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# FINITE ELEMENT ANALYSIS FOR THE ELASTIC STABILITY OF THIN WALLED OPEN SECTION COLUMNS UNDER GENERALIZED LOADING 

 by
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## SYNOPSIS

The current interest in collapse characteristics brought about by crashworthiness requirements has shown the need for a better understanding and predictive capability for the thin walled open section structures. In general three possible modes exist in which a loaded thin walled open section column can buckle:

1. they can bend in the plane of one of the principal axes;
2. they can twist about the shear. centre;
3. or they can bend and twist simultaneously.

The following study was undertaken to investigate the general failure of thin walled open section structures. A literature survey was conducted and it prevailed that a basic fundamental theoretical study was vital in describing the behaviour of thin walled structural members.

The following stages of theoretical study have been completed:

1. Formulation of the stiffness matrix to predict the generalised force-displacement relationships assuming the small displacement theory in the linear elastic range.
2. Formulation of the geometric stiffness matrix to predict the buckling criteria under generalised loading and end constraints in the linear elastic range.
3. Formulation of the compound coordinate transformation matrix to relate local and global displacements or forces.
4. Preparation of the associated finite element computer program to solve general thin walled open sections structural problems.

## ACKNOWLEDGEMENTS

Drawing a dividing line between those who have assisted me in this work during the past four years and those who have not, is an impractical, if not impossible, task. Since, as I infer, such a division does not exist, 1 am able to mention only a few of my colleagues and friends; those whose influences have been a major catalyst towards the successful completion of this project.

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At this point authors generally like to thank their typists. I cringe to be an exception! Thank you Janet. Also thank you Paul for the endless cups of coffee and your profound suggestions on various aspects of life.

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## CHAPTER 1

## INTRODUCTION

Thin walled open section structural members are relatively easy to fabricate and erect. They are extensively used in the construction of ships, aircraft, automotive vehicles, civil engineering projects and in a large number of domestic and industrial applications. Due to their use these structural members are subjected to axial and transverse loads and moments producing direct axial and shear stresses.

Under the action of these stresses three possible modes exist in which a loaded thin walled open section column can buckle: (1) they can bend in the plane of one of the principal axes, (2) they can twist about the shear centre axis, or (3) they can bend and twist simultaneously. For any given member depending on its length and geometry of its cross section, one of these three modes will be critical. In the case of thin walled open sections the most obvious reason for the premature failure is that the torsional rigidity of an open section is low as it is approximately proportional to the cube of its wall thickness. Thus as a result of low torsional stiffness they can therefore buckle by twisting at loads well below the flexural buckling loads.

Generally, at the buckling point, the behaviour basically falls into two categories namely, just prior to buckling the stress throughout the structure:

1. is in the linear elastic region,
2. is partially or completely in the plastic region (post yield).

In the case of the linear elastic condition, it is assumed that the non-linearity associated with the deformations are purely due to its geometric conditions. However in the case of plasticity the non-
linearity generally occurs with the combined effect of the geometry and the state of strain in the material, which subsequently leads to the total collapse of the structure.

## LITERATURE SURVEY

Columns have been used for centuries as compression members in buildings. In the early period, columns were designed empirically and their ultimate strength was determined entirely by the crushing strength of the material similar to that of the fracture strength in tension members. However, it was vaguely understood that the strength is somehow related to the column length and the cross-section. Van Musschenbrock ${ }^{(61)}$ (1729) first recognised this and presented an empirical formula for column strength in terms of column length 'l'.

The theory of elastic flexural buckling of concentrically loaded columns was first formulated by Euler. Wagner was the first to introduce open sections with the possibility of torsion. However, in his theory Wagner assumed arbitrarily, that the centre of rotation coincides with the centroid which, in general, is not the case. The exact solution for the torsional flexural buckling of thin-walled open sections was first presented by Bleich ${ }^{(9)}$, followed by Vlasov ${ }^{(62)}$, Goodier and Timoshenko(58). These theories are readily available in well known textbooks by Bleich ${ }^{(9)}$ and Timoshenko and Gere ${ }^{(58)}$. The analysis of thin-walled structural systems was first approached using ' the method of transfer matrices, Vlasov ${ }^{(62)}$, and the force (flexibility) method, Bazant (6). These methods are however much less versatile than the displacement (stiffness) method. The stiffness matrix of a thin walled beam element, seems to have been first presented by Krahula ${ }^{(36)}$, for the case without the geometric (initial stress) matrix. Krajinovic ${ }^{(37)}$ later included initial bending moments and initial axial force in his derivation.

Barsoum and Gallagher ${ }^{(4)}$ derived the stiffness and geometric stiffness matrices with the inclusion of axial compression, biaxial bending and torsional effects with the aid of fundamental differential equations
derived by Bleich ${ }^{(9)}$ and Timoshenko and Gere ${ }^{(58)}$. Rajasekaran ${ }^{(52)}$ appears to be the first to derive the same stiffness and geometric stiffness matrices of Barsoum and Gallagher ( 4 ) by considering the second order terms in the strain equation while maintaining the thinwalled open section beam assumptions of Vlasov( ${ }^{(62)}$. Rajasekaran's work ${ }^{(53)}$ was well recognised and attempts soon followed by Bazant and El-Nimeri ${ }^{(6) \text {, Ehouney and Kirby (17) and Kasemest and Nishino and }}$ Lee ${ }^{(33)}$ to develop the concept further. Bazant and El-Nimeri ${ }^{(6)}$ suggested the development of a curved element, with the inclusion of the geometric stiffness matrix, such that initially curved members may be analysed while maintaining the geometric compatibility at nodes, whose angles of intersections are less than $180^{\circ}$.

Recently Gaafar and Tidbury ${ }^{(19)}$ ) observed experimentally that a centroidally end-loaded cantilevered channel section in large displacement work, failed always at a predetermined distance from the fixed end. Further they also observed that the torque could vary considerably as the rotational displacement gets larger along the axial direction in an approximately linear manner. This was a useful finding as previous investigators assumed that the torsional loading remained unaltered in their analyses as the structure deformed progressively.

## PROBLEM FORMULATION

Considerable attention was paid to the two analytical techniques presented by Barsoum and Gallagher ${ }^{(4)}$ and Rajasekaran ${ }^{(52)}$. Let us first consider the technique developed by Rajasekaran ${ }^{(53)}$. He used Vlasov's ${ }^{(62)}$ assumptions for thin walled open section prismatic columns and obtained the expressions for the corresponding axial and shear strains. The second order terms were included in the strain equation in order to represent the geometric non-linearity. Finally the strain equation was substituted into the virtual work expression and by considering the state of equilibrium, the compatibility, sectional properties, the associated material properties and Vlasov's hypothesis, Rajasekaran ${ }^{(52)}$ established the loading and reaction
vectors. This suggests that Rajasekaran was now left with very little freedom to allow the external loading vector to vary between the nodes of an element, as Geafar and Tidbury (19) suggested earlier. See also equation 12.45 of ref. (19).

Unlike Rajasekaran, the method developed by Barsoum and Gallagher(4) considered that the total potential was made up of two constituents. This method utilised the total strain energy equation which was developed by Bleich (9) and the differential equation of lateral and torsional buckling of Timoshenko ${ }^{(58)}$ to form the total potential for the columns under consideration. Initially the potential due to each loading component is formulated separately and simultaneously. By the use of the theory of superposition the total potential for the entire column is established.

As mentioned previously, Gaafar and Tidbury ${ }^{(19)}$ observed an unusual failure of beams, due to the increase of torsional and thus also resulted warping loadings, due to large angular displacements. The reason for this is that Vlasov assumed that the applied torsional load remained unaltered during loading (see equation 3 of ref. 19). This assumption is incorrect in large displacement analysis. It is suggested that the external torsional and warping loadings, may therefore be approximated as linear functions between the nodes.

In comparison to the two techniques described above, the method developed by Barsoum and Gallagher $(4)$ provides flexibility and better facilities to introduce variable external loads and therefore appears to be easier to adopt than the alternative method developed by Rajasekaran.

Unfortunately, Barsoum and Gallagher (4) made several errors and misjudgements in their analysis, and they are as follows:

1. From equation (4.30.0) of Chapter 4, it can be seen that when the axial force $F_{X}$ is applied to an arbitrary point on the cross section, the resultant bending in two principal planes and the
torsion about the longitudinal axis are coupled. However, the result and the state of equilibrium cannot be treated as if the columns were loaded by two uncoupled separate bending moments and a torque acting simultaneously. Regardless of this fact Barsoum and Gallagher ( ) allowed the force $F_{X}$ to act arbitrarily on the cross section and a solution was formulated as if the force $F_{x}$ was acting at the shear centre, thereby making bending and torsion actions independent of each other as shown in equation (4.32.0).
2. The warping moment generated by the axial load acting at the point of application on the cross section was ignored.
3. Any external warping moment loading on the system was not considered.
4. It was also assumed that the torsional and also the warping moment loads remained equal in magnitude at the two nodes of the column during loading. This is an invalid assumption, according to the experiments conducted by Gaafar and Tidbury ${ }^{(19)}$. This should also become evident by referring to equations (4.53.0) and (4.55.0).

Furthermore none of the researchers considered examining the nature of the displacements and associated forces when a beam element is placed arbitrarily orientated in space. By referring to the published work it is evident that the work carried out up to now was mainly devoted to cases where the local axial direction of the element was conveniently parallel to any one of the global axes. In order to analyse large assemblies of elements, it is crucial to investigate columns when they are loaded arbitrarily in space.

The purpose of this thesis is to investigate the fundamental effect of large displacements and to predict the failure mode of columns arbitrarily placed in space (i.e. buckling loads and their associated modes) with the inclusion of arbitrary torsional and warping moment loads into the generalized loading vector.

Because of the versatile nature of the finite element method it is desirable to develop a relevant procedure for solving large displacement problems of beams and frames, so that a structure with arbitrary geometry, complex loading and boundary conditions can be treated. Although the finite element formulation can be based on either assumed stress or displacement fields, most often the displacement based finite element formulation is applied in practice. Hence in this thesis a displacement model is used to arrive at the force displacement relationship for a beam-column element, by considering the principle of total potential energy and also the application of calculus of variations to obtain the equations of equilibrium and the state of stability. The conventional stiffness formulation for small deflections is modified by the effect of axial shortening due to (a) direct compression, (b) lateral bending, (c) flexural-torsion. This is incorporated in a geometric (incremental) stiffness formulation. The non-linear load displacement path is traced by a linearized mid-point tangent predictor procedure together with coordinate transformation at every increment in the load.

## CHAPTER 2

## PHYSICAL DIMENSIONAL BOUNDARIES OF THIN WALLED BEAMS

Prior to the study of the behaviour of beam-columns, the basic governing equations for biaxial bending and torsion of thin-walled elastic beam-columns are established first. The dimensional bounds (i.e. aspect ratios of $\frac{D}{L}$ and $\frac{B}{L}$ ), assumptions and hypotheses of which the generalized theory are based and formulated, are as follows:

1. Material is elastic and homogeneous.
2. The column is long and prismatic, typically:

$$
\begin{align*}
& \frac{D}{L}<0.1  \tag{2.1.0}\\
& \frac{B}{L}<0.1
\end{align*}
$$

3. The cross-section is thin-walled, typically:

$$
\begin{align*}
& \frac{t}{D}<0.1  \tag{2.2.0}\\
& \frac{t}{B}<0.1
\end{align*}
$$

4. The shear deformation is neglected.
5. No distortion in cross section can exist.


The adopted sign convention:


6

FIGURE 2.2.0

The analytical formulation presented here does not strictly follow the conventional right handed rule, shown in Figure 2.2.0. In Figure 2.2.2 the rotation on the $x-y$ plane, has been reversed in order to keep the analysis in it as simple as possible. In doing so, the same displacement function can be used to describe the behaviour of $x-y$, $y-z$ and $z-x$ planes and also the torsion warping characteristic along the $x$-axis.

In order to describe the sign convention fully, let us begin with the following example, as illustrated in Figure 2.3.0. The beam is simply supported and carries a uniformly distributed load $q_{y}$, applied in the negative $y$-direction.

The boundary conditions for the problem in Figure 2.3.0 are:

$$
V(x=0)=\frac{d^{2} V}{d x^{2}}(x=0)=V(x=l)=\frac{d^{2} V}{d x^{2}}(x=l)=0
$$

From Macaulay's principle, the deflection curve for the problem under consideration is given by,

$$
\begin{equation*}
V=\frac{q y^{x}}{24 E I}\left(2 \ell x^{2}-x^{3}-\ell^{3}\right) \tag{2.3.0}
\end{equation*}
$$

ぎ


## FIGURE 2.3.0

By differentiating,

$$
\begin{align*}
& v^{\prime}=\frac{q y}{24 E I} \cdot\left(6 \ell x^{2}-4 x^{3}-\ell^{3}\right)  \tag{2.4.0}\\
& v^{\prime \prime}=\frac{-q}{2 E I} \times(\ell-x)  \tag{2.5.0}\\
& v^{\prime \prime}=\frac{-q y}{2 E I}(\ell-2 x)  \tag{2.6.0}\\
& v^{\prime \prime \prime}=\frac{q y}{E I} \tag{2.7.0}
\end{align*}
$$

The five functions derived so far are now plotted as shown in Figure 2.4.0:



Fig. 2.4.2


Fig. (2.4.3)
Fig. (2.4.1)

Fig. (2.4.2)

Fig. (2.4.4)

Fig. (2.4.5)

Let us consider a similar sign convention for the case of pure torsion. The angle of twist $\theta_{x}$ and its derivatives for a thin-walled open cross section resemble the same set of equations (see reference (35)) as the one derived for bending, provided the system is subjected to a similar set of boundary conditions as shown below,

$$
\theta_{x}(x=0)=\theta_{x}^{\prime \prime}(x=0)=\theta_{x}(x=\ell)=\theta_{x}^{\prime \prime}(x=\ell)=0
$$

In this analogy, the torsional moment $M_{x}$ corresponds to the transverse shear forces $Q_{y}$ or $Q_{z}$ while the warping moment $M_{x x}$ corresponds to the bending moment in the simple beam.

By using equations (2.4.0), (2.5.0), (2.6.0), (2.7.0) and Figures (2.4.1), (2.4.2), (2.4.3), (2.4.4) and (2.4.5) the following chart is tabulated:

|  | Subjected to lateral load $Q_{y}$ | Subjected to torsional load $m_{x}$ | Subjected to lateral load $Q_{Z}$ |
| :---: | :---: | :---: | :---: |
| Displacement | $y$ | ${ }^{\text {x }}$ | $z$ |
| Slope | $\theta_{z}=-y^{\prime}$ | $\theta_{x}^{\prime}$ | $\theta_{y}=-z^{\prime}$ |
| Bending moment | $M_{z}=E I_{y y}{ }^{\text {y }}{ }^{\prime}$ | $M_{x x}=E \Gamma \theta_{x}^{\prime \prime}$ | $M_{y}=E I_{z z}{ }^{\text {® }}{ }^{\prime}$ |
| Warping moment | $=E I_{y y} y^{\prime \prime}$ |  | $=\mathrm{EI}_{z z^{\prime \prime}}$ |
| Shearing forces | $Q_{y}=M_{z}^{\prime}$ | $M_{x}=M_{x x}^{\prime}$ | $Q_{z}=M_{y}^{1}$ |
| and Torsional | $=-E I_{y y} \theta_{z}^{\prime \prime}$ | $=-E r \theta_{x}^{\prime \prime}$ | $=-E I_{z z}{ }^{\text {en }}$ |
| moments | $=-E I_{y y} y^{\prime \prime \prime}$ |  | $=-\mathrm{EI}_{\mathrm{zz}} \mathrm{z}^{\text {I' }}$ |
| Load | $q_{y}=-Q_{y}^{\prime}$ | $m_{x}=-M_{x}^{\prime}$ | $\mathrm{q}_{\mathrm{z}}=-\mathrm{Q}_{z}^{\prime}$ |
|  | $=\mathrm{EI}_{\mathrm{yy}}{ }^{\text {® }}{ }^{\prime \prime}$ | $=E r 0_{X}^{\prime \prime \prime \prime}$ | $=\mathrm{EI}_{2 z{ }^{\prime \prime} \mathrm{y}^{\prime}}$ |
|  | = EIyy ${ }^{\prime \prime \prime \prime \prime}$ |  | $=\mathrm{EI}_{\mathrm{zz}} \mathrm{z}^{\prime \prime \prime}$ |

## TORSIONAL BEHAVIOUR OF NON-CIRCULAR BARS

Let us begin the analysis by examining the torsional behaviour of prismatic bars. In the case of torsion of prismatic bars not circular in cross section, the argument that a plane cross section remains plane during deformation is no longer valid. To illustrate the argument, consider the behaviour of elements $A, B, C$ and $D$ of the bar shown in the following example:


Fig.2.5.2

FIGURE 2.5 .0

For simplicity, elements $A$ and $D$ are shown with right angled corners. As no shearing stresses exist on these elements, any component of shearing stress on element $A$, for example, would require stresses to be developed on the outside surface of the bar, which is generally not admissible.

Elements $B$ and $C$, however, can develop shearing stresses, so long as they are parallel to the boundary line ad. Again, this is because shearing stresses directed normal to any boundary, are inadmissible on the outside surface of the bar (Figure 2.5.2). It follows that when the sides of elements $B$ and $C$ acquire a shearing strain $\gamma$, elements $A$ and $D$ must undergo rigid body rotations for the sake of continuity. Hence the outside corners of elements $A$ and $B$ are displaced out of the plane of the cross sections, as indicated in Figure 2.5.3. Out of plane displacements such as those discussed above are the warping displacements. Other than the fact that the cross section warps, perhaps one of the most obvious characteristics of the behaviour, is the absence of normal stresses. No external forces or bending moments are present, and since no end constraints exist, the only stress components needed to provide the equilibrium of any transverse segment are shearing stresses in the cross sectional plane.


FIGURE 2.6.0

From the three components of the shearing stresses, only $\tau_{x y}$ and $\tau_{x z}$ can result in a twisting moment. The component $\tau_{y z}$ is zero which can be verified by examining plane areas parallel to the bar axis.

Thus it is concluded for the non-circular prismatic bar in torsion that,

$$
\begin{equation*}
\sigma_{x}=\sigma_{y}=\sigma_{z}=\tau_{y z}=0, \varepsilon_{x}=\varepsilon_{y}=\varepsilon_{z}=\gamma_{y z}=0 \tag{2.8.0}
\end{equation*}
$$

The fact that $\tau_{y z}$ is zero implies that cross sections do not distort in their own planes.

In other words, the angle between any two lines on a cross section is not changed during the deformation of the bar. This means that deformations of an element such as that indicated in Figure 2.5.0, do not exist and that a point in the zy plane merely rotates about a centre of twist.


FIGURE 2.7 .0

Application to thin-walled beam cross-section:


FIGURE 2.8.0

In thin-walled sections (closed or open), the warping displacement relative to the origin can be expressed as (see page 53 of reference 48 )

$$
\begin{equation*}
U=\frac{1}{G} \int^{\tau} x s d s-2 \theta \omega \tag{2.9.1}
\end{equation*}
$$

$t$

$$
\begin{equation*}
\gamma_{x s}=\frac{\partial u}{\partial s}+\frac{\partial \eta}{\partial x} \tag{2.9.2}
\end{equation*}
$$

where: $u$ is warping in the $x$-direction
$\eta$ is the warping in the tangential (s)- direction
$\theta$ is the twist per unit length
$\omega$ is the warping function
${ }^{\tau} \mathrm{xs}$ is the shear stress in the $s$-direction

However, in the case of an open section it is necessary to pay closer attention to the definition of $\bar{u}$. The shearing stress $\tau_{x s}$ in the case of an open section, varies linearly throughout the wall thickness and is zero at the centre line of the wall. This means that S cannot be measured along the centre line, or else the integral is zero. If, on the other hand, the maximum value of $\tau_{x s}$ is used and $S$ is measured along paths on the outside and then the inside periphery to two points on opposite sides of the wall, the integral would represent the difference between the warping displacement of these points, which is a very small quantity for thin walled sections. It appears that for open sections the integral in equation 2.9.1 is negligible in comparison with the term $2 \theta \omega$. Physically it is true that open sections are much more flexible than the closed sections. In fact, the torsional stiffness of a closed section is often an order of magnitude greater than that of an open section of similar dimensions. Thus the influence of shearing strain $\gamma_{x s}$ is negligible, and the warping displacements are due almost exclusively to the twist of the section.

$$
\therefore \quad \bar{u}=-2 \theta \omega
$$

This follows that,

$$
\begin{equation*}
\frac{\partial u}{\partial s}+\frac{\partial \eta}{\partial x}=0 \tag{2.11.0}
\end{equation*}
$$

More elaborate analysis is found in reference (48). The physical interpretation to the last expression is as follows. The tangent to the middle line at every point on the outline of the cross section remains perpendicular to the associated longitudinal fibre after torsion. This means that the bending and torsional stresses in the beam can be assumed to be uncoupled, if the beam is subjected to siumultaneous bending and torsional loadings (i.e. the behaviour of an open sectioned beam is the same as that of a solid beam).

## CHAPTER 3

## HARPING THEORY FOR THIN HALLED CROSS-SECTIONS



| where $C(0,0)$ | is the centroid |
| :--- | :--- |
| $O\left(e_{y}, e_{z}\right)$ | is the shear centre |
| $D$ | is an arbitrary point, conveniently placed to <br> measure distance $S$ |
| a $\quad$is the sliding radius |  |

FIGURE 3.1.0

In Figure 3.1.0, point ' $C$ ' represents the origin of the coordinate axes and 0 represents the point of rotation of a typical thin-walled cross-section. The objective is to establish the displacement of a point on the middle surface of the cross-section, in terms of $U_{y o}, U_{z 0}$ and $U_{x c}$ which are the transverse displacements of the shear centre, and the axial displacement of the origin respectively. The kinematics of the cross-section are illustrated in Figure 3.2.0

where: $\begin{aligned} & N \text { is the initial position } \\ & N^{\prime} \text { is the intermediate position } \\ & N^{\prime \prime} \text { is the final position } \\ & U_{y o} \text { is the translation of } 0 \text { in } y \text { direction } \\ & U_{z o} \text { is the translation of } 0 \text { in } z \text { direction }\end{aligned}$

FIGURE 3.2 .0

The displacement of an arbitrary point on the plane of cross-section can be obtained by decomposing its displacements to a biaxial translation of the shear centre followed by some rotational displacement of the cross-section about the shear centre. This can be illustrated as follows:

The rigid body translational displacement of point $N, U_{y}$ and $U_{z}$ is given by,

$$
\begin{align*}
& U_{y}=U_{y 0}-a \cos \beta\left(1-\cos \theta_{x}\right)-a \cdot \sin \beta \sin \theta_{x}  \tag{3.1.1}\\
& U_{z}=U_{z 0}+a \cos \beta \sin \theta_{x}-a \sin \beta\left(1-\cos \theta_{x}\right) \tag{3.1.2}
\end{align*}
$$

See Figure 3.2.0.
Also by referring to Figure 3.1 .0 it can be seen that,

$$
\begin{align*}
& a \cos \beta=y-e_{y}  \tag{3.2.1}\\
& a \sin \beta=z-e_{z} \tag{3.2.2}
\end{align*}
$$

By substituting equations (3.2.1) and (3.2.2), in equations (3.1.1) and (3.1.2) yields,

$$
\begin{align*}
& U_{y}=U_{y 0}-\left(z-e_{z}\right) \sin \theta_{x}-\left(y-e_{y}\right)\left(1-\cos \theta_{x}\right)  \tag{3.3.1}\\
& U_{z}=U_{z 0}+\left(y-e_{y}\right) \sin \theta_{x}-\left(z-e_{z}\right)\left(1-\cos \theta_{x}\right) \tag{3.3.2}
\end{align*}
$$

However, for small magnitudes of twisting rotations,

$$
\begin{align*}
& \sin \theta_{x} \simeq \theta_{x}  \tag{3.4.1}\\
& \cos \theta_{x} \simeq 1 \tag{3.4.2}
\end{align*}
$$

Thus it follows that equations (3.3.1) and (3.3.2) are now simplified to,

$$
\begin{align*}
& U_{y}=U_{y o}-\left(z-e_{z}\right) \theta_{x}  \tag{3.4.1}\\
& U_{z}=U_{z o}+\left(y-e_{y}\right) \theta_{x} \tag{3.4.2}
\end{align*}
$$

Equations (3.4.1) and (3.4.2) can now be used to describe the transverse displacements $U_{y}$ and $U_{z}$ of an arbitary point.

To establish the expression for the axial displacement $U_{x}$, of an arbitrary point, it is essential to investigate the tangential displacement of point $N$ in the direction of the mid-surface contour. This may be written as,

$$
\begin{equation*}
U_{t}=U_{z 0} \cos \alpha-U_{y o} \sin \alpha+p_{0} \theta_{x} \tag{3.5.0}
\end{equation*}
$$

See Figure 3.1.0 and Figure 3.2.0.

Using the assumption that no shear deformation exists in the mid plane of the cross-section:

$$
\begin{equation*}
\frac{\partial U_{t}}{\partial x}+\frac{\partial U_{x}}{\partial s}=0 \tag{3.6.0}
\end{equation*}
$$

Substitution of equation (3.5.0) in equation (3.6.0):

$$
\begin{equation*}
\frac{\partial U_{x}}{\partial s}=-U_{z 0}^{\prime} \cos \alpha+U_{y 0}^{\prime} \sin \alpha-\rho_{0} \theta_{x}^{\prime} \tag{3.7.0}
\end{equation*}
$$

$\because$
in which primes denote differentiation with respect to x . Equation (3.7.0) is now:

$$
\begin{equation*}
\left[U_{x}\right]_{D}^{s}=-U_{z 0}^{\prime} \int_{D}^{s} \cos \alpha d s+U_{y o}^{\prime} \int_{D}^{s} \sin \alpha d s-\theta_{x}^{\prime} \int_{D}^{s} \rho_{0} d s \tag{3.8.0}
\end{equation*}
$$

By referring to Figure (3.1.0) it can also be seen that,

$$
\begin{align*}
& \frac{d y}{d s}=-\sin \alpha  \tag{3.9.1}\\
& \frac{d z}{d s}=\cos \alpha
\end{align*}
$$

By substituting equations (3.9.1) and (3.9.2) into equation (3.8.0):

$$
\begin{equation*}
U_{x}(x, y, z)=U_{x D}-\left(z-z_{D}\right) U_{z o}^{\prime}-\left(y-y_{D}\right) U_{y o}^{\prime}-\omega_{D S^{\theta}}^{0}{ }_{x}^{\prime} \tag{3.10.0}
\end{equation*}
$$

where $\omega_{D S}^{0}$ is defined as the sectorial coordinate of point $S$ based on shear centre 0 and the reference point $D$ Thus,

$$
\omega_{D S}^{0}=\int_{D}^{s} \rho_{0} d s
$$

In general it is the term $\omega_{D S}^{0}$ which gives rise to warping, when structures are subjected to torsional loadings. The assumption that the plane section remains plane during biaxial bending and rotation of the plane section about the shear centre due to torsional loadings, is based on the principle of rigid body translation and rotation. Thus, although point $C$ is not on the contour, the longitudinal displacement at $C$ may be obtained by visualizing a connection to it from any point on the contour and by substituting the coordinates in equation (3.10.0) to those of the origin. Thus,

$$
\begin{equation*}
u_{x C}=u_{x D}+z_{D} u_{z o}^{\prime}+y_{D} u_{y o}^{\prime}-\omega_{D C}^{0} \theta_{x}^{\prime} \tag{3.11.0}
\end{equation*}
$$

By combining equations (3.10.0) and (3.11.0), the expression for the axial displacement equation (3.12.0) is found,

$$
\begin{equation*}
u_{x}(x, y, z)=u_{x c}-z u_{z o}^{1}-y u_{y 0}^{\prime}-\left(\omega_{D c}^{0}-\omega_{D s}^{0}\right) \theta_{x}^{\prime} \tag{3.12.0}
\end{equation*}
$$

At this point Rajasekaran (51) made the following statement. 'At a suitably selected reference point $D$ on the outline of the crosssection, the quantity $\omega_{D C}^{0}$ could vanish'. This assumption is incorrect and it can be seen by investigating the sectorial properties of the following exampie,


$$
\begin{aligned}
& d=\frac{b^{2} t_{1}}{h t+2 t_{1} b} \\
& I_{W}=\frac{b^{3} t_{1} h^{2}}{12} \cdot \frac{2 h t+b t_{1}}{h t+\cdot 2 b t_{1}}
\end{aligned}
$$

FIGURE 3.3.0

In the example shown above in Figure (3.3.0), the sectorial coordinate of the centroid is given by $-\frac{d h}{2}$. Furthermore, point $D$ is not an arbitrary point as Rajasekaran suggested, it is a unique point of any given section and its position can only be found by the unique method illustrated in reference (66) Section 2.2.1.

By differentiating equation (3.12.0) twice with respect to $x$ and neglecting all higher order terms, the expression for the total axial strain can be found, and this is shown in equation (3.13.0)

$$
\begin{equation*}
\varepsilon=u_{x c}^{\prime}-y u_{y O}^{\prime \prime}-z u_{z o}^{\prime \prime}-\left(\omega_{D c}^{0}-\omega_{D s}^{0}\right) \theta_{x}^{\prime \prime} \tag{3.13.0}
\end{equation*}
$$

where: $\varepsilon$ is the total axial strain at a point on the cross-section defined by $(y, z)$
$U_{C}^{\prime}$ is the axial strain due to axial load only
$U_{\text {yo }}^{i i}$ is the curvature about the $z$-axis
$U_{z 0}^{\prime \prime}$ is the curvature about the $y$-axis
$\theta_{x}^{\prime \prime}$ is the warping curvature about the $x$-axis

Since in the assumptions in linear elasticity only uniaxial stress is significant,

$$
\begin{equation*}
\sigma=E \varepsilon \tag{3.14.0}
\end{equation*}
$$

where $\sigma$ is the stress resulting due to $\operatorname{strain} \varepsilon$ $E$ is the Young's modulus of elasticity

By considering the state of equilibrium under the loading condition, stress and stress resultants can be found as follows:

$$
\begin{align*}
& f_{y}=\int_{A} \tau y x d A  \tag{3.15.1}\\
& f_{z}=\int_{A} \tau z x d A  \tag{3.15.2}\\
& f_{x}=\int_{A} \sigma d A  \tag{3.15.3}\\
& m_{y}=\int_{A} \sigma z d A  \tag{3.15.4}\\
& m_{x} \int_{A} \sigma y \cdot d A  \tag{3.15.5}\\
& m_{w}=\int_{A} \sigma \omega_{D s}^{0} d A
\end{align*}
$$

where $f_{y}$ is the shear force in the $y$-direction
$f_{z}$ is the shear force in the $z$-direction
$f_{x}$ is the direct force in the axial direction
$m_{y}$ is the bending moment about the $y$-axis
$m_{z}$ is the bending moment about the z-axis
$m_{w}$ is the warping moment about the $x$-axis
$\tau_{y x}$ is the shear stress in the $y$-direction
$\tau_{z x}$ is the shear stress in the $z$-direction
$A$ is the area of the entire cross-section

By combining the relationship developed in equations (3.14.0) and (3.15.0), the following matrix representation can be developed

By selecting coordinate axes $y$ and $z$ to be the principal axes, $C$ and 0 as the centroid and the shear centre and also point $D$ for the principal radius, the off-diagonal terms in equation (3.16.0) will vanish. Thus equation (3.16.0) finally reduces to,

$$
\begin{align*}
& f_{x}=\left(\int_{A} E_{t} d A\right) u_{x c}^{\prime}  \tag{3.17.1}\\
& m_{y}=-\left(\int_{A} E_{t} z^{2} d A\right) u_{z o}^{\prime \prime}  \tag{3.17.2}\\
& m_{z}=-\left(\int_{A} E_{t} y^{2} d A\right) u_{y o}^{\prime \prime}  \tag{3.17.3}\\
& m_{\omega}=-\left(\int_{A} E_{t}\left(\omega_{D C}^{0}-\omega_{D s}^{0}\right)^{2} d A\right) \theta_{x}^{\prime \prime} \tag{3.17.4}
\end{align*}
$$

Thus, by considering equation (3.17.0) shown above, the reaction forces required for a thin-walled cross-section under the generalized load condition can be found.

## CHAPTER 4

## BUCKLING ANALYSIS OF AN ELASTIC BEAM COLUMN

## INTRODUCTION

A typical configuration displaced of a thin-walled cross-section is illustrated in Figure 3.2.0 in Chapter 3. These displacements are generally obtained by independent or simultaneous application of axial and transverse moments and torsional and warping forces. As was explained in the previous chapter, during loading the shear centre is subjected to translational displacements, followed by rotation of the entire cross section about the shear centre. This suggests that it will make the analysis far simpler if all displacements and associated forces are referred to a set of axes which, in general are parallel to the principal axes and pass through the shear centre. However, referring forces to an axis system at the shear centre generally generates additional moments and warping moments. These secondary forces can easily be accounted for by a suitable transformation matrix, as described in equation (8.14.0).

With the assumptions and definitions made so far, the idealized element under consideration can be described as illustrated in Figure (4.1.0)


FIGURE 4.1 .0

An orthogonal coordinate system $x, y, z$ was chosen, so that the $y$ - and z-axes coincided with the principal axes of the cross-section and the axial direction $x$ coincided with the undeformed centroidal axis $C_{1}-C_{2}$.

Let $v, w$ denote the displacements of the centroidal axes in the $y$ - and $z$-directions respectively. $\phi$ is the angle of twist, and $u$ is the axial displacement. The quantities $u, v, w$ and $\phi$ are all assumed to be of a small order so that the square of each quantity is negligible when compared with their individual magnitudes.

The assumption that no shear deformation can exist in the middle surface (i.e. the fourth assumption in Chapter 2), leads to the definition of the angular displacements, in terms of the first derivatives of the transverse displacements (i.e. exactly the same as in the case of a solid beam under bending loading).

$$
\therefore \quad \begin{align*}
\theta_{z} & =-\frac{d V}{d x} \\
\theta_{y} & =-\frac{d w}{d x} \tag{4.1.1}
\end{align*}
$$

The element shown in Figure (4.1.0) is loaded by an axial force $F_{x}$, which acts at the shear centre, whose coordinates $y_{0}, z_{0}$ are measured from the centroid, the end shear forces and end moments, $Q_{y 1}, Q_{y 2}$, $Q_{z 1}, Q_{z 2}, M_{y 1}, M_{y 2}, M_{z 1}$ and $M_{z 2}$ acting along axes parallel to the principal axes through the shear centre. The torque $M_{x 1}$ and $M_{x 2}$ and the warping moments $M_{x x 1}$ and $M_{x x 2}$ are acting along an axis through the shear centre.

The total potential energy $\pi_{p}$ for the structural system illustrated in Figure (4.1.0) is given by,

$$
\begin{equation*}
\pi_{p}=U-V \tag{4.2.0}
\end{equation*}
$$

where $U$ is the strain energy stored during the deformation of the structure, and
$V$ is the potential (work done) by applied loads moving in the direction of associated degrees of freedom.

The method of calculating strain energy during deformation of structures is well known and the corresponding expression for the problem illustrated in Figure (4.1.0) is as shown in equation (4.3.0). The origin and proof of this expression can be found on page 158 of reference 4.

$$
U=\frac{1}{2} \int_{\ell}\left[E I_{z} V^{\prime \prime 2}+E I_{y} W^{\prime \prime 2}+E \Gamma \theta_{x}^{\prime \prime 2}+G J \theta_{x}^{\prime 2}+E A U^{2}\right] d x(4.3 .0)
$$

where, single prime denotes the first derivative $\left(\frac{d}{d x}\right)$ and double prime denotes the second derivative ( $\frac{d^{2}}{d x^{2}}$ )
$E \quad$ is the elastic modulus
G is the shear modulus
$I_{y}, I_{z}$ are second moments of area of the cross-section about its principal axes $y$ and $z$ respectively
$\Gamma$ is the torsion warping constant of the section
$J \quad$ is the St Venant torsion constant, and
A is the cross-sectional area of the section concerned.

The quantities may be identified as:
$E I_{Z} V^{\prime \prime 2}=$ the strain energy due to bending in the $y-x$ plane
$E I_{y} W^{\prime \prime 2}=$ the strain energy due to bending in the $z-x$ plane
$E r \theta_{x}{ }^{\prime 2}=$ the strain energy due to warping about the $x$-axis EAU'2 $=$ the strain energy due to axial loading.

The potential of the applied loads in buckling analysis consists of two constituents. The first part $V$, is the work done by the applied loads in the direction of associated degrees of freedom (i.e. prebuckling deformations) and thus can be shown as in equations (4.4.0)

Therefore

$$
\begin{align*}
V= & F_{x}\left(U_{1}-U_{2}\right)+Q_{y 1} V_{1}+Q_{y 2} V_{2}+Q_{z 1} W_{1}+Q_{z 2} W_{2}+M_{y 1} \theta_{y 1}+M_{y 2} \cdot \cdot y 2+ \\
& M_{z 1} \theta_{z 1}+M_{z 2}{ }^{\theta} z 2+M_{x 1^{\prime} \theta_{x 1}}+M_{x 2} \theta_{x 2}+M_{x x 1} \theta_{x 1}^{\prime}+M_{x x 2} \theta_{x 2}^{\prime} \quad \text { (4.4.0) } \tag{4.4.0}
\end{align*}
$$

See Figure (4.1.0).
where $F_{x}, U_{1}, U_{2}$ are the axial load and the displacements at nodes 1 and 2
$Q_{y 1}, Q_{y 2}, V_{1}, V_{2}$ are the transverse shear forces and the associated displacements in the $y$-direction at nodes 1 and 2
$\mathrm{Q}_{\mathrm{Z1}}, \mathrm{Q}_{\mathrm{Z2}}, \mathrm{~W}_{1}, \mathrm{~W}_{2}$ are the transverse shear forces and the associated displacements in the z-direction at nodes 1 and 2
$M_{y 1}, M_{y 2}, \theta_{y 1}, \theta_{y 2}$ are the moments and associated angular displacements in the $y$-direction at nodes 1 and 2
$M_{z 1}, M_{z 2}, \theta_{z 1}, \theta_{z 2}$ are the moments and associated angular displacements in the $z$-direction at nodes 1 and 2 $M_{x 1}, M_{x 2}, \theta_{x 1}, \theta \times 2$ are the torsional moments and associated angular displacements in the $x$-direction at nodes 1 and 2.
$M_{x x 1}, M_{x \times 2}, \theta_{x 1}^{\prime}, \theta_{x 2}^{\prime}$ are the warping moments and the associated twist gradient in the $x$-direction at nodes 1 and 2.

The utilization of the potential energy concept in the establishment of relationships for instability analysis presumes that pre-buckling deformations have occurred and are followed by buckling deformations. However, the buckling deformations are mainly due to flexural and torsional action only, and hence there must be a proportion of work done by the applied loadings in their associated buckling deformations and are given by $V$ in equation (4.5.0).
$\therefore \cdot$

$$
\begin{equation*}
\tilde{v}=\sum_{i} V_{(i)} \tag{4.5.0}
\end{equation*}
$$

The potential in the buckling deformations is formed by the following constituents:
$V_{1}$ is the bending in the $x-y$ plane due to axial load $F_{x}$
$V_{2}$ is the bending in the $x-z$ plane due to axial load $F_{x}$
$V_{3}$ is the twist about the $x$-axis due to axial load (torsional buckling)
$V_{4}$ is the bending in the $x-y$ plane due to transverse forces $Q_{y 1}$ and $Q_{y 2}$
$V_{5}$ is the bending in the $x-z$ plane due to transverse forces $Q_{z 1}$ and $Q_{z 2}$
$V_{6}$ is the bending in the $x-y$ plane due to moments $M_{z 1}$ and $M_{z 2}$
$V_{7}$ is the bending in the $x-z$ plane due to moments $M_{y 1}$ and $M_{y 2}$
$V_{8}$ is the bending in the $x-y$ and $x-z$ planes due to torsional moments $M_{x 1}$ and $M_{x 2}$
$V_{g}$ is the bending in the $x-y$ and $x-z$ planes due to warping moments $M_{x x 1}$ and $M_{x x 2}$.

Let us now begin by analysing each component independently.

Potential energy formulation for buckling in the $x-y$ and $x-z$ planes due to axial load $F_{x}$ :


Consider the column loaded by an axial thrust $F_{x}$, as shown in Figure (4.2.0). The bend in the beam causes $F_{X}$ to create a bending action as well as the compressive stress. This is a geometric non-linearity rather than a non-linearity caused by the stress-strain relationship for the material, when the yield point is exceeded.

To apply the energy method to the problem, it is necessary to consider the work done by $F_{x}$ in travelling through the distance $\Delta$, as well as $F_{X}$ in travelling through the distance $U$, calculated by direct compression. Thus the total potential energy of the system can now be expressed as follows:

$$
\begin{equation*}
V=\int_{0}^{\ell} \frac{E I z}{2}\left(V^{\prime \prime}\right)^{2} d x-F_{x} \Delta-F_{x} U \tag{4.6.0}
\end{equation*}
$$

where $\frac{E I_{z}}{2} V^{\prime \prime 2}$ is the strain energy in bending in the $x-y$ plane
$F_{x} \Delta$
$F_{x} U$$\quad$ is the potential energy due to buckling deformation, and

Assuming that the distance $\Delta$ generated by the bending is far greater than the axial shortening $U$, then the final length of the bowed beam, will be, $\ell$. Therefore,

$$
\begin{equation*}
\ell=\int d s \tag{4.7.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\ell-\Delta=\int d x \tag{4.7.2}
\end{equation*}
$$

By considering a small length ds in the bowed beam,

$$
d s^{2}=d v^{2}+d x^{2}
$$

i.e. $d s=\left[1+\left(\frac{d v}{d x}\right)^{2}\right]^{\frac{1}{2}} d x$
and by using the binomial expansion and neglecting the higher order terms, equation (4.8.1) is simplified to,

$$
\begin{equation*}
d s \simeq\left[1+\frac{1}{2}\left(\frac{d V}{d x}\right)^{2}\right] d x \tag{4.8.2}
\end{equation*}
$$

Substitution of equation (4.8.2) into equation (4.7.2) and also by integrating:

$$
\begin{equation*}
\Delta=\frac{1}{2} \int\left(\frac{d V}{d x}\right)^{2} d x \tag{4.9.0}
\end{equation*}
$$

Now by substituting equation (4.9.0) in equation (4.6.0), the potential due to application of axial load $F_{x}$, causing bending in the $x-y$ plane is obtained, thus,

$$
\begin{equation*}
V=\frac{1}{2} \int_{0}^{\ell} E I_{z} V^{\prime \prime 2} d x-\frac{1}{2} \int_{0} F_{x} V^{\prime 2} d x-F_{x} U \tag{4.10.0}
\end{equation*}
$$

Similarly, for bending in the $x-z$ plane, the corresponding equation is given by,

$$
\begin{equation*}
V=\frac{1}{2} \int_{0}^{\ell} E I_{y} W^{\prime \prime 2} d x-\frac{1}{2} \int_{0}^{l} F_{x} W^{\prime 2} d x-F_{x} U \tag{4.11.0}
\end{equation*}
$$

where $F_{X} U$ is the potential of the pre-bukcling deformation and $\frac{1}{2} F_{x} \int_{0}^{\ell} V^{2} d x, \frac{1}{2} \int_{0}^{\ell} F_{x} W^{\prime 2} d x$ are the potential of the buckling deformations in $x-y$ and $x-z$ planes respectively.

Potential Energy Formulation in torsional buckling due to application of axial load, $F_{X}$ :


FIGURE 4.3.0
Let the column shown in Figure (4.3.0) be loaded by an axial thrust $F_{x}$. The differential equation governing the equilibrium is well established (see page 2 of reference 58 ) and is given by,


Now consider the cruciform sectioned column under uniform compression, as shown in Figure (4.4.0). The column has four identical flanges of width $b$ and thickness $t$ and also $y$ and $z$ are the axes of symmetry of the cross-section. Under the compression, the load $F_{x}$, causes torsional buckling as shown in Figure (4.4.0). During deformation, the axis of the bar remains straight, while each flange buckles by rotating about the $x$-axis.

For the purpose of determining the compressive force, which produces torsional buckling, it is necessary to consider the deflections of the flanges during buckling. However, the deflection criterion of the flange is identical to the deflections shown in the pin ended column in Figure (4.3.0).

Therefore, the deflection curve of the strut shown in Figure (4.4.0) and the corresponding bending stress can be found by assuming that the strut is loaded by a fictitious lateral load of intensity,

$$
\begin{equation*}
F_{x} \frac{d^{2} V}{d x^{2}} \tag{4.13.0}
\end{equation*}
$$

Let us now consider an element mn in the form of a thin strip of length $d x$, located at distance, $\rho$, from the $x$-axis and having crosssectional area $t d \rho$. Owing to torsional buckling, the deflection of this element in the $y$-direction is,

$$
\begin{equation*}
V=\rho \phi \tag{4.14.0}
\end{equation*}
$$

The compressive force acting on the rotated ends of the element $m n$ is otdp, where $\sigma=\frac{F_{X}}{A}$ denotes the initial compressive stress. These compressive forces are statically equivalent to a lateral load of,

$$
\begin{equation*}
(\sigma t d \rho) \frac{d^{2} V}{d x^{2}} \tag{4.15.0}
\end{equation*}
$$

which can be written in the form of,

$$
\begin{equation*}
\sigma t \rho d \rho \frac{\mathrm{~d}^{2} \phi}{\mathrm{dx}^{2}} \tag{4.16.0}
\end{equation*}
$$

Therefore, the moment about the x-axis of the fictitious lateral load acting on the element $m n$ is given by,

$$
\begin{equation*}
\frac{\sigma d^{2} \phi}{d x^{2}} d x t^{2} d \rho \tag{4.17.0}
\end{equation*}
$$

In the general case of a column of thin-walled open cross-section, buckling failure usually occurs by a combination of torsion and bending. In order to investigate this type of buckling, consider the assymmetrical cross-section shown in Figure (4.5.0):

fig. 6.5.1


Fig 452

The $y$ - and $z$-axes are the principal centroidal axes of the crosssection and $e_{y}, e_{z}$ are the coordinates of the shear centre 0 . During buckling, the cross-section will undergo translational and rotational displacements. The translational displacements are defined by the deflections $V$ and $W$ in the $y$ - and z-directions of the shear centre 0 . Therefore the final deflection of the centroid $C$ during buckling, according to equations (3.4.1) and (3.4.2) in Chapter 3 is given by,

$$
\begin{align*}
& y_{c}=V+e_{z}{ }^{\theta} x  \tag{4.18.0}\\
& z_{c}=W-e_{y} \theta^{\theta} x
\end{align*}
$$

If the only load acting on the column is a central thrust $F_{x}$, as in the case of a pin ended column, then the bending moments with respect to the principal axes at any cross-section are,

$$
\begin{align*}
& M_{z}=-F_{x}\left(V+e_{z} \theta_{x}\right) \quad(4.19 .1)  \tag{4.19.0}\\
& M_{y}=-F_{x}\left(W-e_{y} \theta_{x}\right) \quad(4.19 .2)
\end{align*}
$$

Therefore, the differential equation for the deflection curve of the shear centre is given by,

$$
\begin{align*}
& E I_{y} \frac{d^{2} V}{d x^{2}}=-F_{x}\left(W-e_{y} \theta x\right)  \tag{4.20.0}\\
& E I_{z} \frac{d^{2} W}{d x^{2}}=-F_{x}\left(V+e_{z} \theta^{x}\right)
\end{align*}
$$

To obtain the equation for the angle of twist ${ }^{\theta} x$, the following argument can be made. Consider a longitudinal strip of cross-section tds defined by coordinates ( $y, z$ ) in the plane of the cross-section.

The components of its deflection in the $y$ - and $z$-directions during buckling as stated previously in Chapter 3, are

$$
\begin{align*}
& U_{y}=V-\left(z-e_{z}\right)^{\theta} x \quad(4.21 .1)  \tag{4.21.0}\\
& U_{z}=W+\left(y-e_{y}\right)^{\theta} x \quad(4.21 .2)
\end{align*}
$$

Taking the second derivative of these expressions with respect to $x$ and again considering an element of length $d x$, the compressive forces $\sigma_{t d s}$ acting on slightly rotated ends of the element, produce forces in the $y$ - and z-directions with intensities of,

$$
\begin{align*}
& F_{y}=-(\sigma t d s) \frac{d^{2}}{d x^{2}}\left[V-\left(z-e_{z}\right) \theta_{x}\right]  \tag{4.22.0}\\
& F_{z}=-(\text { otds }) \frac{d^{2}}{d x^{2}}\left[W-\left(y-e_{y}\right) \theta_{x}\right](4.22 .1)
\end{align*}
$$

By taking moments about the shear centre axis of the above forces, the torque per unit length of the bar is given by,

$$
\begin{gather*}
d m_{x}=-(\sigma t d s)\left(z-e_{z}\right)\left[\frac{d^{2} v}{d x^{2}}-\left(z-e_{z}\right) \frac{d^{2} \theta}{d x^{2}} x\right]+(\sigma t d s)\left(y-e_{y}\right) \\
{\left[\frac{d^{2} W}{d x^{2}}+\left(y-e_{y}\right) \frac{d^{2} \theta}{d x^{2}} x\right]} \tag{4.23.0}
\end{gather*}
$$

Integrating over the entire cross-sectional area $A$ and observing that,

$$
\begin{align*}
& \sigma \int_{A} t d s=F_{x}, \quad \int_{A} z t d s=\int_{A} y t d s=0  \tag{4.24.0}\\
& \int_{A} y^{2} t d s=I_{z}, \int_{A} z^{2} t d s=I_{y}
\end{align*}
$$

and defining $I_{0}=I_{z}+I_{y}+A\left(e_{z}{ }^{2}+e_{y}{ }^{2}\right)$. The equation (4.23.0) is simplified to,

$$
\begin{equation*}
M_{x}=\int_{A} d m_{x}=F_{x}\left[e_{z} \frac{d^{2} V}{d x^{2}}-e_{y} \frac{d^{2} W}{d x^{2}}\right]+\frac{I_{0}}{A} F_{x} \frac{d^{2} \theta_{x}}{d x^{2}} \tag{4.25.0}
\end{equation*}
$$

where $I_{0}$ is the polar moment of area, referred to the shear centre.

In the previous section, the buckling of columns, was subjected only to centrally applied compressive loads. The following analysis intends to discuss columns subjected to the action of bending couples $M_{y}$ and $M_{z}$ at the ends, combined with the central compression $F_{X}$. It is also assumed in this analysis, that the effect of $F_{x}$ on bending stresses are negligible. Therefore the normal stress at any point in the bar is assumed to be independent of $x$ and is given by the following equation:

$$
\begin{equation*}
\sigma_{x}=-\frac{F_{x}}{A}-\frac{M_{z} y}{I_{z}}-\frac{M_{y} z}{I_{y}} \tag{4.26.0}
\end{equation*}
$$

where $y$ and $z$ are the centroidal principal axes of the cross-section. Furthermore, the initial deflection of the bar due to the couples $M_{y}$ and $M_{z}$ will be considered as very small in comparison to the geometry of the column. As before, due to buckling deformation, the components of deflection of any longitudinal fibre of the bar can be defined by coordinates $y$ and $z$. Thus,

$$
\begin{aligned}
& U_{y}=V-\left(z-e_{z}\right) \theta_{x} \\
& U_{z}=W+\left(y-e_{y}\right) \theta_{x}
\end{aligned}
$$



FIGURE 4.6 .0

Following the same argument used in the previous section it can also be said that the intensities of the fictitious lateral loads and distributed torque are given by,

$$
\begin{align*}
& q_{y}=-(\sigma t d s) \frac{d^{2}}{d x^{2}}\left[V-\left(z-e_{z}\right) \theta_{x}\right] \quad(4.27 .1) \\
& q_{y}=-(\sigma t d s) \frac{d^{2}}{d x^{2}}\left[W+\left(y-e_{y}\right) \theta_{x}\right] \quad(4.27 .2)  \tag{4.27.0}\\
& d m_{x}=-\int_{A}(\sigma t d s)\left(z-e_{z}\right) \frac{d^{2}}{d x^{2}}\left[V=\left(z-e_{z}\right) \theta_{x}\right]+ \\
& \cdot \int_{A}(\sigma t d s)\left(y-e_{y}\right) \frac{d^{2}}{d x^{2}}\left[W+\left(y-e_{y}\right) \theta_{x}\right] \\
& (4.27 .3)
\end{align*}
$$

Substituting equations (4.26.0), (4.27.1) and (4.27.2) into equation (4.27.3) and performing the integration, the following relationship is obtained,

$$
\begin{align*}
& d m_{x}=\int_{A}\left(-(\sigma t d s)\left(z-e_{z}\right) \frac{d^{2}}{d x^{2}}\left[V+\left(z-e_{z}\right) \theta x\right]+(\sigma t d s)\left(y-e_{y}\right) \frac{d^{2}}{d x^{2}}\left[W-\left(y-e_{y}\right) \theta x\right]\right. \\
& \left.=-(\sigma t d s)\left(z-e_{z}\right)\left[\frac{d^{2} V}{d x^{2}}+\left(z-e_{z}\right) \frac{d^{2} \theta}{d x^{2}}\right]+(\sigma t d s)\left(y-e_{y}\right)\left[\frac{d^{2} w}{d x^{2}}-\left(y-e_{y}\right) \frac{d^{2} \theta}{d x^{2}}\right]\right) \\
& =\int_{A}\left(-\left\{\frac{F_{x} M_{y} z M_{z} y}{A}-\frac{I_{y}}{I_{z}}\right\} \operatorname{tds}\left(z-e_{z}\right)\left[\frac{d^{2} v}{d x^{2}}+\left(z-e_{z}\right) \frac{d^{2} \theta x}{d x^{2}}\right]+\left\{-\frac{F_{x} M_{y} z M_{z} y}{A} I_{y}^{I_{z}}\right\} \operatorname{tds}\left(y-e_{y}\right)\right. \\
& \left.\left[\frac{d^{2} W}{d x^{2}}-\left(y-e_{y}\right) \frac{d^{2} \theta}{d x^{2}}\right]\right) \\
& =\int\left[\left\{\frac{F}{A}\left(z-e_{z}\right)+\frac{M}{I_{y}}\left(z e_{z}-z^{2}\right)+\frac{M}{I_{z}}\left(e_{z} y-z y\right)\right\} \frac{d^{2} v}{d x^{2}}+\left\{\frac{F}{A}\left(e_{z}{ }^{2}-2 z e_{z}+z^{2}\right)\right.\right. \\
& +\frac{M y}{I_{y}}\left(z e_{z}^{2}-2 z^{2} e_{z}+z^{3}\right) \\
& \left.+\frac{M}{I_{z}}\left(y e_{z}^{2}-2 z y e_{z}+y z^{2}\right)\right\} \frac{d^{2} \theta}{d x^{2}}-\left\{\frac{F^{x}}{A}\left(e_{y}-y\right)+\frac{M y}{I_{y}}\left(z e_{y}-z y\right)\right. \\
& \left.+\frac{M_{z}}{I_{z}}\left(y e_{y}-y^{2}\right)\right\} \frac{d^{2} w}{d x^{2}}+ \\
& \left.\left\{\frac{F x}{A}\left(e_{y}{ }^{2}-2 y e_{y}+y^{2}\right)+\frac{M y}{I}\left(z e_{y}{ }^{2}-2 y z e e_{y}+z y^{2}\right)+\frac{M}{I}\left(y e_{y}{ }^{2}-2 y^{2} e_{y}+y^{3}\right)\right\} \frac{d^{2} \theta}{d x^{2}}\right] \tag{4.28.0}
\end{align*}
$$

Noting that,

$$
\begin{aligned}
& \int_{A} t d s=A, \quad \int_{A} z t d s=\int_{A} y t d s=0 \\
& \int_{A} \dot{y}^{2} t d s=I_{z}, \int_{A} z^{2} t d s=I_{y}
\end{aligned}
$$

and defining

$$
I_{0}=I_{z}+I_{y}+A\left(e_{z}^{2}+e_{y}^{2}\right)
$$

where $I_{0}$ is the polar moment of area referred to the shear centre. Thus equation (4.28.0) is simplified to,

$$
\begin{align*}
d m_{x}= & \left\{e_{z} F_{x}-M_{y}\right\} \frac{d^{2} V}{d x^{2}}\left\{\left(e_{y} F_{x}+M_{z}\right\} \frac{d^{2} w}{d x^{2}}+\left\{F_{x} I_{0}+M_{y}\left[\sum_{y} \frac{1}{d}\left(\int_{A}^{3} d A+\int_{A} z y^{2} d A\right)-2 e_{z}\right]+\right.\right. \\
& \left.M_{z}\left[I_{z}\left(\int_{A} y^{3} d A+\int_{A} y z^{2} d A\right)-2 e_{y}\right]\right\} \frac{d^{2} \theta}{d x^{2}} \tag{4.29.0}
\end{align*}
$$

Let us now define the following quantities where,

$$
\begin{equation*}
S_{0}=\frac{I_{0}}{A}+e_{z} \beta_{1}+e_{y} \beta_{2} \tag{4.30.1}
\end{equation*}
$$

in which,

$$
\begin{align*}
& \beta_{1}=\frac{1}{I_{y}}\left(\int_{A} z^{3} d A+\int_{A} z y^{2} d A\right)-2 z_{0}  \tag{4.30.2}\\
& \beta_{2}=\frac{1}{I_{z}}\left(\int_{A} y^{3} d A+\int_{A} y z^{2} d A\right)-2 y_{0} \tag{4.30.3}
\end{align*}
$$

Therefore, equation (4.29.0) can now be rewritten as,

$$
\begin{equation*}
d m_{x}=\left\{\cdot e_{z} F_{x}-M_{y}\right\} \frac{d^{2} V}{d x^{2}}-\left\{e_{y} F_{x}+M_{z}\right\} \frac{d^{2} w}{d x^{2}}+\left\{F_{x} \frac{I_{0}}{A}+M_{y}{ }^{\beta} 1^{+M} M_{z}{ }^{\beta}\right\} \frac{d^{2} \theta}{d x^{2}} \tag{4.30.0}
\end{equation*}
$$

Now consider the open section element shown in Figure (4.7.0) where force $F_{x}$ is applied at a point on the cross-section whose coordinates are ( $a_{y}, a_{z}$ ) from the centroid.


FIGURE 4.7 .0

The system shown above is equivalent to:

$$
M_{y}=F_{x} \cdot a_{z}, \quad M_{z}=F_{x} \cdot a_{y}
$$

and a force $F_{x}$ at the centroid. Applying the values of $M_{y}$ and $M_{z}$ in equation (4.27.1) and (4.27.2) would result,

$$
\begin{align*}
& q_{y}=-F_{x} \frac{d^{2} w}{d x^{2}}-F_{x}\left(a_{y}-e_{y}\right) \frac{d^{2} \theta}{d x^{2}} \\
& q_{z}=-F_{x} \frac{d^{2} V}{d x^{2}}+F_{x}\left(a_{z}-e_{z}\right) \frac{d^{2} \theta_{x}}{d x^{2}} \\
& d m_{x}=-F_{x}\left(a_{z}-e_{z}\right) \frac{d^{2} V}{d x^{2}}+F_{x}\left(a_{y}-e_{y}\right) \frac{d^{2} w}{d x^{2}}+F_{x}\left(\frac{I_{0}}{A}+a_{z} \beta_{1}+a_{y} \beta_{z}\right) \frac{d^{2} \theta_{x}}{d x^{2}} \tag{4.31.3}
\end{align*}
$$

This equation becomes very simple if the force $F_{x}$ acts along the shear centre axis i.e. $a_{z}=e_{z}$ and $a_{y}=e_{y}$.

Then equations (4.31.1), (4.31.2) and (4.31.3) are simplified to
and

$$
\begin{gather*}
q_{z}=-F_{x} \frac{d^{2} w}{d x^{2}} \quad(4.32 .1) \\
q_{y}=-F_{x} \frac{d^{2} V}{d x^{2}} \quad(4.32 .2)  \tag{4.32.0}\\
d m_{x}=-F_{x}\left(\frac{I_{0}}{A}+e_{z} \beta_{1}+e_{y} \beta_{2}\right) \frac{d^{2} \theta x}{d x^{2}} \tag{4.32.3}
\end{gather*}
$$

As Timoshenko ${ }^{(58)}$ pointed out, with the assumptions made previousiy, equations $(4.32 .1),(4.32 .2)$ and (4.32.3) become totally independent of one another. In this instance, lateral buckling in the two principal planes and torsional buckling may occur independently. The first two equations give the usual Euler conditions providing critical buckling loads under bending conditions, while the third equation gives the critical load corresponding to pure torsional buckling of the column.

Furthermore, for any other location of the axial load on the crosssection, the nature of buckling criterion is not independent and generally occurs by the combined effect of bending and torsion.

Barsoum and Gallagher(4) had probably realised the consequence of placing the axial load at points other than the shear centre. In spite of warnings given by Timoshenko ${ }^{(58)}$ they analysed the problem of arbitrary axial load applications, as though the axial force was acting at the shear centre. This is incorrect with respect to equation (4.31.3). Rajasekaran also ignored this effect and placed the axial force at the centroid. Except for Bleich ${ }^{9}$ ), no other authors produced correct results for problems with wide flanges. (For details see ref. (9)).

One of the main objectives in this analysis was to provide the facility for placing the axial force anywhere on the cross-sectional plane. The additional bending moment and the warping moments (i.e. $M_{x x}$ $=F_{x} \cdot{ }^{\omega}$ ) generated are carefully taken into account in a suitable transformation matrix and which is explained in equation (8.14.0) in Chapter 8.

Potential due to axial load:


FIGURE 4.8 .0

Consider the thin-walled beam column shown in Figure (4.8.0) subjected to an axial thrust $F_{x^{*}}$ As a result, under this load, the column sections produced translational and rotational displacement (see equations (4.21.1) and (4.21.2)). Let us assume the angle of twist at distance $x$, to be $\theta_{x}$. Thus, at distance $(x+\Delta x)$, the twist $=\theta_{x}+\Delta \theta_{x}$. Therefore the work done by the torque $m_{x}$ in increasing the twist from $\theta_{x}$ to $\left(\theta+\Delta \theta_{X}\right)$ is given by,

$$
\begin{equation*}
\delta W=m_{x} \delta \theta x d x \tag{4.33.0}
\end{equation*}
$$

but the increment of angular displacement $\delta \theta_{x}$ can be expressed as,

$$
\begin{equation*}
\delta \theta x=\frac{d \theta}{d x} d x \tag{4.34.0}
\end{equation*}
$$

Therefore the work done $\delta W$,

$$
\begin{equation*}
\delta W=m_{x}\left(\int \frac{d \phi}{d x} d x\right) d x \tag{4.35.0}
\end{equation*}
$$

but from equation (4.32.3),

$$
m_{x}=F_{x}\left(e_{y}^{\beta} 1+e_{z}^{\beta} 2+\frac{I_{0}}{A}\right) \frac{d^{2} \theta_{x}}{d x^{2}}
$$

Substituting the value of $m_{x}$ in equation (4.32.3):

$$
\begin{equation*}
\delta w=\left[\left\{F_{x}\left(e_{y}{ }^{\beta} 1+e_{z}{ }^{\beta} 2+\frac{I_{0}}{A}\right)^{d^{2} \theta} \frac{x}{d x^{2}}\right\} \int \frac{d \theta}{d x} d x\right] d x \tag{4.36.0}
\end{equation*}
$$

Thus the total potential energy in torsional buckling is given by,

$$
\begin{equation*}
W=\int_{0}^{l}\left[\left\{F_{x}\left(e_{y} \beta_{1}+e_{z} \beta_{2}+\frac{I_{0}}{A}\right) \frac{d^{2} \theta}{d x^{2}}\right\} \int \frac{d \theta}{d x} d x\right] d x \tag{4.37.0}
\end{equation*}
$$

but from equation (4.30.1), $S_{0}=e_{y}{ }^{\beta} 1+e_{z}{ }^{\beta} 2+\frac{I_{0}}{A}$
Thus,

$$
\begin{equation*}
W=\int_{0}^{\ell}\left[\left\{\cdot F_{X} S_{0} \frac{d^{2} \theta}{d x^{2}}\right\} \int \frac{d \theta}{d x} d x\right] d x \tag{4.38.0}
\end{equation*}
$$

Since,

$$
\begin{equation*}
\frac{d}{d x}\left[\left(\frac{d \theta}{d x}\right)^{2}\right]=2\left(\frac{d \theta}{d x}\right)^{d^{2} \theta} \frac{x}{d x^{2}} \tag{4.39.1}
\end{equation*}
$$

Therefore, $\quad \frac{1}{2} \int\left(\frac{d \theta}{d x}\right)^{2}=\int \frac{d^{2} \theta}{d x^{2}} \frac{d \theta}{d x} d x$

The integral function on the right hand side of equation (4.39.1) is the same as the integral function of equation (4.39.2). Therefore, by substituting equation (4.39.2) in equation (4.38.0):

$$
\begin{equation*}
W=\frac{1}{2} \int_{0}^{l} F_{x} S_{0}\left(\frac{d \theta}{d x}\right)^{2} d x \tag{4.40.0}
\end{equation*}
$$

Equation (4.40.0) now provides the necessary relationship in calculating the potential for columns under axial thrust, providing the column is at pure torsional buckling.

Since equations (4.32.0) are independent of one another, thus uncoupled lateral buckling in two principal planes and torsional buckling may occur independently.

The three modes of uncoupled buckling due to application of an axial load $F_{X}$ can now be fully described by the equation (4.32.0). The potential comprises three quantities and by using the result of equations (4.10.0) and (4.11.0), are as follows:

1. Due to uncoupled ${ }_{\ell}$ lateral buckling about the $y$-axis and moment equivalent to $\frac{1}{2} \int_{0}^{l} F_{X} V^{\prime 2} d x$.
2. Due to uncoupled lateral bucklings about the $z$ axis and moment equivalent to $\frac{1}{2} \int_{0}^{l} F_{x} W^{\prime 2} d x$.
3. Due to purely torsional buckling about the $x$-axis and moment equivalent to $\frac{1}{2} \int_{0}^{l} F_{x} S_{0}\left(\frac{d \phi}{d x}\right) 2 d x$.

Potential due to applied torque:

Application of a torsional load to a column with constrained ends will generally produce an axial stress field. This phenomenon has already been explained in the introduction to non-uniform torsion. Due to the resulting axial stress field, the column may reach the unstable
condition and will produce large displacements about the $x$-axis and also in the $x-y$ and $x-z$ planes. Since these deflections are generally coupled with large angles of twist, the final collapse will always take place in the form of lateral torsional buckling.


FIGURE 4.9.0

Consider the configuration of a displaced column which is loaded by a torsional load about the column axis. Let us also assume that, prior to loading, the axial direction of the column coincided with the global x-direction. To define the orientation of the column in space fully, let us define an orthogonal coordinate system in which the axis $\xi$ represents the local $x$-axis while the $\eta$ and $\zeta$ axes represent the instantaneous positions of the principal axes.

However, the displaced configuration in Figure (4.9.0) is solely due to the applied torque, and hence the relationship between the moving and the fixed coordinate systems can be expressed as:
$\left\{\begin{array}{l}\xi \\ n \\ \zeta\end{array}\right\}=\left[\begin{array}{lll}\cos (\xi x) & \cos (\xi y) & \cos (\xi z) \\ \cos (n x) & \cos (n y) & \cos (n z) \\ \cos (\zeta x) & \cos (\zeta y) & \cos (\zeta z)\end{array}\right]\left\{\begin{array}{l}x \\ y \\ z\end{array}\right\}$

Where the quantities in the square matrix are the direction cosines between the two axis systems. Let us also denote the small rotation angles of $\xi, \eta, \zeta$ axes about the $x-, y$ - and $z$ axes by $\theta_{x}, \theta_{y}$ and $\theta_{z}$ respectively as shown in the following table.

The axis pair under consideration


The small angle

TABLE 4.T. 0

Then, by inserting the values in Table (4.T.0) in equation (4.41.0) and performing the right handed rule of vector multiplication, the following relationship can be found

$$
\left\{\begin{array}{l}
\xi  \tag{4.42.0}\\
\eta \\
\zeta
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \theta_{y} \cos \theta_{z} & \sin \theta_{z} & \sin \theta_{y} \\
-\sin \theta_{z} & \cos \theta_{x} \cos \theta_{z} & \sin \theta_{x} \\
-\sin \theta_{y} & -\sin \theta_{x} & \cos \theta_{x} \cos \theta_{y}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

Now let us consider an arbitarily chosen point $P$, in the $x-y$ and $x-z$ planes, as shown in Figures (4.10.1) and (4.10.2).


Fig. 4.10 .1


Fig. 4.10 .2

FIGURE 4.10 .0

By referring to Figures (4.10.1) and (4.10.2) the slopes of the column to the $x$-axis, with the inclusion of axial shortenings are,

$$
\begin{align*}
& \tan \theta_{y}=-V^{\prime}  \tag{4.43.1}\\
& \tan \theta_{z}=-W^{\prime} \tag{4.43.2}
\end{align*}
$$

By considering bending only and ignoring possible axial shortening, equations (4.43.1) and (4.43.2) become:

$$
\begin{align*}
& \tan \theta_{z}=\frac{V^{\prime}}{1+\varepsilon_{0}}  \tag{4.44.1}\\
& \tan \theta_{y}=\frac{W^{\prime}}{1+\varepsilon_{0}} \tag{4.44.2}
\end{align*}
$$

since, $\cos \theta=\frac{1}{\sqrt{1+\tan ^{2} \theta}}$ and $\sin \theta=\frac{\tan \theta}{\sqrt{1+\tan ^{2} \theta}}$,

Then the square matrix $[R]$ in equation (4.41.0), now yields to

If we assume the deformations are small in magnitude compared to the original dimensions of the column, with respect to unity, the quantity $\varepsilon_{0}$, tends to zero. For small angles of $\theta_{x}, \sin \theta_{x}$ tends to $\theta_{x}$ and $\cos \theta_{x}$ tends to 1. Thus the square matrix $[R]$ now simplifies to:

$$
[R]=\left[\begin{array}{ccc}
1 & V^{\prime} & -W^{\prime}  \tag{4.46.0}\\
-V^{\prime} & 1 & \theta^{\prime} x \\
W^{\prime} & -\theta_{x} & 1
\end{array}\right]
$$

Substituting equation (4.46.0) in equation (4.41.0) yields,

$$
\left\{\begin{array}{l}
\xi  \tag{4.47.0}\\
n \\
\zeta
\end{array}\right\}=\left[\begin{array}{ccc}
1 & v^{\prime} & -W^{\prime} \\
-v^{\prime} & 1 & \theta x \\
W^{\prime} & -\theta x & 1
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

(Note, for orientations only, the absolute position of ( $X, Y, Z$ ) to ( $x, y, z$ ) is immaterial).

Since the curvature of the deflected column has the same orientation as the local axis system, the relationship between the local and global curvatures is:

$$
\left\{\begin{array}{l}
\theta_{\xi}^{\prime}  \tag{4.48.0}\\
\theta_{\eta}^{\prime} \\
\theta_{\zeta}^{\prime}
\end{array}\right\}=\left[\begin{array}{ccc}
1 & v^{\prime} & -W^{\prime} \\
-V^{\prime} & 1 & \theta x \\
W^{\prime} & -\theta_{x} & 1
\end{array}\right]\left\{\begin{array}{l}
\theta^{\prime} x \\
W^{\prime \prime} \\
v^{\prime \prime}
\end{array}\right\}
$$

where $\theta_{\xi}^{\prime \prime}$ is the local warping curvature
is the global warping curvature
$\theta_{\eta}^{\prime}$ is the local bending curvature in the $\eta$ plane
$W^{\prime \prime}$ is the global bending curvature in the $y$ plane
$\theta_{\zeta}^{\prime}$ is the local bending curvature in the plane
$v^{\zeta}$ is the global bending curvature in the $z$ plane

Thus, by applying matrix multiplication to equation (4.48.0),

$$
\begin{equation*}
\theta_{\xi}^{\prime}=\theta_{x}^{\prime}+v^{\prime} W^{\prime \prime}-W^{\prime} v^{\prime \prime} \tag{4.49.0}
\end{equation*}
$$

By differentiating both sides again, with respect to $\xi$, x :

$$
\begin{equation*}
\theta_{\xi}^{\prime \prime}=\theta_{x}^{\prime \prime}+v^{\prime} W^{\prime \prime \prime}-W^{\prime} v^{\prime \prime \prime} \tag{4.50.0}
\end{equation*}
$$

According to the sign convention adopted, warping curvature $f^{\prime}$ is negative, (see page 11). Then, the potential due to the gradually applied torque $M_{x}$ is given by,

$$
\begin{equation*}
W_{T}=-\frac{1}{2} M_{x} \int_{\ell}\left(V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime}\right) d x \tag{4.51.0}
\end{equation*}
$$

By referring to equation (4.51.0), it can be seen that it is the same as the result shown in ref. ( 4 ). Barsoum and Gallagher, emphasised that equation (4.51.0) was strictly valid only to the buckling of closed circular shafts. By referring to Figure (4.9.0) and also equation (4.51.0), it can be seen quite clearly that the only assumption made in developing the theory was that the column under torsional load will produce warping displacements at any cross-section and also by restraining those warping displacements it will produce an axial stress field, thereby exposing the column to a possible instability. It was also shown in Chapter 2, in equation (2.5.0), that a closed circular shaft can never produce warping displacements under torsional loadings since their cross sections are axisymmetric. Thus the statement made in ref. (4) is contrary to the physical behaviour of thin-walled structures in general.

Recently, Gaafar and Tidbury (19) observed experimentally that at large angles of twist, the torsional load along the column varied considerably and approximated this with a linear function. By referring to Figure (4.9.0) and equation (4.49.0) it is fairly obvious that because at large angles of twist, the translational and rotational displacements are significant. The shape of the column under consideration, is now bent, and hence the final equilibrium position for any cross-sectional plane is now affected by the components of axial, flexural and rotational forces rather than the applied torsional load on its own, as was the case at the beginning of loading.

Thus at significant angles of twist the torsional load $M_{x}$, as suggested by Gaafar and Tidbury is given by,

$$
\begin{equation*}
M_{x}=M_{x 1}\left(1-\frac{x}{\ell}\right) \times \frac{x}{\ell} M_{x 2} \tag{4.52.0}
\end{equation*}
$$

where $M_{x 1}$ and $M_{x 2}$ are the torsional loads at ends 1 and 2 respectively, andlength of the beam column between nodes 1 and 2 .

Referring back to the original problem, the potential due to applied torque is,

$$
\begin{equation*}
\delta W=M_{x} \delta \theta_{x} \tag{4.52.1}
\end{equation*}
$$

where $\delta W$ is the potential generated in applying external torque $M_{x}$ at an elemental length $\delta_{x}$
$\delta \theta_{x}$ is the difference of the angle of twist, at the two ends.

But

$$
\delta \theta_{x}=\frac{d \theta}{d x} \cdot d x
$$

also from equation (4.49.0),

$$
\begin{equation*}
\frac{d \theta}{d x}=\theta_{x}^{\prime}+v^{\prime} w^{\prime \prime}-W^{\prime} v^{\prime \prime} \tag{4.52.2}
\end{equation*}
$$

The term, $\theta_{x}^{\prime}$ in the equation shown above, was considered in equation (4.4.0), as contributory deformation due to torsional load in prebuckling displacements. Thus for the buckling deformations only, equation (4.52.1) is simplified to,

$$
\begin{equation*}
\frac{d \theta}{d x}=V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime} \tag{4.53.0}
\end{equation*}
$$

Now, by applying equation (4.53.0) in equation (4.52.1),

$$
\begin{align*}
W & =\frac{1}{2} \int_{0}^{\ell} M_{x} \frac{d \theta}{d x} d x  \tag{4.54.0}\\
& =\frac{1}{2} \int_{0}^{\ell}\left[M_{x 1}\left(1-\frac{x}{\ell}\right)+\frac{x}{\ell} M_{x 2}\right]\left(V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime}\right) d x \tag{4.55.0}
\end{align*}
$$

Thus, equation (4.55.0) can now be used to estimate the potential due to applied torque.

Potential due to warping moment:

By referring to Table (2.T.0), the torque exerted on the column due to application of the warping moment can be expressed as,

$$
\begin{align*}
& M_{x}=-E r \frac{d^{3} \theta x}{d x^{2}}  \tag{4.56.1}\\
& M_{x x}=E \Gamma \frac{d^{2} \theta x}{d x^{2}} \tag{4.56.2}
\end{align*}
$$

where $M_{x x}$ is the applied warping moment
$M_{x}$ is the torque generated.
By recalling equation (4.54.0), the potential generated due to the applied torque is,

$$
\begin{equation*}
\delta W=M_{x} \frac{d \theta}{d x} d x \tag{4.54.0}
\end{equation*}
$$

By applying equation (4.56.1) in equation (4.54.0) yields,

$$
\begin{align*}
\delta W & =-\frac{d M_{x x}}{d x} \cdot \frac{d \theta}{d x} d x \\
& =-\left(\frac{d M_{x x}}{d x} d x\right) \frac{d \theta}{d x} \\
\delta W & =-\delta M_{x x} \cdot \frac{d \theta}{d x} \tag{4.57.0}
\end{align*}
$$

Thus, the potential due to the gradually applied warping moment, $M_{x x}$ is,

$$
\begin{equation*}
W=-\frac{1}{2} \int_{0}^{l} M_{x x} \frac{d \theta}{d x} d x \tag{4.58.0}
\end{equation*}
$$

This theory can also be explained qualitatively with the aid of the following example:


FIGURE 4.11 .1

The base of the $H$ section column shown in Figure (4.11.0) is fixed while the upper end is free and subjected to a torsional load $T$ in the positive direction.

A warping moment will be generated due to the restrained condition at the lower end and consequently the two flanges will be distorted as shown in Figure (4.11.0). This specific example and its phenomenon is found well described in reference (58).

Now consider the state of equilibrium of an element at a distance $x$ from the base in Figure (4.11.0).
The displacement of the web on the $y-z$ plane at distance $x=e_{1}$. The displacement of the web on the $y-z$ plane at distance $x+d x=e_{2}$.

Within the distance $d x$ the angular deflection of the web is Since $e_{1}=b \theta_{x}$

$$
e_{2}=b\left(\theta_{x}+\delta \theta_{x}\right)
$$

therefore, $=\frac{e_{2}-e_{1}}{d x}=b \frac{d \theta_{x}}{d x}$

But the work done by the bending moment $\delta M$ acting on element $\delta_{x}$, to produce a rotational displacement $\delta \alpha$, at a distance $x$ is given by,

$$
\begin{aligned}
\delta W_{1} & =\delta M \frac{d x}{d x} \\
& =\delta M b \frac{d \theta}{d x}
\end{aligned}
$$

Since there are two webs involved in warping, the final work done, $\delta W$, of the element $\delta_{x}$ is,

$$
\delta W=2 \delta M b \frac{d \theta_{x}}{d x}
$$

According to reference (66), the warping moment for an $H$ section is defined as $\delta M_{x x}=2 \delta M b$

$$
\therefore \quad \delta W=\delta M_{x x} \frac{d \theta x}{d x}
$$

Thus, the total work done along the entire length of the column is,

$$
W=\frac{1}{2} \int_{0}^{\ell} M_{x x} \frac{d \theta}{d x} d x
$$

which is the same as the result shown in equation (4.58.0).

Since the warping moment is proportional to the applied torque, from equation (4.52.0), the applied warping moment is,

$$
\begin{equation*}
M_{x x}=M_{x x_{1}}\left(1-\frac{x}{l}\right)+\frac{x}{l} M_{x x_{2}} \tag{4.59.0}
\end{equation*}
$$

and also from equation (4.50.0), the warping curvature is,

$$
\frac{d^{2} \theta x}{d x^{2}}=\theta_{x}^{\prime \prime}+v^{\prime} w^{\prime \prime \prime}-w^{\prime} v^{\prime \prime \prime}
$$

Now by applying equation (4.59.0) in equation (4.50.0), as before, without the pre-buckling warping displacement $\theta_{x}^{\prime \prime}$; the final expression for the potential due to the warping load becomes

$$
\begin{equation*}
W=\frac{1}{2} \int_{0}^{\ell}\left[M_{x x 1}\left(1-\frac{x}{\ell}\right)+\frac{x}{\ell} M_{x x 2}\right]\left(V^{\prime} W^{\prime \prime \prime}-W^{\prime} V^{\prime \prime \prime}\right) d x \tag{4.60.0}
\end{equation*}
$$

Potential due to lateral bending and end shear loads:

Two possible modes of instability were discussed previously in equations (4.27.1), (4.27.2) and (4.29.0). They are briefly:

1. Bending in the plane of one of the principal axes
2. Twisting followed by bending, in the lowest plane of rigidity.

So far, the buckling mode mentioned in category 1 has been discussed in detail in equations (4.10.0) and (4.11.0) and the buckling mode in category 2 due to application of axial load $F_{x}$, was discussed in equation (4.32.3).

However, the application of forces $Q_{y}, Q_{z}$ and moments $M_{y}, M_{z}$ could also cause instability, and, depending upon their end constraints and loading condition, the associated mode of buckling could be either the bending in the plane of one of the principal axes or the twisting followed by bending, in the lowest plane of rigidity.

Let us now consider the stability of the beam column shown in Figure (4.14.0). The potential of the forces can be obtained by considering the equilibrium of the beam column with two cantilevers mounted back to back as shown in Figure (4.12.1). The appropriate signs used in the two cantilevers are also shown in Figure (4.12.0) and (4.13.0), according, to the chosen right handed sign convention (se Table (2.T.0).

Let us assume that the instability had occurred at the beam column. The associated lateral and rotational deformations which correspond to the two cantilevers under consideration are as shown by figures (4.15.1), (4.15.2) and Figures (4.16.1), (4.16.2) respectively.

The potential of a differential element $d x$, due to application of the transverse force $Q_{y}$ and moment $M_{z}$ is as follows:


FIGURE 4.12.1


Fig. 4.12 .2


Fig. 4.12 .3


Fig. 4.12.4


Fig. 4.12 .5

FIGURE 4.12 .0


Fig. 4.13.1


Fig. 4.13.2


Fig. 4.13 .3


Fig. 4.13 .4

FIGURE 4.13 .0


FIGURE 4.14.0


Fig. 4.15.1


Fig. 4.15.2


Fig. 4.15.3


Fig. 4.15.4

For the cantilever beam shown in Figure (4.14.0) the appropriate sign convention can be obtained by referring to Figure (2.4.0) and Table (2.T.0). Corresponding results are illustrated in Figures (4.12.0) and (4.13.0). Thus, the curvature $\left(\frac{d^{2} y}{d x^{2}}\right)$ for the cantilever shown in Figure (4.14.0) is negative.

From Figure (4.15.2), the angle of rotation due to the applied load Qy2 is,

$$
\begin{equation*}
\theta_{z x}=\frac{d x}{R}=-v^{\prime \prime} d x \tag{4.61.0}
\end{equation*}
$$

where $\theta_{z x}$ is the angle rotated by an element of length $d x$ at a distance $x$ from the origin in the positive direction. Now, considering triangle $A B C$, (see Figure (4..15.2),

$$
\begin{equation*}
\frac{\overline{d V}_{2}}{\ell-x}=\tan \theta_{z x} \simeq \theta_{z x} \tag{4.62.0}
\end{equation*}
$$

for small angles ${ }^{2}$ z .
Therefore, by substituting equation (4.62.0) in equation (4.61.0):

$$
\begin{equation*}
d \bar{V}_{2}=-V^{\prime \prime}(\ell-x) d x \tag{4.63.0}
\end{equation*}
$$

Figure (4.15.3) illustrates the case of the deflected element (shown in Figure (4.15.2)) twisted by an angle $\theta_{x}$. Due to this rotation point $2^{\prime}$ has now moved to $2^{\prime \prime}$.

Thus,

$$
\begin{equation*}
d V_{2}=-V^{\prime \prime} \theta_{x}(\ell-x) d x \tag{4.64.0}
\end{equation*}
$$

and this is shown in Figure (4.15.3) and (4.15.4).

Since it was assumed that plane sections remain plane during bending, the rotation of a beam on the $x-z p l a n e ~ c a n ~ b e ~ e x p r e s s e d ~ a s: ~$

$$
\begin{equation*}
d \theta_{z 2}=\frac{d w_{2}}{\ell-x}=-V^{\prime \prime} \theta_{x} d x \tag{4.65.0}
\end{equation*}
$$

Since the displacement $d V_{2}$ and $d \theta_{2}$ are known, the potential (work done) can now be calculated.

Thus,

$$
\begin{align*}
& d V_{Q 2}=-Q_{y 2} V^{\prime \prime} \theta_{x}(l-x) d x  \tag{4.66.0}\\
& d V_{m 2}=M_{z 2} V^{\prime \prime} \theta_{x} d x \tag{4.67.0}
\end{align*}
$$

where $\mathrm{dV}_{\mathrm{Q} 2}$ and $d V_{m 2}$ are the potentials due to application of $\mathrm{Q}_{\mathrm{y} 2}$ and $M_{z 2}$ respectively.

Thus, the total work done in deforming the entire length (i) is given by,

$$
\begin{equation*}
V_{2}=-Q_{y 2} \int_{\ell} V^{\prime \prime} \theta_{x}(l-x) d x+M_{z 2} \int_{\ell} V^{\prime \prime} \theta_{x} d x \tag{4.68.0}
\end{equation*}
$$

In order to account for unequal end moments, it is necessary to consider the case where point 2 is fixed and point 1 is free as shown in Figure (4.16.1). Therefore, using the same argument, the work done by $Q_{y 1}$ and $M_{z 1}$ is given by,

$$
\begin{equation*}
v_{1}=-Q_{y 1} \int_{\ell} V^{\prime \prime} \theta_{x} x d x-M_{z 1} \int_{\ell} V^{\prime \prime} \theta_{x} d x \tag{4.69.0}
\end{equation*}
$$



Fig. 4.16 .1


Fig. 4.16.2


FIGURE 4.16 .0

Assuming all the loads are gradually applied (i.e. $Q_{y 1}, Q_{y 2}, M_{z 1}$ and $M_{z 2}$ were initially set to zero), the work done in the general motion of the element is given by,

$$
\begin{align*}
V_{\text {final }}= & \frac{1}{2}\left(V_{1}+V_{2}\right) . \\
= & -\frac{1}{2} Q_{y 1} \int_{\ell} V^{\prime \prime} \theta_{x} x d x-\frac{1}{2} \theta_{y 2} \int V^{\prime \prime} \theta_{x}(\ell-x) d x- \\
& \frac{1}{2} M_{z 1} \int_{\ell} V^{\prime \prime} \theta_{x} d x+\frac{1}{2} \int_{\ell} M_{z 2} V^{\prime \prime} \theta_{x} d x \tag{4.70.0}
\end{align*}
$$

Now, applying the result shown in equation (4.70.0) to the case of lateral bending in the $x-z$ plane, the following result can be directly obtained. See Figure (4.17.0) and equation (4.71.0).


FIGURE 4.17 .0

$$
\begin{align*}
V_{\text {final }}= & \frac{1}{2}\left(V_{1}+V_{2}\right) \\
= & -\frac{1}{2} Q_{z 1} \int_{\ell} W^{\prime \prime} \theta_{x} x d x-\frac{1}{2} Q_{z 2} \int_{\ell} W^{\prime \prime} \theta_{x}(\ell-x) d x- \\
& \frac{1}{2} M_{y 1} \int_{\ell} W^{\prime \prime} \theta_{x} d x+\frac{1}{2} \int_{\ell} M_{y 2} W^{\prime \prime} \theta_{x} d x \tag{4.71.0}
\end{align*}
$$

Total potential due to buckling deformations:

Since all the constituents of potential in buckling deformations are established, equation (4.5.2) can now be written in its explicit form as follows:

$$
\begin{align*}
& V=\frac{1}{2} \int_{0}^{\ell} F_{x} V^{\prime 2} d x+\frac{1}{2} \int_{0}^{\ell} F_{x} W^{\prime 2} d x+\frac{1}{2} \int_{0}^{\ell} F_{x} S_{0}\left(\theta_{x}^{\prime}\right)^{2} d x-\frac{1}{2} Q_{y} \int_{0}^{l .} V^{\prime \prime} \theta x^{x d x} \\
& -\frac{1}{2} Q_{y 2} \int_{0}^{\ell} V^{\prime \prime} \theta_{x}(l-x) d x-\frac{1}{2} M_{z 1} \int_{0}^{\ell} V^{\prime \prime} \theta x d x+\frac{1}{2} \int_{0}^{\ell} M_{z 2} V^{\prime \prime} \theta x d x \\
& -\frac{1}{2} Q_{z 1} \int_{0}^{\ell} W^{\prime \prime} \theta_{x} x d x-\frac{1}{2} Q_{z 2} \int_{0}^{\ell} W^{\prime \prime} \theta_{x}(l-x) d x-\frac{1}{2} M_{y 1} \int_{0}^{l} W^{\prime \prime} \theta_{x} d x \\
& +\frac{1}{2} \int_{0}^{l} M_{y 2} V^{\prime \prime} \theta x d x-\frac{1}{2} \int_{0}^{l}\left[M_{x 1}\left(1-\frac{x}{l}\right)+\frac{x}{l} M_{x 2}\right]\left(V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime}\right) d x \\
& -\frac{1}{2} \int_{0}^{\ell}\left[M_{x x 1}\left(1-\frac{x}{l}\right)+\frac{x}{l} M_{x x 2}\right]\left(V^{\prime} W^{\prime \prime}-W^{\prime} v^{\prime \prime}\right) d x \tag{4.72.0}
\end{align*}
$$

Now, by combining equations (4.3.0), (4.4.0) and (4.72.0), the final form of the total potential $\pi p$, can be established. However, the total potential of a conservative system can be expressed as:

$$
\begin{equation*}
\Pi p=U-V \tag{4.73.0}
\end{equation*}
$$

where, $U$ is the strain energy stored in the structure
$V$ is the potential due to pre-buckling and buckling deformations.

Therefore, by combining equations (4.3.0), (4.4.0) and (4.72.0) in their explicit form, equation (4.74.0) is obtained:

$$
\begin{align*}
\text { IIP }= & \frac{1}{2} \int_{0}^{\ell}\left[E I_{z} V^{\prime \prime 2}+E I_{y} W^{\prime \prime 2}+E \Gamma \theta_{x}^{\prime \prime 2}+G J \theta_{x}^{\prime 2}+E A U^{\prime 2}\right] d x-\frac{1}{2} \int_{0}^{\ell} F_{x} V^{\prime 2} d x \\
& -\frac{1}{2} \int_{0}^{\ell} F_{x} W^{\prime 2} d x-\frac{1}{2} \int_{0}^{\ell} F_{x} S_{0}^{\prime}\left(\theta_{x}^{\prime}\right)^{2} d x+\frac{1}{2} Q_{y 1} \int_{0}^{\ell} V^{\prime \prime} \theta_{x} x d x+\frac{1}{2} Q_{y 2} \int_{0}^{\ell} V^{\prime \prime} \theta_{x}(\ell-x) d x \\
& +\frac{1}{2} M_{z 1} \int_{0}^{\ell} V^{\prime \prime} \theta_{x} d x-\frac{1}{2} M_{z 2} \int V^{\prime \prime} \theta_{x} d x+\frac{1}{2} Q_{z 1} \int_{0}^{\ell} W^{\prime \prime} \theta_{x} x d x+ \\
& \frac{1}{2} Q_{z 2} \int_{0}^{\ell} W^{\prime \prime} \theta_{x}(\ell-x) x d x+\frac{1}{2} M_{y!} \int_{0}^{\ell} W^{\prime \prime} \theta_{x} d x-\frac{1}{2} \int_{0}^{\ell} M_{y 2} W^{\prime \prime} \theta_{x} d x+ \\
& \frac{1}{2} \int_{0}^{\ell}\left[M_{x 1}\left(1-\frac{x}{\ell}\right)+\frac{x}{\ell} M_{x 2}\right]\left(V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime}\right) d x+ \\
& \frac{1}{2} \int_{0}^{\ell}\left[M_{x x 1}\left(1-\frac{x}{\ell}\right)+\frac{x}{\ell} M_{x x 2}\right]\left(V^{\prime} W^{\prime \prime}-W^{\prime} V^{\prime \prime}\right) d x-F_{x}\left(u_{1}+u_{2}\right) \\
- & Q_{y 1} V_{1}-Q_{y 2} V_{2}-Q_{z 1} W_{1}-Q_{z 2^{2}} W_{2}-M_{y 1} \theta_{y 1}-M_{y 2} \theta_{y 2}-M_{z 1} \theta_{z 1}-M_{z 2} \theta_{z 2} \\
- & M_{x 1} \theta_{x 1}-M_{x 2} \theta_{x 2}-M_{x 1} \theta_{x 1}^{\prime}-M_{x x 2^{\theta} \theta_{x 2}} \tag{4.74.0}
\end{align*}
$$

Substitution of equations (5.7.0), (5.11.0) and (5.12.0), into equation (4.74.0):

$$
\begin{aligned}
& \left.\pi p=\left[\frac{1}{2} \int_{0}^{\ell}\left[E I_{z} L V \theta_{z}\right]\left[\left\{f_{V}^{\prime \prime}\right\}\left[f_{V}^{\prime \prime}\right] d x\right]\left\{\theta_{z}^{V}\right\}+E I_{y} L W \theta_{y}\right]\left[\left\{f_{W}^{\prime \prime}\right\} L f_{W}^{\prime \prime}\right] d x\right]\left\{\theta_{y}^{W}\right\}+ \\
& \left.E\left[L \theta_{x} \theta_{x}^{\prime}\right]\left[\left\{f f_{x}^{\prime \prime}\right\} L f \theta_{x}^{\prime \prime}\right] d x\right]\left\{\left\{_{\theta_{x}^{\prime}}^{\theta_{x}^{\prime}}\right\}+G J L \theta_{x} \theta_{x}^{\prime}\right]\left[\left\{f^{\prime} \theta_{x}\left\{L f^{\prime} \dot{x}_{x}^{\prime}\right] d x\right] f_{\theta_{x}^{\prime}}^{\theta_{x}}\right\}+ \\
& \left.\left.E A L u]\left[\left\{f_{u}^{\prime}\right\}\left[f_{u}^{\prime} J d x\right]\{d\}\right]\right]-\frac{1}{2} \int_{0}^{\ell} F_{x} L v \theta_{z}\right]\left[\left\{f_{v}^{\prime}\right\} L f_{v}^{\prime} J d x\right]\left\{\begin{array}{l}
v \\
\theta_{z}
\end{array}\right\} \\
& \left.\left.-\frac{1}{2} \int_{0}^{\ell} F_{x} L W \theta_{j}\right]\left[\left\{f_{w}^{\prime}\right] L f_{W}^{\prime}\right] d x\right]\left\{\begin{array}{c}
W
\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \left.\left.-\frac{1}{2} \int_{0}^{\ell}\left[M_{x 1}\left(1-\frac{x}{\ell}\right)+\frac{x}{l} M_{x 2}\right] L W_{\theta}\right]\left[\left\{f_{w}^{1}\right\} L f_{v}^{\prime \prime}\right] d x\right]\left\{\begin{array}{l}
v \\
\theta_{z}
\end{array}\right\}+ \\
& \frac{1}{2} \int_{0}^{l}\left[M_{x x]}\left(1-\frac{x}{l}\right)+\frac{x}{l} M_{x x 2}\right]\left[v_{z} \frac{1}{2}\left[\left\{f_{v}^{\prime}\right] L f_{w}^{\prime \prime \prime]} d x\right]\left\{_{\theta_{y}}^{w}\right\}-\frac{1}{2} \int_{0}^{\ell}\left[M_{x x 1}\left(1-\frac{x}{l}\right)+\frac{x}{l} M_{x x 2}\right] \cdot v \theta\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.-L_{\theta}{ }_{x}{ }^{\theta}{ }_{x}^{\prime}\right\rfloor\left\{\begin{array}{l}
\left.M_{x x}^{x}\right\} \\
M_{x}
\end{array}\right. \tag{4.75.0}
\end{align*}
$$

Equation (4.75.0) can now be rewritten in its very compact form:

$$
\pi p=\frac{1}{2} L d J^{\top}[K]\{d\}-\frac{1}{2} L d J^{\top}[N]\{d\}-L d J^{\top}\{F\} \text { (4.76.0) }
$$

where $\left\lfloor d J^{\top}=L u v w{ }^{\top} x{ }^{\theta} y \theta^{\theta} z \theta_{x}^{\prime} ل^{\top}\right.$

$$
\{F\}=\left\{\begin{array}{l}
F_{x} \\
Q_{y} \\
Q_{z} \\
M_{x} \\
M_{y} \\
M_{z} \\
M_{x x}
\end{array}\right\}
$$

where the first term in equation (4.76.0) is the strain energy stored in the structure by deforming, and the second and third terms are the potentials of the applied loads in buckling and pre-buckling deformations. Furthermore the square matrices $[K]$ and $[N]$ in equation (4.76.0) are generally known as the system stiffness and geometric stiffness matrices.

## Equations of Equilibrium:

It has been shown previously that the total potential of a system can be expressed as:

$$
\left.\left.\pi p=\frac{1}{2} L d\right\rfloor^{\top}[K]\{d\}-\frac{1}{2} L d\right\rfloor^{\top}[N]\{d\} \nleftarrow d ل^{\top}\{F\}
$$

where Ld $\rfloor^{\top}$ is the appropriate d.o.f.
[K] is the system stiffness matrix
[N] is the geometric stiffness matrix
$\{F\}$ is the applied loading vector acting on the system.

See equation (4.76.0).

Using calculus of variations, it can be shown that for a stable equilibrium:

$$
\frac{\partial \pi p}{d d_{1}}=0
$$

$$
\underline{\partial \pi p}=0
$$

$$
\partial d_{2}
$$

$$
\cdot
$$

- 

1. $\quad \frac{\partial \pi p}{\partial d_{n}}=0$

See reference (47).

Thus, the above equation of total potential is simplified to:

$$
\begin{equation*}
\{F\}=[K][d\}-[N]\{d\} \tag{4.77.0}
\end{equation*}
$$

## Buckling Analysis:

The equation of total potential (4.76.0) would not be valid for structural systems which are not in equilibrium. However, stable equilibrium conditions can be achieved by deforming the overall structure under applied loads in which case the work done is stored simultaneously as the strain energy in the structure and the relationship remains valid up to the point of buckling.

Mathematically this phenomenon can be explained as the function $\pi$ (total potential) reaching the neutral condition. Thus, for the neutral condition,

$$
\begin{equation*}
\frac{\partial^{2} \Pi}{\partial \Delta^{2}}=0 \tag{4.78.0}
\end{equation*}
$$

Since,

$$
\begin{equation*}
\frac{\partial^{2} \pi}{\partial \Delta^{2}}=\operatorname{det}|[K]-[N]| \tag{4.79.0}
\end{equation*}
$$

Therefore for buckling $\quad \operatorname{Det}|[K]-[N]|=0$

## Solution Procedure:

The solution to equation (4.80.0) can be found if matrix [N] can be factorized with a single parameter $\lambda$. A close examination of the terms in in matrix [ N ] indicates that the special condition mentioned above may be achieved as follows,
a) by applying axial load only
b) by applying $M_{y}$ only
c) by applying $M_{z}$ only.

Under these circumstances, the standard eigenvalue approach may be used in extracting buckling coefficient $\lambda$, along with associated modes.

However, in the case of a generalized problem, matrix [N], consists of terms associated with the generalized degrees of freedom, as shown in equation (4.48.0). Under these circumstances, equation (4.78.0) becomes non-linear, and hence the straightforward eigenvalue analysis becomes impractical. An alternative approach is proposed, which is commonly known as the mid-point tangent technique in which the loads are predicted along with the associated displacement history.

## CHAPTER 5

## UTILIZATION OF FINITE ELEMENT METHOD FOR BUCKLING ANALYSIS

The finite element method is nothing more than a procedure in which a generally continuous problem is discretized by a series of equivalent elements, providing the compatibility and the continuity of the nodal functions and the element boundaries are maintained. This can usually be achieved by selecting suitable admissible displacement functions for each element under consideration, providing the chosen functions and their derivatives can be used in predicting all rigid body modes for the elements and appropriate displacements.

Let us begin now by applying this technique to the problem in equation (4.78.0). With respect to Table (2.T.0), it is seen that the solution used for the problem in torsion is very similar to the solution used for the problems of bending in the $x-y$ and $x-z$ planes. Consider the problem of bending in the $x-y$ plane. The nodal information $V_{1},{ }_{z 1}$, $V_{2}, \theta_{z 2}$ are used in deriving the displacement (shape) function and this is shown in equation (5.1.2). Similar functions are developed for axial, torsion, torsion warping and bending in the $x-z$ plane and they are shown by equations (5.1.1), (5.1.4), (5.1.5) and (5.1.3) respectively.

$$
\begin{align*}
& \left.U=\left(1-\frac{x}{\ell}\right) U_{1}+\frac{x}{\ell} U_{2}=L F_{u}\right\rfloor\{U\}  \tag{5.1.1}\\
& V=f_{1} V_{1}+f_{2}{ }_{z 1}+f_{3} V_{2}+f_{4}{ }_{z 2}=L f_{v}\left\{\left\{_{\theta_{z}}^{v}\right\}\right.  \tag{5.1.2}\\
& W=f_{1} W_{1}+f_{2} \theta y 1+f_{3} W_{2}+f_{4} \theta_{y 2}=L f_{w}\left\{\left\{_{\theta_{y}}^{W}\right\}\right. \tag{5.1.3}
\end{align*}
$$

$$
\begin{align*}
& \left.{ }^{\theta} x=f_{1}^{\theta} x 1+f_{2}^{\theta} x_{x 1}^{\prime \prime}+f_{3}^{\theta} x 2+f_{4}^{\theta_{x 2}^{\prime}}=L f_{x}\right\rfloor\left\{_{\theta_{x}^{\theta}}^{\theta}\right\}  \tag{5.1.4}\\
& \theta_{x}^{\prime}=f_{1}^{\prime} \theta_{x 1}+f_{2}^{\prime} \theta_{x 1}^{\prime}+f_{3}^{\prime} \theta_{x 2}+f_{4}^{\prime} \theta_{x 2}^{\prime}=L f_{\theta x}\left\{_{\theta}^{\left\{_{x}^{\theta} x\right\}}\right. \tag{5.1.5}
\end{align*}
$$

where, suffices 1 and 2 denote the corresponding nodal displacements for every degree of freedom at nodes 1 and 2 respectively.


Fig. 5.1.1


Fig. 5.1.2

FIGURE 5.1.0

Figure (5.1.1) illustrates the general displacements of a beam column under axial, end shear and moment loads. With respect to the sign convention described in Figure (2.4.0) and Table (2.T.0) the generalized displacements at node 1 and node 2 can be expressed as

Node 1
Linear, $V_{1}$
Rotational, $\theta_{z 1}=-\left[\frac{d V}{d x}\right]_{x=0}$

Node 2
Linear, $V_{2}$
Rotational, $\theta_{z 2}=-\left[\frac{d V}{d x}\right]_{x=2}$

By using matrix notation, the displacement vector $\Delta$ can be defined as,

$$
\{\Delta\}=\left\{\begin{array}{l}
v_{1} \\
\theta_{z 1} \\
v_{2} \\
\theta_{z 2}
\end{array}\right\}
$$

As in the case of an axial member it is possible to choose a suitable polynomial to describe the displacement field $\Delta$. However, the degree of freedom (d.o.f) for this particular problem is four.

Thus a cubic polynomial is adequate to describe the entire displacement field effectively. Hence:

$$
\begin{equation*}
V=a_{1} x^{3}+a_{2} x^{2}+a_{3} x+a_{4} \tag{5.2.0}
\end{equation*}
$$

where $a_{1}, a_{2}, a_{3}$ and $a_{4}$ are some unknown constants. By differentiating equation (5.2.0) with respect to $x$,

$$
\begin{equation*}
\frac{d V}{d x}=3 a_{1} x^{2}+2 a_{2} x+a_{3} \tag{5.3.0}
\end{equation*}
$$

as described previously,

$$
\theta_{z 1}=-\left[\frac{d V}{d x}\right]_{x=0}=-a_{3}
$$

and

$$
\theta_{z 2}=-\left[\frac{d V}{d x}\right]_{x=\ell}=-3 a_{1} \ell^{2}-2 a_{2} \ell-a_{3}
$$

Similarly, substituting the boundary condition, $x=0$ and $x=\ell$ in equation (5.2.0) yields,

$$
v_{1}=a_{4}
$$

and.

$$
\begin{aligned}
& v_{2}=a_{1} l^{3}+a_{2} l^{2}+a_{3} l+a_{4} \\
& v_{1}=a_{4} . \\
& \theta_{z 1}=-a_{3} \\
& v_{2}=a_{1} l^{3}+a_{2} l^{2}+a_{3} l+a_{4} \\
& \theta_{z 2}=-3 a_{1} l^{2}-2 a_{2} l-a_{3}
\end{aligned}
$$

This can be rewritten as:

$$
\left\{\begin{array}{l}
v_{1}  \tag{5.4.0}\\
\theta_{z 1} \\
v_{2} \\
\theta_{z 2}
\end{array}\right\}=\left[\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\ell^{3} & \ell^{2} & i & 1 \\
-3 \ell^{2} & -2 \ell & -1 & 0
\end{array}\right]\left\{\begin{array}{l}
a_{1} \\
a_{2} \\
a_{3} \\
a_{4}
\end{array}\right\}
$$

The inverse of the square matrix in equation (5.4.0) is as follows:
$\left\{\begin{array}{l}a_{1} \\ a_{2} \\ a_{3} \\ a_{4}\end{array}\right\}=\frac{1}{\ell^{3}}\left[\begin{array}{cccc}a & -\ell & -2 & -\ell \\ -3 \ell & 2 i^{2} & 3 \ell & \ell^{2} \\ 0 & -\ell^{3} & 0 & 0 \\ \ell^{3} & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{c}v_{1} \\ \theta z_{1} \\ v_{2} \\ \theta_{z 2}\end{array}\right\}$

Now by substituting the result of equation (5.5.0) in equation (5.2.0):
$V=L x^{3} x^{2} x 1 J \frac{1}{\ell^{3}}\left[\begin{array}{cccc}2 & -\ell & -2 & -\ell \\ -3 \ell & 2 \ell^{2} & 3 \ell & \ell^{2} \\ 0 & -\ell^{3} & 0 & 0 \\ \ell^{3} & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{c}v_{1} \\ \theta_{z 1} \\ v_{2} \\ \theta_{z 2}\end{array}\right\}$
which can be rewritten as:
where,

$$
\begin{align*}
L f\rfloor & =L f_{1} f_{2} f_{3} f_{4} ل^{\top}  \tag{5.8.0}\\
f_{1} & =1-3\left(\frac{x}{l}\right)^{2}+2\left(\frac{x}{l}\right)^{3}  \tag{5.8.1}\\
f_{2} & =-x\left(\left(\frac{x}{l}\right)-1\right)^{2}  \tag{5.8.2}\\
f_{3} & =-2\left(\frac{x}{l}\right)^{3}+3\left(\frac{x}{l}\right)^{2}  \tag{5.8.3}\\
f_{4} & =-x\left(\left(\frac{x}{l}\right)^{2}-\left(\frac{x}{l}\right)\right)
\end{align*}
$$

The sign convention and the cubic polynomial used in describing the displacement for bending in the $x-y$ plane is similar for both bending in the $2-x$ plane and the torsion warping problem about the $x$ axis. Therefore the corresponding displacement functions in terms of nodal values are given by:
$\left.W=L x^{3} x^{2} \times 1\right\lrcorner \frac{1}{\ell^{3}}\left[\begin{array}{cccc}2 & 1 & -2 & 1 \\ -3 \ell & -2 \ell^{2} & 3 \ell & -\ell^{2} \\ 0 & \ell^{3} & 0 & 0 \\ \ell^{3} & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{l}W_{1} \\ \theta_{y 1} \\ W_{2} \\ \theta_{y 2}\end{array}\right\}(5.9 .0)$
$\left.{ }^{\theta} x=L x^{3} x^{2} \times 1\right\lrcorner \frac{1}{\ell^{3}}\left[\begin{array}{cccc}2 & 1 & -2 & 1 \\ -3 \ell & -2 \ell^{2} & 3 \ell & -\ell^{2} \\ 0 & e^{3} & 0 & 0 \\ l^{3} & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{c}\theta_{x 1} \\ \theta^{\prime} \times 1 \\ \theta_{x 2} \\ { }^{2} \times 2 \\ \theta_{x 2}^{\prime}\end{array}\right\}$

Similarly, the equations corresponding to equations (5.9.0) and (5.10.0) also given by:

$$
\begin{align*}
& W=L f J\left\{\Delta_{z}\right\}  \tag{5.11.0}\\
& \theta_{x}=L f j\left\{\Delta_{\theta \dot{x}}\right\} \tag{5.12.0}
\end{align*}
$$

where values of $L f\rfloor$ in equations (5.11.0) and (5.12.0) are identical to values of Lf $\rfloor$ in equation (5.6.0).

From the previous section, the shape function which describes the displacements in the $x-y$ plane is as follows:

$$
\begin{equation*}
V=L f\rfloor\left\{\Delta_{y}\right\} \tag{5.7.0}
\end{equation*}
$$

where,

$$
\begin{aligned}
& L f\rfloor=L f_{1} f_{2} f_{3} f_{4} ل^{\top} \\
& f_{1}=1-3\left(\frac{x}{l}\right)^{2}+2\left(\frac{x}{l}\right)^{3} \\
& f_{2}=-x\left(\left(\frac{x}{l}\right)-1\right)^{2} \\
& f_{3}=-2\left(\frac{x}{l}\right)^{3}+3\left(\frac{x}{l}\right)^{2} \\
& f_{4}=-x\left(\left(\frac{x}{l}\right)^{2}-\left(\frac{x}{l}\right)\right)
\end{aligned}
$$

and

$$
\left\{\Delta_{y}\right\}=\left\{\begin{array}{l}
v_{1} \\
\theta_{z 1} \\
v_{2} \\
\theta_{z 2}
\end{array}\right\}
$$

By differentiating equation (5.7.0) with respect to $x$ :

$$
\begin{align*}
& \left.V^{\prime}=L f^{\prime}\right\rfloor\left\{\Delta_{y}\right\}  \tag{5.13.0}\\
& \left.f^{\prime}=L f_{1}^{\prime} f_{2}^{\prime} f_{3}^{\prime} f_{4}^{\prime \prime}\right]^{\top} \\
& f_{1}^{\prime}=-\frac{6 x}{\ell^{2}}+\frac{6 x^{2}}{\ell^{3}}  \tag{5.13.1}\\
& f_{2}^{\prime}=\frac{1}{\ell^{2}}\left(-3 x^{2}+4 x-1\right)  \tag{5.13.2}\\
& f_{3}^{\prime}=-\frac{6 x}{\ell^{3}}+\frac{6 x}{\ell^{2}}  \tag{5.13.3}\\
& f_{4}^{\prime}=-\frac{3 x^{2}}{\ell^{2}}+\frac{2 x}{\ell} \tag{5.13.4}
\end{align*}
$$

where,
and

Similarly by differentiating equation (5.13.0) with respect to $x$

$$
\begin{equation*}
\left.V^{\prime \prime}=L f^{\prime \prime}\right\lrcorner\left\{\Delta_{y}\right\} \tag{5.14.0}
\end{equation*}
$$

where,

$$
\left.\left.L f^{\prime \prime}\right\rfloor=L f_{1}^{\prime \prime} f_{2}^{\prime \prime} f_{3}^{\prime \prime} f_{4}^{\prime \prime}\right\rfloor^{\top}
$$

and

$$
\begin{align*}
& f_{1}^{\prime \prime}=-\frac{6}{\ell^{2}}+\frac{12 x}{\ell^{3}}  \tag{5.14.1}\\
& f_{2}^{\prime \prime}=\frac{1}{\ell^{2}}(-6 x+4)  \tag{5.14.2}\\
& f_{3}^{\prime \prime}=-\frac{12 x}{\ell^{3}}+\frac{6}{\ell^{2}}  \tag{5.14.3}\\
& f_{4}^{\prime \prime}=-\frac{6 x}{\ell^{2}}+\frac{2}{\ell} \tag{5.14.4}
\end{align*}
$$

Finally by differentiating equation (5.14.0) with respect to x :

$$
\begin{equation*}
V " '=L f " ل\left\{\Delta_{y}\right\} \tag{5.15.0}
\end{equation*}
$$

where,

$$
\left.\left.L f^{\prime \prime \prime}\right\rfloor=L f_{1}^{\prime \prime \prime} f_{2}^{\prime \prime \prime} f_{3}^{\prime \prime \prime} f_{4}^{\prime \prime \prime}\right\rfloor^{\top}
$$

and

$$
\begin{align*}
& f_{1}^{\prime \prime}=\frac{12}{\ell^{3}}  \tag{5.15.1}\\
& f_{2}^{\prime \prime \prime}=-\frac{6}{\ell^{2}}  \tag{5.15.2}\\
& f_{3}^{\prime \prime}=-\frac{12}{\ell^{3}}  \tag{5.15.3}\\
& f_{4}^{\prime \prime}=-\frac{6}{\ell^{2}} \tag{5.15.4}
\end{align*}
$$

The sign convention used for bending in the $x-y, y-z$ planes and the torsion warping problem about the $x$ axis are identical to one another
(see Table 2.T.0). Consequently, the same relationship must exist for the shape functions and their derivatives.

By substitution of the shape functions and their derivatives into the corresponding terms of equation (4.75.0) and integrating within the element boundaries the associated potentials can be calculated.

Thus, the corresponding term for bending in the $x-y$ plane in equation (4.75.0):
$\left.\frac{1}{2} \int_{0}^{\ell} E I_{z} L V \theta_{z}\right\lrcorner\left[\left\{\left[f_{v}^{\prime \prime}\right\} L f_{V}^{\prime \prime}\right\rfloor d x\left\{\begin{array}{c}v \\ \theta_{z}\end{array}\right\}\right.$


Hence the potential becomes:
$=E I_{Z} L V_{1} \theta_{z 1} V_{2}{ }^{\theta} z_{2}-\left[\begin{array}{lll}\frac{12}{\ell^{3}} & & \text { Symmetric } \\ \frac{-6}{\ell^{2}} & \frac{4}{\ell} & \\ \frac{-12}{\ell^{3}} & \frac{6}{\ell^{2}} & \frac{12}{\ell^{3}} \\ \frac{-6}{\ell^{2}} & \frac{2}{\ell} & \frac{6}{\ell^{2}} \\ \frac{4}{\ell}\end{array}\right]\left\{\begin{array}{l}V_{1} \\ \theta_{z 1} \\ V_{2} \\ \theta_{z 2}\end{array}\right\}$

By a similar process the following contributions may be found:

$$
\begin{equation*}
\left.\left.\frac{1}{2} \int_{0}^{l} E I_{y} L W \theta_{y}\right]\left[\left\{f_{w}^{\prime \prime}\right\} L f_{w}^{\prime \prime \prime}\right] d x\right]\left\{\left\{_{\theta_{y}}^{W}\right\}\right. \tag{5.17.0}
\end{equation*}
$$

$\left.=E I_{y} L W_{1} \theta_{y 1} W_{2} \theta_{y 2}\right\lrcorner\left[\begin{array}{cccc}\frac{12}{\ell^{3}} & & \text { Symmetric } \\ \frac{-6}{\ell^{2}} & \frac{4}{l} & \\ \frac{-12}{\ell^{3}} & \frac{6}{l^{2}} & \frac{12}{l^{3}} & \\ \frac{-6}{l^{2}} & \frac{2}{l} & \frac{6}{l^{2}} & \frac{4}{l}\end{array}\right]\left\{\begin{array}{l}W_{1} \\ \theta_{y 1} \\ W_{2} \\ \theta_{y 2}\end{array}\right\}$

$$
\begin{equation*}
\left.\frac{1}{2} \int_{0}^{l} E \Gamma L \theta_{x}^{\theta} x_{x}^{\prime}\right]\left[\left\{f \theta_{x}^{\prime \prime}\right\}\left[f_{\theta x}^{\prime \prime}\right] d x\right]\left\{\dot{\theta}_{x}^{\theta} x_{x}^{\prime}\right\} \tag{5.18.0}
\end{equation*}
$$


Similarly, for torsion about the $x$ axis the corresponding term in equation (4.75.0):

$$
\begin{align*}
& \left.\frac{1}{2} \int_{0}^{\ell} G \mathcal{L} \theta_{x}{ }_{x}^{\prime}\right]\left[\left\{f_{\theta x}^{\prime}\right\}\left[f_{\theta x}^{\prime}\right] d x\right]\left\{\begin{array}{l}
\theta x \\
\theta \\
x
\end{array}\right\} \tag{5.19.0}
\end{align*}
$$

Similarly, the term which corresponds to the direct action due to axial force in equation (4.75.0) is:

$$
\begin{align*}
& \frac{1}{2} \int_{0}^{\ell} E A L U J\left[\left\{f_{u}^{\prime}\right\} L f_{u}^{\prime} d d x\right]\{u\}  \tag{5.20.0}\\
& \left.=\frac{E A}{\ell} L U_{1} U_{2}\right\lrcorner\left[\begin{array}{cc}
1^{b} & -1 \\
-1 & 1
\end{array}\right]\left\{\begin{array}{l}
U_{1} \\
U_{2}
\end{array}\right\} \tag{5.20.1}
\end{align*}
$$

The integrated terms calculated so far can now be used in preparing the conventional elastic stiffness matrix $[K]$ and this is found in chart (5.1) of Chapter 5.

Similarly, the term which corresponds to buckling in the $x-y$ plane due to axial force $F_{X}$ in equation (4.75.0) is:

$$
\left.-\frac{1}{2} \int_{0}^{\ell} F_{x} L V \theta_{z}\right\rfloor\left[\left\{f_{v}^{\prime}\right\} L_{v}^{\prime} d d x\right]\left\{\begin{array}{l}
v  \tag{5.21.0}\\
\theta_{z}
\end{array}\right\}
$$

$=F_{x} L V_{1}{ }^{\theta} z 1 V_{2}{ }^{\theta} z 2$
$\left[\begin{array}{c}\frac{-6}{5 \ell} \\ \frac{1}{10} \\ \frac{6}{5 \ell} \\ \frac{1}{10}\end{array}\right.$

Likewise the term which corresponds to buckling in the $x-z$ plane due to axial force $F_{X}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\left.-\frac{1}{2} \int_{0}^{\ell} F_{x} L W \theta_{y}\right]\left[\left\{f_{W}^{\prime}\right\} L f_{w}^{\prime}\right] d x\right] \cdot\left\{_{\theta_{y}}^{W}\right\} \tag{5.22.0}
\end{equation*}
$$

$$
\left.=F_{x} L W_{1} \quad \theta_{y 1} W_{2} \theta_{y 2}\right\lrcorner\left[\begin{array}{cccc}
\frac{-6}{5 \ell} & & \text { Symmetric }  \tag{5.22.1}\\
\frac{1}{10} & \frac{-2 \ell}{15} & & \\
\frac{6}{5 \ell} & \frac{-1}{10} & \frac{-6}{5 \ell} & \\
\frac{1}{10} & \frac{\ell}{30} & \frac{-1}{10} & \frac{-2 \ell}{15}
\end{array}\right]\left\{\begin{array}{l}
W_{1} \\
\theta_{y 1} \\
W_{2} \\
\theta_{y 2}
\end{array}\right\}
$$

The term which corresponds to torsional buckling about the x axis due to axial force $F_{X}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\left.-\frac{1}{2} \int_{0}^{\ell} F_{x} S_{0} L \theta x_{x}^{\theta}{ }_{x}^{\prime}\right]\left[\left\{f_{\theta x}^{\prime}\right\} L f_{\theta x}^{\prime}\right] d x\right]\left\{_{\theta}^{\theta} \frac{x}{1}\right\} \tag{5.23.0}
\end{equation*}
$$

$\left.=F_{x} S^{\prime} \mathcal{L}^{\theta} \times 1^{\theta} \times 1^{\theta} \times 2^{\theta} \times 2\right]\left[\begin{array}{ccc}\frac{-6}{5 \ell} & \text { Symmetric } \\ \frac{1}{10} & \frac{-2 \ell}{15} & \\ \frac{6}{5 \ell} & \frac{-1}{10} & \frac{-6}{5 \ell} \\ \frac{1}{10} & \frac{\ell}{30} & \frac{-1}{10} \\ \frac{-2 \ell}{15}\end{array}\right]\left\{\begin{array}{l}\theta_{x 1} \\ \theta^{\prime}{ }_{x 1} \\ \theta_{\times 2} \\ \theta^{\prime} \times 2\end{array}\right\}$
The term which corresponds to lateral torsional buckling due to shear force $Q_{z 1}$ in equation (4.75.0):

$$
\left.\frac{1}{2} \int_{0}^{\ell} Q_{z 1} L V \theta_{z} J\left\{f_{v}^{\prime \prime}\right\} L f_{\theta x} J x d x\right]\left\{\begin{array}{c}
\theta_{x}  \tag{5.24.0}\\
\theta_{x}^{\prime}
\end{array}\right\}
$$

$\left.=q_{z 1} L V_{1} \theta_{z 1} V_{2} \theta_{z 2}\right\rfloor\left[\begin{array}{cccc}\frac{-1}{10} & 0 & \frac{11}{10} & \frac{\ell}{10} \\ \frac{\ell}{5} & \frac{-\ell^{2}}{30} & \frac{-\ell}{5} & 0 \\ \frac{1}{10} & 0 & \frac{-11}{10} & \frac{-\ell}{10} \\ \frac{-\ell}{10} & \frac{\ell^{2}}{30} & \frac{-9 \ell}{10} & \frac{-\ell^{2}}{10}\end{array}\right]\left\{\begin{array}{c}\theta_{x 1} \\ \theta_{x 1}^{\prime} \\ \theta_{x 2} \\ \theta_{x 2}^{\prime}\end{array}\right\}$

The solution shown above cannot be used directly in the geometric stiffness matrix [N], since the matrix [N] should be symmetric by definition. However, by making the following changes, the symmetry is achieved while maintaining the absolute value unaffected and this is shown by equation (5.24.2):

$$
\begin{align*}
& =Q_{z 1} L V_{1}{ }_{21} V_{2}{ }_{2} 2^{\theta} \times 1^{\theta^{\prime}} \times 1^{\theta} \times 2{ }^{\theta} \times 2 \perp \\
& -\frac{1}{20} \quad \frac{\ell}{10} \quad \frac{1}{20}-\frac{\ell}{20} \\
& \left.-\frac{1}{20} \quad 0 \quad \frac{11}{20} \quad \frac{\ell}{20}\right]\left[\begin{array}{l}
v_{1}
\end{array}\right. \\
& \frac{\ell}{10}-\frac{\ell^{2}}{60}-\frac{\ell}{10} 0 \\
& \frac{1}{20} \quad 0-\frac{11}{20}-\frac{\ell}{20} \\
& 0-\frac{\ell^{2}}{60} \quad 0 \quad \frac{\ell^{2}}{60} \\
& \frac{11}{20}-\frac{\ell}{10}-\frac{11}{20}-\frac{9 \ell}{20} \\
& \frac{\ell}{20} \quad 0-\frac{\ell}{20}-\frac{\ell^{2}}{20}  \tag{5.24.2}\\
& \text { The term which corresponds to lateral torsional buckling due to shear }
\end{align*}
$$ force $Q_{z 2}$ in equation (4.75.0) is:

$$
\left.\left.\frac{1}{2} \int_{0}^{\ell} Q_{z 2} L V \theta_{z}\right\lrcorner\left\{F_{v}^{\prime \prime}\right\} L F_{\theta x}\right\lrcorner(l-x) d x\left\{\begin{array}{l}
\theta_{x}  \tag{5.25.0}\\
\theta_{x}^{\prime} \\
x
\end{array}\right\}
$$


and the term which corresponds to lateral torsional buckling due to moments $M_{y 1}, M_{y 2}$ in equation (4.75.0) is:

$$
\left.\left.\frac{1}{2} \int_{0}^{\ell}\left(M_{y 1}-M_{y 2}\right) L V \theta_{z}\right]\left[\left\{f_{v}^{\prime \prime}\right\} L f_{\theta x}\right] d x\right]\left\{\begin{array}{l}
\theta_{x}  \tag{5.26.0}\\
\theta_{x}^{\prime}
\end{array}\right\}
$$



and the term which corresponds to lateral torsional buckling due to
(5.26.7) shear force $Q_{y 1}$, in equation (4.75.0) is:

$$
\left.\frac{1}{2} \int_{0}^{\ell} Q_{y 1} L \omega_{y y}\right\lrcorner\left\{f_{w}^{\prime \prime}\right\}\left\llcorner f_{\theta x} \perp x d x\left\{\begin{array}{l}
\theta  \tag{5.27.0}\\
\theta x \\
x
\end{array}\right\}\right.
$$



The term which corresponds to lateral torsional buckling due to shear (5.27.1) force $Q_{y 2}$ in equation (4.75.0) is:

$$
\begin{align*}
& \left.\frac{1}{2} \int_{0}^{l} Q_{y 2}\left[w \theta_{y}\right] \quad\left[\left\{f_{w}^{\prime \prime}\right\} \quad L f_{\theta x}\right\rfloor(l-x) d x\right]\left\{\begin{array}{l}
\theta_{x} \\
\theta_{x}^{\prime} \\
x_{x}
\end{array}\right\}  \tag{5.28.0}\\
& =Q_{y 2} L w_{1}{ }_{y} 1^{w_{2}}{ }^{\theta} y 2^{\theta} \times 1^{\theta^{\prime}} \times 1^{\theta} \times 2^{\theta} \times 2- \\
& {\left[\begin{array}{cccc}
-\frac{11}{20} & \frac{9 \ell}{20} & \frac{11}{20} & \frac{\ell}{10} \\
\frac{l}{20} & -\frac{\ell^{2}}{20} & -\frac{\ell}{20} & 0 \\
\frac{1}{20} & \frac{\ell}{20} & -\frac{1}{20} & -\frac{\ell}{10} \\
0 & \frac{l^{2}}{60} & 0 & -\frac{\ell^{2}}{60}
\end{array}\right.} \\
& \text { (5.28.1) }
\end{align*}
$$

and the term which corresponds to lateral torsional buckling due to $M_{z 1}, M_{z 2}$ in equation (4.75.0) is:

$$
\left.\frac{1}{2} \int_{0}^{l}\left(M_{z 1}-M_{z 2}\right) L w_{\theta y}\right\lrcorner\left[\left\{f_{W}^{\prime \prime}\right\} L f_{\theta x} J d x\right]\left\{\begin{array}{c}
\theta_{x}  \tag{5.29.0}\\
\theta_{x}^{\prime}
\end{array}\right\}
$$

The first term which contributes to lateral torsional buckling, due to torsional load $M_{x l}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\frac{1}{2} \int_{0}^{l} M_{x 1}\left[V_{\theta}\right\lrcorner\left[\left\{f_{v}^{\prime}\right\} L f_{w}^{\prime \prime}\right\rfloor\left(1-\frac{x}{l}\right) d x\right]\left\{\left\{_{\theta_{y}}^{w}\right\}\right. \tag{5.30.0}
\end{equation*}
$$


(5.30.1)

The first term which contributes to lateral torsional buckling due to $M_{x 2}$ in equation (4.75.0) is:
(5.31.1)

The second term which contributes to lateral torsional buckling due to torsional load $M_{x 1}$ in equation (4.7.50) is:

$$
\left.-\frac{1}{2} \int^{\ell} M_{x} L W \theta_{y}\right]\left[\left\{f_{w}^{\prime}\right\} L f_{v}^{\prime \prime \prime}\left(1-\frac{x}{\ell}\right) d x\right]\left\{\begin{array}{l}
v  \tag{5.32.0}\\
\theta_{z}
\end{array}\right\}
$$



The second term which contributes to lateral torsional buckling due to torsional load $M_{x 2}$ in equation (4.75.0) is:


The first term which contributes to lateral torsional buckling due to warping moment $M_{x x 1}$ in equation (4.75.0) is:

$$
\begin{align*}
& \left.\left.\frac{1}{2} \int_{0}^{\ell} M_{x x 1} L V \theta_{z}\right]\left[\left\{f_{v}^{\prime}\right\} L f_{W}^{\prime \prime \prime}\right]\left(1-\frac{x}{\ell}\right) d x\right]\left\{\begin{array}{c}
W \\
\theta_{y}
\end{array}\right\}  \tag{5.34.0}\\
& \left.M_{x x 1} L V_{1} \theta_{z 1} V_{2} \theta_{z 2} W_{1} \theta_{y 1} W_{2} \theta_{y 2}\right\lrcorner \Gamma \\
& \begin{array}{l}
\frac{-6}{2 \ell^{3}}-\frac{1}{2 \ell^{2}} \frac{6}{2 \ell^{3}} \frac{1}{2 \ell^{2}} \\
\frac{3}{2 \ell^{2}} \frac{1}{4 \ell}-\frac{3}{2 \ell^{2}}-\frac{1}{4 \ell} \\
\frac{6}{2 \ell^{3}} \\
\frac{3}{2 l^{2}}-\frac{6}{2 \ell^{3}}-\frac{1}{2 \ell^{2}} \\
\frac{1}{2 \ell^{2}} \\
\frac{1}{4 l}-\frac{3}{2 l^{3}}-\frac{1}{4 l}
\end{array}
\end{align*}
$$

The first term which contributes to lateral torsional buckling due to warping moment $M_{x x 2}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\frac{1}{2} \quad M_{x x 2} L V_{-}\left[\left\{f_{v}^{\prime}\right\} L f_{w}^{\prime \prime \prime}\right] \frac{x}{\ell} d x\right]\left\{\left\{_{\theta_{y}}^{W}\right\}\right. \tag{5.35.0}
\end{equation*}
$$



The second term which contributes to lateral torsional buckling due to warping moment $M_{x x 1}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\left.-\frac{1}{2} \int_{0}^{\ell} M_{x x 1} L W \theta_{g}\right]\left[\left\{f_{w}^{\prime}\right\} L f_{v}^{\prime \prime \prime}\right]\left(1-\frac{x}{l}\right) d x\right]\left\{\left\{_{\theta_{z}}^{v}\right\}\right. \tag{5.36.0}
\end{equation*}
$$

$M_{x \times 1} L W_{1}{ }_{y 1}{ }_{y 1} H_{2} \theta_{y} v_{1} v_{1} \theta_{21} V_{2} \theta_{z 2}-$

The second term which contributes to lateral torsional buckling due to warping moment $M_{x \times 2}$ in equation (4.75.0) is:

$$
\begin{equation*}
\left.\left.-\frac{1}{2} \int_{0}^{l} M_{x \times 2} L w_{y}\right\rfloor\left[\left\{f_{w}^{\prime}\right\} L f_{v}^{\prime \prime \prime}\right\rfloor \frac{x}{l} d x\right]\left[\varepsilon_{\theta_{z}}^{v}\right\} \tag{5.37.0}
\end{equation*}
$$



Equations (5.16.2), (5.17.1), (5.18.1), (5.19.1), (5.20.1) (5.21.1), (5.22.1), (5.23.1), (5.24.1), (5.25.1), (5.26.1), (5.27.1), (5.28.1), (5.29.1), (5.30.1), (5.31.1), (5.32.1), (5.33.1), (5.34.1), (5.35.1), (5.36.1) and (5.37.1) are now in their explicit form representing individual potentials and are used in preparing the matrix [N] for the thin-walled element shown in Figure (4.1.6). The final form of the geometric stiffness matrix is illustrated in Chart (5.2) of Chapter 5.

The Linear Stiffness Matrix:

where,
$a=\frac{1}{\ell} E A$
$g=\frac{6}{\ell^{2}} E I_{z}$
$b=\frac{12}{\ell^{3}} E I_{z}$
$h=\frac{6}{\ell^{2}} E C_{W}+\frac{1}{10} G k$
$c=\frac{12}{\ell^{3}} E I_{y}$
$i=\frac{2}{\ell} E I_{y}$
$d=\frac{12}{\ell^{3}} E C_{W}+\frac{6}{5 \ell} G k$
$j=\frac{2}{\ell} E I_{z}$
$e=-\frac{6}{\ell^{2}} E I_{y}$
$k=\frac{2}{\ell} E C_{W}-\frac{\ell}{30} G k$
$f=\frac{4}{\ell} E I_{y}$
$m=\frac{4}{\ell} E C_{W}+\frac{2}{15} \ell G k$

## The Geometric Stiffness Matrix:

where, $a=\frac{-6}{5 \ell} F_{x}$

$$
b=\frac{-2}{15} \ell F_{x}
$$

$$
\begin{aligned}
e & =\frac{3}{20 \ell} M_{x 2}+\frac{7}{20 \ell} M_{x 1}+\frac{1}{10 \ell} M_{x 2} \\
& +\frac{4}{10 \ell} M_{x 1}+\frac{3}{2 \ell^{2}}\left(M_{x x 2}+M_{x x 1}\right)
\end{aligned}
$$

$$
c=\frac{1}{20} Q_{z 1}-\frac{11}{20} Q_{z 2}-\frac{6}{10 \ell}\left(M_{y 1}-M_{y 2}\right)
$$

$$
+\frac{1}{2 \ell^{2}}\left(M_{x x 2^{2}}-M_{x x 1}\right)
$$

$$
d=\frac{1}{20} Q_{y 1}+\frac{11}{20} Q_{y 2}+\frac{6}{10 \ell}\left(M_{z 1}-M_{z 2}\right)
$$

$$
S_{0} a=-S_{0} \frac{6}{5 \ell} F_{x}
$$

$$
\begin{aligned}
e^{\prime} & =\frac{7}{20 \ell} M_{x 2}+\frac{3}{20 \ell} M_{x 1}+\frac{4}{10 \ell} M_{x 2} \\
& +\frac{1}{10 \ell} M_{x 1}+\frac{3}{2 \ell^{2}}\left(M_{x x 2}+M_{x x 1}\right) \\
& +\frac{1}{2 \ell^{2}}\left(M_{x \times 2}-M_{x x 1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& f=\frac{1}{10} F_{x} \\
& g=\frac{\ell}{10} Q_{y 1}+\frac{9}{20} \ell Q_{y 2}+\frac{11}{20}\left(M_{z 1}-M_{z 2}\right) \\
& h=\frac{1}{10 \ell} M_{x 2}+\frac{4}{10 \ell} M_{x 1}+\frac{3}{20 \ell} M_{x 2}+\frac{7}{20 \ell} M_{x 1}+\frac{1}{2 \ell^{2}}\left(M_{x x 2}-M_{x x 1}\right)-\frac{3}{2 \ell^{2}}\left(M_{x x 2^{2}}+M_{x x 1}\right) \\
& h^{\prime}=\frac{1}{10 \ell} M_{x 1}+\frac{4}{10 \ell} M_{x 2}+\frac{3}{20 \ell} M_{x 1}+\frac{7}{20 \ell} M_{x 2}+\frac{1}{2 \ell^{2}}\left(M_{x x 2}-M_{x x 1}\right)+\frac{3}{2 \ell^{2}}\left(M_{x x 2^{2}}+M_{x x 1}\right) \\
& i=\frac{\ell}{10} Q_{z 1}+\frac{9}{20} \ell Q_{z 2}+\frac{11}{20}\left(M_{y 1}-M_{y 2}\right) \\
& i^{\prime}=\frac{9}{20} \quad l Q_{z 1}+\frac{\ell}{10} Q_{z 2}+\frac{11}{20}\left(M_{y 1}-M_{y 2}\right) \\
& j=\frac{1}{20} \quad \ell Q_{z 2}+\frac{1}{20}\left(M_{y 1}-M_{y 2}\right) \\
& k=\frac{\ell}{20} \quad Q_{y 2}+\frac{1}{20}\left(M_{z 1}-M_{z 2}\right) \\
& k^{\prime}=\frac{\ell}{20} Q_{y 1}+\frac{1}{20}\left(M_{z 1}-M_{z 2}\right) \\
& S_{0} f=S_{0} \frac{1}{10} F_{x} \\
& I^{\prime}=-\frac{\ell^{2}}{60} Q_{y 1}-\frac{\ell^{2}}{20} Q_{y 2}-\frac{2 \ell}{30}\left(M_{z 1}-M_{z 2}\right) \\
& 1^{\prime \prime}=-\frac{\ell^{2}}{20} Q_{y 1}-\frac{\ell^{2}}{60} Q_{y 2}-\frac{2 \ell}{30}\left(M_{z 1}-M_{z 2}\right) \\
& m=-\frac{\ell^{2}}{60} Q_{z 1}-\frac{\ell^{2}}{20} Q_{z 2}-\frac{2 \ell}{30}\left(M_{y 1}-M_{y 2}\right) \\
& m^{\prime}=-\frac{l^{2}}{20} Q_{z 1}-\frac{l^{2}}{60} Q_{z 2}-\frac{2 \ell}{30}\left(M_{y 1}-M_{y 2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& S_{0} b=-S_{0} \frac{2 \ell}{15} F_{x} \\
& n=-\frac{\ell}{20} Q_{z 2}-\frac{1}{20}\left(M_{y 1}-M_{y 2}\right) \\
& n^{\prime}=-\frac{\ell}{20} Q_{z 1}-\frac{1}{20}\left(M_{y 1}-M_{y 2}\right) \\
& 0=\frac{11}{20} Q_{z 1}+\frac{1}{20} Q_{z 2}+\frac{6}{10 \ell}\left(M_{y 1}-M_{y 2}\right) \\
& p=\frac{11}{20} Q_{y 1}+\frac{1}{20} Q_{y 2}+\frac{6}{10 \ell}\left(M_{z 1}-M_{z 2}\right) \\
& q=-\frac{\ell}{10} Q_{y 1}+\frac{\ell}{20} Q_{y 2}-\frac{1}{20}\left(M_{z 1}-M_{z 2}\right) \\
& r=-\frac{\ell}{10} Q_{z 1}+\frac{\ell}{20} Q_{z 2}-\frac{1}{20}\left(M_{y 1}-M_{y 2}\right) \\
& s^{\prime}=\frac{\ell^{2}}{60} Q_{z 1}+\frac{\ell}{60}\left(M_{y 1}-M_{y 2}\right) \\
& s^{\prime \prime}=\frac{\ell^{2}}{60} Q_{z 2}+\frac{\ell}{60}\left(M_{y 1}-M_{y 2}\right) \\
& u=\frac{\ell}{30} F_{x} \\
& S_{0} u=S_{0} \cdot \frac{\ell}{30} F_{x} \\
& v=-\frac{\ell}{30} M_{x 2}-\frac{7}{60} M_{x 1}-\frac{7}{60} M_{x 2}-\frac{\ell}{10} Q_{y 2}+\frac{1}{20}\left(M_{z 1}-M_{z 2}\right) \\
& L_{x 1}-\frac{1}{4 \ell}\left(M_{x x 2}-M_{x x 1}\right)-\frac{1}{4 \ell}\left(M_{x x 2}-M_{x x 1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& v^{\prime}=\frac{7}{60} M_{x 2}+\frac{4}{30} M_{x 1}+\frac{4}{30} M_{x 2}+\frac{7}{60} M_{x 1}+\frac{1}{4 \ell}\left(M_{x x 2}-M_{x x 1}\right)+\frac{1}{4 \ell}\left(M_{x x 2}-M_{x x 1}\right) \\
& w=\frac{\ell^{2}}{60} Q_{y 1}+\frac{\ell}{60}\left(M_{z 1}-M_{z 2}\right) \\
& x=-\frac{9 \ell}{20} Q_{y 1}-\frac{\ell}{10} Q_{y 2}-\frac{11}{20}\left(M_{z 1}-M_{z 2}\right) \\
& y=-\frac{\ell}{20} Q_{z 1}+\frac{\ell}{10} Q_{z 2}+\frac{1}{20}\left(M_{y 1}-M_{y 2}\right)
\end{aligned}
$$

CHART (5.2)

## CHAPTER 6

## THE MID POINT TANGENT METHOD

In Chapter 4, the difficulty in adopting the uşe of a straightforward eigenvalue approach to solve buckling problems which are subjected to generalized loadings and end constraints was shown. However, if the structure is loaded incrementally, the non-linear path may be traced by a suitable method which solves the conditions for equilibrium at every load step. One such method is known as the mid point tangent method.

The solution procedure is as follows. At the first loading increment, the geometric stiffness matrix $[N]$ makes no contribution to the state of equilibrium of equation (4.78.0). This is mainly due to the fact that the displacements are assumed to be small. Thus, equation (4.78.0) is simplified to:

$$
\begin{equation*}
\{F\}=[K]\{\Delta\} \tag{6.1.0}
\end{equation*}
$$

Initially, the terms in the stiffness matrix [K] are calculated from the parameters of the problem under consideration (i.e. length, cross sectional properties etc). Then, by solving equation (6.1.0) for a specific loading incremental vector $\left\{\Delta_{F}\right\}$, the unknown displacement vector $\{\delta \Delta\}$ and the associated reaction force vector $\left\{\Delta_{R}\right\}$ are calculated.

Prior to the next loading increment, the new geometry of the structure is obtained by combining appropriately, the displacement vector $\{\delta \Delta\}$ with the undeformed geometry of the structure. This enables us to form the new stiffness matrix. Similarly the new geometry, along with the reaction force vector $\left\{\Delta_{R}\right\}$, is also used in formulating the geometric
stiffness matrix [N]. Subsequently, the two matrices [K] and [N] are inserted in equation ( $4.7,7.0$ ) to examine the equilibrium contribution for the next loading increment $\left\{\Delta_{F}\right\}$.

Mathematically, for the $i^{\text {th }}$ iteration, the method described above can be expressed as:

$$
\begin{equation*}
\{\Delta F\}_{i}=[K]_{i-1}\{\Delta\}_{i}-[N]_{i-1}\{\Delta\}_{i} \tag{6.2.0}
\end{equation*}
$$

where $\{\Delta F\}_{i}$ is the $i^{\text {th }}$ loading increment
$[K]_{i-1}$ is the stiffness matrix whose calculation is based on the structural geometry and the $(i-1)^{\text {th }}$ step
$[N]_{i-1}$ is the geometric stiffness matrix obtained by using the accumulated reaction forces up to the (i-1) increment
and $\{\Delta\}_{i} \quad i s$ the displacement vector corresponding to the $i^{\text {th }}$ loading increment.

However, equation (6.2.0) does not accurately describe the equilibrium condition at the $i^{\text {th }}$ loading increment. This is mainly due to the fact that both [K] and [N] are calculated with the aid of the column length and orientation of the (i-1) increment. However, a better approximation can be made if the column lengths and the orientations are estimated, based upon the average values of the $i^{\text {th }}$ and $i-1^{\text {th }}$ steps, i.e.

$$
\begin{equation*}
\{\Delta\}_{\mathfrak{i}}=\beta\{\Delta\}_{\mathfrak{i}-1} \tag{6.3.0}
\end{equation*}
$$

where, $\beta$ is the ratio of the load increment at step $\mathbf{i}$ and that at step i-1.

The displacements at the $i^{\text {th }}$ increment can now be estimated by using the average stiffness values between the $i^{\text {th }}$ and $i-1^{\text {th }}$ increments.

With respect to reference ${ }^{(65)}$ it can be seen that a better estimation for the stiffness and the geometric stiffness matrices is found by predicting the structural geometry half way through the $i-1{ }^{\text {th }}$ and $i^{\text {th }}$ increments. This approach of modifying the structural stiffness halfway between two increments is known as the mid-point tangent technique.

The above method must be implemented with carefully chosen loading steps, and the collapse or the unstable condition of the structure is observed by comparing the displacements calculated at the $i^{\text {th }}$ increment to what was calculated at the $(i-1)^{\text {th }}$ increment. Complete discussion of the results obtained from the method is found in Chapter 10.

## Implementation of the Mid-point Tangent Technique:

The non-linear behaviour of the structure is now determined by solving a sequence of linear problems. The total load is applied as a sequence of sufficiently small increments such that during loading the structure is assumed to respond linearly. This can be understood with the aid of the diagram shown in Figure (6.1.0).


Fig. 6.1.0

FIGURE 6.1.0

The step by step approach required in implementing the mid-point tangent technique is as follows:

1. Assuming the load increment, $\Delta \mathrm{p}_{1}$, is in the linear elastic region, the equation,

$$
\begin{equation*}
\left\{\Delta p_{1}\right\}=[K]_{0}\left\{\Delta \dot{u}_{1}\right\} \tag{6.4.0}
\end{equation*}
$$

is solved.
2. By predicting the new structural geometry half way through the next increment, the nodal coordinates are updated. Thus,

$$
\begin{equation*}
(\text { coord })_{1}^{\prime}=(\text { coord })_{1}+\frac{\Delta u_{1}}{2} \frac{\Delta p_{2}}{\Delta p_{1}} \tag{6.5.0}
\end{equation*}
$$

3. By recording the reaction forces calculated in the first increment and inserting them in the geometric stiffness matrix appropriately, the equation,

$$
\begin{equation*}
\left\{\Delta p_{2}\right\}=[K]_{1}\left\{\Delta u_{2}\right\}-[N]_{1}\left\{\Delta u_{2}\right\} \tag{6.6.0}
\end{equation*}
$$

is solved.
4. By predicting the new structural geometry half way through the following increment, the nodal coordinates are updated. Thus,

$$
\begin{align*}
& (\text { coord })_{2}=(\text { coord })_{1}+\Delta u_{2}  \tag{6.7.0}\\
& (\text { coord })_{2}^{\prime}=(\text { coord })_{2}+\frac{\Delta u_{2}}{2} \frac{\Delta p_{3}}{\Delta p_{2}} \tag{6.8.0}
\end{align*}
$$

or in general

$$
\begin{aligned}
& (\text { coord })_{i}=(\text { coord })_{i-1}+\Delta u_{i} \\
& (\text { coord })_{i} \\
& =(\text { coord })_{i}+\frac{\Delta u_{i}}{2} \frac{\Delta p_{i+1}}{\Delta p_{i}}
\end{aligned}
$$

$\rightarrow$
Equation (6.10.0) is used to modify the structural geometry and by solving equation (6.11.0), the required displacements at the $(i+1)^{\text {th }}$ loading increment are calculated

$$
\begin{equation*}
\left\{\Delta p_{i+1}\right\}=[K]_{i}\left\{\Delta u_{i+1}\right\}-[N]_{i}\left\{\Delta u_{i+1}\right\} \tag{6.11.0}
\end{equation*}
$$

The procedure is repeated until the desired convergence of the load displacement history is obtained. However, this method does not iterate within an increment for equilibrium. Thus the solution is dependent on sufficiently small increments.

## CHAPTER 7

## ORIENTATION OF THE LOCAL AXES SYSTEM TO THE GLOBAL AXIS SYSTEM

The basic beam element is a three dimensional object. Thus, to define a beam element fully in space, a minimum of three nodes are required. In Figure (7.2.0) the axial direction and the plane of cross-section may be obtained by defining the coordinates of node 1 , node 2 and node 3.

Under a generalized loading condition a thin wall beam will have seven degrees of freedom per node when referred to the local axis system (see equation (4.78.0)) or nine degrees of freedom per node if a global axis system is chosen (see equation (8.14.0). The solution procedure utilized in the mid-point tangent method (see equation (6.11.0)) presumes an overall definition of the updated coordinates of nodes 1,2 and 3 prior to the application of the next loading increment. The method of obtaining the updated positions for nodes 1 and 2, has already been established in Chapter 5 and in this chapter proposes the method to establish the position of the third node.

Let us recall the theory developed in Chapter 3 for predicting the deformation of thin walled cross sections under generalized loads.


FIGURE 7.1 .0

In Figure (7.1.0) point $C$ is the origin of the coordinate axes systems at node 1. Point 0 is the shear centre and $N$ is an arbitrarily chosen point on the cross section. Due to loading, the cross section will undergo translational and rotational displacements. The magnitudes of these displacements may be estimated by inserting appropriate conditions into equations (3.4.1) and (3.4.2):
s.

$$
\begin{align*}
& U_{\eta}=U_{n 0}-\left(\zeta-e_{\zeta}\right) \theta_{\xi}  \tag{3.4.1}\\
& U_{\zeta}=U_{\zeta 0}+\left(\eta-e_{\eta}\right) \theta_{\xi} \tag{3.4.2}
\end{align*}
$$

(Note: equations (3.4.1) and (3.4.2) represent the combined translational rotational displacement of an arbitrary point ( $n, \zeta$ ), defined with respect to axes $\eta$ and $\zeta$ ).


FIGURE 7.2 .0

Let us examine the kinematic behaviour of thin walled beams in space. In Figure (7.2.0), the displacement of point $N$ can be decomposed into its components. The final position of point $N^{\prime}$ is obtained by the axial displacement $\Delta_{x}$ followed by translational displacements $U_{\zeta} N^{\prime}$ and $U_{n N}$, parallel to the $\zeta$ and $\eta$ axes respectively. Note also that the effect of axial twist $\theta_{\xi}$ is automatically accounted for in equations (3.4.1) and (3.4.2). The aim of the exercise is to predict the position of $N^{\prime \prime}$, which is an arbitrary point on the plane normal to $C_{1}$ $C_{2}$ in which $C_{2}$ is the new position of node 2 relative to node 1. Thus, if point $N$ is conveniently chosen on the $\zeta$ axis, by connecting $C_{2}$ and $N^{\prime \prime}$, the new position of the principal axis is predicted. Simultaneously $N^{\prime \prime}$ becomes the new third node.

The coordinates of node 1 with respect to ( $\xi \square \zeta$ ) system is given by,

$$
\begin{equation*}
U_{g, 2}=\Delta x \tag{7.1.0}
\end{equation*}
$$

Substituting $\eta=0,5=0$ in equations (3.4.1), (3.4.2)

$$
\begin{align*}
& u_{n 1,2}=u_{n 0}+e_{\zeta} \theta_{\zeta}  \tag{7.2.0}\\
& u_{\zeta 1,2}=u_{\zeta 0}-e_{\eta} \theta_{\zeta} \tag{7.3.0}
\end{align*}
$$

As explained previously, let us place point $N^{\prime \prime}$ conveniently on the $\zeta$ axis. Then the coordinates of $N$, prior to any form of deformation has taken place, are given by,

$$
\begin{aligned}
& { }^{\xi_{N}}=\ell \\
& { }^{n_{N}}=0 \\
& \zeta_{N}=\ell
\end{aligned}
$$

where, $\ell$ is the length of the element before deformation. By applying the above coordinates into equations (3.4.1) and (3.4.2):

$$
\begin{align*}
& \xi_{N^{\prime}}=\ell+\Delta_{x}  \tag{7.4.0}\\
& U_{n N^{\prime}}=U_{n 0}-\left(\ell-e_{\zeta}\right) \theta_{\xi}  \tag{7.5.0}\\
& U_{\zeta N^{\prime}}=U_{\zeta 0}-e_{n} \theta_{\xi} \tag{7.6.0}
\end{align*}
$$

The vectorial movement of $N$ " with respect to the axis system ( $\xi, n, \zeta$ ) at node 1 is illustrated in Figure (7.3.0)


FIGURE 7.3 .0

The unit vector $\underset{\sim}{\hat{n}}$, in the direction of $C_{1} C_{2}$ is defined as,

$$
\underset{\sim}{n}=\frac{1}{\alpha}\left\{\begin{array}{l}
\ell+\Delta_{x}  \tag{7.7.0}\\
u_{n 1,2} \\
u_{\zeta 1,2}
\end{array}\right\}
$$

where, $\alpha=\left(\ell+\Delta_{x}^{2}+U_{n 1,2}^{2}+U_{\zeta 1,2}^{2}\right)^{\frac{1}{2}}$

Applying vector algebra to the triangle $\mathrm{C}_{2} \mathrm{~N}^{\prime \prime} \mathrm{N}^{\prime}$,

$$
\overrightarrow{C_{2}} N^{\prime}=\overrightarrow{C_{2}} N^{\prime \prime}+\overrightarrow{N^{\prime \prime}} N^{\prime}
$$

$$
\begin{equation*}
=\overrightarrow{C_{2}} N^{\prime \prime}+\left(\overrightarrow{C_{2}} N^{\prime} \cdot \tilde{n}^{\hat{n}}\right) \hat{n} \tag{7.8.0}
\end{equation*}
$$

but $\overrightarrow{C_{2}} N^{\prime}=\left\{\begin{array}{l}\ell+\Delta_{x}-\left(\ell+\Delta_{x}\right) \\ U_{n N^{\prime}}-U_{n 1,2} \\ U_{\zeta N^{\prime}}-U_{\zeta 1,2}\end{array}\right\}, \quad \overrightarrow{C_{2} N^{\prime \prime}}\left\{\begin{array}{l}\xi-\left(\ell+\Delta_{x}\right) \\ n-U_{n 1,2} \\ \zeta-U_{\zeta 1,2}\end{array}\right\}$
by substituting $\overrightarrow{C_{2}} N^{\prime}, \overrightarrow{C_{2}} N^{\prime \prime}$ and $\hat{\sim} \hat{n}$ in equation (7.8.0),

$$
\left\{\begin{array}{l}
\ell+\Delta_{x} \\
u_{n N^{\prime}} \\
u_{\zeta N^{\prime}}
\end{array}\right\}=\left\{\begin{array}{l}
\xi \\
\eta \\
\zeta
\end{array}\right\}+\left[\left\{\begin{array}{c}
0 \\
u_{n N^{\prime}}-u_{n 1,2} \\
u_{\zeta N^{\prime}}-u_{\zeta 1,2}
\end{array}\right\} \cdot\left\{\begin{array}{l}
u_{n 1,2} \\
u_{\zeta 1,2}
\end{array}\right]\right] \cdot \frac{1}{\alpha^{2}}\left\{\begin{array}{l}
\ell+\Delta_{x} \\
u_{n 1,2} \\
u_{\zeta 1,2}
\end{array}\right]
$$



Note that $U_{n l, 2}, U_{51,2}, U_{n N^{\prime}}$ and $U_{\zeta N^{\prime}}$ are obtained by inserting the corresponding coordinates into equations (3.4.1) and (3.4.2).

Thus, by substitution of $U_{n 1,2}, U_{n, 2}, U_{n N^{\prime}}, U_{\zeta N^{\prime}}$ along with $\alpha^{2}$ into equation (7.9.0), the vector

$$
\left\{\begin{array}{l}
\xi \\
n \\
\zeta
\end{array}\right\}
$$

can be calculated.
The vector $\left\{\begin{array}{l}\xi \\ n \\ \zeta\end{array}\right\}$ now provides the new location of the third node for the next loading increment on the local $\zeta$ axis.

By considering the global coordinates of node 1 (point $C_{1}$ ) and the direction cosines used in the local-global transformation matrix (equation 8.14.0) used in the previous load increment, the new position of node 3 with respect to the global system is established and this is illustrated as shown in equation (7.10.0)

$$
\left\{\begin{array}{l}
x  \tag{7.10.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{\top}[T G 2]^{\top}\left\{\begin{array}{l}
\xi \\
\eta \\
\zeta
\end{array}\right\}+[T G 1]^{\top}\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

where, the vector $\left\{\begin{array}{l}x \\ y \\ z\end{array}\right\}$ is the equivalent representation of the local vector $\left\{\begin{array}{l}\xi \\ n \\ \xi\end{array}\right]$. $\quad\{$

For details of equation (7.10.0), see proof of equation (9.16.0) in Chapter 9.

## Assembly of Finite Element Equations:

The element stiffness and the geometric stiffness matrices shown in equation (4.78.0) have been evaluated with respect to nodal displacement, referred to the local coordinate system $\xi, \eta$ and $\zeta$. However, in practice, structures are constructed with a variety of shapes and lengths in arbitrary orientations. Nevertheless, if a global axis system is chosen to measure nodal forces and displacements, then the solution becomes a relatively simple task.


FIGURE 7.4 .0

Figure (7.4.0) describes a typical arrangement of a structural assembly, in which the orientation of the two local axes $\left(\xi_{1}, \eta_{1}, \zeta_{1}\right)$ and $\left(\xi_{2}, \eta_{2}, \zeta_{2}\right)$ are defined with respect to the global axis ( $x, y$, z). Initially, the element and the geometric stiffness matrices for the system shown above are calculated with respect to their local axis system. However, the combined effect of both elements cannot be analysed unless both elements are referred to the common set of axes ( $x, y, z$ ). Let us assume that a relationship between the two axis systems can be obtained with a suitable transformation matrix [T], as shown below.

Thus

$$
\begin{equation*}
\left\{U_{E}\right\}_{L}=[T]\left\{U_{E}\right\}_{G} \tag{7.11.0}
\end{equation*}
$$

or

$$
\begin{equation*}
L U_{E-L}^{-}{ }_{L}^{\top}=L U_{E}-_{G}^{\top}[T]^{\top} \tag{7.12.0}
\end{equation*}
$$

where $\left\{U_{E}\right\}_{L}$ is the displacement referred to the local axis system $\left\{U_{E}\right\}_{G}$ is the displacement referred to the global axis system
[T] is the transformation matrix

Since the energy in deforming the structure under applied loads is independent of both local and global axis systems, then the work done in both systems is:

$$
\begin{equation*}
\frac{1}{2} L U_{E} ل_{L}^{\top}\left\{F_{L}\right\}=\frac{1}{2} L U_{E} ل_{G}^{\top}\left\{F_{G}\right\} \tag{7.13.0}
\end{equation*}
$$

but from equation (4.78.0)

$$
\left\{F_{L}\right\}=[K]\left\{U_{E}\right\}_{L}-[N]\left\{U_{E}\right\}_{L}
$$

or

$$
[[K]-[N]]\left\{U_{E}\right\}_{L}
$$

Now substituting $\left\{F_{L}\right\}$ in equation (7.13.0)

$$
\begin{equation*}
\left.L U_{E}\right]_{L}^{\top}[[K]-[N]]\left\{U_{E}\right\}_{L}=U_{E} \cdot{ }_{G}^{T}\left\{F_{G}\right\} \tag{7.14.0}
\end{equation*}
$$

Substituting equation (6.12.0) for $\left.L U_{E}\right]_{L}^{\top}$ in equation (7.13.0) yields

$$
\begin{equation*}
L U_{E} ل_{G}^{\top}[T]^{\top}[[K]-[N]]\left\{U_{E}\right\}_{L}=L U_{E} ل_{G}^{\top}\left\{F_{G}\right\} \tag{7.15.0}
\end{equation*}
$$

By comparing the multiplications of $L U_{E} ل_{G}^{\top}$ on both sides of the equation (7.15.0) yields,

$$
\begin{equation*}
[T]^{T}[[K]-[N]]\left\{U_{E}\right\}_{L}=\left\{F_{G}\right\} \tag{7.16.0}
\end{equation*}
$$

from equation (6.11.0), $\left\{U_{E}\right\}_{L}=[T]\left\{U_{E}\right\}_{G}$

$$
\therefore \quad\left\{F_{G}\right\}=[T]^{\top}[[K]-[N]][T]\left\{U_{E}\right\}_{G}
$$

or

$$
\begin{equation*}
\left\{F_{G}\right\}=[K]_{G}\left\{U_{E}\right\}_{G} \tag{7.17.0}
\end{equation*}
$$

where,

$$
\begin{equation*}
[K]_{G}=[T]^{\top}[[K]-[N]][T] \tag{7.18.0}
\end{equation*}
$$

The term $[K]_{G}$ in equation (7.18.0), provides the necessary transformation between the local and global systems.

## CHAPTER 8

## FORMULATION OF THE LOCAL GLOBAL TRANSFORMATION MATRIX

In Chapter 7 the relationship between the local and global coordinate systems was discussed. In Figure (8.1.0) a special case is illustrated, in which the local axial direction $\xi$ of the element is parallel to the global axis $x^{\prime}$. During loading of the structure, the axial load was placed at ' $A$ ' while the transverse forces and moments were applied at point B. The location of points $A$ and $B$ were referred with respect to the global system as shown in Figure (8.1.0).


FIGURE 8.1.0
where, $\left(y_{c}^{\prime}, z_{c}^{\prime}\right),\left(\bar{y}^{\prime}, \bar{z}^{\prime}\right)$ are the coordinates of the centroid and the shear centre of the cross section, measured with respect to the global axis system. Also ( $e_{\eta}, e_{\zeta}$ ) is the position of the shear centre referred to the local coordinate system, situated at the centroid.

As established in equations (3.4.1), (3.4.2) and (3.12.0), the displacement of an arbitrary point with respect to the local axis system is given by:

$$
\begin{aligned}
& U_{n}=U_{n 0}-\left(\zeta-e_{\zeta}\right) \theta_{\xi} \\
& U_{\zeta}=U_{\zeta 0}+\left(\eta-e_{n}\right) \theta_{\zeta} \\
& U_{\zeta}=U_{\zeta}-\left(\zeta-e_{\zeta}\right) \theta_{n 0}-\left(n-e_{n}\right) \theta_{\zeta 0}-\left(\omega_{D S}^{0}-\omega_{D O}^{0}\right) \theta_{\xi}^{\prime}(3.12 .0)
\end{aligned}
$$

By differentiating equations (3.4.1) and (3.4.2) with respect to $\xi$

$$
\begin{align*}
& \theta_{\eta}=\theta_{\eta 0}+\left(\eta-e_{\eta}\right) \theta_{\zeta}^{\prime}  \tag{8.1.0}\\
& \theta_{\zeta}=\theta_{\zeta 0}+\left(\zeta-e_{\zeta}\right) \theta_{\zeta}^{\prime} \tag{8.2.0}
\end{align*}
$$

If the loading is applied at points $A$ and $B$, then a rigid connection must exist between the load points and the cross-section.

As shown below, in predicting deformation let us use the global system instead of the local system. Thus, the displacement in the global $y^{\prime}$ direction becomes:

$$
\therefore \quad U_{y^{\prime}}=U_{y^{\prime} 0}-\left(z^{\prime}-\bar{z}^{\prime}\right)^{\theta} x^{\prime}
$$

by applying the coordinates of point B , (i.e. the point at which the transverse loads and moments are applied)

$$
\begin{equation*}
U_{y^{\prime} B}=U_{y^{\prime} 0}-\left(b_{z}{ }^{\prime}-\bar{z}\right) \theta_{x^{\prime}} \tag{8.4.0}
\end{equation*}
$$

Since the local and global axis systems are parallel to each other,

$$
\begin{equation*}
U_{y^{\prime} 0}=U_{n_{0}} \tag{8.5.0}
\end{equation*}
$$

$U_{y^{\prime} B}=U_{n O}-\left(b_{z^{\prime}}-\vec{z}\right)_{x^{\prime}}$

$$
\begin{equation*}
u_{n_{0}}=u_{y^{\prime}}-\left(\bar{z}^{\prime}-b_{z^{\prime}}\right) \theta_{x^{\prime}} \tag{8.7.0}
\end{equation*}
$$

From now on let us denote the displacement in global $y^{\prime}$ as $u_{y^{\prime}}$ instead of $U_{y^{\prime} A}$. Similarly, the displacement in the local $\zeta$ direction is,

$$
\begin{equation*}
u_{\zeta 0}=u_{z^{\prime}}+\left(\bar{y}^{\prime}-b_{y}^{\prime}\right)^{\prime} x^{\prime} \tag{8.8.0}
\end{equation*}
$$

and equations (7.1.0) and (7.2.0) become,

$$
\begin{align*}
& \theta_{\zeta 0}=\theta_{z^{\prime}}-\left(\overline{z^{\prime}}-b_{z}^{\prime}\right) \theta_{x^{\prime}}  \tag{8.9.0}\\
& \theta_{n 0}=\theta_{y^{\prime}}+\left(\overline{y^{\prime}}-b_{y}^{\prime}\right) \theta_{x^{\prime}} \tag{8.10.0}
\end{align*}
$$

Finally, the axial displacement $U_{g o}$ (i.e. the point where the axial load is applied) is,

$$
\begin{equation*}
U_{\xi_{C}}=U_{x^{\prime}}-.\left(z_{c}^{\prime}-e_{z^{\prime}}\right) \theta_{y^{\prime}}-\left(y_{c}^{\prime}-e_{y^{\prime}}\right) \theta_{z^{\prime}}-\left(\theta_{D C} c_{G}-\omega_{D A} c_{G}\right) \theta_{x}^{\prime} \tag{8.11.0}
\end{equation*}
$$

where ${ }^{\omega_{D C}} C_{g}$ and ${ }^{\omega_{D A}} C_{G}$ are the sectorial coordinates of the centroid and point $A$, referred to the global origin $C_{G}$ and the reference point $D$ on the cross section.

Let us concentrate on the general case, where the two axis systems are independent to one another as shown in Figure (8.2.0). Furthermore the basic parameters shown in Figure (8.2.0) are the same as that illustrated in Figure (8.1.0) except in this situation the axial direction of the local system is not parallel to that of the global system.


Point Coordinates with respect to
$x, y, z$ system

| $A$ | $\left(e_{x}, e_{y}, e_{z}\right)$ |
| :--- | :--- |
| $B$ | $\left(b_{x}, b_{y}, b_{z}\right)$ |
| $C$ | $\left(x_{c}, y_{c}, z_{c}\right)$ |
| 0 | $(\bar{x}, \bar{y}, \bar{z})$ |

FIGURE 8.2.0

By using the direction cosines, the relationship between the ( $x, y, z$ ) and the ( $x^{\prime}, y^{\prime}, z^{\prime}$ ) system is found to be:

$$
\left\{\begin{array}{l}
x^{\prime}  \tag{8.12.0}\\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=\left[\begin{array}{lll}
\cos \alpha_{x^{\prime} x} & \cos \alpha_{x^{\prime} y} & \cos \alpha_{x^{\prime} z} \\
\cos \alpha_{y^{\prime} x} & \cos \alpha_{y^{\prime} y} & \cos \alpha_{y^{\prime} z} \\
\cos \alpha_{z^{\prime} x} & \cos \alpha_{z^{\prime} y} & \cos \alpha_{z^{\prime} z}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

Since the axis system ( $x^{\prime}, y^{\prime}, z^{\prime}$ ) was drawn parallel to the $(\xi, \eta, \zeta)$ system,

$$
\left\{\begin{array}{l}
\xi  \tag{8.13.0}\\
\eta \\
\zeta
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \alpha_{\xi x} & \cos \alpha_{\xi y} & \cos \alpha_{\xi z} \\
\cos \alpha_{n} x & \cos \alpha_{n y} & \cos \alpha_{\eta z} \\
\cos \alpha_{\zeta x} & \cos \alpha_{\zeta y} & \cos \alpha_{\zeta z}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

(Note that equation (8.13.0) only describes the relationship of direction cosines and not the absolute value of coordinates).

By substituting the result of equation (8.13.0) in equations (7.7.0), (8.8.0), (8.9.0), (8.10.0) and (8.11.0), the final form of the transformation matrix is established and this is shown in equation (8.14.0). The square matrix $(9 \times 9)$ in equation (8.14.0) represents the transformation of displacements at one node only. However, the generalized beam element always consists of two nodes. Thus, the overall transformation of the displacements at nodes 1 and 2 is:

$$
\left\{\begin{array}{l}
\Delta_{1}  \tag{8.15.0}\\
\Delta_{2}
\end{array}\right\}=\left[\begin{array}{ll}
{\left[T_{11}\right]} & {[0]} \\
{[0]} & {\left[T_{11}\right]}
\end{array}\right]\left\{\begin{array}{l}
\Delta_{1} \\
\Delta_{2}
\end{array}\right\}
$$



$H 1=(E 1-A 1) \cos a_{n x}-(C 1-F I) \cdot \cos a_{5 x}$
$H 2=(E 1-A 1) \cos a_{n y}-(C 1-F 1) \cos a_{r y}$
$H 3=(E 1-A 1) \cos a_{n Z}-(C 1-F 1) \cos { }_{52}$
$11=B 1-A 1$
where the terms in $\left[T_{11}\right]$ are given by the square matrix ( $9 \times 9$ ) in equation (8.14.0).
 nodes 1 and 2 respectively.

An equivalent form of the transformation matrix for forces can be found by considering the energy at both local and global levels. This is explained in the following analysis.

Since the energy remaining is fixed at both local and global levels, then,

$$
\begin{equation*}
\left.\left.\frac{1}{2} L U_{E}\right\lrcorner_{L}^{\top}\left\{F_{E}^{F}\right\}_{L}=\frac{1}{2} L U_{E}\right\lrcorner_{G}^{\top}\left\{\left\{_{E}^{F}\right\}_{G}\right. \tag{8.16.0}
\end{equation*}
$$

is valid
where, $\left\{U_{E}\right\}$ are the displacements $\left\{F_{G}\right\}$ are the forces
and $L$ and $G$ are the identities for local and global systems.

From equation (7.11.0)

$$
\left\{U_{E}\right\}_{L}=[T]\left\{U_{E}\right\}_{G} \quad \text { or } L U_{E} J^{\top}=L U_{E} J^{\top}[T]^{\top}
$$

Substituting equation (7.11.0) in equation (8.16.0) yields,

$$
\begin{equation*}
L U_{E} J_{G}^{\top}[T]^{\top}\left\{F_{E}\right\}_{L}=L U_{E} J_{G}^{T}\left\{F_{E}\right\}_{G} \tag{8.17.0}
\end{equation*}
$$

By considering the terms to the right of $L U_{E} J_{G}^{\top}$ in both sides of the equation (8.17.0),

$$
\begin{equation*}
\left\{F_{E}\right\}_{G}=[T]^{T}\left\{F_{E}\right\}_{L} \tag{8.18.0}
\end{equation*}
$$

The equation shown above provides the necessary transformation of the forces from the local to the global axes system.

Application of the solution in equation (8.18.0) to a specific problem:


FIGURE 8.3.0

Assumptions:

1. Local and global axis systems are parallel to each other
2. Transverse loads are applied at point $B$ on the cross section
3. Axial forces are applied at $A$ on the cross section.

From assumption (1), the angles,

$$
\alpha_{\xi x}=\alpha_{n y}=\alpha_{\zeta z}=0^{0}, \alpha_{\zeta x}=\alpha_{n x}=\alpha_{n z}=\gamma_{\zeta y}=\gamma_{\xi y}=90^{0}
$$

Thus, the square matrix in equation (8.14.0) is simplified to,
$A_{1}=\bar{z}, B_{1}=b_{z}, c_{1}=\bar{y}, D_{1}=b_{y}, E_{1}=e_{z}, F_{1}=e_{y}, H_{1}=z_{c}, H_{2}=y_{c}$
$G_{1}=\omega_{D C}^{0}-\omega_{D A}^{0}-\bar{y}\left(e-z_{c}\right)-\bar{z}\left(y_{c}-e_{y}\right)$
(For definitions of $A_{1}, B_{1}, C_{1}, D_{1}, E_{1}, F_{1}$ and $G_{1}$ see subtitles of equation (8.14.0).

Substitution of these values in the corresponding elements of the square matrix in equation (8.14.0) yields,
$\left[\begin{array}{l}u_{50} \\ u_{n 0} \\ u_{50} \\ \theta_{50} \\ \theta_{n 0} \\ \theta_{50} \\ \theta_{\overline{50}}^{\prime} \\ 0 \\ 0\end{array}\right]=\left[\begin{array}{lllllllll}1 & 0 & 0 & 0 & \left(e_{z}-z_{c}\right)\left(e_{y}-y_{c}\right) & \left(\omega_{D C}^{0} \omega_{D A}^{0}-\bar{y}\left(e_{z}-z_{c}\right)-\bar{z}\left(y_{c}-e_{y}\right)\right. & 0 & 0 \\ 0 & 1 & 0 & \left(b_{z}-\bar{z}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \left(\bar{y}-b_{y}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \left(\bar{y}-b_{y}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \left(b_{z}-\bar{z}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]\left\{\begin{array}{l}u_{x} \\ u_{y} \\ u_{y} \\ u_{z} \\ \theta_{x} \\ \theta_{y} \\ \theta_{z} \\ \theta_{x}^{\prime} \\ \theta_{x} \\ \theta_{y}^{\prime} \\ \theta_{z}^{\prime}\end{array}\right\}$
(8.19.0)

Substitution of the transpose of the above square matrix in equation (8.18.0) yields the corresponding transformation matrix for the forces. This is shown in equation (8.20.0).

(8.20.0)

The elements of the square matrix in equation (8.20.0) is in agreement with the result shown in reference (26), except for the term ( $\omega_{D C}$ $\left.\omega_{D A}^{0}-y\left(e_{z}-z_{c}\right)-z\left(y_{c}-e_{y}\right)\right)$, which is the generation of the warping moment due to application of the axial load. Many authors in the past seem to have ignored the effect of the warping moment due to application of axial load and transverse moments. Furthermore, what is shown in equations (8.19.0) and (8.20.0) is a special case of the more generalized condition shown in equation (8.14.0). Up to the time of writing this thesis, no information was found regarding the existence of a generalized transformation matrix.

## Calculation of Sectorial Coordinates Referred to any Arbitrary Point Other than the Shear Centre:

The sectorial coordinate can be described as an entity which always represents the amount of warping displacement along the periphery of the cross-section. However, to estimate the value of the sectorial coordinates at a particular point, the following information must be provided:
a) Linear coordinates referred to the centroidal principal axis
b) The shear centre is the point of rotation and also the origin of the sliding radius
c) The principal radius is found by referring to a specific point on the cross-section whereby any sectorial coordinate referred from this point would become principal sectorial coordinates.

The sectorial coordinate values in equation (3.12.0) and in the transformation matrix in equation (8.14.0) are referred to the global reference point. A relationship must therefore be established such that the sectorial coordinates referred to any other arbitrary point can be conveniently transformed from its known principal sectorial coordinate.

Let us consider the diagram shown in Figure (8.4.0). The local and global axes are placed parallel to each other and they are displaced as in Figure (8.1.0). The sign convention used in defining sectorial coordinates is the same as the right handed convention illustrated in Table (2.T.0).

In Figure (8.4.0) $B$ is an arbitrarily chosen point on the cross section. Two normals are projected to the tangent at point $B$, from the local and global origin whose instantaneous lengths are defined as $\rho_{0}$ and $\rho$ respectively.

By definition, the sectorial coordinate with respect to the local axis system, is


FIGURE 8.4.0

$$
\omega_{0}=\int_{0}^{S} \rho_{0} d S
$$

where $D$ is the principal reference point and $O D$ is the principal radius.

Similarly, the sectorial coordinate with respect to the global axis system, is

$$
\omega=\int_{0}^{S} \rho d S
$$

From Figure (8.4.0)

$$
\begin{align*}
\frac{1}{2} d \omega_{0} & =\frac{1}{2} \rho_{0} d S=0_{0}^{Q}= \\
& =\frac{1}{2}\left(n+d_{n}\right)(\zeta+d \zeta)+\frac{1}{2}(2 \zeta+d \zeta)(-d n)-\frac{1}{2} n \zeta \\
& =\frac{1}{2}[n \zeta+\zeta d n+n d \zeta-2 \zeta d n-n \zeta]=\frac{1}{2}(n d \zeta-\zeta d n) \tag{8.21.0}
\end{align*}
$$

However, the two linear coordinate systems are related by the following relationship as follows,

$$
\begin{equation*}
\eta=y^{\prime}-\bar{y}^{\prime} \quad \text { and } \quad \zeta=z^{\prime}-\bar{z}-1 \tag{8.22.0}
\end{equation*}
$$

Therefore, the sectorial coordinate $\omega_{D B}^{0}$ is:

$$
\begin{align*}
\omega_{D B}^{0} & =\int_{0}^{S} \rho_{0} d S=\left\{\left(y^{\prime}-\overline{y^{-1}}\right) d z^{\prime}-\left(\bar{z}-\overline{z^{\prime}}\right) d y^{\prime}\right\} \\
\omega_{D B}^{0} & =\int_{0}^{S}\left\{y^{\prime} d z^{\prime}-\bar{y}^{\prime} d z^{\prime}-z^{\prime} d y^{\prime}+\bar{z}^{\prime} d y^{\prime}\right\} \\
& =\int_{0}^{s}\left(y^{\prime} d z^{\prime}-z^{\prime} d y^{\prime}\right) d s-\bar{y}^{\prime}\left(z^{\prime}-z_{D}\right)+\bar{z}^{\prime}\left(y^{\prime}-y_{D}\right) \\
\omega_{D B}^{0} & =\int_{0}^{s}\left(y^{\prime} d z^{\prime}-z^{\prime} d y^{\prime}\right) d s-\bar{y}^{\prime} z^{\prime}+\bar{y}^{\prime} z_{D}^{\prime \prime}+\bar{z}^{\prime} y^{\prime}-\bar{z}^{\prime} y_{D} \tag{8.23.0}
\end{align*}
$$

Now, apply the relationship found in equation (8.23.0) to the problem in Figure (8.1.0), where the axial load is acting arbitrarily, but normal to the plane of cross section.

Equation (8.23.0) can be re-written as,

$$
\begin{equation*}
\omega_{D B}^{0}=\omega_{D B}^{C_{G}} \quad-\bar{y}^{\prime} z^{\prime}+\bar{y}^{\prime} z_{D}^{\prime}+\bar{z}^{\prime} y^{\prime}-\bar{z}^{\prime} y_{D}^{\prime} \tag{8.24.0}
\end{equation*}
$$

By applying this result in equation (8.14.0) to the centroid and to point $A$ :

$$
\begin{align*}
& \omega_{D C}^{D}=\begin{array}{cc}
C_{G C} & -\bar{y}^{\prime} z_{c}^{\prime}+\bar{z}^{\prime} y_{c}{ }_{c}^{\prime}+\bar{y}^{\prime} z_{D}^{\prime}-\bar{z}^{\prime} y_{D}^{\prime}
\end{array}  \tag{8.25.0}\\
& \omega_{D B}^{o}=\omega_{D B}^{C_{G}} \quad-\bar{y}^{\prime} e_{z}^{\prime}+\bar{z}^{\prime} \cdot e_{y}^{\prime}+\bar{y} z_{D}^{\prime}-\bar{z}^{\prime} y_{D}^{\prime} \tag{8.26.0}
\end{align*}
$$

The definitions for $\left(e_{y}^{\prime}, e_{z}^{\prime}\right),\left(y_{c}^{\prime}, z_{c}^{\prime}\right)$ are found in Figure (8.1.0).
Therefore, by subtracting equation (8.26.0) from equation (8.25.0) and rearranging:

$$
\begin{equation*}
\left(\omega_{D C}{ }^{C_{G}}-\omega_{D B}{ }^{C_{G}}\right)=\left(\omega_{D C}^{0}-\omega_{D B}^{0}\right)+\overline{y^{\prime}}\left(z_{C}^{\prime}-e_{z}^{\prime}\right)-\overline{z^{\prime}}\left(y_{C}^{\prime}-e_{y}^{\prime}\right) \tag{8.27.0}
\end{equation*}
$$

Apply the results developed in equation (8.27.0) to the more general problem shown in Figure (8.5.0)


The relationship between the ( $x, y, z$ ) system and the ( $x^{\prime}, y^{\prime}, z^{\prime}$ ) system, as before, is given by:

$$
\left\{\begin{array}{l}
x^{\prime}  \tag{8.28.0}\\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \alpha x^{\prime} x & \cos \alpha x^{\prime} y & \cos \alpha x^{\prime} z \\
\cos \alpha y^{\prime} x & \cos \alpha y^{\prime} y & \cos \alpha y^{\prime} z \\
\cos \alpha z^{\prime} x & \cos \alpha z^{\prime} y & \cos \alpha z^{\prime} z
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

See also Figure (8.2.0).

Since the ( $x^{\prime}, y^{\prime}, z^{\prime}$ ) system is parallel to the ( $\xi, \eta, \zeta$ ) system, then it follows that,

$$
\left\{\begin{array}{l}
x^{\prime}  \tag{8.29.0}\\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \alpha_{\xi x} & \cos \alpha_{\xi y} & \cos \alpha_{\xi z} \\
\cos \alpha_{n x} & \cos \alpha_{n y} & \cos \alpha_{n z} \\
\cos \alpha_{\zeta x} & \cos \alpha_{\zeta y} & \cos \alpha_{\zeta z}
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

(See also the footnote for equation (8.13.0).

Substituting the result of equation (8.29.0) in equation (8.27.0):

$$
\begin{align*}
\mathrm{C}_{G}{ }^{C_{G}} & =\left(\omega_{D C}{ }^{-\omega_{D A}}\right) \\
& =\left(\omega_{D C}^{0}-\omega_{D A}^{0}\right)+\left(\bar{x} \cos \alpha_{n x}+\bar{y} \cos \alpha_{n y}+\bar{z} \cos \alpha_{n z}\right)\left\{\left(x_{c} \cos \alpha_{\zeta x}+y_{c} \cos \alpha_{\zeta y}+\right.\right. \\
& \left.\left.+z_{c} \cos \alpha_{\zeta z}\right)-\left(e_{x} \cos \alpha_{\zeta x}+e_{y} \cos \alpha_{\zeta y}+e_{z} \cos \alpha_{\zeta z}\right)\right\}- \\
& -\left(\bar{x} \cos \alpha_{\zeta x}+\bar{y} \cos \alpha_{\zeta y}+\bar{z} \cos \alpha_{\zeta z}\right)\left\{\left(x_{c} \cos \alpha_{n x}+y_{c} \cos \alpha_{n y}+z_{c} \cos \alpha_{n z}\right)\right.  \tag{8.30.0}\\
& \left.-\left(e_{x} \cos \alpha_{n x}+e_{y} \cos \alpha_{n y}+e_{z} \cos \alpha_{n z}\right)\right\}
\end{align*}
$$

This result ( $\omega_{D C}{ }^{C_{G}}-{ }^{\omega_{D A}}{ }_{G}$ ) can now be used in equation (8.11.0) to calculate the term G1 of the transformation matrix shown in equation (8.14.0)

## CHAPTER 9

## EVALUATION OF THE DIRECTION COSINES AND LINEAR DIMENSIONS REQUIRED IN THE LOCAL GLOBAL TRANSFORMATION MATRIX

In Chapter 8 the development of the local global transformation matrix and its use in large structural assemblies was obtained. To compute the transformation matrix in equation (8.14.0) one must provide the direction cosines between the local and global axis systems. In practice the choice of structural sections is virtually unlimited and also in complex asemblies, one may decide to join specific points on the cross-sections of the structural elements which are not necessarily the centroidal points, but are chosen mainly.from the physical design constraints. Under these circumstances one may conveniently choose a point such as 'p' in Figure (9.1.0) to define the global nodal geometry. Simultaneously the local axis is also placed at $P$ so that the cross-sectional geometry can be defined locally.


The axial nodes of the structural element shown in Figure (9.1.0) are $P\left(x_{1}, y_{1}, z_{1}\right)$ and $Q\left(x_{2}, y_{2}, z_{2}\right)$. The third node $R\left(x_{3}, y_{3}, z_{3}\right)$ is placed on the local $z^{\prime}$ axis so that $x^{\prime}, y^{\prime} z^{\prime}$ are also orthogonal.

The axis $X, Y, Z$ is drawn parallel to the $x, y, z$ system and point $P$ such that,

$$
\left\{\begin{array}{l}
x  \tag{9.1.0}\\
y \\
z
\end{array}\right\}=\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

The position and orientation of the centroid and the principal axis was calculated and placed at point $C\left(x_{a}^{\prime}, y_{a}^{\prime}, z_{a}^{\prime}\right)$ with respect to the local axis system $x^{\prime}, y^{\prime}, z^{\prime}$. A fourth set of axes parallel to $x^{\prime}$, $y^{\prime}, z^{\prime}$ was also drawn at the centroid, and is defined by $X^{\prime}, Y^{\prime}, Z^{\prime}$. From the conditions mentioned so far, the following relationships can be obtained:

$$
\begin{align*}
& \left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=\left[\begin{array}{lll}
\cos \alpha x^{\prime} x & \cos \alpha x^{\prime} y & \cos \alpha x^{\prime} z \\
\cos \alpha y^{\prime} x & \cos \alpha y^{\prime} y & \cos \alpha y^{\prime} z \\
\cos \alpha z^{\prime} x & \cos \alpha z^{\prime} y & \cos \alpha z^{\prime} z
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}  \tag{9.2.0}\\
& \left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=[T G 1]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\} \tag{9.3.0}
\end{align*}
$$

or

$$
\left\{\begin{array}{l}
x  \tag{9.4.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{\top} \quad\left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}
$$

This can only be written since matrix [TG1] is orthogonal. By substituting for $\left\{\begin{array}{l}X \\ Z\end{array}\right\}$ in equation (9.1.0):

$$
\left.\left\{\begin{array}{l}
x^{\prime}  \tag{9.5.0}\\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=[T G 1]\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right\}-\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}\right)
$$

Similarly, by considering the axis systems ( $X^{\prime}, Y^{\prime}, Z^{\prime}$ ) and ( $\xi, \eta$, ૬), the following can be written:

$$
\left\{\begin{array}{l}
\xi  \tag{9.6.0}\\
\eta \\
\zeta
\end{array}\right\}=\left[\begin{array}{ccc}
\cos \alpha_{\xi X^{\prime}} & \cos \alpha_{\xi \gamma^{\prime}} & \cos \alpha_{\xi} Z^{\prime} \\
\cos \alpha_{\eta X^{\prime}} & \cos \alpha_{\eta} \gamma^{\prime} & \cos \alpha_{\eta} Z^{\prime} \\
\cos \xi_{\zeta X^{\prime}} & \cos \alpha_{\zeta} \gamma^{\prime} & \cos \alpha_{\zeta} Z^{\prime}
\end{array}\right]\left\{\begin{array}{l}
x^{\prime} \\
\gamma^{\prime} \\
Z^{\prime}
\end{array}\right\}
$$

This can be re-written as:

$$
\left\{\begin{array}{l}
\xi  \tag{9.7.0}\\
n \\
\zeta
\end{array}\right\}=[\text { TG2 }] \quad\left\{\begin{array}{l}
X^{\prime} \\
y^{\prime} \\
Z^{\prime}
\end{array}\right\}
$$

or

$$
\left\{\begin{array}{l}
X^{\prime}  \tag{9.8.0}\\
Y^{\prime} \\
Z^{\prime}
\end{array}\right\}=[T G 2]^{\top}\left\{\begin{array}{l}
\xi \\
\eta \\
\zeta
\end{array}\right\}
$$

This can only be achieved since matrix [TG2] is orthogonal. The position of the centroid is also defined with respect to the axis system ( $x^{\prime}, y^{\prime}, z^{\prime}$ ).
Therefore,

$$
\left\{\begin{array}{l}
x^{\prime}  \tag{9.9.0}\\
y^{\prime} \\
z^{\prime}
\end{array}\right\}=\left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}+\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}
$$

By substituting for $\left\{\begin{array}{l}X_{\prime}^{\prime} \\ Y^{\prime} \\ Z^{\prime}\end{array}\right\}$ in equation (9.7.0),

$$
\left\{\begin{array}{l}
\boldsymbol{\xi}  \tag{9.10.0}\\
n \\
\zeta
\end{array}\right\}=[T G 2]\left(\left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}-\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}\right)
$$

Now, substituting $\left\{\begin{array}{l}x^{\prime} \\ y^{\prime} \\ z^{\prime}\end{array}\right\}$ in equation (9.10.0):

$$
\left\{\begin{array}{l}
\xi  \tag{9.11.0}\\
n \\
\zeta
\end{array}\right\}=[T G 2]\left([T G 1]\left(\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}-\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}\right)-\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}\right)
$$

or

$$
\left\{\begin{array}{l}
\xi  \tag{9.12.0}\\
\eta \\
\zeta
\end{array}\right\}=[T G 2][T G 1]\left\{\begin{array}{l}
x-x_{1} \\
y-y_{1} \\
z-z_{1}
\end{array}\right\}-[T G 2]\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}
$$

By substituting $\left\{\begin{array}{l}x \\ y \\ z\end{array}\right\}$ in equation (9.4.0) into equation (9.1.0):

$$
\left\{\begin{array}{l}
x  \tag{9.13.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{\top}\left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

and by substituting equation (9.8.0) in equation (9.13.0):

$$
\left\{\begin{array}{l}
x  \tag{9.14.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{T}\left(\left\{\begin{array}{l}
x^{\prime} \\
y^{\prime} \\
z^{\prime}
\end{array}\right\}+\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}\right)+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

Substituting equation (9.8.0) in equation (9.14.0):

$$
\left\{\begin{array}{l}
x  \tag{9.15.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{\top}\left([T G 2]^{\top}\left\{\begin{array}{l}
\xi \\
n \\
\zeta
\end{array}\right\}+\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime} \\
a_{1}
\end{array}\right\}\right)+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

By rearranging the above equation, it can be written as,

$$
\left\{\begin{array}{l}
x  \tag{9.16.0}\\
y \\
z
\end{array}\right\}=[T G 1]^{\top}[T G 2]^{\top}\left\{\begin{array}{l}
\xi \\
n \\
\zeta
\end{array}\right\}+[T G 1]^{\top}\left\{\begin{array}{l}
x_{a}^{\prime} \\
y_{a}^{\prime} \\
z_{a}^{\prime}
\end{array}\right\}+\left\{\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right\}
$$

Calculation of the terms in the square matrix [TG1]:

By referring to Figure (9.1.0),

$$
\underset{\sim}{P R}=\left(\begin{array}{l}
x_{3}-x_{1} \\
y_{3}-y_{1} \\
z_{3}-z_{1}
\end{array}\right)=\left(x_{3}-x_{1}\right) \underset{\sim}{i}+\left(y_{3}-y_{1}\right){\underset{\sim}{m}}_{\text {(9.17.0) }}^{j+\left(z_{3}-z_{1}\right) \underset{\sim}{k}}
$$

$$
\underset{\sim}{P Q}=\left(\begin{array}{l}
x_{2}-x_{1}  \tag{9.18.0}\\
y_{2}-y_{1} \\
z_{2}-z_{1}
\end{array}\right)=\left(x_{2}-x_{1}\right) \underset{\sim}{i}+\left(y_{2}-y_{1}\right) \underset{\sim}{j}+\left(z_{2}-z_{1}\right) \underset{\sim}{k}
$$

By taking the vector product between $\underset{\sim}{P R}$ and $P Q_{\sim}$, vector $P N$ is obtained. Thus,

$$
\underset{\sim}{P R} \times \underset{\sim}{P Q}=\underset{\sim}{P N}=\left|\begin{array}{ccc}
\underset{\sim}{i} & \left.\underset{\sim}{i}{ }^{i} x_{1}\right) & \left(y_{3}-y_{1}\right) \\
\left(z_{3}-z_{1}\right) \\
\left(x_{2}-x_{1}\right) & \left(y_{2}-y_{1}\right) & \left(z_{2}-z_{1}\right)
\end{array}\right|
$$

Evaluation of the determinant in the expression above yields,

$$
\begin{align*}
\underset{\sim}{P N} & =\left(\left(y_{3}-y_{1}\right)\left(z_{2}-z_{1}\right)-\left(y_{2}-y_{1}\right)\left(z_{3}-z_{1}\right)\right) \underset{\sim}{i} \\
& -\left(\left(x_{3}-x_{1}\right)\left(z_{2}-z_{1}\right)-\left(x_{2}-x_{1}\right)\left(z_{3}-z_{1}\right)\right){ }_{\sim}^{j} \\
& +\left(\left(x_{3}-x_{1}\right)\left(y_{2}-y_{1}\right)-\left(x_{2}-x_{1}\right)\left(y_{3}-y_{1}\right)\right){\underset{\sim}{n}}^{2} \tag{9.19.0}
\end{align*}
$$

more precisely equivalent to,

$$
\underset{\sim}{P N}=a_{1} \underset{\sim}{i}+b_{1} \dot{\eta}^{i}+c_{1}^{k}
$$

where the absolute values of $a_{1}, b_{1}$ and $c_{1}$ are given in equation (8.19.0) .

Now, by taking the scalar products of $P Q, P_{\sim}^{N}$ and $P R$ with the unit vectors in the directions of $\underset{\sim}{x}, \underset{\sim}{y}$ and $\underset{\sim}{Z}$ respectively, the direction cosines required in matrix [TG1], are calculated. This can be described as follows. Let us define the unit vectors in $\underset{\sim}{x}, \underset{\sim}{x}$ and $\underset{\sim}{z}$ to be $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$.

Then, the direction cosines are:

$$
\left\{\begin{array}{l}
\cos \alpha^{\prime} x  \tag{9.20.0}\\
\cos x^{\prime} y \\
\cos \alpha^{\prime} z
\end{array}\right\}=\left\{\begin{array}{l}
i \\
j \\
k
\end{array}\right\} \cdot\left\{\begin{array}{l}
\left(x_{2}-x_{1}\right) i \\
\left(y_{2}-y_{1}\right) j_{\sim}^{j} \\
\left(z_{2}-z_{1}\right){ }_{\sim}^{2}
\end{array}\right\}
$$

By solving equation (9.20.0):

$$
\left\{\begin{array}{l}
\cos \alpha x^{\prime} x  \tag{9.21.0}\\
\cos \alpha x^{\prime} y \\
\cos \alpha x^{\prime} z
\end{array}\right\}=\left\{\begin{array}{l}
\left(x_{2}-x_{1}\right) / A 1 \\
\left(y_{2}-y_{1}\right) / A 1 \\
\left(z_{2}-z_{1}\right) / A 1
\end{array}\right\}
$$

where $A 1=\left(\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}\right)^{\frac{1}{2}}$
Similarly, $\left\{\begin{array}{l}\cos \alpha y^{\prime} x \\ \cos \alpha y^{\prime} y \\ \cos \alpha y^{\prime} z\end{array}\right\} \quad$ and $\quad\left\{\begin{array}{l}\cos \alpha z^{\prime} x \\ \cos z^{\prime} y \\ \cos \alpha z^{\prime} z\end{array}\right\} \quad$ are calculated.

The matrix [TG1] is now fully defined.

It was pointed out that both the principal axes $\eta, 5$ and axes $y^{\prime}, z^{\prime}$ are on the same plane as that of the cross-section, and thus, the orientation between the two sets of axes can be found by satisfying the condition shown below:

$$
\begin{equation*}
\tan 2 \alpha=\frac{2 I_{y^{\prime} z^{\prime}}}{I_{y^{\prime} y^{\prime}}-I_{z^{\prime} z^{\prime}}} \tag{9.22.0}
\end{equation*}
$$

Equation (9.22.0) gives the condition required for the principal axes, in which $I_{y^{\prime} z^{\prime}}$ is the product moment of area and $I_{y^{\prime} y^{\prime}}, I_{z^{\prime} z^{\prime}}$ are the second moment of area about $y^{\prime}$ and $z^{\prime}$ axes.


FIGURE 9.2.0

Similarly, axes $Y^{\prime}$, $Z^{\prime}$ are parallel to axes $n-\zeta$ and also axis $X^{\prime}$ is overlapping with axis $\xi$, then,

$$
\begin{aligned}
& \alpha_{\xi X^{\prime}}=0^{0} \quad{ }_{\xi Y^{1}}^{\alpha}=\alpha_{\zeta Z^{\prime}}^{\alpha}={ }_{\eta X^{\prime}}^{\alpha}={ }_{\zeta X^{\prime}}^{\alpha}=90^{\circ} \\
& \alpha_{\eta Y^{\prime}}=\alpha, \alpha_{\zeta Z^{\prime}}=\alpha, \alpha_{\eta Z}=90-\alpha, \alpha_{\zeta Y^{\prime}}=90+\alpha
\end{aligned}
$$

See Figures (8.1.0) and (8.2.0).

Thus applying the conditions above in to equation (8.6.0) yields to,

$$
[T G 2]=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{9.23.0}\\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{array}\right]
$$

By making a close examination of equation (9.12.0), it can be seen
that for the orientation of the two axes systems,

$$
\left\{\begin{array}{l}
\xi  \tag{9.24.0}\\
n \\
\zeta
\end{array}\right\}=[T G 2][T G 1] \quad\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\} .
$$

(Note that the relationship in equation (9.24.0) is only valid for the calculation of angles between the axis only and for the coordinate transformation one must strictly use equation (9.12.0)).

Let us define, [TG3] = [TG2][TG1].
Since matrices [TG2] and [TG1] are fully known, matrix [TG3] can be calculated, and the terms in matrix [TG3] can now be used. as a direction cosine required in the global/local transformation matrix of equation (8.14.0).

Calculation of the positions of the shear centre with respect to the global axis system:


FIGURE 9.3.0

In Figure (9.3.0), point ' 0 ' is the position of the shear centre ( $e_{n}$, $e_{\zeta}$ ), defined by the principal axes $\eta$ and $\zeta$. Let us also define another point $P\left(p_{n}, p_{\zeta}\right)$, used in predicting shear centre coordinates as expressed by the equation shown below,

$$
\begin{align*}
& a_{n}=p_{\eta}+\frac{1}{I_{n \eta}} \int_{A} \omega_{\text {pos }} \zeta d A  \tag{9.25.1}\\
& a_{\zeta}=p_{\zeta}-\frac{1}{I_{\zeta \zeta}} \int_{A} \omega_{\text {pos }} n d A \tag{9.25.2}
\end{align*}
$$

where $\left\{\omega_{\text {pos }} \zeta d A\right.$ and $\left\{\omega_{\text {pos }} n d A\right.$ are the linear sectorial moments about the $\eta$ and $\zeta$ axes.

By inserting ( $a_{\eta}, a_{\zeta}$ ) in equation (9.16.0) the position of the shear centre with respect to the global axis can now be calculated. By simultaneous application of $A\left(b_{y^{\prime}}, b_{z^{\prime}}\right)$ and $B\left(e_{y^{\prime}}, e_{z^{\prime}}\right)$ in equation (9.10.0) followed by equation (9.16.0), the position of the axial and transverse load applications, with respect to the global axis, can also be calculated (see Figure (8.2.0)).

The coordinates of the shear centre, axial and transverse force applications with respect to the global axis system, and also the direction cosines between the principal and global axis are used in the local/global transformation matrix of equation (8.14.0). The application of the theory into a programming concept is illustrated in the Appendix 1 .

## CHAPTER 10

## SOLUTION TO PRACTICAL PROBLEMS

The basic theory for buckling of thin-walled structural members has been extensively discussed in the earlier chapters and some practical problems to which alternative solutions ăre available in the literature are discussed in this chapter.

## Solutions for Small Displacement Theory:

Let us concentrate on a beam element with the cross-section shown in Figure 10.1 to investigate the behaviour for:

1. bending on the $x-y$ plane
2. bending on the $x-z$ plane
3. torsion and torsion warping about the axial direction.



View on section $x-x$

Note:-
all dimensions are in mm
thickness 2.0 mm all over
FIGURE 10.1

The deflection caused by bending due to load $F$ at the free end of the cantilever was initially calculated using the Euler-Bernoulli theory leading to equation 10.1.0. The results were compared for single element solutions as shown in Table 10.T.1:

$$
\begin{array}{ll}
\delta y=\frac{F y e^{3}}{3 E I_{z z}} & 10.1 .1 \\
\delta_{z}=\frac{E y e^{3}}{3 E I_{y y}} & 10.1 .2
\end{array}
$$

The cases for torsion and torsion warping depend upon the boundary conditions used in the differential equation (10.2). The solution to St Venant torsion is obtained with free warping throughout the beam while for the case of the combination of torsion warping and St Venant, the solution is found by introducing the condition of warping restrained at the fixed end. The generalized solution for the torsion problems shown in equation (10.2) is presented in equation (10.3):

$$
\begin{equation*}
M_{x}=T_{S . v}+T_{w}=G J \theta_{z}^{\prime}-E \Gamma \theta_{x}^{\prime \prime \prime} \tag{10.2}
\end{equation*}
$$

Hence,

$$
\theta_{x}=\frac{M_{x}}{\lambda^{3} E \Gamma}\left(C_{1}+c_{2} \cosh \lambda x+c_{3} \sinh \lambda x+\lambda x\right)(10.3)
$$

where,

$$
\lambda=\downarrow \frac{G J}{E \Gamma}
$$

St Venant torsion problem:


$$
\begin{aligned}
& \theta_{x}=0 \\
& \theta_{x}^{\prime \prime}=0
\end{aligned}
$$

thickness 9.6 mm all over


Note:-
all dimensions are in mm

| Force in y direction (N) | Force in z direction (N) | Second moment of area about $\begin{gathered} y-y \\ \left(m^{4}\right) \end{gathered}$ | Second moment of area about $\left(\begin{array}{c} z-z \\ \left(m m^{4}\right) \end{array}\right.$ | ```Deflection y direction (one element-FE) (mm)``` | ```Deflection z direction (one element-FE) (mm)``` | Exact solution (y\&z) (mm) | $\begin{aligned} & \text { Error } \\ & \% \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | $0.4 \times 10^{3}$ | $0.534 \times 10^{5}$ | $0.341 \times 10^{6}$ | 0.0 | $0.961 \times 10^{2}$ | $0.961 \times 10^{2}$ | 0 |
| $0.1 \times 10^{4}$ | 0.0 | $0.534 \times 10^{5}$ | $0.341 \times 10^{6}$ | $0.376 \times 10^{2}$ | 0.0 | $0.376 \times 10^{2}$ | 0 |

TABLE (10.T.1)

The boundary condition shown in Figure (10.2) yields $C_{1}=C_{2}=C_{3}=0$ in equation (10.3). Thus equation (10.3) is simplified to,

$$
\theta_{x}=\frac{M_{x} \cdot x}{\lambda^{2} E \Gamma}=\frac{1}{G J} M_{x}
$$

t.

Therefore at $x=\ell$, the twist at the free end,

$$
\begin{equation*}
\theta_{x \ell}=M_{\ell} \cdot \frac{\ell}{G J} \tag{10.10.4}
\end{equation*}
$$

Torsion warping problem:


At $x=0$,


View on section $x-x$

Note:-

$$
\theta_{x}=0
$$

all dimensions are in mm
thickness 1.6 mm all ower

$$
\theta_{x}^{\prime}=0
$$

FIGURE 10.3

The boundary condition shown in Figure (10.3) yields $C_{1}=-C_{2}=$ $-\operatorname{Tanh}(\lambda \ell)$ and $C_{3}=-1$ in equation (10.2). Thus equation (10.2) is simplified to,

$$
\theta_{x}=\frac{M_{x}}{\lambda^{3} E \Gamma}(-\tanh \lambda l+\tanh \lambda l \cdot \cosh \lambda x-\sinh \lambda x+\lambda x)
$$

At $x=\ell$, the twist at the free end:

$$
\begin{equation*}
\theta_{r, \ell}^{\ell}=\frac{M_{\ell}}{\lambda^{3} E \Gamma}(-\tanh \lambda \ell+\lambda \ell) \tag{10.10.5}
\end{equation*}
$$

As in the previous case, the results obtained by the F.E. method for a single element are shown in Table (10.T.2).

## Generalized transformation matrix:

The transformation matrix developed in this thesis has the following features:

1. It transfers deformations and forces between the local and global axis systems,
2. It takes into account the effects of offset loading and moments.

Two tests were conducted independently to examine the accuracy of the bending, torsion and torsion warping properties of the transformation matrix developed in Chapter 9.

## Bending Test:

Node 1 of the cantilever shown in Figure (10.4) was constrained fully at the global origin while maintaining the axial direction parallel to the global xaxis. The structure was loaded at the free end, in the global $z$ direction,

The cantilever in Figure 10.4 was then shifted to a new position as shown in Figure 10.5 and the constraints applied at node 1 to the same degrees of freedom as in the previous test. An equivalent loading
system can be found such that the problem illustrated in Figure (10.4) and Figure (10.5) are identical.

## Note:-


all dimensians are in mm
thickness 1.6 mm all over


FIGURE 10.5
The equivalent system:

| $F_{x}(N)$ | $F_{y}(n)$ | $F_{z}(N)$ | Resultant $(N)$ |
| :--- | :--- | :--- | :--- |
| 25.0 | 25.0 | -50.0 | 61.237 in the local z- <br> direction (i.e. same as in <br> the previous case) |


TABLE (10.T.2)

The displacements predicted for both tests were compared as shown in Table (10.T.3) and found to be identical.


FIGURE 10.7


View on section $X-X$

FIGURE 10.6

A similar test was conducted for torsion and torsion warping problems using the channel cross-section beam, shown in Figure (10.7). The reason for this that channel section beams produce warping displacements, whereas the cruciform section beam does not. As before, the two cantilever systems shown in Figures (10.6) and (10.7) were tested. The agreement was found to be identical to within the accuracy of the machine and can be seen in Table (10.T.4).

The non-linear theory:

Generally, the buckling of a thin-walled structural member may be induced by:
a) Bending caused on either of the principal planes due to application of the axial load, commonly known as Euler buckling

|  |  | 1 |
| :---: | :---: | :---: |
| Test Description | Cantilever placed coincided with the global axes system loaded at the free end with a transverse force in the $z$ direction | Cantilever placed in the space and loaded at the free end in $x, y$ and $z$ directions |
| Node 1 <br> $x$ coord <br> y coord <br> $z$ coord | 0.0 0.0 0.0 | 25.0 25.0 25.0 |
| Node 2 <br> $x$ coord <br> y coord <br> $z$ coord | 303.0 0.0 0.0 | $\begin{aligned} & 200.0 \\ & 200.0 \\ & 200.0 \end{aligned}$ |
| $\begin{aligned} & \ell(\mathrm{mm}) \\ & \operatorname{Iyy}\left(\mathrm{mm}^{4}\right) \\ & \operatorname{Izz}\left(\mathrm{mm}^{4}\right) \\ & E\left(\mathrm{~N} / \mathrm{mm}^{2}\right) \\ & \mathrm{F}_{x}(\mathrm{~N}) \\ & \mathrm{F}_{y}(\mathrm{~N}) \\ & \mathrm{F}_{\mathrm{z}}(\mathrm{~N}) \\ & \mathrm{dx}(\mathrm{~mm}) \\ & \operatorname{dy}(\mathrm{mm}) \\ & \operatorname{dz}(\mathrm{mm}) \\ & \operatorname{Rot}(\mathrm{rad}) \\ & \operatorname{Roty}(\mathrm{rad}) \\ & \operatorname{Rotz}(\mathrm{rad}) \end{aligned}$ <br> Resultant linear displacement at node 2 (mm) | 303.0 | 303.0 |
|  | $0.534 \times 10^{4}$ | $0.534 \times 10^{4}$ |
|  | $0.534 \times 10^{4}$ | $0.534 \times 10^{4}$ |
|  | $0.208 \times 10^{6}$ | $0.208 \times 10^{6}$ |
|  | 0.0 | 25.0 The components of forces |
|  | 0.0 | 25.0 in sulh a manner that the |
|  | 612.0 | -50.0 ${ }^{\text {local }}$ positive 2 -direction |
|  | 0.0 | 0.209 |
|  | 0.0 | 0.209 |
|  | 0.512 | -0.418 |
|  | 0.0 | -0.179 $\times 10^{-2}$ |
|  | $-0.253 \times 10^{-2}$ | $0.179 \times 10^{2}$ |
|  | 0.0 | 0.0 |
|  | 0.512 mm in the positive local 2 direction | 0.512 mm |
| Resultant rotational displacement at node 2 (rad) | $-0.253 \times 10^{-2}$ radians in the $x-z$ plane | -0.253 $\times 10^{-2}$ radians |

TABLE (10.T.3)

|  |  |  |
| :---: | :---: | :---: |
| Test Description | Cantilever placed coincided with the global axes system loaded at the free end with a torsional moment in the positive $x$ direction | Cantilever placed in the space and loaded at the free end with moments in $x, y$ and $z$ directions |
| Node 1 <br> $x$ coord <br> $y$ coord <br> $z$ coord | $\begin{aligned} & 0.0 \\ & 0.0 \\ & 0.0 \end{aligned}$ | $\begin{aligned} & 25.0 \\ & 25.0 \\ & 25.0 \end{aligned}$ |
| Node 2 <br> $x$ coord <br> $y$ coord <br> $z$ coord | $\begin{array}{r} 303.0 \\ 0.0 \\ 0.0 \end{array}$ | $\begin{aligned} & 200.0 \\ & 200.0 \\ & 200.0 \end{aligned}$ |
| $\begin{aligned} & \ell(\mathrm{mm}) \\ & I_{y y}\left(\mathrm{~mm}^{4}\right) \\ & I_{z z}\left(\mathrm{~mm}^{4}\right) \\ & \left(\mathrm{mm}^{6}\right) \\ & E\left(\mathrm{~N} / \mathrm{mm}^{2}\right) \\ & G\left(\mathrm{~N} / \mathrm{mm}^{2}\right) \\ & M_{x}(\mathrm{Nmm}) \\ & M_{y}(\mathrm{Nmm}) \\ & m_{z}(\mathrm{Nmm}) \\ & d x(\mathrm{~mm}) \\ & d y(\mathrm{~mm}) \\ & d z(\mathrm{~mm}) \\ & \operatorname{Rotx}(\mathrm{rad}) \\ & \operatorname{Roty}(\mathrm{rad}) \\ & \operatorname{Rotz}(\mathrm{rad}) \end{aligned}$ | $\begin{aligned} & 303.0 \\ & 0.454 \times 10^{5} \\ & 0.711 \times 10^{4} \\ & 0.241 \times 10^{7} \\ & 0.208 \times 10^{6} \\ & 0.832 \times 10^{5} \\ & 520.0 \\ & 0.0 \\ & 0.0 \\ & 0.0 \\ & 0.0 \\ & 0.0 \\ & 0.557 \times 10^{-2} \\ & 0.0 \\ & 0.0 \end{aligned}$ | 303.0 $0.454 \times 10^{5}$ <br> $0.711 \times 10^{4}$ <br> $0.241 \times 10^{7}$ <br> $0.208 \times 10^{6}$ <br> $0.832 \times 10^{3}$ <br> 300.0 The components of moments $n_{x}$. $h_{y}$ and $x_{z}$ are chosen in 300.0 sich a maner, that the resultant monemt is eliays 300.0 direction in the local positive $x-$ $\begin{array}{r} 0.677 \times 10^{-13} \\ -0.574 \times 10^{-13} \\ -0.974 \times 10^{-14} \\ 0.321 \times 10^{-2} \\ 0.321 \times 10^{-2} \\ 0.321 \times 10^{-2} \end{array}$ |
| Resultant linear displacement at node 2 (rad) | 0.0 | $0.8929 \times 10^{-14}$ |
| Resultant rotational displacement at node 2 (rad) | $0.557 \times 10^{-2}$ radians about the $x$ axis | $0.557 \times 10^{-2}$ radians |

TABLE (10.T.4)
b) Torsional buckling of the cross-sectional plane due to application of the axial load
c) Lateral torsional buckling of structural members at the plane of the least second moment of area when loaded arbitrarily on the cross-section.

In Chapter 6 it was also mentioned that there are two basic approaches to buckling analysis as follows

1. Eigen value approach
2. Large displacement approach.

In the first approach the geometric stiffness matrix in equation (6.2.0) is factorized with a single multiplier and by the use of the Eigen value method the force required for the buckling condition is evaluated. Mathematically, this is defined as the condition for the singularity of the assembly stiffness matrix, i.e.

$$
\begin{equation*}
\operatorname{det}([k]-[N])=0 \tag{10.6}
\end{equation*}
$$

where, $k$ is the linear stiffness matrix
$N$ is the geometric stiffness matrix.

## :

However, the terms in the geometric stiffness matrix cannot be represented by a single multiplier (i.e. all reaction forces in the geometric stiffness matrix are distributed independently). Thus, the the Eigen value approach is not applicable.

The two classes of nonlinear problems under consideration are:

1. Bifurcation type
2. Large displacement type (i.e. buckling associated with the displacement history).

The computer program developed in this thesis is based on the large displacement approach. Thus, any bifurcation type of problem must be converted to the large displacement type. The conversion is generally done by applying the following techniques:
a) By introducing geometric imperfections (e.g. an Euler buckling problem becomes an imperfect structure)
b) By applying a perturbation force.

In general, the latter method is easier than having to introduce any geometric imperfection to the structure.

Buckling of an imperfect strut:

Noie:-
all dimensions are in mm thickness 2.0 mm all over


View on sextion $X-X$


FIGURE 10.8

The strut illustrated in Figure (10.8) is slightly bowed with a maximum imperfection of 40 mm at the mid-span. A typical cross-section along the span is also shown in Figure (10.8). The structure was then constrained as shown in the table in Figure ( 10.9 ). The theoretical centre deflection at the mid-span may also be calculated using equation 10.7:

$$
\begin{equation*}
V=\frac{b_{1}}{\left(1-\frac{P}{P e}\right)} \tag{10.7}
\end{equation*}
$$

Euler buckling of an imperfect pin ended strut under axial compression

corsuraln Conditions.


Constrala conditiome.

| Mode Mo: | Dx | OY | [10 | Rutx | P017 | 0012 | 40 | 17 | 45 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 8 | 1 | 9 | 1 | 0 | 0 | - | 1 | 1 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 1 | 0 | - | 0 | 1 | 1 |
| 4 | 0 | 0 | - | 1 | - | 0 | 0 | 1 |  |
| 5 | - | 0 | 0 | 1 | - | 0 | 0 | 1 |  |
| 6 | 0 | 0 | 0 | 1 | - | 0 | 0 | 1 |  |
| 7 | 0 | 0 | 0 | 1 | - | - | 0 | 1 |  |
| 8 | 0 | 0 | 0 | 1 | - | 0 | - | 1 |  |
| 9 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |  |
| 10 | 0 | 0 | 0 | 1 | - | - | 0 | 1 |  |
| 11 | 0 | 1 | 1 | 1 | - | - | - | 1 |  |


| No. of elements | $E\left(\mathrm{~N} / \mathrm{mm}^{2}\right)$ | $I_{y y}\left(m m^{4}\right)$ | $I_{z z}\left(m m^{4}\right)$ | Critical buckling force exact solution <br> (N) | Critical buckling force F.E. prediction (N) | $\begin{gathered} \text { Error } \\ \% \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | $.208 \times 10^{6}$ | $.534 \times 10^{4}$ | $.534 \times 10^{4}$ | 685.0 | 700.0 | 2.18 |
| 20 | $.208 \times 10^{6}$ | $.534 \times 10^{4}$ | $.534 \times 10^{4}$ | 685.0 | 685.0 | 0.0 |

TABLE (10.T.5)
where, $b_{1}$ is the magnitude of the initial out of straightness measured at the mid-span
$P$ is the compressive force acting on the structural member $P_{e}$ is the Euler critical buckling load for a pin ended strut
(i.e. $\mathrm{Pe}=\frac{\pi^{2} E I}{\ell^{2}}$ )

The results obtained for 4 and 10 finite elements at the mid-span are shown in Figure (10.9). Finally, the trends shown in Figure (10.9) are as follows,

| Critical buckling <br> load (N) calcula- <br> ted from eq. 10.7 | Critical buckling <br> load (N) using <br> elements | Critical buckling <br> load (N) using 10 <br> elements | The <br> \% <br> Eror |
| :---: | :---: | :---: | :---: |
| 170000 | 180000 | 175000 | 5.8 |
|  |  |  |  |

Flexural buckling of a perfect pin ended strut under axial compression:

Two examples were considered:

1. Cruciform section - see Figure 10.10
2. Flanged cruciform section - see Figure 10.11.

The geometry of the cruciform section strut under consideration is illustrated in Figure (10.10). The solution obtained for 10 finite elements with a perturbation lateral force varying from 0.001 N to 1.0 N is shown in Figure (10.12).

The magnitude of the critical buckling load calculated for each perturbation force is given in Table (10.T.5). By referring to Figure (10.17), it can be seen that, as the magnitude of the perturbation force decreases the corresponding load displacement curve changes from a large displacement to a bifurcation type. It is also evident that
when the perturbation force is $0.001 N$ the accuracy of the solution obtained by the finite element method was almost identical to that of the exact solution.

Nore:-
all dimensions are in mm
thickness 9.0 ma all over


FIGURE 10.10

Let us consider next, the flexural behaviour of a cruciform section under axial compression when flanges are introduced to its radiating members as shown in Figure (10.11):

[^0]

FIGURE 10.11

Flexural Buckling of a pin ended strut.


Consrain Condilions.

 Any degree of frovea is fros. denoled by of

Flexural Buckling of a pin ended strut.


Orminalin Onditions.

| Nose Mo: | Dx | D | 80 | Rer | NY |  | $4 \times$ | YY Mz |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 1 | 1 | 1 | 1 | - | - | 0 | 11 |
| 2 | 0 | 0 | - | 0 | - | 0 | 0 | 11 |
| 3 | 0 | - | 0 | 0 | - | 0 | - | 11 |
| 4 | 0 | 0 | 0 | - | 0 | - | 0 | 11 |
| 5 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1.1 |
| 6 | 0 | - | 0 | 0 | - | 0 | 0 | 11 |
| 7 | 0 | 0 | - | 0 | 0 | - | 0 | 11 |
| 8 | 0 | 0 | 0 | 0 | 0 | - | 0 | 11 |
| 9 | 0 | 0 | 0 | 0 | - | - | 0 | 11 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | - | 11 |
| 12 | 0 | - | 0 | 0 | 0 | 0 | 0 | 18 |
| 13 | 0 | $\bigcirc$ | - | 0 | 0 | 0 | 0 | 18 |
| 9 15 | 0 | - | - | - | 0 | - | 0 | 11 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 |
| 16 | 0 | 0 | - | 0 | 0 | 0 | - | 1 |
| 17 | 0 | 0 | - | 0 | 0 | 0 | - | 18 |
| 18 | 0 | 0 | 0 | - | 0 | 0 | 0 | 11 |
| 19 | 0 | 0 | - | - | - | 0 | 0 | 11 |
| 20 | 0 | - | - | 0 | - | 0 | - | 11 |
| 21 | - | 1 | 1 | 1 | 0 | 0 | 0 | 11 |

 Ary defree of Irvedia if frue. dinoted by 0

## Comparison 10\&20 elements, Flexural Buckling of a pin ended strut.



Constrain Concitions.


Onstrain Onditiona.


Effect of axial load increment on Flexural Buckling


Constrain Condizions.

| Hode Mas | Ix | Or | ER | ROTX | 80TY | 1808 | Lx | $\boldsymbol{Y Y}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 80 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

Note: Ans degree of froedion if fined, denoted by it Any degree of ireedo is free, denoled by 0

Effect of axial load increment on Flexural Buckling .


Constrain condiziona.


Notez hny degree of frwedon is IIzed, denoted by Any degpe of freedsa is free. denoted by 0

| No. of elements | $E\left(N / m^{2}\right)$ | $\mathrm{I}_{\mathrm{yy}}\left(\mathrm{mm}^{4}\right)$ | $\mathrm{I}_{z z}\left(\mathrm{~mm}{ }^{4}\right)$ | Critical buckling force exact solution (N) | Critical buckling force F.E. prediction (N) | $\begin{aligned} & \text { Error } \\ & \% \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | . $208 \times 10^{6}$ | $.224 \times 10^{6}$ | . $224 \times 10^{6}$ | 28740.28 | 26500.0 | 7.79 |
| 20 | $.208 \times 10^{6}$ | $.224 \times 10^{6}$ | $.224 \times 10^{6}$ | 28740.28 | $\cdots 2720010$ | 5.35 |

The example shown above was modelled with 10 finite elements and with the constraint conditions as in the previous example. The structure was then loaded with increments of 300 N and also with a lateral perturbation force. This was repeated for a range of perturbation forces ( 0.01 N to 0.05 N ) and the results are shown in Figure (10.17). As in the previous example the approximate bifurcation condition was achieved at a perturbation force of 1.0 N . The results are compared with the exact solution and shown in Table (10.T.6).

To investigate the effect of a finer discretisation the structural member shown in Figure ( 10.11 ) was re-modelled with 20 elements. However, the results show that there is no significant improvement over that of the 10 element model. By referring to Figures (10.12.3) and (10.12.3), it is also noticed a rapid change of the curve for buckling force around 300N. This is because as the load approaches towards the buckling load the determinant of the assembled stiffness matrix approaches near zero, thereby producing large displacements. However, since the loading increment is finite in magnitude, it is impossible to attain the singularity condition of the assembled stiffness matrix. Similarly, for load increments beyond the fictitious buckling load, the A.S.M. can still be non-singular and will produce displacements which correspond to the shift in origin. This can be seen infigures (10.12) and (10.13). The results obtained for 10 and 20 elements were compared and are shown in Figure (10d4). The effect of the magnitude of the axial load increment on the final result was examined according to the combinations shown in Table (10.T.7).

| Test No. No. of Elements | Magnitude of the <br> axial force (N) | Magnitude of the <br> lateral force (N) |  |
| :---: | :---: | :---: | :---: |
| 1 | 10 | 5.0 | 0.7 |
| 2 | 10 | 10.0 | 0.5 |
| 3 | 20 | 5.0 | 0.5 |
| 4 | 20 | 10.0 | 0.5 |

TABLE 10.T. 7

Fiexural Buckling of a pin ended strut.


Onstrala Donditione.

| Mose Mo: | Ix | Dr | gr | 20] | Torr | $\underline{1078}$ | M $\times$ | Y\% 18 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $y$ | 1 | 1 | - | - | 0 | 11 |
| 2 | 0 | - | * | - | 0 | $\bullet$ | 0 | 18 |
| 3 | 0 | 0 | - | - | 8 | - | 0 | 11 |
| 4 | 0 | 0 | - | - | - | - | 0 | 1 |
| 5 | - | 0 | - | 0 | - 0 | - | 0 | 1 |
| 6 | 0 | 0 | ¢ | - | - | - | 0 | 11 |
| 7 | 0 | 0 | - | - | - | - | 0 | 11 |
| 8 | 0 | 0 | - | - | - | - | 0 | 11 |
| 9 | 0 | 0 | - | 0 | e | - | - | 11 |
| 10 | 0 | 0 | - | - | - | - | 9 | 1 |
| 11 | 0 | 1 | 1 |  |  | - | - |  |

 Ary deove croede is free. decoled by 0

Flexural Buckling of a pin ended strut.


Consurain Oondivions.

| Mode lios: | IXX | Or | IR | 10] | RIT | ETE | 10X | W | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0. | 0 | 1 | 1 |
| 4 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | . |
| 6 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 21 | 0 | 1 | 8 | 1 | 0 | 0 | 0 | 1 | 1 |

Note: Any degree of freedon is fixed, denoled by i
Any defree of froeton is frue, deroued by 0

Comparison 10\&20 elements, Flexural Buckling of a pin ended strut.

corsurais Condivions.

| Mode Mo: | DK | If | \% | ROTX | EJTY | 8012 | 15X | W | I2 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | , |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| 4 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | D | 0 | 0 | 0 | , | 1 |
| 6 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | - | 0 | 0 | 0 | , | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | , | 1 |

Onstrala Conaltions.


Nove: Any degree of freaton is rixed, denoled by 1 Any degree of freedol is free. denoled by 0

Effect of axial load increment on Flexural Buckling .


Ornsuain Conditiona.


Moles Any degree of freedon is flxed, danoted by 1 ory degree of fruedow is free, denoted by 0


Conarnale Dondivions.


Wies Ary detree of fruedon is Ifxed, denoted by 1

The results obtained for the runs shown in Table (10.T.7) can be seen in Figures (10.15) and (10.16).

## Examples in pure torsional buckling:

Basically there exists two classes of torsional buckling problems:

1. St Venant torsion
2. Combined effects of St Venant torsion and torsion due to restrained warping.

Let us begin with St Venant torsion by considering the behaviour of a cruciform section strut under axial compression. The geometry of the strut is shown in Figure ( 10,22 ). The strut was loaded by an axial force and a range of perturbation torsional moments acting simultaneously at the mid-span as shown in Figure (10.22). The result obtained for 10 and 20 element assemblies with an axial load increment of 100 N and perturbation torques varying from 0.01 Nmm to 0.5 Nmm are shown in Figures (10.23) and (10.24). The axial load increment was then changed to 50.0 N and the perturbation torques were varied from 0.01 Nmm to 0.5 Nmm . The results obtained for these runs can also be seen in Figures (10.23) and (10.24). These results are compared with the exact solution shown in equation (10.8). As in the case of flexural buckling, near bifurcation behaviour occurred for smaller magnitudes of perturbation torques. The accuracy of the agreements found was almost perfect as shown in Table (10.8).

$$
\begin{equation*}
P_{c r}=\frac{A}{I_{p}}\left(\frac{\pi^{2} E \Gamma}{l^{2}}+G J\right) \tag{10.8}
\end{equation*}
$$

where $P_{C r}$ is the compressive force under pure torsional buckling
$A$ is the area of the cross-section
$I_{p}$ is the polar second moment of area
$\Gamma$ is the torsion warping constant
$E$ is the Young's elastic modulus
$G$ is the modulus of rigidity in shear
$J$ is the St Venant torsion constant
$\ell$ is the length of the strut


```
Note:-
    all dimensions are in man
    thickness }1.0\textrm{mm}\mathrm{ all over
```

FIGURE 10.22

Torsional buckling of a flanged cruciform section strut:

The flanged cruciform section strut under investigation is shown in Figure (10.28) and has been modelled with 10 and 20 elements. For each mesh the strut was constrained and loaded in the same way as in the previous examples. The magnitude of the increment of the axial force was kept at 100 N and the results obtained for both models can be seen in Figures (10.28) and (10.29).

By referring to equation 10.8 it is apparent that the buckling force under St Venant torsion is a function only of the torsional buckling rigidity $\frac{G J}{I_{p}}$ and is independent of the column length. However, in the case where the warping constant is non-zero, the buckling force will increase by $\frac{\pi^{2} E \Gamma}{I p \ell^{2}}$ and therefore the mode of buckling is somewhat similar to that of flexural buckling. Under these circumstances the magnitude of the perturbation torque was kept at a larger magnitude than in the previous section.

| No. of elements | $\underset{m m}{\mathrm{~J}_{4}}$ | $\mathrm{I}_{\mathrm{y}} \mathrm{y}_{\mathrm{y}}$ | $\mathrm{I}_{\mathrm{I}_{2}}$ | $\mathrm{I}_{\mathrm{m}_{\mathrm{n}}} / \mathrm{A}$ | ${ }_{\text {mm }}{ }_{6}$ | $\underset{\mathrm{N} / \mathrm{mm}^{2}}{\mathrm{E}}$ | $\underset{N / m^{2}}{\mathrm{G}}$ | Critical torsional buckling force (N) exact solution | Critical torsional buckling force (N) FE prediction | Error \% |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 26.7 | $.534 \times 10^{4}$ | $.534 \times 10^{4}$ | 133 | 0.0 | $.205 \times 10^{6}$ | $.832 \times 10^{5}$ | 16702.0 | 16702.0 | 0.0 |
| 20 | 26.7 | $.534 \times 10^{4}$ | . $534 \times 10^{4}$ | 133 | 0.0 | $.205 \times 10^{6}$ | $.832 \times 10^{5}$ | 16702.0 | 16702.0 | 0.0 |

TABLE (10.T.8)

Torsional buckling of a pin ended strut.


Phastrala Condifions.

| Mode fors | mx | Ir | [ | 10] | 80\%Y | P0I2 | WX | $\mathbf{W}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 8 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | - | - | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 |
| $\square$ | - | - | - | - | 0 | 0 | 0 | 1 |
| 5 | 0 | 0 | - | 0 | - 0 | 0 | 0 | 1 |
| 6 | 0 | 0 | - | 0 | - | 0 | 0 | 1 |
| 7 | 0 | 0 | - | - | 0 | 0 | 0 | 1 |
| 8 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 |
| 9 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

 ons deproe af in itom is free, denoled by 0

Torsional buckling of a pin ended strut.
20 element solution


Consurain Opedriagn.


Note: Any degree of froedon is fluat, deroced by Any degrete of frwedte is froe. torored by 0

## Comparison 10\&20 elements,Torsional Buckling of a pin ended strut.



Canstrain Conditiona.


Consurala Condizions.

Node Mor tDX DY 仅 RODX ROTY ROTZ uS UY EZ

| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 21 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

Hole: Any degree bf freadon is fized, denoled by
Any degrete of irveton in ifree, dentued by 0

Effect of axial load increment on Torsional Buckling .


Constrain Orealeiore.

| Mode Mo: | trx | DY | \% | R0PX | notr | 0012 | 4x | WY |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $\stackrel{1}{ }$ | ! | 5 | - | 0 | - | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 |
| 4 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $\bigcirc$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

Move: Ary degree of frwedon is fired, daopled by 1
Avy defve of freedon in free. daried bs

Effect of axial load increment on Torsional Buckling .


Onstrate Dondilions.

| Node Mos | mx | DY | $\underline{\square}$ | Ledx | 2 mP | 3 | Wx | WY | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 9 | 1 | , | \% | - | 0 | 1 | , |
| 2 | 0 | 0 | 0 | 0 | - | - | 0 | - |  |
| 3 | 0 | 0 | 0 | 0 | 9 | 0 | 0 | 1 |  |
| 4 | 0 | 0 | 0 | 0 | - | - | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| 8 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 |  |
| 9 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 |  |
| 10 | 0 | - | - | 0 | - | - | 0 | 1 |  |
| 11 | 0 | 0 | - | D | 0 | 0 | 0 | 1 |  |
| 12 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 |  |
| 13 | 0 | 0 | 0 | 0 | - | - | 0 | 1 |  |
| 14 | 0 | 0 | 0 | - | - | 0 | 0 | 1 |  |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| 16 | 0 | 0 | 0 | 0 | - | 0 | - | 1 | 1 |
| 17 | 0 | 0 | 0 | 0 | - | 0 | 0 | , |  |
| 18 | - | 0 | 0 | 0 | - | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | 0 | 0 | - | 0 | 0 | + |  |
| 20 | 0 | 0 | 0 | - | - | - | 0 | 8 | 1 |
| 21 | 0 | 1 | 1 | 8 | 0 | 0 | 0 | 1 | 1 |

 Any degres of freedore is frue. thentied by 0

Torsional buckling of a pin ended strut.


Conurale Conditiona.

| Mode Mos | EX | Dr | [2 | Porx | 80 | 00 | $1 \times$ | Wr | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | - | - | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | - | 0 | - | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 1 | 1 |
| $\pm$ | 0 | 0 | 0 | 0 | - | 0 | - | 1 | 1 |
| 5 | 0 | 0 | 0 | - | - 0 | - | - | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | - | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | - | 0 | - | 0 | 1 | 1 |
| 11 | 0 | 1 | 1 | 1 | - | 0 | 0 | 1 | 1 |

 Any deque of fremon ta froe. denoted by 0

Torsional buckling of a pin ended strut.


Constrita Craditions.

| Node Jo: | 8x | Ir | E2 | EfR | RUTY | morz | 15 | M | $\underline{18}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 8 |
| 3 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 9 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 12 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 14 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 15 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 16 | 0 | 0 | 0 | - | 0 | 0 | 0 | 1 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 20 | 0 | - | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 21 | 0 | 1 | 1 | 1 | 0 | - | 0 | 1 | 1 |

hove: Ans degre of frmedo is flyed, denoted by i Ary there of freatim is Irre. denoved by 0

Comparison 10\&20 elements, Torsional Buckling of a pin ended strut.


Constrain Conditions.

| Mode Mo: | Ex | Or | [0 | Notx | 807Y | P0I2 | WX | WY | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | - | 0 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 6 | 0 | - | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | . 1 | 1 |

Conrtrain Condilions.


| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 21 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

Mole: Any degree of freedon is fixed, denoled by 1 Any dearee of freedon is frree, denoled by 0

Effect of axial lood increment on Torsional Buckling .


Conerais conditions.

| Mode Mas | Ex | It | $\underline{8}$ | H0x | Ery | 1072 | $1 \times$ | YY |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | - | 0 | 0 | 1 |
| * | 0 | - | 0 | 0 | - | 0 | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 6 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 |
| 7 | 0 | - | 0 | - | 0 | 0 | 0 | 1 |
| B | 0 | - | 0 | 0 | 0 | 0 | 0 | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 10 | 0 | - | 0 | 0 | 0 | 0 | 0 | 1 |
| 18 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

Hole: and deye of fruedore is fired, denored by 1

FISURE 10.31

## Effect of axial load increment on Torsional Buckling .



Consurala Condiziona.

| Mode Mos | EX | Or | me | Mrix |  | 16812 | WX | YY | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |  |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |  |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 8 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 6 | D | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 9 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | , | 1 |
| 82 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | , | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 19 | 0 | 0 | - | 0 | 0 | 0 | 0 | 1 | 1 |
| 20 | 0 | 0 | 0 | - | 0 | 0 | 0 | , | 1 |
| 21 | 0 | 1 | 1 | 1 | - | 0 | 0 | 1 |  |

dotez Aw degrwe of friedon is fired, denoted by any degrete of irvedol is free, denoted by 0
-The results obtained for the case of free torsional buckling is shown in Figures (10.28) and (10.29). It can be seen that up to the value * of the axial thrust of 9000 N , the behaviour of the strut is almost linear. Beyond this value the trend becomes non-linear, leading to buckling. As before, the analysis was extended to lower increments of the axial load and the results obtained are shown in Table (10.T.9).

Elastic Stiffening and Large Deflection Phenomenon of a Cantilever:

The channel section cantilever shown in Figure (10.1) is constrained at its fixed end except for the warping degrees of freedom. It is also constrained for the lateral degrees of freedom at all nodes for the two loading conditions in the $x-y$ and $x-z$ planes, and loaded with a transverse force acting on the shear centre at its free end. The results of the normalized deflection $\left(\frac{\delta}{\ell}\right)$ and the normalized load ( $\frac{P l^{2}}{E I}$ ) at the free end for both the $x-y$ and $x-z$ planes are shown in Figures (10.33), (10.34) and (10.35). These results are compared with the exact solution obtained by D.C. Drucker (see reference 8) and also with a finite element solution by T.Y. Yang (see reference 64). For single element modes, the agreement was quite satisfactory, and using 10 elements the results show an almost perfect agreement with the exact solution.

Large deflection analysis of a thin walled channel section cantilever beam loaded at the centroid:

The arrangement illustrated in Figure (10.35) is similar to that of the previous example, except that the transverse force at the free end acts through the centroid. The warping displacement is constrained at the root. No significant axial twist was computed when the force was applied along the $y$ axis. This is because the line joining the shear centre and the centroid is overlapped by the direction of the transverse force.

| No. of Elements | $\mathrm{mm}^{\mathrm{J}}$ | $\mathrm{I}_{\mathrm{mf}^{2} y_{4}}$ | ${ }_{\mathrm{mm}}^{\mathrm{I}_{4}}$ | $\operatorname{Ip}_{m m^{2}} / A$ | $\Gamma_{\mathrm{mm}}^{6}$ | $\underset{\mathrm{N} / \mathrm{mm}^{2}}{\mathrm{E}}$ | $\underset{\mathrm{N} / \mathrm{mm}^{\mathrm{G}}}{2}$ | Critical torsional buckling force (N) F.E. method | Critical torsional buckling force (N) exact solution | $\begin{gathered} \text { Error } \\ \% \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 164.0 | $.224 \times 10^{6}$ | $.224 \times 10^{6}$ | 933.0 | $.341 \times 10^{8}$ | $.208 \times 10^{6}$ | $.832 \times 10^{5}$ | 19314.0 | 19314.0 | 0.0 |
| 20 | 164.0 | $.224 \times 10^{6}$ | $.224 \times 10^{6}$ | 933.0 | $.341 \times 10^{8}$ | . $208 \times 10^{6}$ | $.832 \times 10^{5}$ | 19314.0 | 19314.0 | 0.0 |

TABLE (10.T.9)

Elastic stiffening and large deflection phenomerion of a contilever


Comstrela Condizions.


Coratraia condizions.

| Node No: | D | Fr | 12 | Rerx | Rodr | ETL | 10x | vy | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 5 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | , |  |
| 6 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |  |
| 7 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |  |
| B | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |  |
| 9 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |  |  |
| 10 | 0 | 0 | , | 1 | 1 | 0 | 0 | 1 |  |
| 11 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 8 |  |

 Awy degree of froetion is ifres. denoted by 0

Elastic stiffening and large deflection phenomenon of a cantilever

casurain Onditioes.
$\begin{array}{llllllllll}\text { Node No: } & \text { ID } & \text { UY } \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1\end{array}$

Corarals prodivons.

| Node Ibs: | 5x | 87 | \% 12 | Fix | Y-7\% | Perz | UX | H\% | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | , |
| 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | , |
| 5 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| - | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 9 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |
| 10 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 11 | 0 | 1 | - | 1 | 0 | 1 | 0 | 1 | 8 |

Molez Any degrow of froedon is gixed, denoved by 1 Any degree of froudon is frwe. denoled by 0

Comparison of elastic stiffening on XY\&XZ planes with published work.
10 element solution comparison


Coastrais amdiliocs.

| Node Mos | Ex | Dr | IT | HoxX | $\underline{4}$ | 2072 | 1 mx | Y | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | ; |
| 4 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |
| 8 | 0 | 1 | 0 | 8 | - | 1 | 0 | 1 | 1 |
| 9 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | , |
| 80 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 11 | 0 | 1 | 0 | 1 | - | 1 | 0 | 1 | 1 |



Note:-
t.
all dimensions are in mm thicknoss 2.0 mm all over


FIGURE 10.36

A transverse force of 5.0 N was applied along the $z$ direction through the centroid and a significant twist along the span was observed as shown in Table (10.T.10).


View on section $X-X$


FIGURE 10. 37
all dimensions are in mn
thickness 9.6 mm all over
In this situation, where displacements are small, the twist produced due to the offset transverse force is in agreement with that of the exact solution. The structure was then loaded with 5.0 N load increments. The axial twists produced were compared with experimental and the continuum solutions produced by Gaafar and Tidbury (see reference 19).

The cantilever was remodelled with meshes of 9 and 18 elements and the results obtained are shown in Figures (10.35), (10.39) and (10.40). A discrepancy was noted for transverse forces greater than 200 N and the maximum errors are shown in Table (10.T.11).

| No of Elements | Distance from the <br> fixed end, $x(\mathrm{~mm})$ | The error when the load <br> at the free end is $700 \mathrm{~N}, \%$ |
| :---: | :---: | :---: |
| 9 | 360.0 | 33.0 |
| 9 | 560.0 | -30.0 |
| 9 | 700.0 | 16.5 |
| 18 | 360.0 | 0.0 |
| 18 | 560.0 | -8.16 |
| 18 | 700.0 | 20.8 |

## TABLE 10.T. 11

## Note:

If the angle of twist predicted by the computer program is greater than the experimental observations, then the error is positive.

## Large displacement analysis of a shallow truss:

The structure shown in Figure (10.42) has been analysed by Williams (64) and is used in several publications as a benchmark. When loaded at the apex the behaviour is similar to a toggle, and represents a very severe test to any non-linear solution technique.


View on section $X-X$

Note:-
all dimensions are in inches
thickness 0.243 in all over

$\sin ^{-1}(0.0247)=2.425(\mathrm{deg})$

|  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| Node 1 |  |  |  |  |
| $x$ oord | 0.0 | 0.0 | 0.0 | 0.0 |
| $y$ coord | 0.0 | 0.0 | 0.0 | 0.0 |
| $z$ coord | 0.0 | 0.0 | 0.0 | 0.0 |
| Node 2 |  |  |  |  |
| $x$ coord | 700.0 | 700.0 | 700.0 | 700.0 |
| $y$ coord | 0.0 | 0.0 | 0.0 | 0.0 |
| $z$ coord | 0.0 | 0.0 | 0.0 | 0.0 |
|  |  |  |  | i |
| $\ell(m m)$ | 700.0 | 700.0 | 700.0 | 700.0 |
| Iyy ( $\mathrm{mm}^{4}$ ) | $0.454 \times 10^{5}$ | $0.454 \times 10^{5}$ | $0.454 \times 10^{5}$ | $0.454 \times 10^{5}$ |
| $\mathrm{Izz}\left(\mathrm{mm}^{4}\right)$ | $0.711 \times 10^{4}$ | $0.711 \times 10^{4}$ | $0.711 \times 10^{4}$ | $0.711 \times 10^{4}$ |
| $\Gamma\left(\mathrm{mm}^{6}\right)$ | $0.241 \times 10^{7}$ | $0.241 \times 10^{7}$ | $0.241 \times 10^{7}$ | $0.241 \times 10^{7}$ |
| $E\left(\mathrm{~N} / \mathrm{mm}^{2}\right)$ | $0.208 \times 10^{6}$ | $0.208 \times 10^{6}$ | $0.208 \times 10^{6}$ | $0.208 \times 10^{6}$ |
| $G\left(N / m^{2}\right)$ | $0.832 \times 10^{5}$ | $0.832 \times 10^{5}$ | $0.832 \times 10^{5}$ | $0.832 \times 10^{5}$ |
| Fx (N) | 0.0 | 0.0 | 0.0 | 0.0 |
| Fy (N) | 0.0 | 0.0 | 10.0 | -10.0 |
| Fz (N) | 10.0 | -10.0 | 0.0 | 0.0 |
| dx (mm) | 0.0 | 0.0 | 0.0 | 0.0 |
| dy (mm) | 0.0 | 0.0 | 0.733 | -0.773 |
| dz (mm) | 0.211 | -0.211 | 0.0 | 0.0 |
| Rotx ( rad ) | $0.656 \times 10^{-2}$ | $-0.656 \times 10^{-2}$ | 0.0 | 0.0 |
| Roty (rad) | $-0.432 \times 10^{-3}$ | $0.432 \times 10^{-3}$ | 0.0 | 0.0 |
| Rotz ( rad ) | 0.0 | 0.0 | -0.166×10-2 | $0.166 \times 10^{-2}$ |

TABLE (10.T.10)

Comparison of axial twist of $9 \& 18$ elements at $x=360$ with published work


Constrain canditions.

| Node Mo: | ID | Dr | 12 | ITIX | ITY | R0072 | 15 X | WY | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 5 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 9 | 0 | 1 | 0 | 0 | - | 1 | 0 | 1 | 1 |
| 10 | 0 | 1 | 0 | 0 | - | 1 | 0 | 1 | 8 |
| 11 | 0 | 1 | 0 | 0 | - | 1 | 0 | 1 | 1 |

Constraia Conditions.

| Mode Mo: | EX | Dr | [10 | mux | Ear | ReJI | vx | $\boldsymbol{Y}$ | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 5 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 8 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 9 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 10 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 11 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 12 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 13 | 0 | $9^{1}$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 14 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 15 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 16 | 0 | 1 | - | 0 | 0 | 1 | 0 | 1 | 1 |
| 17 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | , |
| 18 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 19 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 20 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 21 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |

Wbie: Any degree of freedor is flaed. denoled by I
Any decree of freedon is free. denoled by 0

Comparison of axial twist of $9 \& 18$ elements at $x=560$ with published work


Corsiruin Denditions.

| Mode Mo: | dx | Dr | I2 | 5 | vary | $\underline{0}$ | ux | w | uz |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | - | 1 | - | 0 | - | , | 0 | 1 | 1 |
| 3 | 0 | , | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | - | 0 | , | 0 | 1 | 1 |
| 5 | 0 | 1 | - | - | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | ! | 0 | 0 | - | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 8 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 9 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 10 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 11 | 0 | 1 | 0 | 0 | 0 | , | 0 | 1 | 1 |

Constrain Condizfors.

| Mode Mo: | 50 | Dr | 71\% | ROTX | VIT | 8012 | $12 \times$ | $\mathbf{Y}$ | LZ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 5 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 7 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 6 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 5 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 10 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 4 |
| 11 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 12 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 13 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 14 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 15 | D | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 16 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 17 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 18 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 19 | 0 | 1 | 0 | 0 | 0 | , | 0 | 1 | 1 |
| 20 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 21 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |

Mote: Any degree of freedos is tixed, denoled by I Any degree of freedos is fret. denoled by 0

Comparison of axial twist of $9 \& 18$ elements at $x=700$ with published work


Consrain Condizions．

| Mode Mo： | Ex | Dr | R | eng | ETY | R002 | $4 \times$ | UY | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ！ | 1 | ！ | 1 | ： | 1 | 1 | 0 | 1 | 1 |
| 2 | － | 1 | 0 | － | － | 1 | 0 | 1 | 1 |
| 3 | 0 | 1 | 0 | － | 。 | 1 | － | 1 | 1 |
| － | － | 1 | 0 | － | 0 | 1 | 0 | 1 | ， |
| 5 | － | 1 | 0 | － | 0 | ， | 0 | 1 | 1 |
| 6 | － | 1 | － | － | － | 1 | 0 | － | 1 |
| 7 | 0 | 1 | 0 | － | － | 1 | － | 1 | 1 |
| 6 | 0 | 1 | 0 | － | － | 1 | － | 1 | 1 |
| ， | 0 | 1 | 0 | － | － | ， | 0 | 1 | 1 |
| 10 | 0 | 1 | － | 0 | － |  | 0 | － | 1 |
| 11 | － | 1 | 0 | 0 | － | 1 | 0. | 1 | 1 |

Consireia Concitions．

| Node Nor | tox | Dr | 吅 | RJTX | ETY | ect | vox | ur | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
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Elastic stiffening and large deflection phenomenon of a cantilever

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The structure shown in Figure (10.42) was modelled with two meshes of 10 and 20 elements with an initial load increment of 1.0 lb . The load displacement history predicted is shown in Figure (10.43) and compared with the theoretical and experimental results of Williams. To improve the accuracy, the problem under consideration was remodelled with the load increment halved ( 0.5 lb ). No significant change in the result was found and in both tests a maximum error of approximately $22 \%$ was recorded at a maximum load of 70 lbs.

Large deflection of a rigidly fixed toggle loaded transversely at apex


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## CHAPTER 11

## dISCUSSION AND CONCLUSIONS

In the majority of the literature published in the past, two independent reference axis systems are considered in deriving the governing differential equations of thin walled open section structural members.

1. For displacements related to axial and flexural loads an axis system parallel to the principal axis located at the centroid
2. For displacements related to torsional and warping moments an axis system parallel to the principal axis located at the shear centre.

This situation leads to some confusion as, in practice, the joint construction normally dictates the position of the loading, and also in many cases, the centroid and the shear centre are not coincident. Furthermore for the sake of simplicity, generally the authors in the past only considered cases where the local elemental axial direction was parallel to one of the global axes. Unfortunately however, this option certainly restricted the classes of problem that are likely to be solved by the established theory. However in this thesis, the situation has been overcome by formulating a transformation matrix, as shown in equation (8.14.0), while taking into account any of the secondary forces that are likely to develop due to offset loadings. The work conducted by Gaafar and Tidbury (19) was carefully formulated into a finite element approach and included in the total potential formulation as shown in equation (4.72.0). Without this modification, solving large displacement structural problems would not have not been possible and instead the analysis would have been rather limited to just bifurcation type of problems. Briefly the refinements and developments that are introduced into the present theory of the thin walled beam structural members are as follows:

1. The coupling effects of the resultant bending in two principal planes, torsion and torsion warping, due to an arbitrarily placed axial load $F_{X}$ on the cross-sectional plane. For details see equations (4.31.3), (4.32.3) and (8.14.0);
2. Inclusion of unequal torsion and warping moment loads as suggested by Gaafar and Tidbury (19) to the generalised finite element theory of buckling. For details see equation (4.72.0);
3. Method of transformation of sectorial properties from the local cross-sectional plane to the global axis system in the space. For details see equations (8.14.0) and (8.30.0);
4. Development of local-global transformation matrix to encounter secondary forces and moments developed due to application of offset forces and moments. For details see Figure 8.2 .0 and equation (8.14.0);
5. Theoretical prediction of the displaced position of the third node and the local axis system relative to the global axis system due to previous load increments. For details see equation (7.9.0);

A series of tests were carried out to examine the accuracy of the computer predictions made by the following parts of the theory with the published work:
a) the linear elastic theory
b) the local-global transformation matrix
c) the non-linear elastic theory.

The linear theory of bending for a channel section beam was tested for both principal planes as illustrated in Figure 10.1. The local axis system of the structure was purposely placed coincident with the global axis system so that no contribution from the transformation matrix was assumed to be made to the analysis. The
critical buckling load predicted for a single element was compared with the Euler-Bernoulli theory and found to be exact as shown in Table 10.T.1. Under a similar geometric condition the structure was next tested for St Venant torsion and torsion warping problems, as shown by Figures 10.2 and 10.3. The results obtained for both cases were compared with the exact solution shown in equation (10.3) and found to be exact as shown in Table 10.T.2.

To examine the behaviour of the transformation matrix under bending only, a cruciform sectioned beam was placed in the space (note under torsion cruciform sections do not warp) as shown by Figures 10.4 and 10.5. At first the beam was loaded in the $z$ direction for the case where the local and the global axes are coincident as shown in Figure 10.4. The beam was next placed in the space, all degrees of freedom were constrained at node 1 and loaded at node 2 in the $x, y$ and $z$ directions so that the resultant force at node 2 still amounted to 61.237 N as shown in Figure 10.5. The displacements produced for both test cases were compared, as shown in Table 10.T.3 and found to be exact. Therefore from this test it can be seen that the bending feature of the transformation matrix appears to function satisfactorily. A very similar test was carried out for torsion and torsion warping effects of a channel section cantilever, as shown in Figures 10.6 and 10.7. As before initially the structure was placed coincident with the global axis system, constrained at node 1 and loaded with a torsional moment at node 2. Then the beam was placed $i n^{-3}$ the space, constrained at node 1 , and loaded with moments $M_{x}, M_{y}$ and $M_{z}$ as shown in Figure 10.7. The results produced for both cases were compared and found to be exact as shown in Table 10.T.4.

To examine the performance of the transformation matrix for offset loading, the channel section cantilever was next loaded at the centroid along the $y$ and $z$ directions as shown in Figures 10.36 and 10.37. By intuition it can be seen that there should not be any torsional displacement developed when loaded at the centroid in the $y$ direction. However for the case when loaded along the $z$ direction, the induced torsional moment amounts to $\mathrm{F}_{\mathrm{z}} \cdot{ }^{e} y$ and thus
must produce a significant torsional displacement. The direction of the loads was also reversed to observe the sign and the accuracy of the results generated and was found to be very satisfactory with the expected results, as shown in Table 10.T.10. However the exact solution for this particular problem is rather complicated as pointed out by Gaafar and Tidbury (19) and thus the agreement of the finite element result to that of the exact solution is found by comparison with the experimental results shown in Figures 10.38, 10.39 and 10.40.

The theory developed for non-linear analysis in Chapter 5 was tested for several classes of problems and these are as follows:

1. The channel section column illustrated in Figure 10.8 has a maximum imperfection of 40 mm at the mid-span and varies gradually over a span of 2000 mm . The column was tested for flexural buckling under an axial load with an increment of 400.00 N . The computed results of buckling load for 4 and 10 element meshes are shown in Figure 10.9. The errors estimated for both the meshes are listed in the table shown on page 151. The solution produced for the 10 element model was found in fairly good agreement with a maximum error of $2.9 \%$, that of the exact solution in equation (10.7).

2, a) A perfect cruciform section strut of 400 mm span was tested for flexural buckling under an axial load, as shown in Figure 10.10. The results were obtained for 10 and 20 element meshes with a lateral perturbation force ranging from 0.001 N to 1.0 N as shown in Figures 10.12 and 10.13. With the 10 element solution the predicted buckling load became the same as the Euler load when the perturbation force was 0.001 N , whereas for the 20 element solution, the critical buckling load became exact with the Euler load when the perturbation force was $0.01 N$ (i.e. ten times greater than that of the 10 element solution). By referring to Figure 10.15 it can also be seen that as the load increment
is reduced by $50 \%$ the prediction rapidly became much closer to the exact solution.
b) Generally any strut under an axial load produces some degree of cross-sectional rotation about the shear centre and this has been described in detail in Chapter 4 (page 33). Therefore in order to investigate the flexural buckling of a flanged cruciform strut it is essential to prevent any cross-sectional rotation caused by an axial load, as shown in Figure 10.17. However flanged cruciform sections under axial rotation produce warping displacements and therefore care must be taken not to restrain the warping displacements, as shown in Figure 10.17. However due to relatively large flexural stiffness, the magnitude of the perturbation force used for the flanged cruciform section strut was greater than that for the straightforward cruciform section strut. As before the strut was tested for 10 and 20 element meshes with a range of lateral perturbation forces and axial load increments. For the 10 element solution with 300 N load increment the predicted buckling load was found to be within $7.79 \%$ that of the Euler load, whereas for the 20 element mesh under similar loading conditions, a critical buckling load was found within 5.35\% that of the exact solution as shown in Table 10.T.6.
3. The exact solution for torsional buckling under axial load is shown in equation (10.8). In this test two types of structural sections were used:
a) a cruciform section with zero torsion warping constant
b) a flanged cruciform section with $0.341 \times 10^{8} \mathrm{~mm}^{6}$ of torsion , warping constant
a) As before the cruciform section strut was tested for 10 and 20 element meshes. The structure was also tested for a range of axial load increments ranging between 50 to 100 N with a perturbation torque acting at the mid-span ranging
between 0.001 Nmm to 0.5 Nmm . The results obtained for these tests are shown in Figures $10.23,10.24,10.25,10.26$ and 10.27. The results were compared with the exact solution and found to be in perfect agreement, as shown in Table 10.T.8.
b) The tests carried out for the torsional buckling of the flanged cruciform section are shown in Figures 10.28, 10.29, 10.30, 10.31 and 10.32. The axial load increment was kept at 100 N or 150 N . As in the case of flexural buckling due to relatively large torsional rigidity (see equation 10.8), the magnitude of the perturbation torque was kept between 1.0 Nmm to 25.0 Nmm . The results predicted were compared with the exact shown in equation (10.8) and found to be in perfect agreement as shown in table 10.T.9.
4. The problems discussed so far have only dealt with flexural and torsional buckling of struts due to application of an axial load. In the following example, the criterion of elastic stiffening, rather than its "inverse" i.e. buckling, of a cantilever was investigated. The cantilever beam shown in Figure 10.33 was constrained laterally for all out-of-plane displacements, as shown in Figure 10.33. The structure was loaded transversely at the tip and the normalized in plane tip displacement and the applied load was plotted for meshes with 1 : $z$ and 10 elements. The tests were repeated for both the $y$ and $z$ directions as shown in figures 10.33 and 10.34 . The results produced by both tests were compared with the exact solution produced by Bishop (8) and the non-coupling finite element solutions produced by Yang (65) and found to be in close agreement.

The cantilever was then loaded at the centroid in the $z$ direction and all in plane displacements constrained as shown in Figure 10.38. The axial twist produced due to increasing transverse load acting at the centroid, was examined at distances of $360 \mathrm{~mm}, 560 \mathrm{~mm}$ and 700 mm from the fixed end, as

Shown in Figures 10.38, 10.39 and 10.40. By referring to Figure 10.38 it can also be seen that for the axial twist at 360 mm from the fixed node the displacements produced by the theory developed tend to show a better agreement to the experimental results than the exact solution produced by Gaafar and Tidbury (19). However at 560 mm for transverse load around 800 N , the theory developed in the thesis tends to produce much greater torsional stiffening than the exact solution developed by Gaafar and Tidbury. A maximum error of $8.16 \%$ was also estimated against the experimental results at a force magnitude of 800 N . The torsional displacement produced for the 20 element mesh at 700 mm showed the highest early torsional stiffening. However with the 10 element mesh the theory developed in the thesis produced much less stiffer torsional displacements than the results due to theory and the experiments conducted by Gaafar and Tidbury. A maximum error of $20 \%$ was estimated for torsional displacement at 800 N against the experimental results.
5. Finally a highly nonlinear structure of a shallow truss was chosen to complete the analysis. By referring to the geometry constrained and the loading condition shown for the structure in Figure 10.43 it can be seen that this particular test should examine the theory for flexural as well as the inherited togglelike behaviour of the structure under consideration. The predicted results for the test showed a general trend similar to the results produced by Williams (64). However, as in the previous case, some early stiffening was also noticed for loads beyond 25 ib . By using a much refined mesh no significant change was made to the results and a maximum error of $22 \%$ was estimated at a maximum loading of 70 lb .

With regard to the discussion made so far, it can be concluded that for most problems the results of the linear analysis were found to be in perfect agreement with that of the theory developed. However, the accuracy of the computer predictions to the non-linear problems shows a favourable agreement with their exact solutions and, as it
was noticed for more complex problems like:

1. Large rotational twist of a cantilever when loaded transversely at the centroid
2. The shallow truss with toggle effect loaded vertically down at the apex

Some early stiffening and discrepancies were observed.

The theory and the computer program developed in the thesis are believed to have a much wider scope for solving general assemblies of space frameworks with mixed thin walled beam structural members than in the past. To solve such problems by using the program developed, however, needs more verification of the results that it predicts. The satisfactory results produced by the program for the test problems dealt with so far do not necessarily provide a justification for its general use. Therefore for wider applications it is suggested that the program be thoroughly tested against analytical and/or experimental results for a range of typical highly nonlinear industrial problems prior to its possible general use. Once the program is thoroughly tested it can be used in:

1. Linear analysis of assemblies of beam elements in space frames
2. Nonlinear analysis of assemblies of beam elements in space
$\because$ frames.

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## APPENDIX 1

Structural programming of the finite element technique is well established and therefore no attempt has been made in this thesis to discuss each stage and techniques that are fairly common to any ordinary finite element program. Instead it is intended to emphasise the unusual aspect of the parts of the program which are unique in automatic data preparation for the thin walled beam column element.

As mentioned previously in Chapter 6, the non-linear behaviour is determined by solving a sequence of linear problems. The load is also applied with a sequence of sufficiently small increments so that during loading the structure is assumed to respond linearly. For each load increment the displacement increment is computed by solving equation 4.78.0. At each increment of loading the geometry of the structure is also modified and updated by using equation 6.10.0, which basically implements the midpoint tangent technique. (For details of the method see Chapter 6).

Similarly a subsequent increment in load was also applied and the process is repeated until the desired large displacement or the collapse criterion is achieved and this is illustrated in the following flow chart.

Read for each element the coordinate of the nodal points the node number and the cross-sectional geometry

Calculation of semi bandwidth

Read structural constraints and the loading vector

Setting up element stiffness matrix

For each element call subroutine 'GLOLO' and calculate,
a) The direction cosines of the local principal axis to the global reference axis system.
b) Calculations of shear centre and centroidal coordinates with respect to the global axis system.
c) Calculation of coordinates of the axial load, transverse loads and moments applications with respect to the global axis system.
d) Calculation of sectorial coordinate of the axial load application.
Subroutine 'GLOLO' internally calls up subroutine 'SPRO' which
calculates the following for each cross-section of the element
a) Position of the centroid and the shear centre
b) Position of the principal axis and the second moment area corresponds to each principal axis
c) St Venent torsion constant, torsion warping constant and also for each segmental area of the section,
i. the sectorial coordinate
ii. linear sectorial moment of area and the statical moment of area about the two principal axis system
d) Calculation of Timoshenko torsional buckling constants

Formulation of element stiffness matrix subroutine BFW3D if the number of the run is greater than 1 then the formulation of the geometric stiffness matrix subroutine BTW4D


Accumulation of the forces in the global direction and convert them to corresponding components in the local direction which is to be used in the geometric stiffness matrix in the next loading increment.

Accumulation of the displacement in the global direction.

Prepare next loading increment, predict new geometry and repeat the process from setting up element stiffness matrix.

If the buckling is achieved, the assembled stiffness matrix becomes singular or the displacement produced becomes excessive. However in examples like the elastic problem of a cantilever they become progressively stiffer.

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## APPENDIX 2

## THE STRUCTURE OF SUBROUTINE 'SPRO'

Let us consider a general cross-section of a thin walled beam, as illustrated in Figure A.2.1. In general, the geometry of the cros.ssection can be complicated by branching as in Figure Á2.1. The section is divided into a number of manageable sizes of trapezoidal segments as shown. The origin of the axis (, ) which defines the geometry is also arbitrary.


Let us estimate the position of the centroid and the segmental properties for a typical segment shown in Figure A.2.2.


FIGURE A.2.2

The geometry of the element mentioned is as follows:

|  | Node 1 | Node 2 |
| :--- | :---: | :---: |
| Coordinates | $\left(\eta_{1}, \zeta_{1}\right)$ | $\left(\eta_{2}, 5_{2}\right)$ |
| Thickness | $t_{1}$ | $t_{2}$ |

and $i$ denotes the $i^{\text {th }}$ segment
length of the element $=L$
distance to the local centroid from node $1=\eta_{L, i}$.

Therefore from basic geometry,
The area of the segment $=A_{i}=\frac{L_{i}}{2}\left(t_{1, i}+t_{2, i}\right)$

Distance to the local centroid from node $1=\eta_{L, i}=\frac{L_{i}}{3} \frac{\left(2 t_{2, i}+t_{1, i}\right)}{\left(t_{1, i}+t_{2, i}\right)}$

The second moment of area about $\eta_{L}=I_{n L}$

$$
\begin{equation*}
=\frac{L_{i}}{48}\left(t_{2, i}^{3}+t_{2, i}^{2} t_{1, i}+t_{2, i} t_{1, i}^{2}+t_{1, i}^{3}\right) \tag{A.2.3}
\end{equation*}
$$

The second moment of area about $\zeta_{L}=I_{\zeta_{L}}$

$$
=\frac{L_{i}^{3}}{36}\left(t_{2, i}^{2}+4 t_{1, i} t_{2, i}+t_{1, i}^{2}\right) \quad(A .2 .4)
$$

The product moment of area $I_{n L \zeta_{L}}=0$
Let us also define the inclination of the segments to the $\eta$ axis as, $\theta$

Thus,

$$
\begin{equation*}
\tan \theta=\frac{\zeta_{2, i}-\zeta_{1, i}}{n_{2, i}-n_{1, i}} \tag{A.2.5}
\end{equation*}
$$

However, by using the parallel axis theorem,

The second moment of area about the $\eta$ axis

$$
\begin{equation*}
=I_{n \eta}=\frac{1}{2}\left(I_{n L}+I_{\zeta L}\right)+\frac{1}{2}\left(I_{n L}-I_{\zeta L}\right) \cos 2 \theta+A_{i} \overline{\zeta_{i}} \tag{A.2.6}
\end{equation*}
$$

Similarly, the second moment of area about the 5 axis

$$
\begin{equation*}
=I_{\zeta \zeta}=\frac{1}{2}\left(I_{n L}+I_{\zeta L}\right)-\frac{1}{2}\left(I_{n L}-I_{\zeta L}\right) \cos 2 \theta+A_{i} \frac{2}{\eta_{i}} \tag{A.2.7}
\end{equation*}
$$

The product moment of area $=A_{\mathbf{j}} \overline{n \zeta}$
where,

$$
\begin{align*}
& \bar{n}_{i}=n_{1}+n_{L, i} \cos \theta  \tag{A.2.8}\\
& \bar{\zeta}_{i}=\zeta_{1}+n_{L, i} \sin \theta
\end{align*}
$$

Thus to obtain the properties for the entire cross-section by summing the segmental properties yields,

$$
\begin{align*}
& \bar{n}=\sum_{i=1}^{n} \frac{A_{i} \bar{\eta}_{\mathbf{i}}}{\sum_{i=1}^{n} A_{i}}  \tag{A.2.9.1}\\
& \bar{\zeta}=\sum_{i=1}^{n} \frac{A_{i} \bar{\zeta}_{\mathbf{i}}}{\sum_{i=1}^{n} A_{i}} \tag{A.2.9.2}
\end{align*}
$$

where, $(\bar{n}, \bar{\zeta})$ is the coordinates of the centroid of the entire crosssection from the arbitrarily chosen axes $\eta$ and $\zeta$.
$n$ is the number of finite segments that the cross-section is composed of. Thus, for the entire cross-section,

$$
\begin{aligned}
& \left.I_{\zeta \zeta}=\sum_{i=1}^{n}\left(A_{i}\left(\bar{n}_{i}-\bar{n}\right)\right)^{2}+I_{n L i}\right) \\
& =\sum_{i=1}^{n} A_{i}\left(\bar{n}_{1}^{2}-2 \bar{n} \bar{n}_{i}+\bar{n}^{2}\right)+\Sigma I_{n L i} \\
& =\sum_{i=1}^{n}\left(A_{i} \bar{n}_{1}^{2}-2 \bar{n} A_{i} \bar{n}_{i}+A_{i} \bar{n}^{2}\right)+\Sigma I_{n L} \\
& =\sum_{i=1}^{n} A_{i} \bar{\eta}_{i}^{2}-\frac{2 \sum_{i=1}^{n} A_{i} \bar{\eta}_{i}}{\sum_{i=1}^{n} A_{i}} \cdot \sum_{i=1}^{n} A_{i} \bar{\eta}_{i}+\frac{\left(\sum_{i=1}^{n} A_{i} \bar{\eta}_{i}\right)^{2}}{\sum_{i=1}^{n} A_{i}}+\sum_{i=1}^{n} I n_{L i} \\
& \left.I_{\zeta \zeta}=\sum_{i=1}^{n} A_{i} \bar{n}_{i}^{2}-\frac{\left(\sum_{i=1}^{j=1} A_{i} \bar{\eta}_{i}\right)^{2}}{\sum_{i=1}^{n} A_{i}}=\sum_{i=1}^{n} A_{i} \bar{\eta}_{i}^{2}-\left(\sum_{i=1}^{n} A_{i}\right) \bar{n}^{2}+\sum_{i=1}^{n} I_{n_{L}, i}\right) \\
& \text { Similarly, } I_{n \eta}=\sum_{i=1}^{n} A_{i} \bar{\zeta}_{i}^{2}-\left(\sum_{i=1}^{n} A_{i}\right) \bar{\zeta}^{2}+\sum_{i=1}^{n} I_{\zeta L i} \quad \text { (A.2.10.2) }
\end{aligned}
$$

The product moment of area $I_{n \zeta}=\sum_{i=1}^{n} A_{i}\left(\bar{n}_{i}-\bar{n}\right)\left(\bar{\zeta}_{i}-\bar{\zeta}\right)+\sum_{i=1}^{n} I_{n_{L} \zeta} L$

$$
\text { but } \sum_{i=1}^{n} I_{n_{L} \zeta_{L}}=0 \text { (see equation (A.2.10.1)). }
$$

Therefore,

$$
\begin{aligned}
I_{n \zeta} & =\sum_{i=1}^{n} A_{i}\left(\bar{n}_{i} \bar{\zeta}_{i}-\bar{n}_{i} \bar{\zeta}-\bar{\zeta}_{i} \bar{n}+\overline{n \zeta}\right) \\
& =\sum_{i=1}^{n} A_{i} \bar{n}_{i} \bar{\zeta}_{i}-\bar{\zeta} \sum_{i=1}^{n} A_{i} \bar{n}_{i}-\bar{n} \sum_{i=1}^{n} A_{i} \bar{\zeta}_{i}+\bar{n} \bar{\zeta} \sum_{i=1}^{n} A_{i}
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{i=1}^{n} A_{i} \bar{n}_{i} \bar{\zeta}_{i}-\frac{\sum_{i=1}^{n} A_{i} \bar{\zeta}_{i} \cdot \sum_{i=1}^{n} A_{i} \bar{n}_{i}}{\sum_{i=1}^{n}}-\frac{\sum_{i=1}^{n} A_{i} \bar{n}_{i} \cdot \sum_{i=1}^{n} A_{i} \bar{\zeta}_{i}}{\sum_{i=1}^{n} A_{i}}  \tag{A.2.11}\\
& =\sum_{i=1}^{n} A_{i} \bar{n}_{i} \bar{\zeta}_{i}-\sum_{i=1}^{n}\left(A_{i}\right) \bar{n} \bar{\zeta}
\end{align*}+\frac{\sum_{i=1}^{n} A_{i} \frac{n_{i}}{n} \cdot \sum_{i=1}^{n} A_{i} \bar{\zeta}_{i}}{\sum_{i=1}^{n} A_{i}}
$$

## Calculation of Sectorial properties:

The sectorial coordinates of a point with respect to a pole and a reference point is defined as the net area traced by the sliding radius, as it moves away from the reference point to the point of interest as shown in Figure (A.2.3)


Fig. 2.2.3
where, $D$ is a reference point placed arbitrarily on the outline of the section
$C$ is a pole point placed arbitrarily on the ( $n-(t)$ plane
$P$ is a point of interest defined by $(n, r)$.
If pole $C$ is replaced by the shear centre and $C D$ is the principal radius, then the sectorial coordinate at point ' $p$ ' becomes unique and defined as the principal sectorial coordinate. A good detafled analysis of principal sectorial coordinates is found on pages 20 and 21 of reference (66).

The position of the shear centre with respect to the $(\eta, 5)$ axis is given by,

$$
\begin{align*}
& a_{n}=b_{n}+\frac{I_{\zeta \zeta} \int_{A} \omega_{C D P} \zeta d A-I_{n \zeta} \int_{A} \omega_{C D P} n d A}{I_{n n} I_{\zeta \zeta}-I_{n \zeta}^{2}} \\
& a_{\zeta}=b_{\zeta}-\frac{I_{n n} \int_{A} \omega_{C D P} n d A-I_{n \zeta} \int_{A} \omega_{C D P} \zeta d A}{I_{n n} I_{\zeta \zeta}-I_{n \zeta}^{2}} \tag{A.2.12}
\end{align*}
$$

Sectorial coordinate with respect to an arbitrary pole and reference point:


FIGURE A.2.4

The sectorial coordinate anywhere between the two nodes can be found by approximating a linear variation of the nodal values as shown in Figure (A.2.4).

Therefore the sectorial coordinate $\omega\left(\eta_{L}\right)=\omega_{1, i}+\left(\omega_{2, i}-\omega_{1, i}\right) \frac{n_{L}}{L}$ The finite area,

$$
\begin{align*}
d A\left(n_{L}\right)= & d n_{L} \cdot \frac{1}{2}\left[t_{1, i}+\left(t_{2, i}-t_{1, i}\right) \frac{n_{i}}{L}+t_{1, i}\right. \\
& \left.+\left(t_{2, i}-t_{1, i}\right) \frac{n_{L}+d n_{L}}{L}\right] . \\
d A\left(n_{L}\right)= & d n_{L}\left[t_{1, i}+\left(t_{2, i}-t_{1, i}\right) \frac{n_{i}}{L}+\frac{\left(t_{2, i}-t_{1, i}\right)}{2 L} d n_{L} .\right. \tag{A.2.13}
\end{align*}
$$

By ignoring second order terms,

$$
\begin{equation*}
d A\left(n_{L}\right)=\left[t_{i, 1}+\left(t_{i, 2}-t_{i, 1}\right) \frac{n_{L}}{L}\right] d n_{L} \tag{A.2.14}
\end{equation*}
$$

Therefore the linear sectorial moment of area,

$$
\begin{align*}
\int_{A} \omega d A= & \int_{0}^{L}\left(\omega_{1, i}+\left(\omega_{2, i}-\omega_{1, i}\right) \frac{\eta_{L}}{L}\right)\left(t_{1, i}+\left(t_{2, i}-t_{1, i}\right)^{n_{L}}\right) d_{\eta_{L}} \\
\int_{A} \omega d A= & L\left[\omega_{1, i} t_{1, i}+\frac{\omega_{1}\left(t_{2, i}-t_{1, i}\right)+t_{1, i}\left(\omega_{2, i}-\omega_{1, i}\right)}{2}\right. \\
& \left.+\frac{\left(\omega_{2, i} \omega_{1, i}\right)\left(t_{2, i}-t_{1, i}\right)}{3}\right] \tag{A.2.15}
\end{align*}
$$

The linear sectorial moment of area about the axis is given by,

$$
\int_{A} \tau \omega d A
$$

Thus,

$$
\begin{aligned}
\int_{A} \dot{\zeta} \omega d A= & \int_{0}^{L}\left(\zeta_{1, i}+\left(\zeta_{2, i}-\zeta_{1, i}\right) \frac{\eta_{L}}{L}\right)\left(\omega_{1, i}+\left(\omega_{2, i}-\omega_{1, i}\right) \frac{n^{\prime}}{L}\right) \\
& \left(t_{1, i}+\left(t_{2, i}-t_{1, i}\right) \frac{\eta_{L}}{L}\right) d_{n}
\end{aligned}
$$

$$
\begin{align*}
& =L\left[\zeta_{1, i}{ }^{\omega} 1, i{ }^{t} 1, i+\frac{{ }^{\zeta} 1, i^{\omega} 1, i^{D T_{i}}{ }^{+t} 1, i^{\left\{\zeta_{1, i}\right.}{ }^{D \omega_{i}+D \gamma_{i}{ }^{\omega} 1, i}{ }^{\}}}{2}\right. \\
& +\frac{D T_{i}\left\{y_{1,}{ }_{j} D \omega_{i}+D Y_{i} \omega_{1, j}\right\}+\cap Y_{j} \cap \omega_{i}{ }_{i}{ }_{i, i}}{3}+\frac{D Y_{i} D \omega_{i} D \tau_{i}}{4} \tag{A.2.16}
\end{align*}
$$

Similarly the linear sectorial moment of area about the $\zeta$ axis is given by $\int_{A}: n \omega d A$

$$
\begin{align*}
\therefore A^{\int_{n \omega d A}} & =L\left[n_{1, i} \omega_{1, i} t_{1, i}+\frac{n_{1, i}\left(\omega_{1, i} D T_{i}+t_{1, i} D \omega_{i}\right)+D n_{i} \omega_{1, i} t_{1, i}}{2}\right. \\
& \left.+\frac{n_{1, i} \omega_{i} D T_{i}+D n_{i}\left(\omega_{1, i} D T_{i}+t_{1, i} D \omega_{i}\right)}{3}+\frac{D n_{i} D \omega_{i} D T_{i}}{4}\right] \quad \text { (A.2. } \tag{A.2.17}
\end{align*}
$$

where, $\omega={ }^{\omega}$ CDP

$$
\begin{aligned}
& D T_{i}=t_{2, i}-t_{1, i} \\
& D{ }_{i}=\omega_{2, i}-\omega_{1, i} \\
& D n_{i}=n_{2, i}-n_{1, i} \\
& D \zeta_{\mathfrak{j}}=\zeta_{2, i}-\zeta_{1, i}
\end{aligned}
$$

 cross-section in equations (A.2.12.1) and (A.2.12.2) the position of the shear centre ( $a_{n}, a_{\zeta}$ ) can be calculated.

## Calculation of principal sectorial coordinate:

To calculate the principal sectorial coordinates, the following conditions must be met:

1. Shear centre as the pole point
2. Principal radius, which is defined by the principal reference point chosen on the outline of the section and the shear centre.

Since the absolute position of the principal reference point is not relevant at this stage, we calculate the sectorial coordinate. In general this is done by using the shear centre as the pole point and also an arbitrary point on the outline as the reference point.

Thus by definition the condition that is required to satisfy principal sectorial coordinates yields,

$$
\begin{equation*}
\omega_{0}=\frac{A^{\oint \omega d A}}{\oint d A} \tag{A.2.18}
\end{equation*}
$$

where $\omega_{0}$ is the sectorial coordinate of the principal reference point on the outline of the section, providing that the shear centre is chosen as the pole point and any other arbitrary point is as the reference point.

Therefore the principal sectorial coordinate at the $i^{\text {th }}$ node of the section is given by,

$$
\omega_{p, i}=\omega_{i}-\omega_{0}
$$

Also, by definition the torsion warping constant $\Gamma$ is given by,

$$
\begin{align*}
r & =\sum_{i=1}^{n} \int_{A}\left(\omega_{P, i}-\omega_{0}\right)^{2} d A \\
& =\sum_{i=1}^{n} \int_{A}\left(\omega_{P, i}^{2}-2 \omega_{P, i} \omega_{0}+\omega_{0}{ }^{2}\right) d A \\
& =\sum_{i=1}^{n} \int_{A}\left(\omega_{P, i}^{2} d A-2 \omega_{0} \sum_{i=1}^{n} \int_{A} \omega d A+\omega_{0}^{2} \sum_{i=1}^{n} A_{i}\right. \tag{A.2.19}
\end{align*}
$$

where $\oint \omega d A$ is the principal linear sectorial moment of area and can be calculated by substituting the principal sectorial coordinate into equation (A.2.15), where $\sum_{i=1} A_{i}$ is the net area of the complete section.

## Calculation of the torsion warping constant:

As before the distribution of the principal sectorial coordinate of the $\mathfrak{i}^{\text {th }}$ segment,

$$
\begin{equation*}
\omega_{n_{L, i}}=\omega_{1, i}+\left(\omega_{2, i}-\omega_{1, i}\right) \frac{n_{L, i}}{L_{i}} \tag{A.2.20}
\end{equation*}
$$

Similarly the distribution of the thickness of the segment,

$$
t_{i}=t_{1, i}+\left(t_{2, i}-t_{1, i}\right) \frac{\eta_{L, i}}{L_{i}}
$$

(See also equation (A.2.14)).
Therefore the torsion warping constant $\Gamma$, for the complete section,

$$
\begin{align*}
\Gamma=\oint \omega^{2} d A=\sum_{i=1}^{n} \int_{A} \omega_{i}^{2} d A= & \sum \int_{0}^{L}\left\{\left(\omega_{1, i}+\left(\omega_{2, i}-\omega_{1, i}\right) \frac{n_{L}, i}{L}\right\}^{2} .\right. \\
& \left\{t_{1, i}+\left(t_{2, i}-t_{1, i}\right)^{n_{L, i}} L_{i}\right\} d_{n}
\end{aligned} \quad \begin{aligned}
\Gamma=\sum_{i=1}^{n} L_{i}\left\{\omega_{1, i}^{2} t_{1, i}\right. & +\omega_{1, i} \frac{\left(\omega_{1, i} D T_{i}+2 D \omega_{i} t_{1, i}\right)}{2}+\frac{2 D \omega_{i} \omega_{1, i} D T_{i}+D \omega_{i}^{2} t_{1, i}}{3} \\
& +\frac{D \omega_{i}^{2} D T_{i}}{4}
\end{align*}
$$

where, as before, $D T_{i}=t_{2, i}-t_{1, i}$

$$
D_{\omega_{i}}=\omega_{2, i}-\omega_{1, i}
$$

## Calculation of Timoshenko torsional buckling constant:

The phenomenon of torsional lateral buckling under axial load was described in Chapter 4 and the corresponding equations which govern the buckling constants are as follows:

$$
\begin{align*}
& B_{1}=\frac{1}{I_{y}}\left(\int^{3} d A+\int_{A} z y^{2} d A\right)-2 z_{0}  \tag{A.2.21.1}\\
& B_{2}=\frac{1}{I_{z}}\left(\int_{A} y^{3} d A+\int_{A} y z^{2} d A\right)-2 y_{0}  \tag{A.2.21.1}\\
& S_{0}=\frac{I_{0}}{A}+e_{z} B_{1}+e_{y} \beta_{2} \tag{A.2.21.3}
\end{align*}
$$

where, $I_{y}$ and $I_{z}$ are the second moments of area about the principal axes, $I_{0}$ is the polar moment of area at the shear centre defined by,

$$
\begin{equation*}
I_{0}=I_{z}+I_{y}+A\left(e_{z}^{2}+e_{y}{ }^{2}\right) \tag{A.2.21.3.1}
\end{equation*}
$$

where, $e_{y}$ and $e_{z}$ are the coordinates of the shear centre with respect to the principal axes system.

The position of the shear centre and the terms $I_{y}, I_{z}$ are already established and by substituting into equation A2.27.3.1, the polar moment of area at the shear centre is tabulated. The method of estimating terms $\beta_{1}$ and $\beta_{2}$ is as follows,


FIGURE A.2.5

As before let us begin by assuming a linear variation of coordinates along the element as shown in Figure (A.2.5). Thus the coordinate of an arbitrary point on the element can be expressed as,

$$
\begin{align*}
& n_{i}=n_{i, 1}+\left(n_{2, i}-n_{1, i}\right) \frac{n_{L, i}}{L_{i}}  \tag{A.2.22.1}\\
& \zeta_{i}=\zeta_{1, i}+\left(\zeta_{2, i}-\zeta_{1, i}\right) \frac{n_{L, i}}{L_{i}} \tag{A.2.22.2}
\end{align*}
$$

Let us first establish the term $\oint \zeta^{3} d A$.
From equation (A.2.22.1) the $\zeta$ coordinate of an arbitrary point is given by,

$$
\begin{equation*}
\zeta_{i}=\zeta_{1, i}+\left(\zeta_{2, i}-\zeta_{1, i}\right) \frac{\eta_{L, i}}{L_{i}} \tag{A.2.23}
\end{equation*}
$$

By substituting equation (A.2.23), for $\zeta_{i}$ in the expression above,

$$
\begin{equation*}
\int \zeta_{i}^{3}=\left\{\zeta_{1, i}+\left(\zeta_{2, i}-\zeta_{1, i}\right) \frac{n_{L, i}}{L_{i}}\right\}^{3} d A \tag{A.2.24}
\end{equation*}
$$

Also by substituting equation (A.2.13) for $d A$ in equation (A.2.24) yields,

$$
\int \zeta_{i}{ }^{3} d A=\zeta_{1, i}^{3} t_{1, j} L_{i}+\frac{3}{2} \zeta_{1, i}^{2} t_{1, i} D \zeta_{i} L_{i}+\zeta_{1, i} D \zeta_{i}^{2} t_{1, i} L_{i}^{2}
$$

$$
\begin{aligned}
& +\frac{1}{4} D \zeta_{i}^{3} t_{1, i} L_{i}+\frac{1}{2} D t_{i} \zeta_{1, i}^{3} L_{i} \\
& +\zeta_{1, i}^{2} D \zeta_{i} D T_{i} L_{i}+\frac{3}{4} \zeta_{1, i} D \zeta_{i}^{2} D t_{i} L_{i}+\frac{1}{5} D \zeta_{j}^{3} D t_{i} L_{i}
\end{aligned}
$$

(A.2.25)

Finally the results of the equation shown above now provides the $\int \zeta^{3} d A$ for a finite segment shown in Figure (A.2.1). Thus for the entire section,

$$
\begin{aligned}
& \int \zeta_{i}^{3} d A=\int_{i}\left\{\zeta_{1, i}+\left(\zeta_{2, i}-\zeta_{1, i}\right) \frac{\eta_{L, i}}{L_{i}}\right\}\left\{t_{i, 1}+\left(t_{2, i}-t_{1, i}\right) \frac{n_{L, i}}{L_{i}}\right\} d n_{L, i} \\
& =\int_{L_{i}}\left\{\zeta_{1, i}^{3} t_{1, i}+3 \zeta_{1, i}^{2} t_{1, i} D \zeta_{i} \frac{L_{L}}{L_{i}}+3 \zeta_{1, i} D \zeta_{i}^{2} t_{1, i}\left(\frac{L_{L, i}}{L_{i}}\right)^{2}\right. \\
& +D \zeta_{i}^{3} t_{1, i}\left(\frac{n_{L, i}}{L_{i}}\right)^{3}+\zeta_{1, i}^{3} D t_{i} \frac{n_{L, i}}{L_{i}} \\
& +3 \zeta_{1, i}^{2} D \zeta_{i} D t_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{2}+3 \zeta_{1, i} D \zeta_{i}^{2} D t_{i}\left(\frac{n L, i}{L_{i}}\right)^{3} \\
& \left.+D \zeta_{i}^{3} D T_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{4}\right\} d \eta_{L, i}
\end{aligned}
$$

$$
\begin{align*}
\oint \zeta_{1}^{3} d A= & \sum_{i=1}^{n}\left(\zeta_{1, i}^{3} t_{1, i} L_{i}+\frac{3}{2} \zeta_{1, i}^{2} t_{1, i} D \zeta_{i} L_{i}+\zeta_{1, i} D \zeta_{i}^{2} t_{1, i} L_{i}\right. \\
& +\frac{1}{4} D \zeta_{i}^{3} t_{1, i} L_{i}+\frac{1}{2} D t_{i} \zeta_{1, i}^{3} L_{i} \\
& \left.+\zeta_{1, i}^{2} D \zeta_{j} D t_{i} L_{i}+\frac{3}{4} \zeta_{1, i} D \zeta_{i}^{2} D \tau_{i} L_{i}+\frac{1}{5} D \zeta_{i}^{3} D t_{i} L_{i}\right) \tag{A.2.26}
\end{align*}
$$

where, $D t_{i}=t_{2, i}-t_{1, i}$

$$
D \zeta_{i}=\zeta_{2, i}-\zeta_{1, i}
$$

A similar expression was also developed for the term $\oint \eta^{3} d A$ :

$$
\begin{align*}
\phi n_{i}^{3} d A= & \sum_{i=1}^{n}\left(n_{1, i}^{3} t_{1, i} L_{i}+\frac{3}{2} n_{1, i}^{2} t_{1, i} D n_{i} L_{i}+n_{1, i} D n_{j}^{2} t_{1, i} L_{i}^{2}\right. \\
& +\frac{1}{4} D n_{i}^{3} t_{1, i} L_{i}+\frac{1}{2} D t_{i} n_{1, i}^{3} L_{i} \\
& \left.+n_{1, i}^{2} D n_{i} D t_{i} L_{i}+\frac{3}{4} n_{1, i} D n_{j}^{2} D t_{i} L_{i}+\frac{1}{5} D n_{i}^{3} D t_{i} L_{i}\right) \tag{A.2.27}
\end{align*}
$$

Let us now concentrate on the term $\oint \underset{2}{\zeta} n^{2} d A$. As before the corresponding term for the $i^{\text {th }}$ segment $\int \zeta_{i} \eta_{j} d A$

$$
\begin{gathered}
\int \zeta_{i} \eta_{i}^{2} d A=\int_{i}\left\{\zeta_{1, i}+\left(\zeta_{2, i-\zeta_{1, i}}\right)^{n_{L, i}} L_{i}\left\{n_{1, i}+\left(n_{2, i}-n_{1, i}\right)^{n_{L, i}} L_{i}^{2}\right.\right. \\
\\
\left\{t_{i, 1}+\left(t_{2, i}-t_{1, i}\right)^{n_{L, i}} L_{i}\right\} d n_{L, i}
\end{gathered}
$$

$$
\begin{aligned}
& =\int_{L_{i}}\left\{n_{1, i}^{2} \zeta_{1, i} t_{i, 1}+2 n_{1, i} D n_{i} \frac{n_{L, i}}{L_{i}} \zeta_{1, i} t_{1, i}\right. \\
& +D_{n}^{2}\left(\frac{n_{i, i}}{L_{i}}\right)^{2} \zeta_{1, i} t_{1, i}+n_{1, i}^{2} D \zeta_{i} \frac{n_{L, i}}{L_{i}} t_{1, i} \\
& +2 n_{1, i} D n_{i} D \zeta_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{2} t_{1, i}+D \zeta_{i} D n_{i}^{2}\left(\frac{n_{L, i}}{L_{i}}\right)^{3} t_{1, i} \\
& +n_{1, i}^{2} \zeta_{1, i} D t_{i} \frac{n_{L, i}}{L_{i}}+2 n_{1 ; i} \zeta_{1, i} D n_{i} D t_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{2} \\
& +\zeta_{1, i} D n_{i}^{2} D t_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{3}+n_{1, i}^{2} D s_{i} D t_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{2} \\
& +2 \eta_{1, i} D n_{i} D s_{i} D t_{i}\left(\frac{n_{L}, i}{L_{i}}\right)^{3}+D n_{i}^{2} D \zeta_{i} D t_{i}\left(\frac{n_{L, i}}{L_{i}}\right)^{4\}} d n_{L}, i
\end{aligned}
$$

After integrating:

$$
\begin{align*}
\phi \zeta_{i} \eta_{i}^{2} d A & =\sum_{i=1}^{n}\left\{n_{1, i}^{2} \zeta_{1, i} t_{i, 1} L_{i}+n_{1, i} D n_{i} L_{i} \zeta_{1, i} t_{1, i}\right. \\
& +\frac{1}{3} D n_{i}^{2} L_{i} \zeta_{1, i} t_{1, i}+\frac{1}{2} n_{1, i}^{2} D \zeta_{i} L_{i} t_{1, i} \\
& +\frac{2}{3} n_{1, i} D n_{i} D \zeta_{i} L_{i} t_{1, i}+\frac{1}{4} D \zeta_{i} D n_{i}^{2} L_{i} t_{1, i}+\frac{1}{2} n_{1, i}^{2} \zeta_{1, i} D t_{i} L_{i} \\
& +\frac{2}{3} n_{1, i} \zeta_{1, i} D n_{i} D t_{i} L_{i}+\frac{1}{4} \zeta_{1, i} D n_{i}^{2} D t_{i} L_{i} \\
& +\frac{1}{3} n_{1, i}^{2} D \zeta_{i} D t_{i} L_{i}+\frac{1}{2} n_{1, i} D n_{i} D \zeta_{i} D t_{i} L_{i} \\
& \left.+\frac{1}{5} D n_{i}^{2} D \zeta_{i} D t_{i} L_{i}\right\} \tag{A.2.30}
\end{align*}
$$

By substituting equations (A.2.26), (A.2.27), (A.2.28) with the coordinates of the shear section into equations (A.2.21.1) and (A.2.21.2) the parameters $\beta_{1}$ and $\beta_{2}$ are established. Finally, further substitution of $\beta_{1}, \beta_{2}$ and the polar moment of area about the shear centre from equation (A.2.21.3.1) into equation (A.2.21.3) yields the Timoshenkian torsional buckling constant.

↔.

## Calculation of the St Venent torsion constant:

By referring to page 3 of reference (66) the generalized expression of the torsion constant for the section shown in Figure (A.2.1) is as follows:

$$
\begin{equation*}
K=\sum_{i=1}^{n} \frac{1}{3} L_{i} t_{i}^{3} \tag{A.2.31}
\end{equation*}
$$

where, $L_{i}$ is the length of the $i^{\text {th }}$ segment
$t_{i}=\left(t_{i, 2}+t_{i, 1}\right) / 2$ is the average thickness of the $\mathfrak{i}^{\text {th }}$ segment
$n$ designates the total number of segments of which the section is comprised.


[^0]:    Note:-
    all dimensions are in mm thickness 9.0 mm all over

