COUPLED-WAVE THEORY APPROACH TO UNDERSTANDING RESONANT VIBRATIONS OF NON-CIRCULAR CYLINDRICAL SHELLS

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1 INTRODUCTION

According to the finite element analysis of flexural vibrations in some simplified models of vehicle body structures built up of non-circular cylindrical shells (see e.g. Figure 1), resonant vibration modes of such structures can be represented as symmetric and anti-symmetric combinations of vibrations of their quasi-flat parts taken separately [1]. In the current paper it is proposed to model such modes analytically using the coupled-wave theory approach utilising the concept of coupled-waveguide propagation in shells of the same non-circular shape but having an infinite length (depth). The concept of coupled waveguides has been successfully applied earlier to studying Rayleigh wave propagation on the surfaces of solid bodies having complex cross-sectional shapes [2]

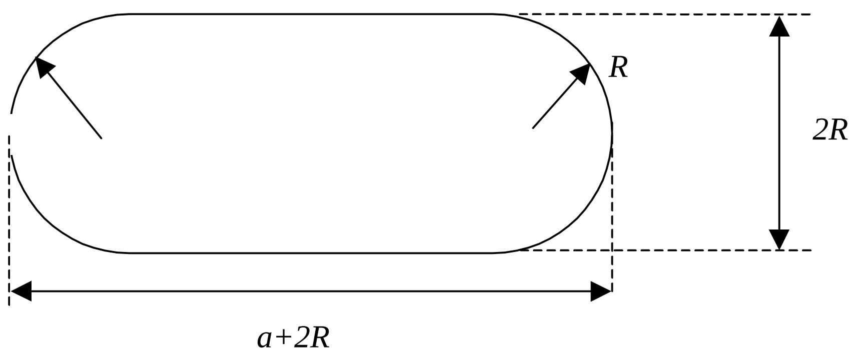

Figure 1. Profile of the non-circular cylindrical shell used for experimental modelling of structure-borne vehicle interior noise [3,4].

The physical reason for waveguide propagation along quasi-flat parts of non-circular cylindrical shells is the difference between flexural wave velocities in their flat and curved areas. In particular, for waveguide propagation it is necessary that the velocity of flexural waves in the flat area is lower than their velocity in the adjacent circular cylinders. This is always the case for flexural wave propagation in near-axial directions of circular cylindrical shells.

In the following sections, we will employ the approximate expressions for flexural wave velocities (or wavenumbers) in circular shells with different radii of curvature to derive the dispersion equations for waves propagating in the waveguides comprising infinitely long flat plates (strips) bounded by fragments of two circular shells. The coupling between two neighbouring waveguides and the coupled waveguide modes will be discussed as well. After that, considering shells of finite length, the transition will be made from coupled guided modes to coupled resonant vibrations of quasi-flat parts of the shells. It will be demonstrated that resonant frequencies and spatial distributions of the resultant vibration modes are in good agreement with the experiments [3,4] and with the results of finite element calculations [1]. The discussed coupled-wave theory approach can be useful not only for understanding the results of numerical calculations, but also for quick prediction of resonant vibration modes in numerous complex structures.

2. WAVEGUIDE PROPAGATION IN A PLATE BOUNDED BY TWO CIRCULAR CYLINDRICAL SHELLS

Let us consider waveguide propagation of flexural waves in an infinitely long flat plate (strip) of thickness h and width a bounded by fragments of two cylindrical shells with equal radii R (Figure 2). It is assumed that the shells are of the same thickness h as the flat plate. Such a structure can be considered as a three-layered anisotropic medium for flexural waves, the middle layer (a flat isotropic strip) being characterised by a lower velocity of flexural waves in comparison with flexural wave velocities in the near-axial direction in the adjacent cylindrical shells. We remind the reader that we assume initially that the above-mentioned structure is infinite in z-direction ($L = \infty$). Waveguide propagation in such a structure can be described by the well-known from Acoustics and Optics general dispersion equation for three-layered media that should be modified for the considered case of shell-induced anisotropic side layers [2,5]:

$$\frac{a}{2}\left[k_{pl}^{\ 2} - \gamma^2\right]^{1/2} = m\frac{\pi}{2} + \tan^{-1}\left[\frac{\gamma^2 - k_{sh}^{\ 2}(\varphi)}{k_{pl}^{\ 2} - \gamma^2}\right]^{1/2}. \tag{1}$$

Here γ is yet unknown wavenumber of a guided wave propagating in the above-mentioned three-layered system, $k_{pl} = (\omega^2 \rho_s h/D)^{1/4}$ is the wavenumber of flexural waves in a flat plate, where ρ_s is the mass density of the plate material and D is flexural rigidity, $k_{sh}(\varphi)$ is the angular-dependent wavenumber of flexural waves in a circular cylindrical shell, and $m = 0,1,2,3\ldots$ Note that

$$\gamma = k_{pl}\cos\varphi, \tag{2}$$

where φ is the angle of propagation of two plane waves comprising a guided mode (see Figure 2). The guided wave field propagating in the flat plate area is described as

$$w_{pl} = \cos[(k_{pl}^2 - \gamma^2)]^{1/2} x] \exp(i\gamma z - i\omega t), \tag{3}$$

and the guided wave field penetrating into the adjacent shells is

$$w_{sh} = C \exp[-(\gamma^2 - k_{sh}^2(\varphi(\gamma)))^{1/2} |x_{sh}|] \exp(i\gamma z - i\omega t). \tag{4}$$

Here x_{sh} is the surface curvilinear coordinate that extends the coordinate x from the plate to the shell area, and C is the constant determined from the continuity condition at the boundaries between the plate and the shells.

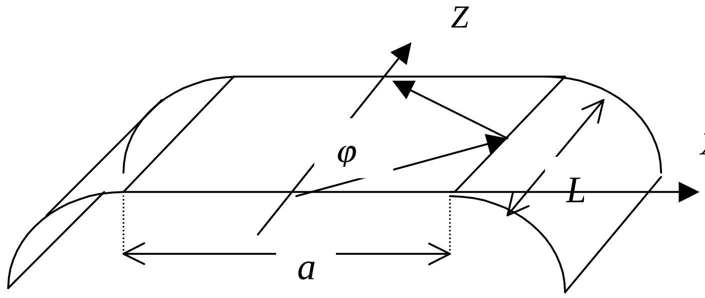

Figure 2. Waveguide propagation of flexural waves in an infinitely long flat plate (strip) bounded by two circular cylindrical shells.

To find γ from the dispersion equation (1) one has to use appropriate analytical expressions for the wavenumber of flexural waves in the shell $k_{sh}(\varphi)$ propagating at arbitrary angle φ in respect of the axial direction of the shell. It is well known [6-9], that propagation of flexural waves in shells is governed by bending and membrane effects, which makes the expressions for flexural wavenumbers rather complex in different practical situations. Their functional appearance depends on the characteristic parameters of the shell, in particular on its ring frequency $\omega_r = c_l'/R$, where c_l' is the velocity of a quasi-longitudinal wave in a thin flat plate (plate wave velocity), and R is radius of the shell's curvature.

Let us first consider the case of wave propagation in a circular shell at frequencies ω that are essentially higher than the ring frequency ω_r, i.e. consider values of the non-dimensional parameter $\Omega = \omega/\omega_r$ being much larger than unity. Starting from the general dispersion equation for flexural wave propagation in a circular shell given in Reference 7 (Eqn. (4.167)), one can easily derive the following approximate expression for the wavenumber of a flexural wave propagating in such a shell:

$$k_{sh}(\varphi) = k_{pl}\left[1 - \frac{1}{4\Omega^2}\cos^4\varphi\right]. \tag{5}$$

Keeping in mind that $\cos\varphi = 1 - \varphi^2/2$ for small φ and substituting Eqns. (2) and (5) into Eqn. (1), one can derive the following simplified equation versus φ:

$$\frac{a}{2}k_{pl}\varphi = m\frac{\pi}{2} + \tan^{-1}\left[\frac{1}{2\Omega^2\varphi} - 1\right]^{1/2}. \tag{6}$$

Considering for simplicity the propagation of the lowest order mode only ($m = 0$), this equation can be reduced to a biquadratic one. Assuming that $ak_{pl} >> \Omega$, one can obtain the approximate solution of this equation $\varphi^2 = 2^{1/2}/\Omega k_{pl}a$, which describes a weak waveguide effect. Using Eqn. (2) and this solution, one can obtain the following expression for γ:

$$\gamma = k_{pl}\left[1 - \frac{1}{2^{1/2}\Omega k_{pl} a}\right]. \tag{7}$$

As it could be expected for waveguide propagation in a three-layered system, the velocity of guided wave $c = \omega/\gamma$ in the case under consideration is higher than the velocity in the 'slow' region (flat plate), but lower than the velocity in the 'fast region' (circular shell).

Let us now turn to waveguide propagation at frequencies lower than the ring frequency. In this case the expressions for $k_{sh}(\varphi)$ are generally too complex to be described analytically [6,7]. For illustration purposes we limit ourselves, as above, with the case of small wave propagation angles φ, for which $k_{sh}(\varphi)$ is known to be extremely small. To simplify things even further, we assume that $k_{sh}(\varphi) = 0$ for all φ in the range of interest. In this case the dispersion equation (1) can be rewritten in the form

$$\frac{a}{2}\left[k_{pl}{}^{2} - \gamma^{2}\right]^{1/2} = m\frac{\pi}{2} + \tan^{-1}\left[\frac{\gamma^{2}}{k_{pl}{}^{2} - \gamma^{2}}\right]^{1/2}. \tag{8}$$

Keeping in mind that $\cos\varphi = 1 - \varphi^2/2$ for small φ and substituting Eqn. (2) into Eqn. (8), one can derive the following simplified equation versus φ:

$$\frac{a}{2}k_{pl}\varphi = m\frac{\pi}{2} + \tan^{-1}\left[\frac{1-\varphi^{2}}{\varphi^{2}}\right]^{1/2}. \tag{9}$$

Noticing that for small φ the last term in the right hand side of Eqn. (9) is approximately equal to $\pi/2$, one can reduce Eqn. (9) to

$$\frac{a}{2}k_{pl}\varphi = (m+1)\frac{\pi}{2}, \tag{10}$$

from which it follows that $\varphi = (m + 1)\pi/ak_{pl}$. Substituting this solution into Eqn. (2), one can derive the following approximate expression for the wavenumbers of guided flexural waves propagating in the above system at frequencies that are below the ring frequency:

$$\gamma = k_{pl}\left[1 - \frac{(m+1)^{2}\pi^{2}}{2a^{2}k_{pl}{}^{2}}\right]. \tag{11}$$

In this case the waveguide effect is rather strong, and the energy of a guided wave is almost entirely concentrated in the flat plate area.

3. RESONANT VIBRATIONS OF A FLAT PLATE BOUNDED BY TWO CIRCULAR SHELLS

In this section we consider a finite value of the length L of the above-mentioned plate/shell system and assume for simplicity that at $z = 0$ and at $z = L$ the system is subject to simply supported boundary conditions. Then the distribution of the resulting elastic field along z-axis formed by incident and reflected guided waves can be expressed in the form $sin(\gamma z)$. Using the condition $sin(\gamma L) = 0$, one can obtain that $\gamma L = \pi n$, where $n = 1,2,3\ldots$

Let us first analyse resonant vibrations for the case of frequencies that are higher than the ring frequency ω_r. Then, expressing k_{pl} in Eqn. (7) as $k_{pl} = \omega/c_{pl}$, where $c_{pl} = \omega^{1/2}(D/\rho_s h)^{1/4}$, and using Eqn (7) with the condition $\gamma L = \pi n$, one can derive the simple equation versus resonant frequencies ω_{0n}. Solving this equation by perturbation method, one can derive the following expression for the resonant frequencies ω_{0s} (we recall that in this case we consider only modes with $m = 0$):

$$\omega_{0n} = \frac{\pi^2 n^2}{L^2}\left(\frac{D}{\rho_s h}\right)^{1/2}\left[1 + \frac{2^{1/2} c_l^{'} \rho_s^{1/2} h^{1/2} L^3}{\pi^3 n^3 D^{1/2} Ra}\right]. \tag{12}$$

Note that Eqn. (12) describes the increase in resonant frequencies of the plate with waveguiding properties in comparison with the case of one-dimensional wave propagation in a flat plate of the same length L but with infinitely large value of the width a. We recall that distribution of the guided wave field in the transverse direction is described by Eqns. (3) and (4).

Acting in a similar way in the case of frequencies that are lower than the ring frequency, one can obtain the following equation for the system's resonant frequencies:

$$\omega_{mn} = \left(\frac{D}{\rho_s h}\right)^{1/2}\left[\frac{\pi^2(m+1)^2}{a^2} + \frac{\pi^2 n^2}{L^2}\right]. \tag{13}$$

Note that Eqn. (13) is very similar to the well-known expression for resonant frequencies of simply supported plates having the dimensions L and a. In the limit of large m Eqn. (13) gives asymptotically the same result as does the formula for simply-supported plates. This reflects the fact that at frequencies lower than the ring frequency the waveguide effect created by two adjacent circular shells is very strong and almost the whole vibration energy is concentrated in the flat plate area. One should keep in mind, however, that Eqn. (13) is valid for $n >> m$, which corresponds to low values of the propagation angle φ considered in the above-mentioned approximate solution.

4. EFFECT OF WAVE COUPLING IN THE NEIGHBOURING STRIP/SHELL WAVEGUIDES

Let us now return to the full cylindrical non-circular shell shown on Figure 1 and consider it as a system of two coupled identical parallel strip/shell waveguides separated by the distance $d = a + \pi R$ between their central lines measured along the surface. If $u_n = u_n(z)$ is a slowly varying amplitude factor which characterises the field in the n-th waveguide, where $n = 1,2$ in the case considered, then for the shell having a gap on one side, as shown on Figure 1, the coupling takes place on the opposite side only. In such a case the amplitude factors u_n should satisfy the system of two simultaneous coupled equations (see, e.g. References 2,5,10):

$$\frac{\partial u_1}{\partial z} - i\gamma u_1 + i\kappa u_2 = 0, \tag{14}$$

$$\frac{\partial u_2}{\partial z} - i\gamma u_2 + i\kappa u_1 = 0. \tag{15}$$

Here γ represents the wavenumber of any chosen guided mode in each waveguide taken separately (uncoupled waveguides), and κ is the wave coupling coefficient for this particular mode, which depends on the strength of mutual penetration of the acoustic fields from the neighbouring waveguides. In the case under consideration κ can be written dawn as [2,5].

$$\kappa = \frac{({k_{pl}}^2 - \gamma^2)(\gamma^2 - {k_{sh}}^2(\gamma))}{\gamma[1 + (\gamma^2 - {k_{sh}}^2(\gamma))^{1/2} a/2]({k_{pl}}^2 - {k_{sh}}^2(\gamma))} \exp[-(\gamma^2 - {k_{sh}}^2(\gamma))^{1/2}(d-a)]. \tag{16}$$

It is seen from Eqn. (16) that the coupling coefficient κ is determined decisively by the exponential factor that describes the decay law of the field outside the waveguides.

Seeking the solution of the simultaneous coupled equations (14), (15) in the form $u_n = A_n \exp(ikz)$, where $A_n = const$, one can obtain the following solution for the values of the wavenumber mismatch $\Delta\gamma = k - \gamma$ and for the corresponding relationships between A_1 and A_2 :

$\Delta\gamma = \kappa$; corresponding to the anti-symmetric mode of the coupled system ($A_1 = -A_2$),

$\Delta\gamma = -\kappa$; corresponding to the symmetric mode of the coupled system ($A_1 = A_2$).

If the gap on the left-hand side of the shell shown on Figure 1 is bridged, then both waveguides interact with each other on two sides and the resulting coupling coefficient becomes two times larger. The values of the mismatch $\Delta\gamma = k - \gamma$ in this case can be obtained from the above-written expressions if to replace κ by 2κ.

Considering propagation of anti-symmetric and symmetric modes in the coupled system of the finite length L, one can easily derive the expressions for resonant frequencies of the above-mentioned finite non-circular shell. Obviously, the coupling is stronger at frequencies that are higher than the ring frequency and weaker at frequencies that are below the ring frequency, where there is a little energy escape from the flat plate area (see also Eqn. (16)). In the case of higher frequencies one can use Eqn. (7) for γ and derive the following expression for resonant frequencies of symmetric and anti-symmetric modes (signs '+' and '-' in front of κ respectively):

$$\omega_{0n} = \left(\frac{\pi n}{L} \pm \kappa\right)^2 \left(\frac{D}{\rho_s h}\right)^{1/2} \left[1 + \frac{2^{1/2} c_l{}' \rho_s{}^{1/2} h^{1/2} L^3}{\pi^3 n^3 D^{1/2} R a}\right]. \tag{17}$$

In the case of frequencies that are lower than the ring frequency Eqn. (11) must be used, and the expression for resonant frequencies of symmetric and anti-symmetric modes (signs '+' and '-' in front of κ respectively) will be as follows:

$$\omega_{mn} = \left(\frac{D}{\rho_s h}\right)^{1/2} \left[\frac{\pi^2 (m+1)^2}{a^2} + \left(\frac{\pi n}{L} \pm \kappa\right)^2\right]. \tag{18}$$

Note that in the latter case the values of κ are extremely small and can be easily ignored in Eqn. (18), since the energy of vibrations is almost entirely confined in the flat plate area. Nevertheless, the symmetric and anti-symmetric coupled modes do exist even at negligibly small coupling, albeit their resonant frequencies are almost indistinguishable. This agrees well with the recent experiments [3,4] and with the finite element calculations carried out by the present authors [1]. These calculations show that resonant frequencies of symmetric and anti-symmetric modes of the non-circular cylindrical shells under consideration are practically the same (a small difference appears only in the fifth digit, and the meaning of such very small difference is not clear). In the same time, the numerical routine clearly identifies the symmetric and anti-symmetric modes in the calculated distribution of vibrations over the shell cross-section (see Figure 3).

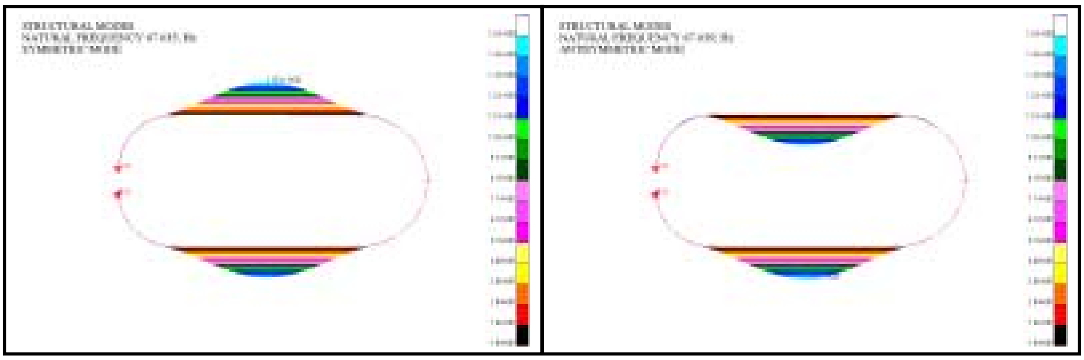

Figure 3. Numerically calculated distributions of vibration fields for lowest order symmetric and anti-symmetric modes of the non-circular cylindrical shell made of steel and having two quasi-flat surfaces [1] (see also Figure 1); geometrical parameters of the shell are: radius $R = 125\ mm$, the total length $a + 2R = 550\ mm$, the width $L = 300\ mm$, and the shell thickness $h = 1.2\ mm$; resonant frequencies of the symmetric and anti-symmetric modes are 67.035 Hz and 67.039 Hz respectively.

Although in the present paper we analysed a non-circular shell comprising only two coupled plate areas, the results of the analysis can be easily extended to arbitrary number of coupled components, in particular to structures with 3, 4 or more flat surfaces. For example, in a cylindrical shell with the shape of rounded rectangle (see Figure 4) one should consider the same quasi-flat single waveguides that have been analysed in the previous sections and then take into account wave coupling in the system of four equal waveguides, using the general approach applicable for arbitrary number of coupled waveguides [2].

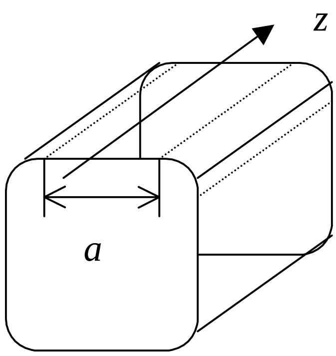

Figure 4. A cylindrical shell with the shape of rounded rectangle that can be considered as a system of four coupled strip/shell waveguides.

The obvious advantage of this procedure lies in the fact that it is much easier to make quick estimations of the resonant frequencies and modal shapes in such structures, having studied the behaviour of single quasi-flat waveguides as their basic structural components.

5. CONCLUSIONS

In the present paper it has been demonstrated that resonant vibrations of cylindrical non-circular shells can be described analytically using the coupled-wave theory approach utilising the concept of coupled-waveguide propagation in shells of the same non-circular shape, but having an infinite length. The physical reason for waveguide propagation along quasi-flat parts of such shells is the difference between flexural wave velocities in their flat and curved areas.

Using simple approximations for wavenumbers of flexural waves propagating in circular shells with different radii of curvature, the expressions for resonant frequencies have been derived for both uncoupled and coupled finite shell systems. In the case of very weak coupling, which is typical for vibrations at frequencies lower than the ring frequencies of adjacent circular shells, the values of the resultant resonant frequencies are almost entirely determined by resonant properties of the flat plate areas. This agrees well with the experiments and with the results of finite element calculations.

The developed coupled-wave theory approach to studying flexural vibrations of non-circular cylindrical shells can be useful not only for understanding the results of numerical calculations and experiments, but also for quick prediction of resonant vibration modes in numerous thin-walled structures containing quasi-flat surfaces.

6. REFERENCES

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