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‘Warrant’ revisited: Integrating mathematics teachers’ pedagogical and epistemological considerations into Toulmin’s model for argumentation

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Abstract: In this paper we propose an approach to analysing teacher arguments that takes into account field-dependence–namely, in Toulmin’s sense, the dependence of warrants deployed in an argument on the field of activity to which the argument relates. Freeman, to circumvent issues that emerge when we attempt to determine the field(s) that an argument relates to, proposed a classification of warrants (*a-priori*, *empirical*, *institutional* and *evaluative*). Our approach to analysing teacher arguments proposes an adaptation of Freeman’s classification that distinguishes between: *epistemological* and *pedagogical a-priori* warrants; *professional* and *personal empirical* warrants; *epistemological* and *curricular institutional* warrants; and, *evaluative* warrants. Our proposition emerged from analyses conducted in the course of a written-response and interview study that engages secondary mathematics teachers with classroom scenarios from the mathematical areas of Analysis and Algebra. The scenarios are hypothetical, grounded on seminal learning and teaching issues, and likely to occur in actual practice. To illustrate our proposed approach to analysing teacher arguments here we draw on the data we collected through the use of one such scenario, the Tangent Task. We demonstrate how teacher arguments, not analysed for their mathematical accuracy only, can be reconsidered, arguably more productively, in the light of other teacher considerations and priorities: pedagogical, curricular, professional and personal.

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Keywords: *teacher knowledge and beliefs, Freeman's classification of warrants, Toulmin's model for argumentation, practical rationality of teaching, visualisation, example use, tangent line*

Research into the complex nature of teacher knowledge and beliefs

One of the aims of the study we draw on here is to explore teachers' knowledge and beliefs, and how these transform into pedagogical practice. This transformation has been described before by concepts such as Chevallard's (1985) *transposition didactique*, Lampert's seminal work on teachers' dilemmas and commitments (e.g. 1985), Shulman's (1986, 1987) *pedagogical content knowledge* and Hill and Ball's (2004) *mathematical knowledge for teaching*. Over the years these concepts have evolved–see, for example, the refinements of Shulman's typology by Ball and her colleagues (e.g. Hill & Ball, 2004; Ball, Thames, & Phelps, 2008; etc.). Our work relates to some of these refinements and we return to these in the concluding parts of the paper.

Our analyses of teacher knowledge and beliefs are at some distance from certain research in this area which is often conducted within a deficit paradigm, namely mainly reporting inconsistencies between teachers' beliefs and their actions, or inadequacies of their mathematical knowledge. While much of this research has been contributing careful accounts of, for example, the limitations in the teachers' subject knowledge with regard to mathematical reasoning (e.g. Harel & Sowder, 2007; Knuth, 2002), our aim is to tread beyond the identification of such limitations. In this sense we have been influenced by perspectives such as that of Leatham (2006) whose framework conceptualises teachers' belief systems as "inherently sensible" (p. 91) rather than "inconsistent" (ibid).

Our aims resonate with those in recent works that have addressed the complex set of considerations that teachers seem to take into account when they determine their actions. Herbst and colleagues' (e.g. Herbst & Chazan, 2003; Miyakawa & Herbst, 2007) notion of *practical rationality of teaching* describes the complex set of teachers' considerations in a way that is highly relevant to the perspective of our study.

As Herbst and Chazan (2003) write, "individual practitioners can build their own mathematics teaching against the backdrop of their personal commitments and the demands of the institutional contexts where they work" (p. 2). "[S]chool mathematical activity", they

observe, "is shaped by various (institutional, situational, epistemological, temporal, material) contexts where it unfolds […] as much as it is shaped by the individual characteristics of the agents who facilitate it" (p. 3). Our analyses aim to contribute to the discussion of "how the interplay of agency and structure allows (and perhaps produces) very different 'kinds' of mathematics teaching…" (p. 3). Specifically we focus on what Herbst and Chazan (2003) call *practical rationality of teaching*, "a network of dispositions activated in specific situations" (p. 13). *Dispositions*, a term that Herbst and his colleagues adapt from Bourdieu, are categories of perception and appreciation "that help a scholar reconstruct the regulatory mechanisms of action in context" (p. 12).

In agreement with Herbst and Chazan's postulate that "conversations about a specific event activate certain dispositions in practitioners" (pp. 12-13), in the study we draw on here we invite teachers' comments on classroom scenarios (Biza, Nardi, & Zachariades, 2007) in order to discern the influence of the teachers' beliefs and knowledge on the didactical contract (Brousseau, 1997) they are likely to offer their students–see, for example, our account (Biza, Nardi, & Zachariades, 2009) of the multiple didactical contracts on the role of visualisation in mathematics and mathematical learning that teachers are likely to offer their students under those influences. From their utterances we elicit what we see as the arguments they put forward to support or steer away from certain pedagogical actions.

In this paper we propose a particular approach to analysing teacher arguments, that emerged in the course of our data analysis. This approach involves an adaptation of Toulmin's model of argumentation (1958) and Freeman's (2005a) refinement of parts of the Toulmin model. In what follows first we briefly introduce Toulmin's model and Freeman's refinement. We then describe our adaptation, and how it came to be, and illustrate its employment in a sample of our data. Finally we conclude with a brief discussion of how our proposed approach fits in with other works in this area.

Toulmin's model of argumentation and Freeman's classification of warrants

Toulmin's (1958) model describes the structure and semantic content of an informal argument. The model consists of six basic types of statement, each of which plays a particular role in an argument. The *conclusion* (C) is the statement of which the arguer wishes to convince an audience. The *data* (D) are the foundations on which the argument is based; this includes

evidence relevant to the claim being made. The *warrant* (W) justifies the connection between data and conclusion; warrants include appealing to a definition, a rule, an example, or an analogy. The warrant is supported by the *backing* (B), which presents further evidence, justifications or reasons. The *modal qualifier* or *qualifier* (Q) qualifies the conclusion by expressing degrees of the arguer's confidence. Finally, the *rebuttal* (R) consists of potential refutations of the conclusion; rebuttals include exceptions to the conclusion or citing the conditions under which the conclusion would not hold. Not all of these six statements are always explicit in the presentation of an argument.

Toulmin's model has been employed by researchers in mathematics education mainly in order to analyse student arguments. These studies (e.g.: Krummheuer, 1995, 2007; Yackel, 2001, 2002; Whitenack & Knipping, 2002; Stephan & Rasmussen, 2002; Evens & Houssart, 2004; Hoyles & Küchemann, 2002; Weber & Alcock, 2005; Pedemonte, 2005, 2007) span across educational levels, focus largely on arguments produced collectively and use reduced versions of Toulmin's model (comprised of: conclusion, data, warrant, and backing; or, conclusion, data and warrant). Recently researchers (Inglis, Mejia-Ramos, & Simpson, 2007; Inglis & Mejia-Ramos, 2008; Giannakoulias, Mastoridis, Potari, & Zachariades, 2010) have argued for the importance of employing Toulmin's full model. Inglis et al. (2007) have also elaborated the model by offering a classification of warrants–inductive, structural-intuitive and deductive–employed by the participants in their studies.

With regard to potential variations within warrants and backings Toulmin himself refers to field-dependence, namely the dependence of the warrant and the backing deployed in an argument on the field of activity to which the argument relates. In different fields, he stresses, warrants will be backed in different ways (1958, p. 104). In the examples:

> A whale will be (i.e. is classifiable as) a mammal.
> A Bermudan will be (in the eyes of the law) a Briton.
> A Saudi Arabian will be (found to be) a Muslim.

the words in parentheses indicate some of these different ways. The first warrant is backed by an accepted natural history classification. The second warrant appeals to the law determining the nationality of people born in a British colony. The third warrant may rely on statistical information regarding the distribution of religious affiliations amongst different nationalities.

The notion of field-dependence, while welcomed in principle, has also been met with concern by some authors. For example, Freeman (2005a) has found Toulmin's notion of field-dependence problematic for several reasons. It is beyond the scope of this paper to delve into a discussion of *field* but we refer the interested reader to Freeman's work (e.g. 2005a, b). However, in a nutshell, the debate there and elsewhere often turns to the concern that the notion of field is perhaps too vague as it does not distinguish between "generally recognized as authority-conferring" (Freeman, 2005a, p. 333) bodies of knowledge and those that are not. Also it does not facilitate our capacity to "assess whether a warrant is properly backed", especially if we accept that warrants in different fields are backed in different ways. Would the canons for such assessment be themselves "field-dependent or field-transcendent?" (p. 333), Freeman (2005a) wonders.

To circumvent some of the aforementioned concerns Freeman proposes focusing on "grasping different sorts of connections, suggested or discovered in different ways, and backed or justified by different sorts of considerations" (p. 342). To this purpose he suggests replacing Toulmin's notion of warrants belonging to fields with warrants classified according to the type of intuition, belief or prior understanding that gave rise to them. His proposed classification is for four types: *a priori*, *empirical*, *institutional*, and *evaluative* (p. 342). For brevity we do not cite here the elaboration and examples that Freeman poses as test cases for his classification scheme (2005a, p. 343-4). We note however that this classification emerged in parallel to his claim for *necessary*, *empirical*, *institutional* and *evaluative* generalizations, developed in detail in Chapters 6-9 by Freeman (2005b).

In Freeman's (2005a) words:

> This classification preserves Toulmin's insight on the field dependency of warrants but without the problematic notion of field. Different warrants will be justified or backed in different ways, and we must look to the type of warrant to determine how this is done properly. But we do not classify warrants according to fields. […] If several different types of warrants are used within an argument, we need not puzzle over what field is involved. (p. 342)

Freeman contends that considering warrants in the light of such classification circumvents all of the aforementioned concerns with the notion of field distinction and dependence – crucially the one regarding the assignment of a warrant to a particular field. This analysis, he adds, may on occasion "show that warrants are of mixed type" (p. 343). It is this

variation within warrants that our proposed approach to analysing teacher arguments aims to elaborate upon.

A proposed approach to analysing teacher arguments

In this paper we consider arguments put forward by secondary mathematics teachers in the context of their evaluation of students' written responses to a mathematical problem, and of their feedback to these students. Our analysis aims to discern, differentiate and discuss the range of influences (epistemological, pedagogical, curricular, professional and personal) on the arguments teachers put forward in their scripts and interviews. We focus particularly on the warrants of these arguments in the light of Toulmin's (1958) field-dependence account and our own adaptation of Freeman's (2005a) classification of warrants:

- an *a priori* warrant is, for example, the resorting to a mathematical theorem or definition (*a priori-epistemological)*; or the resorting to a pedagogical principle (*a priori-pedagogical*);
- an *institutional* warrant is, for example, a justification of a pedagogical choice on the grounds of it being recommended or required in a textbook (*institutional-curricular*); or on the grounds that it reflects the standard practices of the mathematics community (*institutional-epistemological*);
- an *empirical* warrant is, for example, the citation of a frequent occurrence in the classroom (according to the arguer's teaching experiences, *empirical-professional*) or the resorting to personal learning experiences in mathematics (*empirical-personal*);
- an *evaluative* warrant is a justification of a pedagogical choice on the grounds of a personally held view, value or belief.

We explain how this adaptation of Freeman's classification emerged and provide specific examples of these four categories later in the paper. First we outline briefly the proposition that this paper aims to put forward in the language of the above theoretical foundations.

Our work is akin to recent efforts by mathematics education researchers to revisit classic theoretical constructs from a more socially and institutionally aware perspective (such as Bingolbali and Monaghan's (2008) revisiting of the *concept image-concept definition* (Tall &

Vinner, 1981) construct). This approach has been put robustly forward also by Chevallard's Anthropological Theory of Didactics (1985). We aim at an analogous *revisiting* of a theoretical construct, Toulmin's model, originating in epistemological studies of informal argumentation, that has been attracting increasing interest by mathematics educators (see examples in the opening parts of the previous section).

Our point is relatively simple: teachers' acceptance, skepticism or rejection of students' mathematical utterances–as expressed in their evaluation of these utterances and their feedback to the students–does not have exclusively mathematical (epistemological) grounding. Their grounding is broader and includes a variety of other influences, most notably of a pedagogical, curricular, professional and personal nature. When a mathematics teacher, say, accepts what appears to be a mathematically dubious utterance by a student, the teacher might have dubious mathematical foundations. But s/he might as well have a perspective on the student's utterance that takes into account other issues that are not necessarily purely mathematical. In previous analyses (e.g. Biza et al., 2009) we saw this occurring time and again in the case of teachers' treatment of students' attempts at a visually-based argument. In what we cite later as illustrative samples from our data we identify and discuss these teacher perspectives in the light of the above classification of warrants (*a priori*, *institutional*, *empirical* and *evaluative*) in order to illuminate and elaborate those other issues.

With our adaptation of Freeman's classification of warrants we aim to argue that uses of Toulmin's model in mathematics education contexts must acknowledge the broader warrants that teachers employ when they determine and justify their actions. This acknowledgement of the breadth and scope of these warrants may render necessary the effort to re-define our criteria for evaluating teachers' arguments in a pedagogical context. Excluding a consideration of such a context, the use of Toulmin's model could risk becoming yet another cog in the wheel of deficit discourse on teacher knowledge and beliefs. Recent considerations of the model (e.g. by Inglis and colleagues) have been in favour of examining the variation of warrants. We too are concerned with variations of warrants, albeit in the different context of arguments put forward by secondary mathematics teachers. Our refinement aims to be better attuned to the needs of studies of the *practical rationality of teaching* (Herbst & Chazan, 2003). Our intentions bode well with Krummheuer's (1995) distinction between *substantial* and *analytic* argumentation, where the former contrasts with the latter in its inclusion of

elements that are not purely logical; and with his statement that substantial argumentation "has a right by itself" (p. 236).

In the following we outline how our adaptation of Freeman's classification of warrants came to be and offer examples of all types of warrants in this adaptation. Before doing so we outline briefly the study during the analyses of which our adaptation emerged.

The study: exploring teacher knowledge and beliefs through situation-specific tasks

As much of the research into the relationship between teachers' beliefs and pedagogical practice (e.g. Thompson, 1992; Leder, Pehkonen, & Törner, 2002) acknowledges, there is often an overt discrepancy between theoretically and out-of-context expressed teacher beliefs about mathematics and pedagogy and actual practice. Therefore teacher knowledge is likely to be better explored in situation-specific contexts (Biza et al, 2007). In this sense our study's aims and rationale resonate with those of several other researchers: Herbst and colleagues' (e.g. Herbst & Chazan, 2003; Miyakawa & Herbst, 2007) eliciting teachers' *practical rationality* through discussions of video-taped lesson episodes; Kennedy's (2002) examination of teachers' "reasons for doing particular things at particular moments" (p. 357); Jacobs and Morita's (2002) examination of teachers' "ideas about what constitutes effective mathematics pedagogy" (p. 154) through their commenting on videos of colleagues' lessons; etc..

In this study we engaged mathematics teachers with classroom scenarios which are hypothetical–yet grounded on learning and teaching issues that previous research and experience have highlighted as seminal–and likely to occur in actual practice. The mathematically and pedagogically specific situations that we invite teachers to engage with are in the form of tasks (Biza et al., 2007) with the following structure:

- Solving (and reflecting upon the learning objectives within) a mathematical problem
- Examining flawed (fictional) student solution(s)
- Describing, in writing, feedback to the student(s)

Elsewhere we have elaborated the potential of these tasks both in research and teacher education (e.g. Biza et al., 2007), particularly when coupled with post-Task interviews (e.g. Biza et al., 2009).

Here we draw on one of the Tasks we have used. The Tangent Task–see Appendix 1–was amongst the questions in a written examination taken by 91 candidates for a Masters in Mathematics Education programme in Greece. All were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about eighty hours (we note that in Greece, at the time of writing, there was no teacher education programme that teachers are required to attend prior to their appointment).

On the basis of a first-level analysis of the 91 scripts we selected eleven of the participating teachers for interview. Our selection aimed to ensure a reasonable range with regard to the following: years of teaching experience, strength of mathematical background (on the basis of their mathematics degree classification), and correctness or not of response to the mathematical problem in the Task. In selecting interviewees we also considered whether there was sufficient substance/complexity/ambivalence of the written response to the Task to trigger further investigation in the interview. Their individual interview schedules were tailored to the analysis of their written responses. Interviews lasted between twenty and thirty-five minutes, were audio-recorded and then fully transcribed.

The mathematical problem within the Tangent Task aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (e.g. Biza, Christou, & Zachariades, 2008; Castela, 1995) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The aforementioned studies attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point *A*, the problem provides an opportunity to investigate the two beliefs about tangency mentioned above. Under the influence of the first belief, Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line as

tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding the claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

With regard to the Greek educational context in which the study was conducted we note that:

- Students encounter the concept of tangent in Year 10 in Euclidean Geometry as the circle tangent; in Year 11 in the context of conic sections (Analytical Geometry); and, in Year 12 in the context of Analysis (derivative and slope)
- The official syllabus is taught through the use, across the country, of a Ministry distributed textbook.
- At the end of Year 12 students sit a national examination the results of which determine their admission to university. In this examination students are expected to provide proof or at least some detailed justification of their answers to the set questions.

The data presented here are translated from Greek. In order to help the reader see the data excerpts from the eleven interviewed teachers in the wider context of the total set of participants, we note that of the 91 teachers: 38 had no problem recognizing the line as a tangent; 25 stated that the line is not a tangent; 18 offered an ambivalent response which tended towards rejection of the line as a tangent; and, 10 gave responses that were too unclear or brief to be classified. Information on the graduation year, degree class, professional status, teaching experience and type of response to the Task of the eleven teachers is available in Table 1, Appendix 2.

We now offer an account of how our adaptation of Freeman's classification of warrants came to be in the course of the analyses of the teachers' written responses to the Tangent Task and the interview transcripts.

Emergence of our adaptation of Freeman's classification of warrants

The proposition that we put forward in this paper emerged during the analysis of a section of the data that concerned the participating teachers' beliefs and practices about the role of

visualization in mathematical learning (e.g. Biza et al., 2009). Specifically in that analysis we had focused on a substantial number of participants' responses who appeared to embrace, often fervently, Student B's visual approach to the problem. The teachers' grounds for this endorsement were that a visual approach reflects deeper understanding of the problem and also stays clear of the algebraic approach (which they described as the mechanistic reproduction of routines typically used in the classroom). As an indicator of the student's growth in learning they thus valued the former more than the latter. At the same time, as mathematicians themselves, they expressed support for the more comprehensive and more readily acceptable as mathematical proof–in the mathematics community as well as in the Year 12 national examination in Greece–algebraic approach. At the time of conducting that analysis the case of Spyros came to embody many of these thoughts:

> …Spyros's statement is clear: while he cannot accept a graph-based argument as proof, he recognises graph-based argumentation as part of the learning trajectory towards the construction of proof. He seems to approach visual argumentation from three different and interconnected perspectives: the *restrictions of the current educational setting*, in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof within the mathematical community; and, finally, the *pedagogical role* of visual argumentation as a means towards the construction of formal mathematical knowledge.
>
> These three perspectives reflect **three roles that a mathematics teacher needs to balance: *educator*** (responsible for facilitating students' mathematical learning), ***mathematician*** (accountable for introducing the normal practices of the mathematical community) and ***professional*** (responsible for preparing candidates for one of the most important examinations of their student career). Spyros' awareness of these roles, and their delicate interplay, is evidence of the multi-layered didactical contract he appears to be able to offer to his students. (Biza et al., 2009, p. 34)

So, in the course of that initial analysis, we started to notice that the teachers do not rely on logical and mathematical reasons only for the preferences and priorities that they state with regard to their pedagogical practices (in those cases, regarding visualisation). They have other, mainly pedagogically inclined, reasons–or, in Krummheuer's (1995) terms, their considerations started to appear to us as not merely *logical* but more broadly *rational* (p. 229), namely also bound by what is best in a certain situation, subject to negotiation etc.. In Toulmin's sense, the warrants for the claims these teachers put forward started to appear to us as not exclusively mathematical.

We examined these warrants more closely through an analysis of the scripts and interview transcripts of the eleven interviewed teachers as follows. We scrutinised the data for

claims and their respective warrants, we highlighted the relevant data excerpts and we inserted warrant-type characterisations (and brief justifications for these characterisations) in the margins of the transcript's (or script's) text. In a series of team meetings we contrasted and compared the above work (produced by each one of us independently). This examination of the warrants used in the teachers' arguments led to our noticing of certain groupings that these warrants could be seen as belonging to. The warrant-types we list earlier in the paper emerged from this work. In the following we offer illustrative examples of these warrant-types.

Examples of our adaptation of Freeman's classification of warrants

In the examples that follow W stands for Warrant, C stands for Conclusion and *italicized text* indicates the Warrant type:

- Elias claims that the most appropriate method for determining whether the line in the Task is a tangent is the Analysis method, not the Geometry method ("So I don't think we can treat this [here] geometrically."). He grounds his claim on the statement that the Geometry method covers a few cases only, such as the circle, not any function ("In Geometry there is a circle, it's fixed, there is a line, it will either intersect at two, one or no points […]"). His train of thought appears to be the following: the Geometry method for determining tangency covers the cases of circle and conic sections, and not the function in question (W); I therefore propose that we use the Analysis method (C). To warrant his claim Elias seems to resort to a part of mathematical theory, that which concerns the range of cases covered by the Geometry method. For this reason we labeled his warrant as *a priori epistemological*.
- Fotis claims that an appropriate response to Student A is to offer the student a counterexample. He grounds this claim through highlighting the need to provide feedback to the student with "refuting what [the student] writes" and "basically cause what we call, more or less, cognitive conflict". His train of thought appears to be the following: in order to refute the student's claim we need to cause cognitive conflict and a counterexample can cause cognitive conflict (W); I therefore propose that we use a counterexample (C). At the heart of Fotis' warrant seems to lie an endorsement of the

pedagogical potential of cognitive conflict and of the capacity of a counterexample to cause such a conflict. For this reason we labeled his warrant as *a priori pedagogical*.

- Spyros claims that in his classroom a visual argument would not be acceptable as a full argument. He grounds his claim on the statement "because this is what we mathematicians have learned so far, to ask for precision". His train of thought appears to be the following: within the mathematics community, only precise, formal arguments are acceptable and a visual argument is not precise (W); I therefore would not accept a visual argument as a full argument in my classroom (C). To ground his claim Spyros draws on the characteristics of the mathematics community to which he sees himself as belonging. For this reason we labeled his warrant as *institutional-epistemological,* where the referenced *institution* is the mathematics community.
- Marios claims that he "would accept an argument, regarding continuity, […] based on the graph" in his classroom. He grounds this claim through referring to the textbook that has "exercises saying that the function is continuous from the graph and it gave a graph…". His train of thought appears to be the following: the textbook has exercises in which continuity of a function is established via the presentation of its graph (W); I therefore would accept a graph-based argument regarding the continuity of a function in my classroom (C). Marios grounds his claim about the acceptability of a graph-based argument, at least in the case of continuity, on the textbook's approach to this matter. For this reason we labeled his warrant as *institutional-curricular*, where the referenced *institution* is the upper secondary mathematics curriculum and its enactment in the Year 12 textbook.
- Takis claims that the example of the function $f(x)=x\sin 1/x$, and its tangent $y=x$, is an appropriate example in a discussion about tangency in his classroom ("...with this example I wish to show them that…"). He grounds his claim on the observation of how frequently the students' perception of a tangent as a line that keeps the curve on the same side and has one common point with the curve appears in his lessons ("in school I also observe that…"). He believes that the example he has chosen has the capacity to change the students' perception as $y=x$ is tangent to the curve at more than one point, in fact at an infinite number of points, and also splits the curve in two. His train of thought

appears to be the following: in my lessons I have observed certain student perceptions and the example of $x\sin 1/x$ contradicts these perceptions (W); I therefore propose that I deploy the example of $xsin1/x$ (C). Because the basis of Takis' warrant is empirical–his own teaching experience–we labeled his warrant as *empirical-professional*.

- Marios claims that, even at this early stage, students need to learn how to distinguish cases in which a visual approach is adequate and when it is not: "With a figure you can understand continuity, you can understand something else, but differentiation is, I think, a rather difficult process, to understand if [a function] is differentiable through a figure. We have to check other things too". He grounds his claim on his own practice as a learner of mathematics: "I remember myself as a student I used a lot of figures but I didn't accept everything on the figure". His train of thought appears to be the following: as a student I didn't rely for all my arguments on the figure (W); I therefore propose that the students learn how to distinguish cases in which a visual approach is adequate. Because the basis of Marios' warrant is empirical–his own learning experience as a student–we labeled his warrant as *empirical-personal.*
- Finally, Christos claims that his approach to discussing tangency through inviting students to construct examples themselves is less and less welcome and effective ("I used to be able to ask them…"). He grounds his claim on the belief that students now have a more passive approach to mathematical learning ("these days students want a buttered piece of bread straight into their mouth"). His train of thought appears to be the following: I believe that today students have a more passive approach to their mathematical learning (W); I am therefore less inclined to engage them with construction of their own examples than I used to be (C). Because Christos' warrant appears to be hued by some of his broader beliefs about contemporary students' study habits, we labeled it as *evaluative*.

We note that it is not always straightforward to identify a claim and characterise its warrant in the teachers' utterances. Katia, for example, warrants her claim that she "wouldn't mark [Student B's response] with zero" (C) with "I believe that he knows how to do the calculations, that is he is not a student that [deserves such a mark]" (W). It was rather hard to discern Katia's grounding for this decision. Is it grounded on a broad *evaluative* basis

according to which a student who can "do calculations" deserves some acknowledgement, and perhaps "marking with zero" is a generally undesirable teacher tactic? Or, is it grounded on a pedagogical principle (*a priori-pedagogical* warrant), according to which 'doing the calculations' is a skill that precedes the superior ability to reason on the basis of a function's graph? On several occasions the warrants for the teachers' claims were too implicit, too deeply embedded in their statements–and perhaps not always sufficiently probed in the course of the interview–to allow our confident characterisation of them. This tacitness is an issue discussed more widely in studies of *organisational knowledge* (e.g. Cook & Brown, 1999), as well as with specific reference to mathematics teachers' knowledge (e.g. Jacobs & Morita, 2002; Krummheuer, 1995). We return to this issue in the concluding part of the paper.

We also note that in the course of our analysis we identified many occasions where a teacher's claim appeared to be grounded on a multiplicity of warrants. To illustrate one such occasion we sample from the data of one participant, Elias. To help the reader's comprehension of this particular section of the data we begin with a factual summary of Elias' interview up to that point. We then zoom in on the part of the data that records that occasion.

A teacher's many and varied warrants: the case of Elias

Early in his interview Elias recognises that Student A's conclusion is correct–the line is a tangent–but also stresses that the student's justification is not complete as, according to the norms of mathematical theory, the student should explain why the line $y=2$ is a tangent. "The student's intuition is correct", Elias stresses, and some probing into the origins of this intuition is necessary. Elias lists the questions he would ask the student for the purpose of such probing. Elias adds that the student is by now, at Year 12, familiar with the textbook definitions and methods for identifying and checking tangency via the derivative. Therefore an appropriate response to the student would be a detailed step by step exposition of how this checking could be done. Elias presents this exposition in detail. His exposition seems to aim at shifting the student away from a 'geometric' view of tangency (according to which one common point between a line and a conic section necessarily implies tangency). Given this student's persistent geometric images, and to change the student's perceptions, Elias suggests demonstrating a counterexample, and doing so graphically. "To go with the formula" would be of no benefit in this case, he says. Before demonstrating the complete, analytic approach to

the student, we must first change his mind, he then adds. Ultimately, though, we must conclude with a discussion of the complete Analysis approach, he stresses–see also an excerpt from Elias' interview in the list of warrant-type examples, under *a priori-epistemological*. Asked about his view on whether we can nevertheless 'treat' some problems in Analysis in a 'geometric' way, Elias lists proofs, such as the Intermediate Value Theorem and Rolle's Theorem, that are highly dependent on visualisation.

We now zoom in on a particular section of Elias' interview which starts with the interviewer's question on whether Elias would accept an argument from a student who claims that a function is continuous on the grounds of what the function's graph looks like. Elias claims that we would not, as "we must go via the lemmas, not via the graph". His argument is that a student's response based on a graph is not acceptable (C), as an acceptable response should be based on mathematical theory (W). We see this warrant as *a priori-epistemological.* He then continues that, while the student's "intuition might be correct", this intuitive approach would not be acceptable "in the way in which we teach mathematics in Years 10-12". The same claim, that a student's response based on a graph is not acceptable (C), is based now on the fact that graphical solutions are not acceptable in the upper secondary school practice (W). We see his warrant as *institutional-curricular*, with the referenced *institution* being the prescribed Year 10-12 mathematics curriculum and pedagogy. Furthermore, Elias continues, "neither strictly mathematically" would this intuitive approach be acceptable. Therefore, the same claim, that a student's response based on a graph is not acceptable (C), is now grounded on its non-acceptability within the mathematics community (W). We see his warrant as *institutional-epistemological*, with the referenced *institution* being the wider mathematics community and its standards of rigour.

But then Elias makes a further distinction, this time with regard to his use of "mathematically" and the potential acceptability of the graphical solution: "mathematically in the sense that you will enjoy [the fact that the student] thought of this intuitively and knows it and has engaged...to me internally it would be acceptable"

Elias claims that the student's response might be seen as acceptable (C) as this type of solution is evidence of the student's engagement and understanding (W). Elias thus marks a distance between the formal conventions of the mathematics community and his own, personally-held views on what constitutes an acceptable, *personally convincing* argument,

convincing both mathematically and as evidence of the student's engagement and cognitive growth. With some trepidation we see here an *evaluative* warrant for the claim that the student's response might be seen as acceptable but we do not have further evidence from Elias' interview to pursue this speculation much further.

Elias then concludes his response to the question about the acceptability of visual arguments with a reaffirmation of his requirement for a complete analytical approach: "I don't think it is…because [if we accept this type of response] we will mix up Geometry and Calculus, and it will aaaall become…" [his tone suggesting a muddle].

At this point Elias returns to his claim that a student's response based on a graph is not acceptable (C), as this will (con)fuse the analytic with the geometric approach (W). We see his warrant as *a priori-epistemological* as he seems to perceive the canons of rigour within each one of these mathematical domains as firmly distinct.

Despite this clear preference for the analytic approach Elias also acknowledges that "visualisation can be of great help" in Analysis, and more generally in mathematical learning:

> In the Year 11 syllabus, where [the students] learn about vectors and analysis of vectors […] that the children do not understand this very well, when they make the graph is the first step to understand and be persuaded, accept it […]. Because even though it is easy, they do not see it as easy.

Elias claims that visualisation can really help students overcome difficulty in Year 11 Analytic Geometry (C), as suggested by his experience of teaching topics such as Vector Analysis which students begin to grasp when offered a graphical representation (W). We see this warrant as *empirical-professional* as it draws heavily on Elias' teaching experience.

Elias then concludes this discussion of his views on the role of visualisation with a more general, and critical, exposition on what he sees as typical teaching of this part of the curriculum. Due to time pressures, Elias claims, an elaborate introduction to a tangent as a limiting position of secants–which he strongly prefers–is not always possible. The procedural, algebraic approach to teaching this part of the curriculum, he continues, is restricting. And he concludes with listing several cases of tangency that bring "havoc" to students' minds (e.g. a vertical tangent, tangency at cusp points, etc.).

We posit that the examination of Elias' data from the above warrant-classification perspective allowed us to trace and discuss in some detail several multi-layered aspects of his arguments: for example, his distinguishing between what he, the mathematics community and

the Greek curricular specifications see as an acceptable response to a mathematical problem; or, his distinction between what he considers appropriate mathematically (eventually, the analytic approach) and pedagogically (at least to start with, a visually rich approach).

Overall we posit that this approach to analysing teacher arguments allows us insight into the range of considerations and priorities, often at a distance from purely logical or mathematical ones, that underlie teachers' choices. Our analyses, sampled here for the purpose of illustrating the potential of our proposed approach, highlighted many occasions in which a wealth of mainly *a priori, institutional* and *empirical* warrants are put forward by teachers to support their choices (of interpretation of students' responses, of preferred pedagogical practice etc.). We conclude with a summary discussion of the potential we see in our proposed approach and the ways in which our approach relates to, and elaborates, works in this area. We also suggest how we plan to take our current work forward.

Analysing teacher arguments through an adaptation of Freeman's classification of warrants

The approach to analysing teacher arguments we propose in this paper has been initiated by an urge to explore and discuss the range of influences on teachers' views and actions, particularly those influences of a pedagogical and epistemological nature. Our proposal is for deploying an enriched version of one of the components of the Toulmin model, the Warrant. This enrichment was inspired by our reading of Freeman's classification of warrants (*a priori*, *empirical*, *institutional* and *evaluative*) and took the shape we present here through its trial on, and substantiation from, our data. Our proposed approach to analysing teacher arguments employs an adaptation of Freeman's classification that distinguishes between: *epistemological* and *pedagogical a priori* warrants; *professional* and *personal empirical* warrants; *epistemological* and *curricular institutional* warrants; and, *evaluative* warrants.

Why, one may ask, use this approach to analyse teacher arguments? Our intention is to operationalise to some extent what Toulmin himself saw as crucial in this type of informal argumentation: integrate social, cultural, pedagogical, contextual, psychological etc. considerations into some of the model's components (mainly the warrant); and, explore how our perspective on these components changes under the influence of these newly introduced considerations. So, as evident in the examples from the data we cite here (from Elias' data as

well as those from other participants) this approach allowed us to discern and examine the layers of justification behind the teachers' varied approaches to visualization.

Our approach steers a path between approaches that focus on individual characteristics of teacher knowledge and those that focus on collective characteristics. Through inviting teachers' comments on plausible classroom scenarios, much like Herbst and colleagues (Herbst & Chazan, 2003; Miyakawa & Herbst, 2007), we elicit the teachers' *dispositions* (for example, towards the employment of visualisation in their mathematics lessons) that constitute the *practical rationality* of their teaching. And, through our classification of warrants we identify constituting elements of these dispositions, originating both in their individual characteristics and experiences as well as the institutional contexts in which they teach.

Our approach resonates with approaches such as that of Jacobs and Morita (2002), whose use of teacher *idea units* can be seen alongside our use of teacher *arguments*, and who explore both individual (teachers' "openness to alternative pedagogical methods", p. 154) and more institutionally-inclined ("the extent to which opinions are shared among teachers", p. 155) characteristics of teachers' decision making. Of particular resonance is their analysis of coded transcripts of teacher comments: what they record as the teachers' *explanations* for the preferences, suggestions etc. that the teachers offer during the interviews–see, for example, the table in (ibid, p. 161)–could be analysed further in the light of the warrant-classification we propose.

Of course, much like colleagues before us in this area of research, we are aware that, however focused, subtle and persistent our questioning in the tasks and interviews is, warrants and backings of teachers' arguments in most cases remain essentially tacit. Characterising warrants etc. of teachers' arguments is a "matter of inference" and some "hypothesising" (Jacobs & Morita, 2002, p. 163) on our part. Like other components of Toulmin's model, warrants "cannot, in general, be recognised on the surface of spoken formulations; they must be identified by an appropriate analysis of interaction" (Krummheuer, 1995, p. 247). With this awareness to the forefront, we now outline how we see our proposed approach fitting with some other uses of the model.

Our proposed approach follows suit from the work of Inglis and colleagues (Inglis et al, 2007) who also proposed a classification of warrants (though with reference to their analysis of purely mathematical arguments) and emphasised the need to deploy the complete version of

Toulmin's model. Our work so far focuses exclusively on the Warrant. In the course of our analyses though at least two forward-looking observations emerged, concerning two other components of the model, the Backing and the Qualifier.

With regard to the first, as our analyses of the teachers' arguments was developing, evidence of what Krummheuer (1995, p. 252) describes as "infinite regression" (the warrant for a claim being backed by an argument which in turn is warranted by backing … etc.) started to surface. For the sake of clarity and simplicity of this first attempt on our part to map the interviewed teachers' arguments, we tried to distil 'claim grounded on warrant' cases that were as clear-cut as possible. Our current analyses–and extent of our data–are unlikely to afford us the possibility of further, subtler warrant-and-backing analyses. An application of our task/interview method, that focuses tightly on this type of aim, most likely would.

With regard to the second, one observation that emerged in the course of our analyses was that the strength of conviction with which teachers put forward their arguments is certainly germane to the stability and stealth of the ways in which they are processing prior experience, policy guidelines, professional development and training. Again, a research design that would focus on eliciting from participants more on, for example, their degree of certainty about certain arguments, would make the inclusion of considerations regarding the Qualifier possible.

More widely, we envisage our work as relating to seminal investigations of teacher knowledge such as those by Shulman (1986, 1987) and Ball and colleagues (e.g. Ball et al., 2008). Shulman (1987) wrote of his set of seven categories as a "'blueprint' for the knowledge base of teaching" which "has many cells or categories with only the most rudimentary place-holders, much like the chemist's periodic table a century ago" (p. 12). In the years that followed, Ball et al. (2008) responded to Shulman's call for refinement with their further searches that now allow researchers "to fill in some of the rudimentary 'periodic table' of teacher knowledge." (p. 396). With the help of these refinements–and the additional apparatus provided by our classification of warrants that steers a path between individual and collective aspects of teacher knowledge–we propose starting to build up the *compounds* of these elements. So our analyses, for example, allow us to examine the varied grounds (*empirical-personal, empirical-professional, institutional-curricular, institutional-epistemological*, *a priori-epistemological, a-priori pedagogical)* of teachers' views and preferred practices. We

thus gain more systematic, situation-specific and natural insight into teachers' *pedagogical content knowledge*–and its variations (e.g. Ball et al., 2008)–and *curriculum knowledge* (Shulman, 1986). We also gain insight into how different sources of teacher knowledge–as, for example, listed by Shulman (1987) and, more recently, Kennedy (2002)–shape teachers' arguments. We aim that our accumulation and discussion of such insight furthers understanding of the dynamic and complex character of teacher knowledge.

Acknowledgement

The research reported in this paper is supported by an EU Erasmus Staff Mobility Bilateral Agreement between the University of East Anglia in the UK and the University of Athens in Greece and by ELKE at the University of Athens.

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Appendix 1: The Tangent Task

Year 12 students, specialising in mathematics, were given the following exercise:
'Examine whether the line with equation $y = 2$ is tangent to the graph of function f, where $f(x) = 3x^3 + 2$.'
Two students responded as follows:

Student A

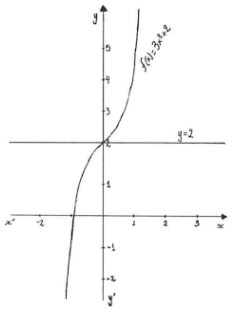
'I will find the common points between the line and the graph solving the system:

$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.
The line is tangent of the graph at point A because they have only one common point (which is A).'

Student B

'The line is not tangent to the graph because,
even though they have one common point,
the line cuts across the graph, as we can see
in the figure.'



a. In your view what is the aim of the above exercise?
b. How do you interpret the choices made by each of the students in their responses above?
c. What feedback would you give to each of the students above with regard to their response to the exercise?

Appendix 2: Background information about the 11 interviewed participants

Pseudonym	Graduation Year[1]	Degree Class[2]	Teacher[3]	Experience[4]	Line[5]
Christos	1988	Excellent	Y	H	Y1
Fotis	1988	Good	Y	H	Y1
Spyros	1989	Good	Y	H	Y1
Takis	1983	Good	Y	H	Y1
Georgia	G	N/A	N	M	Y1
Marios	G	N/A	N	M	Y2
Katia	2002	Very good	N	M	Y2
Elias	1989	Good	Y	H	Y2
Nikos	2006	Very good	N	M	N
Anna	G	N/A	N	M	N
Dimos	2005	Good	N	M	N

1. Year of graduation from mathematics department.
   Or G: graduating at the time of interview.
2. Degree class (out of 10):
   Excellent: 8.5-10.
   Very good: 6.5-8.5.
   Good: 5-6.5.
   Or N/A: not available yet at the time of interview.
3. In-service secondary mathematics teacher at the time of interview:
   Y (Yes) or N (No).
4. Teaching experience according to length of service as secondary mathematics teacher and/or private tuition experience (in individual tutorials or as employee of a private tuition institution) prior and/or after graduation:
   H (high, more than 10 years) or M (moderate to high, 5-10 years).
5. Is the line in the Task a tangent at point (0,2)?
   Y1 (Yes, according to unambiguous statement both in written response and in the interview) or
   Y2 (Yes, according to ambiguous written response but clear statement in the interview).
   N (No, according to unambiguous statement in written response).