more generally, given a net $(H_\varepsilon, (\cdot|\cdot)_{H_\varepsilon})_\varepsilon$, the quotient of the corresponding moderate nets over negligible nets (Proposition 2.7).

With the intent of developing a topological and functional analytic theory of Hilbert $\widetilde{\mathbb{C}}$-modules, we start in Subsection 2.2 by investigating the notion of projection on a suitable subset C of a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. This requires some new assumptions on C, such as being reachable from a point u of $\mathcal{G}$, the property of being edged, i.e., reachable from any u, and a formulation of convexity in terms of

$\widetilde{\mathbb{R}}$-linear combinations which resembles the well-known classical definition for subsets of a vector space but differs from the $\widetilde{\mathbb{C}}$-convexity introduced in [7]. In detail, we prove that if C is a closed nonempty subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $C + C \subseteq 2C$ and it is reachable from $u \in \mathcal{G}$, i.e., the set $\{\|u - w\|, w \in C\}$ has a close infimum in $\widetilde{\mathbb{R}}$, then the projection $P_C(u)$ of u on C exists. The operator P_C is globally defined and continuous when C is edged and is $\widetilde{\mathbb{C}}$-linear when $C = M$ is a closed and edged $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$. We also see, by means of a counterexample, that the condition of being edged is necessary for the existence of P_M and that this operator allows us to extend any continuous $\widetilde{\mathbb{C}}$-linear map with values in a topological $\widetilde{\mathbb{C}}$-module from M to the whole of $\mathcal{G}$. In this way we obtain a version of the Hahn-Banach theorem where the fact that M is edged is essential. Moreover, closed and edged submodules of $\mathcal{G}$ can be characterized as those submodules M for which $M + M^\perp = \mathcal{G}$.

Section 3 gives a closer look at edged submodules of a Hilbert $\widetilde{\mathbb{C}}$-module. For the sake of generality we work in the context of $\widetilde{\mathbb{K}}$-modules, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, and we state many results in the framework of Banach $\widetilde{\mathbb{K}}$-modules. In our investigation on submodules we distinguish between cyclic submodules, i.e., generated by one element, and submodules generated by $m > 1$ elements. In particular, we prove that when a submodule is finitely generated the property of being edged is deeply related to topological closedness and to some structural properties of the generators. We carefully comment our statements by providing explanatory examples and counterexamples.

The main topic of Section 4 is the formulation of a Riesz representation theorem for continuous $\widetilde{\mathbb{C}}$-linear functionals acting on a Hilbert $\widetilde{\mathbb{C}}$-module. We prove that a functional T can be written in the form $T(u) = (u|c)$ if and only if there exists a closed and cyclic $\widetilde{\mathbb{C}}$-submodule N such that $N^\perp \subseteq \operatorname{Ker} T$. In particular, on $\mathcal{G}_H$, the Riesz representation theorem gives necessary and sufficient conditions for a $\widetilde{\mathbb{C}}$-linear functional to be basic. The structural properties of continuous $\widetilde{\mathbb{C}}$-sesquilinear forms on Hilbert $\widetilde{\mathbb{C}}$-modules are investigated by making use of the previous representation theorem.

In Section 5 we concentrate on continuous $\widetilde{\mathbb{C}}$-linear operators acting on a Hilbert $\widetilde{\mathbb{C}}$-module. In detail, we deal with isometric, unitary, self-adjoint and projection operators obtaining the following characterization: T is a projection operator (i.e., self-adjoint and idempotent) if and only if it is the projection P_M on a closed and edged $\widetilde{\mathbb{C}}$-submodule M.

A version of the Lax-Milgram theorem valid for Hilbert $\widetilde{\mathbb{C}}$-modules and $\widetilde{\mathbb{C}}$-sesquilinear forms is proved in Section 6 for forms of the type $a(u, v) = (u|g(v))$, when we assume that the range of g is edged and that a satisfies a suitable coercivity condition. This theorem applies to any basic and coercive $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G}_H$ and plays a relevant role in the applications of the last section of the paper.

Section 7 concerns variational inequalities involving a continuous $\widetilde{\mathbb{R}}$-bilinear form in the framework of Hilbert $\widetilde{\mathbb{R}}$-modules. Under suitable hypotheses on the set $C \subseteq \mathcal{G}$ we prove that the problem

$$a(u, v - u) \geq (f|v - u)\,, \qquad \text{for all } v \in C$$

is uniquely solvable in C if a is a symmetric, coercive and continuous $\widetilde{\mathbb{R}}$-bilinear form and the functional $I(u) = a(u, u) - 2\,(f|u)$ has a close infimum on C in $\widetilde{\mathbb{R}}$.

This applies to the case of basic and coercive forms on $\mathcal{G}_H$ when C is internal and can be extended to basic $\widetilde{\mathbb{R}}$-sesquilinear forms which are nonsymmetric via some contraction techniques. The theorems of Section 7 are one of the first examples of existence and uniqueness theorems in the Colombeau framework obtained in an intrinsic way via topological and functional analytic methods.

The paper ends by discussing some concrete problems coming from partial differential operators with highly singular coefficients, which in variational form can be solved by making use of the theorems on variational equalities and inequalities of Section 7. The generalized framework within which we work allows us to approach problems which are not solvable classically and to obtain results consistent with the classical ones when the latter exist.

1. Basic notions

This section of preliminary notions provides some topological background necessary for the comprehension of the paper. Particular attention is given to Colombeau spaces of generalized functions, locally convex topological $\widetilde{\mathbb{C}}$-modules and topological $\widetilde{\mathbb{C}}$-modules with $\widetilde{\mathbb{R}}$-seminorms. The main references are [7, 8, 10].

1.1. **Colombeau spaces of generalized functions and topological $\widetilde{\mathbb{C}}$-modules.**

1.1.1. *First definitions, valuations and ultra-pseudo-seminorms.* Let E be a locally convex topological vector space topologized through the family of seminorms $\{p_i\}_{i\in I}$. The elements of

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall i \in I\ \exists N \in \mathbb{N}\quad p_i(u_\varepsilon) = O(\varepsilon^{-N})\ \text{as}\ \varepsilon \to 0\},$$
$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall i \in I\ \forall q \in \mathbb{N}\quad p_i(u_\varepsilon) = O(\varepsilon^{q})\ \text{as}\ \varepsilon \to 0\},$$

are called E-moderate and E-negligible, respectively. The space of *Colombeau generalized functions based on* E is defined as the quotient $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$.

The rings $\widetilde{\mathbb{C}} = \mathcal{E}_M/\mathcal{N}$ of *complex generalized numbers* and $\widetilde{\mathbb{R}}$ of *real generalized numbers* are obtained by taking $E = \mathbb{C}$ and $E = \mathbb{R}$, respectively. One can easily see that for any locally convex topological vector space E (on $\mathbb{C}$), the space $\mathcal{G}_E$ has the structure of a $\widetilde{\mathbb{C}}$-module. We use the notation $u = [(u_\varepsilon)_\varepsilon]$ for the class u of $(u_\varepsilon)_\varepsilon$ in $\mathcal{G}_E$. This is the usual way adopted in the paper to denote an equivalence class.

$\widetilde{\mathbb{C}}$ is trivially a module over itself and it can be endowed with a structure of a topological ring. This is done by defining the *valuation* v of a representative $(r_\varepsilon)_\varepsilon$ of $r \in \widetilde{\mathbb{C}}$ as $\sup\{b \in \mathbb{R} : |r_\varepsilon| = O(\varepsilon^b)\ \text{as}\ \varepsilon \to 0\}$. By observing that $\mathrm{v}((r_\varepsilon)_\varepsilon) = \mathrm{v}((r'_\varepsilon)_\varepsilon)$ for all representatives $(r_\varepsilon)_\varepsilon, (r'_\varepsilon)_\varepsilon$ of r, one can let v act on $\widetilde{\mathbb{C}}$ and define the map

$$|\cdot|_{\mathrm{e}} := \widetilde{\mathbb{C}} \to [0, +\infty) : u \to |u|_{\mathrm{e}} := \mathrm{e}^{-\mathrm{v}(u)}.$$

The properties of the valuation on $\widetilde{\mathbb{C}}$ make the coarsest topology on $\widetilde{\mathbb{C}}$ for which the map $|\cdot|_{\mathrm{e}}$ is continuous compatible with the ring structure. It is common in the already existing literature [19, 22, 23, 24] to use the adjective "sharp" for such a topology.

A *topological* $\widetilde{\mathbb{C}}$*-module* is a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ endowed with a $\widetilde{\mathbb{C}}$-linear topology, i.e., with a topology such that the addition $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (u, v) \to u + v$ and the product $\widetilde{\mathbb{C}} \times \mathcal{G} \to \mathcal{G} : (\lambda, u) \to \lambda u$ are continuous. A *locally convex topological* $\widetilde{\mathbb{C}}$*-module* is a topological $\widetilde{\mathbb{C}}$-module whose topology is determined by a family of *ultra-pseudo-seminorms*. As defined in [7, Definition 1.8] an ultra-pseudo-seminorm on $\mathcal{G}$ is a map $\mathcal{P} : \mathcal{G} \to [0, +\infty)$ such that

(i) $\mathcal{P}(0) = 0$,
(ii) $\mathcal{P}(\lambda u) \le |\lambda|_{\mathrm{e}} \mathcal{P}(u)$ for all $\lambda \in \widetilde{\mathbb{C}}$, $u \in \mathcal{G}$,
(iii) $\mathcal{P}(u + v) \le \max\{\mathcal{P}(u), \mathcal{P}(v)\}$.

Note that since $|[(\varepsilon^{-a})_\varepsilon]|_{\mathrm{e}} = |[(\varepsilon^{a})_\varepsilon]|_{\mathrm{e}}{}^{-1}$, from (ii) it follows that

(ii)$'$ $\mathcal{P}(\lambda u) = |\lambda|_{\mathrm{e}} \mathcal{P}(u)$ for all $\lambda = [(c\varepsilon^a)_\varepsilon]$, $c \in \mathbb{C}$, $a \in \mathbb{R}$, $u \in \mathcal{G}$.

The notion of *valuation* can be introduced in the general context of $\widetilde{\mathbb{C}}$-modules as follows: a valuation on $\mathcal{G}$ is a function $\mathrm{v} : \mathcal{G} \to (-\infty, +\infty]$ such that

(i) $\mathrm{v}(0) = +\infty$,
(ii) $\mathrm{v}(\lambda u) \ge \mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}(u)$ for all $\lambda \in \widetilde{\mathbb{C}}$, $u \in \mathcal{G}$,
(iii) $\mathrm{v}(u + v) \ge \min\{\mathrm{v}(u), \mathrm{v}(v)\}$.

As above, from (ii) it follows that

(ii)$'$ $\mathrm{v}(\lambda u) = \mathrm{v}_{\widetilde{\mathbb{C}}}(\lambda) + \mathrm{v}(u)$ for all $\lambda = [(c\varepsilon^a)_\varepsilon]$, $c \in \mathbb{C}$, $a \in \mathbb{R}$, $u \in \mathcal{G}$.

Any valuation generates an ultra-pseudo-seminorm by setting $\mathcal{P}(u) = \mathrm{e}^{-\mathrm{v}(u)}$. An ultra-pseudo-seminorm $\mathcal{P}$ such that $\mathcal{P}(u) = 0$ if and only if $u = 0$ is called an *ultra-pseudo-norm*. The topological dual of a topological $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is the set $\mathcal{L}(\mathcal{G}, \widetilde{\mathbb{C}})$ of all continuous and $\widetilde{\mathbb{C}}$-linear functionals on $\mathcal{G}$. A thorough investigation of $\mathcal{L}(\mathcal{G}, \widetilde{\mathbb{C}})$ can be found in [7, 8], together with interesting examples coming from Colombeau theory.

The family of seminorms $\{p_i\}_{i \in I}$ on E equips $\mathcal{G}_E$ with the structure of a locally convex topological $\widetilde{\mathbb{C}}$-module by means of the valuations

$$\mathrm{v}_{p_i}([(u_\varepsilon)_\varepsilon]) := \mathrm{v}_{p_i}((u_\varepsilon)_\varepsilon) := \sup\{b \in \mathbb{R} : \quad p_i(u_\varepsilon) = O(\varepsilon^b) \text{ as } \varepsilon \to 0\}$$

and the corresponding *ultra-pseudo-seminorms* $\{\mathcal{P}_i\}_{i \in I}$, where $\mathcal{P}_i(u) = \mathrm{e}^{-\mathrm{v}_{p_i}(u)}$.

1.1.2. *Basic $\widetilde{\mathbb{C}}$-linear maps and $\widetilde{\mathbb{C}}$-sesquilinear forms.* Consider locally convex topological $\widetilde{\mathbb{C}}$-modules $(\mathcal{G}, \{\mathcal{P}_i\}_{i \in I})$ and $(\mathcal{F}, \{\mathcal{Q}_j\}_{j \in J})$. Theorem 1.16 and Corollary 1.17 in [7] prove that a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{F}$ is continuous if and only if it is continuous at the origin, if and only if for all $j \in J$ there exists a constant $C > 0$ and a finite subset I_0 of I such that the inequality

$$\mathcal{Q}_j(Tu) \le C \max_{i \in I_0} \mathcal{P}_i(u) \tag{1.1}$$

holds for all $u \in \mathcal{G}$.

In the particular case of $\mathcal{G} = \mathcal{G}_E$ and $\mathcal{F} = \mathcal{G}_F$, we recall that a $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_E \to \mathcal{G}_F$ is *basic* if there exists a net $(T_\varepsilon)_\varepsilon$ of continuous linear maps from E to F fulfilling the continuity-property

$$\forall j \in J\, \exists I_0 \subseteq I \text{ finite } \exists N \in \mathbb{N}\, \exists \eta \in (0,1]\, \forall u \in E\, \forall \varepsilon \in (0, \eta] \quad q_j(T_\varepsilon u) \le \varepsilon^{-N} \sum_{i \in I_0} p_i(u), \tag{1.2}$$

and such that $Tu = [(T_\varepsilon(u_\varepsilon))_\varepsilon]$ for all $u \in \mathcal{G}_E$. It is clear that (1.2) implies (1.1), and therefore any basic map is continuous.

This notion of basic structure can be easily extended to multilinear maps from $\mathcal{G}_{E_1} \times \ldots \times \mathcal{G}_{E_n} \to \mathcal{G}_F$. In this paper we will often work with basic $\widetilde{\mathbb{C}}$-sesquilinear forms. A *basic* $\widetilde{\mathbb{C}}$-sesquilinear form a on $\mathcal{G}_E \times \mathcal{G}_F$ is a $\widetilde{\mathbb{C}}$-sesquilinear map a from $\mathcal{G}_E \times \mathcal{G}_F \to \widetilde{\mathbb{C}}$ such that there exists a net $(a_\varepsilon)_\varepsilon$ of continuous sesquilinear forms on $E \times F$ fulfilling the continuity-property

$$\exists I_0 \subseteq I \,\text{finite}\, \exists J_0 \subseteq J \,\text{finite}\, \exists N \in \mathbb{N}\, \exists \eta \in (0,1]\, \forall u \in E\, \forall v \in F\, \forall \varepsilon \in (0,\eta] \quad |a_\varepsilon(u,v)| \le \varepsilon^{-N} \sum_{i \in I_0} p_i(u) \sum_{j \in J_0} q_j(v) \tag{1.3}$$

and such that $a(u,v) = [(a_\varepsilon(u_\varepsilon, v_\varepsilon))_\varepsilon]$ for all $u \in \mathcal{G}_E$ and $v \in \mathcal{G}_F$.

1.1.3. *Internal subsets of* $\mathcal{G}_E$. A subset $A \subseteq \mathcal{G}_E$ is called internal [21] if there exists a net $(A_\varepsilon)_{\varepsilon\in(0,1]}$ of subsets $A_\varepsilon \subseteq E$ such that

$$A = \{u \in \mathcal{G}_E : \exists \text{ representative } (u_\varepsilon)_\varepsilon \text{ of } u\, \exists \varepsilon_0 \in (0,1]\, \forall \varepsilon \le \varepsilon_0\, u_\varepsilon \in A_\varepsilon\}.$$

If all $A_\varepsilon \neq \emptyset$, then we can take $\varepsilon_0 = 1$ in the previous definition without loss of generality. The internal set corresponding to the net $(A_\varepsilon)_\varepsilon$ is denoted by $[(A_\varepsilon)_\varepsilon]$. Let E be a normed vector space and A an internal subset of $\mathcal{G}_E$. Then the following hold [21]:

(i) A is closed.
(ii) Let $u \in \mathcal{G}_E$. If A is not empty, then there exists $v \in A$ such that $\|u - v\| = \min_{w\in A} \|u - w\|$ [21].

1.1.4. *Some properties of the ring of real generalized numbers.* We finally concentrate on the ring $\widetilde{\mathbb{R}}$ of real generalized numbers. It can be equipped with the order relation $\le$ given by $r \le s$ if and only if there exist $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ representatives of r and s respectively such that $r_\varepsilon \le s_\varepsilon$ for all $\varepsilon \in (0,1]$. We say that $r \in \widetilde{\mathbb{R}}$ is nonnegative iff $0 \le r$. We write $r > 0$ if and only if $r \ge 0$ and $r \neq 0$. Equipped with this order, $\widetilde{\mathbb{R}}$ is a partially ordered ring. One can define the square root of a nonnegative generalized number $r \in \widetilde{\mathbb{R}}$ by setting $r^{\frac{1}{2}} = [(r_\varepsilon^{\frac{1}{2}})_\varepsilon]$ for any representative $(r_\varepsilon)_\varepsilon$ of r such that $r_\varepsilon \ge 0$ for all ε. We leave it to the reader to check that $|r^2|_{\mathrm{e}} = |r|_{\mathrm{e}}^2$ for all $r \in \widetilde{\mathbb{R}}$ and that $|r^{\frac{1}{2}}|_{\mathrm{e}} = |r|_{\mathrm{e}}^{\frac{1}{2}}$ for all $r \ge 0$. In the sequel we collect some further properties concerning the order relation in $\widetilde{\mathbb{R}}$ which will be useful in the course of the paper.

Proposition 1.1. *Let a, b, b_n, r be real generalized numbers. The following assertions hold:*

(i) *$r \ge 0$ if and only if there exists a representative $(r_\varepsilon)_\varepsilon$ of r such that $r_\varepsilon \ge 0$ for all $\varepsilon \in (0,1]$ if and only if for all representatives $(r_\varepsilon)_\varepsilon$ of r and for all $q \in \mathbb{N}$ there exists $\eta \in (0,1]$ such that $r_\varepsilon \ge -\varepsilon^q$ for all $\varepsilon \in (0,\eta]$;*
(ii) *if $a \ge 0$, $b \ge 0$ and $a^2 \le b^2$, then $a \le b$;*
(iii) *if $a \le b_n$ for all $n \in \mathbb{N}$ and $b_n \to b$ as $n \to \infty$, then $a \le b$.*

Let $S \subseteq (0,1]$. We denote by $e_S \in \widetilde{\mathbb{R}}$ the generalized number with the characteristic function $(\chi_S(\varepsilon))_\varepsilon$ as representative, and $S^c = (0,1] \setminus S$. Then clearly, $e_S \neq 0$ iff $0 \in \overline{S}$ and $e_S \neq 1$ iff $0 \in \overline{S^c}$.

Let $z \in \widetilde{\mathbb{C}}$ and $S \subseteq (0,1]$ with $e_S \neq 0$. Then z is called invertible w.r.t. S if there exists $z' \in \widetilde{\mathbb{C}}$ such that $zz' = e_S$; z is called zero w.r.t. S if $ze_S = 0$. The following holds [25]:

Let $(z_\varepsilon)_\varepsilon$ be a representative of z.

(i) z is zero w.r.t S iff $(\forall m \in \mathbb{N})(\exists \eta > 0)(\forall \varepsilon \in S \cap (0, \eta))(|z_\varepsilon| \le \varepsilon^m)$ iff $(\forall T \subseteq S$ with $e_T \ne 0)(z$ is not invertible w.r.t. $T)$;

(ii) z is invertible w.r.t. S iff $(\exists m \in \mathbb{N})(\exists \eta > 0)(\forall \varepsilon \in S \cap (0, \eta))(|z_\varepsilon| \ge \varepsilon^m)$ iff $(\forall T \subseteq S$ with $e_T \ne 0)(z$ is not zero w.r.t. $T)$.

Finally we have the following characterizations of the zero divisors and the idempotent elements of $\widetilde{\mathbb{C}}$. Let $z, z' \in \widetilde{\mathbb{C}}$ such that $zz' = 0$. Then, there exists $S \subseteq (0, 1]$ such that $ze_S = 0$ and $z'e_{S^c} = 0$ [25]. If $z = z^2$, then there exists $S \subseteq (0, 1]$ such that $z = e_S$ [2].

1.1.5. *Infima in* $\widetilde{\mathbb{R}}$. Let $A \subseteq \widetilde{\mathbb{R}}$. As in any partially ordered set, $\delta \in \widetilde{\mathbb{R}}$ is a lower bound for A iff $\delta \le a$, for each $a \in A$. The infimum of A, denoted by $\inf A$, if it exists, is the greatest lower bound for A. As the set of lower bounds of $\overline{A}$ is equal to the set of lower bounds of A, $\inf A$ exists iff $\inf \overline{A}$ exists, and in that case, $\inf A = \inf \overline{A}$. The following proposition gives a characterization of the infimum.

Proposition 1.2. *Let $A \subseteq \widetilde{\mathbb{R}}$. Let $\delta \in \widetilde{\mathbb{R}}$ be a lower bound for A. The following are equivalent:*

(i) $\delta = \inf A$;

(ii) $\forall m \in \mathbb{N}\ \ \forall S \subseteq (0, 1]$ *with* $e_S \ne 0\ \ \exists a \in A\ \ ae_S \ngeq (\delta + [(\varepsilon)_\varepsilon]^m)e_S$;

(iii) $\forall m \in \mathbb{N}\ \ \forall S \subseteq (0, 1]$ *with* $e_S \ne 0\ \ \exists T \subseteq S$ *with* $e_T \ne 0\ \ \exists a \in A\ \ ae_T \le (\delta + [(\varepsilon)_\varepsilon]^m)e_T$.

Proof. *(i)* $\Rightarrow$ *(ii)*: Suppose there exists $m \in \mathbb{N}$ and $S \subseteq (0, 1]$ with $e_S \ne 0$ such that for each $a \in A$, $ae_S \ge (\delta + [(\varepsilon)_\varepsilon]^m)e_S$. Then also $a \ge ae_{S^c} + (\delta + [(\varepsilon)_\varepsilon]^m)e_S \ge \delta + [(\varepsilon)_\varepsilon]^m e_S$. So $\delta + [(\varepsilon)_\varepsilon]^m e_S$ is a lower bound for A. As $e_S \ne 0$, $\delta \ne \inf A$.

(ii) $\Rightarrow$ *(iii)*: If $ae_S \ngeq (\delta + [(\varepsilon)_\varepsilon]^m)e_S$, then there exists $T \subseteq S$ with $e_T \ne 0$ such that $ae_T \le (\delta + [(\varepsilon)_\varepsilon]^m)e_T$.

(iii) $\Rightarrow$ *(i)*: Let $\rho \in \widetilde{\mathbb{R}}$ be a lower bound for A. Suppose $\rho \nleq \delta$. Then there exists $S \subseteq (0, 1]$ with $e_S \ne 0$ and $m \in \mathbb{N}$ such that $\rho e_S \ge (\delta + [(\varepsilon)_\varepsilon]^m)e_S$. By hypothesis, there exists $T \subseteq S$ with $e_T \ne 0$ and $a \in A$ such that $ae_T \le (\delta + [(\varepsilon)_\varepsilon]^{m+1})e_T$. Hence $(\delta + [(\varepsilon)_\varepsilon]^m)e_T \le \rho e_T \le (\delta + [(\varepsilon)_\varepsilon]^{m+1})e_T$, which contradicts the fact that $e_T \ne 0$. $\square$

The infimum of A is called *close* if $\inf A \in \overline{A}$. In this case we use the notation $\overline{\inf} A$. Unlike $\mathbb{R}$, an infimum in $\widetilde{\mathbb{R}}$ is not automatically close.

Example 1.3. Let $T \subseteq (0, 1]$ with $e_T \ne 0$ and $e_{T^c} \ne 0$ and let $A = \{e_T + [(\varepsilon)_\varepsilon]^m e_{T^c} : m \in \mathbb{N}\} \cup \{e_{T^c} + [(\varepsilon)_\varepsilon]^m e_T : m \in \mathbb{N}\}$. Clearly, 0 is a lower bound for A. Let $\delta \in \widetilde{\mathbb{R}}$ be a lower bound for A. Then $\delta \le \lim_{n\to\infty}(e_T + [(\varepsilon)_\varepsilon]^n e_{T^c}) = e_T$; hence $\delta e_{T^c} \le 0$ and similarly $\delta e_T \le 0$. So $\delta = \delta e_{T^c} + \delta e_T \le 0$ and $\inf A = 0$. On the other hand, $|e_T + [(\varepsilon)_\varepsilon]^m e_{T^c}|_\mathrm{e} = |e_{T^c} + [(\varepsilon)_\varepsilon]^m e_T|_\mathrm{e} = 1$, for each $m \in \mathbb{N}$. Hence $0 \notin \overline{A}$.

The close infimum can be easily characterized as follows.

Proposition 1.4. *Let $A \subseteq \widetilde{\mathbb{R}}$. Let $\delta \in \widetilde{\mathbb{R}}$ be a lower bound for A. Then δ is a close infimum iff*

$$\forall m \in \mathbb{N}\ \exists a \in A \qquad a \le \delta + [(\varepsilon)_\varepsilon]^m.$$

Clearly, if A reaches a minimum, then $\inf A = \min A$ and the infimum is close.

1.2. **$\widetilde{\mathbb{C}}$-modules with $\widetilde{\mathbb{R}}$-seminorms.** We introduce the notion of an $\widetilde{\mathbb{R}}$-seminorm on a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. This determines a special kind of topological $\widetilde{\mathbb{C}}$-module: $\widetilde{\mathbb{C}}$-modules with $\widetilde{\mathbb{R}}$-seminorms. In the sequel, given $\lambda = [(\lambda_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$ we define $|\lambda|$ as the equivalent class of $(|\lambda_\varepsilon|)_\varepsilon$ in $\widetilde{\mathbb{R}}$.

Definition 1.5. Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module. An $\widetilde{\mathbb{R}}$-seminorm on $\mathcal{G}$ is a map $p : \mathcal{G} \to \widetilde{\mathbb{R}}$ such that

(i) $p(0) = 0$ and $p(u) \geq 0$ for all $u \in \mathcal{G}$;
(ii) $p(\lambda u) = |\lambda| p(u)$ for all $\lambda \in \widetilde{\mathbb{C}}$ and for all $u \in \mathcal{G}$;
(iii) $p(u + v) \leq p(u) + p(v)$ for all $u, v \in \mathcal{G}$.

An $\widetilde{\mathbb{R}}$-seminorm p such that $p(u) = 0$ if and only if $u = 0$ is called an $\widetilde{\mathbb{R}}$-norm.

From the properties which define an $\widetilde{\mathbb{R}}$-seminorm we easily see that the coarsest topology which makes a family $\{p_i\}_{i\in I}$ of $\widetilde{\mathbb{R}}$-seminorms on $\mathcal{G}$ continuous equips $\mathcal{G}$ with the structure of a topological $\widetilde{\mathbb{C}}$-module. Hence, any $\widetilde{\mathbb{C}}$-module with $\widetilde{\mathbb{R}}$-seminorms is a topological $\widetilde{\mathbb{C}}$-module. More precisely we have the following result.

Proposition 1.6. *Any $\widetilde{\mathbb{R}}$-seminorm p on $\mathcal{G}$ generates an ultra-pseudo-seminorm $\mathcal{P}$ by setting $\mathcal{P}(u) := |p(u)|_{\mathrm{e}} = \mathrm{e}^{-\mathrm{v}(p(u))}$. The $\widetilde{\mathbb{C}}$-linear topology on $\mathcal{G}$ determined by the family of $\widetilde{\mathbb{R}}$-seminorms $\{p_i\}_{i\in I}$ coincides with the topology of the corresponding ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$.*

Proof. The fact that $\mathcal{P}$ is an ultra-pseudo-seminorm follows from the properties of p combined with the defining conditions of the ultra-pseudo-norm $|\cdot|_{\mathrm{e}}$ of $\widetilde{\mathbb{R}}$. The families $\{p_i\}_{i\in I}$ and $\{\mathcal{P}_i\}_{i\in I}$ generate the same topology on $\mathcal{G}$ since for all $\eta > 0$, $\delta > 0$ and $u \in \mathcal{G}$ we have that

$$\begin{aligned}\{u \in \mathcal{G} :\ p_i(u) \leq [(\varepsilon^{-\log\eta})_\varepsilon]\} &\subseteq \{u \in \mathcal{G} :\ \mathcal{P}_i(u) \leq \eta\}\\ &\subseteq \{u \in \mathcal{G} :\ p_i(u) \leq [(\varepsilon^{-\log\eta-\delta})_\varepsilon]\}.\end{aligned}$$

□

In the particular case of $\mathcal{G} = \mathcal{G}_E$, where $(E, \{p_i\}_{i\in I})$ is a locally convex topological vector space, one can extend any seminorm p_i to an $\widetilde{\mathbb{R}}$-seminorm on $\mathcal{G}_E$. This is due to the fact that if $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$, then $(p_i(u_\varepsilon))_\varepsilon \in \mathcal{E}_M$ and if $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathcal{N}_E$, then $|p_i(u_\varepsilon) - p_i(u'_\varepsilon)| \leq p_i(u_\varepsilon - u'_\varepsilon) = O(\varepsilon^q)$ for all $q \in \mathbb{N}$. Proposition 1.6 says that the sharp topology on $\mathcal{G}_E$ can be regarded as the topology of the $\widetilde{\mathbb{R}}$-seminorms $p_i(u) := [(p_i(u_\varepsilon))_\varepsilon]$ as well as the topology of the ultra-pseudo-seminorms $\mathcal{P}_i(u) = |p_i(u)|_{\mathrm{e}}$.

Proposition 1.7. *Let $(\mathcal{G}, \{p_i\}_{i\in I})$, $(\mathcal{F}, \{q_j\}_{j\in J})$ and $(\mathcal{H}, \{r_k\}_{k\in K})$ be topological $\widetilde{\mathbb{C}}$-modules with $\widetilde{\mathbb{R}}$-seminorms.*

(i) *A $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{F}$ is continuous if and only if the following assertion holds: for all $j \in J$, there exist a finite subset I_0 of I and a constant $C \in \widetilde{\mathbb{R}}$ such that*

$$q_j(Tu) \leq C \sum_{i\in I_0} p_i(u) \tag{1.4}$$

for all $u \in \mathcal{G}$.

(ii) *A $\widetilde{\mathbb{C}}$-sesquilinear map a from $\mathcal{G}\times\mathcal{F}$ to $\mathcal{H}$ is continuous if and only if for all $k\in K$ there exist finite subsets I_0 and J_0 of I and J respectively and a constant $C\in\widetilde{\mathbb{R}}$ such that*

$$r_k(a(u,v))\le C\sum_{i\in I_0}p_i(u)\sum_{j\in J_0}q_j(v), \tag{1.5}$$

for all $u\in\mathcal{G}$ and $v\in\mathcal{F}$.

Proof. If the inequality (1.4) holds, then the $\widetilde{\mathbb{C}}$-linear map T is continuous, since from (1.4) we have that

$$\mathcal{Q}_j(Tu)\le |C|_{\mathrm{e}}\max_{i\in I_0}\mathcal{P}_i(u).$$

This characterizes the continuity of T as proved by Corollary 1.17 in [7]. Assume now that T is continuous at 0. Hence, for all $j\in J$ and for all $c\in\mathbb{R}$ there exist $b\in\mathbb{R}$ and a finite subset I_0 of I such that $q_j(Tu)\le[(\varepsilon^c)_\varepsilon]$ if $\sum_{i\in I_0}p_i(u)\le[(\varepsilon^b)_\varepsilon]$. Let $q\in\mathbb{N}$. For any $u\in\mathcal{G}$ we have that $[(\varepsilon^b)_\varepsilon]u/(\sum_{i\in I_0}p_i(u)+[(\varepsilon^q)_\varepsilon])$ belongs to the set of all $v\in\mathcal{G}$ such that $\sum_{i\in I_0}p_i(v)\le[(\varepsilon^b)_\varepsilon]$. Thus,

$$q_j(Tu)\le[(\varepsilon^{c-b})_\varepsilon]\biggl(\sum_{i\in I_0}p_i(u)+[(\varepsilon^q)_\varepsilon]\biggr).$$

Letting q go to ∞ we conclude that (1.4) is valid for $C=[(\varepsilon^{c-b})_\varepsilon]$.

The proof of the second assertion of the proposition is similar and therefore left to the reader. □

We now consider the framework of Colombeau spaces of generalized functions based on a normed space, and we provide a characterization for continuous $\widetilde{\mathbb{C}}$-linear maps given by a representative. We recall that a representative $(T_\varepsilon)_\varepsilon$ of a $\widetilde{\mathbb{C}}$-linear map $T:\mathcal{G}_E\to\mathcal{G}_F$, if it exists, is a net of linear maps from E to F such that $(T_\varepsilon u_\varepsilon)_\varepsilon\in\mathcal{M}_F$ for all $(u_\varepsilon)_\varepsilon\in\mathcal{M}_E$, $(T_\varepsilon u_\varepsilon)_\varepsilon\in\mathcal{N}_F$ for all $(u_\varepsilon)_\varepsilon\in\mathcal{N}_E$ and $Tu=[(T_\varepsilon u_\varepsilon)_\varepsilon]$ for all $u=[(u_\varepsilon)_\varepsilon]\in\mathcal{G}_E$.

Proposition 1.8. *Let E, F be normed spaces and let $(T_\varepsilon)_\varepsilon$ be a net of linear maps from E to F such that $(T_\varepsilon u_\varepsilon)_\varepsilon\in\mathcal{M}_F$ for each $(u_\varepsilon)_\varepsilon\in\mathcal{N}_E$. Then $(T_\varepsilon u_\varepsilon)_\varepsilon\in\mathcal{M}_F$ for each $(u_\varepsilon)_\varepsilon\in\mathcal{M}_E$ and $(T_\varepsilon u_\varepsilon)_\varepsilon\in\mathcal{N}_F$ for each $(u_\varepsilon)_\varepsilon\in\mathcal{N}_E$.*

Proof. Let $(u_\varepsilon)_\varepsilon\in\mathcal{M}_E$, i.e., there exists $N\in\mathbb{N}$ such that $\|u_\varepsilon\|\le\varepsilon^{-N}$, for sufficiently small ε. Suppose that $(T_\varepsilon u_\varepsilon)_\varepsilon\notin\mathcal{M}_F$. Then we can find a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_n\varepsilon_n=0$ such that $\|T_{\varepsilon_n}u_{\varepsilon_n}\|\ge\varepsilon_n^{-n}$. Let $v_{\varepsilon_n}=u_{\varepsilon_n}\varepsilon_n^{n/2}$, $\forall n\in\mathbb{N}$ and let $v_\varepsilon=0$ if $\varepsilon\notin\{\varepsilon_n:n\in\mathbb{N}\}$. Then for any $M\in\mathbb{N}$, $\|v_\varepsilon\|\le\|u_\varepsilon\varepsilon^M\|\le\varepsilon^{M-N}$ for sufficiently small ε, but for each $n\in\mathbb{N}$, $\|T_{\varepsilon_n}(v_{\varepsilon_n})\|=\varepsilon_n^{n/2}\|T_{\varepsilon_n}(u_{\varepsilon_n})\|\ge\varepsilon_n^{-n/2}$. Hence $v_\varepsilon\in\mathcal{N}_E$, but $T_\varepsilon(v_\varepsilon)\notin\mathcal{M}_E$, which contradicts the hypotheses.

Similarly, let $(u_\varepsilon)_\varepsilon\in\mathcal{N}_E$, i.e., for each $m\in\mathbb{N}$, $\|u_\varepsilon\|\le\varepsilon^m$ as soon as $\varepsilon\le\eta_m\in(0,1]$. Suppose that $(T_\varepsilon u_\varepsilon)_\varepsilon\notin\mathcal{N}_F$. Then we can find $m\in\mathbb{N}$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_n\varepsilon_n=0$, such that $\varepsilon_n\le\eta_n$ and $\|T_{\varepsilon_n}(u_{\varepsilon_n})\|\ge\varepsilon_n^m$, $\forall n\in\mathbb{N}$. Let $v_{\varepsilon_n}=u_{\varepsilon_n}\varepsilon_n^{-n/2}$, $\forall n\in\mathbb{N}$ and let $v_\varepsilon=0$ if $\varepsilon\notin\{\varepsilon_n:n\in\mathbb{N}\}$. Then for each $n\in\mathbb{N}$, $\|v_{\varepsilon_n}\|\le\|u_{\varepsilon_n}\|\varepsilon_n^{-n/2}\le\varepsilon_n^{n/2}$, but $\|T_{\varepsilon_n}(v_{\varepsilon_n})\|=\|T_{\varepsilon_n}(u_{\varepsilon_n})\|\varepsilon_n^{-n/2}\ge\varepsilon_n^{m-n/2}$. Hence $v_\varepsilon\in\mathcal{N}_E$, but $T_\varepsilon(v_\varepsilon)\notin\mathcal{M}_E$, which contradicts the hypotheses. □

Inspired by a similar result in [21] we obtain the following proposition.

Proposition 1.9. *Let E and F be normed spaces and $T : \mathcal{G}_E \to \mathcal{G}_F$ a $\widetilde{\mathbb{C}}$-linear map. If T has a representative $(T_\varepsilon)_\varepsilon$, then it is continuous.*

Proof. We prove that if $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$ implies $(T_\varepsilon u_\varepsilon)_\varepsilon \in \mathcal{N}_F$, then

$$\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \exists \varepsilon_0 \in (0,1]\, \forall \varepsilon \in (0,\varepsilon_0)\, \forall u \in E \qquad \|u\|_E \le \varepsilon^m \;\Rightarrow\; \|T_\varepsilon u\|_F \le \varepsilon^n. \tag{1.6}$$

Indeed, if we negate (1.6), then we can find some $n' \in \mathbb{N}$, a decreasing sequence ε_m converging to 0 and some $u_{\varepsilon_m} \in E$ with $\|u_{\varepsilon_m}\|_E \le \varepsilon_m^m$ such that $\|T_{\varepsilon_m} u_{\varepsilon_m}\|_F > \varepsilon_m^{n'}$. Now let $u_\varepsilon = u_{\varepsilon_m}$ for $\varepsilon \in [\varepsilon_m, \varepsilon_{m-1})$ and $u_\varepsilon = 0$ for $\varepsilon \in [\varepsilon_0, 1]$. By construction we have that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$ and $\|T_{\varepsilon_m} u_{\varepsilon_m}\|_F > \varepsilon_m^{n'}$ for all m. This is in contradiction with $(T_\varepsilon u_\varepsilon)_\varepsilon \in \mathcal{N}_F$.

The assertion (1.6) says that for all $n \in \mathbb{N}$ there exists a neighborhood $U = \{u \in \mathcal{G}_E : \|u\|_E \le [(\varepsilon^m)_\varepsilon]\}$ of 0 which has image $T(U)$ contained in the neighborhood $V = \{v \in \mathcal{G}_F : \|v\|_F \le [(\varepsilon^n)_\varepsilon]\}$. Hence, the map T is continuous at 0 and thus continuous from $\mathcal{G}_E$ to $\mathcal{G}_F$. □

Proposition 1.10. *Let E and F be normed spaces, and let T be a continuous $\widetilde{\mathbb{C}}$-linear map from $\mathcal{G}_E$ to $\mathcal{G}_F$ with a representative and $C \ge 0$ in $\widetilde{\mathbb{R}}$. Then, the following assertions are equivalent:*

(i) $\|Tu\|_F \le C\|u\|_E$ *for all* $u \in \mathcal{G}_E$;

(ii) *for all representatives $(T_\varepsilon)_\varepsilon$ of T, for all representatives $(C_\varepsilon)_\varepsilon$ of C and for all $q \in \mathbb{N}$ there exists $\eta \in (0,1]$ such that*

$$\|T_\varepsilon u\|_F \le (C_\varepsilon + \varepsilon^q)\|u\|_E \tag{1.7}$$

for all $u \in E$ and $\varepsilon \in (0,\eta]$;

(iii) *for all representatives $(T_\varepsilon)_\varepsilon$ of T there exists a representative $(C_\varepsilon)_\varepsilon$ of C and $\eta \in (0,1]$ such that*

$$\|T_\varepsilon u\|_F \le C_\varepsilon \|u\|_E \tag{1.8}$$

for all $u \in E$ and $\varepsilon \in (0,\eta]$.

Proof. From Proposition 1.7 we have that the continuity of T is equivalent to (i). In order to prove that (i) implies (ii), we begin by observing that (i) is equivalent to $e_S\|Tu\|_F \le Ce_S\|u\|_E$ for all $S \subseteq (0,1]$. We want to prove that the negation of (ii) implies that there exists a subset S of $(0,1]$ and some $u \in \mathcal{G}_E$ such that $e_S\|Tu\|_F > Ce_S\|u\|_E$. From

$$\exists (T_\varepsilon)_\varepsilon\, \exists (C_\varepsilon)_\varepsilon\, \exists q \in \mathbb{N}\, \forall \eta \in (0,1]\, \exists \varepsilon \in (0,\eta]\, \exists u \in E \quad \|T_\varepsilon u\|_F > (C_\varepsilon + \varepsilon^q)\|u\|_E$$

we have that there exists a decreasing sequence $(\varepsilon_k)_k \subseteq (0,1]$ converging to 0 and a sequence $(u_{\varepsilon_k})_k$ of elements of E with norm 1 such that

$$\|T_{\varepsilon_k}(u_{\varepsilon_k})\|_F > (C_{\varepsilon_k} + \varepsilon_k^q).$$

Let us fix $x \in E$ with $\|x\|_E = 1$. The net $u_\varepsilon = u_{\varepsilon_k}$ when $\varepsilon = \varepsilon_k$ and $u_\varepsilon = x$ otherwise generates an element $u = [(u_\varepsilon)_\varepsilon]$ of $\mathcal{G}_E$ with $\widetilde{\mathbb{R}}$-norm 1. Now let $S = \{\varepsilon_k : k \in \mathbb{N}\}$. By construction we have that

$$e_S\|Tu\|_F = [(\chi_S T_{\varepsilon_k}(u_{\varepsilon_k}))_\varepsilon] \ge e_S(C + [(\varepsilon^q)_\varepsilon]) > e_S C.$$

This contradicts (i). It is easy to prove that (ii) implies (iii). Indeed, by fixing representatives $(T_\varepsilon)_\varepsilon$ and $(C'_\varepsilon)_\varepsilon$ of T and C respectively, we can extract a decreasing sequence $(\eta_q)_{q\in\mathbb{N}}$ tending to 0 such that $\|T_\varepsilon(u)\|_F \le (C'_\varepsilon + \varepsilon^q)\|u\|_E$ for all $u \in E$ and $\varepsilon \in (0,\eta_q]$. The net $n_\varepsilon = \varepsilon^q$ for $\varepsilon \in (\eta^{q+1}, \eta^q]$ is negligible, and therefore

$C_\varepsilon = C'_\varepsilon + n_\varepsilon$ satisfies (1.8) on the interval $(0, \eta_0]$. Finally, it is clear that (iii) implies (i). □

Note that from the previous propositions we have that if T is given by a representative $(T_\varepsilon)_\varepsilon$, then it is a basic map.

2. Hilbert $\widetilde{\mathbb{C}}$-modules

2.1. **Definition.** This section is devoted to the definition and the first properties of the class of topological $\widetilde{\mathbb{C}}$-modules which are the mathematical core of the paper: the *Hilbert* $\widetilde{\mathbb{C}}$*-modules*. With the intent of developing a topological and functional analytic theory of Hilbert $\widetilde{\mathbb{C}}$-modules, we start in Subsection 2.2 by investigating the notion of projection on suitable subsets of a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$. This requires the new concept of an *edged* subset of $\mathcal{G}$ and a formulation of convexity, which unlike the $\widetilde{\mathbb{C}}$-convexity introduced in [7], resembles the well-known classical definition for subsets of a vector space.

Definition 2.1. Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module. A scalar product $(\cdot|\cdot)$ is a $\widetilde{\mathbb{C}}$-sesquilinear form from $\mathcal{G} \times \mathcal{G}$ to $\widetilde{\mathbb{C}}$ satisfying the following properties:

(i) $(u|v) = \overline{(v|u)}$ for all $u, v \in \mathcal{G}$,
(ii) $(u|u) \in \widetilde{\mathbb{R}}$ and $(u|u) \geq 0$ for all $u \in \mathcal{G}$,
(iii) $(u|u) = 0$ if and only if $u = 0$.

In the sequel we denote $\sqrt{(u|u)}$ by $\|u\|$.

Since $\widetilde{\mathbb{C}}$ is not a field, the following proposition is not immediate.

Proposition 2.2. *Let $\mathcal{G}$ be a $\widetilde{\mathbb{C}}$-module with scalar product $(\cdot|\cdot)$. Then for all $u, v \in \mathcal{G}$ the Cauchy-Schwarz inequality holds:*

$$|\,(u|v)\,| \leq \|u\|\|v\|. \tag{2.1}$$

Proof. Let $\alpha \in \widetilde{\mathbb{C}}$. By definition of a scalar product we know that $\|u + \alpha v\|$ is a positive generalized real number. Hence, the $\widetilde{\mathbb{C}}$-sesquilinearity of $(\cdot|\cdot)$ yields

$$0 \leq \|u + \alpha v\|^2 = \|u\|^2 + \alpha\overline{(u|v)} + \overline{\alpha}\,(u|v) + |\alpha|^2\|v\|^2. \tag{2.2}$$

We will derive the Cauchy-Schwarz inequality (2.1) from (2.2) by choosing a suitable sequence of $\alpha \in \widetilde{\mathbb{C}}$. In detail, let $\alpha_n := -\,(u|v)\,/(\|v\|^2 + [(\varepsilon^n)])$. The equality (2.2) combined with $\|v\|^2 \leq \|v\|^2 + [(\varepsilon^n)]$ yields

$$\begin{aligned} 0 \leq \|u\|^2 - \frac{|\,(u|v)\,|^2}{\|v\|^2 + [(\varepsilon^n)]} - \frac{|\,(u|v)\,|^2}{\|v\|^2 + [(\varepsilon^n)]} + \frac{|\,(u|v)\,|^2}{(\|v\|^2 + [(\varepsilon^n)])^2}\|v\|^2 \\ \leq \|u\|^2 - 2\frac{|\,(u|v)\,|^2}{\|v\|^2 + [(\varepsilon^n)]} + \frac{|\,(u|v)\,|^2}{\|v\|^2 + [(\varepsilon^n)]}. \end{aligned}$$

Hence,

$$0 \leq \|u\|^2(\|v\|^2 + [(\varepsilon^n)]) - |\,(u|v)\,|^2$$

for all n, and since the sequence $(\|v\|^2 + [(\varepsilon^n)])_n$ tends to $\|v\|^2$ in $\widetilde{\mathbb{R}}$, it follows that the Cauchy-Schwarz inequality (2.1) holds. □

We use the Cauchy-Schwarz inequality in proving the following proposition.

Proposition 2.3. *The map* $\|\cdot\| : \mathcal{G} \to \widetilde{\mathbb{R}} : u \to \|u\| := (u|u)^{\frac{1}{2}}$ *is an* $\widetilde{\mathbb{R}}$*-norm on* $\mathcal{G}$*, and the map* $\mathcal{P} : \mathcal{G} \to [0, +\infty) : u \to |\,(u|u)^{\frac{1}{2}}\,|_{\mathrm{e}} = |\,(u|u)\,|_{\mathrm{e}}^{\frac{1}{2}}$ *is an ultra-pseudo-norm on* $\mathcal{G}$*.*

Proof. The third property of Definition 2.1 ensures that $\|u\| = 0$ if and only if $u = 0$. Let us now take $\lambda \in \widetilde{\mathbb{C}}$. From the homogeneity of the scalar product we have that

$$\|\lambda u\| = (\lambda u|\lambda u)^{\frac{1}{2}} = \left(|\lambda|^2 \|u\|^2\right)^{\frac{1}{2}} = |\lambda| \|u\|.$$

Finally, we write $\|u + v\|^2$ as $\|u\|^2 + 2\Re\,(u|v) + \|v\|^2$, and since $\Re\,(u|v) \le |\,(u|v)\,|$ we obtain from the Cauchy-Schwarz inequality (2.1) that

$$\|u + v\|^2 \le \|u\|^2 + \|v\|^2 + 2|\,(u|v)\,| \le (\|u\| + \|v\|)^2.$$

It follows that $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in \mathcal{G}$. Thus, $\|\cdot\|$ is an $\widetilde{\mathbb{R}}$-norm on $\mathcal{G}$. Proposition 1.6, combined with the fact that $|\lambda^{\frac{1}{2}}|_{\mathrm{e}} = |\lambda|_{\mathrm{e}}^{\frac{1}{2}}$, allows us to conclude that $\mathcal{P}$ is an ultra-pseudo-norm. □

From Proposition 1.6 we have that a $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ with scalar product $(\cdot|\cdot)$ can be endowed with the topology of the $\widetilde{\mathbb{R}}$-norm $\|\cdot\|$ generated by $(\cdot|\cdot)$ or equivalently with the topology of the ultra-pseudo-norm $\mathcal{P}(u) = |\,(u|u)\,|_{\mathrm{e}}^{\frac{1}{2}}$. This means that any $\widetilde{\mathbb{C}}$-module with a scalar product is a $\widetilde{\mathbb{C}}$-module with an $\widetilde{\mathbb{R}}$-norm and hence a topological $\widetilde{\mathbb{C}}$-module. Proposition 2.2, combined with Proposition 2.3, yields the following continuity result.

Proposition 2.4. *Let* $\mathcal{G}$ *be a* $\widetilde{\mathbb{C}}$*-module with scalar product* $(\cdot|\cdot)$*, topologized through the ultra-pseudo-norm* $\mathcal{P}(u) = |\,(u|u)\,|_{\mathrm{e}}^{\frac{1}{2}}$*. The scalar product is a continuous* $\widetilde{\mathbb{C}}$*-sesquilinear map from* $\mathcal{G} \times \mathcal{G}$ *to* $\widetilde{\mathbb{C}}$*.*

Definition 2.5. A Hilbert $\widetilde{\mathbb{C}}$-module is a $\widetilde{\mathbb{C}}$-module with scalar product $(\cdot|\cdot)$ which is complete when endowed with the topology of the corresponding ultra-pseudo-norm $\mathcal{P}$.

Since a closed subset of a complete topological $\widetilde{\mathbb{C}}$-module is complete, we have that a closed $\widetilde{\mathbb{C}}$-submodule of a Hilbert $\widetilde{\mathbb{C}}$-module is itself a Hilbert $\widetilde{\mathbb{C}}$-module.

Example 2.6.

(i) A first example of a Hilbert $\widetilde{\mathbb{C}}$-module is given by $\mathcal{G}_H$, where $(H, (\cdot|\cdot))$ is a Hilbert space. The scalar product on $\mathcal{G}_H$ is obtained by letting $(\cdot|\cdot)$ act componentwise on the representatives of the generalized functions in $\mathcal{G}_H$ as follows: $(u|v) = [((u_\varepsilon|v_\varepsilon))_\varepsilon]$. By Proposition 3.4 in [7] one can omit the assumption of completeness on H and still obtain that $\mathcal{G}_H$ is complete with respect to the sharp topology induced by the scalar product.

(ii) The topological structure on $\mathcal{G}_H$ determined by the scalar product of H can be equivalently generated by any continuous $\widetilde{\mathbb{C}}$-sesquilinear form a on $\mathcal{G}_H \times \mathcal{G}_H$ such that $a(u, v) = \overline{a(v, u)}$ for all $u, v \in \mathcal{G}_H$, $a(u, u) \ge 0$ for all $u \in \mathcal{G}_H$ and the following bound from below holds:

$$\exists C \in \widetilde{\mathbb{R}}, C \ge 0, \text{ invertible}, \ \forall u \in \mathcal{G}_H \qquad a(u, u) \ge C\|u\|^2 \tag{2.3}$$

(see also Definition 6.1). Since a satisfies the conditions of Definition 2.1, it is a scalar product on $\mathcal{G}_H$ and the corresponding Cauchy-Schwarz inequality is valid. Hence, $\|u\|_a := a(u, u)^{\frac{1}{2}}$ is an $\widetilde{\mathbb{R}}$-norm. Combining the continuity of a with the estimate (2.3) we have that $\|\cdot\|_a$ is equivalent to the usual $\widetilde{\mathbb{R}}$-norm $\|\cdot\|$. This means that there exist $C_1, C_2 \geq 0$ real generalized numbers such that

$$C_1\|u\| \leq \|u\|_a \leq C_2\|u\|$$

for all $u \in \mathcal{G}_H$.

A further example of a Hilbert $\widetilde{\mathbb{C}}$-module is provided by the following proposition.

Proposition 2.7. *Let $(H_\varepsilon, (\cdot|\cdot)_{H_\varepsilon})_\varepsilon$ be a net of vector spaces with scalar product and let $\mathcal{G}$ be the $\widetilde{\mathbb{C}}$-module obtained by factorizing the set*

$$\mathcal{M}_{(H_\varepsilon)_\varepsilon} = \{(u_\varepsilon)_\varepsilon : \forall\varepsilon \in (0,1]\, u_\varepsilon \in H_\varepsilon \quad \text{and} \quad \exists N \in \mathbb{N}\, \|u\|_{H_\varepsilon} = O(\varepsilon^{-N})\}$$

of moderate nets with respect to the set

$$\mathcal{N}_{(H_\varepsilon)_\varepsilon} = \{(u_\varepsilon)_\varepsilon : \forall\varepsilon \in (0,1]\, u_\varepsilon \in H_\varepsilon \quad \text{and} \quad \forall q \in \mathbb{N}\, \|u\|_{H_\varepsilon} = O(\varepsilon^{q})\}$$

of negligible nets. Let $(\cdot|\cdot) : \mathcal{G}\times\mathcal{G} \to \widetilde{\mathbb{C}}$ be the $\widetilde{\mathbb{C}}$-sesquilinear form defined as follows:

$$(u|v) = [((u_\varepsilon|v_\varepsilon)_{H_\varepsilon})_\varepsilon]. \tag{2.4}$$

Then, $(\cdot|\cdot)$ is a scalar product on $\mathcal{G}$ which equips $\mathcal{G}$ with the structure of a Hilbert $\widetilde{\mathbb{C}}$-module.

Proof. Applying the Cauchy-Schwarz inequality componentwise in any Hilbert space H_ε we have that (2.4) is a well-defined $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G} \times \mathcal{G}$ such that the properties $(i), (ii), (iii)$ of Definition 2.1 are fulfilled. Let $\mathcal{G}$ be endowed with the topology of this scalar product, i.e., with the topology of the $\widetilde{\mathbb{R}}$-norm $\|u\| = [(\|u_\varepsilon\|_{H_\varepsilon})_\varepsilon]$. We want to prove that any Cauchy sequence in $\mathcal{G}$ is convergent. If $(u_n)_n$ is a Cauchy sequence, then we can extract a subsequence $(u_{n_k})_k$ and a corresponding subsequence $((u_{n_k,\varepsilon})_\varepsilon)_k$ of representatives such that $\|u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon}\|_{H_\varepsilon} \leq \varepsilon^k$ for all $\varepsilon \in (0, \varepsilon_k)$, with $\varepsilon_k \searrow 0$, $\varepsilon_k \leq 2^{-k}$ for all $k \in \mathbb{N}$. Arguing as in the proof of [7, Proposition 3.4] we set $h_{k,\varepsilon} = u_{n_{k+1},\varepsilon} - u_{n_k,\varepsilon}$ for $\varepsilon \in (0, \varepsilon_k)$ and $h_{k,\varepsilon} = 0$ for $\varepsilon \in [\varepsilon_k, 1]$. Obviously, $(h_{k,\varepsilon})_\varepsilon \in \mathcal{M}_{(H_\varepsilon)_\varepsilon}$ and $\|h_{k,\varepsilon}\|_{H_\varepsilon} \leq \varepsilon^k$ on the whole interval $(0, 1]$. Now let

$$u_\varepsilon := \sum_{k=0}^{\infty} h_{k,\varepsilon} + u_{n_0,\varepsilon}.$$

This sum is locally finite and moderate, since

$$\|u_\varepsilon\|_{H_\varepsilon} \leq \sum_{k=0}^{\infty} \|h_{k,\varepsilon}\|_{H_\varepsilon} + \|u_{n_0,\varepsilon}\|_{H_\varepsilon} \leq \sum_{k=0}^{\infty} \varepsilon_k^k + \|u_{n_0,\varepsilon}\|_{H_\varepsilon} \leq \sum_{k=0}^{\infty} 2^{-k} + \|u_{n_0,\varepsilon}\|_{H_\varepsilon}.$$

Hence, $(u_\varepsilon)_\varepsilon$ generates an element of $\mathcal{G}$. By construction the sequence $(u_{n_k})_k$ converges to u. Indeed, for all $\overline{k} \geq 1$ we have that

$$\begin{aligned}
\|u_{n_{\overline{k}},\varepsilon} - u_\varepsilon\|_{H_\varepsilon} &= \Big\| u_{n_{\overline{k}},\varepsilon} - u_{n_0,\varepsilon} - \sum_{k=0}^{\infty} h_{k,\varepsilon} \Big\|_{H_\varepsilon} \\
&= \Big\| - \sum_{k=\overline{k}}^{\infty} h_{k,\varepsilon} \Big\|_{H_\varepsilon} \leq \varepsilon^{\overline{k}-1} \sum_{k=\overline{k}}^{\infty} \varepsilon_k \leq \varepsilon^{\overline{k}-1} \sum_{k=\overline{k}}^{\infty} 2^{-k},
\end{aligned}$$

and the proof is complete. □

Proposition 2.8. *The Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G} = \mathcal{M}_{(H_\varepsilon)_\varepsilon}/\mathcal{N}_{(H_\varepsilon)_\varepsilon}$ defined in the previous proposition is (algebraically and isometrically) isomorphic with an internal submodule of a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}_H$ for some pre-Hilbert space H.*

Proof. Let $H = \bigoplus_{\lambda\in(0,1]} H_\lambda$ be the direct sum of the pre-Hilbert spaces H_λ, which is by definition the set of all nets $(u_\lambda)_{\lambda\in(0,1]}$, where $u_\lambda \in H_\lambda$ for each λ, which satisfy $\sum_{\lambda\in(0,1]} \|u_\lambda\|^2_{H_\lambda} < +\infty$. This is a pre-Hilbert space [17, Section 2.6] for the componentwise algebraic operations and the inner product

$$((u_\lambda)_\lambda|(v_\lambda)_\lambda) = \sum_{\lambda\in(0,1]} (u_\lambda|v_\lambda)_{H_\lambda}.$$

(When all H_λ are Hilbert spaces, the direct sum is actually a Hilbert space ([17, Section 2.6]).) Each H_λ is canonically (algebraically and isometrically) isomorphic with a submodule $\widetilde{H}_\lambda$ of H by the embedding ι_λ: $H_\lambda \to H$: $\iota_\lambda(u) = (u_\mu)_\mu$ with $u_\lambda = u$, $u_\mu = 0$ if $\mu \neq \lambda$. Hence we can consider the internal subset $[(\widetilde{H}_\varepsilon)_\varepsilon] \subseteq \mathcal{G}_H$. Now let ι: $\mathcal{G} \to \mathcal{G}_H$ be defined on representatives by $\iota((u_\varepsilon)_\varepsilon) = (\iota_\varepsilon(u_\varepsilon))_\varepsilon$. Since $\|u_\varepsilon\|_{H_\varepsilon} = \|\iota_\varepsilon(u_\varepsilon)\|_H$ for each ε, $(\iota_\varepsilon(u_\varepsilon))_\varepsilon$ belongs to $\mathcal{M}_H$, resp. $\mathcal{N}_H$, iff $(u_\varepsilon)_\varepsilon$ belongs to $\mathcal{M}_{(H_\varepsilon)_\varepsilon}$, resp. $\mathcal{N}_{(H_\varepsilon)_\varepsilon}$. Hence ι is well-defined and injective. Clearly, the image of ι is contained in $[(\widetilde{H}_\varepsilon)_\varepsilon]$. Conversely, each $v \in [(\widetilde{H}_\varepsilon)_\varepsilon]$ has a representative $(v_\varepsilon)_\varepsilon$ with each $v_\varepsilon \in \widetilde{H}_\varepsilon$. So $v_\varepsilon = \iota_\varepsilon(u_\varepsilon)$, for some $u_\varepsilon \in H_\varepsilon$. Again by $\|u_\varepsilon\|_{H_\varepsilon} = \|\iota_\varepsilon(u_\varepsilon)\|_H$, the net $(u_\varepsilon)_\varepsilon$ belongs to $\mathcal{M}_{(H_\varepsilon)_\varepsilon}$, so it represents $u \in \mathcal{G}$ with $\iota(u) = v$. □

We see from the previous proposition that there is no loss of generality by considering the $\widetilde{\mathbb{C}}$-modules $\mathcal{G}_H$ instead of the factors $\mathcal{M}_{(H_\varepsilon)_\varepsilon}/\mathcal{N}_{(H_\varepsilon)_\varepsilon}$.

2.2. **Projection on a subset C.**

Definition 2.9. Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module and C a nonempty subset of $\mathcal{G}$. We say that C is *reachable* from $u \in \mathcal{G}$ if

$$\overline{\inf}_{w\in C}\|u - w\|$$

exists in $\widetilde{\mathbb{R}}$. C is called *edged* if it is reachable from any $u \in \mathcal{G}$.

From the definition it is clear that if C is edged, then $u + C$ is edged as well for all $u \in \mathcal{G}$. Since $\overline{\inf}_{w\in\overline{C}}\|u - w\| = \overline{\inf}_{w\in C}\|u - w\|$ we have that C is edged if and only if $\overline{C}$ is edged.

Theorem 2.10. *Let C be a closed nonempty subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $C + C \subseteq 2C$. If C is reachable from $u \in \mathcal{G}$, then there exists a unique $v \in C$ such that*

$$\|u - v\| = \inf_{w\in C} \|u - w\|.$$

The element v is called the projection of u on C and denoted by $P_C(u)$.

Proof. Note that when $\overline{\inf}_{w\in C}\|u - w\|$ exists in $\widetilde{\mathbb{R}}$ one has that $\overline{\inf}_{w\in C}\|u - w\|^2 = (\overline{\inf}_{w\in C}\|u - w\|)^2$. As the properties of C are translation invariant, we can assume $u = 0$. We set $\overline{\inf}_{w\in C}\|w\|^2 = \delta$ in $\widetilde{\mathbb{R}}$. By definition of close infimum we can extract a sequence w_n in C such that $\|w_n\|^2 \to \delta$. The fact that $C + C \subseteq 2C$ implies that $\frac{w_n+w_m}{2}$ belongs to C for all $n, m \in \mathbb{N}$. So,

$$0 \leq \|w_n - w_m\|^2 = -4\|\frac{w_n + w_m}{2}\|^2 + 2\|w_n\|^2 + 2\|w_m\|^2 \leq -4\delta + 2\|w_n\|^2 + 2\|w_m\|^2. \tag{2.5}$$

From $\|w_n\|^2 \to \delta$ it follows that $(w_n)_n$ is a Cauchy sequence in C and therefore it is convergent in $\mathcal{G}$ to an element v of C. By continuity of the $\widetilde{\mathbb{R}}$-norm we have that $\|v\|^2 = \delta$. Finally, if we assume that there exists another $v' \in C$ such that $\|v'\|^2 = \delta$, the inequality (2.5) is valid for $v - v'$ and proves that $v = v'$ in $\mathcal{G}$. □

Corollary 2.11. *Let C be a closed edged subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $C + C \subseteq 2C$. Then, for all $u \in \mathcal{G}$ there exists a unique $v \in C$ such that*

$$\|u - v\| = \inf_{w \in C} \|u - w\|.$$

The following example shows that the hypothesis of close infimum is necessary in the assumptions of the previous theorem.

Example 2.12. There exists a nonempty closed subset C of $\widetilde{\mathbb{C}}$ with $\lambda C + (1-\lambda)C \subseteq C$ for each $\lambda \in \widetilde{[0,1]} := \{x \in \widetilde{\mathbb{R}} : 0 \le x \le 1\}$ for which $\inf_{c \in C} |c|$ exists, but which is not reachable from $0 \in \widetilde{\mathbb{C}}$.

Proof. Let for each $n \in \mathbb{N}$, $S_n \subseteq (0,1]$ with $e_{S_n} \neq 0$ and $S_n \cap S_m = \emptyset$ if $n \neq m$. Let $\mathcal{T} = \{T \subseteq (0,1] : e_T \neq 0 \text{ and } e_T e_{S_n} = 0, \forall n \in \mathbb{N}\} \cup \{S_n : n \in \mathbb{N}\}$. Let $A = \{e_{T^c} : T \in \mathcal{T}\}$. We show that $\inf A = 0$. Let $\rho \in \widetilde{\mathbb{R}}$, $\rho \le e_{T^c}$ for each $T \in \mathcal{T}$. Suppose that $\rho \not\le 0$. Then there exist $U \subseteq (0,1]$ with $e_U \neq 0$ and $m \in \mathbb{N}$ such that $\rho e_U \ge [(\varepsilon)_\varepsilon]^m e_U$. Then also $[(\varepsilon)_\varepsilon]^m e_U e_{S_n} \le \rho e_U e_{S_n} \le e_{S_n^c} e_U e_{S_n} = 0$, so $e_U e_{S_n} = 0$, $\forall n$. Hence $U \in \mathcal{T}$, and $[(\varepsilon)_\varepsilon]^m e_U \le \rho e_U \le e_{U^c} e_U = 0$, which contradicts $e_U \neq 0$.

Now let $B = \{\lambda_1 a_1 + \cdots + \lambda_m a_m : m \in \mathbb{N}, a_j \in A, \lambda_j \in \widetilde{[0,1]}, \sum_{j=1}^m \lambda_j = 1\}$. Then also $\inf B = 0$ and $\lambda B + (1-\lambda)B \subseteq B$ for each $\lambda \in \widetilde{[0,1]}$. We show that 0 is not a close infimum for B.

Let $\lambda_1 a_1 + \cdots + \lambda_m a_m \in B$. Fix representatives $\lambda_{j,\varepsilon}$ of λ_j and let

$$U_j = \{\varepsilon \in (0,1] : \lambda_{j,\varepsilon} = \max(\lambda_{1,\varepsilon}, \dots, \lambda_{m,\varepsilon})\}.$$

Then $e_{U_j} = \sum_{i=1}^m \lambda_i e_{U_j} \le m\lambda_j e_{U_j}$ for $j = 1, \dots, m$. So $\lambda_1 a_1 + \cdots + \lambda_m a_m \ge \frac{1}{m}(e_{U_1} a_1 + \cdots + e_{U_m} a_m)$ with $\bigcup U_j = (0,1]$. Let $a_j = e_{T_j^c}$, $T_j \in \mathcal{T}$. By the definition of $\mathcal{T}$, there exists $n \in \mathbb{N}$ such that $e_{T_1} e_{S_n} = \cdots = e_{T_m} e_{S_n} = 0$. Then $\lambda_1 a_1 + \cdots + \lambda_m a_m \ge \frac{1}{m}(e_{U_1} e_{T_1^c} + \cdots + e_{U_m} e_{T_m^c}) e_{S_n} \ge \frac{1}{m} e_{S_n}$. Hence $|\lambda_1 a_1 + \cdots + \lambda_m a_m|_{\mathrm{e}} \ge 1$. Consequently, $0 \notin \overline{B}$.

Finally, let $C = \overline{B}$. □

Under the hypotheses of Corollary 2.11 we can define the map P_C as the map which assigns to each $u \in \mathcal{G}$ its projection on C. A careful investigation of the properties of the map P_C requires the following lemma, which is obtained by observing the proof of Theorem 2.10.

Lemma 2.13. *Let C be a closed nonempty subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $C + C \subseteq 2C$, u an element of $\mathcal{G}$ such that C is reachable from u and $(v_n)_n$ a sequence of elements of C. If $\|u - v_n\| \to \inf_{w \in C} \|u - w\| = \|u - P_C(u)\|$ in $\widetilde{\mathbb{R}}$, then $v_n \to P_C(u)$ in $\mathcal{G}$.*

Proposition 2.14. *Let C be a closed edged subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $C + C \subseteq 2C$. The operator P_C has the following properties:*

(i) *$P_C(u) = u$ if and only if $u \in C$;*
(ii) *$P_C(\mathcal{G}) = C$;*

(iii) $P_C^2 = P_C$;
(iv) P_C *is a continuous operator on* $\mathcal{G}$.

Proof. (i) It is obvious that u belongs to C if it coincides with its projection. Conversely, if $u \in C$, then $\|u - P_C(u)\| = \inf_{w\in C} \|u - w\| = 0$, and therefore $u = P_C(u)$. The assertion (ii) is trivial, and from (i) it follows that the operator P_C is idempotent. Let us now prove that P_C is continuous. Since $\mathcal{G}$ is a metric space it is sufficient to prove that P_C is sequentially continuous, i.e., $u_n \to u$ implies $P_C(u_n) \to P_C(u)$. This is guaranteed by Lemma 2.13 if we prove that the sequence $\|u - P_C(u_n)\|$ converges to $\|u - P_C(u)\|$ in $\widetilde{\mathbb{R}}$. The triangle inequality, valid in $\widetilde{\mathbb{R}}$ for $\|\cdot\|$, combined with the fact that $\|u_n - P_C(u_n)\| \le \|u_n - P_C(u)\|$, leads to

$$\|u - P_C(u_n)\| \le \|u - u_n\| + \|u_n - P_C(u_n)\| \le 2\|u - u_n\| + \|u - P_C(u)\|.$$

It follows that

$$0 \le \|u - P_C(u_n)\| - \|u - P_C(u)\| \le 2\|u - u_n\|.$$

Since $u_n \to u$, we conclude that $\|u - P_C(u_n)\| \to \|u - P_C(u)\|$ in $\widetilde{\mathbb{R}}$. □

Proposition 2.15. *Let* C *be a closed nonempty subset of the Hilbert* $\widetilde{\mathbb{C}}$*-module* $\mathcal{G}$ *such that* $\lambda C + (1-\lambda)C \subseteq C$ *for all real generalized numbers* $\lambda \in \{[(\varepsilon^q)_\varepsilon]\}_{q\in\mathbb{N}} \cup \{\frac{1}{2}\}$. C *is reachable from* $u \in \mathcal{G}$ *if and only if there exists* $v \in C$ *such that*

$$\Re\,(u - v|w - v) \le 0 \tag{2.6}$$

for all $w \in C$. *In this case* $v = P_C(u)$.

Proof. We begin by assuming that C is reachable from u. Then, $P_C(u) \in C$ and $\|u - P_C(u)\|^2 = \inf_{w\in C} \|u - w\|^2$. Let $w \in C$. By the hypotheses on C we know that $(1 - [(\varepsilon^q)_\varepsilon])P_C(u) + [(\varepsilon^q)_\varepsilon]w$ belongs to C. Hence,

$$\begin{aligned}\|u - P_C(u)\|^2 &\le \|u - P_C(u) - [(\varepsilon^q)_\varepsilon](w - P_C(u))\|^2\\ &\le \|u - P_C(u)\|^2 - 2[(\varepsilon^q)_\varepsilon]\Re\,(u - P_C(u)|w - P_C(u)) + [(\varepsilon^q)_\varepsilon]^2\|w - P_C(u)\|^2.\end{aligned}$$

By the previous inequality and the invertibility of $[(\varepsilon^q)_\varepsilon]$ we obtain

$$\Re\,(u - P_C(u)|w - P_C(u)) \le \frac{[(\varepsilon^q)_\varepsilon]}{2}\|w - P_C(u)\|^2.$$

Letting q tend to ∞ we conclude that $\Re\,(u - P_C(u)|w - P_C(u)) \le 0$.

Now assume that $v \in C$ and $\Re\,(u - v|w - v) \le 0$ for all $w \in C$. By the properties of a scalar product we can write

$$\|u - v\|^2 - \|u - w\|^2 = \|u - v\|^2 - \|u - v\|^2 + 2\Re\,(u - v|w - v) - \|w - v\|^2 \le 0$$

for all $w \in C$. This means that $\|u - v\|^2 \le \inf_{w\in C} \|u - w\|^2$, and since $v \in C$ we conclude that $\|u - v\|^2 = \min_{w\in C} \|u - w\|^2$. Thus, $v = P_C(u)$. □

In Proposition 3.11 we will prove that under the assumptions of the previous theorem, in fact $\lambda C + (1-\lambda)C \subseteq C$ for all $\lambda \in \widetilde{[0,1]} = \{x \in \widetilde{\mathbb{R}} : 0 \le x \le 1\}$.

Corollary 2.16. *Let* M *be a closed* $\widetilde{\mathbb{C}}$*-submodule of* $\mathcal{G}$. *Then, defining the* $\widetilde{\mathbb{C}}$*-submodule* $M^\perp := \{w \in \mathcal{G} : \forall v \in M\ \ (w|v) = 0\}$, *we have that* M *is reachable from* $u \in \mathcal{G}$ *if and only if there exists* $v \in M$ *such that* $u - v \in M^\perp$ *if and only if* $u \in \mathcal{G}$ *can be uniquely written in the form* $u = u_1 + u_2$, *where* $u_1 \in M$ *and* $u_2 \in M^\perp$.

Proof. Since M is a closed $\widetilde{\mathbb{C}}$-submodule, (2.6) is equivalent to $(u-v|w)=0$ for all $w \in M$. Indeed, $\Re\,(u-v|w)=0$, and from $(u-v|-iw)=0$ we have $\Im\,(u-v|w)=0$. Hence, $u=v+(u-v)$, where $v \in M$ and $u-v \in M^{\perp}$ are uniquely determined by the scalar product on $\mathcal{G}$ since $M \cap M^{\perp}=\{0\}$. □

Corollary 2.17. *Let M be a $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$. Then,*

(i) *M is closed and edged if and only if $\mathcal{G}=M \oplus M^{\perp}$, i.e., $\mathcal{G}=M+M^{\perp}$ and $M \cap M^{\perp}=\{0\}$.*

If M is closed and edged, the following holds:

(ii) *if $M^{\perp}=\{0\}$, then $M=\mathcal{G}$;*
(iii) *$M^{\perp\perp}=M$;*
(iv) *the projection P_M is a $\widetilde{\mathbb{C}}$-linear operator on $\mathcal{G}$;*
(v) *$(P_M(u)|u) \geq 0$ for all $u \in \mathcal{G}$;*
(vi) *$M^{\perp}$ is closed and edged, and $P_{M^{\perp}}(u)=u-P_M(u)$.*

Proof. (i) $\Rightarrow$: is clear from Corollary 2.16. Let us assume that $\mathcal{G}=M+M^{\perp}$ and that $u \in \overline{M}$. It follows that $u=u_1+u_2$ with $u_1 \in M$ and $u_2 \in M^{\perp}$. So, $u-u_1=u_2 \in \overline{M} \cap \overline{M}^{\perp}=\{0\}$. Hence, $u \in M$ and M is closed. From Corollary 2.16 we have that M is edged.

(ii) Now assume that $M^{\perp}=\{0\}$. Then M has to coincide with $\mathcal{G}$. This follows from the fact that any $u \in \mathcal{G} \setminus M$ can be written as u_1+u_2, where $u_1 \in M$ and $u_2 \neq 0$ belongs to $M^{\perp}$.

(iii) By construction $M \subseteq M^{\perp\perp}$. From the first assertion of this proposition we know that any $u \in M^{\perp\perp}$ can be written as u_1+u_2, where $u_1 \in M \subseteq M^{\perp\perp}$ and $u_2 \in M^{\perp}$. Hence, $u_2=u-u_1 \in M^{\perp\perp}$, and since $u_2 \in M^{\perp}$ we obtain that $(u_2|u_2)=0$. It follows that $u \in M$.

(iv) The $\widetilde{\mathbb{C}}$-linearity of the operator P_M is due to the uniqueness of the decomposition $u_1+u_2=P_M(u_1)+P_M(u_2)+(u_1+u_2-P_M(u_1)-P_M(u_2))$, where $P_M(u_1)+P_M(u_2) \in M$ and $u_1+u_2-P_M(u_1)-P_M(u_2) \in M^{\perp}$. Analogously, for all $\lambda \in \widetilde{\mathbb{C}}$ one has that $\lambda P_M(u) \in M$, $\lambda(u-P_M(u)) \in M^{\perp}$ and $\lambda u=\lambda P_M(u)+\lambda(u-P_M(u))$.

(v) We write u as the sum of $u-P_M(u) \in M^{\perp}$ and $P_M(u) \in M$. It follows that

$$(P_M(u)|u)=(P_M(u)|u-P_M(u))+(P_M(u)|P_M(u))=\|P_M(u)\|^2.$$

(vi) It is clear that if M is a closed $\widetilde{\mathbb{C}}$-submodule, then $M^{\perp}$ is a closed $\widetilde{\mathbb{C}}$-submodule, too. We want to prove that $M^{\perp}$ is edged, i.e., it is reachable from every element of $\mathcal{G}$. By Corollary 2.16 we know that every element u of $\mathcal{G}$ can be uniquely wriiten as $P_M(u)+(u-P_M(u))$, where $u-P_M(u) \in M^{\perp}$. By assertion (iii) we have that $P_M(u) \in M=M^{\perp\perp}$. So, again by Corollary 2.16 we conclude that $M^{\perp}$ is reachable from u. □

Remark 2.18. In [25] it is shown that for $\mathcal{G}=\widetilde{\mathbb{C}}$ and M a maximal ideal (in particular a closed submodule) of $\widetilde{\mathbb{C}}$, $M^{\perp}=\{0\}$ and thus $M^{\perp\perp}=\{0\}$. Hence, the condition that M is edged cannot be dropped in the statements (ii) and (iii) of the previous corollary.

The proof of Corollary 2.20 makes use of the following lemma.

Lemma 2.19. *Let $a,b,c \in \widetilde{\mathbb{R}}$ with $a,b,c \geq 0$. If $b \leq a$ and $ab \leq ac$, then $b \leq c$.*

Proof. Fix a representative $(a_\varepsilon)_\varepsilon$ of a. For each $n \in \mathbb{N}$, let $S_n = \{\varepsilon \in (0,1] : |a_\varepsilon| \ge \varepsilon^n\}$. Since a is invertible with respect to S_n, we have that $be_{S_n} \le ce_{S_n} \le c$. Further, $0 \le b - be_{S_n} = be_{S_n^c} \le ae_{S_n^c} \to 0$, so $b = \lim_n be_{S_n} \le c$. □

Note that Lemma 2.19 allows us to deduce for positive real generalized numbers a and c that $a^2 \le ac$ implies $a \le c$ without involving any invertibility assumption on a.

Corollary 2.20. *Let C be a closed edged subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $\lambda C + (1-\lambda)C \subseteq C$ for all real generalized numbers $\lambda \in \widetilde{[0,1]}$. Then,*

$$\|P_C(u_1) - P_C(u_2)\| \le \|u_1 - u_2\|$$

for all $u_1, u_2 \in \mathcal{G}$.

Proof. By Proposition 2.15, the inequalities $\Re\,(u_1 - P_C(u_1)|P_C(u_2) - P_C(u_1)) \le 0$ and $\Re\,(u_2 - P_C(u_2)|P_C(u_1) - P_C(u_2)) \le 0$ hold for all $u_1, u_2 \in \mathcal{G}$. Thus,

$$\begin{aligned}
&-\Re\,(u_1 - P_C(u_1)|P_C(u_1) - P_C(u_2)) + \Re\,(u_2 - P_C(u_2)|P_C(u_1) - P_C(u_2))\\
&\qquad = \Re\,(u_2 - u_1 + P_C(u_1) - P_C(u_2)|P_C(u_1) - P_C(u_2))\\
&\qquad\quad = \Re\,(u_2 - u_1|P_C(u_1) - P_C(u_2)) + \|P_C(u_1) - P_C(u_2)\|^2 \le 0.
\end{aligned}$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\|P_C(u_1) - P_C(u_2)\|^2 &\le \Re\,(u_1 - u_2|P_C(u_1) - P_C(u_2))\\
&\le \|u_1 - u_2\|\|P_C(u_1) - P_C(u_2)\|.
\end{aligned}$$

Lemma 2.19 allows us to deduce that $\|P_C(u_1) - P_C(u_2)\| \le \|u_1 - u_2\|$. □

When we work on the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}_H$ and the set $C \subseteq \mathcal{G}_H$ is internal, the projection operator P_C and the set $C^\perp$ have the following expected properties.

Proposition 2.21.

(i) *Let H be a Hilbert space, $(C_\varepsilon)_\varepsilon$ a net of nonempty convex subsets of H and $C := [(C_\varepsilon)_\varepsilon]$. If $C \neq \emptyset$, then it is closed and edged and $P_C(u) = [(P_{\overline{C}_\varepsilon}(u_\varepsilon))_\varepsilon]$ for all $u \in \mathcal{G}_H$.*

(ii) *In particular, if $(C_\varepsilon)_\varepsilon$ is a net of closed subspaces of H, then $C^\perp = [({C_\varepsilon}^\perp)_\varepsilon]$.*

Proof. (i) Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_H$. Working at the level of representatives we have that $\|u_\varepsilon - P_{\overline{C}_\varepsilon}(u_\varepsilon)\| = \inf_{w \in \overline{C}_\varepsilon} \|u_\varepsilon - w\|$. Let v be an arbitrary element of C. Then there exists a representative $(v_\varepsilon)_\varepsilon$ such that $v_\varepsilon \in C_\varepsilon$ for all ε and $\|u_\varepsilon - P_{\overline{C}_\varepsilon}(u_\varepsilon)\| \le \|u_\varepsilon - v_\varepsilon\|$. Since $\|P_{\overline{C}_\varepsilon}(u_\varepsilon)\| \le \|P_{\overline{C}_\varepsilon}(u_\varepsilon) - u_\varepsilon\| + \|u_\varepsilon\| \le \|u_\varepsilon - v_\varepsilon\| + \|u_\varepsilon\|$, the net $(\|P_{\overline{C}_\varepsilon}(u_\varepsilon)\|)_\varepsilon$ is moderate . It follows that $\|u - [(P_{\overline{C}_\varepsilon}(u_\varepsilon))_\varepsilon]\| \le \|u - v\|$ for all $v \in C$. Since, as proved in [21], the set C is closed and edged, by Corollary 2.11 we have that $[(P_{\overline{C}_\varepsilon}(u_\varepsilon))_\varepsilon]$ coincides with $P_C(u)$.

(ii) The inclusion $[({C_\varepsilon}^\perp)_\varepsilon] \subseteq C^\perp$ is clear. If $u \in C^\perp$, then $P_C(u) = 0$, and from the first assertion of this proposition the net $(\|P_{C_\varepsilon}(u_\varepsilon)\|)_\varepsilon$ is negligible. So, $(u_\varepsilon - P_{C_\varepsilon}(u_\varepsilon))_\varepsilon$ is another representative of u and $u_\varepsilon - P_{C_\varepsilon}(u_\varepsilon)$ belongs to $C_\varepsilon^\perp$ for each ε. □

We conclude this section with a version of the Hahn-Banach theorem for operators acting on Hilbert $\widetilde{\mathbb{C}}$-modules.

Theorem 2.22. *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module, M a closed and edged $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$ and $\mathcal{H}$ a topological $\widetilde{\mathbb{C}}$-module. Let $f : M \to \mathcal{H}$ be a continuous $\widetilde{\mathbb{C}}$-linear map. Then, f can be extended to a continuous $\widetilde{\mathbb{C}}$-linear map on $\mathcal{G}$.*

Proof. Take the projection operator P_M. From Corollary 2.17 we know that $P_M : \mathcal{G} \to M$ is $\widetilde{\mathbb{C}}$-linear and continuous and that $P_M(u) = u$ when $u \in M$. Thus, $f \circ P_M : \mathcal{G} \to \mathcal{H}$ is a continuous $\widetilde{\mathbb{C}}$-linear extension of f. □

Since there exists a (without loss of generality, closed) submodule M of $\widetilde{\mathbb{C}}$ and a continuous $\widetilde{\mathbb{C}}$-linear functional $T : M \to \widetilde{\mathbb{C}}$ which cannot be extended to the whole of $\widetilde{\mathbb{C}}$ [25], we see that the condition that M is edged cannot be dropped in the previous theorem.

3. Edged submodules

In this section, we take a closer look at edged submodules of a Hilbert $\widetilde{\mathbb{C}}$-module (cf. Definition 2.9). In the case of finitely generated submodules, edged submodules can be characterized by a topological condition (Theorem 3.16). Some of the results hold for more general $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{K}}$-modules (here $\widetilde{\mathbb{K}}$ denotes either $\widetilde{\mathbb{R}}$ or $\widetilde{\mathbb{C}}$) fulfilling the following *normalization property*.

Definition 3.1. An $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{K}}$-module $\mathcal{G}$ fulfills the normalization property if for all $u \in \mathcal{G}$ there exists v in $\mathcal{G}$ such that $v\|u\| = u$.

Proposition 3.2. *Let $\mathcal{G}$ be an $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{K}}$-module. Then $\mathcal{G}$ has the normalization property iff for each $u \in \mathcal{G}$ and $\lambda \in \widetilde{\mathbb{K}}$, the following holds: if $\|u\| \le C\,|\lambda|$ for some $C \in \widetilde{\mathbb{R}}$, then there exists $v \in \mathcal{G}$ such that $u = \lambda v$.*

Proof. $\Rightarrow$: By absolute convexity of ideals in $\widetilde{\mathbb{K}}$, there exists $\mu \in \widetilde{\mathbb{K}}$ such that $\|u\| = \mu\lambda$. By the normalization property, there exists $v \in \mathcal{G}$ such that $\|u\|v = u$. Hence $u = \lambda(\mu v)$.

$\Leftarrow$: choose $\lambda = \|u\|$. □

We observe that the Colombeau space $\mathcal{G}_E$ of generalized functions based on the normed space E fulfills the normalization property.

Proposition 3.3. *Let E be a normed space. The $\widetilde{\mathbb{K}}$-module $\mathcal{G}_E$ fulfills the normalization property.*

Proof. Let $u \in \mathcal{G}_E$ with representative $(u_\varepsilon)_\varepsilon$. We define v_ε as $u_\varepsilon/\|u_\varepsilon\|$ when $\|u_\varepsilon\| \neq 0$ and 0 otherwise. The net $(v_\varepsilon)_\varepsilon$ is clearly moderate, and $v_\varepsilon\|u_\varepsilon\| = u_\varepsilon$ for all ε. This defines an element $v \in \mathcal{G}_E$ such that $v\|u\| = u$. □

Note that Definition 2.9 can clearly be stated in the more general context of $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{K}}$-modules. We recall that a Banach $\widetilde{\mathbb{K}}$-module is a complete ultra-pseudo-normed $\widetilde{\mathbb{K}}$-module [7, 9].

Proposition 3.4. *Let M be a closed submodule of a Banach $\widetilde{\mathbb{K}}$-module $\mathcal{G}$ and let $\mathcal{G}/M$ be endowed with the usual quotient topology.*

(i) *$\mathcal{G}/M$ is a Banach $\widetilde{\mathbb{K}}$-module.*
(ii) *If $\mathcal{G}$ is $\widetilde{\mathbb{R}}$-normed and M is edged, then $\mathcal{G}/M$ is $\widetilde{\mathbb{R}}$-normed.*
(iii) *If $\mathcal{G}$ is a Hilbert $\widetilde{\mathbb{C}}$-module and M is edged, then $\mathcal{G}/M$ is a Hilbert $\widetilde{\mathbb{C}}$-module.*

(iv) *If $\mathcal{G}$ is a Hilbert $\widetilde{\mathbb{C}}$-module satisfying the normalization property and M is edged, then M has the normalization property.*

Proof. (i) By [7, Example 1.12], the relative topology on $\mathcal{G}/M$ is generated by one ultra-pseudo-seminorm. It is easy to check that this ultra-pseudo-seminorm is an ultra-pseudo-norm if M is closed. By Proposition 4.25 in [9] we have that $\mathcal{G}/M$ is complete.

(ii) For $u \in \mathcal{G}$, let $\bar{u} := u + M \in \mathcal{G}/M$. We define $\|.\|\colon \mathcal{G}/M \to \widetilde{\mathbb{R}}$: $\|\bar{u}\| = \inf_{w\in M} \|u - w\|$. As M is edged, the infimum exists. It is easy to see that $\|\bar{u}\|$ does not depend on the representative $u \in \mathcal{G}$, that $\|\bar{u}\| \geq 0$ and $\|\bar{0}\| = 0$. If $\|\bar{u}\| = \inf_{w\in M} \|u - w\| = 0$, then there exists a sequence $(w_n)_n$ with $w_n \in M$ and $u = \lim_n w_n$. Hence $u \in \overline{M} = M$ and $\bar{u} = 0$. Let $u_1, u_2 \in \mathcal{G}$ and $w_1, w_2 \in M$. Then

$$\begin{aligned}\|\bar{u}_1 + \bar{u}_2\| &= \inf_{w\in M} \|u_1 + u_2 - w\| \\ &\leq \|u_1 + u_2 - (w_1 + w_2)\| \leq \|u_1 - w_1\| + \|u_2 - w_2\|.\end{aligned}$$

Taking the infimum over $w_1 \in M$ and $w_2 \in M$, we obtain $\|\bar{u}_1 + \bar{u}_2\| \leq \|\bar{u}_1\| + \|\bar{u}_2\|$. Now let $u \in \mathcal{G}$ and $\lambda \in \widetilde{\mathbb{K}}$. Then

$$\begin{aligned}\|\lambda\bar{u}\| &= \inf_{w\in M} \|\lambda u - w\| \leq \inf_{w\in M} \|\lambda u - \lambda w\| \\ &= \inf_{w\in M} |\lambda|\, \|u - w\| = |\lambda| \inf_{w\in M} \|u - w\| = |\lambda|\, \|\bar{u}\|.\end{aligned}$$

If $\lambda = 0$, the converse inequality trivially holds. If $\lambda \neq 0$, let $S \subseteq (0, 1]$ with $e_S \neq 0$ such that λ is invertible w.r.t. S, say $\lambda\mu = e_S$, and let $w \in M$. Then

$$\begin{aligned}\|\lambda e_S u - w\| &\geq \|\lambda e_S u - w\| e_S = \|\lambda e_S u - \lambda e_S(\mu w)\| \\ &\geq \inf_{w\in M} \|\lambda e_S u - \lambda e_S w\| = |\lambda|\, e_S \|\bar{u}\|.\end{aligned}$$

Fix a representative $(\lambda_\varepsilon)_\varepsilon$ of λ and let $S_n = \{\varepsilon \in (0, 1] : |\lambda_\varepsilon| \geq \varepsilon^n\}$ for each $n \in \mathbb{N}$. Then $e_{S_n} \neq 0$ and λ is invertible w.r.t. S_n for sufficiently large n. As $\lambda = \lim_n \lambda e_{S_n}$, by the continuity of the $\widetilde{\mathbb{R}}$-norm,

$$\|\lambda u - w\| = \lim_n \|\lambda e_{S_n} u - w\| \geq \lim_n |\lambda|\, e_{S_n} \|\overline{u}\| = |\lambda|\, \|\overline{u}\|.$$

Taking the infimum over $w \in M$, we obtain $\|\lambda\overline{u}\| \geq |\lambda|\, \|\overline{u}\|$. So $\|.\|$ is an $\widetilde{\mathbb{R}}$-norm on $\mathcal{G}/M$. By the continuity of the sharp norm $|.|_{\mathrm{e}}$ on $\widetilde{\mathbb{R}}$ and the fact that $|.|_{\mathrm{e}}$ is increasing on $\{x \in \widetilde{\mathbb{R}} : x \geq 0\}$, the corresponding ultra-pseudo-norm

$$\mathcal{P}(\overline{u}) = |\|\overline{u}\||_{\mathrm{e}} = \left|\inf_{w\in M} \|u - w\|\right|_{\mathrm{e}} = \inf_{w\in M} |\|u - w\||_{\mathrm{e}} = \inf_{w\in M} \mathcal{P}(u - w)$$

is the usual quotient ultra-pseudo-norm.

(iii) The map $f\colon \mathcal{G}/M \to M^\perp\colon u + M \mapsto P_{M^\perp}(u)$ is well defined, since for $v \in \mathcal{G}$ with $u + M = v + M$, $P_{M^\perp}(u) - P_{M^\perp}(v) = P_{M^\perp}(u - v) = 0$. Further, f is $\widetilde{\mathbb{C}}$-linear and surjective and $\|u + M\| = \inf_{w\in M} \|u - w\| = \|u - P_M(u)\| = \|P_{M^\perp}(u)\|$, so f is an algebraic and isometric isomorphism. Hence $\mathcal{G}/M$ is a Hilbert $\widetilde{\mathbb{C}}$-module for the scalar product $(u + M|v + M)_{\mathcal{G}/M} := (P_{M^\perp}(u)|P_{M^\perp}(v))_{\mathcal{G}}$.

(iv) Let $u \in M$. If there exists $v \in \mathcal{G}$ such that $\|u\|v = u$, then $P_M(v) \in M$, and by the linearity of the projection operator, $\|u\|P_M(v) = P_M(\|u\|v) = P_M(u) = u$. $\square$

3.1. **Cyclic submodules.**

Definition 3.5. Let $\mathcal{G}$, $\mathcal{H}$ be $\widetilde{\mathbb{K}}$-modules with $\widetilde{\mathbb{R}}$-norm $\|.\|$. A map ϕ: $\mathcal{G} \to \mathcal{H}$ is an isometry iff $\|\phi(u) - \phi(v)\| = \|u - v\|$ for each $u, v \in \mathcal{G}$.

The submodules considered in the sequel are always $\widetilde{\mathbb{K}}$-submodules. We recall that a $\widetilde{\mathbb{K}}$-module M is called cyclic iff it is generated by one element, i.e., there exists $u \in M$ such that $M = u\widetilde{\mathbb{K}}$. An ideal I of $\widetilde{\mathbb{K}}$ (for short $I \trianglelefteq \widetilde{\mathbb{K}}$) which is generated by one element is said to be principal. Before proving Proposition 3.8 we collect some results concerning the ideals of $\widetilde{\mathbb{K}}$ which will be used later. Detailed proofs can be found in [25].

Proposition 3.6. *Let $I \trianglelefteq \widetilde{\mathbb{K}}$.*

(i) *I is absolutely order convex, i.e., if $x \in I$, $y \in \widetilde{\mathbb{K}}$ and $|y| \le |x|$, then $y \in I$.*
(ii) *If $x \in \overline{I}$ is invertible w.r.t. $S \subseteq (0,1]$, then $e_S \in I$.*
(iii) *A principal ideal I of $\widetilde{\mathbb{K}}$ is closed if and only if there exists $S \subseteq (0,1]$ such that $I = e_S\widetilde{\mathbb{K}}$.*

Theorem 3.7. *For an ideal I of $\widetilde{\mathbb{K}}$, the following statements are equivalent:*

(i) *I is internal.*
(ii) *I is closed and edged.*
(ii′) *I is edged.*
(iii) *I is a direct summand of $\widetilde{\mathbb{K}}$, i.e., there exists an ideal J of $\widetilde{\mathbb{K}}$ such that $I + J = \widetilde{\mathbb{K}}$ and $I \cap J = \{0\}$.*
(iv) *$(\exists S \subseteq (0,1])(I = e_S\widetilde{\mathbb{K}})$.*

Proof. $(i) \Rightarrow (ii)$: holds for any nonempty internal set of $\widetilde{\mathbb{K}}$ [21].

$(ii) \Rightarrow (iii)$: by Corollary 2.17, $I + I^\perp = \widetilde{\mathbb{K}}$ and $I \cap I^\perp = \{0\}$.

$(iii) \Rightarrow (iv)$: by hypothesis, $1 = a + b$ with $a \in I$ and $b \in J$. As $ab \in I \cap J$, $ab = 0$. Let $x \in I$. Then $xb \in I \cap J$, so $xb = 0$ and $x = x(a + b) = xa$. Therefore, $I = a\widetilde{\mathbb{K}}$. As $a = a(a + b) = a^2$, a is idempotent; hence $a = e_S$ for some $S \subseteq (0,1]$.

$(iv) \Rightarrow (i)$: let $I_\varepsilon = \mathbb{K}$ if $\varepsilon \in S$ and $I_\varepsilon = \{0\}$, otherwise. Then $I = [(I_\varepsilon)_\varepsilon]$.

$(ii) \Leftrightarrow (ii')$: let I be edged. As $\overline{I}$ is closed and edged, the previous equivalences show that $\overline{I} = e_S\widetilde{\mathbb{K}}$ for some $S \subseteq (0,1]$. But if $e_S \in \overline{I}$, then $e_S \in I$ by Proposition 3.6, so $I = \overline{I}$. □

Proposition 3.8. *Let $M = u\widetilde{\mathbb{K}}$ be a cyclic submodule of an $\widetilde{\mathbb{R}}$-normed $\widetilde{\mathbb{K}}$-module $\mathcal{G}$.*

(i) *M is isometrically isomorphic with the ideal $\|u\|\widetilde{\mathbb{K}} \trianglelefteq \widetilde{\mathbb{K}}$.*
(ii) *$\overline{M}$ is isometrically isomorphic with an ideal $I \trianglelefteq \widetilde{\mathbb{K}}$ and $\|u\|\widetilde{\mathbb{K}} \subseteq I \subseteq \overline{\|u\|\widetilde{\mathbb{K}}}$. If $\mathcal{G}$ is a Banach $\widetilde{\mathbb{K}}$-module, then $I = \overline{\|u\|\widetilde{\mathbb{K}}}$.*
(iii) *If $v \in \overline{M}$ and $\|v\|$ is invertible w.r.t. S, then $ve_S \in M$.*
(iv) *If $v \in \overline{M}$ and $\|v\| \le c\|u\|$, for some $c \in \widetilde{\mathbb{R}}$, then $v \in M$.*
(v) *If $v \in M$ and $\|u\| \le c\|v\|$, for some $c \in \widetilde{\mathbb{R}}$, then $M = v\widetilde{\mathbb{K}}$.*
(vi) *If there exists $w \in \mathcal{G}$ and $S \subseteq (0,1]$ such that $M = w\widetilde{\mathbb{K}}$ and $\|w\| = e_S$ (or equivalently, $\|u\|$ is invertible w.r.t. S and zero w.r.t. S^c), then M is closed.*
(vii) *If $\mathcal{G}$ is a Banach $\widetilde{\mathbb{K}}$-module, then M is closed iff there exists $w \in \mathcal{G}$ and $S \subseteq (0,1]$ such that $M = w\widetilde{\mathbb{K}}$ and $\|w\| = e_S$.*

(viii) *If M is closed, then any edged submodule N of M is closed and cyclic.*
(ix) *If $\mathcal{G}$ has the normalization property, then M is contained in a closed cyclic submodule of $\mathcal{G}$.*
(x) *If $\mathcal{G}$ has the normalization property and M is edged, then M is closed.*
(xi) *If $\mathcal{G}$ is a Hilbert $\widetilde{\mathbb{K}}$-module, $v \in \mathcal{G}$ and $\|v\| \le c\|u\|$, for some $c \in \widetilde{\mathbb{R}}$, then there exists $P_M(v) \in M$, which is both the unique element of M such that $\|v - P_M(v)\| = d(v, M)$ and the unique element of M such that $(v|u) = (P_M(v)|u)$.*
(xii) *If $\mathcal{G}$ is a Hilbert $\widetilde{\mathbb{K}}$-module and M is closed, then M is edged. If u is a generator of M with idempotent norm, then for any $v \in \mathcal{G}$, $P_M(v) = (v|u)\, u$.*
(xiii) *If $\mathcal{G}$ is a Hilbert $\widetilde{\mathbb{K}}$-module with the normalization property, then $M^{\perp\perp} = \overline{M}$.*

Proof. (*i*) Define ϕ: $M \to \widetilde{\mathbb{K}}$: $\phi(\lambda u) = \lambda\|u\|$ ($\lambda \in \widetilde{\mathbb{K}}$). Then the equality $\|\lambda u - \mu u\| = |\phi(\lambda u) - \phi(\mu u)|$ shows that ϕ is well defined and isometric (hence also injective). It is easy to check that ϕ is $\widetilde{\mathbb{K}}$-linear and $\phi(M) = \|u\|\widetilde{\mathbb{K}}$.

(*ii*) We extend ϕ: $M \to \widetilde{\mathbb{K}}$ to a map $\overline{M} \to \widetilde{\mathbb{K}}$ by defining $\phi(\lim_n \lambda_n u) := \lim_n \phi(\lambda_n u)$. Because $(\lambda_n u)_n$ is a Cauchy-sequence, $(\phi(\lambda_n u))_n$ is also a Cauchy-sequence in $\widetilde{\mathbb{K}}$, and hence convergent in $\widetilde{\mathbb{K}}$. To see that ϕ is well defined, let $\lim_n \lambda_n u = \lim_n \mu_n u$. Then also the interlaced sequence $(\lambda_1 u, \mu_1 u, \dots, \lambda_n u, \mu_n u, \dots)$ is a Cauchy-sequence. Hence also $(\phi(\lambda_1 u), \phi(\mu_1 u), \dots, \phi(\lambda_n u), \phi(\mu_n u), \dots)$ is convergent to $\lim_n \phi(\lambda_n u) = \lim_n \phi(\mu_n u)$. It is easy to check that also the extended ϕ is linear and isometric and that $\phi(\overline{M})$ is an ideal of $\widetilde{\mathbb{K}}$ such that $\|u\|\widetilde{\mathbb{K}} \subseteq \phi(\overline{M}) \subseteq \overline{\|u\|\widetilde{\mathbb{K}}}$. If $\mathcal{G}$ is complete, we find that the image under ϕ^{-1} of any convergent sequence in $\|u\|\widetilde{\mathbb{K}}$ (say to $\lambda \in \overline{\|u\|\widetilde{\mathbb{K}}}$) is a Cauchy sequence and hence convergent to an element $v \in \overline{M}$. By definition of the extended map ϕ, we have that $\phi(v) = \lambda$ and therefore $\phi(\overline{M}) = \overline{\|u\|\widetilde{\mathbb{K}}}$.

(*iii*) As ϕ is an isometry, $|\phi(v)|$ is invertible w.r.t. S. As $\phi(v) \in \overline{\|u\|\widetilde{\mathbb{K}}}$, by Proposition 3.6(*ii*), $e_S \in \|u\|\widetilde{\mathbb{K}}$. Hence also $\phi(ve_S) = \phi(v)e_S \in \phi(M)$. So $ve_S \in M$ by the injectivity of ϕ.

(*iv*) As ϕ is an isometry, $|\phi(v)| = \|v\| \le c\|u\|$. By absolute order convexity of ideals in $\widetilde{\mathbb{K}}$, $\phi(v) \in \|u\|\widetilde{\mathbb{K}} = \phi(M)$. So $v \in M$ by the injectivity of ϕ.

(*v*) As $\|u\| \le c\|v\| = c\,|\phi(v)|$, by absolute order convexity of ideals in $\widetilde{\mathbb{K}}$, $\|u\| = \mu\phi(v)$ for some $\mu \in \widetilde{\mathbb{K}}$. So $\phi(u) = \|u\| = \phi(\mu v)$, and $u \in v\widetilde{\mathbb{K}}$ by the injectivity of ϕ. It follows that $M = v\widetilde{\mathbb{K}}$.

(*vi*) Let us assume that $\|u\|$ is invertible w.r.t. S and zero w.r.t. S^c. Then, there exists $\lambda \in \widetilde{\mathbb{R}}$ such that $\lambda\|u\| = e_S = \|\lambda u\|$ and $\|u\|e_{S^c} = 0$. It follows that $ue_{S^c} = 0$ or equivalently $u = ue_S$. Hence, $u\widetilde{\mathbb{K}} = \lambda u\widetilde{\mathbb{K}}$, and we can choose $w = \lambda u$. Now, $\phi(M) = \|w\|\widetilde{\mathbb{K}} = e_S\widetilde{\mathbb{K}}$ is closed in $\widetilde{\mathbb{K}}$, and hence complete, so M is also complete, and hence closed.

(*vii*) Let M be closed. As a closed submodule of a Banach $\widetilde{\mathbb{K}}$-module, M is complete. By part (*i*), $\|u\|\widetilde{\mathbb{K}}$ is also a complete, hence closed, principal ideal of $\widetilde{\mathbb{K}}$. By Proposition 3.6(*iii*) this implies that $\|u\|\widetilde{\mathbb{K}} = e_S\widetilde{\mathbb{K}}$ for some $S \subseteq (0, 1]$.

(*viii*) By part (*vii*), we may assume that $\|u\| = e_S$ for some $S \subseteq (0, 1]$. If N is edged, then it is reachable from any element of M. By the isometry, $\phi(N)$ is also reachable from every element of $\phi(M) = e_S\widetilde{\mathbb{K}}$. Hence, for each $\lambda \in \phi(N) \subseteq e_S\widetilde{\mathbb{K}}$

and $\mu \in \widetilde{\mathbb{K}}$,

$$|\mu - \lambda| = |\mu - \lambda|\, e_{S^c} + |\mu - \lambda|\, e_S = |\mu|\, e_{S^c} + |\mu e_S - \lambda|\,,$$

so $\phi(N)$ is also reachable from μ. This implies that $\phi(N)$ is an edged ideal of $\widetilde{\mathbb{K}}$ and by Theorem 3.7 that $\phi(N)$ is closed (hence complete) and principal. So N is closed and cyclic.

(ix) Consider $v \in \mathcal{G}$ with $\|u\|v = u$. Then $\|u\|\,\|v\| = \|u\|$, so $\|u\|(1-\|v\|) = 0$. By a characterization of zero divisors in $\widetilde{\mathbb{R}}$, there exists $S \subseteq (0,1]$ such that $\|u\|e_{S^c} = 0$ and $\|v\|e_S = e_S$. Let $w = ve_S$; then $\|w\| = e_S$. Hence the cyclic submodule $w\widetilde{\mathbb{K}}$ is closed by part (vi). Further, $u = ue_S = \|u\|w$, so $M = u\widetilde{\mathbb{K}} \subseteq w\widetilde{\mathbb{K}}$.

(x) By parts $(viii)$ and (ix).

(xi) By the Cauchy-Schwarz inequality (2.1), $|(v|u)| \le \|u\|\,\|v\| \le c\|u\|^2$, so by absolute order convexity of ideals in $\widetilde{\mathbb{K}}$, there exists $\lambda \in \widetilde{\mathbb{K}}$ such that $(v|u) = \lambda\|u\|^2$. We show that $P_M(v) = \lambda u$. First, for $\mu \in \widetilde{\mathbb{K}}$, $(\mu u|u) = (v|u)$ iff $(\mu - \lambda)\|u\|^2 = 0$. It follows that $|\mu - \lambda|^2\,\|u\|^2 = \|(\mu - \lambda)u\|^2 = 0$ as well, so $\mu u = \lambda u$, and $P_M(v)$ is the unique element in M such that $(v|u) = (P_M(v)|u)$. From this equality, it follows that $\|v - P_M(v) + \mu u\|^2 = \|v - P_M(v)\|^2 + \|\mu u\|^2$, which is only minimal if $\mu u = 0$.

(xii) By part (vii), we can suppose that $\|u\| = e_S$ for some $S \subseteq (0,1]$. In particular, $\|e_{S^c}u\| = 0$, so $u = e_S u$. Let $v \in \mathcal{G}$ and let $p = (v|u)\,u \in M$. Then $(v - p|u) = (v|u) - (v|u)\,(u|u) = (v|e_S u) - (v|u)\,e_S = 0$. It follows from Corollary 2.16 that M is reachable from v. So M is edged and $p = P_M(v)$.

$(xiii)$ By parts (vii), (ix) and (xii), $M \subseteq w\widetilde{\mathbb{K}}$ with $w\widetilde{\mathbb{K}}$ closed and edged, and $\|w\| = e_S$ for some $S \subseteq (0,1]$.

Let $v \in M^{\perp\perp}$. If $\lambda \in \widetilde{\mathbb{K}}$ and $(u|\lambda w) = 0$, then $(v|\lambda w) = 0$. By Corollary 2.17, $M^{\perp\perp} \subseteq (w\widetilde{\mathbb{K}})^{\perp\perp} = w\widetilde{\mathbb{K}}$. Let ϕ be the isometric embedding $w\widetilde{\mathbb{K}} \to \widetilde{\mathbb{K}}$: $\phi(\lambda w) = \lambda e_S$. Since ϕ is a $\widetilde{\mathbb{K}}$-linear isometry, ϕ also preserves the scalar product. For $a \in e_S\widetilde{\mathbb{K}}$, $(a|\phi(\lambda w)) = a\bar{\lambda}e_S = a\bar{\lambda}$. So if $\lambda \in \widetilde{\mathbb{K}}$ and $\phi(u)\bar{\lambda} = 0$, then $\phi(v)\bar{\lambda} = 0$, i.e., $\phi(v)$ is orthogonal to any $\lambda \in \phi(M)^\perp$ (orthogonal complement in $\widetilde{\mathbb{K}}$). Hence $\phi(v) \in \phi(M)^{\perp\perp} = \overline{\phi(M)}$ since $\phi(M)$ is a principal ideal of $\widetilde{\mathbb{K}}$ [25]. By part (ii), $\overline{\phi(M)} = \phi(\overline{M})$. By the injectivity of ϕ, $v \in \overline{M}$. The converse inclusion holds for any submodule. $\square$

The following example shows that the normalization property cannot be dropped in Proposition 3.8(x).

Example 3.9. For each $m \in \mathbb{N}$, let $S_m \subseteq (0,1]$ with $0 \in \overline{S}_m$ and such that $S_n \cap S_m = \emptyset$ if $n \ne m$.

Let $\beta_\varepsilon = \varepsilon^m$ for each $\varepsilon \in S_m$, and $\beta_\varepsilon = 0$ for $\varepsilon \in (0,1] \setminus \bigcup S_n$. Let $\beta \in \widetilde{\mathbb{R}}$ be the element with representative $(\beta_\varepsilon)_\varepsilon$.

Then $\mathcal{G} = \overline{\beta\widetilde{\mathbb{K}}}$, the closure in $\widetilde{\mathbb{K}}$ of $\beta\widetilde{\mathbb{K}}$, is a Hilbert $\widetilde{\mathbb{K}}$-module and $\beta\widetilde{\mathbb{K}} \ne \overline{\beta\widetilde{\mathbb{K}}}$ as it is proven in [25]. Then $M = \beta\widetilde{\mathbb{K}}$ is an edged cyclic submodule of $\mathcal{G}$, since for each $u \in \mathcal{G}$, $\overline{\inf}_{v\in M}|u - v| = 0$; yet $\beta\widetilde{\mathbb{K}}$ is not closed in $\mathcal{G}$.

The following example shows that Proposition 3.8(i) does not hold for a general Banach $\widetilde{\mathbb{K}}$-module $\mathcal{G}$. In particular it provides an example of a Banach $\widetilde{\mathbb{K}}$-module which is not $\widetilde{\mathbb{R}}$-normed and proves that a quotient of a Hilbert $\widetilde{\mathbb{K}}$-module over a closed but not edged submodule is not necessarily a Hilbert $\widetilde{\mathbb{K}}$-module itself. We recall that for $\gamma \in \widetilde{\mathbb{K}}$, $\mathrm{Ann}(\gamma)$ denotes the set of all $x \in \widetilde{\mathbb{K}}$ such that $x\gamma = 0$.

Example 3.10. Let $\beta \in \widetilde{\mathbb{R}}$ as in Example 3.9. Then $\mathcal{G} = \widetilde{\mathbb{K}}/\overline{\beta\widetilde{\mathbb{K}}}$ is a cyclic Banach $\widetilde{\mathbb{K}}$-module, yet $\mathcal{G}$ is not algebraically isomorphic with an ideal of $\widetilde{\mathbb{K}}$.

Proof. By Proposition 3.4, $\mathcal{G}$ is a Banach $\widetilde{\mathbb{K}}$-module. For $x \in \widetilde{\mathbb{K}}$, we denote by $\bar{x}$ the class $x + \overline{\beta\widetilde{\mathbb{K}}} \in \mathcal{G}$. Then $\mathcal{G}$ is generated by the element $\bar{1} \in \mathcal{G}$. Suppose that $\mathcal{G} \cong I$ (as a $\widetilde{\mathbb{K}}$-module) for some $I \trianglelefteq \widetilde{\mathbb{K}}$. Then there exists $a \in \widetilde{\mathbb{K}}$ such that $I = a\widetilde{\mathbb{K}}$. By the algebraic isomorphism, $x\bar{1} = 0$ iff $xa = 0$, $\forall x \in \widetilde{\mathbb{K}}$. So the annihilator ideal $\mathrm{Ann}(a) = \mathrm{Ann}(\bar{1}) = \overline{\beta\widetilde{\mathbb{K}}}$. But $\mathrm{Ann}(a)$ is either principal or is not the closure of a countably generated ideal, whereas $\overline{\beta\widetilde{\mathbb{K}}}$ is the closure of a countably generated ideal but is not principal [25]. $\square$

By means of Proposition 3.8, we are now able to prove that the formulation of convexity on C given in Proposition 2.15 automatically holds for all the values of λ in $\widetilde{[0,1]} = \{x \in \widetilde{\mathbb{R}} : 0 \le x \le 1\}$.

Proposition 3.11. *Let C be a closed edged subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ such that $\lambda C + (1-\lambda)C \subseteq C$ for all $\lambda \in \{[(\varepsilon)_\varepsilon]^q\}_{q\in\mathbb{N}} \cup \{\frac{1}{2}\}$. Then $\lambda C + (1-\lambda)C \subseteq C$ for all $\lambda \in \widetilde{[0,1]}$.*

Proof. Let $u, u' \in C$ and $\lambda \in \widetilde{[0,1]}$. We show that $v = \lambda u + (1-\lambda)u' \in C$. As the properties of C are translation invariant, we may suppose that $u' = 0$ (so $v = \lambda u$). If $\|u\| = 0$, then trivially $v = 0 \in C$. So, without loss of generality $\|u\| \ne 0$. Let $S \subseteq (0,1]$ with $e_S \ne 0$ such that $\|u\|$ is invertible w.r.t. S. Then $\|ue_S\| = \|u\|e_S$ is invertible w.r.t. S and zero w.r.t. S^c, so $M = ue_S\widetilde{\mathbb{C}}$ is a closed, edged submodule by Proposition 3.8. Let $P_M(P_C(v)e_S) = (\mu + i\kappa)ue_S$ for some $\mu, \kappa \in \widetilde{\mathbb{R}}$. Then $P_C(v)e_S = (\mu + i\kappa)ue_S + w$, with $(u|w) = (ue_S|w) = 0$. Fix representatives $(\lambda_\varepsilon)_\varepsilon$ of λ and $(\mu_\varepsilon)_\varepsilon$ of μ. Let $T = \{\varepsilon \in S : \lambda_\varepsilon \le \mu_\varepsilon\}$; then $0 \le \lambda e_T \le \mu e_T$. By Proposition 2.15,

$$\begin{aligned}0 \ge \Re\,(v - P_C(v)| - P_C(v))\,e_T &= -\Re\,(v|P_C(v)e_T) + \|P_C(v)\|^2 e_T\\ &= -\lambda\mu\|u\|^2 e_T + (\mu^2+\kappa^2)\|u\|^2 e_T + \|w\|^2 e_T,\end{aligned}$$

so

$$0 \le \|w\|^2 e_T + \kappa^2\|u\|^2 e_T \le (\lambda-\mu)\mu\|u\|^2 e_T \le 0.$$

By the invertibility of $\|u\|$ w.r.t. S, $(\lambda-\mu)\mu e_T = we_T = \kappa e_T = 0$. Then also $0 \le (\lambda-\mu)^2 e_T = (\lambda-\mu)\lambda e_T \le 0$ and $\lambda e_T = \mu e_T$.

Denoting $U = S \setminus T$, we have $\mu e_U \le \lambda e_U \le e_U$. Again by Proposition 2.15,

$$\begin{aligned}0 &\ge \Re\,(v - P_C(v)|u - P_C(v))\,e_U\\ &= \Re\,(v|u)\,e_U - \Re\,(v|P_C(v)e_U) - \Re\,(P_C(v)|u)\,e_U + \|P_C(v)\|^2 e_U\\ &= \lambda\|u\|^2 e_U - \lambda\mu\|u\|^2 e_U - \mu\|u\|^2 e_U + (\mu^2+\kappa^2)\|u\|^2 e_U + \|w\|^2 e_U,\end{aligned}$$

so

$$0 \le \|w\|^2 e_U + \kappa^2\|u\|^2 e_U \le (\lambda-\mu)(\mu-1)\|u\|^2 e_U \le 0.$$

Hence, as before, $(\lambda-\mu)(\mu-1)e_U = we_U = \kappa e_U = 0$. Then also $0 \le (\lambda-\mu)^2 e_U \le (\lambda-\mu)(1-\mu)e_U = 0$ and $\lambda e_U = \mu e_U$.

Together, this yields $w = we_S = 0$, $\kappa e_S = 0$ and $\lambda e_S = \mu e_S$. It follows that $P_C(v)e_S = \mu u e_S = \lambda u e_S = v e_S$. Now fix a representative $(\|u\|_\varepsilon)_\varepsilon$ of $\|u\|$ and consider $S_n = \{\varepsilon \in (0,1] : \|u\|_\varepsilon \geq \varepsilon^n\}$ for $n \in \mathbb{N}$. Since $\|u\| \neq 0$, $e_{S_n} \neq 0$ for sufficiently large n. As $\|u\|$ is invertible w.r.t. S_n, $P_C(v)e_{S_n} = ve_{S_n}$ for sufficiently large n. Further, as $0 \in C$, $\|P_C(v)\| \leq \|P_C(v) - v\| + \|v\| \leq 2\|v\| \leq 2\,|\lambda|\,\|u\|$. As $\lim_n \|u\| e_{S_n^c} = 0$, then also $\lim_n \|P_C(v)\| e_{S_n^c} = \lim_n \|v\| e_{S_n^c} = 0$, so $v = \lim_n v e_{S_n} = \lim_n P_C(v) e_{S_n} = P_C(v) \in C$. □

Theorem 3.12.

(i) *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{K}}$-module with the normalization property. Then a cyclic submodule is edged iff it is closed iff it is generated by an element with idempotent $\widetilde{\mathbb{R}}$-norm.*

(ii) *Let $\mathcal{G}_E$ be a Banach $\widetilde{\mathbb{K}}$-module constructed by means of a Banach space E. Then a cyclic submodule is edged iff it is closed iff it is generated by an element with idempotent norm iff it is internal.*

Proof. (i) Follows by Proposition 3.8, assertions (vii), (x), and (xii).

(ii) By Proposition 3.3, $\mathcal{G}_E$ has the normalization property. So, by Proposition 3.8 we already have the implications edged $\Longrightarrow$ closed $\Longrightarrow$ generated by an element with idempotent $\widetilde{\mathbb{R}}$-norm.

Let $M = u\widetilde{\mathbb{K}}$ be a submodule of $\mathcal{G}_E$, and suppose that $\|u\| = e_S$ for some $S \subseteq (0,1]$. We show that M is internal. Let $(u_\varepsilon)_\varepsilon$ be a representative of u. As $ue_{S^c} = 0$, we may suppose that $u_\varepsilon = 0$ for each $\varepsilon \in S^c$. Let $A_\varepsilon = u_\varepsilon \mathbb{K}$ for each $\varepsilon \in (0,1]$. If $v = \lambda u$ for some $\lambda \in \widetilde{\mathbb{K}}$, then there exist representatives such that $v_\varepsilon = \lambda_\varepsilon u_\varepsilon$, so $v \in [(A_\varepsilon)_\varepsilon]$. For the converse inclusion, if $v \in [(A_\varepsilon)_\varepsilon]$, we find $\lambda_\varepsilon \in \mathbb{K}$ such that, on representatives, $v_\varepsilon = \lambda_\varepsilon u_\varepsilon$. We may assume that $\lambda_\varepsilon = 0$ for $\varepsilon \in S^c$. Then, denoting by χ_S the characteristic function of S, the net $(|\lambda_\varepsilon|)_\varepsilon = (\frac{\|v_\varepsilon\|}{\|u_\varepsilon\|}\chi_S(\varepsilon))_\varepsilon$ is moderate (since $\|u\|$ is invertible w.r.t. S). So $(\lambda_\varepsilon)_\varepsilon$ represents $\lambda \in \widetilde{\mathbb{K}}$ and $v = \lambda u$.

Finally, any internal set in $\mathcal{G}_E$ is edged [21]. □

In spite of the obtained results, some elementary operations on cyclic modules appear not to preserve the property of being edged. Even in $\widetilde{\mathbb{R}}^2$, neither intersections nor projections, nor sums preserve this property.

Example 3.13. Let $\beta \in \widetilde{\mathbb{R}}$ as in Example 3.9. Then $(1,\beta)\widetilde{\mathbb{R}} \cap (1,0)\widetilde{\mathbb{R}} = \mathrm{Ann}(\beta) \times \{0\}$ is not edged in $\widetilde{\mathbb{R}}^2$ (since $\mathrm{Ann}(\beta)$ is not edged in $\widetilde{\mathbb{R}}$ [25]). Since $\|(1,\beta)\|$ is invertible by Theorem 3.12 we have that $(1,\beta)\widetilde{\mathbb{R}}$ is edged.

Example 3.14. Let $\beta \in \widetilde{\mathbb{R}}$ as in Example 3.9. Let $M = (1,0)\widetilde{\mathbb{R}} \subseteq \widetilde{\mathbb{R}}^2$. Then $P_M((\beta,1)\widetilde{\mathbb{R}}) = (\beta,0)\widetilde{\mathbb{R}}$ is not edged in $\widetilde{\mathbb{R}}^2$ (since $\beta\widetilde{\mathbb{R}}$ is not generated by an idempotent [2]).

This also gives an example of a projection of a closed submodule on a closed submodule which is not closed.

Example 3.15. Let $\beta \in \widetilde{\mathbb{R}}$ as in Example 3.9. Let $M = (1,\beta)\widetilde{\mathbb{R}} + (1,0)\widetilde{\mathbb{R}} \subset \widetilde{\mathbb{R}}^2$. As $\|(1,\beta)\|$ and $\|(1,0)\|$ are invertible, M is the sum of cyclic edged submodules.

Yet M is not edged, since $M = (0,\beta)\widetilde{\mathbb{R}} + (1,0)\widetilde{\mathbb{R}}$, so

$$\overline{\inf}_{v\in M}\|(0,a)-v\| = \overline{\inf}_{\lambda,\mu\in\widetilde{\mathbb{R}}}(|\mu|^2 + |a-\lambda\beta|^2)^{1/2} = \overline{\inf}_{\lambda\in\widetilde{\mathbb{R}}}|a-\lambda\beta|$$

does not exist for some $a\in\widetilde{\mathbb{R}}$, since $\beta\widetilde{\mathbb{R}}$ is not edged.

This also gives an example of two submodules M, N with $\overline{M}+\overline{N}\neq\overline{M+N}$. Concerning *direct* sums of edged submodules, see Theorem 3.20 below.

3.2. **Submodules generated by $m \geq 1$ elements.**

Theorem 3.16. *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{K}}$-module and M a submodule of $\mathcal{G}$ generated by m elements. Then*

(i) *M is a direct sum of m mutually orthogonal cyclic modules ('interleaved Gram-Schmidt').*
(ii) *M is isometrically isomorphic with a submodule M' of $\widetilde{\mathbb{K}}^m$.*
(iii) *$\overline{M}$ is isometrically isomorphic with $\overline{M'}$ (closure in $\widetilde{\mathbb{K}}^m$).*
(iv) *M is closed iff M is a direct sum of m mutually orthogonal closed cyclic modules.*
(v) *If M is closed, then M is edged.*
(vi) *If M is closed, any edged submodule N of M is closed and finitely generated.*
(vii) *If $\mathcal{G}$ has the normalization property and M is a direct sum of mutually orthogonal cyclic modules M_1, …, M_m, then there exist mutually orthogonal closed cyclic modules N_j such that $M_j\subseteq N_j$ for $j=1,\dots, m$.*
(viii) *If $\mathcal{G}$ has the normalization property and M is edged, then M is closed.*

Proof. (i) We proceed by induction on m. The case $m=1$ is trivial.

Let $M = u_1\widetilde{\mathbb{K}}+\cdots+u_m\widetilde{\mathbb{K}}$. Fix representatives $(\|u_j\|_\varepsilon)_\varepsilon$ of $\|u_j\|$ and define recursively for $j=1,\dots,m$

$$S_j = \bigl\{\varepsilon\in(0,1] : \|u_j\|_\varepsilon \geq \max_{k\neq j}\|u_k\|_\varepsilon\bigr\}\setminus\{S_1,\dots,S_{j-1}\}.$$

Then $e_{S_j}e_{S_k}=0$ if $j\neq k$, and $e_{S_1}+\cdots+e_{S_m}=1$. By Proposition 3.8, we can project $u_je_{S_1}$ on $N = u_1e_{S_1}\widetilde{\mathbb{K}}$, obtaining $\tilde{u}_j = u_je_{S_1} - P_N(u_je_{S_1})$ with $(u_1|\tilde{u}_j)=0$ and $\tilde{u}_j=\tilde{u}_je_{S_1}$ $(j>1)$. With $N' = \tilde{u}_2\widetilde{\mathbb{K}}+\cdots+\tilde{u}_m\widetilde{\mathbb{K}}$, we also have $Me_{S_1} = u_1e_{S_1}\widetilde{\mathbb{K}}+N'$ and $u_1\in N'^{\perp}$. By induction, there exist mutually orthogonal generators $v_2^{(1)}$, …, $v_m^{(1)}$ of N'. Since $N'e_{S_1}=N'$, $v_j^{(1)} = v_j^{(1)}e_{S_1}$ for all j. With $v_1^{(1)} = u_1e_{S_1}$ we obtain m mutually orthogonal generators of Me_{S_1}. Similarly, we obtain m mutually orthogonal generators $v_1^{(k)},\dots,v_m^{(k)}$ of Me_{S_k} $(k=1,\dots,m)$ (in particular satisfying $v_j^{(k)} = v_j^{(k)}e_{S_k}$). Then $v_j=\sum_{k=1}^m v_j^{(k)}$ $(j=1,\dots,m)$ are mutually orthogonal generators of M. By orthogonality, it follows that the sum is a direct sum: if $\sum_j\lambda_jv_j=0$ for some $\lambda_j\in\widetilde{\mathbb{K}}$, then $0=(\sum_j\lambda_jv_j|\sum_j\lambda_jv_j)=\sum_j\|\lambda_jv_j\|^2$, so each $\lambda_jv_j=0$.

(ii) By part (i), $M=v_1\widetilde{\mathbb{K}}+\cdots+v_m\widetilde{\mathbb{K}}$, with v_j mutually orthogonal. Define ϕ: $M\to\widetilde{\mathbb{K}}^m$: $\phi(\sum_j\lambda_jv_j) = (\lambda_1\|v_1\|,\dots,\lambda_m\|v_m\|)$ $(\lambda_j\in\widetilde{\mathbb{K}})$. Then, by the orthogonality,

$$\Big\|\phi\Big(\sum_j \lambda_j v_j\Big) - \phi\Big(\sum_j \mu_j v_j\Big)\Big\|^2 = \sum_j \|(\lambda_j - \mu_j)v_j\|^2$$
$$= \Big\|\sum_j(\lambda_j - \mu_j)v_j\Big\|^2 = \Big\|\sum_j \lambda_j v_j - \sum_j \mu_j v_j\Big\|^2,$$

which shows that ϕ is well defined and isometric (hence also injective). It is easy to check that ϕ is $\widetilde{\mathbb{K}}$-linear.

(iii) We extend ϕ: $M \to \widetilde{\mathbb{K}}^m$ to a map $\overline{M} \to \widetilde{\mathbb{K}}^m$ by defining $\phi(\lim_n w_n) := \lim_n \phi(w_n)$ $(w_n \in M)$. Because $(w_n)_n$ is a Cauchy-sequence, $(\phi(w_n))_n$ is also a Cauchy-sequence in $\widetilde{\mathbb{K}}^m$, and hence convergent in $\widetilde{\mathbb{K}}^m$. To see that ϕ is well defined, let $\lim_n w_n = \lim_n w_n'$. Then also the interlaced sequence $(w_1, w_1', \dots, w_n, w_n', \dots)$ is a Cauchy-sequence. Hence also $(\phi(w_1), \phi(w_1'), \dots, \phi(w_n), \phi(w_n'), \dots)$ is convergent to $\lim_n \phi(w_n) = \lim_n \phi(w_n')$. It is easy to check that the extended ϕ is linear and isometric as well. As $\mathcal{G}$ is complete, we find that the image under ϕ^{-1} of any convergent sequence in $\phi(M)$ (say to $\xi \in \overline{\phi(M)}$) is a Cauchy sequence, and hence convergent to an element $w \in \overline{M}$. By definition of the extended map ϕ, $\phi(w) = \xi$. So $\overline{\phi(M)} \subseteq \phi(\overline{M})$. The converse inclusion holds by continuity of ϕ.

(iv), (v) Let M be closed. By part (i), $M = u_1\widetilde{\mathbb{K}} + \dots + u_m\widetilde{\mathbb{K}}$ with u_j mutually orthogonal. Let $w \in \overline{u_1\widetilde{\mathbb{K}}}$, so $w = \lim_n \lambda_n u_1$ for some $\lambda_n \in \widetilde{\mathbb{K}}$. As M is closed, $w = \sum_j \mu_j u_j$ for some $\mu_j \in \widetilde{\mathbb{K}}$. By the continuity of the scalar product, $(w|u_j) = \lim_n \lambda_n (u_1|u_j) = 0$ for $j > 1$. So $0 = (w|u_j) = \sum_k \mu_k (u_k|u_j) = \mu_j \|u_j\|^2$ for $j > 1$. So also $\|\mu_j u_j\|^2 = 0$ for $j > 1$ and $w = \mu_1 u_1 \in u_1\widetilde{\mathbb{K}}$. Similarly, $u_j\widetilde{\mathbb{K}}$ is closed ($j = 1$, $\dots$, m).

Conversely, let $M = u_1\widetilde{\mathbb{K}} + \dots + u_m\widetilde{\mathbb{K}}$, with $u_j\widetilde{\mathbb{K}}$ closed and u_j mutually orthogonal. By Proposition 3.8(xii), we know that $M_j = u_j\widetilde{\mathbb{K}}$ are edged. Let $v \in \mathcal{G}$ and let $p = \sum P_{M_j}(v) \in M$. Then by orthogonality, $(v - p|u_j) = \big(v - P_{M_j}(v)|u_j\big) = 0$. So, by Corollary 2.17$(i)$ it follows that M is closed and edged.

(vi) Let $M = u_1\widetilde{\mathbb{K}} + \dots + u_m\widetilde{\mathbb{K}}$. As $\overline{N}$ is edged and closed, by the linearity of the projection operator $P_{\overline{N}}$, $\overline{N} = P_{\overline{N}}(M) = v_1\widetilde{\mathbb{K}} + \dots + v_m\widetilde{\mathbb{K}}$ with $v_j = P_{\overline{N}}u_j$. By part (iv), we may suppose that $\|v_j\| = e_{S_j}$ for some $S_j \subseteq (0, 1]$ and that v_j are mutually orthogonal. Therefore,

$$0 = \overline{\inf}_{\sum_j \mu_j v_j \in N}\Big\|v_1 - \sum_j \mu_j v_j\Big\| = \overline{\inf}_{\sum_j \mu_j v_j \in N}\big(|1 - \mu_1|^2 e_{S_1} + \sum_{j>1}\|\mu_j v_j\|^2\big)^{1/2},$$

so for each $m \in \mathbb{N}$ there exist $\sum_j \mu_j v_j \in N$ with $|1 - \mu_1| e_{S_1} \le [(\varepsilon^m)_\varepsilon]$, $\|\mu_j v_j\| \le [(\varepsilon^m)_\varepsilon]$. For sufficiently large m, this implies that μ_1 is invertible w.r.t. S_1. Let $\lambda_1\mu_1 = e_{S_1}$ with $\lambda_1 = \lambda_1 e_{S_1}$. Then, $|\mu_1|e_{S_1} \ge (1 - [(\varepsilon^m)_\varepsilon])e_{S_1} \ge \frac{1}{2}e_{S_1}$. Hence, $|\lambda_1| \le 2e_{S_1}$. So for each m, there exist $v_1 + \sum_{j\ne 1}\mu_j v_j \in N$ with $\|\mu_j v_j\| \le [(\varepsilon^m)_\varepsilon]$. Similarly, for each m, there exist $v_k + \sum_{j\ne k}\mu_j^{(k)} v_j \in N$ with $\|\mu_j^{(k)} v_j\| \le [(\varepsilon^m)_\varepsilon]$ ($k = 1$, $\dots$, m). Then there also exist linear combinations

$$\Big(v_1 + \sum_{j\ne 1}\mu_j^{(1)} v_j\Big) - \mu_2^{(1)} e_{S_2}\Big(v_2 + \sum_{j\ne 2}\mu_j^{(2)} v_j\Big) = (1 - \mu_2^{(1)} e_{S_2}\mu_1^{(2)} e_{S_1})v_1 + \sum_{j\ne 1,2}\mu_j' v_j \in N,$$

with $\|\mu_j' v_j\|$ arbitrarily small. As $1-\mu_2^{(1)}e_{S_2}\mu_1^{(2)}e_{S_1}$ is invertible (for m sufficiently large), this also implies that there exist $v_1+\sum_{j\ne 1,2}\mu_j v_j\in N$ with $\|\mu_j v_j\|$ arbitrarily small, and so on. We conclude that $v_1, \dots, v_m \in N$. So $N=\overline{N}$ is closed and finitely generated.

(*vii*) Let $M=u_1\widetilde{\mathbb{K}}+\cdots+u_m\widetilde{\mathbb{K}}$ with u_j mutually orthogonal. By the normalization property, there exists $v_1\in\mathcal{G}$ with $\|u_1\|v_1=u_1$. As in Proposition 3.8(*ix*), there exists $S_1\subseteq(0,1]$ such that $\|u_1\|e_{S_1^c}=0$ and $\|v_1\|e_{S_1}=e_{S_1}$. As $(u_1|u_j)=0$ for $j>1$, then also $\|u_1\|\,(v_1|u_j)=0$, so by a characterization of zero divisors in $\widetilde{\mathbb{K}}$, there exist $S_j\subseteq(0,1]$ such that $\|u_1\|e_{S_j^c}=0$ and $(v_1|u_j)\,e_{S_j}=0$. Let $w_1=v_1e_{S_1}\cdots e_{S_m}$. Then $\|w_1\|=e_{S_1}\cdots e_{S_m}$ is idempotent, and hence $w_1\widetilde{\mathbb{K}}$ is closed by Proposition 3.8(*vi*). Further, $u_1=u_1e_{S_1}\cdots e_{S_m}=\|u_1\|w_1$, so $u_1\widetilde{\mathbb{K}}\subseteq w_1\widetilde{\mathbb{K}}$. Finally, $(w_1|u_j)=(v_1|u_j)\,e_{S_1}\cdots e_{S_m}=0$ for $j>1$. Similarly, we find $w_2\in\mathcal{G}$ such that $w_2\widetilde{\mathbb{K}}$ is closed, $u_2\widetilde{\mathbb{K}}\subseteq w_2\widetilde{\mathbb{K}}$ and $(w_2|w_1)=(w_2|u_3)=\cdots=(w_2|u_m)=0$, and so on.

(*viii*) By parts (*i*) and (*vii*), $M=u_1\widetilde{\mathbb{K}}+\cdots+u_m\widetilde{\mathbb{K}}$ and there exist w_j with $w_j\widetilde{\mathbb{K}}$ closed and with $u_j\in w_j\widetilde{\mathbb{K}}$ and w_j mutually orthogonal. As $\mathcal{H}=w_1\widetilde{\mathbb{K}}$ is closed, it is itself a Hilbert $\widetilde{\mathbb{K}}$-module. We show that $u_1\widetilde{\mathbb{K}}$ is an edged submodule of $\mathcal{H}$.

So let $\lambda\in\widetilde{\mathbb{K}}$. Since M is edged in $\mathcal{G}$, $\overline{\inf}_{v\in M}\|\lambda w_1-v\|$ exists. So by orthogonality,

$$\overline{\inf}_{\mu_j\in\widetilde{\mathbb{K}}}\Big\|\lambda w_1-\sum_j\mu_j u_j\Big\|=\overline{\inf}_{\mu_j\in\widetilde{\mathbb{K}}}\Big(\|\lambda w_1-\mu_1u_1\|^2+\sum_{j>1}\|\mu_ju_j\|^2\Big)^{1/2}$$
$$=\overline{\inf}_{\mu_1\in\widetilde{\mathbb{K}}}\|\lambda w_1-\mu_1u_1\|;$$

hence $u_1\widetilde{\mathbb{K}}$ is edged in $\mathcal{H}$. By Proposition 3.8(*viii*), $u_1\widetilde{\mathbb{K}}$ is closed in $\mathcal{H}$ and, by completeness, also in $\mathcal{G}$. Similarly, $u_j\widetilde{\mathbb{K}}$ is closed ($j=1,\dots,m$). By the fourth assertion of this theorem, M is closed. $\square$

Theorem 3.17. *Let $\mathcal{G}_H$ be a Hilbert $\widetilde{\mathbb{K}}$-module constructed by means of a Hilbert space H. Then a finitely generated submodule M of $\mathcal{G}_H$ is edged iff M is closed iff M is a finite direct sum of mutually orthogonal closed cyclic modules iff M is internal.*

Proof. Let M be a finite direct sum of mutually orthogonal closed cyclic modules, so $M=u_1\widetilde{\mathbb{K}}+\cdots+u_m\widetilde{\mathbb{K}}$ with u_j mutually orthogonal and $\|u_j\|=e_{S_j}$ for some $S_j\subseteq(0,1]$. We show that M is internal.

Fix representatives $(u_{j,\varepsilon})_\varepsilon$ of (u_j). By interleaved Gram-Schmidt process at the level of representatives, we may suppose that $(u_{j,\varepsilon}|u_{k,\varepsilon})=0$ for $j\ne k$. As $u_je_{S_j^c}=0$, we may also suppose that $u_{j,\varepsilon}=0$ for each $\varepsilon\in S_j^c$. Let $A_\varepsilon=u_{1,\varepsilon}\mathbb{K}+\cdots+u_{m,\varepsilon}\mathbb{K}$ for each $\varepsilon\in(0,1]$. If $v\in M$, looking at representatives, $v\in[(A_\varepsilon)_\varepsilon]$. Conversely, if $v\in[(A_\varepsilon)_\varepsilon]$, we find $\lambda_{j,\varepsilon}\in\mathbb{K}$ such that, on representatives, $v_\varepsilon=\sum_j\lambda_{j,\varepsilon}u_{j,\varepsilon}$. We may assume that $\lambda_{j,\varepsilon}=0$ for $\varepsilon\in S_j^c$. Then $(v_\varepsilon|u_{j,\varepsilon})=\lambda_{j,\varepsilon}\,(u_{j,\varepsilon}|u_{j,\varepsilon})$, so $(\lambda_{j,\varepsilon})_\varepsilon$ are moderate (since $\|u_j\|$ are invertible w.r.t. S_j). So $(\lambda_{j,\varepsilon})_\varepsilon$ represent $\lambda_j\in\widetilde{\mathbb{K}}$ and $v=\sum_j\lambda_ju_j\in M$.

Further, any nonempty internal set in $\mathcal{G}_H$ is edged [21].

Since $\mathcal{G}_H$ has the normalization property, the other equivalences follow by the previous theorem. $\square$

Theorem 3.18.

(i) *Let M be a finitely generated submodule of $\widetilde{\mathbb{K}}^d$. Then M is generated by d elements.*

(ii) *Let M be a submodule of a Hilbert $\widetilde{\mathbb{K}}$-module $\mathcal{G}$ that is generated by m elements. Then any finitely generated submodule of M is generated by m elements.*

Proof. (i) Let $M = u_1\widetilde{\mathbb{K}} + \cdots + u_m\widetilde{\mathbb{K}}$ with $m > d$. Applying interleaved Gram-Schmidt process at the level of representatives, we can obtain representatives $(u_{j,\varepsilon})_\varepsilon$ of u_j such that for each ε, $(u_{j,\varepsilon}|u_{k,\varepsilon}) = 0$ if $j \neq k$. Define recursively for $j = 1, \dots, m$

$$S_j = \left\{\varepsilon \in (0,1] : \|u_j\|_\varepsilon \leq \max_{k\neq j} \|u_k\|_\varepsilon\right\} \setminus \{S_1, \dots, S_{j-1}\}.$$

Then $e_{S_j}e_{S_k} = 0$ if $j \neq k$ and $e_{S_1} + \cdots + e_{S_m} = 1$. Let $\varepsilon \in S_1$. Should $u_{1,\varepsilon} \neq 0$, then also $u_{j,\varepsilon} \neq 0$ for all j. So we would obtain $m > d$ orthogonal (hence linearly independent) elements of $\mathbb{K}^d$, a contradiction. So $u_1 e_{S_1} = 0$, and $Me_{S_1} = v_1^{(1)}\widetilde{\mathbb{K}} + \cdots + v_{m-1}^{(1)}\widetilde{\mathbb{K}}$ for $v_j^{(1)} = u_{j+1}e_{S_1}$. Similarly, $Me_{S_k} = v_1^{(k)}\widetilde{\mathbb{K}} + \cdots + v_{m-1}^{(k)}\widetilde{\mathbb{K}}$ for some $v_j^{(k)} \in \widetilde{\mathbb{K}}^d$ satisfying $v_j^{(k)} = v_j^{(k)} e_{S_k}$ $(k = 1, \dots, m)$. Then $v_j = \sum_{k=1}^m v_j^{(k)}$ $(j = 1, \dots, m-1)$ are $m-1$ generators of M.

(ii) Follows from part (i) and Theorem 3.16. $\square$

Theorem 3.19. *Let M, N be edged submodules of a Hilbert $\widetilde{\mathbb{K}}$-module $\mathcal{G}$. If $M \perp N$, then $M + N$ is edged and $\overline{M} + \overline{N} = \overline{M + N}$.*

Proof. First, by the continuity of the scalar product in $\mathcal{G}$, if $M \perp N$, then also $\overline{M} \perp \overline{N}$.

Let $v \in \mathcal{G}$. For each $u \in \overline{M}$,

$$(v - (P_{\overline{M}}(v) + P_{\overline{N}}(v))|u) = ((v - P_{\overline{M}}(v)) + P_{\overline{N}}(v)|u) = 0$$

by the properties of the $P_{\overline{M}}$ and the fact that $\overline{M} \perp \overline{N}$. Switching roles of M and N, we obtain that $v-(P_{\overline{M}}(v)+P_{\overline{N}}(v)) \in (\overline{M}+\overline{N})^\perp$. As also $P_{\overline{M}}(v)+P_{\overline{N}}(v) \in \overline{M}+\overline{N}$, it follows from Corollary 2.17 that $\overline{M}+\overline{N}$ is closed and edged. As $M+N \subseteq \overline{M}+\overline{N} \subseteq \overline{M+N}$ and $\overline{M} + \overline{N}$ is closed, $\overline{M} + \overline{N} = \overline{M+N}$. $\square$

Theorem 3.20. *Let M, N be submodules of a Hilbert $\widetilde{\mathbb{K}}$-module $\mathcal{G}$. Let M be closed and finitely generated, and let N be closed and edged. If $M \cap N = \{0\}$, then $M + N$ is closed and edged.*

Proof. We proceed by induction on the number m of generators of M.

First, let $M = u\widetilde{\mathbb{K}}$ be cyclic. By Proposition 3.8, we may suppose that $\|u\| = e_S$ for some $S \subseteq (0,1]$. Now suppose that $\|u - P_N(u)\|$ is not invertible w.r.t. S. Then there exists $T \subseteq S$ with $0 \in \overline{T}$ such that $\|ue_T - P_N(u)e_T\| = \|u - P_N(u)\|e_T = 0$, so $ue_T = P_N(u)e_T \in M \cap N$, and $\|ue_T\| = e_S e_T = e_T \neq 0$, which contradicts $M \cap N = \{0\}$. As $0 \leq \|P_N(u)\| \leq \|u\|$, then also $\|u - P_N(u)\|e_{S^c} = 0$, and $M' = (u - P_N(u))\widetilde{\mathbb{K}}$ is also closed, hence edged by Proposition 3.8 parts (vi) and (xii). Since $M' \perp N$, by Theorem 3.19, $M + N = M' + N$ is closed and edged.

Now let M be generated by m elements. By Theorem 3.16, M is a direct sum of a closed cyclic module M_1 and a closed module M_2 generated by $m-1$ elements. By induction, as $M_2 \cap N = \{0\}$, $M_2 + N$ is closed and edged. As also $M_1 + (M_2 + N)$

is a direct sum, then by the first part of the proof $M + N = M_1 + (M_2 + N)$ is closed and edged. $\square$

4. A Riesz-representation theorem for continuous $\widetilde{\mathbb{C}}$-linear functionals on $\mathcal{G}$

In this section we consider Hilbert $\widetilde{\mathbb{C}}$-modules with the normalization property and we prove a Riesz representation theorem for the corresponding continuous $\widetilde{\mathbb{C}}$-linear functionals.

Theorem 4.1. *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module with the normalization property and let T be a continuous $\widetilde{\mathbb{C}}$-linear functional on $\mathcal{G}$. The following assertions are equivalent:*

- (i) *there exists a closed edged $\widetilde{\mathbb{C}}$-submodule M of $\operatorname{Ker} T$ and a subset S of $(0,1]$ such that*
  - (a) *there exists $u_1 \in M^\perp$ with $\|u_1\| = e_S$;*
  - (b) *$\|u\| = e_S\|u\|$ for all $u \in M^\perp$;*
  - (c) *$T(u)v - T(v)u \in M$ for all $u, v \in M^\perp$;*
- (ii) *there exists a closed, cyclic (and hence edged) $\widetilde{\mathbb{C}}$-submodule N of $\mathcal{G}$ such that $N^\perp \subseteq \operatorname{Ker} T$;*
- (iii) *there exists a unique $c \in \mathcal{G}$ such that $T(u) = (u|c)$.*

Proof. $(i) \Rightarrow (iii)$ Let $u_1 \in M^\perp$ satisfying the condition (a) and $u \in M^\perp$. From (c) we get that $T(u)u_1 - T(u_1)u \in M$ and thus $T(u)e_S = T(u)\|u_1\|^2 = T(u_1)\,(u|u_1)$. Since T is continuous there exists $C > 0$ such that $|T(u)| \le C\|u\|$. It follows that $|T(u)|e_{S^c} \le C\|u\|e_{S^c} = 0$ because of property (b). So $T(u) = T(u)e_S = (u|c)$, where $c = \overline{T(u_1)}u_1$.

Now let $u \in \mathcal{G}$. By Corollary 2.17 we know that $u = (u - P_M(u)) + P_M(u)$, where $u - P_M(u) \in M^\perp$ and $P_M(u) \in M$. Since $T(u) = T(u - P_M(u))$, by the previous case we have that $T(u) = (u - P_M(u)|c) = (u|c)$.

$(iii) \Rightarrow (ii)$ By the normalization property and the assertions (ix) and (xii) of Proposition 3.8, we know that there exists a closed cyclic and edged $\widetilde{\mathbb{C}}$-module N such that $c\widetilde{\mathbb{C}} \subseteq N$. Thus, $N^\perp \subseteq (c\widetilde{\mathbb{C}})^\perp = \operatorname{Ker} T$.

$(ii) \Rightarrow (i)$ From the assertion (vii) of Proposition 3.8 we have that N is generated by an element $w \in \mathcal{G}$ such that $\|w\| = e_S$ for some $S \subseteq (0,1]$. Let us define $M = N^\perp$. By Corollary 2.17 M is closed and edged. (a) and (b) are straightforward, and (c) follows from the fact that $M^\perp$ is cyclic. $\square$

Remark 4.2. Note that $(i) \Rightarrow (iii)$ is valid without assuming the normalization property on $\mathcal{G}$. In the first assertion of Theorem 4.1 we assume the existence of an edged and closed $\widetilde{\mathbb{C}}$-submodule M contained in $\operatorname{Ker} T$ because in general the kernel of a continuous $\widetilde{\mathbb{C}}$-linear functional is not edged. Indeed, taking β as in 3.9 and the functional $T : \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}} : z \to \beta z$, we have that $\operatorname{Ker} T = \operatorname{Ann}(\beta)$ is not edged [25].

The following example shows that there are continuous $\widetilde{\mathbb{C}}$-linear functionals on Hilbert $\widetilde{\mathbb{C}}$-modules for which the Riesz representation theorem does not hold.

Example 4.3. By [25] there exists a submodule (=ideal) M of $\widetilde{\mathbb{C}}$ and a continuous $\widetilde{\mathbb{C}}$-linear map T: $M \to \widetilde{\mathbb{C}}$ that cannot be extended to a $\widetilde{\mathbb{C}}$-linear map $\widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$. Let $\mathcal{G} = \overline{M}$ (the topological closure of M in $\widetilde{\mathbb{C}}$). Then M is a Hilbert $\widetilde{\mathbb{C}}$-module as a closed submodule of a Hilbert $\widetilde{\mathbb{C}}$-module. By continuity, T can be uniquely extended to a continuous $\widetilde{\mathbb{C}}$-linear map $\widetilde{T}$: $\mathcal{G} \to \widetilde{\mathbb{C}}$. Suppose that there exists $c \in \mathcal{G}$ such that $\widetilde{T}(u) = (u|c)$. Then the $\widetilde{\mathbb{C}}$-linear map $\widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$: $u \mapsto (u|c)$ would be an extension of T, a contradiction.

Proposition 4.4. *Let H be a Hilbert space and $\mathcal{G}_H$ the corresponding Hilbert $\widetilde{\mathbb{C}}$-module. A continuous $\widetilde{\mathbb{C}}$-linear functional T on $\mathcal{G}_H$ is basic if and only if it fulfills the equivalent properties of the previous theorem.*

Proof. Apply the Riesz theorem at the level of representatives, noting that $T_\varepsilon(u) = (u|c_\varepsilon)$ with $\|c_\varepsilon\| = \|T_\varepsilon\|$. □

Conjecture. *there exists a Hilbert space H (necessarily infinitely dimensional) and a continuous $\widetilde{\mathbb{C}}$-linear functional that is not basic.*

We now investigate the structural properties of continuous $\widetilde{\mathbb{C}}$-sesquilinear forms on Hilbert $\widetilde{\mathbb{C}}$-modules by making use of the previous representation theorem.

Theorem 4.5. *Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules with $\mathcal{H}$ satisfying the normalization property, and let $a : \mathcal{G} \times \mathcal{H} \to \widetilde{\mathbb{C}}$ be a continuous $\widetilde{\mathbb{C}}$-sesquilinear form. The following assertions are equivalent:*

(i) *for all $u \in \mathcal{G}$ there exists a closed and cyclic $\widetilde{\mathbb{C}}$-submodule N_u of $\mathcal{H}$ such that $N_u^\perp \subseteq \{v \in \mathcal{H} : a(u, v) = 0\}$;*

(ii) *there exists a unique continuous $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$ such that $a(u, v) = (Tu|v)$ for all $u \in \mathcal{G}$ and $v \in \mathcal{H}$.*

Proof. $(i) \Rightarrow (ii)$ Let $u \in \mathcal{G}$. We consider the continuous $\widetilde{\mathbb{C}}$-linear functional $a_u : \mathcal{H} \to \widetilde{\mathbb{C}} : v \to \overline{a(u, v)}$. Since $\mathrm{Ker}\, a_u = \{v \in \mathcal{H} : a(u, v) = 0\}$ contains the orthogonal complement of a closed and cyclic $\widetilde{\mathbb{C}}$-submodule N_u of $\mathcal{H}$, by Theorem 4.1 there exists a unique $c \in \mathcal{H}$ such that $\overline{a(u, v)} = (v|c)$. We define $T : \mathcal{G} \to \mathcal{H} : u \to c$. By construction, $a(u, v) = (Tu|v)$. We leave it to the reader to check that the map T is $\widetilde{\mathbb{C}}$-linear. By definition of the operator T we have that

$$\|T(u)\|^2 = (Tu|Tu) = a(u, Tu) \leq C\|u\|\|Tu\|, \tag{4.1}$$

where the constant $C \in \widetilde{\mathbb{R}}$ comes from the continuity of a. Applying Lemma 2.19 to (4.1) we have that $\|T(u)\| \leq C\|u\|$ for all u. This shows that T is continuous.

$(ii) \Rightarrow (i)$ Let us fix $u \in \mathcal{G}$. Since $v \mapsto \overline{a(u, v)} = (v|Tu)$ is a continuous $\widetilde{\mathbb{C}}$-linear functional on $\mathcal{H}$ satisying the assertion (iii) of Theorem 4.1, we find a subset N_u as desired. □

Proposition 4.6. *Let H and K be Hilbert spaces and let a be a basic $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G}_H \times \mathcal{G}_K$. Then the hypotheses of Theorem 4.5 are satisfied. Moreover, the map $T : \mathcal{G}_H \to \mathcal{G}_K$ such that $a(u, v) = (Tu|v)$ is basic.*

Proof. By Proposition 3.3 the $\widetilde{\mathbb{C}}$-module $\mathcal{G}_K$ has the normalization property. If a is basic, then for any fixed $u \in \mathcal{G}_H$ the $\widetilde{\mathbb{C}}$-linear functional $\mathcal{G}_K \to \widetilde{\mathbb{C}} : v \to \overline{a(u, v)}$ is basic, too. Hence, from Theorem 4.1 there exists a closed and cyclic $\widetilde{\mathbb{C}}$-submodule N_u of $\mathcal{G}_K$ such that $N_u^\perp \subseteq \{v \in \mathcal{G}_K : a(u, v) = 0\}$. It remains to prove that the continuous $\widetilde{\mathbb{C}}$-linear map $T : \mathcal{G}_H \to \mathcal{G}_K$, that we know to exist from Theorem 4.5, has a basic structure. Let us take a net $(a_\varepsilon)_\varepsilon$ representing the $\widetilde{\mathbb{C}}$-sesquilinear form a. By fixing $u \in H$ we obtain from the continuity of a_ε that there exist a net

$(c_\varepsilon)_\varepsilon$ of elements of K and a net $t_\varepsilon(u) = c_\varepsilon$ of linear maps from H to K such that $\overline{a_\varepsilon(u,v)} = (v|c_\varepsilon)$ for all $v \in K$. Since for some $N \in \mathbb{N}$ and $\eta \in (0,1]$ the inequality

$$\|t_\varepsilon(u)\|^2 = a_\varepsilon(u, t_\varepsilon(u)) \le \varepsilon^{-N} \|u\| \, \|t_\varepsilon(u)\|$$

holds for all $u \in H$ and $\varepsilon \in (0,\eta]$, we obtain that $(t_\varepsilon)_\varepsilon$ defines a basic map $T' : \mathcal{G}_H \to \mathcal{G}_K$ such that $a(u,v) = (T'u|v)$. By Theorem 4.5 there exists a unique continuous $\widetilde{\mathbb{C}}$-linear map from $\mathcal{G}_H$ to $\mathcal{G}_K$ having this property. It follows that $T' = T$ and that T is basic. □

5. Continuous $\widetilde{\mathbb{C}}$-linear operators on a Hilbert $\widetilde{\mathbb{C}}$-module

In this section we focus on continuous $\widetilde{\mathbb{C}}$-linear operators acting on a Hilbert $\widetilde{\mathbb{C}}$-module. In particular we deal with isometric, unitary, self-adjoint and projection operators obtaining an interesting characterization for the projection operators.

5.1. **Adjoint.**

Definition 5.1. Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules and let $T : \mathcal{G} \to \mathcal{H}$ be a continuous $\widetilde{\mathbb{C}}$-linear map. A continuous $\widetilde{\mathbb{C}}$-linear operator $T^* : \mathcal{H} \to \mathcal{G}$ is called adjoint of T if

$$(Tu|v) = (u|T^*v) \tag{5.1}$$

for all $u \in \mathcal{G}$ and $v \in \mathcal{H}$.

Note that if there exists an operator T^* satisfying (5.1), then it is unique.

The following proposition characterizes the existence of the adjoint T^* under suitable hypotheses on the spaces $\mathcal{G}$ and $\mathcal{H}$.

Proposition 5.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules with $\mathcal{G}$ satisfying the normalization property, and let $T : \mathcal{G} \to \mathcal{H}$ be a continuous $\widetilde{\mathbb{C}}$-linear map. The adjoint $T^* : \mathcal{H} \to \mathcal{G}$ exists if and only if for all $v \in \mathcal{H}$ there exists a closed and cyclic $\widetilde{\mathbb{C}}$-submodule N_v of $\mathcal{G}$ such that $N_v^\perp \subseteq \{u \in \mathcal{G} : (v|Tu) = 0\}$.*

Proof. The proof is clear by applying Theorem 4.5 to the continuous $\widetilde{\mathbb{C}}$-sesquilinear form $a : \mathcal{H} \times \mathcal{G} \to \widetilde{\mathbb{C}} : (v,u) \to (v|Tu)$. □

Proposition 5.3. *If H and K are Hilbert spaces and T is a basic $\widetilde{\mathbb{C}}$-linear map from $\mathcal{G}_H$ to $\mathcal{G}_K$, then the hypotheses of the previous proposition are fulfilled. In particular the operator $T^* : \mathcal{G}_K \to \mathcal{G}_H$ is basic.*

Proof. It suffices to observe that the $\widetilde{\mathbb{C}}$-sesquilinear form $(v|Tu)$ is basic and to apply Proposition 4.6. □

Proposition 5.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules and $S, T : \mathcal{G} \to \mathcal{H}$ be continuous $\widetilde{\mathbb{C}}$-linear maps having an adjoint. The following properties hold:*

(i) $(S+T)^* = S^* + T^*$;
(ii) $(\lambda T)^* = \overline{\lambda} T^*$ *for all* $\lambda \in \widetilde{\mathbb{C}}$;
(iii) $(T^*(v)|u) = (v|Tu)$ *for all* $u \in \mathcal{G}$ *and* $v \in \mathcal{H}$;
(iv) $T^{**} = T$;
(v) $T^*T = 0$ *if and only if* $T = 0$;
(vi) $(ST)^* = T^*S^*$;
(vii) *if* $M \subseteq \mathcal{G}$, $N \subseteq \mathcal{H}$ *and* $T(M) \subseteq N$, *then* $T^*(N^\perp) \subseteq M^\perp$;

(viii) *if* $M \subseteq \mathcal{G}$ *and* N *is a closed and edged* $\widetilde{\mathbb{C}}$*-submodule of* $\mathcal{H}$*, then* $T(M) \subseteq N$ *if and only if* $T^*(N^\perp) \subseteq M^\perp$.

Proof. We omit the proof of the first seven assertions of the proposition because they are elementary.

$(viii)$ From assertion (vii) we have that $T(M) \subseteq N$ implies $T^*(N^\perp) \subseteq M^\perp$. Conversely, assume that $T^*(N^\perp) \subseteq M^\perp$ and apply (vii) to T^*. It follows that $(T^*)^*(M^{\perp\perp}) \subseteq N^{\perp\perp}$. By (iv) we can write $T(M) \subseteq T(M^{\perp\perp}) \subseteq N^{\perp\perp}$, and from Corollary 2.17(iii) we have that $T(M) \subseteq N$. □

Proposition 5.5. *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be Hilbert* $\widetilde{\mathbb{C}}$*-modules and* $T : \mathcal{G} \to \mathcal{H}$ *be a continuous* $\widetilde{\mathbb{C}}$*-linear map. Assume that the adjoint of* T *exists. The following equalities hold:*

(i) $\operatorname{Ker} T = (T^*(\mathcal{H}))^\perp$;

(ii) $\operatorname{Ker} T^* = (T(\mathcal{G}))^\perp$;

(iii) *if* $T^*(\mathcal{H})$ *is a closed and edged* $\widetilde{\mathbb{C}}$*-submodule of* $\mathcal{G}$*, then* $(KerT)^\perp = T^*(\mathcal{H})$;

(iv) *if* $T(\mathcal{G})$ *is a closed and edged* $\widetilde{\mathbb{C}}$*-submodule of* $\mathcal{H}$*, then* $(KerT^*)^\perp = T(\mathcal{G})$.

Proof. An application of Proposition 5.4(vii) and (iii) to T and T^* yields

$$T^*(\mathcal{H}) \subseteq (\operatorname{Ker} T)^\perp, \tag{5.2}$$

$$T(\mathcal{G}) \subseteq (\operatorname{Ker} T^*)^\perp, \tag{5.3}$$

$$(T(\mathcal{G}))^\perp \subseteq \operatorname{Ker} T^*, \tag{5.4}$$

$$(T^*(\mathcal{H}))^\perp \subseteq \operatorname{Ker} T. \tag{5.5}$$

(5.2), combined with (5.5), entails the first assertion, while (5.4), combined with (5.3), entails the second assertion. The assertions (iii) and (iv) are obtained from (i) and (ii) respectively, making use of Corollary 2.17(iii). □

5.2. **Isometric, unitary, self-adjoint and projection operators.**

Definition 5.6. Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules. A continuous $\widetilde{\mathbb{C}}$-linear operator $T : \mathcal{G} \to \mathcal{H}$ is said to be isometric if $\|Tu\| = \|u\|$ for all $u \in \mathcal{G}$.

Lemma 5.7. *Any* $\widetilde{\mathbb{C}}$*-sesquilinear form* $a : \mathcal{G} \times \mathcal{G} \to \widetilde{\mathbb{C}}$ *is determined by its values on the diagonal, in the sense that*

$$a(u,v) = \frac{1}{4}\big[a(u+v,u+v) - a(u-v,u-v) + ia(u+iv,u+iv) - ia(u-iv,u-iv)\big]$$

for all $u, v \in \mathcal{G}$.

Proposition 5.8. *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be Hilbert* $\widetilde{\mathbb{C}}$*-modules and* $T : \mathcal{G} \to \mathcal{H}$ *be a continuous* $\widetilde{\mathbb{C}}$*-linear operator with an adjoint. The following assertions are equivalent:*

(i) T *is isometric;*

(ii) $T^*T = I$;

(iii) $(Tu|Tv) = (u|v)$ *for all* $u, v \in \mathcal{G}$.

Proof. $(i) \Rightarrow (ii)$ By definition of an isometric operator and an adjoint operator we have that $(u|u) = (Tu|Tu) = (T^*Tu|u)$. Hence, $(T^*Tu - Iu|u) = 0$. Since the form $(u,v) \to (T^*Tu - Iu|v)$ is $\widetilde{\mathbb{C}}$-sesquilinear, from Lemma 5.7 we conclude that $(T^*Tu - Iu|v) = 0$ for all u, v; that is, $T^*T = I$. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are immediate. □

More generally, from Lemma 5.7 we have that (i) is equivalent to (iii) for any continuous $\widetilde{\mathbb{C}}$-linear operator $T : \mathcal{G} \to \mathcal{H}$ even when the adjoint T^* does not exist.

Proposition 5.9. *The range of an isometric operator $T : \mathcal{G} \to \mathcal{H}$ between Hilbert $\widetilde{\mathbb{C}}$-modules is a closed $\widetilde{\mathbb{C}}$-submodule of $\mathcal{H}$.*

Proof. Let $v \in \overline{T(\mathcal{G})}$. There exists a sequence $(u_n)_n$ of elements of $\mathcal{G}$ such that $Tu_n \to v$ in $\mathcal{H}$. By definition of isometric operator we obtain that $(u_n)_n$ is a Cauchy sequence in $\mathcal{G}$ and therefore it is convergent to some $u \in \mathcal{G}$. It follows that $v = Tu$. □

Proposition 5.10. *Let $T\colon \mathcal{G} \to \mathcal{H}$ be a continuous $\widetilde{\mathbb{C}}$-linear operator between Hilbert $\widetilde{\mathbb{C}}$-modules. If T^* exists and there exists a continuous $\widetilde{\mathbb{C}}$-linear operator $S\colon \mathcal{H} \to \mathcal{G}$ such that $T^*TS = T^*$, then $T(\mathcal{G})$ is closed and edged. Moreover, $P_{T(\mathcal{G})} = TS$.*

Proof. Let $u \in \mathcal{H}$. Then $T^*(u - TSu) = T^*u - T^*u = 0$, so by Proposition 5.5, $u = (u - TSu) + TSu \in \operatorname{Ker} T^* + T(\mathcal{G}) = T(\mathcal{G})^\perp + T(\mathcal{G})$. By Corollary 2.17, $T(\mathcal{G})$ is closed and edged, and $P_{T(\mathcal{G})} = TS$. □

Corollary 5.11. *Let $\mathcal{G}$ and $\mathcal{H}$ be Hilbert $\widetilde{\mathbb{C}}$-modules and T be an isometric operator with adjoint. Then, $T(\mathcal{G})$ is closed and edged.*

Proof. Apply Proposition 5.10 to T with $S = T^*$. □

Example 5.12. A basic operator $T : \mathcal{G}_H \to \mathcal{G}_H$ given by a net of isometric operators $(T_\varepsilon)_\varepsilon$ on H is clearly isometric on $\mathcal{G}_H$. In particular, by the corollary above the range $T(\mathcal{G}_H)$ is a closed and edged $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}_H$.

Definition 5.13. Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module and $T : \mathcal{G} \to \mathcal{G}$ be a continuous $\widetilde{\mathbb{C}}$-linear operator with an adjoint. T is unitary if and only if $T^*T = TT^* = I$.

Proposition 5.14. *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module and $T : \mathcal{G} \to \mathcal{G}$ be a continuous $\widetilde{\mathbb{C}}$-linear operator with an adjoint. The following conditions are equivalent:*

(i) *T is unitary;*
(ii) *T^* is unitary;*
(iii) *T and T^* are isometric;*
(iv) *T is isometric and T^* is injective;*
(v) *T is isometric and surjective;*
(vi) *T is bijective and $T^{-1} = T^*$.*

Proof. By Proposition 5.8 it is clear that (i), (ii) and (iii) are equivalent. Since any isometric operator is injective we have that (iii) implies (iv).

$(iv) \Rightarrow (v)$ By Corollary 5.11 we know that $T(\mathcal{G})$ is a closed and edged $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$ and that $\operatorname{Ker} T^* = \{0\}$. Hence, by Proposition 5.5(iv) we have that $\{0\}^\perp = (\operatorname{Ker} T^*)^\perp = T(\mathcal{G})$, which means that $\mathcal{G} = T(\mathcal{G})$.

$(v) \Rightarrow (vi)$ T is isometric and surjective. Thus, it is bijective. Moreover, $T^*T = I = TT^{-1}$. Thus, $T^* = T^*(TT^{-1}) = (T^*T)T^{-1} = T^{-1}$. The fact that (vi) implies (i) is clear. □

Definition 5.15. Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module and $T : \mathcal{G} \to \mathcal{G}$ be a continuous and $\widetilde{\mathbb{C}}$-linear operator. T is said to be self-adjoint if $(Tu|v) = (u|Tv)$ for all $u, v \in \mathcal{G}$.

If T is self-adjoint, then the adjoint operator T^* exists and coincides with T.

Proposition 5.16. *The following conditions are equivalent:*

(i) T *is self-adjoint;*
(ii) $(Tu|u) = (u|Tu)$ *for all* $u \in \mathcal{G}$*;*
(iii) $(Tu|u) \in \widetilde{\mathbb{R}}$ *for all* $u \in \mathcal{G}$*.*

Proof. We prove that (iii) implies (i). By Lemma 5.7 we can write $(Tu|v)$ as

$$\frac{1}{4}\big[\,(T(u+v)|u+v) - (T(u-v)|u-v)\,\big] + i\frac{1}{4}\big[\,(T(u+iv)|u+iv) - (T(u-iv)|u-iv)\,\big].$$

Since each scalar product belongs to $\widetilde{\mathbb{R}}$ and therefore $(Tw|w) = (w|Tw)$ for all $w \in \mathcal{G}$, we obtain that $(Tu|v) = (u|Tv)$. □

We leave it to the reader to prove the following proposition.

Proposition 5.17. *Let* $\mathcal{G}$ *be a Hilbert* $\widetilde{\mathbb{C}}$*-module and let* $S, T : \mathcal{G} \to \mathcal{G}$ *be continuous* $\widetilde{\mathbb{C}}$*-linear operators.*

(i) *If* S, T *are self-adjoint, then* $S + T$ *is self-adjoint;*
(ii) *if* T *is self-adjoint and* $\alpha \in \widetilde{\mathbb{R}}$*, then* αT *is self-adjoint;*
(iii) *if* T^* *exists, then* T^*T *and* $T + T^*$ *are self-adjoint;*
(iv) *if* S *and* T *are self-adjoint, then* ST *is self-adjoint if and only if* $ST = TS$*.*

Note that Proposition 5.5 can be stated for self-adjoint operators on a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ by replacing T^* with T.

Example 5.18. There are self-adjoint operators whose range is not edged. Indeed, let $\beta \in \widetilde{\mathbb{R}}$ be as in Example 3.9 and $T : \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}} : u \to \beta u$. T is self-adjoint, but $T(\widetilde{\mathbb{C}}) = \beta\widetilde{\mathbb{C}}$ is not edged [25].

Definition 5.19. A continuous $\widetilde{\mathbb{C}}$-linear operator $T : \mathcal{G} \to \mathcal{G}$ on a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is called a projection if it is self-adjoint and $T = TT$.

Note that when M is a closed and edged $\widetilde{\mathbb{C}}$-submodule of $\mathcal{G}$, then the corresponding P_M is a projection in the sense of Definition 5.19. Indeed, by Proposition 2.14(iii) P_M is idempotent, and combining Corollary 2.17(v) with Proposition 5.16 we have that P_M is self-adjoint. We prove the converse in the following proposition.

Proposition 5.20. *If* T *is a projection, then* $T(\mathcal{G})$ *is an edged and closed* $\widetilde{\mathbb{C}}$*-submodule of* $\mathcal{G}$ *and* $T = P_{T(\mathcal{G})}$*.*

Proof. We apply Proposition 5.10 with $S = I$. □

6. Lax-Milgram theorem for Hilbert $\widetilde{\mathbb{C}}$-modules

As in the classical theory of Hilbert spaces we prove that for any $f \in \mathcal{G}$ the problem

$$a(u, v) = (v|f)\,, \qquad \text{for all } v \in \mathcal{G},$$

can be uniquely solved in $\mathcal{G}$ under suitable hypotheses on the $\widetilde{\mathbb{C}}$-sesquilinear form a. In this way, we obtain a Lax-Milgram theorem for Hilbert $\widetilde{\mathbb{C}}$-modules.

Definition 6.1. A $\widetilde{\mathbb{C}}$-sesquilinear form a on a Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}$ is coercive if there exists an invertible $\alpha \in \widetilde{\mathbb{R}}$ with $\alpha \geq 0$ such that

$$a(u,u) \geq \alpha \|u\|^2 \tag{6.1}$$

for all $u \in \mathcal{G}$.

Theorem 6.2. *Let $\mathcal{G}$ be a Hilbert $\widetilde{\mathbb{C}}$-module and $g : \mathcal{G} \to \mathcal{G}$ be a $\widetilde{\mathbb{C}}$-linear continuous map such that $g(\mathcal{G})$ is edged. Let a be the $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G}$ defined by $a(u,v) = (u|g(v))$. If a is coercive, then for all $f \in \mathcal{G}$ there exists a unique $u \in \mathcal{G}$ such that*

$$a(v,u) = (v|f)$$

for all $v \in \mathcal{G}$.

Proof. We want to prove that the map g is an isomorphism on $\mathcal{G}$. We begin by observing that the coercivity of a, combined with the Cauchy-Schwarz inequality, yields for all $u \in \mathcal{G}$,

$$\alpha \|u\|^2 \leq |a(u,u)| = |\,(u|g(u))\,| \leq \|u\| \|g(u)\|. \tag{6.2}$$

By applying Lemma 2.19 it follows that

$$\alpha \|u\| \leq \|g(u)\|. \tag{6.3}$$

This means that g is an isomorphism of $\mathcal{G}$ onto $g(\mathcal{G})$. It remains to prove that g is surjective. The $\widetilde{\mathbb{C}}$-submodule $g(\mathcal{G})$ is closed. Indeed, if $g(u_n) \to v \in \mathcal{G}$, then from (6.3) we have that $(u_n)_n$ is a Cauchy sequence in $\mathcal{G}$ converging to some $u \in \mathcal{G}$. Since g is continuous we conclude that $v = g(u)$. In addition, $g(\mathcal{G})$ is edged by assumption and (6.2) entails $g(\mathcal{G})^\perp = \{0\}$. Hence, by Corollary 2.17, $g(\mathcal{G})$ coincides with $\mathcal{G}$. Now let $f \in \mathcal{G}$. We have proved that there exists a unique $u \in \mathcal{G}$ such that $f = g(u)$. Thus, $a(v,u) = (v|g(u)) = (v|f)$ for all $v \in \mathcal{G}$. $\square$

Note that when C is a subspace of H the corresponding space $\mathcal{G}_C$ of generalized functions based on C is canonically embedded into $\mathcal{G}_H$.

Lemma 6.3. *Let H be a Hilbert space, C be a subspace of H, $\alpha \in \widetilde{\mathbb{R}}$ be positive and invertible, and a be a basic $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G}_H$. The following assertions are equivalent:*

(i) *$a(u,u) \geq \alpha \|u\|^2$ for all $u \in \mathcal{G}_C$;*

(ii) *for all representatives $(a_\varepsilon)_\varepsilon$ of a and $(\alpha_\varepsilon)_\varepsilon$ of α and for all $q \in \mathbb{N}$ there exists $\eta \in (0,1]$ such that*

$$a_\varepsilon(u,u) \geq (\alpha_\varepsilon - \varepsilon^q) \|u\|^2$$

for all $u \in C$ and $\varepsilon \in (0,\eta]$;

(iii) *for all representatives $(a_\varepsilon)_\varepsilon$ of a there exists a representative $(\alpha_\varepsilon)_\varepsilon$ of α and a constant $\eta \in (0,1]$ such that*

$$a_\varepsilon(u,u) \geq \alpha_\varepsilon \|u\|^2$$

for all $u \in C$ and $\varepsilon \in (0,\eta]$.

Proof. It is clear that (iii) implies (i). We begin by proving that (ii) implies (iii). Let $(\alpha'_\varepsilon)_\varepsilon$ be a representative of α. Assume that there exists a decreasing sequence $(\eta_q)_q$ tending to 0 such that $a_\varepsilon(u,u) \geq (\alpha'_\varepsilon - \varepsilon^q)\|u\|^2$ for all $u \in C$ and $\varepsilon \in (0,\eta_q]$. The net $n_\varepsilon = \varepsilon^q$ for $\varepsilon \in (\eta^{q+1}, \eta^q]$ is negligible, and therefore $\alpha_\varepsilon = \alpha'_\varepsilon - n_\varepsilon$ satisfies the inequality of the assertion (iii) on the interval $(0,\eta_0]$.

Note that the first assertion is equivalent to $e_S a(u,u) \geq \alpha\|u\|^2 e_S$ for all $S \subseteq (0,1]$. We now want to prove that if

(6.4)
$$\exists (a_\varepsilon)_\varepsilon\, \exists (\alpha_\varepsilon)_\varepsilon\, \exists q \in \mathbb{N}\, \forall \eta \in (0,1]\, \exists \varepsilon \in (0,\eta]\, \exists u \in C \qquad a_\varepsilon(u,u) < (\alpha_\varepsilon - \varepsilon^q)\|u\|^2,$$

then we can find $S \subseteq (0,1]$ and $u \in \mathcal{G}_C$ such that $e_S a(u,u) < \alpha\|u\|^2 e_S$. From (6.4) it follows that there exists a decreasing sequence $(\varepsilon_k)_k \subseteq (0,1]$ converging to 0 and a sequence $(u_{\varepsilon_k})_k$ of elements of C with norm 1 such that

$$a_{\varepsilon_k}(u_{\varepsilon_k}, u_{\varepsilon_k}) < \alpha_{\varepsilon_k} - \varepsilon_k^q.$$

Let us fix $x \in C$ with $\|x\| = 1$. The net $v_\varepsilon = u_{\varepsilon_k}$ when $\varepsilon = \varepsilon_k$, and $v_\varepsilon = x$ otherwise generates an element $v = [(v_\varepsilon)_\varepsilon]$ of $\mathcal{G}_C$ with $\widetilde{\mathbb{R}}$-norm 1. Now let $S = \{\varepsilon_k : k \in \mathbb{N}\}$. By construction we have that

$$e_S a(v,v) = [(\chi_S a_{\varepsilon_k}(u_{\varepsilon_k}, u_{\varepsilon_k}))_\varepsilon] \leq e_S(\alpha - [(\varepsilon^q)_\varepsilon]) < e_S \alpha \|v\|^2.$$

This contradicts assertion (i). □

Proposition 6.4. *Let H be a Hilbert space, a be a basic coercive $\widetilde{\mathbb{C}}$-sesquilinear form on $\mathcal{G}_H$ and f be a basic functional on $\mathcal{G}_H$. Then there exists a unique $u \in \mathcal{G}_H$ such that $a(v,u) = f(v)$ for all $v \in \mathcal{G}_H$.*

Proof. By applying Proposition 4.6 to the $\widetilde{\mathbb{C}}$-sesquilinear form $b(u,v) := \overline{a(v,u)}$, there exists a basic map $g : \mathcal{G}_H \to \mathcal{G}_H$ such that $a(u,v) = (u|g(v))$. In order to apply Theorem 6.2 it remains to prove that $g(\mathcal{G}_H)$ is edged. By the continuity of g and the inequality (6.3), we find by Proposition 1.10 and Lemma 6.3 that $C = [(C_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$ and an invertible $\alpha = [(\alpha_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$, $\alpha \geq 0$, for which

$$\alpha_\varepsilon \|u\| \leq \|g_\varepsilon(u)\| \leq C_\varepsilon \|u\|, \forall u \in H, \forall \varepsilon \leq \eta. \tag{6.5}$$

Let us call H_ε the Hilbert space H provided with the scalar product $(u|v)_\varepsilon := (g_\varepsilon(u)|g_\varepsilon(v))$ and consider the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G} = \mathcal{M}_{(H_\varepsilon)_\varepsilon}/\mathcal{N}_{(H_\varepsilon)_\varepsilon}$ as in Proposition 2.7. By equation (6.5), a net $(u_\varepsilon)_\varepsilon$ of elements of H is moderate (resp. negligible) in $\mathcal{G}_H$ iff it is moderate (resp. negligible) in $\mathcal{G}$. Hence the map $\tilde{g}$: $\mathcal{G} \to \mathcal{G}_H$: $\tilde{g}([(u_\varepsilon)_\varepsilon]) = [g_\varepsilon(u_\varepsilon)_\varepsilon]$ is a well-defined isometric $\widetilde{\mathbb{C}}$-linear operator with $\tilde{g}(\mathcal{G}) = g(\mathcal{G}_H)$. Let $\tilde{g}_\varepsilon$: $H_\varepsilon \to H$: $\tilde{g}_\varepsilon(u) = g_\varepsilon(u)$. As $\tilde{g}_\varepsilon$ is a continuous linear map, there exists $\tilde{g}_\varepsilon^*$: $H \to H_\varepsilon$ such that $(\tilde{g}_\varepsilon(u)|v) = (u|\tilde{g}_\varepsilon^*(v))_\varepsilon$, $\forall u \in H_\varepsilon$, $\forall v \in H$ and with $\|\tilde{g}_\varepsilon^*\| = \|\tilde{g}_\varepsilon\|$. Hence the map $\tilde{g}^*$: $\mathcal{G}_H \to \mathcal{G}$: $\tilde{g}^*([(u_\varepsilon)_\varepsilon]) = [\tilde{g}_\varepsilon^*(u_\varepsilon)_\varepsilon]$ is a well-defined continuous $\widetilde{\mathbb{C}}$-linear map and is the adjoint of $\tilde{g}$. By Corollary 5.11, $\tilde{g}(\mathcal{G}) = g(\mathcal{G}_H)$ is edged in $\mathcal{G}_H$. □

7. Variational inequalities in Hilbert $\widetilde{\mathbb{R}}$-modules

In the framework of Hilbert $\widetilde{\mathbb{R}}$-modules we now study variational inequalities involving a continuous and $\widetilde{\mathbb{R}}$-bilinear form. We will make use of the results proved in the previous sections in the context of Hilbert $\widetilde{\mathbb{C}}$-modules which can be easily seen to be valid for Hilbert $\widetilde{\mathbb{R}}$-modules. We begin with a general formulation in Theorem 7.1, and then we concentrate on some internal versions in Proposition 7.3 and Theorem 7.5.

Theorem 7.1. *Let $a(u,v)$ be a symmetric, coercive and continuous $\widetilde{\mathbb{R}}$-bilinear form on a Hilbert $\widetilde{\mathbb{R}}$-module $\mathcal{G}$. Let C be a nonempty closed subset of $\mathcal{G}$ such that $\lambda C + (1-\lambda)C \subseteq C$ for all real generalized numbers $\lambda \in \{[(\varepsilon^q)_\varepsilon]\}_{q \in \mathbb{N}} \cup \{\frac{1}{2}\}$. For all $f \in \mathcal{G}$*

such that the functional $I(u) = a(u,u) - 2\,(f|u)$ has a close infimum on C in $\widetilde{\mathbb{R}}$, there exists a unique solution $u \in C$ of the following problem:

$$a(u, v-u) \geq (f|v-u) \qquad \textit{for all } v \in C. \tag{7.1}$$

Proof. Let d be the close infimum of the functional I on C and $(u_n)_n \subseteq C$ be a minimizing sequence such that $d \leq I(u_n) \leq d + [(\varepsilon^n)_\varepsilon]$. By means of the parallelogram law and the assumptions on C we obtain that

$$\begin{aligned}
\alpha\|u_n - u_m\|^2 &\leq a(u_n - u_m, u_n - u_m)\\
&= 2a(u_n,u_n) + 2a(u_m,u_m) - 4a(\frac{u_n+u_m}{2}, \frac{u_n+u_m}{2})\\
&= 2I(u_n) + 2I(u_m) - 4I(\frac{u_n+u_m}{2})\\
&\leq 2(d + [(\varepsilon^n)_\varepsilon] + d + [(\varepsilon^m)_\varepsilon] - 2d)\\
&\leq 2[(\varepsilon^{\min(m,n)})_\varepsilon].
\end{aligned}$$

Since α is invertible, it follows that $(u_n)_n$ is a Cauchy sequence, and therefore it is convergent to some $u \in C$ such that $I(u) = \lim_{n\to\infty} I(u_n) = d$.

For any $v \in C$ let us take $w = u + \lambda(v-u)$ with $\lambda = [(\varepsilon^q)_\varepsilon]$. By the properties of C we know that $w \in C$ and $I(w) \geq I(u)$. It follows that

$$\begin{aligned}
&a(u+\lambda(v-u), u+\lambda(v-u)) - 2\,(f|u+\lambda(v-u)) - a(u,u) + 2\,(f|u)\\
&\qquad = \lambda a(u, v-u) + \lambda a(v-u, u) + \lambda^2 a(v-u, v-u) - 2\lambda\,(f|v-u) \geq 0,
\end{aligned}$$

and since λ is invertible,

$$a(u, v-u) \geq (f|v-u) - \frac{1}{2}\lambda a(v-u, v-u).$$

Letting $\lambda = [(\varepsilon^q)_\varepsilon]$ tend to 0 in $\widetilde{\mathbb{R}}$ we conclude that $a(u, v-u) \geq (f|v-u)$ for all $v \in C$, or in other words that u is a solution of our problem.

Finally, assume that u_1, u_2 are both solutions in C of the variational inequality problem (7.1). Then, $a(u_1, u_1 - u_2) \leq (f|u_1 - u_2)$, $-a(u_2, u_1-u_2) \leq -(f|u_1-u_2)$ and

$$\alpha\|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0.$$

This means that $u_1 = u_2$ and that the problem (7.1) is uniquely solvable in C. $\square$

Corollary 7.2. *Let $a(u,v)$ be a symmetric, coercive and continuous $\widetilde{\mathbb{R}}$-bilinear form on a Hilbert $\widetilde{\mathbb{R}}$-module $\mathcal{G}$. For all $f \in \mathcal{G}$ such that the functional $I(u) = a(u,u) - 2\,(f|u)$ has a close infimum in $\widetilde{\mathbb{R}}$, there exists a unique solution $u \in \mathcal{G}$ of the problem*

$$a(u,v) = (f|v) \qquad \textit{for all } v \in \mathcal{G}.$$

Proof. Since Theorem 7.1 applies to the case of $C = \mathcal{G}$, we have that there exists a unique $u \in \mathcal{G}$ such that $a(u, v-u) \geq (f|v-u)$ for all $v \in \mathcal{G}$. This implies that $a(u,v) = (f|v)$ for all $v \in \mathcal{G}$. $\square$

Note that unlike Theorem 6.2, Corollary 7.2 does not require the particular structure $(u|g(v))$ for the symmetric $\widetilde{\mathbb{R}}$-bilinear form $a(u,v)$.

As a particular case of Theorem 7.1 we obtain the following result for basic, symmetric and coercive $\widetilde{\mathbb{R}}$-bilinear forms on $\mathcal{G}_H$.

Proposition 7.3. *Let H be a real Hilbert space, $(C_\varepsilon)_\varepsilon$ be a net of convex subsets of H, $C = [(C_\varepsilon)_\varepsilon]$ and a be a basic, symmetric and coercive $\widetilde{\mathbb{R}}$-bilinear form on $\mathcal{G}_H$. If $C \neq \emptyset$, then for all basic functionals f on $\mathcal{G}_H$ there exists a unique solution $u \in C$ of the problem*

$$a(u, v-u) \geq f(v-u) \qquad \text{for all } v \in C.$$

Proof. By Proposition 4.4 and Theorem 4.1, there exists $b \in \mathcal{G}_H$ such that $f(v) = (b|v)$, $\forall v \in \mathcal{G}_H$. Since C is a closed and edged subset of $\mathcal{G}_H$ such that $\lambda C + (1-\lambda)C \subseteq C$ for all real generalized numbers $\lambda \in \{[(\varepsilon^q)_\varepsilon]\}_{q\in\mathbb{N}} \cup \{\frac{1}{2}\}$, in order to apply Theorem 7.1 it suffices to prove that the functional $I(u) = a(u,u) - 2\,(b|u)$ has a close infimum on C in $\widetilde{\mathbb{R}}$. We fix representatives $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ of a and b respectively, and we denote the corresponding net of functionals by $(I_\varepsilon)_\varepsilon$. From the coercivity of a and Lemma 6.3 it follows that for each sufficiently small ε the inequality

$$I_\varepsilon(w) \geq \alpha'_\varepsilon \|w\| - 2c_\varepsilon \|w\| = \Big(\sqrt{\alpha'_\varepsilon}\|w\| - \frac{c_\varepsilon}{\sqrt{\alpha'_\varepsilon}}\Big)^2 - \frac{1}{\alpha'_\varepsilon}{c_\varepsilon}^2 \geq -\frac{1}{\alpha'_\varepsilon}{c_\varepsilon}^2 \tag{7.2}$$

holds on H, where $\alpha'_\varepsilon = \alpha_\varepsilon - \varepsilon^q$, $(c_\varepsilon)_\varepsilon$ is a representative of $c = \|b\|$ and $q \in \mathbb{N}$. Hence, I_ε has an infimum d_ε on C_ε such that $-{c_\varepsilon}^2/\alpha'_\varepsilon \leq d_\varepsilon$. Let $v_\varepsilon \in C_\varepsilon$ be such that $I_\varepsilon(v_\varepsilon) \leq d_\varepsilon + \varepsilon^{1/\varepsilon}$. From (7.2) we see that for every moderate net of real numbers $(\lambda_\varepsilon)_\varepsilon$ there exists a moderate net $(\mu_\varepsilon)_\varepsilon$ such that $I_\varepsilon(u_\varepsilon) \geq \lambda_\varepsilon$ as soon as $u_\varepsilon \in H$ and $\|u_\varepsilon\|^2 \geq \mu_\varepsilon$. Applying this to $\lambda_\varepsilon = 1 + d_\varepsilon$, we conclude that the net $(\|v_\varepsilon\|^2)_\varepsilon$ is moderate and $v = [(v_\varepsilon)_\varepsilon] \in C$. It follows that the functional I reaches its minimum $d = [(d_\varepsilon)_\varepsilon]$ on C in v. The uniqueness of the solution follows as in the proof of Theorem 7.1. □

Remark 7.4. Note that Proposition 7.3 makes use of the completeness of $\mathcal{G}_H$ which holds even if H is not complete (see [7, Proposition 3.4]).

We extend now Proposition 7.3 to $\widetilde{\mathbb{R}}$-bilinear forms which are not necessarily symmetric by making use of some contraction techniques.

Theorem 7.5. *Let H be a real Hilbert space, $(C_\varepsilon)_\varepsilon$ be a net of convex subsets of H, $C = [(C_\varepsilon)_\varepsilon]$ and a be a basic and coercive $\widetilde{\mathbb{R}}$-bilinear form on $\mathcal{G}_H$. If $C \neq \emptyset$, then for all basic functionals f on $\mathcal{G}_H$ there exists a unique solution $u \in C$ of the problem*

$$a(u, v-u) \geq f(v-u) \qquad \text{for all } v \in C.$$

Proof. By Proposition 4.4 and Theorem 4.1, there exists $c \in \mathcal{G}_H$ such that $f(v) = (c|v)$, $\forall v \in \mathcal{G}_H$. By Proposition 4.6 and Theorem 4.5, there exists a basic $\widetilde{\mathbb{R}}$-linear map T: $\mathcal{G}_H \to \mathcal{G}_H$ such that $a(u,v) = (Tu|v)$, $\forall u, v \in \mathcal{G}_H$. We look for $u \in C$ satisfying the inequality

$$(Tu|v-u) \geq (c|v-u)\,, \qquad \forall v \in C.$$

For any $\rho \in \widetilde{\mathbb{R}}$, with $\rho \geq 0$ invertible, the inequality is equivalent with

$$((\rho c - \rho Tu + u) - u|v-u) \leq 0, \qquad \forall v \in C.$$

By Proposition 2.21(i), C is closed and edged; further, $\lambda C + (1-\lambda)C \subseteq C$, $\forall \lambda \in \widetilde{\mathbb{R}}$ with $0 \leq \lambda \leq 1$. So by Proposition 2.15 we look for $u \in \mathcal{G}_H$ with $u = P_C(\rho c - \rho Tu + u)$ (for a suitable ρ that will be determined below).

By the basic structure of T we know that there exists a moderate net $(M_\varepsilon)_\varepsilon$ and $\eta_1 \in (0,1]$ such that $\|T_\varepsilon u\| \leq M_\varepsilon \|u\|$, $\forall u \in H$, $\forall \varepsilon \in (0,\eta_1]$. By coercivity of a, there exists a moderate net $(\alpha_\varepsilon)_\varepsilon$ and $m \in \mathbb{N}$ with $\alpha_\varepsilon \geq \varepsilon^m$, $\forall\varepsilon$ and there exists $\eta_2 \in (0,1]$, such that $(T_\varepsilon u|u) \geq \alpha_\varepsilon \|u\|^2$, $\forall u \in H$, $\forall\varepsilon \in (0,\eta_2]$ (Lemma 6.3). Let $\eta = \min(\eta_1,\eta_2)$. Fix $\varepsilon \in (0,\eta]$. Let $\rho_\varepsilon = \frac{\alpha_\varepsilon}{M_\varepsilon^2}$ and

$$S_\varepsilon : \overline{C}_\varepsilon \to \overline{C}_\varepsilon : S_\varepsilon(v) = P_{\overline{C}_\varepsilon}(\rho_\varepsilon c_\varepsilon - \rho_\varepsilon T_\varepsilon v + v).$$

For v_1, $v_2 \in \overline{C}_\varepsilon$, by the properties of $P_{\overline{C}_\varepsilon}$,

$$\|S_\varepsilon(v_1) - S_\varepsilon(v_2)\| \leq \|(v_1 - v_2) - \rho_\varepsilon(T_\varepsilon v_1 - T_\varepsilon v_2)\|,$$

so

$$\begin{aligned}\|S_\varepsilon(v_1) - S_\varepsilon(v_2)\|^2 &\leq \|v_1 - v_2\|^2 - 2\rho_\varepsilon\,(T_\varepsilon v_1 - T_\varepsilon v_2|v_1 - v_2) + \rho_\varepsilon^2\|T_\varepsilon v_1 - T_\varepsilon v_2\|^2\\ &\leq (1 - 2\rho_\varepsilon\alpha_\varepsilon + \rho_\varepsilon^2 M_\varepsilon^2)\|v_1 - v_2\|^2 = \left(1 - \frac{\alpha_\varepsilon^2}{M_\varepsilon^2}\right)\|v_1 - v_2\|^2.\end{aligned}$$

So S_ε is a contraction. Let $w \in C$ with representative $(w_\varepsilon)_\varepsilon$, $w_\varepsilon \in C_\varepsilon$, $\forall\varepsilon$. Denoting the contraction constant by $k_\varepsilon = \big(1 - \frac{\alpha_\varepsilon^2}{M_\varepsilon^2}\big)^{1/2}$, by the properties of a contraction, $\|S_\varepsilon^n(w_\varepsilon) - w_\varepsilon\| \leq \frac{1}{1-k_\varepsilon}\|S_\varepsilon(w_\varepsilon) - w_\varepsilon\|$, $\forall n \in \mathbb{N}$. In particular, for the fixed point u_ε of S_ε in $\overline{C}_\varepsilon$, then also $\|u_\varepsilon - w_\varepsilon\| \leq \frac{1}{1-k_\varepsilon}\|S_\varepsilon(w_\varepsilon) - w_\varepsilon\|$. Hence

$$\|u_\varepsilon\| \leq \|w_\varepsilon\| + \frac{1}{1-k_\varepsilon}\|S_\varepsilon(w_\varepsilon) - w_\varepsilon\|.$$

Now there exists $m \in \mathbb{N}$ such that $k_\varepsilon^2 \leq 1 - \varepsilon^m$, $\forall\varepsilon$; hence $k_\varepsilon \leq \sqrt{1-\varepsilon^m} \leq 1 - \frac{\varepsilon^m}{2}$, $\forall\varepsilon$. Further, if $\rho = [(\rho_\varepsilon)_\varepsilon]$, then $[(S_\varepsilon(w_\varepsilon))_\varepsilon] = P_C(\rho c - \rho T w + w)$ by Proposition 2.21. So $(\|u_\varepsilon\|)_\varepsilon$ is a moderate net, and hence $(u_\varepsilon)_\varepsilon$ represents some $u \in C$. Similarly, as $S_\varepsilon(u_\varepsilon) = u_\varepsilon$, $\forall\varepsilon \leq \eta$, we have $u = P_C(\rho c - \rho T u + u)$ for $\rho = \big[\big(\frac{\alpha_\varepsilon}{M_\varepsilon^2}\big)_\varepsilon\big] \in \widetilde{\mathbb{R}}$ with $\rho \geq 0$ and invertible, as required.

The uniqueness of the solution follows as in the proof of Theorem 7.1. $\square$

8. Applications

We conclude the paper by applying the theorems on variational equalities and inequalities of Section 7 to some concrete problems coming from partial differential operators with highly singular coefficients. The generalized framework within which we work allows us to approach problems which are not solvable classically and to get results consistent with the classical ones when the latter exist.

8.1. **The generalized obstacle problem.** In the sequel $\Omega \subseteq \mathbb{R}^n$ is assumed to be open, bounded and connected with smooth boundary $\partial\Omega$. We consider a net $(\psi_\varepsilon)_\varepsilon \in H^1(\Omega)^{(0,1]}$ such that $\psi_\varepsilon \leq 0$ a.e. on $\partial\Omega$ for all ε, and we define the set

$$C_\psi = \{[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{H_0^1(\Omega)} : \forall\varepsilon \quad u_\varepsilon \geq \psi_\varepsilon \text{ a.e. on } \Omega\}.$$

One can easily see that C_ψ is a nonempty internal subset of the Hilbert $\widetilde{\mathbb{C}}$-module $\mathcal{G}_{H_0^1(\Omega)}$ given by a net of convex subsets of $H_0^1(\Omega)$. Note that the net $(\psi_\varepsilon)_\varepsilon$ can be generated by a highly singular obstacle ψ regularized via convolution with the mollifier φ_ε, where $\varphi \in \mathcal{C}_c^\infty(\Omega)$, $\int \varphi\,dx = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. For instance, on $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ one can take an arbitrary $\psi \in \mathcal{E}'(\Omega)$. From the structure theorem for distributions with compact support we obtain that there exists some $\eta \in (0,1]$ such that $(\psi * \varphi_\varepsilon)_{\varepsilon\leq\eta}$ is a $H^1(\Omega)$-moderate net.

Let $(a_{i,j,\varepsilon})_\varepsilon$ be moderate nets of L^∞-functions on Ω such that

$$\lambda_\varepsilon^{-1}\xi^2 \le \sum_{i,j=1}^{n} a_{i,j,\varepsilon}(x)\xi_j\xi_i \le \lambda_\varepsilon\xi^2 \tag{8.1}$$

holds for some positive and invertible $[(\lambda_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$ and for all $(x,\xi) \in \Omega \times \mathbb{R}^n$. From (8.1) it follows that

$$a(u,v) = \left[\left(\int_\Omega \sum_{i,j=1}^{n} a_{i,j,\varepsilon}(x)\partial_{x_j}u_\varepsilon(x)\partial_{x_i}v_\varepsilon(x)\,dx\right)_\varepsilon\right] \tag{8.2}$$

is a well-defined basic $\widetilde{\mathbb{R}}$-bilinear form on $\mathcal{G}_{H_0^1(\Omega)}$. Before proceeding we recall that from [7, Proposition 3.22] the space $\mathcal{G}_{H^{-1}(\Omega)}$ coincides with the set of basic functionals in $\mathcal{L}(\mathcal{G}_{H_0^1(\Omega)}, \widetilde{\mathbb{C}})$.

We are now ready to state the following theorem.

Theorem 8.1. *Let a be as in* (8.2). *For any $f \in \mathcal{G}_{H^{-1}(\Omega)}$ there exists a unique solution $u \in C_\psi$ of the problem*

$$a(u, v-u) \ge f(v-u) \qquad \text{for all } v \in C_\psi.$$

Proof. In order to apply Theorem 7.5 we have to prove that the $\widetilde{\mathbb{R}}$-bilinear form a is coercive, in the sense of Definition 6.1. The condition (8.1) on the coefficients of a and the Poincaré inequality yield that

$$a(v,v) \ge \lambda^{-1}\|v\|^2_{H_0^1(\Omega)}$$

is valid for all $v \in \mathcal{G}_{H_0^1(\Omega)}$. This completes the proof. □

Remark 8.2. When the obstacle ψ and the coefficients $a_{i,j}$ are classical, for any $f \in H^{-1}(\Omega)$ the problem $a(u, v-u) \ge f(v-u)$ can be classically settled in $H_0^1(\Omega)$ by looking for a solution u in $C_\psi^{\mathrm{cl}} := \{u \in H_0^1(\Omega) : u \ge \psi \text{ a.e. on } \Omega\}$. Let $u_0 \in C_\psi^{\mathrm{cl}}$ such that

$$a(u_0, v-u_0) \ge f(v-u_0) \tag{8.3}$$

for all $v \in C_\psi^{\mathrm{cl}}$. Note that by embedding $H^{-1}(\Omega)$ into $\mathcal{G}_{H^{-1}(\Omega)}$ by means of $f \to [(f)_\varepsilon]$, we can study the previous obstacle problem in the generalized context of $\mathcal{G}_{H_0^1(\Omega)}$. By Theorem 8.1 we know that there exists a unique $u \in C_\psi := \{[(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{H_0^1(\Omega)} : \forall\varepsilon \quad v_\varepsilon \ge \psi \text{ a.e. on } \Omega\}$ such that

$$a(u, v-u) \ge [(f)_\varepsilon](v-u) \tag{8.4}$$

for all $v \in C_\psi$. By the fact that (8.4) is uniquely solvable it follows that u coincides with the classical solution, i.e., $u = [(u_0)_\varepsilon]$. Indeed, since for any $v \in C_\psi$ we can find a representative $(v_\varepsilon)_\varepsilon$ such that $v_\varepsilon \in C_\psi^{\mathrm{cl}}$ for all ε, from (8.3) we have that

$$a(u_0, v_\varepsilon - u_0) \ge f(v_\varepsilon - u_0)$$

for all $v = [(v_\varepsilon)_\varepsilon] \in C_\psi$.

Example 8.3. When the coefficients $a_{i,j}$ are not bounded, the obstacle problem is in general not solvable in the Sobolev space $H_0^1(\Omega)$. In this case one can think of regularizing the coefficients by convolution with a mollifier φ_ε and looking for a generalized solution in some subset of $\mathcal{G}_{H_0^1(\Omega)}$. For instance, let μ_i be finite measures on $\mathbb{R}^n$ with $\mu_i \ge c\chi_V$, $i = 1, \dots, n$, where V is a neighbourhood of $\overline{\Omega}$, χ_V denotes the characteristic function of V and $c \in \mathbb{R}$, $c > 0$. Let us take $a_{i,j} = 0$ when $i \neq j$ and $a_{i,i} = \mu_i$ for $i, j = 1, \dots, n$. If φ is a nonnegative function in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\int \varphi = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, we obtain for sufficiently small ε and $x \in \Omega$ that

$$c \leq \mu_i * \varphi_\varepsilon(x) \leq \mu_i(\mathbb{R}^n)\|\varphi_\varepsilon\|_{L^\infty} \leq c'\varepsilon^{-n}$$

for some constant $c' \in \mathbb{R}$ depending on φ and the measures μ_i. It follows that setting $a_{i,i,\varepsilon}(x) = \mu_i * \varphi_\varepsilon(x)$, the net

$$\int_\Omega \sum_{i=1}^n a_{i,i,\varepsilon}(x)\partial_{x_i}u(x)\partial_{x_i}v(x)\,dx$$

of bilinear forms on $H_0^1(\Omega)$ generates a basic and coercive $\widetilde{\mathbb{R}}$-bilinear form on $\mathcal{G}_{H_0^1(\Omega)}$. For a generalized C_ψ as at the beginning of this subsection and any $f \in \mathcal{G}_{H^{-1}(\Omega)}$, the corresponding obstacle problem is uniquely solvable.

8.2. **A generalized Dirichlet problem.** We want to study the homogeneous Dirichlet problem

$$-\nabla \cdot (A\nabla u) + a_0 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{8.5}$$

where $A = [(A_\varepsilon)_\varepsilon]$ and $a_0 = [(a_{0,\varepsilon})_\varepsilon]$ are $\mathcal{G}_{L^\infty(\Omega)}$ generalized functions satisfying the following conditions. There exist some positive moderate nets $(\lambda_\varepsilon)_\varepsilon$, with moderate inverse $(\lambda_\varepsilon^{-1})_\varepsilon$, and $(\mu_\varepsilon)_\varepsilon$ such that for all $x \in \Omega$ and $\varepsilon \in (0,1]$,

$$\lambda_\varepsilon^{-1} \leq A_\varepsilon(x) \leq \lambda_\varepsilon \tag{8.6}$$

and

$$0 \leq a_{0,\varepsilon}(x) \leq \mu_\varepsilon. \tag{8.7}$$

Let $f \in \mathcal{G}_{H^{-1}(\Omega)}$. We formulate the problem (8.5) in $\mathcal{G}_{H^{-1}(\Omega)}$. Its variational formulation is given within the Hilbert $\widetilde{\mathbb{R}}$-module $\mathcal{G}_{H_0^1(\Omega)}$ in terms of the equation

$$a(u,v) = f(v) \qquad \text{for all } v \in \mathcal{G}_{H_0^1(\Omega)}, \tag{8.8}$$

where

$$a(u,v) = \left[\left(\int_\Omega A_\varepsilon(x)\nabla u_\varepsilon(x)\cdot\nabla v_\varepsilon(x)\,dx + \int_\Omega a_{0,\varepsilon}(x)u_\varepsilon(x)v_\varepsilon(x)\,dx\right)_\varepsilon\right].$$

From (8.6), (8.7) and the Poincaré inequality it follows that a is a basic and coercive $\widetilde{\mathbb{R}}$-bilinear form on $\mathcal{G}_{H_0^1(\Omega)}$. Recalling that f is a basic functional on $\mathcal{G}_{H_0^1(\Omega)}$, an application of Proposition 6.4 yields the desired solvability in $\mathcal{G}_{H_0^1(\Omega)}$.

Theorem 8.4. *For any $f \in \mathcal{G}_{H^{-1}(\Omega)}$ the variational problem* (8.8) *is uniquely solvable in $\mathcal{G}_{H_0^1(\Omega)}$.*

Example 8.5. Let us consider the one-dimensional Dirichlet problem given by the equation

$$-(H(x)u')' + \delta u = f \tag{8.9}$$

in some interval $I = (-a, a)$. One can think of approximating the singular coefficients which appear in (8.9) by means of moderate nets of L^∞ functions which satisfy the conditions (8.6) and (8.7). Let $(\nu_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]}$ be a positive moderate net with moderate inverse $(\nu_\varepsilon^{-1})_\varepsilon$ such that $\nu_\varepsilon \to 0$ if $\varepsilon \to 0$. We easily see that $A_\varepsilon(x)$, equal to 1 for $x \in (0,a)$ and to ν_ε for $x \in (-a, 0]$, fulfills (8.6), while $a_{0,\varepsilon}(x) = \varphi_\varepsilon(x)$, $x \in I$, with $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ as in Example 8.3, has the property (8.7). From Theorem

8.4 we have that for any $f \in \mathcal{G}_{H^{-1}(I)}$ the variational problem associated to (8.9) is uniquely solvable in $\mathcal{G}_{H_0^1(I)}$. It is clear that a similar result can be obtained for other functions $h \geq 0$ with zeroes instead of the Heaviside-function H.

In a similar way we can deal with the inhomogeneous problem

$$-\nabla \cdot (A\nabla u) + a_0 u = f \quad \text{in } \Omega, \qquad\qquad u = g \quad \text{on } \partial\Omega, \tag{8.10}$$

where we assume that $g \in \mathcal{G}_{\mathcal{C}^1(\partial\Omega)}$. Let $(g_\varepsilon)_\varepsilon$ be a representative of g and $(\widetilde{g_\varepsilon})_\varepsilon$ be a net in $\mathcal{M}_{H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}$ such that $g_\varepsilon = \widetilde{g_\varepsilon}$ on $\partial\Omega$. Defining,

$$C = \{v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^1(\Omega)} :\ v_\varepsilon - \widetilde{g_\varepsilon} \in H_0^1(\Omega)\},$$

a variational formulation of (8.10) is

$$a(u, v) = f(v) \qquad\qquad \text{for all } v \in C. \tag{8.11}$$

Under the assumptions (8.6) and (8.7) for A and a_0 we obtain the following result.

Theorem 8.6. *For any $f \in \mathcal{G}_{H^{-1}(\Omega)}$ the variational problem* (8.11) *is uniquely solvable in C.*

References

[1] A. Antonevich and Y. Radyno. A general method for constructing algebras of generalized functions. *Soviet Math. Dokl.*, 43(3):680–684, 1991. MR1126090 (92h:46054)

[2] J. Aragona, S. O. Juriaans, O. R. B. Oliveira, and D. Scarpalezos. Algebraic and geometric theory of the topological ring of Colombeau generalized functions. *Proc. Edin. Math. Soc.* (2), 51(3):545–564, 2007. MR2465924

[3] J. F. Colombeau. *New Generalized Functions and Multiplication of Distributions*. North-Holland, Amsterdam, 1984. MR738781 (86c:46042)

[4] J. F. Colombeau. *Elementary Introduction to New Generalized Functions*. North-Holland Mathematics Studies 113. Elsevier Science Publishers, 1985. MR808961 (87f:46064)

[5] A. Delcroix, M. F. Hasler, S. Pilipović, and V. Valmorin, *Generalized function algebras as sequence space algebras*, Proc. Amer. Math. Soc., 132:2031–2038, 2004. MR2053975 (2004m:46009)

[6] E. Farkas, M. Grosser, M. Kunzinger, and R. Steinbauer. On the foundations of nonlinear generalized functions I and II. *Mem. Amer. Math. Soc.*, 153(729), 2001. MR1848157 (2002f:46066)

[7] C. Garetto. Topological structures in Colombeau algebras: Topological $\widetilde{\mathbb{C}}$-modules and duality theory. *Acta. Appl. Math.*, 88(1):81–123, 2005. MR2163747 (2006h:46035)

[8] C. Garetto. Topological structures in Colombeau algebras: Investigation of the duals of $\mathcal{G}_c(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$. *Monatsh. Math.*, 146(3):203–226, 2005. MR2184224 (2006h:46036)

[9] C. Garetto. Closed graph and open mapping theorems for topological $\widetilde{\mathbb{C}}$-modules and applications. Math. Nachr., 282(8), 1159–1188, 2009. MR2547714

[10] C. Garetto. Microlocal analysis in the dual of a Colombeau algebra: Generalized wave front sets and noncharacteristic regularity. *New York J. Math.*, 12: 275–318, 2006. MR2259242 (2008d:46051)

[11] C. Garetto, T. Gramchev, and M. Oberguggenberger. Pseudodifferential operators with generalized symbols and regularity theory. *Electron. J. Diff. Eqns.*, 2005 (116), 2005. MR2174548 (2007f:35312)

[12] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions*, volume 537 of *Mathematics and its Applications*. Kluwer, Dordrecht, 2001.

[13] M. Grosser, M. Kunzinger, R. Steinbauer, and J. Vickers. A global theory of algebras of generalized functions. *Adv. Math.*, 166:50–72, 2002. MR1882858 (2003a:46108)

[14] G. Hörmann. First-order hyperbolic pseudodifferential equations with generalized symbols. *J. Math. Anal. Appl.*, 293(1):40–56, 2004. MR2052530 (2005f:35347)

[15] G. Hörmann and M. V. de Hoop. Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients. *Acta Appl. Math.*, 67:173–224, 2001. MR1848743 (2002g:35006)

[16] G. Hörmann, M. Oberguggenberger, and S. Pilipovic. Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients. *Trans. Amer. Math. Soc.*, 358:3363–3383, 2006. MR2218979 (2007a:35017)

[17] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume I. Academic Press, Inc., 1983. MR719020 (85j:46099)

[18] M. Kunzinger and R. Steinbauer. Generalized pseudo-Riemannian geometry. *Trans. Amer. Math. Soc.*, 354(10):4179–4199, 2002. MR1926870 (2003g:46046)

[19] M. Nedeljkov, S. Pilipović, and D. Scarpalézos. *The Linear Theory of Colombeau Generalized Functions*. Pitman Research Notes in Mathematics 385. Longman Scientific & Technical, 1998. MR1638310 (99m:46096)

[20] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations*. Pitman Research Notes in Mathematics 259. Longman Scientific & Technical, 1992. MR1187755 (94d:46044)

[21] M. Oberguggenberger and H. Vernaeve. Internal sets and internal functions in Colombeau theory. *J. Math. Anal. Appl.*, 341(1):649–659, 2008. MR2394113 (2009a:46075)

[22] D. Scarpalézos. Topologies dans les espaces de nouvelles fonctions généralisées de Colombeau. $\widetilde{\mathbb{C}}$-modules topologiques. Université Paris 7, 1992.

[23] D. Scarpalézos. Some remarks on functoriality of Colombeau's construction; topological and microlocal aspects and applications. *Integral Transform. Spec. Funct.*, 6(1–4):295–307, 1998. MR1640520 (99e:46053)

[24] D. Scarpalézos. Colombeau's generalized functions: Topological structures; microlocal properties. A simplified point of view. *I. Bull. Cl. Sci. Math. Nat. Sci. Math.*, 25:89–114, 2000. MR1842816 (2002f:46068)

[25] H. Vernaeve. Ideals in the ring of Colombeau generalized numbers. *Comm. Alg.*, 38(6):2199–2228, 2010. MR2675536

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