The Principle of Economy in the Learning and Teaching of Mathematics

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To Laurinda and Patrick
for the many discussions about writing,
without which this might never have been written

Abstract

This thesis looks at the learning and teaching of mathematics through the issue of economy. Here, economy is concerned with the personal time and effort given by a learner to achieve some desired learning. The study sets out to establish that the principle of economy informs the learning and teaching of mathematics, and to establish a list of principles which can assist an economic approach to the teaching of mathematics.

The study is carried out within the Discipline of Noticing and is based on the development of theory from significant events building on the work carried out by Caleb Gattegno on the subordination of teaching to learning. An *account of* these events are given, followed by *accounting for* them, and linking the generality contained within these isolated events with everyday learning experiences. At times, the reader is asked to carry out simple tasks which assist in drawing their attention, through a personal experience, to the points being developed.

The learning process which turns something newly met into something which can be done with little conscious attention, is analysed and called *functionalisation*. The analysis of this process produces the idea of *practice through progress*, where the learner's attention is placed in a task which requires the desired learning to be subordinated to it. Particular attention is given to the learning of young children before entering school, since this is impressive in terms of economy. This study identifies powers children use in their early learning, and how these link in with root notions in mathematics called *mathematical essences*. A list of principles of economy are developed which provide guide-lines for approaches to teaching to make use of children's powers and utilise mathematical essences. A computer program, *GRID Algebra*, is developed to demonstrate how the principles of economy can be incorporated into a resource.

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1 Beginnings

1.1 Introduction

The title of this thesis includes the word *economy* which has financial associations. However, I am not using the word to refer to the finances of learning and teaching mathematics. Instead I am concerning myself with other resources - human resources. These are the resources of both learners and teachers. Children in the UK spend 11 years of their life in compulsory schooling. This is a considerable commitment of time and thus a resource of great potential. Time is the first of the human resources which both teacher and learner have at their disposal. The potential of this resource is concerned with what can be gained in terms of learning as that period of time transforms the future into the past. I agree with Bruner (1966, p1) that *Instruction is, after all, an effort to assist or to shape growth*, and thus a teacher is employed for two reasons: firstly to influence *what* is learned over this period of time (shaping growth); and secondly, to influence *how* something is learned over this period of time (assisting growth). It is the second of these two roles which is of interest to me in this thesis. The influence of *how* the *what* is taught can influence the quantity and the quality of learning over a period of time.

It is sometimes said that young children spend most of their time *playing*. Indeed, many activities in their early schooling could be described as educational play. By the time a child becomes an adolescent, the emphasis on schooling is concerned with *work*, with play being relegated to lunch and break times. I have not found it helpful to contrast work and play since learning can be taking place in both cases. As Rousseau said in about 1760:

> Work or play are all one to him [Emile], his games are his work; he knows no difference. He brings to everything the cheerfulness of interest, the charm of freedom, and he shows the bent of his own mind and the extent of his knowledge. (1986, p126)

It is the amount of *effort* put into either work or play which is of interest to me in this thesis. Effort is the result of a commitment of human energy, and this is the second resource which both teacher and learner have at their disposal. The amount of energy, in terms of effort, is limited for both teacher and learner, and so brings the question of what is gained, in terms of learning, by either party increasing or decreasing their effort, or directing it differently.

Bruner includes economy in his factors which affect a learner gaining mastery of a domain of knowledge:

> The structure of any domain of knowledge may be characterized in three ways, each affecting the ability of any learner to master it: the *mode of representation* in which it is put, its *economy*, and its effective *power*. (1966, p44)

With regard to economy, Bruner states:

> Economy in representing a domain of knowledge relates to the amount of information that must be held and processed to achieve comprehension. The more items of information one must carry to understand something or deal with a problem, the more successive steps one must take in processing that information to achieve a conclusion, and the less the economy. (1966, p45)

From this I understand that Bruner is thinking mainly of economy in terms of representations. The examples he goes on to give are concerned with whether a *representation* is economical, and whether that representation is powerful. Regarding power, Bruner says:

> The power of a representation can also be described as its capacity, in the hands of a learner, to connect matters that, on the surface, seem quite separate. (1966, p48)

His use of economy and power all relates to the idea of gaining as much productive learning as possible with the least amount of personal time and effort involved. However, he focuses on representations whereas I am interested in all the dynamics involved in the process of teaching and learning, of which representations are one part. For example, as well as the consideration of a particular representation, I am also interested in what aspect of that representation might be stressed at any given time and what might be ignored. Where attention is placed can also be a factor in the amount of time and effort required to achieve some learning. Thus, I widen the use of the word *economy* and shift from economy *of representation* to economy *of personal time and effort*. Thus, the principle of economy is concerned with the management of the human resources of time and effort, in relation to the quality and quantity of learning achieved. I will relate this to the learning and teaching of mathematics in particular.

The following section outlines some experiences which led me to an interest in the principle of economy, and develops some of the key ideas which are involved in this study. This background leads up to the questions which I had when embarking on this study, which I articulate in section 1.3. The questions, in their turn, drive the issue of methodology which is discussed in chapter 2. It is also in this second chapter that the remaining structure of the thesis is described since its structure is influenced by methodological issues.

1.2 Background

In this section I will give the background of my own experiences, influences and thinking which led me to work at the notion of economy in the learning and teaching of mathematics.

From 1973 to 1976 I studied mathematics at a university. I remember working hard in the first year in that I allocated a reasonable amount of time for my studies and put in some effort into understanding the courses I took. During the second year of the course I began to question the way in which I was being asked to spend my time. Lectures lasted one hour, and during that time it was not unusual for the lecturer to pay little or no attention to whether anything being written or said made any sense to the students who were present. Several lectures involved the lecturer spending less than a minute reminding us which lemma number we had got up to in the previous lecture, and then proceeding to start writing on the first of nine blackboards. The writing continued for one hour and was often accompanied by a verbal rendering of what was being written. I gave up any attempt to understand what was being written since, if I stopped for a while to consider a particular statement, I would find myself several blackboards behind. The pressure on me was simply to copy down what was being written on the boards so that I could work at trying to understand it later.

If it was intended that lectures were simply a means of passing on written notes to students, then why not hand out the notes in the first place? It seemed to me that the lectures were a complete waste of time for both the students and the lecturers. This was the first time I can recall being aware of the fact that my time was a valuable resource and that lectures were not a productive way of spending my time. I stopped attending lectures.

As well as becoming aware of time as a resource, I began to ask myself questions as to the nature of teaching. I felt that what went on in most of the lectures I attended was not teaching. What might someone do so that their actions would be appropriately described as teaching? I had learned that teaching was something different to passing on information. My interest in teaching was greatly assisted by the poor lectures I attended.

During my PGCE year in 1978-79, my tutor, Denis Crawforth, asked me *when did you first operate mathematically?* This was a question which was to stay with me for a long time. I began to see mathematics as something far more than that which is learned in schools. I became sensitive to a wide variety of mathematical activity in the everyday learning of a young child. In the first years of my teaching I tried to discover what students were able to achieve when presented with mathematical situations to explore. I learned, from the students I worked with, that they were able to formulate their own mathematical questions, develop systems in their approach to a question, spot patterns, make conjectures and justify their statements. I learned that the students were already operating as mathematicians. However, I still had my question of what someone might do which I would regard as teaching.

Following my PGCE course, I went to Bristol to teach and got involved in a writing group at the Resources for Learning Development Unit (RLDU). There, I met some people with whom I was to continue meeting and discussing ideas over the years which followed. The group included Laurinda Brown, Jo Waddingham, Pat Evans, John Chatley, Eileen Billington and Christine Hopkins. We met on a regular basis, sometimes weekly, throughout the first half of the 1980's. It was within this group that many ideas and experiences concerning teaching and learning mathematics were shared and discussed. This group was influential in my development as a teacher, through the sharing of ideas and the mutual desire to explore and develop new approaches to teaching. The group was also influential in my development as someone who tries to articulate reasons for my actions in a classroom, through being challenged and questioned in a supportive atmosphere at meetings.

In November 1981, I attended a week long seminar given by Caleb Gattegno at Charney Manor in Oxfordshire. One evening, Gattegno worked with a group of us using his computer program, *Infused reading in Spanish*[1], designed to teach illiterate speakers of Spanish to read Spanish. After 30 minutes, we were all reading Spanish out loud. We did not understand what we were saying but then we were not Spanish speakers to start with. However, we could read the written words on the screen and pronounce them with a good Spanish accent. Along with most other people who were in that room, I gained a sense of the potential of the program in helping someone gain the reading of their own native language. Furthermore, it only needed 30 minutes. In fact, this last statement is not true, since more practice would be required to help the learning become something which would remain for life. However, this did not change how impressed I felt about

[1] Available for the Apple II computer from Educational Solutions, New York.

how much had been achieved in only 30 minutes, and further developed my interest in economy.

In the following years until Gattegno's death in 1988, I attended a number of Gattegno seminars in Bristol and London and was influenced by his work. The notion of economy was central to much of his work, not only in mathematics but also in a variety of other areas of learning, notably reading and foreign languages. I was inspired by the seminars I attended, to explore and develop new ways of working in my own classroom. In particular, I was attempting not only to observe students' abilities as mathematicians, but also to ask what actions I might take as a teacher which could assist students using these abilities to gain a more substantial understanding and command over a given mathematics syllabus. Thus, my attention shifted from consideration of activities I might provide to allow students to exercise their mathematical abilities, to what actions I, as a teacher, might make to encourage more productive use of those abilities within the given mathematics syllabus. The shift was from prepared activities to teacher's actions. In particular, I was interested in what was the minimal contribution I could make in order to help a student make a significant shift in their learning. What type of contribution might this be? verbal? visual?...

Alongside the development of my teaching, came an increasing awareness of talents that students had outside the classroom. Through conversations with students I learned that many of them were involved in activities which required a high level of skill and knowledge. At the same time, these same students were sometimes struggling with their work inside classrooms. I felt that we must be failing to some extent not to make more use within the classroom of the talents students display outside classrooms.

Another contrast of which I became aware was that of the learning achieved by young children before school age and the learning achieved by adolescents in mathematics classrooms. Several of my friends had children and as a consequence I began to have contact with babies and young children. I was, and am, fascinated by the achievements of young children and began considering how children manage to learn so much in their early years with little explicit teaching taking place. Furthermore, I felt that, in comparison, very little was being achieved in the learning of mathematics by the students in my classroom. For example, so much of what young children learn in the first five years of their life stays with them for the rest of their life; whereas, so much of what I attempted to teach my students in the five years of their attendance in my mathematics classroom was being forgotten before they even left school. These contrasts meant that I paid as much attention to learning happening outside classrooms as much as inside classrooms.

At the time of beginning work on this study in January 1987, I was in my third school, as a Head of Department, and I felt I had developed considerably in my teaching style since beginning teaching. However, I was not able to articulate a theoretical basis for what I was doing in the classroom. My actions were mainly based on years of exploration, finding that some things worked and others did not. I continued to develop the things that appeared to be productive and stopped doing the things that did not. I had, what Elliot describes as practical wisdom:

> Practical wisdom as the form of the practitioner's professional knowledge is not stored in the mind as sets of theoretical propositions, but as a reflectively processed repertoire of cases. Theoretical understandings are encapsulated in such cases, but it is the latter which are primarily utilized in attempts to understand current circumstances. Comparisons with past cases illuminate practically relevant features of the present situation. (1991, p53)

I wished to develop a pedagogy of the way in which I was attempting to work in the classroom, and to articulate some of the theoretical understandings which my practice was based on. Additionally, I felt that Gattegno's work could be built on and developed to make more explicit the way in which students could be helped to work more productively in their learning of mathematics.

As a practising teacher, I had classes which I observed on a daily basis and with whom I could develop different ways of working in the classroom. I had set up within the mathematics department a system whereby we all observed each other teach, and spent time after the lesson discussing what we had seen. This meant that I was able to observe other people's classrooms as well as having discussions about my own. I developed this further with a group of teachers from a variety of disciplines within the school, where we met on a regular basis and visited each others' classrooms. Thus, I had opportunities to observe a variety of lessons outside of my own classroom and subject area.

During the beginning of 1990, I spent a few months on secondment to the RLDU in Avon as Mathematics Editor. As part of a project I carried out, I visited a number of mathematics classrooms in different parts of the county. Then, in September 1990, I became a lecturer in mathematics education at the University of Birmingham. This brought me in contact with a variety of schools and mathematics classrooms in the area.

My interest in engaging in this study was partly to provide myself with a discipline to continue developing my skills as a teacher of mathematics, a discipline to devote some time and thought to a theoretical basis for my work which would also feed into my practice as a teacher. The other aspect was that I felt I had something to say and something to contribute to a body of knowledge on the learning and teaching of mathematics. I wanted to devote time to develop what it was I had to say, and to be able to say it in a reasonably coherent form.

1.3 Initial questions

In this section I begin by presenting a series of questions which were on my mind as I approached the commencement of this study.

The question which drove my interest in beginning this study was *how can I become a more effective teacher of mathematics?* Although this was a driving force for much of my work, the question itself was not a useful one for me to work on. It was not a question which indicated a direction for me to look, or provided a focus for my attention. I needed something which, if I attended to it, could help me learn something relevant to the original question. One direction was the consideration of time as a resource. Gattegno talks of the transference of time into experiences:

> While time is given, experience is *created*. Hence, the most primitive generation of wealth is the transformation of time into experience. Time is a universal raw material out of which humans make all the things that are "objectivations of energy" - a pot, a novel, a hypothesis, a theorem. Time that is spent - actively exchanged for experience - leads to objectivations. (1986, p214)

Thus, the transferring of time into experiences could lead to 'objectivations' within mathematics. What sort of experiences are helpful to learning? In considering this question I began to consider my own experiences and reflect on those which I thought had been particularly effective in terms of my own learning. Was there anything these experiences have in common?

Neville talked about *indirect* teaching as being more powerful than direct, traditional methods:

> They [indirect teaching methods] emulate the way in which children and adults 'pick up' information, skills and values more or less unconsciously as they go about the business of living. (1989, p15).

There is so much that we do 'pick up' in the course of our daily living. How do we learn so much apparently unconsciously and yet forget so much that we consciously try to remember?

The activity of young children was of particular interest since so much impressive learning takes place in relatively little time. How is it that children can learn so much when they are young and learn relatively little in mathematics classrooms when they are older? Is there a difference in the way they are being asked to learn in each circumstance? If the impressive learning of young children can be brought into the mathematics classroom, then what sort of mathematical experiences can be called upon?

Kilpatrick said that:

> When a child's mind is viewed from an information-processing perspective, one has a difficult time seeing it as anything like a blank slate. The child comes equipped with wiring already installed and programs already running. Whether one views these programs as microworlds or as domains of subjective experiences, the school-age child is a self-programming being who has already put together many programs for dealing with intellectual tasks. Some of these programs are quite different from the programs that teachers have in mind. (1984, p19)

Was there a mis-match between the 'programs' I had in mind as a teacher, and the 'programs' children are actually equipped with?

There is so much that students learn when they are very young which will stay with them for the rest of their life. In mathematics classrooms, there is so much that students learn that they forget within a matter of a few weeks, hours, or even minutes. Why is this so? What is it about the nature of the relative experiences such that in one case time appears well spent and in the other, time appears to have been wasted?

When do I notice a lot of effort going into some work? What is gained from this effort? Does this commitment of energy lead to learning or only to frustration? When does the effort appear to be fruitful in terms of learning, and when does it not appear so?

Gattegno (1974, pvii) made a powerful statement that *only awareness is educable*. This implies that my attention as a teacher needs to be concerned with awareness. In what ways can I assist the education of a student's awareness in mathematics?

In the end, these questions relate to teaching, which involves decisions and actions to be taken. How are these questions going to relate to the question of how I might act in a classroom so as to assist economic learning of mathematics? What might guide my decisions as to what I do or do not do in the classroom? What can be discovered and learned by students on their own and what needs to be provided by a teacher? Rousseau wrote (in about 1760):

> With our foolish and pedantic methods we are always preventing children from learning what they could learn much better by themselves, while we neglect what we alone can teach them. Can anything be sillier than the pains taken to teach them to walk, as if there were anyone who was unable to walk when he grows up through his nurses neglect? How many we see walking badly all their life because they were ill taught? (1986, p42)

What requires our intervention and what is best left to the students themselves?

I give this flow of questions in order to provide a flavour for the areas of interest I have and the fact that consideration of a question leads to several more questions, perhaps more than any answers. Below, I articulate the main questions which have driven my study. These will incorporate most of the above questions within them:

- What abilities and powers do young children make use of in their productive early learning?

  Are these powers being called upon within mathematics classrooms? How might they be called upon more?

- Why is it that some learning is lost over a period of time? How can this be prevented?

- Are there ways of working with students on mathematics which could be described as economic? What is significant about these ways of working which leads to economy?

  How might I act as a teacher to enhance economic learning?

  What aspects of an activity assist economic learning?

- How can the teaching of mathematics be approached in terms of the education of awareness?

In particular, I set out in this study to

(a) establish that the principle of economy is one which can inform the learning and teaching of mathematics;
(b) establish a list of principles which can assist an economic approach to the teaching of mathematics; and
(c) demonstrate the development of a piece of computer software based on these principles.

2 Methodology

2.1 Introduction

When I began my study I knew I was interested in the economy of learning and teaching mathematics. The issues I raised in the last chapter have indicated that in order to be informed about economy I need to look at places where there are examples of economic learning. One obvious source is the learning of young children before they are socialised into school learning. Another source is learning which takes place outside classrooms, as well as what happens inside classrooms. Thus, I need to look outside classrooms as much as inside classrooms, and I need to pay attention to the learning of young children. Furthermore, my interest is not confined to the learning of mathematics, since economic learning of any kind may well offer some information about ways of learning which could be applied in a mathematics classroom. Indeed, attempts at learning which have not been economic are also of interest, as they can inform what not to do in a classroom.

I consider *all* young children to be impressive learners. Emphasis can be placed on the fact that some children begin walking before others, and that some children (not necessarily the same) can say more words in their native language, by the time they are two years of age, than other children. There are differences between all children. However, there are also some things which are the same for all children. Unless there are particular handicaps or accidents which befall them, all children do learn a language, all children do learn to walk, and all children do learn to count. My interest is not so much in what it is that all children learn, but in the fact that they are all successful learners. So, if a child suffers an accident which means they are unable to make use of their legs in order to walk, all this means is that their impressive learning may be manifested in other ways. I recently saw a child on television who has born with severely deformed limbs, yet this did not prevent the child finding a way to use their limbs to move speedily across a room. This child's form of movement was different to that of most children, but the fact that they learned to use their limbs in order to gain mobility is the same. My interest is in powers children possess and utilise within their learning, whatever form that learning may take. To find out about economic learning does not require study of a particular set of people since what I am looking for is common within all people.

Information valuable to my study is available from a variety of sources, including the daily activities of the people I meet in the normal course of my life, both at work and outside work. I consider that a specially constructed set of circumstances in which to do observations offers no more information relevant to my study than those circumstances

I find myself in during the normal course of my life. Thus, it is not so much a matter of choosing particular situations to observe, or particular people to observe, but more a matter of to what I might attend within any given situation, or any given set of people. It is for these reasons that I reject the idea of carrying out specially organised sets of observations.

I am particularly interested in the personal time and effort involved in learning. Objective measurement of these are fraught with difficulties. If I consider some of my own learning of mathematics, I have sometimes immersed myself in a problem and spent, say, two hours consciously working without achieving any resolution. Then, I have left the problem and carried on with some domestic or social activities. Some time later, perhaps several days, the problem suddenly comes into my conscious thoughts along with a solution. Such scenarios are quite common. Each person to whom I have described such a situation, has been able to describe something similar from their own experience. In one sense, the time taken from being presented with the problem to achieving a solution was several days. However, for all but a few hours, I had been consciously attending to other matters. Yet to say that the time taken was a few hours would not be accurate since I appeared to need time away from consciously thinking about the problem. Time and effort are linked in a personal way where some of the time is involved in consciously applying effort to the problem, whilst other time has conscious effort withdrawn from the problem. Both periods of time are significant. The precise measurement of time and effort is not only difficult, but does not provide information which informs my study further. In the example I have given, what is of interest is the dynamic between time and effort, and the fact that solutions can sometimes appear at times when my conscious effort is not in the problem. Whether I spend two or ten hours originally working consciously on the problem is of little importance; whether it is one day or one month between working on the problem and suddenly knowing a solution is also of little importance. The anecdote contains possibilities for shared experiences about how we learn without the requirement for any objective measurements. In fact the measurements might make us place our attention in something which is not of importance in the study. For this reason there is no attempt to make objective measurements within my study.

The data presented here to be worked on consists of observations of particular incidents of individuals, sometimes not in a classroom setting, sometimes not concerned with mathematics. Yet I wish these observations to have some application to mathematics classrooms. Thus, in making observations, I am interested in noticing events which are representative of behaviour beyond that particular situation. One way I can recognise generality is to see events as examples of behaviour I have seen on other occasions. A second way is to recognise events within my own experience. In both cases, what is important is analysis of the event in terms of the generality. Each particular incident is related to other experiences and described in a manner designed to resonate with the reader's experience. Thus, general principles drawn from particular incidents achieve their generality and validity by making sense of the past and informing the future.

If there are some general statements to be made which apply to us all, as human beings, drawing attention to ways in which we worked as young children and ways we may still work as adults, then I do not have to observe particular groups of people. If there are things that can be said about the learning in our everyday life, in terms of economy, then I am surrounded by potential data if only I choose to notice. Thus, to develop a theory concerning economy, I have plenty of opportunity to make the observations required to inform my theory from within my existing surroundings.

Alongside a development of theory is my practice of teaching mathematics in classrooms. The development of theory informs my practice as a teacher which in turn produces opportunities for noticing events which, in turn, inform my theory. Thus, the development of theory and practice are linked together. This study is also concerned with the development of practice informed by principles of economy.

2.2 Assumptions

The way in which I approach this study depends upon the way in which I view social reality. Burrell and Morgan (1979) identified four sets of assumptions which lie behind our conceptions of the social world: (a) ontological; (b) epistemological; (c) relationships between human beings and their environment; and (d) methodological. As Cohen and Manion (1985, p7-8) indicated, the views held on the first three influence the methodology used in searching for greater understanding of the dynamics of the social world (in my case a subset of the social world: learning and teaching mathematics).

Below, I state some assumptions I bring with me to this study so that the reader is aware of the perspectives I have. They are beliefs, not facts; beliefs I have held during this study:

> (a) objects have an independent existence which is not dependent on people. However, objects have no meaning in their own right. Meaning comes from a person interacting with that object. An example from the world of teaching and learning is that a teacher may attempt to explain verbally an aspect of mathematics. The words exist in terms of sound waves. However, whether a student develops meaning for those sound waves is dependent on the work of the

student. For knowledge to exist it has to be meaningful for some person or persons and as such is dependent on people. I view knowledge as a *potential* set of relationships, not as an external object;

(b) knowledge cannot be transferred in a mechanistic way. A student can only receive waves of sound, photons of light, impacts of kinesthetic energy, molecules of gases, liquids or solids. Whether the student pays attention to these is another matter. Whether the student attempts to relate their own experiences to these is another matter. Thus, I perceive *coming to know* as a personal activity which cannot be done for the student by a teacher. The process of *coming to know* happens as a person develops meaning with the *potentiality* of a set of relationships becoming *actuality*. The two beliefs: (i) that developing meaning is a personal activity, and (ii) that coming to know is gained through developing meaning, has a consequence for my notion of truth. The knowledge someone gains through the process of coming to know, which in turn is gained through the development of meaning, is also personal. Knowledge is personal. Thus, the question of whether the knowledge gained is true is a relative question. For each person, truth lies within internal consistency of experiences. Some of these experiences will come from contact with other people's opinions or practices, and from this there can come a consensus amongst groups of people as to whether something is considered to be true.

(c) people create their own thoughts and make their own decisions about their actions. Children surprise teachers by their thoughts and actions since they have independent and creative minds which can choose to go against expectations and social norms.

As a consequence of these assumptions, I do not consider that the factors affecting the learning and teaching of mathematics are externally measurable. Some of my assumptions link with a constructivist perspective. For example, the notion of truth links with the issue of 'rightness' which von Glasersfeld discusses:

> From a constructivist point of view, it makes no sense to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as "rightness" is assessed by someone else. To become a source of real satisfaction, "rightness" must be seen as the fit with an order one has established *oneself*. (1987, p15)

I entered teaching with a belief that students do not come to know something by passively listening to words of wisdom from their teacher. I was aware that I had to sort out my own understanding of the world around me and that it was the same for my students and the world of mathematics. From my first teaching job, I began exploring ways of working with students which are concerned with discovering and developing the mathematical awarenesses they already have. I am clear that it is the students' responsibility to do their own learning, whilst it is my responsibility to do my own learning of what contributions I can make to assist and enhance the students' learning. Since I have been working in such a way with students for a long time before I heard of constructivism, many of the notions attributed to constructivism have had little impact on me since, if anything, they seem obvious. Noddings talks of pedagogical constructivism, as opposed to cognitive constructivism, as *a way of teaching that acknowledges learners as active knowers* (1990, p10). To this extent someone observing my lessons and talking with me might decide I am a pedagogical constructivist. Also, there are links between my stated beliefs and the following list of cognitive aspects of constructivism which Noddings gives as those things which constructivists generally agree on:

> 1. All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.
>
> 2. There exist cognitive structures that are activated in the processes of construction. These structures account for the construction; that is, they explain the result of cognitive activity in roughly the way a computer program accounts for the output of a computer.
>
> 3. Cognitive structures are under continual development. Purposive activity induces transformation of existing structures. The environment presses the organism to adapt.
>
> (1990, p10)

However, I do not describe my study as coming from a constructivist perspective since (a) it is not a perspective which offers me new insights into the dynamics of my classroom; and (b) I find Gattegno's use of the word *awareness* offers more, in terms of processes involved in learning, than the language associated with *construction*. So, although I am in sympathy with many aspects of constructivism, my study is not written within a constructivist paradigm.

2.3 Methodologies

My study is a qualitative one which is primarily concerned with two aspects: the development of my own teaching, and the development of a theory which addresses the issue of economy in the learning and teaching of mathematics. A standard methodology which relates to the former is that of action research. Elliott says that:

> The fundamental aim of action research is to improve practice rather than to improve knowledge. The production and utilization of knowledge is subordinate to, and conditioned by, this fundamental aim. (1991, p49)

This places my desire to develop theory as subordinate to my desire to improve practice. In fact, although the former was an initial motivational reason for my interest in economy, the study was concerned equally with both. Cohen and Manion say that:

> action research is *situational* - it is concerned with diagnosing a problem in a specific context and attempting to solve it in that context. (1985, p208)

The context for my study is beyond my own classroom. I am interested in how young children, in general, manage to learn so much in such a short space of time. I am interested in general issues about why students in mathematics classrooms might not be learning as productively as they used to learn as young children. I am interested in the role of a teacher within the learning process. I am interested in what can be learned from experiences of learning outside a classroom context. My theory involves much more than the particular situation of my own classroom, although I expect it still to be relevant to my classroom. Thus, my prime interest is not one of my immediate situation. I see my classroom as offering opportunities to learn through noticing and experimenting. My classroom is, in this sense, subordinate to my desire to improve my knowledge which in turn would feed into a theory, which in turn would inform my practice. This is more in line with Carr and Kemmis who say that:

> ... action research involves practitioners directly in the reconstruction and transformation of practice through involving them in planning, acting, observing, and reflecting on practice, and the situation in which practice occurs. In this way, action research helps practitioners to theorize their practice, to revise their theories self-critically in the light of practice, and to transform their practice into praxis (informed, committed action). (1993, p237)

However, this brings in the issue of the structure of action research. Definite stages are proposed based round the idea of a cycle of planning, implementing, evaluating, revising plan, implementing, evaluating, revising plan, and so on. This structure is of particular importance when the research is of a collaborative nature. If several people are involved then there is a need for a structure in order for such things as: identifying issues; discussing problems; co-ordinating and linking what is happening in different classrooms; and sharing observations. The fact that I am working on individual research means that there is no organisational need to operate within a tight structure. Indeed, a tight structure can limit opportunities to carry out the important components of planning, implementing and evaluating at a time when they are most required, rather than following mechanistic procedures. Since teaching is an activity where every single day there is more potential data which can inform my study, it is more appropriate for me to remain flexible. Events which are of importance to my study can happen at any time and not just at times when I plan to carry out careful observation, or plan particular courses of action. Furthermore, my interests involve learning outside of the classroom as well as inside. Thus, events informing my study might happen at any time, at any place, and with any group of people. It is for this reason that action research does not offer me a methodological model which suits my particular needs.

My study involves reflection on my practice as a teacher, and Lerman and Scott-Hodgetts talk of the reflective practitioner as someone who uses reflection for more than on the spot decision making in the classroom:

> ... when we use the term reflective practitioner, we are also describing metacognitive processes of, for instance, recording those special incidents for later evaluation and self-criticism, leading to action research; consciously sharpening one's attention in order to notice more incidents; finding one's experiences resonating with others, and/or the literature, and so on. (1991, p293-294)

They talk of the transition from a researcher-in-action (Schon 1984) to a researcher:

> ... who has a developed critical attention, noticing interesting and significant incidents, and turning these into research questions. (1991, p294)

This raises the role of critical incidents which might inform a research study, and, most importantly, the notion of *noticing* incidents significant to that study. The analysis of critical incidents assumes the fact that a researcher already manages to notice an incident as critical. The fact that I might notice an incident as critical or *significant* can be informative in itself as much as my analysis of that incident. Joy Davis talks of these incidents as *Teaching Moments* and comments that:

> Their [Teaching Moments] significance lies in the fact that they do stand out, that they come readily to mind. (1992, p170)

In fact, analysis may well include the question of why I found events significant by relating the event to the research questions I had at the time of observing.

2.4 The Discipline of Noticing

> To notice something is to become suddenly aware of something, to distinguish it from its surrounding context. The word *noticing* has its roots in the making of a distinction, in stressing some perceived features and consequently ignoring others, and refers to a shift in the focus, locus, or structure of attention.
> Mason with Nevile (1992, p5).

There are many things on a daily basis of which I am aware. However, my interpretation of the above quote is that this awareness distinguishes that something from its surrounding context. Thus, I notice something as an example of a general phenomenon, or I link that something to other noticings I have had. Noticing requires a sensitivity not only to my surroundings but also to my past experiences and awarenesses. For this study, I am interested in developing ways in which I can notice events which relate to my research interest. Thus, initially, I will consider ways in which I can develop my sensitivity to noticing for this study.

There is potential material for my study from my job, my social life, and through reflecting on my own learning. I cannot spend my whole life collecting information. There has to be a way in which the collection of information is both manageable and relevant. To consider this, I relate the following story:

> I was in my mid-twenties when I was walking one autumn with a friend, Pat, in Wales. She commented on how she loved autumn because of the variety of reds and greens in the trees. This was the first time I became consciously aware of the fact that the leaves of some trees turn red in autumn. Following this incident, I have enjoyed the reds and greens of autumns.

This is an example of the fact that the way I view something is related to what I am aware of at the time. Before that autumn in Wales, I still looked at trees and the same frequencies of light entered my retina. However, I did not notice the reds. After that autumn, I noticed the reds. I am naturally selective about what I choose to attend to from my environment, and this is dictated by the awarenesses I have at that time. This is no new idea, Stuart Braham quotes Alphonse Bertillon:

> Bertillon said that one can only see what one observes, and one observes only things which are already in the mind. (1986, p171)

Laurinda Brown made reference to this quote when she wrote the following:

> Having moved into a new house and therefore acquired a new garden to go with it, watching the changes in vegetation was a minor obsession. A ground-cover plant with a central silver stripe to its leaves was new to me and eventually I tracked it down by the name of 'horus'.
>
> On a recent walk around Cliftonwood, our old haunts, I observed a large expanse of horus and said to my companion, 'I'm beginning to see horus now.' We had talked many times about the apparently strange phenomenon of learning a new word and then suddenly seeming to read it everywhere, or meeting a new idea and hearing it again shortly afterwards, or more mystically, thinking of old friends and they arrive!
>
> Shortly after 'horus' walks, the above quote literally leapt out of the page at me. I found it interesting but carried on reading. The next day the quote lived with me and pictures of horus and my use of 'see'. Horus has 'always' been there, as have the dianthus and ivies which were unremarkable to me that day and, no doubt, other varieties of plant which may one day become available to me when I go through a process of coming to know them. (1986, p3)

There are so many things which are part of our visual environment, but what we 'see' has to do with what we are aware of at the time. John Berger comments that *The way we see things is affected by what we know or what we believe.* (1973, p8). Thus, noticing is not an objective activity, what is noticed is determined by the person who is doing the noticing and of what they are aware at the time. Noticing is an individual experience of personally becoming aware of something.

My noticing is individual, although it is not in isolation of those around me. My attention can be affected by other people, through their stressing and ignoring. So, frequently I notice what is also significant within my culture. However, sometimes I do not, as I had not done for so many years with the colours of leaves in Autumn. I may notice things differently to others since my noticing is based on a unique collection of awarenesses and experiences.

As I began my study, there were a number of questions I had on my mind. These questions were representative of some awarenesses I had at the time. Thus, the events which were happening around me as part of my daily encounters, were viewed from a perspective of these awarenesses (amongst, of course, many others). An event would become significant to me, in terms of this study, if I found myself relating to the event with these particular awarenesses. An ability to catch this happening gave me the opportunity to note down an account of the event. Reflecting on the dynamics between the event and the awarenesses which were involved in viewing the event, gave me the opportunity to account for that event in terms of those awarenesses. Such a process not only sheds some light on the event but also sheds light on the awarenesses and hence the questions behind them.

Thus, my questions are developed and refined through the process of noticing. I use the term *noticing* as that process where an event is viewed through a particular set of awarenesses and is analysed in terms of those awarenesses. The process not only gives an account of the event and an account for the event, but also informs the awarenesses. As a consequence of this process, the event is seen as something beyond its immediate surroundings and thus my use of noticing fits alongside that of Mason's (op cit).

In order for the process of noticing to be effective, I need to be steeped in my research questions. By this I mean that the related awarenesses become sufficiently part of me that I am able to view events in terms of those awarenesses. This is not always the case, for example there are a number of questions I have heard on *Gardener's Question Time* but I have not become involved in those questions and as such do not view my garden with any great awareness of what is happening with my plants and flowers. The more steeped I am in my research questions, the more I notice events of interest to my study within my daily life.

When I began my study, I was steeped in my questions as a consequence of wanting to improve my practice as a teacher, and through a desire to articulate some principles behind the way I was trying to work. Thus, the first way in which I selected from my environment was to become steeped in my research questions and to allow the process of noticing to select those events which appeared relevant to my study. These events I describe as *significant events*.

As my initial questions informed my noticing, I began to develop conjectures which led to the development of my practice. Thus, I might try a different way of approaching the teaching of some mathematics, or a different way of responding to a question I am asked by a student. Lessons of this nature I tape-recorded with a microphone hidden under my jumper and a tape machine in my pocket. This was another way in which I gathered information, although very few direct transcriptions have ended up forming part of this thesis.

The noticings informed my awareness, which in turn modified my research questions. The cycle of events continued with the new questions prompting awarenesses which influenced the next collection of noticings,... It was in this way that my theories developed. Mason describes such a cycle as:

> Accounts [of incidents] provide data for detecting threads, and threads detected will tend to influence the subsequent incidents which are noticed.
> Mason with Nevile (1992, p11)

2.5 Validity

Mason states that:

> A world of personal reflection and development supports awareness, noticing, and resonance seeking, and is typical of the noticing paradigm. (1992, p6)

The process I have described involves a considerable amount of personal reflection, with development happening in terms of my research questions and my practice as a teacher. As this process of development continued, I was gradually building a theory which would articulate a number of principles of economy. Issues of validity are connected with the resonance seeking that Mason talked about. There are four ways in which I considered issues of validity regarding my developing theory:

(a) consistency of theory between a number of events;
(b) resonance with my own experiences;
(c) resonance with others through exercises designed to highlight key factors;
(d) my practice being informed by a developing theory.

(a) The theory needs to account for a variety of incidents and not just the events from which it stemmed. I have addressed this through the use of the same key notions and terminology, developed in the theory, which give accounts for a variety of events referred to in this thesis. Joy Davis talks of personal validation involving *laying the strands of your experience alongside each other and comparing them* (1992, p174). I view the strands of my experience as individual events, and the linking is achieved through a theory which places each event within a common generality.

(b) Further to this, I have reflected on my own experiences to see whether the theory, developed to give an account for certain events, also informs my own experience. This includes both my experience of teaching other people mathematics, and my experience of personal learning in everyday life.

(c) Mason says that within the Discipline of Noticing:

> The essence of validation is two fold: selecting and honing descriptions which others instantly recognise; refining tasks which highlight fruitful skills or sensitivities.
> Mason with Nevile (1992, p25)

My undertaking of the first is through descriptions within the thesis of examples taken from everyday experiences which others may be able to recognise within their own experience. My undertaking of the second is through a number of exercises which are designed as opportunities for the reader to relate their own experiences to the issues being raised at that point in the thesis. The exercises frequently involve reflection on simple activities involved in everyday life and as such make few assumptions about the particularity of past experiences the reader may or may not have. On some occasions I include reactions others have had to these activities.

(d) As I have mentioned on several occasions, the development of theory and practice have been inter-connected within this study. Thus, much of the way in which I have worked with students, and to which I have made reference in this thesis, has been influenced by my developing theory. I have made careful observation of the new developments in my classroom practice, by the use of tape-recordings and notes from lessons. Such observations have fed back into the theory. Thus, the development of theory is inter-connected with observations from practice, and the development of practice is informed by my theory. In particular, I present examples of my practice and a computer program I developed which is based on the principles developed from my theory.

As there have been things that I have noticed during my study, so I have attempted to help the reader notice these things within their experience. Something becomes valid for the reader if they can recognise what is being said, within their own experience, and if the reader is led to greater sensitivity to the dynamics of learning and teaching mathematics as a consequence. This is not something which can be guaranteed but is dependent upon the reader as well as the content of the thesis. This is consistent with my assumptions about truth, which I mentioned above, as lying within internal consistency of experiences. The thesis is not written for all possible readers. For example, there are assumptions I make about the reader being someone who has experiences of learning and teaching mathematics, and of observing young children. I make assumptions that the reader can reflect on their own learning experiences and is prepared to engage in exercises of noticing (i.e. engage within the paradigm of noticing for the purpose of reading this thesis). I make assumptions that the reader can read the thesis from the perspective of the ontological, epistemological, and human relationship assumptions outlined above.

2.6 Structure

This approach to my study has resulted in a particular structure to the presentation of my thesis. I start, in chapters 3, 4 and 5, with significant events which have shaped my thinking. These events are classified under the headings of pre-classrooms (with young children before they are old enough to enter school), inside classrooms, and outside classrooms. Within these chapters I give an account of the event, followed by an analysis which accounts for the event in relation to the notion of economy. This analysis also informs my awareness of economy, and the awarenesses gained from each event are articulated in a short statement contained in parentheses and printed in bold. These statements will feed into the following chapters.

Chapter 6 turns attention to the learning of young children in particular, and identifies powers they possess and make use of in their learning. I consider children's learning first since I am following Gattegno's (1971) demand of subordinating teaching to learning. Thus it is these powers of children to which teaching is to be subordinated. I have a brief analysis of some text books, in chapter 7, to demonstrate that few of children's powers are called upon as students attempt to learn mathematics from these books. In contrast to this, the next three chapters give examples of practice which are designed to make good use of children's powers. Alongside these examples, I continue developing my theory but with the attention this time on the act of teaching. This produces a list of principles of economy for the learning and teaching of mathematics which are included in my conclusions in chapter 11 with further questions following my study discussed in chapter 12.

3 Significant events 1: pre-classrooms

3.1 Introduction

There have been many encounters I have had with young children, adolescents and adults. In fact there were many encounters every day during this study. In the next three chapters, I describe 19 events which have been particularly influential in the development of my thinking. This is but a small number of all the encounters I had, and have been chosen because I noticed something within the encounter which related to the awarenesses I had at the time about my research questions. They acted as generic examples round which I began to develop a theory. Sometimes I developed ideas whilst the encounter was taking place and this affected the way in which I responded within the situation. On other occasions it was not until some time after, when I was reflecting on the encounter, that my ideas developed. However, it is interesting to note that in some way all of these encounters affected me significantly, remaining with me so that I was able to recall and reflect on the incidents some time after they happened. I am not always able to re-enter every encounter I have with people, and so in some way these were significant incidents for me whether or not I was aware of this at the time.

I will give a descriptive account of each encounter followed by my reflections after the event. During the reflection, I will attempt to account for what happened and why I found the incident significant. This technique of giving an *account of* followed by giving an *account for* is one of which I became aware from John Mason (1991) and Joy Davis. Joy writes that:

> We distinguish here between an account *of* the moment, by which we mean an account free of judgement and justification, an as-far-as-possible clean account of what took place, and *accounting for* the incident through explanation, justification and judgement. (1992, p170)

I will summarise in a short sentence, or even a single word, key factors which have pedagogic significance for me, which stem from my reflections. These will be put in curly brackets - **{...}** - and will be picked up and explored further in later chapters. At this stage, these summary statements are mainly signals of meaning for me and may not have meaning for the reader until they are developed in later chapters.

Thus the structure for each encounter will be:

anecdote (account of)
reflection (account for)
summary of key pedagogic factors - **{...}**

The bracketed summary statements which have been highlighted during the chapter will feed in to later chapters. There, they will contribute to the development of theories concerned with the powers children have, the nature of learning, and implications for teaching. Section 3.3 will give a list of the summary statements from this chapter and indicate exactly where each statement will be developed in later chapters.

3.2 Encounters with young children

3.2.1 Mark

In 1985, I spent Boxing Day with my brother, his wife and our respective parents. After breakfast, my brother's neighbours came to spend the morning with us. They had a boy, aged two, who I will call Mark.

I brought a BBC computer down and as I was setting it up, Mark was already sitting at the keyboard pressing the keys. I typed a program to produce a random triangle in a random colour whenever a key was pressed. I told him to press a key and he seemed interested in what was happening on the screen. A minute later I found his attention was no longer on the screen but fixed on the keyboard and he was looking delighted with himself. He had found the Shift Lock button which made a light on the computer go on and off. The triangles appeared to be of no further interest.

I had left Mark and was talking with the others when a loud beep interrupted us. We looked over and saw a huge grin. He had found the Copy key. This became the centre of attention for a while, mixed with some repeated pressing of the Shift Lock button. I thought that it would be a good idea to turn the keyboard into a noise making machine with each key having its own sound. He moved over and sat quietly whilst I wrote a new program.

After I'd finished, I told Mark to press a key again. He was very excited. The room remained noisy for some time. The computer's buffer was getting well stocked up and sounds kept booming out even though his fingers were not always on the keys. Then.....silence! He had pressed the Escape key which had stopped the program running. It didn't seem to bother Mark much, he turned his attention back to some practice with the lights.

I quickly added a line in the program, so that the Escape button would just make it sit and wait for the next key to be pressed, and let Mark loose. He appeared to like certain sounds and kept repeating them, but he could also stop them when he wanted. The Escape key now stopped the sound. He practised this time and again, filling the buffer by holding a key down and then deliberately pressing Escape. He also began pressing the keys at regular intervals by waiting until one sound had finished before hitting another key. I was impressed by how regular the rhythm of sound was, even when he included different notes.

Reflections on Mark

The coloured triangles appeared to be an initial attraction for Mark and I was surprised at first that one light on the keyboard, once he had discovered it, took more of his attention than the triangles on the screen. It seemed that the light was something he could gain some control over, by being able to turn it on or turn it off by pressing a particular button. The triangles may have been of interest to watch but after a while the screen became crowded with various coloured triangles and there was less dynamic change each time a new one appeared compared to when the first triangle came on the screen. It seemed to me that Mark was interested when something happened as a consequence of his actions. In this case it was initially a shape appearing on the screen, then a light on the keyboard, and then different sounds. Through his exploration, Mark ended up knowing the effect certain keys had.

He seemed to enjoy pressing keys at the start, but after a while, he became more interested in pressing particular keys because of the effect they had. I relate this to a baby who is exploring the sounds they can make through their mouth; initially they may be involved in making a number of sounds but they might become more interested in sounds which have a particular effect such as their parents showing delight, or the parents repeating the sound back to them. This might increase the likelihood that the baby repeats the sound just as Mark was repeating the pressing of the buttons which produced an interesting effect.

Gattegno describes such a possible event:

> While practicing his *nth* sound and amalgamation of sounds, a baby may be uttering a sequence of *dadada* or *mamama* or *papapa...* when an adult hears him and exclaims, "Listen, he is calling his father or his mother." And the adult may even believe it. Then all the family surrounds the baby and asks for encores,

> uttering the sounds the baby is practicing and is therefore acutely aware of. (1973, p77)

I have observed a number of occasions when an adult has recognised a collection of sounds that a baby has made as one that belongs to their native language. The adult expresses delight and often repeats the word back to the baby. This type of response increases the likelihood of the baby repeating those sounds.

Here, babies experience reactions of others to their own actions. Claxon said:

> ... I cannot learn if I am not attentive to, or aware of the success or failure of my actions at some, not necessarily conscious, level. (1984, p45)

Mark was experiencing the consequences, in terms of lights and sounds, of his actions. This reminds me of a situation with a young boy, Idris (who is the subject of other encounters later), who was less than two years of age. He was confident with walking and was in a kitchen trying to reach up to some biscuits. There was a ball lying on the floor and he went to step on it in order to reach higher. As soon as he put pressure on the ball, it slid away from under his foot and, due to the backspin created, the ball stopped and slowly rolled back to him. In this situation, Idris had not only found out that this may not be the way to reach what he wanted, but he also had the opportunity to become aware of the consequences of pushing down on a ball with his foot. Thus, by being informed of the consequences of his actions, Idris can learn two things at once. He does not need to be informed that he had not reached what he wanted because Idris can look around and see that this was so. He had, however, got the potential richness of pursuing the dynamics of making balls spin. This appeared to become a more relevant challenge to him at that particular time as he turned his attention away from the biscuits and onto the ball. Thus, it is sometimes the case that children switch quickly from one activity to another, before an adult might feel a child has completed the first activity. In many ways, this is something I felt Mark had done, as his attention shifted from the screen to the lights and then to the sounds.

{Experiencing the consequences of their actions}

Mark did not do one thing and then go straight on to the next. He continued exploring the lights and the sounds for some time. He was pressing the key to make the light go on, then off, on, off, on,... Why is it that adults sometimes express the fact that it can be hard to keep a child's attention when there are times such as this when a child becomes seemingly obsessed with something, repeating it again and again (sometimes to the adult's annoyance!)? John Mason recalled to me the fact that his daughter, Lydia, at about 9 years of age could not sit through a meal without getting up and doing a handstand against the wall. Is it that once a child is involved in something, they wish to stay with it until they have gained control and mastery over the situation? *Gaining control* of something can be a way in which you come to know about your surroundings; your understanding of something can be in terms of the way you can exert control over it.

White talks of competence as being a desired aim which motivates much early learning:

> It [competence] is... a suitable word to describe such things as grasping and exploring, crawling and walking, attention and perception, all of which promote an effective - a competent - interaction with the environment... I shall argue that it is necessary to make competence a motivational concept... The behavior that leads to the building up of effective grasping, handling, and letting go of objects, to take one example, is not random behavior that is produced by an overflow of energy. It is a directed, selective, and persistent, not because it serves primary drives, which indeed it cannot serve until it is almost perfect, but because it satisfies an intrinsic need to deal with the environment. (1959, p317-318)

Papousek (1969) carried out a number of experiments. He started out by rewarding babies with some milk if they made certain simple movements. Another experiment made a light appear if certain movements were made. In both cases, he observed that the babies continued to make the movements and showed apparent pleasure even when they were no longer hungry, and so didn't want milk, or did not pay particular attention to the lights. Papousek concluded that it was not primarily the sight of lights involved in his experiment which pleased the children, but rather their success in mastering a skill. Donaldson talks of Papousek's conclusion:

> If he is right in this - and there is a considerable amount of other confirming evidence - then we may conclude that there exists a fundamental human urge to make sense of the world and bring it under deliberate control. (1986, p111)

I ask whether it is more the case of making sense of the world *by* bringing it under deliberate control?

{Gaining control}

3.2.2 Idris 1

In 1988, I was in the back of a car with a 2 year old boy, Idris. He suddenly called out *tractor*. I looked around but could not see any tractor. Idris said it again, and I looked to see where he was looking. Then I saw that in between two houses there was a small gap; through that gap, in the distance, was a field; and in the field, you could just make out a tractor.

Reflections on Idris 1

I was amazed that amongst all the complexity to explore within his possible field of vision, as the car was moving, he was able to notice something so small (due to it being so far away). This was also true for myself as the observer of this event: of all the events that were happening around me at that time, I was able to notice the fact that Idris had noticed the tractor. The ability to extract is one which we are all using in our daily lives.

Borges wrote in 1942 about a fictitious person, Funes, whose perception and memory were infallible. It was as if Funes did not extract from his surroundings, but that all aspects of his surroundings remained with him for the rest of his life. At the same time, Borges portrayed the image of someone who was not very capable of thought:

> To think is to forget a difference, to generalize, to abstract. In the overly replete world of Funes there were nothing but details, almost contiguous details. (1985, p104).

Part of the engagement I had with the story was trying to imagine what life might be like for Funes, since I had not heard of anyone with such abilities. This highlights the fact that people do extract from their surroundings, otherwise the character of Funes would not be so unusual. The ability to extract is a powerful ability as it gives us the possibility to place our attention somewhere in particular rather than keep our attention spread over everything which is before us.

From the encounter with Idris, I realise that children, and adults, are able to turn their attention to one part of all the information that enters their senses. This comes through the stressing of some things and, unlike Funes, the ignoring of others.

{Extraction through stressing and ignoring}

3.2.3 **Idris 2**

In October 1988, I was sitting with a 2 years and 4 months old boy, Idris, whilst he had a number of toys around him on the floor.

There were a number of models of animals, people and bikes over the floor. Idris carefully put a cow on top of a man, looked up at me, saying *the cow is eating the man*. Then he took the cow off, put it on the floor, and put the man on top of it. Idris looked up at me and said *now the man's eating the cow*. Later on he said *the man's going to get on the bike*. And then, *the man's going to get off now*.

Reflections on Idris 2

I could imagine someone describing this situation as 'Idris playing with his models'. It seemed to me that Idris was also playing with language and using the models as a means by which to carry out this language exploration and practice. I had been with Idris on a number of occasions before and watched him gain control over saying many English words. I had also watched him master putting one thing on top of another, and pushing something along the floor. On this day, these were all being subordinated to a new situation. In the first pair of sentences, Idris created a situation with the models such that the subject had become the object and vice versa. This has implications in terms of the language used to describe the situation, and a transformation is required within the descriptive sentence. To become competent at the grammar of a language requires practice. Although Idris's attention appeared to be with the models, I felt he was working at his language ability which he was using in order to describe the situation.

Idris looked at me each time he said a sentence. At the time I felt this was because he wanted to get me involved in his playing and that this was a social desire. This is one aspect, but also I realised that by looking at me, he could observe my reaction to his sentence. This was a way to test whether what he said was acceptable to someone whom he perceives as having command over this world of language he was exploring. If I reacted negatively, by frowning for example, he might learn that there would have to be some changes made to the way he was saying his sentence. If I appeared to accept what he said, he may repeat such structures in his sentences in the future.

The second pair of sentences involve a change in the verb tense to those sentences he said before. He did state each sentence before carrying out the action. Thus, a change in the order within time (action-then-sentence to sentence-then-action) also requires a change in the content of the sentence. The ability to make transformations in sentences is one which is continually used as children engage with the learning of a language. Gattegno says:

> ... we have to acknowledge that children have the power to make transformations, for to learn to speak is to use *transformations* constantly. In every verbal situation in which someone is trying to tell us something, the words are to be used by us as they are by the others. The words cannot simply be repeated.
>
> If someone says to me, "*This* is *my* pen," and I repeated it, I would be wrong; and if we were children we might quarrel. Then perhaps I would see that I have to say something else to be at peace with the other person, and I might learn very quickly. In any case, I will learn to say, "*That* is *your* pen."
>
> Again, if I look at one person, a woman, and I talk to her, I will use the word "you." But if I look at her and talk of a third person, a man, I must say, "he." Such transformations go on all the time. (1971, p9-10)

{Children performing transformations within language}

It seemed that Idris was working hard at quite high level challenges whilst he was 'playing' with his models. I was also struck by the fact that adults might perceive his attention to be with the toys (as indeed I did initially) and yet a development of his learning was taking place within the language. Thus, there may be times when what is learned is different to the apparent external focus of attention.

{Learning is not always taking place where the observable attention is}

As mentioned previously, I had watched Idris acquire many physical skills which were now being used in the placing of one thing on top of another. In this way, old skills were now being subordinated to engage in the acquisition of new skills. There is a sense of progression here: having learned the skills required to place one object on top of another, Idris can practise that by becoming involved in a new language challenge. Learning and practice come together. Practise the old whilst learning the new.

{Subordinate the old to engage in the new}

{Practise the old whilst learning the new}

Watching Idris, I recalled a weekend I had spent learning Spanish in September 1988.

The weekend was run by Laura Guajardo and was taught *The Silent Way* (a method developed by Gattegno, see Gattegno (1972)). On the Sunday, I was creating situations with some resources that were available (books, cuisenaire rods, people,...) in order to test whether I was able to make a statement about the situation. I realised that, although I only had a small vocabulary, I was still able to explore, within that vocabulary, many aspects of the structure of the language. When I watched Idris a month later, I felt that I was watching myself on the Sunday of the Spanish weekend.

Reflecting back on my development throughout the weekend, I realised that the challenges I was engaged in on the Sunday (those primarily of structure and grammar) were unknown to me when I began on the Friday night. At that time, I was involved purely in making sounds which form the basis of the Spanish language. With the sounds came the possibility of combining them to make words. At this time, I did not know whether the words I was saying were part of the language or whether they were simply a combination of sounds. With the combining of sounds, I was aware that there is choice over the emphasis placed on certain sounds rather than others. Intonation needed to be worked on and practised for each word. Knowing some words, and knowing that they were obtained by a process of combining sounds, leads to the possibility of combining words. Thus sentences were formed, and again the dynamic of stress worked on for each one of those.

Once I was able to say a particular sentence, I was told to point at someone whilst looking at someone else, and then say the sentence. For the first time the challenge was now concerned with meaning. From the situations in which I and my fellow students were placed whilst certain sentences were said, I began to develop possible meanings for the sentence, or more appropriately, other situations in which I might be able to say the same sentence. I could test this out and see whether it was acceptable, as I perceived Idris to have done. I was in a position where I had to develop my own meaning.

{It is the learner who develops meaning}

Within similar situations, I noticed that the sentences some people were told to say had similarities and differences to sentences others were told to say. In this way, I became aware of how certain words changed according to whether there were one or many, masculine or feminine, or concerned with me, you or others. I found myself engaged in the structure of the language and ended up using the limited vocabulary that I had in order to explore situations where I could test and develop my skill and understanding.

Reflecting on this experience, along with my observation of Idris, I realised that although I may have begun to gain control over a current challenge, I did not feel as if I had mastered it until I was able to subordinate it successfully in gaining control of a subsequent, higher order challenge.

{Control/mastery comes only when it is subordinated to higher order challenges}

I felt that the understanding and abilities I had accrued were part of me and available to be used rather than memorised. Thus I was not concerned with forgetting because I had not made a particular effort to remember.

{Learning without the need to remember}

3.2.4 Robert

On 7th December 1991, my nephew, Robert, was two years and two weeks old. As I went into the spare bedroom, Chris, my brother, had Robert on his lap. In front of them was a computer. Robert switched it on. He pressed the *n* key which was required in order to progress with the booting up of the machine. He leaned over and got the mouse, moving it until the arrow on the screen was pointing at the appropriate place. Robert proceeded to get into the program *solitaire* which was a version of the card game of *patience*. This had involved a *double-click* on the mouse, an impressive demonstration of his manual dexterity. Then, I was amazed to see him beginning the game.

Robert still needed some help in playing the game and I became fascinated with the way in which Chris helped him. At times Chris placed his hand on top of Robert's hand which, in turn, was holding the mouse. In this way, Chris would guide the mouse so that the arrow was in the appropriate place on the screen. Whilst this guiding was going on I noticed Robert's attention appeared to be with the screen rather than the mouse. Because Robert's hand was over the mouse, he was experiencing the movements required to obtain such a resultant movement on the screen. Chris helped in this way quite rarely; most of the time Robert was moving the mouse on his own.

Robert had learned to click on the pack of cards to reveal a card which then can be used in the game. A card can be placed on another if it is one lower in value and is of a different colour. Chris helped him with this aspect of the game by saying such things as *the red seven will go on the black eight*, accompanying it with a finger pointing at the appropriate places on the screen. Robert responded by moving the mouse over to where Chris had first pointed, clicking the mouse button and holding it down. Then, moving the mouse so that the card was dragged across the screen on top of the second card Chris had pointed to, and releasing the button to drop the card.

At one point an ace appeared and Robert quickly did a *double-click* on the mouse which sent the ace to the top of the screen and started one of the four piles that correspond to the four suits. The object of the game is to build up the four piles from the ace to the king. Robert had clearly recognised the ace as a significant card. A while later, I noticed Robert had moved one card on top of the other when Chris had said *the black five can go on the red six* and not pointed. Later, I learned that Robert was able to press the appropriate number on the keyboard if I told him a particular number from one to nine.

The next day we were downstairs with a pack of cards and I asked him to give me a *six*. He searched through the pack and gave me the first *six* that appeared. At this point in time, Robert did not seem to know *red* and *black*, however I felt it would not be long before he did.

<u>Reflections on Robert</u>

As I watched, I recalled an ex-teacher of physical education, Jean Lyttle, who described to me how she helped someone develop their skill of hitting a rounders ball. She used to put her arms round the student and hold their hands whilst they held the rounders bat. As the ball was thrown, she made sure the student experienced the movements required for a successful hit of the ball. As time passed, the student began to make the necessary movements themself and she gradually began to allow her own hands to take a passive role by going along with the student's movements.

This contrasts with a teaching style I was encouraged to use when I went on a training course to coach squash. I was told to describe in words how the racket should be held and a particular shot played. This was to be followed with a demonstration of playing the shot. Then the students were to have a go themselves with you encouraging them for what they do right and telling them where they went wrong. This is a crude description which does not mention the subtleties of the coaching, however, it was based on explanation and demonstration with verbal encouragement and correction. With such a style, the learner is in a position of having to translate what they see, or hear, into their own muscular movements. With Jean's example, and that of Chris's, the learner has the opportunity of directly experiencing the physical movements and observing the successful consequences of them. No translation is involved. What is required is for the learner to use their muscles in a way which keeps them in harmony with the movements made. This appears an easier task than having to create those movements in the first place. Once harmony is achieved, the teacher can begin to withdraw their dominance and allow the learner to take control.

{Reduce the need for a learner to translate from one medium to another}

Jean and Chris's style also contrasts with that of simply allowing the learner to practise and find their own way of achieving success. Jean could have thrown the rounders ball to the learner a hundred times, giving them a collection of experiences to learn from. In such a case, the learner would be encouraged to develop their own strategies of how they might hold and swing the bat in order to hit the ball successfully. However, she decided to be more interventionist.

A statement, such as *the red seven will go on the black eight*, requires considerable interpretation and understanding if it is to be translated successfully into the required physical action of the mouse. This is particularly so for someone who has only turned two and can only say a few words of their first language. The pointing, that Chris did, provided more for Robert to work with. However, a finger pointing in one position and moving to point to another, still requires considerable interpretation. Since Robert was able to move the card successfully from one place to the other, this demonstrates that some learning had already taken place. I found out later that Chris used to hold Robert's hand on top of the mouse, as mentioned above, and do the necessary pressing of the button and moving. Sufficient learning had taken place by this time as Chris had been able to withdraw his hand completely.

Initially I was not convinced Robert had learned more than being able to move a card from one place to another as indicated by Chris's finger pointing. However when he was able to move cards without Chris's pointing, I was sure more learning had taken place. Robert seemed confident with numbers from one to ten, even though he was unable to say any of the associated words. This, along with the physical skills demonstrated by his use of the mouse, was an impressive piece of learning.

Robert did not know all the rules of the game, yet he was playing it and was involved. When Robert began playing, he knew none of the rules. This contrasts with a pedagogy that says someone can only begin doing something after a teacher has explained what it is they have to do. I have observed so many lessons (including many of my own) where the teacher has begun a lesson with an explanation of what they want the students to do. This may, for example, be the rules of a game, or it may be a method to change an amount of money from one currency to another. This is often followed by a demonstration, as in my training for squash coaching. In classrooms the demonstration is often called an *example*. The problem remains for the learners to translate these explanations and demonstrations into their own personal mental operations. Often, if a learner has difficulty with making a successful translation, the teacher offers further examples which still require the same translation to be made. The learner is treated as someone who needs to be told. Yet nobody is going to explain to Robert the changes he needs to make in the movement of his tongue in order to say *uncle* as others do in the English language. Nobody is going to explain to him that you have to add *ed* at the end of a verb to change it from the present to the past tense. However, despite not being told, Robert will do these things. He will do them because he operates intelligently and is required to do so in a world where he is engaged and is still trying to find out and apply rules.

{Begin, then discover rules through what is accepted and what is not}

{No explanations}

I have stated that Robert will be able to do these things as indeed you have been able to do them. How did you learn to do them?

How do you move your tongue when you say the word 'uncle'?

You may be able to move your tongue in an appropriate way to say *uncle* but are you aware which way that is? There is an issue here about *knowing*. In one sense I can say you must be aware because otherwise you would not be able to do it. In another sense, you are not aware of everything you become able to do. People say at times that they *found themselves doing it*. This indicates they began doing something and became aware of it afterwards. Thus, our knowledge consists of more than that which we might be able to articulate.

{Our knowledge consists of more than that which we might be able to articulate}

Vygotsky refers to this difference between what someone can do and what someone knows they can do, when discussing a child's process of detaching meaning from an object, or a word from an object:

> ... before a child has acquired grammatical and written language, he knows how to do things but does not know that he knows. He does not master these activities voluntarily. In play a child spontaneously makes use of his ability to separate meaning from an object without knowing he is doing it, just as he does not know he is speaking in prose but talks without paying attention to the words. (1978, p99)

I suggest that, immediately before reading this chapter, you were not aware of how you moved your tongue in order to say *uncle* even though you have moved your tongue in such a way on many occasions during your life. Yet I can force[2] this awareness on you by asking you to put your attention in the movement of your tongue whilst saying *uncle* a number of times. I feel confident you will gain this awareness because I have asked you to do nothing else but attend to something you can already do. Trying to explain this to Robert, or trying to explain how you play a *boast* shot in squash to a beginner, is asking them to attend to something they cannot do at present. As Vygotsky states, being able to do something often comes before a conscious awareness of that something:

> ... consciousness and control appear only at a late stage in the development of a function, after it has been used and practiced unconsciously and spontaneously. In order to subject a function to intellectual and volitional control, we must first possess it. (1962, p90)

{Forcing awareness by affecting attention}

{Pay attention to what you can do rather than what you cannot do}

I observe Robert participating in a computer activity he as yet knows nothing about, but his teacher, Chris, does. There is no prerequisite that Robert understands. Understanding comes through participating. Chris ensures that Robert experiences the playing of the game rather than having it demonstrated or explained. This requires Robert to place a level of trust in Chris. Robert does not know, at the outset, what he is required to do yet he is prepared to put trust in his father and allow himself to be taken on a journey into the unknown. Not all learners demonstrate this trust and, as a consequence, they may not allow themselves to be taken on such a journey. Their energies may go on wanting to be told what is to come, thus diverting their attention from being with what is happening at present. In this way, they may fail to participate in the activity since this requires their attention in the present.

{Attention is in the present and not in the unknown future}

Robert learns some abilities, which might be referred to as *content* in an educational syllabus, as a by-product of wanting to participate successfully in the activity. His attention is with the activity and not with the content itself. Indeed, Robert may not be aware that he has learned anything. It may require a separate, reflective, activity for him to become aware of what he has been doing for some time. Indeed, Vygotsky talks of a child having a *pseudo-concept* when their action is consistent with the action of

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[2] I say more about the phrase *forcing awareness* in section 5.2.3.

someone who has a certain concept, yet the child's actions are guided by perceptual likenesses rather than coming from a knowledge of the concept itself.

> ... the child begins to operate with concepts, to practice conceptual thinking, before he is clearly aware of the nature of these operations. This particular genetic situation is not limited to the attainment of concepts; it is the rule rather than an exception in the intellectual development of the child. (1962, p69)

By being able to successfully carry out actions consistent with a certain concept, a child has the opportunity of reflecting on those actions and articulating for themselves what might lay behind the consistency of those actions.

{Successfully do, followed by reflecting on that doing}

<u>3.2.5</u> <u>Helen and Mark</u>

On 8th April 1990 my nephew, Robert, who was 4 months old, was christened. Afterwards, a number of people came back for food and drink. Several children were there and one of them, who I will call Helen (aged 3), was playing with a line of plastic *people* attached to Robert's chair:

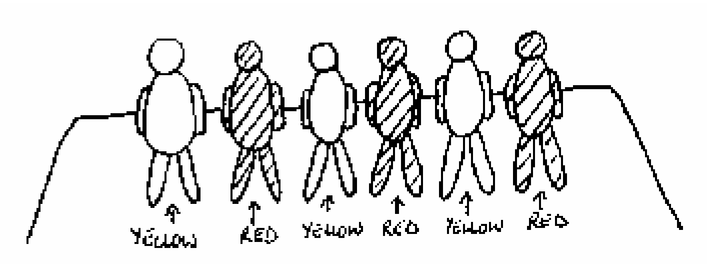

Figure 1: Drawing of attachment to Robert's chair.

Helen started counting as she touched each of the people. My brother, Chris, joined in and counted with her, taking turns in saying the numbers. I felt this was a nice activity that offered the potential to gain experience of odd and even.

Later, I found myself nearby Robert's chair and touched the first person, saying *one*. I waited and Helen said *two*, hitting the second. We continued taking turns until we reached the end of the line with Helen saying *six*. We repeated this a few times, sometimes with me starting and sometimes with her starting. Then, after Helen had said *six*, I hit the first one again and said *seven*. After I kept doing this a few times, Helen went along with me and continued by saying *eight*. She confidently counted up to the mid *-teens* and then, equally confidently, said a variety of numbers beyond, although not in numerical order.

This toy seemed to offer potential as a focus whilst saying numbers, so I decided to explore other word activities. On the next game, I started by saying *twenty-* quietly before saying *one*. Helen replied *two*. I said twenty-*three*. At this point she came back with twenty-*four*. The *twenty* was said tentatively but as we continued counting it was said with growing confidence. We played this game, starting with twenty-one, fifty-one, thirty-one,... At one time, we got to thirty-nine, Helen hesitated and then started again at thirty-one. There seemed to be a natural feeling that the sequence was fine up to *...-nine* but she was not sure what to say after. Sometimes Helen might say *...-ten*. If so, I would not continue at this point.

Helen revealed to me that she understood the activity perfectly well by starting another game with touching the first person and saying *twenty-yellow*. I continued with *twenty-red*... This indicates to me that Helen was aware that all you have to do is say the word *twenty-* beforehand. Children who can say the numbers from one up to nine are also perfectly capable of saying another word in front of each of these. Thus, if they can count from one to nine, then they can also count from twenty-one to twenty-nine, or seventy-one to seventy-nine. There is no need to fully comprehend how these sequences fit together before being in a position to be able to say each sequence *something-one, something-two,...*

Mark, who I mentioned earlier and who was now 6, joined us and I continued this word game with him. I asked him to say *four hundred and twenty-*, which he did successfully. So I started four hundred and twenty-*one*. He continued four hundred and twenty-*two*... We continued by starting with thousands and millions and so on.

Mark's mother came down the stairs and commented that *Mark hasn't done numbers above two hundred yet*.

<u>Reflections on Helen and Mark</u>

There seems to be a belief that the names of numbers are to be learned in numerical order and that the higher they are, the more difficult they are. Walkerdine quotes a teacher of 6 and 7 year olds who discussed the importance of keeping the children to small numbers:

> You see (the headmistress) is very keen on - er - if a child can't, um, sort of analyse mentally beyond six and hasn't much concept of what numbers beyond six are - you know, how they break up and so on - that everything as far as you can should be limited to that - so that they're not trying to grasp hundreds of grammes which - they can't understand what a hundred is if they are only able to grasp, say, six. (1990, p144)

I would not say that I understand what a billion is in terms of being able to picture a billion of something. However, this does not prevent me from using *billion* in my conversations and to know it as a word I say along with other words such as *million* and *thousand*. Should I not have access to the word because I cannot picture that number of something? The experiences with Helen and Mark have shown me that being able to say *eighty-one* or *ninety-one* is no more difficult than saying *ten* or *eleven*. Learning *eighty-* and *ninety-* give you another 18 numbers you can say, whereas *ten* and *eleven* give you just two. It seems that knowing the numbers from one to nine can open up a whole world of numbers at little extra effort. Why stay with the exceptions within the English language of *ten*, *eleven*,... which produce small reward for the effort of learning them?

For a mathematician, an answer to the question *What comes after nine?* may be *ten*, but for a pedagogue I suggest the answer is certainly not *ten*. If a child knows the number-names from *one* to *nine*, then the next word I would consider introducing is not an irregular one such as *ten*. In appendix 1, I give an account of introducing *ty* but another possibility could equally well be *hundred*. Both these allow a child to gain access to a variety of other numbers, whereas *ten* only enables '10' to be said. Thus, although '200' may be viewed as a big number, it is only one word away from '2'. I will argue that learning to say number-names such as *two hundred* or *sixty-four* before *ten* makes sense for pedagogical and economical reasons.

{Rules first, exceptions later}

This observation can be extended to situations where the mathematics curriculum is placed within a hierarchical situation, such as the National Curriculum (NC) in England and Wales. It appears to me that there has been an attempt to place statements of content and processes in levels with the implication that a student achieving something in level

8 is in some way a better mathematician than someone who has only achieved something in level 6. However, the hierarchy is sometimes based on the idea that if something involves larger numbers, then it is more difficult, or that carrying out some symbolic manipulation of an algebraic expression containing letters is harder than if there were only numbers present (see chapters 9 and 10 for discussion of the introduction of algebraic symbols). Although in both cases it may be true that the mathematics is more sophisticated, this does not imply that the mathematics is harder to learn. Bruner quotes an experienced teacher of elementary mathematics, David Page, as saying:

> It is appropriate that we warn ourselves to be careful of assigning an absolute level of difficulty to any particular topic. When I tell mathematicians that fourth-grade students can go a long way into 'set theory' a few of them reply: 'Of course.' Most of them are startled. The latter ones are completely wrong in assuming that 'set theory' is intrinsically difficult. Of course it may be that nothing is intrinsically difficult. We just have to wait until the proper point of view and corresponding language for presenting it are revealed. Given particular subject matter or a particular concept, it is easy to ask trivial questions or to lead the child to ask trivial questions. It is also easy to ask impossibly difficult questions. The trick is to find the medium questions that can be answered and that take you somewhere.
> Bruner (1960, p40)

As I have mentioned above, the number-name *two hundred* is just as easy to learn as *ten*, and it makes more pedagogic sense to learn *sixty-four* before *twelve*. In chapter 10, I will argue that manipulation of algebraic expressions can be just as easy, or difficult, whether or not they contain letters. Thus, there seems a confusion in the hierarchy of the mathematics National Curriculum as to whether it is a hierarchy of mathematical sophistication or a hierarchy of difficulty to learn. The main thrust of my thesis is that the latter is dependent upon pedagogical decisions which each teacher makes, along with the individual variety of experiences each student brings to the mathematics classroom. There are economical ways of approaching some content which enable students to learn quickly and effectively, and there are uneconomic approaches which require considerable work on the part of students for them to learn relatively little. Thus, to attempt to state a national hierarchy based on this is an impossibility. As to a structure based on mathematical sophistication, this may be of academic interest but does not relate to the ease or difficulty someone might have in learning that mathematics. Thus, such a hierarchy is of little use to a teacher.

An additional observation is that in order to learn the number-name *two hundred* or *sixty-four* , it is not necessary to have already learned to say all the numbers less than that number. Applying this to the National Curriculum, it is not necessary to have learned all the content leading up to level 6 in order to learn something which is classified as level 6. In fact it may not be necessary to know any of the content in the earlier levels. I will argue that using the powers that students have may mean that some mathematical content can be approached without the need to know other content. If so, there is no natural order in learning mathematics, some mathematical content can be approached at any stage of a student's mathematical development in school, even though it may be considered sophisticated mathematics. This might add support to Bruner's hypothesis:

> We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of the curriculum. No evidence exists to contradict it; considerable evidence is being amassed that supports it. (1960, p33)

Bruner talks about the spirit behind this bold hypothesis in the preface to the 1977 edition of the same book:

> One matched the problem to the learner's capacities or found some aspect of the problem that could be so matched. That was the spirit behind the dictum. It has sometimes been gravely misinterpreted, as when I am asked, "Do you *really* think the calculus can be taught to six-year-olds?" That is surely not the point. One can certainly get across the idea of limits to the six-year-old, and that is an honest step en route to grasping a basic idea in the calculus. (1977, px)

My interest is to explore how particular mathematical content can be made accessible to students of all sorts of ages and with all sorts of experiences, by using approaches which assume little in terms of previously taught mathematical knowledge. An example of how mathematical statements can be developed, such as

$$\sin 30 = \sin (360n + 30) = \sin (360n + 180 - 30)$$

from very little prior knowledge of mathematical content, see Hewitt (1989a).

{Approach mathematics through powers used in early learning rather than previous mathematical content}

{Mathematical sophistication does not imply learning difficulty}

{Without attending to the powers of children, we may be seduced into basing the order, approach and progression of our teaching on mathematical sophistication}

3.3 Backwards and forwards

The significance of the events contained in this chapter are crystalised in the statements in parentheses. There are many such statements and I give below a summary of what the statements are in this chapter. These statements will feed into the continuing development of a theory of economy of learning and teaching mathematics which continues in later chapters. The number after each statement indicates the chapter and section which continues to develop the ideas behind the statement. For some statements there is more than one place where the idea is developed. Other statements are not explicitly considered, however I include them because they add to the overall sense of what economy is concerned with.

3.2.1 Mark

{Experiencing the consequences of their actions} - 10.3
{Gaining control} - 10.3

3.2.2 Idris 1

{Extraction through stressing and ignoring} - 6.4.1

3.2.3 Idris 2

{Children performing transformations within language} - 6.3
{Learning is not always taking place where the observable attention is} - 6.2; 9.3
{Subordinate the old to engage in the new} - 6.2
{Practise the old whilst learning the new} - 6.2
{It is the learner who develops meaning} - 10.3
{Control/mastery comes only when it is subordinated to higher order challenges} - 6.2
{Learning without the need to remember} - 10.3

3.2.4 Robert

{Reduce the need for a learner to translate from one medium to another} - 8.3
{Begin, then discover rules through what is accepted and what is not}

{No explanations} - 10.3
{Our knowledge consists of more than that which we might be able to articulate} - 9.2
{Forcing awareness by affecting attention}
{Pay attention to what you can do rather than what you cannot do} - 8.3
{Attention is in the present and not in the unknown future}
{Successfully do, followed by reflecting on that doing} - 9.2; 9.3

3.2.5 Helen and Mark
{Rules first, exceptions later} - 6.3; 10.3
{Approach mathematics through powers used in early learning rather than previous mathematical content} - 8.3
{Mathematical sophistication does not imply learning difficulty} - 8.3
{Without attending to the powers of children, we may be seduced into basing the order, approach and progression of our teaching on mathematical sophistication} - 8.3

4 Significant events 2: inside classrooms

4.1 Introduction

I continue describing events which have been particularly influential to my study, this time events which occurred inside classrooms - sometimes my own and sometimes other people's. I maintain the structure of giving a descriptive account of each encounter followed by my reflections after the event. As in the previous chapter, I summarise in a short sentence, or even a single word, key factors which have pedagogic significance for me, which stem from my reflections. These are written in curly brackets - **{...}** - and will be picked up and explored further in later chapters. Section 4.3 will give a list of the summary statements from this chapter and indicate exactly where each statement will be developed in later chapters.

4.2 Inside classrooms

4.2.1 Richard

I was observing a student teacher, who I will call Richard, teaching my year 9 class. This was a class which I had taught for nearly two years and who were used to exploring mathematical situations.

Richard was using a computer program on one computer with the whole class. The students were attentive and looking at the screen. Several of them were making comments which expressed ideas they had as to what was happening on the screen or were making suggestions as to what could be entered on the computer next. My observation at this time was that the class were actively involved and being creative. The students were doing most of the talking in the room and I noticed that several of them were leaning forward in their chairs. Their attention appeared to be on what was happening on the computer screen. Richard was doing little talking and was mainly listening and observing the class. He looked relaxed and took a standing position which was behind the computer and slightly to one side.

Then, one of the students asked Richard why something had happened on the screen. Richard hesitated initially and then replied with a short sentence of explanation. Someone else in the class was not quite clear and said so. Richard elaborated a little. Within this elaboration, there was something Richard said which another person in the class was unclear about. Richard began explaining that.

This continued for some time. After a while, Richard became frustrated that his explanations were not clear enough or not sufficient. He was working very hard at trying to make things as clear as possible. He was becoming flustered, slightly red in the face and his voice was gradually getting louder. I looked at the students and I noticed that their posture had now changed. Many were now leaning back in their chairs, no-one was leaning forward. Their attention no longer appeared to be with the computer screen but was directed at Richard. The only comments they made were questions directed at Richard. Most of the students were silent.

Reflections on Richard

I was aware of a shift having taken place. At the beginning, the students were active and leaning forward, with Richard passive and standing back. After a while the students were passive and leaning back, with Richard active and standing forward. The students had been doing most of the mathematical work in the room, whereas Richard ended up doing most of the work. The beginning seemed to be about going forward with developing understanding of what mathematics was happening on the screen. After a while, most of the talk was about the same mathematical point with a feeling of standing still mathematically. In the beginning, the students seemed to take the main responsibility of finding out about the mathematics. Later, it appeared that the responsibility had been shifted onto Richard.

I felt the atmosphere in the room had become uncomfortable and negative, with the class giving Richard a hard time. I was surprised at this since the students were usually good at exploring mathematics on their own. The change was a gradual one, and initially I could not understand how this shift had happened. On running through the lesson in my mind, and talking with Richard after, the key moment seemed to be his decision to reply to a request for some explanation. Once he had accepted his role as an explainer of the mathematics, then the students appeared to allow him to do all the work. They were no longer in a position to explore, since it would be explained instead. The students shifted from actively working to passively listening.

I call this the *explanation trap*. It is quite easy to fall into, after all it is perfectly reasonable for a teacher to respond to a polite request for explanation about an aspect of mathematics in a mathematics classroom. If a teacher knows why something is so, it is tempting to tell this to one of their students when they ask.

It is not easy to get out of the explanation trap since understanding cannot be given to students. Thus, there is always the likelihood that at least one student will have further questions to ask. The explanations which follow this new enquiry are also likely to contain something which at least one student is not clear about. So the process of further and further explanations continues. Each time, the teacher and students are going further away from the original mathematics.

{The explanation trap}

<u>4.2.2</u> <u>An area activity</u>

There is an area activity (Hewitt (1980)) I have used with 11-12 year olds on a number of occasions. Students are asked to work in groups of four. One student is given a sheet of 1cm squared paper with four curved shapes drawn on. Their first task is to work out the area of each shape. Meanwhile, another student has a sheet of 1cm isometric dot paper with four shapes drawn on. Another has four more on 1cm hexagonal grid paper and finally a fourth has plain paper with four shapes on. After each has decided the area of their shapes, the group task is to find out which, of all sixteen shapes, has the greatest area.

An issue arises of how the group are going to compare areas on the different grids. A variety of methods have been adopted by different groups. On a number of occasions, a group has assumed that two adjoining equilateral triangles, forming a rhombus with sides of 1cm (see Figure 2), have the same area as a 1cm square. They justify this by saying that both the square and the rhombus have sides of length 1cm.

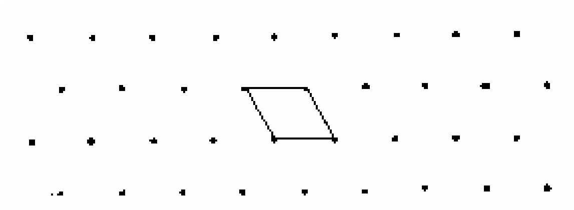

Figure 2: Rhombus drawn on isometric dotty paper.

On several of these occasions, I have produced a rhombus made from four geostrips of the same length. I ask the students to look at the area inside the rhombus and to keep looking as I change the angles at the corners (see Figure 3):

Figure 3: Two positions of a rhombus made from geostrips.

On every occasion I have done this, the students have realised that the area does not remain the same. I ask when the area is greatest as I continue to change the angles. They have replied that the area is greatest when it is a square.

Reflections on An Area Activity

I was faced, as a teacher, with a situation where a group of students were holding a misconception and I felt a desire to provide some input. If I had not done so, I felt they might not have encountered a problem which might have caused them to reconsider their assumption. One option I had was to explain to them that the area of the rhombus was not the same as that of the square. What I chose to do was an alternative to explanation, one where I provided an input which had been carefully chosen to be relevant to their current situation and which offered them an opportunity to gain a new awareness. The students only needed to use their attention and their sense of vision. I am convinced that there are very few students who would say that the area remains the same when they are watching the area disappear and re-appear before their eyes. Thus, there are few requirements in terms of mathematical knowledge for a student to gain this new awareness.

I could have chosen to explain why the two areas are not the same by calculating the height of the rhombus, made from the two equilateral triangles, and seeing that the area cames out to be a number less than one. Thus, with the square having an area of one, the areas would be different. This explanation would have required the students to know about angles inside an equilateral triangle, trigonometry and how to calculate the area of a rectangle.

This is an example of how one mathematical awareness can become known either by an approach which is quick, requires little prior knowledge, and little to be said or shown; or one which takes longer, requires quite sophisticated prior mathematical knowledge, and an amount of explanation from the teacher with understanding from the learner. Thus, the former approach is economic in terms of the time taken and the amount of effort required from the teacher and the learner.

{Economy is concerned with the time and effort spent for the learning gained}

This example is also illustrative of the impossibility of placing a mathematical awareness within a hierarchical structure of 'difficulty', as mentioned in section 3.2.5.

I am intrigued by how obvious the difference in area appeared to the students when I moved the geostrips, and how much they felt sure the areas were the same when they looked at the static drawings. I considered why this might be so. One possibility was that the students' idea of sameness might not be concerned with the areas of the two shapes but about some other similar feature. The students' attention may have been on this feature even though the conversation was about area. However, even if their attention was on the area, there is little noticeable difference between the two areas in the static drawings. With the geostrips there is the opportunity to observe a movement where the area is changing from its maximum to its minimum, and to be aware of where each of the original shapes are placed within this dynamic. Where is the square in this dynamic? Where is the rhombus? With the static drawings, there is only about a 10% difference in both the height and the area of one to the other. This may be difficult to perceive. The movement goes from one extremity to another and so provides an obvious change in the area. Thus, a dynamic which includes extremities and all possibilities allows properties to be noticed which may not be clear within a few examples. In addition, the movement allows the examples to be placed within these noted properties.

{Movement allows all possibilities to be considered and properties noted which might otherwise be difficult within a few particular examples}

{The given examples can be positioned within the noted properties}

4.2.3 Telephone numbers

I have given classes a list of telephone numbers to remember:

3 4 1 2
5 2 1 0
8 4 3 2
6 6 3 6
9 5 4 5

After some time, they close their books and I ask if they can finish off the number that starts 8 4 . . , 3 4 . . , etc.

There is a limited amount of success. I add further numbers to the list:

3 7 2 1
9 2 1 8
4 5 2 0

As I add more, I get complaints from the students that there are too many to remember. However, I observe them concentrating and trying to memorise the numbers. As I add more numbers, I give them some time and then test them.

Suddenly someone says they can remember them all. The class tests this person by starting off a number, 5 2 . . , and are amazed when all the replies are correct. The list may now include 10 or 15 such numbers. The students cannot understand how someone is able to do this when they are having so much difficulty in remembering them.

Reflections on Telephone numbers

The success comes when there is no longer a reliance on memory:

3 x 4 = 1 2
5 x 2 = 1 0

Given the first two numbers, there is no need to remember the remaining ones, there is now just a calculation required. The difference in the amount of effort required was very noticeable in the students. I observed them concentrating hard and becoming frustrated with the enormity of the task. The fact that they complained about the quantity of numbers I was adding to the list, indicated the amount of effort they felt it would take to memorise the list. The fact that they complained when I said I was going to test them indicates that they needed more time in order to memorise them. The amount of time and effort required to be successful was considerable.

This was in stark contrast with the person who was able to remember them all. They looked relaxed, seemed to be making little or no effort, even when more numbers were added to the list. The amount of time and effort required to be successful was minimal.

I learn from this that using memory is expensive. Furthermore, the students are aware of this on one level since otherwise they would not have sought other ways of being successful at the task I set them.

I explored this idea further and devised the following activities described in Figures 4 and 5.

> Look at the table below for three minutes. After that time, you will be tested. In the test, you will be told two numbers and asked to find the answer after they are combined as indicated in the table. Thus:
>
> 5 # 2 = 26
> 2 # 5 = 35
> 3 # 4 = 14
> etc

#	**1**	**2**	**3**	**4**	**5**	**6**
1	0	12	16	12	30	30
2	11	20	23	18	35	34
3	12	22	20	14	30	28
4	3	14	13	0	15	12
5	14	26	26	14	20	16
6	5	18	19	8	15	0

Figure 4: Instructions and table of information for *Table Group*.

Look at the instructions below for three minutes. After that time, you will be tested. In the test, you will be told two numbers and asked to find the answer:

Given two numbers, e.g. 2 and 5, they are combined as follows:

2 # 5

(How many are prime?) x 10

2 x 10 = 20

(difference between them) x second number

3 x 5 = 15

add them together

20 + 15 = 35

So 2 # 5 = 35

Figure 5: Instructions with information for *Rule Group*.

Test 2

1) 1 # 4	2) 4 # 6	3) 2 # 3
4) 3 # 6	5) 3 # 2	6) 2 # 2
7) 6 # 4	8) 5 # 4	9) 1 # 2
10) 6 # 6		

Figure 6: Test for both groups.

On 4th February 1991, I asked a group of seven MEd students (all practising teachers or advisory teachers) to carry out this activity. To three of them, who I will call Andy, Jane and Julie, I gave the table to learn from (see Figure 4). To three others, who I will call Bob, Sally, and Stephen, I gave the rule to learn from (see Figure 5). They were told that they had three minutes to learn from the sheets. After this time, they would be tested (see Figure 6). To another, who I will call Anne, I gave a sheet asking her to observe the others throughout the activity.

Following a test of 10 questions, the results were:

(Table group)

Andy - 0

Jane - 0

Julie - 4

(Rule group)

Bob - 9

Sally - 7

Stephen - 8

(Observer)

Anne - 1 (she was guessing answers!)

There are a number of points that came out of this simple activity. Ignoring Anne for the moment, I gave all six of the teachers the same test. The desired outcome was for each person to be able to perform such operations. There was a marked difference in the performance of the two groups, with the Table group having less success. This group commented after the activity that the sheet had appeared daunting when they first looked at it and remained so at the end of the three minutes. They said the numbers in the table appeared quite arbitrary and this made it difficult for them to remember. Essentially, they were being asked to memorise arbitrary facts.

The Rule group were relatively successful. They commented that they did not find the sheet particularly daunting but they did need to have some prior knowledge as to what a prime was. Essentially, they were being asked to memorise an arbitrary rule.

Both groups were being asked to memorise, but the Rule group had to memorise relatively little compared to the Table group. The Rule group had acquired a rule they could apply to a number of different situations. They could answer 8 # 9 just as easily as 2 # 3. The Table group, however, could only tell the answers from 1 # 1 to 6 # 6. They could not apply their knowledge to other situations. It also appeared more difficult for them to acquire that knowledge. Anne's observations seem to bear this out:

In the Table group, Anne said that Julie appeared to get fed up after the first minute. Jane appeared to be working intensely and was looking quite awkward after the three minutes. Andy was starring at his sheet throughout the three minutes.

In the Rule group, Anne observed Bob giving the appearance of having finished after one minute. Sally appeared to have finished after two minutes and was taking down notes of the activity. Stephen appeared to be working throughout.

Memory is an expensive way to ask people to learn. If the underlying structure can be made explicit, then a learner can be creative within that structure and generate their knowledge rather than being asked to memorise it.

{Memory is expensive}

{Generate knowledge rather than memorise it}

We have successfully remembered so many things that we can sometimes forget the fact that we have forgotten so many more things. Along with the request that someone remembers something, comes the opportunity for them to forget it. To help enter into what is required in order to remember something which is arbitrary, try the following exercise:

Remember the word *pimolitel*

That's it, just read it once. Now remember it. Notice the relatively large amount of energy you use repeating it to yourself, trying to picture it,... Have you looked back yet? How long will you remember it for? Will you still be able to recall it tonight? Next week? In a year's time? How many times have you looked back already? Try not to look back from now on. Observe over (a) the next few minutes, (b) the rest of the day, (c) the following day, how much effort you put into trying to remember it and how successful you are. This is only one single word. I have helped you a little by providing a visual image of the word contained within a box, and using an italic type-face to stress it. I could have simply included the word within a normal sentence of text. Yet, still observe how much effort it is taking to try to remember it. Using memory is expensive in terms of our personal effort and time. It is also questionable as to how successful it is.

There is nothing else to do but use memory in order to know the above word. It is an arbitrary word and, although everyone is perfectly capable of generating new words, the chances that you would generate this particular word is minute. Thus, if it were desirable that you know this word, then memory is the only vehicle through which it can be achieved.

Gattegno states:

> We can further see that memory should be relegated to a limited area in education - that it should be used only for that which we cannot invent. (1986, p126)

It seems sensible, given the cost of memory, to relegate the use of it. However, I would prefer to substitute the last sentence with *that it should be used only for that which is arbitrary*.

The telephone numbers consisted of the first two digits which were arbitrary, and the remaining digits which were not. This is why I gave the students what was arbitrary, and asked them to complete the number. They discovered that it was more economic not to use memory in order to know the last part of each number.

{Provide that which is arbitrary}

If I had not given them the start of the number but asked them to tell me which telephone number was fifth in the list, for example, then it would have been arbitrary as to which number was fifth in the list. Thus, memory would have been required.

{The arbitrary requires memory in order for it to be known}

4.2.4 The Ent

In 1989, I read the following extract from Tolkien's *Lord of the Rings* to a class of 12 year olds:

> "The wind's changing," said Merry. "It's turned east again. It feels cool up here."
>
> "Yes," said Pippin; "I'm afraid this is only a passing gleam, and it will all go grey again. What a pity! This shaggy old forest looked so different in the sunlight. I almost felt I liked the place."

> "Almost felt you liked the Forest! That's good! That's uncommonly kind of you," said a strange voice. "Turn round and let me have a look at your faces. I almost feel that I dislike you both, but do not let us be hasty. Turn round!" A large knob-knuckled hand was laid on each of their shoulders, and they were twisted round, gently but irresistibly; then two great arms lifted them up.
>
> They found that they were looking at a most extraordinary face. It belonged to a large Man-like, almost Troll-like, figure, at least fourteen foot high, very sturdy, with a tall head, and hardly any neck. Whether it was clad in stuff like green and grey bark, or whether that was its hide, was difficult to say. At any rate the arms, at a short distance from the trunk, were not wrinkled, but covered with a brown smooth skin. The large feet have seven toes on each. The lower part of the long face was covered with a sweeping grey beard, bushy, almost twiggy at the roots, thin and mossy at the ends. But at the moment the hobbits noticed little but the eyes. These deep eyes were now surveying them, slow and solemn, but very penetrating. They were brown, shot with a green light. Often afterwards Pippin tried to describe his first impression of them.
>
> "One felt as if there was an enormous well behind them, filling up with ages of memory and long, slow, steady thinking; but their surface was sparkling with the present: like sun shimmering on the outer leaves of a vast tree, or on the ripples of a very deep lake. I don't know, but it felt as if something that grew in the ground - asleep, you might say, or just feeling itself as something between roof-tip and leaf-tip, between deep earth and sky had suddenly waked up, and was considering you with the same slow care that it had given its own inside affairs for endless years."...
>
> Pippin, though still amazed, no longer felt afraid. Under those eyes he felt a curious suspense, but not fear. "Please," he said, "who are you? And what are you?" (1972, p484-485)

This is the story of the hobbits' first encounter with an Ent. I asked the students to draw a picture of this Ent. Some of the drawings I received are shown in Figure 7.

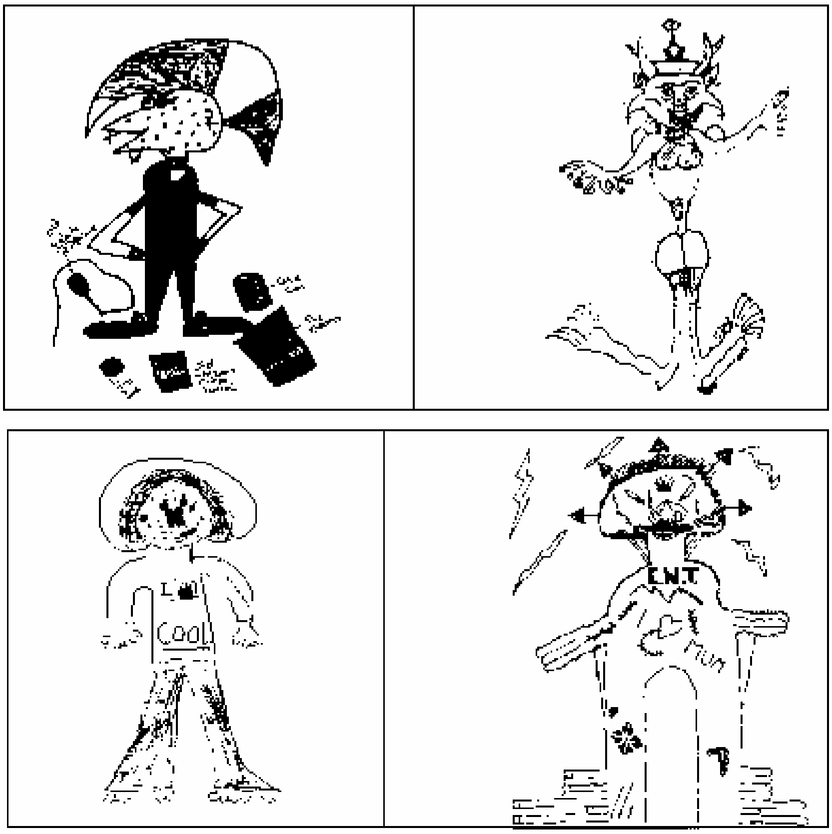

Figure 7: Students' drawings of an Ent.

Reflections on The Ent

When I work with images, I am working with people. Their images are unique to them as their previous experiences are brought to bear in creating their image. Their previous experiences have not been the same as mine or yours or anyone else's. They are unique, as no-one has led the same life as anyone else. In this description, Tolkien did not mention skateboards, earrings, bones, cigarettes or punk hair styles which appeared in some of the drawings.

{Images are unique}

These drawings are not the same as each other, even though each student heard the same story and description of the Ent. The students were creative in that they acted on their own initiative to introduce ideas and images from their past experience and combine them with the images gained from listening to the story. The way in which they were combined was of their own choosing. Thus, I am aware that students are creative. This is not something which has to be taught to them, merely encouraged.

{Students are already creative}

The fact that these experiences or images were used in their drawings indicates that the images each student has are available to be used by them. None of this group of 12 year olds physically went and looked during the lesson at a cigarette or earrings or all the other items that were involved in their drawings, so the images must have already been available to them. They had not memorised these images, in the sense of consciously deciding to commit them to memory. I suggest that no-one says *I must remember what a cigarette looks like*, but that images such as these are retained rather than memorised. Gattegno says that *Children are retaining systems, not memorizing tapes* (1973, p127).

The difference between my use of memorising and retaining is to do with whether there is a conscious effort used at the time in order to hope that something will be available later. A personal price in terms of effort is not required to be used for images to be available. That is not to say that all images you have in your life are available to you (although a surprising amount might be), but that the ones that are have not required any conscious effort on your part to remember them.

A friend of mine, Steve, was at his girl friend, Gail's, house when he wanted to make a telephone call. It was to someone whom he telephoned quite regularly, yet he could not remember the number. He realised it was due to the fact that Gail's telephone had a dial on it whereas Steve was used to using a push-button phone. Whilst Steve was trying to remember, Gail drew the buttons, in their correct position for a push-button phone, on a piece of paper and stuck the drawing on the wall next to her telephone. Steve immediately pressed his finger on the appropriate 'buttons' as if he was telephoning his friend. Then, he observed which 'buttons' his fingers had pressed and made note of the numbers.

There is more than one way to gain access to a particular piece of information. One way requires the use of memory and the ability to remember, and another way, the use of images and the ability to recall. As far as Steve was concerned, even though he telephoned this number regularly, he knew there was no need for him to remember the number, the image was available to him to recall whenever he required it. Having recalled the movements his hand made, both Gail and Steve had an association with the relative position of his finger and the numbers from 0 to 9.

{Images are retained and recalled, rather than memorised and remembered}

The use of images was more economical for Steve. I can make this statement since he did not have the image before the number. When he was first informed of his friend's number, he knew it as a number and not as a movement of his hand. Thus knowledge of the number came first. I argue that he would not choose to forget the number and learn the image of the movement if it was more difficult for him and required more effort. Quite the contrary, he chose not to put the effort into remembering the number since it required less effort to stay with the image which was formed without conscious effort through the practice of frequent telephone calls. I do not claim that this choice was made at a conscious level, nevertheless, I claim it was a choice in that he ended up doing one thing rather than another and that it was not arbitrary. As humans we are economical about the time and effort required to do many things in our life. We choose to keep the toothpaste in the bathroom and not in a cupboard in the lounge. If we began doing the latter, we would soon change once we became aware of the relatively large amount of time and effort needed to brush our teeth each day compared to keeping the toothbrush in the bathroom.

Images are powerful in the sense that they are effective (hence your ability to recall) and efficient in terms of the little effort required from you to recall them (this effort being so minimal that you might say *they came to me*).

> Imagine walking from your house to the nearest shop. If you are like me, with a shop across the road from your house, you may take this walk slowly; if your shop is some distance away, then you may speed it up. As you make your walk, look either side of you and consider what you see, both in the foreground and in the background.

Unless you have powers of which I am not aware, you did not know, prior to reading the above, I was going to ask this and so prepare by making this walk and committing it to memory. These are images you have retained and are now available to you.

Images do not have to conform to the laws of physics, or be confined only to those things which you have seen before. You can combine different images you have seen to produce something you have never seen:

> Imagine odd plants that have never existed.
>
> Imagine taking someone's nose, pulling it until it is very long, and then twisting it round like a corkscrew.
>
> Imagine floating up from where you are at this moment, looking down on your surroundings.
>
> Imagine doing some aerobatics.

Images do not have to be visual:

> Without moving: Stroke a cat's fur... taste some chocolate... smell fresh coffee... listen to a close friend of yours talk.

All this is available to you without any conscious effort on your part to remember.

{Images are available from all senses}

{Images require minimal effort}

4.2.5 Demonstration lesson

In 1984, I gave a demonstration lesson to some students with a number of teachers watching. In my attempt to explain what I wanted the students to do, I became quite flustered. The teachers observing reported later that they were quite confused. However, to my surprise, the students had begun work. As I wandered round, I found that each group knew exactly what they were doing, yet no group was doing the same as any other, and none was doing what I had intended.

Reflections on Demonstration lesson

> ...the questions the children were answering were frequently not the questions the experimenter had asked.
> Margaret Donaldson (1986, p49)

There were words I was saying, actions I was doing. There was a blackboard which had writing and diagrams on. All these things were part of my contribution to the start of this lesson. I had intentions, ideas, desires,.. some of which I wanted to convey to the students. Obviously, I was spectacularly unsuccessful at achieving this. I couldn't directly give the students what I wanted, in the sense of opening up the top of their heads and placing that information inside. I had to go through the mediums of words, writings, drawings and actions.

The students could not open the top of my head and take out the information they needed in order to start work. The only things available to gain information from were the words, writings, drawings and actions I offered. These were on offer for all the students, so why did they not all do the same? There must be some other dynamic involved in order for different students to make different decisions about what they were to do. Just because the same material was available to all, this does not mean that any of it was 'received'. In fact the notion of something being 'received' in this way does not seem appropriate. The students may all have 'received' similar rays of light and waves of sound (allowing for their different positions in the room and the different angles that their eyes and ears were pointing). However, giving meaning to these is clearly not a passive activity, and so the notion of 'receiving' seems inappropriate when considering meaning.

As von Glasersfeld has commented:

> Educators share the goal of generating knowledge in their students. However, from the epistemological perspective I have outlined, it appears that knowledge is not a transferable commodity and communication not a conveyance. (1987, p16)

There are different things that each student can choose to attend to. Attention is an active activity. Consider the following exercise:

> Close your eyes.
>
> Listen.
>
> Become aware of three sounds from different sources which you were not aware of before.

These sounds, or some of them, may have been available for you to 'hear' for some time. Yet you had your attention elsewhere and so had not become aware of them, even though you were receiving the waves of sound.

Thus, what each student attends to will be particular to that student. Furthermore, the meaning they give to what they are aware of, is dependent upon the personal experiences and meanings they have already developed in their life. Giving meaning to something is also a personal and active act.

The words, writings, drawings and actions in themselves are hollow. There is no meaning that comes with them. Gattegno has described words as hollow in a number of his seminars and von Glasersfeld states that the idea of words as containing meaning is misguided:

> This notion of words as containers in which the writer or speaker "conveys" meaning to readers or listeners is extraordinarily strong and seems so natural that we are reluctant to question it. Yet, it is a misguided notion. (1987, p6)

Also, St. Augustine was clear about the fact that words do not convey any meaning, and wrote the following in the 4th Century:

> ... we do not learn anything by means of the signs called words. For, as I have said, we learn the meaning of the word - that is, the signification that is hidden in the sound - only after the reality itself which is signified has been recognized, rather than perceive that reality by means of such signification. (1950, p174)

In order to develop some meaning with what I have offered, each student will have to be active with the material which is on offer.

I describe this situation in terms of a *Neutral Zone* - a zone in which I have placed a number of sounds and images to which each student can then choose to attend. It is a zone in which offerings are placed. The material with which each person can potentially work is a subset of the offerings in the zone, and is dependent upon the attention of that person in time. Some offerings may be available <u>over</u> a period of time, such as the visual sentences and drawings on a blackboard. Others are available <u>in</u> time, such as speech.

Thus, my image of the Neutral Zone is one in which the dynamic of how everyone relates to it is indicated by an arrow from each person to the Neutral Zone, and there are no arrows going the other way. The arrows represent human attention rather than physical entities such as photons of light or sound waves. So the arrows have a direction away from each person whether they be a listener or a speaker; watcher or a demonstrator. It is an image for what is personally involved for communication to take place.

The direction of the arrows has implications for the way in which some things are described:

This quote brings to mind...
It brought about...
It summoned in me...
This caused me to think that...
This is what prompted me to...

These imply an arrow flowing *to* me. However, I am proposing that all arrows go *away* from each person. This consolidates my feeling that the image of the Neutral Zone is worth pursuing as it offers a different perspective to that which has shaped parts of the English language.

{Offerings}

{Material}

{Neutral Zone}

4.2.6 Aircraft

Between 1982 and 1987, I taught in a school where my classroom overlooked pleasant fields with rolling hills in the distance. I was not the only one who enjoyed this area: the military did as well. The air force considered it ideal territory to practise low-level flying. I became used to having a class quite involved in their work when there would be a shriek of sound, closely followed by the somewhat less deafening clatter of falling chairs as some of the class rushed to the window. I could not complain because my nose was usually the first to be squashed against the glass.

Reflections on Aircraft

Before, we were involved inside the classroom; after, some were involved with what was outside the classroom. This means that for some, their attention changed from one thing to an event that was not caused by them. Both before and after, the students were learning, but the focus shifted from one thing to another. So I learn that it is possible to cause some disturbance whereby some students may shift the focus of their attention.

{Affecting attention is possible}

The language here is difficult, due to the dynamic of the Neutral Zone; the noise of the aircraft is an offering placed into the Neutral Zone, and some students choose to attend to that offering. In writing this paragraph, I have been tempted to write sentences such as *the noise caused a shift in attention*. However, the noise did not physically get hold of childrens' heads and make them turn in the direction of the window. It was the students themselves who activated the muscles necessary to make such movements. A shift in attention can never be guaranteed. Not everyone got up out of their chair to look at the planes, some stayed and showed no visible signs of having even heard the noise. The students still had control of where they chose to put their attention. I cannot insist that someone shifts their attention, all I can do is offer the possibility. For some students, although they gave the outward appearance of being involved, their work may not have been a relevant challenge for them, consequently the noise of the planes offered potentially richer opportunities for their involvement. Others may have been so engaged in a challenge that it was worthwhile to remain with it, the planes being a possible distraction.

Just because a student is looking at some writing on a blackboard, does not mean that their attention is being put into what that writing means for them. A teacher might have drawn a diagram that they feel clearly illustrates a point they wish to make, and say to a student: *can't you see?* Of course they can see. They might see lines, angles, letters... but there are so many aspects of the drawing the student could be attending to even if their attention is indeed in their sight at the time. They might see that one line is thicker than the others, or that three of the lines are almost horizontal, or that some of the lines seem to form the letter F, or that if you turn it round it looks a bit like a dog,... Of course they can see, but of what they are aware is another matter. What aspect of this drawing is to be stressed and what is to be ignored?

{Affecting attention cannot be guaranteed}

4.2.7 Andy and Kimberley

On 22nd May 1986, I was working with a low ability year 9 class on an activity which involved taking three numbers, such as four, two and seven, and imagine walking four units forward, turning left 90 degrees, walking two units forward, turning left 90 degrees, walking seven units forward, turning left 90 degrees, then returning to the four again, repeating the progress. We had done this for a while in the playground and now the class were exploring what would happen with different sets of three numbers on paper.

One student, who I will call Andy, was exploring five, one and six and feels he was getting it wrong. I asked him to draw it whilst I watched. He drew it perfectly; the drawing being such that he returned to the beginning where the pattern would repeat itself. However, Andy did not stop at this point and continued to draw another line of length five even though it was directly on top of the first line he drew. In fact, he kept going on and on, continuing to draw lines with careful measurement even though he was repeating the pattern already drawn. This carried on for several minutes with no sign of him stopping. He appeared completely absorbed in the activity. Then he turned right instead of left which meant a departure from the established drawing which he had gone over approximately ten times now. I pointed out the fact that he had turned right. He looked and agreed with me but continued with his drawing from this incorrect position.

On 13th November 1992, I observed a girl, who I will call Kimberley, in a year 7 class as she was doing an activity from an SMP booklet. The task was to work out 54 divided by 2, using objects to represent the numbers of tens and units involved. She was using pens to represent a ten and a small piece of paper for a unit. I asked her how she was tackling this question. She explained that she had got five pens and four pieces of paper and divided them into two. She found that she had one pen left over so swopped it for ten units (pieces of paper) and shared those out. I asked her what her answer was and she said 66. I looked down and saw that there were some pieces of paper in the same pile as some pens.

I watched her do the next question, which was 54 divided by 3. She got out five pens and four pieces of paper. She split the pens up into three lots of one and said she would swop the two remaining pens for units. She did not have enough pieces of paper and so had to tear off some more pieces from a sheet. There was some difficulty in counting them due to the smallness of each piece and the lack of space on her desk. Eventually, these were added to the pile with the original four pieces of paper.

Then she started sharing out the pieces by putting one in turn next to each of the three pens. The order in which she went to the pens did not remain the same. There were times when Kimberley had to make sure some pieces stayed where they were because occasionally they would get stuck to her finger. When she finished she found that she had one piece of paper left over.

Kimberley seemed to think that this should not happen and so began counting how many there were in each pile. The first had seven and the next two had eight, so she took one away from each of the last two. She now had three remaining and gave one to each pile. Then Kimberley told me the answer was 18.

<u>Reflections on Andy and Kimberley</u>

I recall feeling surprised that Andy continued drawing over the same lines again and again. I kept thinking that he must stop soon but he kept on carefully measuring and drawing. In fact it was the care he was taking which indicated that the task of drawing was taking up most of his attention. In fact, so much of his attention was taken up with the mechanics of drawing that he appeared not to notice that he was drawing over the same lines, he appeared to be drawing as if there were no lines on the paper. A number of people in the same class had turned to the right rather than the left. This was not surprising since a turn to the right can appear to be a turn to the left unless the exercise book is rotated as turns are made. The fact that Andy had not made such a mistake for such a long time was indicative of the thought and care he was putting into his drawing. When he did make such an error, he was not aware that he had until I pointed it out to him, despite the fact that it meant drawing a line for the first time away from the pattern he had established in his book. This indicates to me that his attention was not with the pattern he was drawing over but with the technicalities of drawing the next line.

It could be that the task of drawing in itself was demanding for Andy; that it took up so much of his attention that there was none left for him to reflect on the results of his drawing at the same time. This indicates that the of amount of energy available to us in the form of attention is limited and finite. Ginsburg talks of whether students might abstract the notion of commutativity from performing several calculations over a period of time:

> Over the years, they work 6+7, 7+6, 7x6, 6x7, 7÷6, 6÷7. At first, the very act of calculation may be so difficult that they focus attention only on individual problems. They concentrate on getting the correct sum for say 6+7 and so they cannot notice that 7+6 yields the same result. Or it is so hard for them to do 7÷6 that they cannot see that 6÷7 gives a different answer. (1977, p164)

If a task is sufficiently difficult that it takes up most of a student's attention, then that attention is not available for the student to reflect on what is happening whilst they are still involved in the task.

{The amount of attention available to us is limited and finite}

Kimberley may have been in a similar situation in the sense that her activity involved the mechanics of dealing with small pieces of paper which took up a considerable amount of her time and attention. She could articulate well what she was doing and on listening to her explanation of the first problem of 52 divided by two, I was surprised to hear her finish by giving me an answer of 66. Her description of what she was doing and why she was doing it gave me the impression that this was someone who knew what they were doing. However, when I looked down at the piles she had on the table, I felt I knew what might have happened. She had swopped one pen for some pieces of paper and shared them out between the pile of pens and the pile of paper. This had changed two pens and two pieces of paper into six of each. Presumably, she had also mis-counted the number of pieces of paper she had swopped for the pen - counting eight instead of ten.

Even though Kimberley appeared to know what to do, the mechanics of doing it was difficult as I believe my description of the events indicates well. There is a danger that so much attention is taken up with the mechanics of a task that there is none left available to make any abstractions from the activity which might lead to mathematical awarenesses. One reason I had asked my year 9 class to do the activity they did was for the possible conjecturing and reasoning which might follow such drawings. With Andy, I realised that his challenge was not with this but was concerned with whether he could draw accurately what was asked of him. Kimberley was so caught up with the counting and sorting of several small pieces of paper that her attention was not with a desired mathematical structure where the pens and the pieces of paper are kept in separate columns. Attention could become available by stopping the activity and reflecting back, since attention would no longer be required to carry out the task itself. This would relate to 3.2.4 where I talked of doing first, followed by reflecting on that doing. With both Andy and Kimberley, I felt that they were still intensely involved in wanting to gain some mastery over the doing and so wanted to continue with the doing. Andy's response, to my comment that he had turned right instead of left, appeared to be sufficiently minimal and appeasing so that he would be able to continue with the drawing. The problem for me was that I felt at the time that there was nothing intrinsically important about the ability to draw lines on paper and keep turning left. Thus, I felt that the activity I presented Andy failed in the desired aim of being a tool to work on some mathematical activity of conjecturing and reasoning. Of course, it may well have satisfied more important needs for Andy.

Since the amount of attention a learner has is finite, it might make a considerable difference where a teacher asks someone to place their attention. Some materials may be designed to assist a learner in dealing with some mathematical task. However, there are times when the mechanics of dealing with those materials can take up so much of the learner's attention that there is none left for them to consider the mathematics which the materials were offered to help with.

{Where attention is placed is crucial to what learning may take place}

The materials are subordinate to the desired mathematics. Thus, Kimberley's attention was placed at a level below that of the mathematics involved in the question and the desired area of learning. I suggest that if someone's attention is placed at a level which is subordinate to a desired area of learning, then there will be little progress made in that desired area. Kimberley will continue to learn about counting small pieces of paper and putting them into piles, she will not develop her ability to deal with such divisions mentally or by a written algorithm; Andy will continue to learn about drawing accurate straight lines and turning left, he will not develop his ability to conjecture and reason.

I will develop this theme of levels of subordination in section 5.2.4.

4.2.8 Placing the point

On 12th September 1986, in a low ability year 8 class, the digits '427051' were written on the blackboard. We were engaged in an activity of putting a decimal point in a particular position in order to meet certain requirements: just less than fifty; a bit more than three hundred;... I asked that the decimal point should be placed so that the digit '1' should be worth one. A girl, who I will call Clare, thought that the point should go before the '1' and wrote '42705.1'. I asked her to say the number in words. Finding that she was able to do this, I asked her to say it again:

Clare: *Forty two thousand, seven hundred and five point one.*
DH: Say the words again but don't say anything before the 'five'.
Clare: *Five point one.*
DH: Now don't say the five.
Clare: *Point one.*
DH: Is that the same as one?
Clare: *No, it's less than one.*

Reflections on Placing the point

When Clare placed the decimal point to have '42705.1', I did not know what to do. The answer was wrong but there was little to be gained by me saying so. I wanted to find a way for her to become aware that it was not correct. I was not sure whether she would be able to say the number but, having discovered that she could, I realised that there was the possibility of an awareness coming through the language of how the number was said.

I wanted Clare to attend to the fact that she had said *point one* and not *one*. Thus I wanted her to become aware of what she had just said. A possible obstacle was that there were a number of other words said in the number-name which were not relevant to this awareness. Thus, I invited her not to say certain parts, which in turn leaves the part I wanted her attention to be with.

Acting as an editor is one way in which I can attempt to affect someone's attention. If I present to you the alphabet:

a b c d e f g h i j k l m n o p q r s t u v w x y z

And attempt to shift your attention onto some specific letters then I might do this by stressing them:

A b c d **E** f g h **I** j k l m n **O** p q r s t **U** v w x y z

Alternatively I might ignore the others:

a e i o u

Both of these are techniques for attempting to shift attention and are thus tools for a teacher to use. Stressing and ignoring offer the metaphor of teacher as amplifier and teacher as editor.

{Teacher as amplifier}

{Teacher as editor}

<u>4.2.9</u> <u>Tyson</u>

During a lesson with a group of 13 year olds, a student who I call Tyson, suddenly said to me *why is it that there is only one decimal point in numbers?* It was unexpected in the sense that decimals did not seem to be particularly relevant to the work we were doing, but, more significantly, I found myself in a situation where I was not sure what to do or what to say. I replied to Tyson that it was a very nice question and continued by giving some of the stories I had for myself. I wanted to see if he could experience some of the ambiguities that might exist and so secretly wrote on a piece of paper the following:

13.495.2

I covered up part of it so all that could be seen was:

495.2

I asked Tyson to say the number to me. He replied *four nine five point two*. Since I knew what I was going to do later, I asked him to say the number as I would. After a little hesitation, he said *four hundred and ninety-five point two*.

DH: Say that one (pointing to '9').
Tyson: *Nine.*
DH: Say the whole number again.
Tyson: *Four hundred and ninety-five point two.*
DH: Say it again but don't say these three (pointing at 4, 5 and 2).
Tyson: *Ninety.*

I asked about the other digits. Then I covered up a different part so that he could only see:

13.495

I asked him to say what each of these digits were. Finally, I revealed all the digits and asked *what now?*

Tyson agreed that there was a problem about this but I felt as though his original question still remained for him.

Reflections on Tyson

Tyson's question has remained with me also. I have found myself re-considering the way in which we interpret our structure for writing numbers. My initial (unsaid) response to Tyson's question was *When have you ever seen a number with more than one decimal point?* However, I recall images of signs at petrol stations, when prices were for gallons, of:

1.63.9

I think of watches that show the time as:

13:25:04

Dates are written as:

24.11.92

On a trip to France, I see prices marked as:

2.357,90

And telephone numbers written as:

48.95.26.46

The present section of this thesis is:

4.2.9

I begin to see a 'decimal point' as a separator, along with other possible symbols which might be used to separate digits.

Say the following out loud: £2.85

When I have asked people to say this, they have replied something similar to *two pounds, eighty five*. I have never heard anyone say something similar to *two pounds, eight five*. The fact that *two* is said as it is, and not *two hundred* or *twenty* or *two thousand* implies a decimal point where it is seen, immediately after the '2'. However, the fact the '85' is said as *eighty five* implies a decimal point after the '5'. Thus, within '£2.85', we have two decimal points - one seen and the other implied.

There are two aspects of this discussion I wish to comment on. The first is that I found myself questioning my understanding of an area of mathematics due to an event which took place in my classroom. I will say more about this questioning after I have explored my second point. This is that I have not considered 'place value' as such, but considered the way in which certain numbers are said. I have not found a single reference to 'place value' in Gattegno's books (1971, 1974, 1988) which deal with the development of number. Tahta says, of Gattegno's treatment of the teaching of number, *In this treatment, 'place value' becomes a totally unnecessary concept as far as teaching is concerned;* (1991, p236). I also question the notion that understanding the written numerals requires ideas of 'place value'. It may be more appropriate to approach an understanding of the written numerals through language.

I am aware that the re-questioning of mathematics I have done is as someone who is not so much a mathematician as a pedagogue. Thus, I look for the roots of an area of mathematics - pedagogical roots - by considering what is at the heart of *learning* this mathematics, and consequently what is at the heart of teaching it.

{Questioning my understanding of an area of mathematics as a pedagogue and not as a mathematician}

{What are the pedagogical roots of a piece of mathematics?}

4.3 Backwards and forwards

The significance of the events contained in this chapter are crystalised in the statements in parentheses. There are many such statements and I give below a summary of what the statements are in this chapter. These statements will feed into the continuing development of a theory of economy of learning and teaching mathematics which continues in later chapters. The number after each statement indicates the chapter and section which continues to develop the ideas behind the statement. For some statements there is more than one place where the idea is developed. Other statements are not explicitly considered, however I include them because they add to the overall sense of what economy is concerned with.

4.2.1 Richard

{The explanation trap} - 10.3

4.2.2 An area activity

{Economy is concerned with time and effort spent for the learning gained} - 11.3

{Movement allows all possibilities to be considered and properties noted which might otherwise be difficult within a few particular examples} - 6.3; 8.2

{The given examples can be positioned within the noted properties} - 8.2

4.2.3 Telephone numbers

{Memory is expensive} - 6.3; 6.4.2; 10.3

{Generate knowledge rather than memorise it} - 8.2

{Provide that which is arbitrary} - 9.3; 10.3

{The arbitrary requires memory in order for it to be known} - 6.3; 6.4.2

4.2.4 The Ent

{Images are unique}

{Students are already creative} - 6.3

{Images are retained and recalled, rather than memorised and remembered} - 6.4.2

{Images are available from all senses} - 6.3

{Images require minimal effort} - 6.3

4.2.5 Demonstration lesson

{Offerings} - 8.3; 9.2

{Material} - 8.3; 9.2

{Neutral Zone} - 8.3; 10.3

4.2.6 Aircraft

{Affecting attention is possible} - 8.3; 9.3

{Affecting attention cannot be guaranteed} - 9.3

4.2.7 Andy and Kimberley

{The amount of attention available to us is limited and finite} - 6.4.1

{Where attention is placed is crucial to what learning may take place} - 9.3

4.2.8 Placing the point

{Teacher as amplifier} - 8.3; 9.2

{Teacher as editor} - 8.3; 9.2

4.2.9 Tyson

{Questioning my understanding of an area of mathematics as a pedagogue and not as a mathematician} - 8.3

{What are the pedagogical roots of a piece of mathematics?} - 8.3

5 Significant events 3: outside classrooms

5.1 Introduction

I continue describing events which have been particularly influential to my study. Observations which contribute to pedagogy need not be based in classrooms. Learning is something we do as part of our daily lives as Holt has stated:

> ... we are *already* learners all our lives. Living is learning. It is impossible to be alive and conscious (and some would say unconscious) without constantly learning things. (1991a, p157)

In this chapter I consider events outside classrooms. I maintain the structure of giving a descriptive account of each encounter followed by my reflections after the event. As in the previous two chapters, I summarise in a short sentence, or even a single word, key factors which have pedagogic significance for me, which stem from my reflections. These are written in curly brackets - **{...}** - and will be picked up and explored further in later chapters. Section 5.3 will give a list of the summary statements from this chapter and indicate exactly where each statement will be developed in later chapters.

5.2 Outside classrooms

5.2.1 John Lennon

When I heard of John Lennon's death, I was living in a small bedsit in Bristol, lying in bed waking up to the radio. I can picture the room clearly from the perspective of lying in bed and recall the shock and disbelief of the news.

Reflection on John Lennon

ϑ−ϖ

Whenever someone talks of John Lennon's death, I recall the images I have of my small bedsit. It seems that with one thing comes another. A number of people talk of the shock they experienced with the news of John F. Kennedy's death, and describe exactly where they were when they heard the news. We have an ability of associating one thing with another, so that the existence of one thing can trigger recall of the other.

ϑ−ϖ

There are many examples of associations we have formed in our lives, including:

- pieces of music and certain memories;
- certain clothes with particular sports;
- smells with particular foods;
- written symbols with particular sounds (e.g. this very sentence!).

{Association}

ϑ−ϖ

A common thread amongst many of these is the fact that two things occur simultaneously:

- I heard the music at the same time as the memory I recalled;
- clothes are worn whilst the sport is being played;
- smells exist whilst I see and eat food;
- as a young child, I notice someone pointing to symbols whilst making particular sounds.

ϑ−ϖ

Thus, if it is desirable for one thing to become associated with another, then whether they occur at the same time may be a significant factor in whether the association is formed successfully or not. Simultaneity in time can provide a link between two things which may be concerned with different senses. For example, taste and smell. When I eat something, the act of bringing it to my mouth also brings it close to my nose, thus I experience the two sensations, with two different senses, at the same time. It is unlikely that someone who smells bacon, will think of the taste of strawberries. However, I suggest that if a new breed strawberries were produced which had the same taste of strawberries but had a smell of bacon, then such an association may become common.

ϑ–ϖ

One day, a friend of mine, Tom, invited a number of people round for tea. To our surprise, he had coloured the tea so that it was green, and the milk was coloured blue. A number of us talked afterwards of feeling reluctant to drink our tea, and that it didn't seem to taste 'usual'. This was so even though we accepted that the colourings would have made no noticeable change in the taste of the tea. This indicates to me that we established a strong association between the colour and taste of tea. This was strong enough for us to have difficulty in perceiving the taste staying the same whilst the colour had changed.

Without time providing the link between senses, additional work would be required from someone who is trying to establish an association between a sensation in one sense with a sensation in another. When considering the notion of economy in establishing such associations, this extra effort on the part of the learner can be avoided by using simultaneity.

The principle of simultaneity within time can be applied to associations formed within the same sense.

Look at the box below:

Take a few seconds before continuing to catch what else you have pictured as well as the box.

As you looked at the box, did you consider symbols inside it? An association has been forming between the box and symbols which appear within it. You have been developing this association whilst you have been reading this section of writing. This association may not be fully established at this juncture, however, a partial establishment is likely to have taken place with you picturing something within the box, the number of symbols inside and/or their vague shape. You have begun to establish this association without the need for me to explicitly ask you to form one.

{Simultaneity}

5.2.2 Gerry

One night I was in a pub with a friend, Gerry. At one point we were talking about our respective jobs and he asked me how I might teach trigonometry. I described the image of a dot moving anti-clockwise round the circumference of a circle and asked him to attend to the height of the dot. I said that the dot will always start on the extreme right point on the circumference, where its height is 0. As the dot moves round, at its highest point the height is 1, and at its lowest, it is -1. I asked through what angle the dot would turn from its starting position to reach a height of 0.5. He replied 45 degrees.

I asked him to imagine the triangle made from the radius going to the dot, the horizontal diameter and the vertical from the dot to this horizontal. When asked about the angles, he said they were 45, 90 and 45 degrees; and the length of the sides, he said the radius was 1 (this had come out earlier), the height 0.5 (as I had asked) and the horizontal was 0.5. Then, I gave him 4 matchsticks, saying each one was of length 0.5, and asked him to make that triangle (see Figure 8).

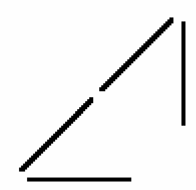

Figure 8: Gerry's attempt at making a matchstick right-angled triangle with sides 1, 0.5, and 0.5.

After a while he commented that such a triangle was not possible. I pushed him further and said that we know the radius is of length 1 and that the height must be 0.5, so could he ensure that this was so. He found himself pushing the radius back clockwise and as he did so stated that the angle must be less than 45 degrees (see Figure 9).

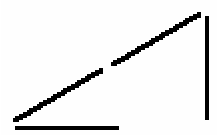

Figure 9: Modification of Gerry's triangle.

Reflections on Gerry

Initially, Gerry saw no problem with the angle being 45 degrees. The matches gave him the opportunity to see the consequences of his decision of having 45 degrees. It was only when he was able to see these consequences that he was in a position of re-evaluating that decision in the light of what he could see now. *Seeing the consequences of his decisions* is an extension of the point already made in 3.2.1 (Seeing the consequences of their actions).

{Seeing the consequences of their decisions}

From my perspective, as taking on the role of a teacher, I was only in a position to suggest the use of matches in the way I did because of what I perceived to be Gerry's image of the triangle. If Gerry had not informed me of the lengths and angles of his triangle, then I would not have been in a position to know whether the matches were appropriate. Thus, I need to find out what my student is thinking and of what they are aware. I was working with my awarenesses of Gerry's awarenesses.

{A teacher is in a position to make pedagogic decisions when they become aware of what their student is aware of}

{Working with awareness}

5.2.3 Xor

One afternoon Laurinda Brown introduced me to a computer program called Xor[3] . The basis of the program is moving a shield round a maze collecting masks as you go. The situation becomes rapidly complex with the introduction of waves, dots, fish, chickens, bombs, dolls and others, all having certain properties that can prevent you reaching a mask. The precise details of the program I will not describe here; what is relevant is that in order to be successful I must know how a fish, for example, behaves, since initially it is stationary and I do not know how it will behave or how I might be able to use it.

[3] Published by Logotron Ltd, Ryman House, 59 Markham street, London.

As I began a maze, I was faced with the situation shown in Figure 10. All I controlled was the movement of the shields, up, down, left or right. I started with the lion one and moved to collect two masks. Then I pushed into the fish, which fell to the bottom (Figure 11). From this, I learned that this is the way the fish behaved.

Later, I moved the striped shield into the chicken and found out that chickens drift to the left when they are not blocked (Figure 12). This information was necessary as I progressed through the game in order to collect further masks.

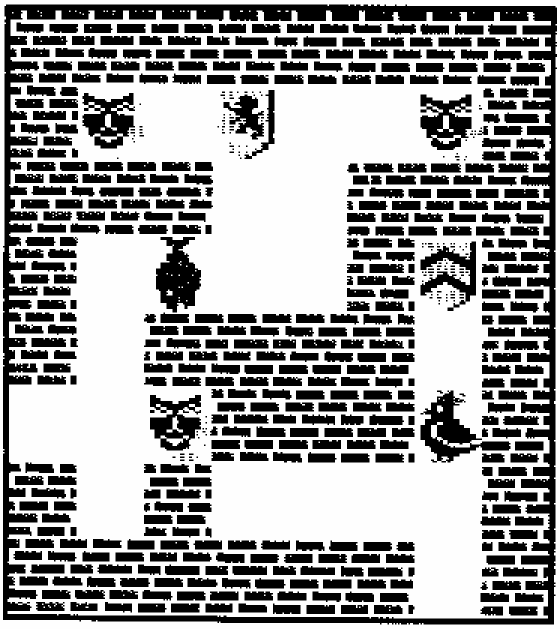

Figure 10: Starting position in Xor maze.

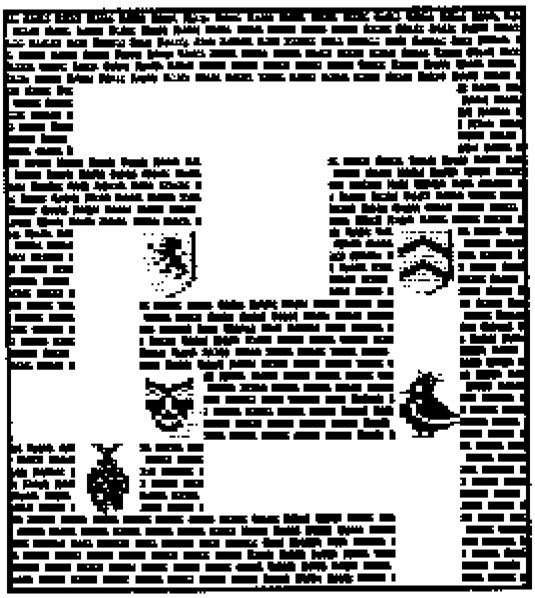

Figure 11: Xor maze after fish has fallen.

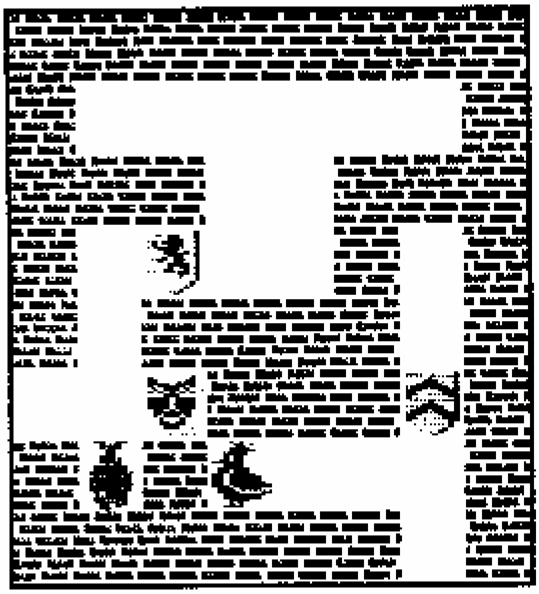

Figure 12: Xor maze after chicken has drifted to the left.

Reflection on Xor

The situation had been carefully created so that, if I was prepared to engage in this activity, I had no choice but to learn what fishes do. After collecting the first two masks, there was nowhere else I could move my shield except into the fish. Thus, the situation was such that I was forced into doing this. I had no idea what might come of this but I had no alternative. As a consequence, I learned how fish behaved. Likewise, when operating the striped shield, I was offered no other choice but to push the chicken down. From this I learned how chickens behaved in this game. Thus my awareness of the behaviour of fish and chickens was forced through the structure of the game. I described this process as *Forcing Awareness* in Hewitt (1989b) using the phrase from Gattegno. Gattegno gives two meanings to this phrase:

> Awareness is neither automatic nor constant. If (sic) fact, most people go through life only aware of a very small fraction of what could have struck them had they been uniformly and constantly watchful.
>
> There are, therefore, two meanings to "forcing awareness." One is concerned with what we do to ourselves, and the other with what can be done to us so that we become aware of what has escaped us, or might escape us. This, in turn, can become a matter of doing it to others, in particular if we become educators. (1987, p210)

{Forcing awareness}

It is the second of these meanings which I refer to in describing Xor. The devisor of the software was the person who made decisions which meant that I was forced to become aware of how fish and chickens behave. Gattegno (1987) gives examples of forcing awareness by text and with reference to some computer software, but he does not expand on the pedagogic techniques involved in forcing awareness in others. I will expand on one such technique.

The software guaranteed that I learned about fish and chickens quickly. If I had had wider choice, I might not have discovered these properties for some time. I still had control over the situation, I was the one who moved the shield and I was the one who decided to push the fish. Therefore, what I learned was a consequence of my own actions not someone else's. Control was never taken away from me, it was only the limits within which I was able to explore that had been imposed. The end effect is a subtle difference in how I become informed about fish and chickens. The attempt was not to try to do my learning for me, by telling me, but by providing a situation in which I would definitely encounter their behaviour in an activity which I had full control over.

The behaviour of fish and chickens in this game was arbitrary, thus I needed to be informed. If I had been informed by being told, via an explanation on the screen for example, I would have been a passive recipient of some knowledge which I would then have to remember. As it turned out, I was an active participant that experienced the same knowledge in the course of my own actions.

{Experiencing knowledge of the arbitrary in the course of my actions}

There is a difference in where my attention lay in each case: in the former my attention would have been with the piece of knowledge to be learned and taken away from the game for that period of time; with the latter it was not, I was attending to the maze and trying to make my shield explore other parts of it. Thus, I gained knowledge of the arbitrary without the need to take my attention away from the game.

{Attention remains with the activity rather than the need to remember arbitrary knowledge}

After gaining the experience of a fish falling, I carried on through the maze and encountered a room where there was a mask for me to collect. However, this time there were fish around. Now I had to use my existing knowledge, along with my recent experience of fish, in order to be successful at obtaining the mask. I was practising what I knew about fish, but the practice was not practice for its own sake, it was required in order to achieve something else, in this case collecting a particular mask. I did not learn about fishes because I had one experience of how they behaved, I learned because I was continually having to make use of this knowledge in achieving higher order tasks working out how I can collect particular masks. My attention was with these higher order tasks, not the practising of knowing about fishes. As a consequence, I may have learned without realising that I had, because my attention had been elsewhere.

{I am not always aware of the learning taking place within me}

My past experience (of fish) was being integrated in the present by a task where my attention was concerned with a desired future (that of gaining a particular mask). Gattegno talks of integration as *the process by which the future affects that which already exists* (1973, p50). My experience of fish already exists and this is affected in the sense that it does not remain a past memory, but becomes knowledge used in the present, only because my attention is concerned with a future which requires the subordination of this experience of fish. Thus, integration of something is linked to the need for that something to be subordinated to another challenge. Gattegno talks of the law of integration and subordination, saying: *It produces a functioning past ready to yield to the descent of the future* (1987, p221).

Let me consider some crude models of how I might integrate some knowledge. Suppose I have understanding of something which I will call A and wish to learn about B, then one way is to have someone explain B to me and give me practice at this, in order that I might integrate B through practising it. My attention during this time will be with B.

$$A \xrightarrow[\text{Receive Explanation}]{} B \quad \text{Practice}$$

This model will require me to receive knowledge and use memory in order to learn, aided with practice. This is, of course, a frequently used teaching style.

Another model might make use of an investigation where I use what I already know (A) which might lead me to discover B myself. An argument here is that, in exploring, I have become familiar with the area in which B is contained and, consequently, make B my own, since I have discovered it through my own hard work and investigation. Here, my attention is with the area round B.

$$A \xrightarrow[\text{Investigate}]{} B \quad \text{Discover}$$

Here, routine practice of B might not be as common since the emphasis is on discovering B rather than using it. This is a teaching style increasingly used, particularly in the wake of GCSE coursework. Yet the model I have described through the example of Xor is quite different to either of these. In this case, I start work on C, which is a clear task understandable from A but needs me to use B, so that awareness is forced.

A — B	⟶	C
Made aware of	Work at C using A and B	Gain some success

My knowledge of C may or may not be sound at the end of this activity. It may be that I am still struggling with gaining reasonable command, have a clear understanding now but might forget in the future without practice, or have total mastery. However, in each of these cases, I am likely to have integrated B. My attention remains with C throughout, by being concerned with the future: the desire to gain some control over C. This future drives my use of B as it is required in order for me to achieve my desired control over C. In fact, my integration of B will be a partial consequence of the fact that my attention is <u>not</u> with B, but with C. Thus, with this model of forcing awareness, my attention is not where the real learning (integration) is taking place. It is at a higher level in the sense that B is subordinated to work at C, so I describe C to be at a higher level than B on this occasion. It may be possible with a different activity, to subordinate C whilst working at B. As I have mentioned in 3.2.5, a pedagogic hierarchy will depend upon approaches taken.

{Attention is placed at a higher level to that which is to be learned}

<u>5.2.4</u> <u>Sailing</u>

During August 1987, whilst on holiday, I decided to take an hour's sailing lesson in a small boat. The man who was my teacher slowly introduced me to some of the dynamics involved in sailing. At one point I recall feeling as though there were too many aspects to consider and, as a consequence, no longer felt in control. At that moment, I was told to look at the sail. I was to concentrate on keeping it at such an angle that it only just did not flap. After a period of time, I noticed I was successfully steering the boat with less effort expended on my part than before my teacher had asked me to shift my attention onto the sail.

Reflections on Sailing

Before this shift of attention, I was concentrating on manipulating the rudder, ropes and my body position. These I considered as separate in the sense that I attended to one and then another. I was not confident with the effects each had on the overall steering of the boat, consequently, my attention was fully taken up with attending to just one of them. I felt as if I could not consider more than one at a time. This was when I no longer felt in control.

The shift in attention to the sail, brought my attention to something which was affected by each of the separate factors of rudder, ropes and body position. In this sense, my attention was with something which I will refer to as of a higher order, in that the others are subordinate to it.

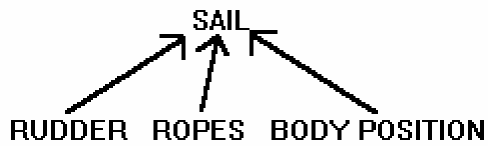

Figure 13: Diagram representing some dynamics of subordination in sailing.

I could not directly affect the sail; I had no direct physical contact with the sail. However, the ropes were attached to the boom which affected the plane the sail was in relative to the direction of the boat; the rudder affected the direction of the boat and consequently the angle in the horizontal plane the sail was in relative to the direction of the wind; and my body position affected the angle which the boat was leaning in the water and consequently the vertical plane the sail was in relative to the direction of the wind.

The shift in my attention, onto the sail, meant I was attending to the consequences of my actions in the lower level (of the rudder, ropes and body position), rather than the individual activities themselves. Thus the higher level contains the consequences of actions in the lower level. For the reason that one level is subordinate to the demands of the next level, I will call these levels *Subordinate levels*.

{Subordinate levels}

The shift in attention meant I was no longer attempting to attend to three dynamics which had little effect on each other. Now, I was attending to just one - the dynamic of the wind in the sail. This helped me feel less frantic and, as a consequence, I used less personal energy. At the same time, I was handling a number of different skills simultaneously which had previously caused me difficulty in their isolation.

I am not saying that my attention was never with the rudder from then on, but my main focus was with the sail. I began to feel as if I was directly moving the angle of the sail although, in reality, I was moving the rudder which, in turn, moved the angle of the sail relative to the wind. However, I began not to be aware that it was the rudder I was moving. The action at this subordinate level became subordinate to my desire for a certain change in the higher subordinate level. This was equally true of the use of ropes and my body position. My attention was with the desired effect rather than the tools I was using to achieve that effect. This is in contrast to the two cases in 4.2.7 where both Andy's and Kimberley's attention was placed at a lower subordinate level to a desired area of learning.

Write down the correct spelling of *suboardinat*

The task I have set you asks you to place your attention in the spelling of a word and to write that down. I suggest that your attention was not with the physical movements required to achieve the writing of that word. You did not attend to the fact that, in order to write down the letter 'b', your pen needs to start relatively high and come down before doing a loop to the bottom right of this line. (You can change the instructions accordingly, if you write your 'b' a different way.) Furthermore, you did not consider the various muscles which needed to be activated in order to achieve these movements. In fact the more skilled you are at something, the less aware you are of the tasks which are subordinate to its achievement.

The difference between this example and my sailing story is that, I was not skilled at manipulating the angle of the sail. I was in a learning situation. I was attempting to gain control over steering a boat, which involved my skill of using the rudder, ropes and altering my body position. My skill in these areas improved considerably when my attention was raised to a higher subordinate level. Thus, learning took place in an subordinate level below the level where my attention was placed. This is a re-wording of some of my comments in 5.2.3 above.

{Attention at a higher subordinate level can drive learning at the levels below}

5.2.5 Computer games

Whilst I was at The Grange School, students were allowed to play games on the computers in the mathematics department after 5pm. I observed them playing on several occasions. One 16 year old, who I will call Shaun, was playing a racing car game. When he began playing, he was struggling to stay on the track and drove slowly in order to do so. When he began to be able to stay on the track, he tried continually to go round a particular corner at full throttle in fifth gear, coming off the track each time. Going round the rest of the track appeared to be of little interest to him.

Another 16 year old, who I will call David, was involved in a game where a number of spacecraft had to be shot down. As is usual with such games, the complexity increased as time went on. There was one particular type of large craft that caused difficulty. When it was shot, it would split up into several smaller craft which began heading straight towards the craft David was controlling. If any of these craft reached him, then his craft would be destroyed. David resolved this difficulty by using a bomb immediately after he had hit the original large craft. If he hesitated too long, the craft would have split up into the smaller craft. Whenever he played this game, he would be ready to use the bomb at this crucial moment. Over a number of these game sessions, David became very good at this game, achieving scores far higher than anyone else.

One evening, I noticed that he was no longer using the bomb but was attempting to shoot down each of the smaller craft which had been created. This increased the complexity immensely. After a few days, David began shooting all of the larger craft, one after the other, which meant that he had about twenty of the smaller craft coming from various directions trying to get him. Now, he appeared to play the game purely to get in this position. He had developed manoeuvring techniques that made these smaller craft form a chain following him, then he would suddenly turn round and be able to shoot several at once. As a consequence of making these choices, one of the smaller craft usually destroyed David's craft and so his scores were considerably lower than before.

Reflections on Computer games

I had my perceptions of what the object of each of these games was. The racing game was about getting round the track as fast as possible and beating the other cars in a race. The spacecraft game was about shooting other craft without having your craft destroyed, and collecting as many points as possible. Both Shaun and David appeared to start out attempting to succeed in these objectives. However, there came a point in time when they both became involved in attempting to do something which ensured they failed in these objectives. With Shaun, his attempts at going round a corner in fifth gear meant he always crashed. With David, his decision not to use a bomb at a crucial moment meant the complexity increased to such an extend that his craft was destroyed and his scores became lower than they had previously been.

As a consequence, I stopped viewing these games as activities with particular objectives, but as worlds in which people can define their own challenges.

{Challenges}

For a while, there had been a competitive environment in the room with people talking about the scores they got. Such talk failed to inform me or others about any of the detail of what was happening in the games. From the talk, I would not know what each game was concerned with, only that there was a related score. However, after they became more familiar with each game, talk became less concerned with scores and more concerned with the detail of each game. For example, it became a topic of conversation whether it was possible to go round this particular corner with full throttle in fifth gear. These conversations included discussions about how you might go into the previous corner and what angle you come out and approach the next. David developed manoeuvring techniques in his game which others became interested in and attempted to develop similar techniques. These conversations and challenges were directly concerned with detailed and precise skills. The atmosphere became less competitive and more one of sharing and assisting others to develop their skills.

{Personal, rather than competitive, development}

I relate this to an experience (see Hewitt (1987a)) I had when I began teaching at Priory school in January 1987. My year 11 class were working towards an examination related to an individual learning scheme. Whenever a student talked to me about what they were doing, they referred to the numbers or names of the booklets they worked from. I would have no idea, from what was said, about what mathematics that student had been involved in. The students talked to each other about which card they were on rather than the mathematics which might be involved in working on the card. To some extent, the students had an opinion of their ability in mathematics, relative to the rest of the class, by which card they were on. The desire of many students was to get on to the next card rather than become skilful within the mathematics involved in a particular card. The way in which the scheme was assessed assisted with this desire.

The booklets were prescriptive in the sense that the students were told what to do, the questions were closed in the sense that there was a precise answer expected. The amount of choice for the student was limited. The computer games offered a situation for someone to explore; the possible variations in what might happen were infinite; at all stages the student was making choices within the restrictions of the game. The students were able, due to them having choices, to re-define what the object of their activity will be. Thus, they could choose to attend to a particular part of the game and ignore the score. This may not be so likely to happen with someone using booklets with limited choice and an external pressure of assessment which places emphasis on particular answers and how many booklets have been completed 'correctly'.

With both Shaun and David, they appeared to define their new challenge at a time when they were beginning to gain control over their previous challenge: Shaun was beginning to be able to get round the track without going off; David was able to play his game successfully and achieve very high scores. Neither appeared content with just continuing to practise the skill they had developed. Rather than continue to practise something they could do, they both created a new challenge that they could not do. There appeared to be a desire for a challenge rather than contentment with practising what they could already do.

Their new challenges were at a higher subordinate level: Shaun's challenge to get round a particular corner at high speed subordinated the skills of steering he was becoming successful at; David's ability to avoid having his craft destroyed and his ability to shoot other craft were subordinated to the new complexity he created. As I continue to watch people playing computer games I am struck by the speed of their learning and skill within the game, and the fact that they continually choose new challenges which are on a subordinate level above that which they are presently operating on.

{Deliberate creation of challenges on a higher subordinate level, rather than successful practice of what can already be done}

5.3 Backwards and forwards

The significance of the events contained in this chapter are crystalised in the statements in parentheses. There are many such statements and I give below a summary of what the statements are in this chapter. These statements will feed into the continuing development of a theory of economy of learning and teaching mathematics which continues in the coming chapters. The number after each statement indicates the chapter and section which continues to develop the ideas behind the statement. For some statements there is more than one place where the idea is developed. Other statements are not explicitly considered, however I include them because they add to the overall sense of what economy is concerned with.

5.2.1 John Lennon

{Association} - 6.3; 6.4.1; 9.3

{Simultaneity} - 9.3

5.2.2 Gerry

{Seeing the consequences of their decisions} - 10.3

{A teacher is in a position to make pedagogic decisions when they become aware of what their student is aware of} - 8.3; 9.2

{Working with awareness} - 9.2

5.2.3 Xor

{Forcing awareness} - 9.3

{Experiencing knowledge of the arbitrary in the course of my actions} - 10.2.11

{Attention remains with the activity rather than the need to remember arbitrary knowledge} - 10.3

{I am not always aware of the learning taking place within me}

{Attention is placed at a higher level to that which is to be learned} - 9.2

5.2.4 Sailing

{Subordinate levels} - 6.2

{Attention at a higher subordinate level can drive learning at the levels below} - 6.2

5.2.5 Computer games

{Challenges}

{Personal, rather than competitive, development}

{Deliberate creation of challenges on a higher subordinate level, rather than successful practice of what can already be done} - 6.2

6 Powers of children

6.1 Introduction

In this chapter I will consider the powers and abilities of which children make use whilst they go about their impressive learning as young children. Reference will be made to several of the summary statements of the previous three chapters and these will help develop a picture of what powers children use in their early learning. When I refer to the statements, they will appear as footnotes with the reference of the section from which they came.

Dewey comments:

> The teacher is a guide and director; he steers the boat, but the energy that propels it must come from those who are learning. The more the teacher is aware of the past experiences of students, of their hopes, desires, chief interests, the better will he understand the forces at work that need to be directed and utilized for the formation of reflective habits. The number and quality of these factors vary from person to person. They cannot therefore be categorically enumerated in a book. But there are some tendencies and forces that operate in every normal individual, forces that must be appealed to and utilized if the best methods for the development of good habits of thought are to be employed. (1933, p36)

Whilst being aware of the individuality of each child with respect to the experiences they gain in their life and their learning, I agree with Dewey that there are commonalities across all individuals, and it is those commonalities of powers employed in early learning that I seek to identify within this chapter. Gattegno called for a similar use of children's strengths:

> In any effort to educate, we need to be aware of the skills and capacities of the learners. What we offer to be learned should be fitted to these skills and capacities. It perhaps is fair to say that no educational system has yet grasped the strengths that all young people bring with them to the classroom. (1986, p124)

After identifying some of children's skills and capacities in this chapter, I will see how our teaching can be fitted to them by developing some principles which can assist a teacher to 'appeal to and utilize' children's powers within mathematics classrooms.

I start with a section recognising the achievements of children and identifying the way in which they have gained these achievements. I continue by considering the powers Gattegno (1971) identified in his book *What We Owe Children. The Subordination of Teaching to Learning*. Following this, I select my own list of powers of children, which are used in everyday activity as a human being rather than being powers especially related to mathematical activity, and present a structure of how they relate to each other. The next section describes the link between the powers of children and fundamental notions of mathematics called essences. This chapter finishes with a summary which also indicates how the next chapter analyses sections of some text books in terms of children's powers.

6.2 The achievements of children

When babies are born, there are certain things they can do: they can move their limbs; they can make sounds; they can open and close their eye lids; their heart beats;... For the first time since conception, the baby meets a new external world that contains new demands and offers new possibilities and opportunities. There are immediate demands, for example, breathing. There are immediate possibilities, for example, increased space for the movement of limbs.

The changes brought about at birth are not always positive. Babies lose their connection with their mother - the umbilical chord. They no longer have the blood and nourishment that that brought. This loss has the consequence that, if nourishment is to be brought into their body, other means are required.

I was in this situation. So were you. Yet neither of us are able to recall this and describe how we dealt with these problems. All we can say is that we were successful; our proof being that we are alive today. Some day, we will meet new problems, and if we do not succeed in averting or solving them, we will die. Although we may not be able to re-enter into our early years, there are still some conclusions we can come to by making simple observations.

We can marvel at what new born babies can do. For the moment I will attend to some of the things they cannot do. New born babies cannot:

- talk in an established language;
- walk;
- control their bowels;
- focus their eyes;
- count;
- ...

If you disagree with any item on my list, then I will be happy to cross it out. Perhaps you would be able to substitute something in its place. New born babies are unable to do so many things. Yet, within three years, most babies will be able to do all of the things I have mentioned.

I asked two people to do the following exercise:

> If you have been privileged to observe babies and young children, then write down five other things that children of three can do that newly born babies cannot do.

Their lists included:

- tell which direction a sound is coming from;
- press 'shift and break' on a BBC computer;
- feed themselves;
- hold objects;
- build things;
- manipulate people;
- be reasoned with;
- recount stories.

If a number of people do this exercise and their lists are combined, we can begin to appreciate the number of achievements made in three years. I can know these as achievements of a three year old even though I may not have known them since their birth. If this particular child was able to do these things at birth, then the daily press would have announced the fact to the world. In this way, I can observe two different people, one new born and the other three years old, and make assumptions about the achievements of that three year old by comparing the two.

Many of these achievements are learned so well that they will never have to be learned again. Such achievements Gattegno has called the *functionings of children*:

> What are the functionings of children? They could all become known to us because we all have been children. We have used these functionings, we have them in us, and we did with each such a good job, mastering it so successfully, that we do not have to do it again (except in an extreme situation, as with an accident that takes, say, half of one's brain, after which one has to learn to use the other half for the functionings involved with the missing half). On the whole, for example, we learned so well to sit that we do not have to learn to do it ever again. Sitting is one of the functionings of children. (1971, p7)

Consider the list, I wrote above, of what children of three can do which newly born babies cannot do.

Which of them have become functionings of children?

Which are likely to be forgotten?

I will take one item from this list which is a functioning of children: walking. Up to one point in time, a child has never walked. Then, they walk. Having walked once does not mean that they can always walk; the next time they try, they may fall down after the first step. Practice is required. Through this practice, there may come a time when they are in a position to be able to walk whenever they wish. It is only at this point that I would say that walking has been learned. It is not necessarily true that walking has become a functioning at this point. Although a child, at a point in time, may be able to walk whenever they wish, it is another matter for walking to be so much part of them that they will never forget. It is one thing for something to be learned so that it can be activated in the short term. It is another thing for that learning to become learned for the rest of one's life. I call the process of something becoming a functioning *functionalisation*, where learning is *learning for life*.

How does a child become so good at walking that it is learned for life? I notice that children, having learned to walk, are not content to continue just walking. They want to walk on walls, walk on kerbs, walk up and down stairs, they want to run[4] . The practice of walking is done by subordinating it to a challenge at a higher subordinate level[5] . Rousseau said in about 1760 that *Before you can practise an art you must first get your tools;* (1986, p90). However, I suggest that *In order to practise your art, you will acquire your tools*. The acquiring of the 'tool' of walking is acquired through engaging in the art of walking - walking up stairs, walking backwards, walking quickly. Bruner said of young children learning to talk:

> Children learn to use a language initially (or its prelinguistic precursors) to get what they want, to play games, to stay connected with those on whom they are dependent... The engine that drives the enterprise is not language acquisition per se, but the need to get on with the demands of the culture. (1983, p103)

The acquisition of language is driven by the fact that it is continually being subordinated to the tasks of trying to get someone to do what you want, to participate in games, etc. Practice and progress are fundamentally linked, with both happening simultaneously at their respective subordinate levels. For example, practice at walking happens simultaneously with progress towards running; practice of language happens simultaneously with progress in participating in and controlling the surrounding environment. It is this *practice through progress* that turns learning into learning for life.

This type of practice has some similarities with what Wittmann (1985) calls *implicit practice*. He describes problems being given to students where skills are required to be practised but at the same time the problems are such that they require decisions, evaluations and deliberations to be continuously made. He goes on to say:

> Thus a great deal of the students' attention is drawn to the solution of the overall problem. The students cannot <u>concentrate</u> on the practice of a certain skill. Therefore I would like to call this form of practice <u>implicit practice</u>. (1985, p15)

A similarity between practice through progress and implicit practice is in the fact that attention is not always on the practising of the skill itself. However, a difference is that with implicit practice, attention can be on the skill itself for periods of time but that

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4 **{Deliberate creation of challenges on a higher subordinate level, rather than successful practice of what can already be done}** - 5.2.5

5 **{Subordinate levels}** - 5.2.4; **{Attention at a higher subordinate level can drive learning at the levels below}** - 5.2.4

concentration on that skill is not possible since higher order decisions are having to be made as well. Thus, I picture attention changing between the practising of skills and the higher order decision making. With practice through progress, attention is kept on a higher order task whilst going through the skill to get to that task. Attention is not swopped between one and the other, it remains with the higher order task. As a consequence the practice is always being subordinated with conscious attention elsewhere. It is this which helps the skill become a functioning. So, I make a difference between *applying* a skill to a problem solving situation, and *subordinating* that skill to a task at a higher subordinate level.

Gattegno describes traditional text books breaking up knowledge into little bits and presenting them piecemeal where Chapter 1 of the first textbook resembles Chapter n of the last.

> In this approach, there is no concern with one of the things that all of us know - that all of us go through - and that is, that practice gives one the capacity to undertake bigger tasks, to be involved in greater challenges. Is this not so? Is it not something that everybody knows, that practice provides us with the capacity to attack bigger tasks?
>
> In my own case, I learned this as an adolescent when I lifted weights. Lifting weights teaches one a lot if one can learn more than lifting weights. I recognized that lifting weights made me have muscles that allowed me to lift bigger weights, and that when I lifted bigger weights, I got bigger muscles which allowed me to lift bigger weights.
>
> But this is not the approach that we have embedded in the curriculum. Instead we work in the same way throughout the entire curriculum and do not take into account that there is a law - *the law of the cumulative effect of learning* - which can be described by saying once you have learned something, once you have mastered something, then you can attack a bigger task. The curriculum should be like a fan, opening up to more and more things, to bigger and bigger things. (1971, p15-16)

Let me consider how this law of the cumulative effect of learning relates to my subordinate levels and the notion of practice through progress. Suppose there are a number of challenges which I will label with letters of the alphabet. Suppose I have just learned A and am subordinating it to a challenge, B, which is on a higher subordinate level. Thus, A is practised through progress at B, I represent this as:

(A)**B**

I highlight B because that is where my attention is. Although I am learning B, I am practising A which is helping to turn A into something I can do so well that I do not need to think much about it. It is here where I can gain mastery over something I have learned[6] .

At some point in time, I become good enough at B in order for me to subordinate it to a new challenge, C, which is at a new higher order subordinate level. I can represent this in a similar way as (B)C. However, what about A? A is subordinated to B which is now subordinated to C. Thus A is also being subordinated to C, from two subordinate levels below. My attention is with C. If I am asked what I am using to work at C, I may be able to reflect on what I am doing and mention B. However, A is beginning to be at a subordinate level so far below C, that I may not think of mentioning it, or I may assume this within my statement of having used B. It is for this reason that I do not include it within the brackets:

A(B)**C**

This process continues with C being subordinated to D, again at a higher subordinate level:

aB(C)**D**

There comes a time when A has been subordinated at so many levels (at this stage it is three subordinate levels below the current challenge) that I manage to do A without even being aware that I am doing it. It is at this stage that I consider A to have become a *functioning*, something that has become a part of me, something which I can do with such little effort that I might not even be conscious of doing it. This is when A has been learned for life. To indicate this, I have turned it from a capital letter to a small letter. As this process continues, the chain may develop as follows:

abcD(E)**F**

Or, perhaps:

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6 **{Learning is not always taking place where the observable attention is}** - 3.2.3; **{Subordinate the old to engage in the new} - 3.2.3; {Practise the old whilst learning the new}** - 3.2.3; **{Control/mastery comes only when it is subordinated to higher order challenges}** - 3.2.3

abcD(E)**F**

In reality, one progression is likely to involve another progression with several such chains inter-linking. However, the principle remains the same, practice is carried out through progress onto challenges at higher subordinate levels. The metaphor Gattegno used of weights relates to the challenges, and that bigger weights relate to challenges at a higher subordinate level. Thus the law of the cumulative effect of learning is that a learner does not remain with the same abilities and awarenesses as they progress, meeting one chapter of a text book in the same way as they meet another chapter; they can be fundamentally changed by the gaining of new functionings as a consequence of practice through progress.

Halmos, in the introduction to his book on *Naive Set Theory*, talks of the desirability to make the mathematics become operational at an unconscious level:

> The student's task in learning set theory is to steep himself in unfamiliar but essentially shallow generalities till they become so familiar that they can be used with almost no conscious effort. In other words, general set theory is pretty trivial stuff really, but, if you want to be a mathematician, you need some, and here it is; read it, absorb it, and forget it. (1960, pvi)

Halmos's closing remark of *forget it* is an invitation to absorb set theory so thoroughly that it becomes something which no longer requires conscious attention. It no longer requires to be remembered. It is available to be used unconsciously instead.

Whitehead said *Civilisation advances by extending the number of important operations which we can perform without thinking about them* (1969, p42). Pimm, in referring to this quote, said:

> To this extent, calculators and computers are liberating devices, freeing the mind of the technical intricacies of computational algorithms, thereby permitting the addressing of more serious strategic problems,... (1987, p174)

The mind can also be freed from the need to think about operations by making these operations become functionings. Bruner considered the talk of young children and commented that:

> One is led to conclude that as the infant masters the routines of one level, enough processing capacity is freed for him to manage the next step forward. (1983, p85)

Practice through progress can force operations, which used to require time and thought, into things which can be done with such little effort that a person may not even be aware they are doing them. This frees the mind to attend to the higher order tasks, which become possible due to the fact that the operations at lower subordinate levels are now integrated and have become functionings. The main difference between my description of practice through progress and Bruner's comment above, is that it is the progress at the higher subordinate level which drives the operations at the lower level into becoming routine. Hence, work on the next step forward needs to begin before the lower levels have been fully mastered. Indeed it is that advancement which creates mastery. I have heard many parents talk of the fact that their young children are often trying to do things that they do not have the necessary skills in order to be successful. However, the drive to engage in a higher order task will drive the skills into becoming functionings through their required subordination to the task.

What have you been doing in the last five minutes which (to stay with my analogy above) was once a bold capital letter at one point in your life, and is now a small letter?

When you were a baby, you were making noises which were not part of an established language. Now you are able to discuss your thoughts and ideas with others. Below, I outline one possible chain of challenges which can take you from the former to the latter situation. The chain is such that each challenge becomes subordinated to a new challenge at a higher subordinate level. The chain is offered to help gain a sense in which practice through progress can lead to learning for life, where the challenges in the lower subordinate levels (here written first in the list) can become functionings through the continual progress of higher order challenges. The chain is not considered to be unique or complete. You may like to consider additional challenges and/or alternative chains.

- What noises can I make with my lungs, mouth, throat, tongue, lips,..?
- Can I make a combination of noises (a word)?
- Can I repeat particular words on command?

- Can I say words which sound similar to the words I hear adults say?
- What words are associated with particular objects or actions?
- How are words joined together (a sentence)?
- How are words and sentences transformed according to time and context?
- Can I express my thoughts and feelings in accepted sentences?

The achievements of children are impressive, not only due to the number of things they have managed to learn in a relatively short period of time, but also because these achievements remain with them for the rest of their lives without the need for re-learning.

6.3 Consideration of Gattegno's powers of children

Children are born into a world which is complex and in which so much is unknown at the outset. Yet, most children successfully engage in the challenges involved in gaining some control over their surroundings and achieving the impressive achievements mentioned in the last section. How they engage and what powers they make use of whilst engaging in those challenges, is the subject of this section.

Gattegno (1971) lists a number of powers children have and I will consider each in turn.

> What do we learn about the mental powers of children from the fact that the ability to make words becomes one of their functionings? We learn first of all that children are equipped - we are all equipped - with the power of *extraction*, which obviously is very competent since it can find what is common among so large a range of variations. (1971, p9)

Gattegno appears to use the word *extraction* to refer to the ability to take one thing out from many. So, for example, from a list of sounds made by an adult (a sentence), a child manages to take out a particular subset of sounds (a word). A friend of mine, Jo, told me something her son, Robert, did as he was learning to talk. He would choose a word, seemingly at random, from a sentence Jo said, and try to repeat it. Thus, Robert demonstrated his power to extract. However, to develop meaning for this word requires further work, since at present it is known only as some sounds. If the word is *chair*, then to gain a meaning for *chair* requires the child to separate some essence from a number of examples. This Gattegno refers to as *abstraction*, another power of children:

> ... everyone who has learned to speak has demonstrated an enormous competence in handling *abstractions*, for no particular word has an exclusive meaning of its own.
>
> Words are signs, arbitrary signs, since each object, for example, can have as many names as there are languages. Not only do children have to extract from the full packages represented by the voices they hear, they must also attach meaning to the words... Children must learn to make their proper abstractions so as to give to words their particular agreed upon meaning, and they do learn. (1971, p10)

Richard Skemp had the following definition:

> *Abstracting* is an activity by which we become aware of similarities (in the everyday, not the mathematical, sense) among our experiences... An *abstraction* is some kind of lasting mental change, the result of abstracting, which enables us to look at new experiences as having the similarities of an already formed class. (1987, p21)

Abstraction is concerned with the development of meaning which is consistent across experiences. Freudenthal (1980, p170) talks of *comprehension* (generalisation from numerous examples) and *apprehension* (grasping the general situation directly from one example). His use of comprehension seems similar to that of Gattegno's and Skemp's use of abstraction, and I argue that apprehension is also an example of abstraction. When someone meets something new, if they try to relate to it, they do so with the experiences they already have. They can be said to be looking for similarities with already formed classes, as Skemp describes. This produces the effect of the person attempting to generalise immediately from the first experience of something new. A generalisation may be made from the one example, but whether it stays the same, adapted, or abandoned will depend upon the consistency of that generalisation with experiences of further examples.

In 1983 I became aware of the word *fickle* for the first time. A friend, Laura, had used it in relation to a cat we were watching. In Hewitt (1988) I described my experiment to develop a meaning for this word, as a young child might have to do, without receiving explanation or definition from a dictionary or another person. I had extracted the word from the collection of sounds Laura made in saying her sentence, and knew it as a word I had no prior meaning for. At that time, I was aware of developing meaning straight away. I noticed that Laura had said the word as the cat was walking away from her towards someone else who has trying to stroke it, and so I considered that the word might have something to do with seeking out what was advantageous or desirable. I chose not to inform Laura of my ignorance but waited for other occasions when the word was said, so that I could abstract something common from the different examples of its use. It took several years for me to encounter sufficient examples before I felt brave enough to use the word myself and see whether my usage was accepted by others. Thus, I developed some meaning by generalising from this one example. The generalisation was through seeking similarities between the cat's actions and behaviour I already had meaning for within my previous experiences. However, the meaning changed when I met other examples of the word usage, and it took several examples before I felt that I had developed a meaning which was consistent with those within my social environment.

Both extraction and abstraction require the ability to stress some things and ignore others. Gattegno goes on to state that stressing and ignoring is abstraction:

> There is one universal functioning without which nothing is noticed. This is the *stressing* and *ignoring* process.
>
> Without stressing and ignoring, we can not see anything. We could not operate at all. And what is stressing and ignoring if not abstraction? We come with this power and use it all the time... To stress and ignore *is* the power of abstraction that we as children use all the time, spontaneously and not on demand, though in its future uses we may learn to call it forth by demand. (1971, p11-12)

I choose to consider them as closely related but not the same, as abstraction requires a particular collection of stressing and ignoring. I consider abstraction to be on a higher subordinate level as it makes use of stressing and ignoring.

All of the powers mentioned so far are functionings, as they are used by all of us every day of our lives yet rarely are we aware that we are using them. Thus, the powers I am listing in this section can be viewed as basic functionings which are used fundamentally in our learning throughout our life.

There are many examples of abstraction being used by children learning their first language. It is often when children say something which does not conform to the language that we can become aware of the work they are doing. On occasions when children say, for example, *I kicked the ball yesterday*, we may not show surprise or notice what a child is doing. However, if the child says, *I throwed the ball yesterday*, we have the opportunity of becoming aware that the child has learned a rule about adding *ed* at the end of a verb when they are talking in the past tense. This is another example of children working at general rules they abstract before concerning themselves with exceptions to those rules[7] . Such observations have been known for some time. Rousseau wrote in about 1760:

> To begin with, they have, so to say, a grammar of their own, whose rules and syntax are more general than our own; if you attend carefully you will be surprised to find how exactly they follow certain analogies, very much mistaken if you like, but very regular; these forms are only objectionable because of their harshness or because they are not recognised by custom. I have just heard a child severely scolded by his father for saying, "Mon pere, irai-je-t-y?" Now we see that this child was following the analogy more closely than our grammarians, for as they say to him, "Vas-y," why should he not say, "Irai-je-t-y?" Notice too the skilful way in which he avoids the hiatus in irai-je-y or y-irai-je? Is it the poor child's fault that we have so unskilfully deprived the phrase of this determinative adverb "y," because we did not know what to do with it? It is an intolerable piece of pedantry and most superfluous attention to detail to make a point of correcting all children's little sins against the customary expression, for they always cure themselves with time. Always speak correctly before them, let them never be so happy with any one as with you, and be sure that their speech will be imperceptibly modelled upon yours without any correction on your part. (1986, p37)

There is a recognition here that children use abstraction to find rules from their surroundings and work on the application of these before dealing with exceptions. Ginsburg comments:

> Young children's errors of speech are quite similar to their counting errors. One example of speech errors that every parent encounters involves the four-year-old who says, "I *goed* to the store" or "He *bringed* the book." Psychologists who have investigated early language point out that mistakes of this type teach us several interesting things. First, children's language is based on rules. Children never hear "goed" or "bringed" and hence cannot just be mimicking use of these words. Instead, the words must be the product of a rule to the effect that past tense is indicated by adding "-ed" to the stem. Of course children are not aware of this rule, just as adult speakers are not aware of many language rules. Nevertheless, rules govern speech.
>
> Second, children's rules are derived in sensible ways from their experience. Children have frequently heard "-ed" used quite

[7] **{Rules first, exceptions later}** - 3.2.5

> correctly to indicate past tense - for example, "helped" or "waited." So their rule is a sensible one, based on real experience. It is not a figment of their imagination. The main mistake children make is in applying it too widely - to all verbs. They need to learn the exceptions, like "went" and "brought," so that they can avoid overgeneralization. *Children's mistakes frequently have a rational basis*.
>
> Third, the mistakes children make are the result of a search for meaning. In learning a language, children do not aim merely to imitate what they have heard. Instead, they look for the underlying structure, for what is really going on. They do not simply repeat strings of words; they try instead to construct rules at a deeper level. Sometimes they are wrong, but their mistakes indicate that they are digging below the surface.
>
> As we shall see, the same holds true in children's mathematics. Their counting mistakes result from an overgeneralized application of rules; the rules reflect children's experiences; and they are constructed as a result of an attempt to understand. In language as in number, children's intellectual learning is in part a creative, intelligent process. Rote "mechanical" factors play a secondary role. (1977, p8)

Children use abstraction as a way of getting to the rules and structures which exist in their first language. If they had to learn their first language without attending to the structure, it would indeed be a daunting prospect as they would have to memorise, not only the different tenses of verbs as if they were new verbs, but also memorise the order words are placed within a sentence for each possible sentence! Without the drive at abstract structure, the task of learning a language would be overwhelming. Thus, the abstraction of structure is a means to economise the effort required to learn a language.

The structure contains a number of transformations, either of the same words within a sentence:

You are reading this sentence.
Are you reading this sentence?

Or of different words substituted within parts of a sentence due to a change in the context:

He is sitting down.
She is sitting down.

The way verbs transform according to their tense is another example. The ability to use transformations[8] is another power that Gattegno mentions. However, transforming uses the fact of keeping some aspects the same whilst making other aspects different. Hence, transforming makes use of sameness/difference which I am not choosing to label as a power of children, and will consider in more detail in section 6.5.

In 6.2, I talked of the mind being freed from the need to think of everything involved in a task, by the fact that several aspects involved can become functionings by practice through progress. Gattegno talks of these functionings as *know-hows*:

> What reader of this book literally remembers his native language? Not one. None of us remembers it, we function in it, we have at our disposal the "know-how" to do it. This is what it means to have a functioning.
>
> The know-how leads to skill, the know-how is what we have within us that does not require conscious recall. It is just there. If I had to remember my speech, I would never be able to talk. Anyone observing himself will see that to have an intention to speak is sufficient for all of one's verbal elements to be available and for finding them adequate for one's intention. An individual need not call in these elements one by one; they come, the intention brings the appropriate words in and excludes the others.
>
> Further - and here we move to another point and another power of children - when the words come out, it is the will that acts upon the speech organ for the words to be spoken in the way that the language expects them to sound. (1971, p12)

Here, there is the notion of the will as the controlling element which activates the other powers. Thus, it is the will which drives the requirement to extract, abstract,... It is through the will that we have attention, that there is something of which we become conscious. If we place our attention in something which is some distance away, and our will is placed in the desire to be near to it quickly, then the know-hows we possess in the functionings of activating muscles, moving our legs, balancing,... (which all are subordinate to the know-how of running) are all called into action. Rousseau commented:

> I want to move my arm and I move it without any other immediate cause of the movement but my own will... In a word,

[8] **{Children performing transformations within language}** - 3.2.3

> no motion which is not caused by another motion can take place, except by a spontaneous, voluntary action; inanimate bodies have no action but motion, and there is no real action without will. This is my first principle. I believe, therefore, that there is a will which sets the universe in motion and gives life to nature. (1986, p234-236)

To see how your will is involved in quite basic functionings, carry out the following exercise which I first introduced to a seminar at the ATM conference in Lancaster in 1989 (it is best done in a darkened room lit only by one light):

Take a sheet of A4 paper with writing on one side and nothing on the other.

Have a light in the room.

Hold the sheet of paper between your eyes, perpendicular to your face, so that the paper comes from your nose dividing your eyes. I will differentiate between your eyes by considering which eye is on the writing side, and which eye is on the blank side of the paper.

Hold up a finger on the blank side so that it can only be seen by one eye.

Have a light in the room such that it lights up the writing on the paper but does not light up the finger. Thus, the writing is in the light and the finger is in the dark.

Focus on your finger with both eyes and from now on do not move your eyes or eyelids.

See your finger with the one eye which is able to.

Use your will to move your attention to the other eye and see the writing (without refocussing).

Put sufficient attention in this eye so that the finger 'disappears'.

Since your eyes did not move and your eyelids did not block off or change the light coming into your eyes, the effect must have been due to work you did internally with the received photons. You have two sets of photons coming from your left and right eyes. The power of your will can force attention from one set of photons to the other, or *see* a mixture of the two. One consequence of your will at work is *attention*, which produces the power of stressing and ignoring as a consequence. You can stress the images of the photons from one eye and ignore the photons from the other. Vygotsky indicated a difference between apes and children in this respect:

> The ape must see his stick in order to pay attention to it; the child may pay attention in order to see. (1978, p36)

In 4.2.4, I mentioned the power of images[9] . This is where the will is able to provoke sensations similar to those we might receive from an outside source via our senses. Gattegno says:

> The eye, instead of receiving photons now from outside, is made the judge (somatically) of the energy alterations that are produced by the will pouring energy into the cells to affect them in ways comparable with the effects caused by the input of photons from the environment. (1973, p37)

This is also possible with the other senses we have[10] . Thus images are another power children possess and, as Gattegno points out, is one which can offer possibilities of achieving much learning:

> Because images are dependent on our will, once we begin deliberately to employ them, we can very soon obtain an awareness that indeed imagery is a power of the mind, and it can yield in a short time vast amounts of insights into fields that become almost sterile when the dynamics are removed from them. (1971, p27-28)

Here, Gattegno considers the fact that images offer the opportunity of changing dynamically in time. Additionally, due to the fact that the images can be controlled by the will, the person concerned can control the way in which the images change. Thus, through images a number of areas of mathematics can be explored by considering the dynamics involved in a particular situation. The use of movement I have indicated as one which can provide the opportunity to notice properties which would otherwise be difficult to note[11] . Examples of possible situations for teachers to work with students' images can be found in the ATM publication *Geometric Images* (Beeney, et al (1982)), for example.

Gattegno mentions other powers:

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[9] **{Images require minimal effort}** - 4.2.4

[10] **{Images are available from all senses} - 4.2.4**

[11] **{Movement allows all possibilities to be considered and properties noted which might otherwise be difficult within a few particular examples}** - 4.2.2

law of cumulative effect of learning;
differences and similarities;
know-hows.

Each of these I either do not consider to be appropriately described as powers, or feel as if they have been included in the ones considered so far. The law of cumulative effect of learning I do not consider to be a power children use but a description of the consequences of them using their powers effectively in the learning process. The ability to notice differences and similarities is a particular use of stressing and ignoring. As I have indicated, I will talk more about this in section 6.5. Know-hows I have already mentioned in this section and I consider to be the accumulation of functionings gained by the productive process of practice through progress.

In addition to considering those Gattegno has identified as powers, I include that of *association*[12] , *creativity*[13] and *memorisation*. Although I have stated that using memory is sometimes an inefficient way to make something known, it remains a process children use to hold information[14] .

Thus, I am choosing to identify the following:

will;
attention;
stressing and ignoring;
imagery;
extraction;
abstraction;

association;
creativity;
memorisation.

Not all of these do I consider to be of the same type. The discussion of this and the choice of which I describe as powers of children follows in the next section.

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[12] **{Association}** - 5.2.1
[13] **{Students are already creative}** - 4.2.4
[14] **{Memory is expensive}** - 4.2.3; **{The arbitrary requires memory in order for it to be known}** - 4.2.3

6.4 Powers and information.

6.4.1 Working with information

I wish to identify the powers children possess, and have used extensively in their early learning, which can, in turn, be called upon by a teacher of older children in mathematics classrooms. These powers are used as part of everyday activity as a human being rather than being concerned with mathematics in particular. Gattegno produced a number of labels that he described as powers of children and, in the previous section, I have decided to attend to a subset of these along with my additions. I have already mentioned that I do not consider these to be all of the same type, for example the will acts as a control to activate other powers. Thus, the will can be viewed as of a different nature to abstraction, for example. Since these are all not of the same type, I consider that a structure containing them will be of some use.

If the will is viewed as an activator, then what it can activate is some personal energy which is manifested by the person paying attention to something. Dewey said *The exercise of will is manifest in the direction of attention* (1975, p8). If you engaged with the last exercise in the previous section, you may have become aware of the fact that your will was required in order to 'see' through one eye and not the other. Furthermore, it required some effort on your part. People I have given this exercise to have commented that, although they were successful, it was hard work to achieve success. Thus, the fact that people have felt tired at the end of this exercise indicates that some personal energy was required in order for them to place their attention in one eye and not the other. Thus, to achieve attention requires some use of personal energy and the will can be viewed as activating that energy.

Once I am attending to something, that implies that I am not attending to something else since the amount of attention available is limited[15] . Attention means that all things are not equal at that point in time. Thus, with attention comes the fact that some things are stressed and other things are ignored. So, stressing and ignoring can be seen as the consequence of attention.

will ----(activates energy)----> attention ----(which produces)----> stressing and ignoring

The will, as an activator, is involved in producing images, memorising, extracting something from complexity, abstracting something common from a selection of events,

[15] **{The amount of attention available to us is limited and finite}** - 4.2.7

and associating things with one another. However, there is more involved. With the production of images, for example, there are an infinite number of possible images which someone might produce. So, there is an element of choice.

Close your eyes and spend a few moments involved in producing some visual images.

The ability to make choices is concerned with creativity, along with awareness that choices can be made. Thus, you have been involved in a creative act in producing the images you did. No one else was making the decisions you made or selecting the images you had. The will and creativity combine to produce images.

Although we may be able to create images, the meanings we give to them is another matter. Gattegno talks of *a sense of truth* guiding us:

> ... acknowledge the existence of *a sense of truth* which guides us all and is the basis of all our knowing...
>
> For a tiger, a carrot has no nutritional value. Not so for a donkey. Our optical eye may produce an optical image on our retina or our brain, but its significance does not necessarily follow. Something else is required to give it significance; in animals, we call it instinct, in man sense... This is our sense of truth, which functions well and independently at the beginning of life and less well in the instances when we have no immediate access to areas under investigation and we are made to lose our independence - the situation in the traditional classroom. (1971, p56-57)

Along with will and creativity, we can consider a sense of truth which acts as a guide. St. Augustine commented in the 4th Century:

> Regarding... all those things which we understand, it is not a speaker who utters sounds exteriorly whom we consult, but it is truth that presides within, over the mind itself; though it may have been words that prompted us to make such consultation. (1950, p177)

All three of will, creativity and a sense of truth, can be involved in the placement of attention. Now, I will consider what can come of attention being placed somewhere and, as a consequence, certain things being stressed and ignored.

As I mentioned in 6.3, there is a need to stress some things and ignore others in order to extract something from the complexity of its environment. Similarly, in order to abstract something from a collection of instances, it is necessary to stress and ignore.

Association can also be formed through a will of stressing some things and ignoring others. Such associations can be consciously employed in order to aid memorising. For example, on meeting someone new, I sometimes try to associate an aspect of their appearance with someone I already know who has the same name. In this way, I may be able to remember the person's name through the association with someone whose name I have already succeeded in remembering. On other occasions an association may have been formed without such conscious consideration (as indeed is most of the extraction and abstraction we do). An example is the association between some traumatic news and the surroundings in which the news was heard, as with John Lennon[16] in 5.2.1. I have noticed further examples when I have played a game with other people where a short extract of music is played and one task is to guess the year when the extract was in the pop charts. Several people have commented that they recall the situations they were in at the time and then try to work out which year they were in those situations. Thus, associations have been formed between the music and the physical situations they were in at the time. I notice that not all songs provide a clear association with a place. On many occasions when there has been an association formed, emotions were recalled as well. The association between music and place can be achieved by stressing the emotion that was present at the same time. Other associations can be accessed by stressing other things, for instance the relative treble and bass balance which can indicate a period or style of the music production, particularly with pop music. Thus, associations are still dependent upon some stressing and ignoring.

Extraction, abstraction, and association can be viewed as manifestations of different uses of stressing and ignoring[17] . Thus, a structure develops, as represented in Figure 14.

[16] **{Association}** - 5.2.1
[17] **{Extraction through stressing and ignoring}** - 3.2.2

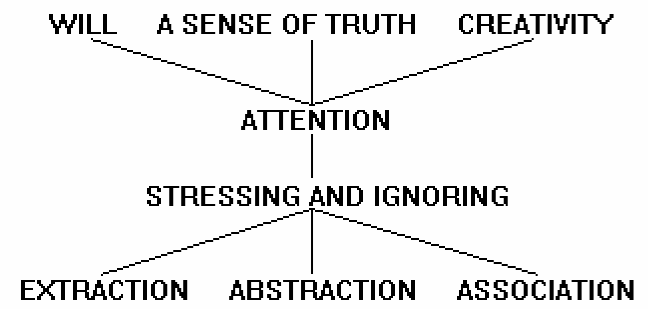


Figure 14: Diagram representing a structure of how one uses oneself when working with information.

This diagram describes a way in which we work with information. I will choose to describe extraction, abstraction and association as powers of children.

6.4.2 Holding information

In addition to working with information, we are able to hold information. There is more than one way in which children can hold information. I have discussed the use of memory and found it to be a relatively expensive way of holding information in terms of the personal energy required to achieve the memorisation of an arbitrary fact[18] . However, I also indicated that there are times when memorisation is required as, for example, in the learning of an arbitrary name[19] . Memorisation is one method for holding information - it is relatively expensive in terms of personal effort and requires practice.

I have indicated that images are another way in which we hold information. Their nature is quite different to that of memorisation. With images, there is often no conscious attempt to remember at the time, images being received through one or more of the senses. The fact that there was little noticeable effort required for them to become held indicates the difference in nature with the activity of memorising. It is for this reason that I say that images are not *remembered* at a later date but *recalled*[20] . Both memory and imagery have the potential for long term holding of information but neither of them are guaranteed to be successful. Memory sometimes requires considerable effort at the time the information is to be held, images do not require this effort but do, along with memory, require some mechanism to access them at a later date. Such mechanisms may involve some of the items listed in the previous section, for example association.

[18] **{Memory is expensive}** - 4.2.3
[19] **{The arbitrary requires memory in order for it to be known}** - 4.2.3
[20] **{Images are retained and recalled, rather than memorised and remembered}** - 4.2.4

The short term holding of information can involve memory or imagery. I may give sufficient effort to remember someone's name for a limited period of time but, if I do not have sufficient practice with using their name, it may be that I will not be able to remember it after a while. With imagery, it is possible for me to recall something that I have just received in my senses and 'play it back'. For example, you may be able to recall times when your attention has been elsewhere when someone was talking to you; you apologise and ask them to repeat what they said, but before they do this, you suddenly recall what they had said. The other person may say that you were listening after all. In fact, you were not listening in the sense that your attention was not with what they were saying. However, their words were still entering your ears as sound waves and it is possible to use your will to 'replay' them, using imagery, and put your attention in them this time.

A third way of holding information is through functionalisation - the generation of functionings, as mentioned in 6.2. This differs from both memorisation and imagery in that it requires a particular type of practice - practice through progress. Additionally, there is no problem about access to the information, in the sense that it has become part of your functionings and as such is available on command. Once something has become a functioning, it is learned for life, in that there is minimal practice required to keep it available for future access.

Thus, I have identified three ways in which information can be held:

MEMORISATION   IMAGERY   FUNCTIONALISATION

6.5 Powers and mathematical essences

The powers of children that have been identified in the section 6.4.1 are human in the sense that they are involved as children go about their life as a human being. There is nothing special to mathematics as such. However, when children use these powers in certain ways, they can enter the world that is labelled mathematics. The use of abstraction, for example, can lead to the notion of sameness. In order for a child to gain an understanding of what a *chair* is, they need to use their power of abstraction (which in turn uses stressing and ignoring) to identify what is the same in the collection of times when people use the word *chair*.

Sameness is also at the heart of much mathematics. Mathematical properties are concerned with what stays the same whilst other things are changing. The mathematical equation:

$$3 + 2 = 5$$

is saying that there is something which is the same about the expressions '3 + 2' and '5'. In many, if not all, areas of mathematics the notion of sameness is fundamental.

The idea of such fundamental notions in mathematics was discussed amongst colleagues in Bristol in the early 1980's, notably Laurinda Brown and John Chatley. In 1984, I devised an exercise for teachers to discover other fundamental notions, such as sameness, which appear to form the root of much of the mathematics curriculum. This was first used in an in-service training course I jointly ran in July 1984, and has been written about in Brown, Hewitt, Mason (1994) where the fundamental notions are called *essences*:

Stage 1:	Choose something in mathematics which you consider to be essential for students to learn. You can decide the age of the students you have in mind and the level of the mathematics.
Stage 2:	Suppose someone has a poor understanding of what you have chosen. Write down three aspects of mathematics you could work on in order to help them improve their understanding of this.
Stage 3:	Decide on one of these and commit yourself to it. You cannot change your mind later on. Go to stage 2.

At each stage, something is being split up, and three more fundamental things considered. As you repeat this process, you may find it hard to keep splitting up each thing you choose. However, it is surprising what relatively fundamental things can be split, so do not give up easily. The task continues until you have arrived at some *unsplittables*, those things which you feel cannot be broken down any more. It is these unsplittables that I refer to as essences.

An example of what one person produced within a group when they had chosen the starting point of Pythagoras was:

<u>Pythagoras:</u>

- Squares and roots
- Addition
- Areas

  <u>Squares and roots:</u>

  - Area
  - Calculators
  - Multiplication

    <u>Area:</u>

    - Multiplication
    - Measuring
    - Counting

      <u>Multiplication:</u>

      - Numbers
      - Tables
      - Multiple adding

        <u>Numbers:</u>

        - Names
        - Order
        - Patterns

          <u>Order:</u>

          - Value
          - Greater/smaller
          - More/less

            <u>Greater/smaller:</u>

            - same/different

These mathematical essences form the basis of much of the mathematics curriculum. In fact the content of the curriculum can be viewed as a collection of different manifestations of essences. It may be that a different starting point and different routes still lead to the same essences. It is my view that the whole mathematics curriculum is based on remarkably few essences. For example, *same/different* is involved in:

classifying and naming;
equivalent fractions;
factorising and multiplying out brackets;
arithmetic;
manipulation of algebraic expressions;
spotting patterns;
conjecturing;
links between decimals, fractions and percentages;
properties of shapes;
...

Since same/different is based on the use of abstraction, which is a power that children have used in their early development, it is not surprising to find that same/different is something which children have had experience of throughout their life in doing activities such as:

developing meaning for words in their first language;
establishing verb endings in their first language;
recognising particular people even though they might be wearing different clothes;
naming things and people;
...

Thus, if I, as a teacher, make use of these essences in designing activities for lessons, then I am asking children to make use of the powers they are familiar with, and have used since being a young child. Additionally, since a root mathematical essence such as sameness is involved in so much of the mathematics curriculum, it will help establish links between areas which might otherwise have appeared disparate.

As I mentioned earlier, it is my belief that there are remarkably few root mathematical essences and listed below are the ones that I personally consider to be of this nature:

sameness/difference;
inverse;
order.

As with sameness/difference, the others are different manifestations of the use of powers identified earlier, and used frequently in children's early learning. With *inverse*, I have observed a young child looking at me and then turning their head to the right to look at something else. When they wish to look at me again, they activate different muscles so that their head turns to the left. In order to know which way to turn, they have to have held an image of me along with associations with link me with other items which were or are in their view. When the child turns their head back, the amount of turning remains the same, but the direction is changed to the 'opposite'. Likewise, when I think of changing:

$$14x+3=9$$

to

$$14x=9-3$$

the amount, '3', stays the same but the operation is changed to the 'opposite'. Thus inverse, as a root mathematical essence, is also known to children as something they have used in their early learning.

Inverse, as a mathematical essence, is involved in:

addition - subtraction;
multiplication - division;
squaring - square roots;
powers - logarithms;
manipulation of equations;
...

And inverse, as a use of the powers of children, is involved in:

putting on, and taking off, clothes;
opening and closing doors;
turning a tap on and off;
looking from one part of the room to another, and back again;
...

Order also has its roots in the early experiences of children. Vygotsky talks of children developing a time field:

> In addition to recognizing the visual-spatial field, the child, with the help of speech, creates a time field that is just as perceptible and real to him as the visual one. The speaking child has the ability to direct his attention in a dynamic way. He can view changes in his immediate situation from the point of view of past activities, and he can act in the present from the viewpoint of the future. (1978, p36)

The establishment of a time field, and of children involving themselves in speech, means that there comes an inherent order, an order created by time. When something is said, it is not said at the same time, only one sound can be made at a time. Hence, with a collection of different sounds, there is always one that is said before another. Particular orders are established (as with the number names, or the sounds that make up the word *mummy*) by an association between the finishing of one word or sound and the beginning of another. To notice such associations, read the words in the box below as quickly as possible:

one, two, three, four, five, six, seven, nine, eight, ten

When I have asked people to do this they talk of finding themselves stopping in mid-flow and having to think carefully about what they are saying. The first series of words are said quite fast, with the last few said much slower. The order of the number names have become a functioning and no longer need to be consciously thought about. The associations are strong - after one comes two,... However, there is not such a strong association between finishing *seven* and saying *nine*. This is even though the individual words themselves are functionings.

Order, as a mathematical essence, is involved in:

counting;
operations;
notation;
functions;
size;

...

Order, as a power, is involved with:

the sounds which make up a word;
the words which make up a sentence;
putting on clothes;
tying a shoelace;
...

6.6 Backwards and forwards

By looking at the achievements of young children, I have identified that much of their learning is learned for life. Skills and ideas that they did not have at one point in their life become functionings and are available for the rest of their life. The process by which this occurs is practice through progress, continually subordinating something to a task at a higher subordinate level. This leads to information (skills, ideas,...) being held as a functioning. Consequently, I have called this process *functionalisation*. Other ways of holding information are through *memorisation* and *imagery*.

The powers that children use in much of their productive early learning are based around the use of *attention* and consequently, *stressing and ignoring*. The particular powers I have chosen to identify are *extraction*, *abstraction* and *association*. One aspect of the use of these powers is the development of fundamental mathematical notions called *essences*. The ones I have chosen to identify are *sameness/difference*, *inverse* and *order*. These essences are developed and used extensively in early childhood.

Freudenthal says that *... a theory of teaching should be the complement of a theory of learning* (1980, p78) and Gattegno (1971, p14) talks about the subordination of teaching to learning. Here, I have attempted to identify some of the dynamics employed by children as they go about the productive learning they do in their early childhood. Gattegno's call is for us to pay attention to this and to adapt our teaching accordingly. My claim is that teaching approaches which call upon the tools children have at their disposal, and have used so successfully in their early learning, can result in older children learning school mathematics faster and more thoroughly than with traditional approaches. In this sense I say that these approaches pay attention to the notion of economy in the learning and teaching of mathematics. It may well be the case that different labels could have been chosen to describe the abilities children have and equally productive teaching approaches based on those. My claim is not to do with uniqueness, it is that it is possible to create teaching approaches which are based on the powers young children demonstrate, and that these approaches can produce learning which is faster and more thorough than traditional methods.

The next chapter will consider ways in which students are being asked to work by the authors of some text books which are currently used in many UK schools. In particular, I will look at what powers are being called upon to work at some of the mathematics presented in the books.

7 Analysis of some text books in terms of the use of children's powers

7.1 Introduction

Holt commented:

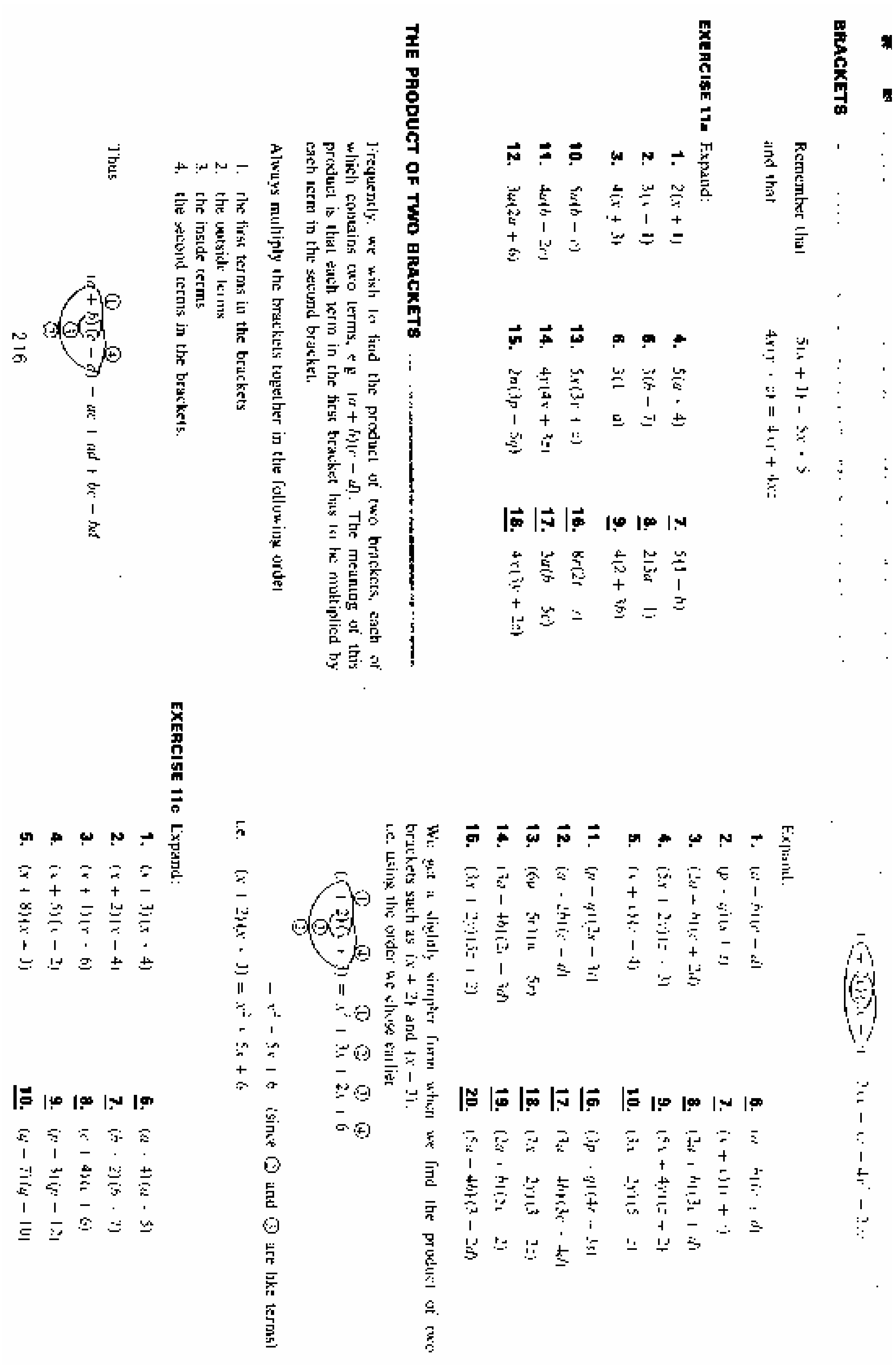
> In short, children have a style of learning that fits their condition, and which they use naturally and well until we train them out of it. We like to say that we send children to school to teach them to think. What we do, all too often, is to teach them to think badly, to give up a natural and powerful way of thinking in favor of a method that does not work well for them and that we rarely use ourselves. (1991b, pvii)

The learning which students achieve in classrooms is not as impressive as the learning they achieve as young children. Holt lays part of the blame on the way in which schools ask students to think. Having identified powers which students use so productively in their early learning, I now turn to some ways students are asked to work in mathematics classrooms when lessons are based on some written materials.

I will look at some examples from three text books. I select these as an exercise to consider how much the powers of children are called upon to be used by students as they engage in their mathematics work. I have chosen some text books which are widely used in schools in the UK at this moment in time. I have chosen pages which I consider to be representative of some ways in which mathematics is presented and, although there were a number of possible pages I could have chosen from each book, I do not wish to imply that these pages are representative of each book.

7.2 Some pages taken from three text books

The first is taken from *ST(P) Mathematics 3A* (Bostock, et al (1985, p216-217)) (see Figure 15).


BRACKETS

Remember that $5(x+1) = 5x + 5$

and that $4x(y+z) = 4xy + 4xz$

EXERCISE 11a Expand:

1. $2(x+1)$ 4. $5(a+4)$ 7. $5(1-b)$
2. $3(x-1)$ 5. $3(b-7)$ 8. $2(3a-1)$
3. $4(x+3)$ 6. $3(1-a)$ 9. $4(2+3b)$
10. $5a(b-c)$ 13. $5x(3r+z)$ 16. $8r(2t-r)$
11. $4a(b-2c)$ 14. $4y(4x+3z)$ 17. $3a(b-5c)$
12. $3a(2a+6)$ 15. $2n(3p-5q)$ 18. $4x(3y+2z)$

THE PRODUCT OF TWO BRACKETS

Frequently, we wish to find the product of two brackets, each of which contains two terms, e.g. $(a+b)(c-d)$. The meaning of this product is that each term in the first bracket has to be multiplied by each term in the second bracket.

Always multiply the brackets together in the following order

1. the first terms in the brackets
2. the outside terms
3. the inside terms
4. the second terms in the brackets.

Thus $(a+b)(c-d) = ac - ad + bc - bd$

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Expand.

1. $(a-b)(c-d)$
5. $(x+y)(z-4)$
11. $(p-q)(2x-3t)$
12. $(a+2b)(c-d)$
16. $(3x+2y)(3z+2)$
20. $(5a-4b)(3-2d)$

We get a slightly simpler form when we find the product of two brackets such as $(x+2)$ and $(x+3)$,

i.e. using the order we chose earlier

$(x+2)(x+3) = x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6$ (since ② and ③ are like terms)

i.e. $(x+2)(x+3) = x^2 + 5x + 6$

EXERCISE 11c Expand:

1. $(x+3)(x+4)$ 6. $(a+4)(a+5)$
2. $(x+2)(x-4)$ 7. $(b+2)(b+7)$
3. $(x+1)(x-6)$ 8. $(c+4)(c+6)$
4. $(x+5)(x-2)$ 9. $(p-3)(p-12)$
5. $(x+8)(x-3)$ 10. $(q-7)(q-10)$

Figure 15: *ST(P) Mathematics 3A*, p216-217.

The section begins by asking a student to *remember* how algebraic expressions with single brackets, such as '5(x+1)' and '4x(y+z)', are expanded. An exercise is given for them to practise this, as it is to be made use of in the following section. Thus, there appears to be an assumption made that students need to be able to perform one piece of mathematical content (multiplying out a single bracket) in order to progress onto the next piece of content (the product of two brackets). Concerning the product of two brackets, students are given a list of instructions as to what to do and in which order. The order is emphasised by the inclusion of *Always* in the statement *Always multiply the brackets together in the following order*. This order may be a convention but it is not offered as such in passing but it is made an object of attention, thus asking the student to *memorise* the order as well as the way of combining the two brackets. There is no encouragement for students to explore possible orders of the multiplication, thus no creativity is being asked for. An *image* is offered with lines drawn to connect the relevant terms to be multiplied. Although I say multiplied, I note that there is no need for me to consider multiplication in this case, since I could consider this to be a way in which letters are placed next to each other. This image is offered as an aid to memory. The example given in the box has some differences to the previous image in that a student is asked to view '2y' as one term and to consider 'x' multiplied with '2y' to be written as '2xy' and not as 'x2y'. Also '2y' multiplied with '2y' has been written as '$4y^2$' and not as '2y2y'. Students are asked to do the same as they have been told in the text and are given an exercise of examples to repeat the process. For a student to be successful at this, they will need to *stress* some things (such as the form of two brackets each consisting of two terms) and *ignore* others (such as the fact that there needs to be those particular letters in the expression). The practice which is offered appears to be at the same subordinate level as the required learning of how to multiply two brackets together. Thus, there is no attempt to drive such learning into becoming a functioning.

The next example is taken from *NMP 2 Mathematics for Secondary Schools* (Blackett, N et al (1987, p36)) (see Figure 16).

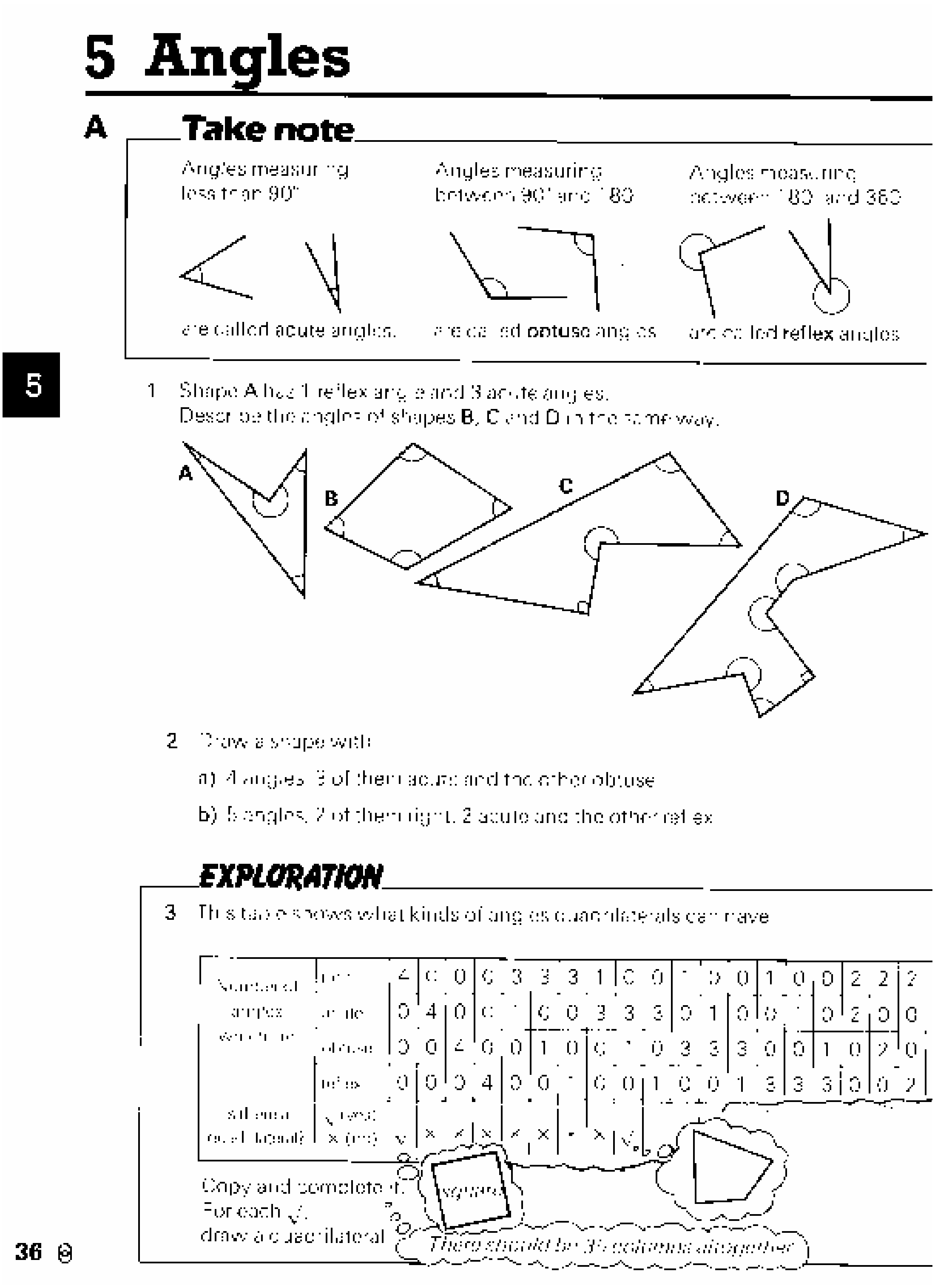
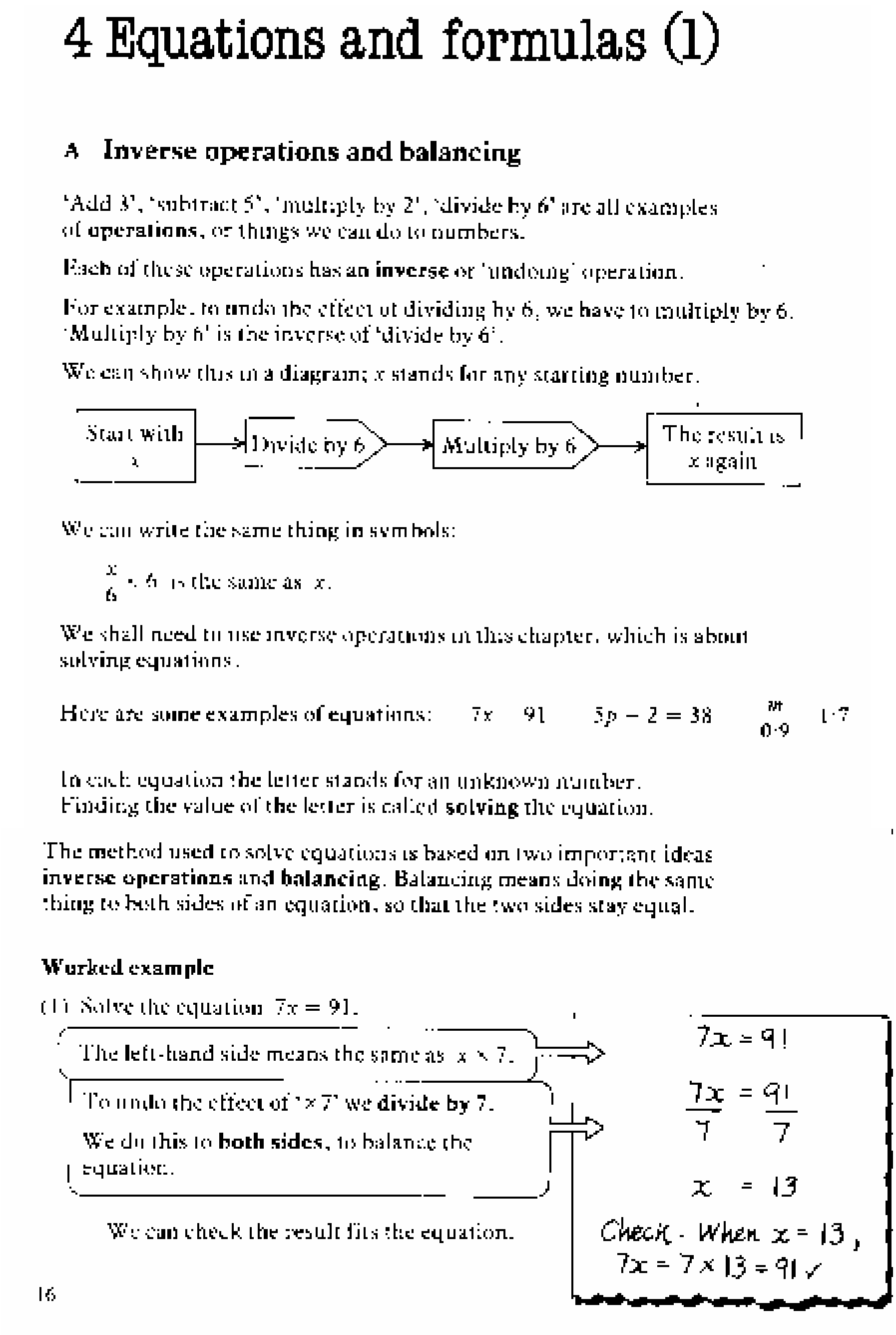


Figure 16: *NMP 2 Mathematics for Secondary Schools*, p36.

The box asks students to use *memory* in order to know the meanings of the words 'acute', 'obtuse' and 'reflex', as well as the words themselves. Some *images* are offered in terms of two examples for each of the words. These examples are of fixed, static drawings and so to been seen as examples, there will have to be certain things *stressed* (for example their relationship to right angles and straight lines) and other things *ignored* (such as the length of the lines). The definition of these words are given in terms of the angles 90^o, 180^o and 360^o. These are given as numbers rather than as drawings. Thus, for each of the drawings to be compared with these numbers of degrees, a student will have to *translate* them from the numbers to mental pictures, or to *translate* the drawings into numbers of degrees. The first question presented asks the students to practise this knowledge. The challenge is at the *same subordinate level* as the original learning. The second question asks the students to use their *creativity* to draw two shapes given the constraints of the types of angles they are to include. In this case their original learning is to be made use of in constructing these shapes. However, these tasks will only be at a higher subordinate level if a student's main attention is with the creation of the shape rather than the naming of the angles themselves. A task is only at a subordinate level above some content, C, if someone's attention is taken away from C whilst C is still subordinate to the new task. I suggest that the exploration offered in question 3 is of a *higher subordinate level* to the naming of angles. However, with 35 situations to be explored, students will be staying at the same subordinate level (albeit a higher subordinate level to the naming of angles) for a long time. Consequently, further progress does not appear to be expected.

The third example comes from *SMP 11-16 Book R2* (see SMP (1986, p16)) (see Figure 17).



4 Equations and formulas (1)

A Inverse operations and balancing

'Add 3', 'subtract 5', 'multiply by 2', '**divide by 6**' are all examples of **operations**, or things we can do to numbers.

Each of these operations has an **inverse** or 'undoing' operation.

For example, to undo the effect of dividing by 6, we have to multiply by 6. 'Multiply by 6' is the inverse of 'divide by 6'.

We can show this in a diagram; x stands for any starting number.

We can write the same thing in symbols:

$\frac{x}{6} \times 6$ is the same as x.

We shall need to use inverse operations in this chapter, which is about solving equations.

Here are some examples of equations: $7x = 91$ $\quad 5p - 2 = 38$ $\quad \frac{m}{0 \cdot 9} = 1 \cdot 7$

In each equation the letter stands for an unknown number. Finding the value of the letter is called **solving** the equation.

The method used to solve equations is based on two important ideas **inverse operations** and **balancing**. Balancing means doing the same thing to both sides of an equation, so that the two sides stay equal.

Worked example

(1) Solve the equation $7x = 91$.

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Figure 17: *SMP 11-16 Book R2*, p16.

In this page, a number of words or phrases are introduced: 'operations', 'inverse', 'equations', 'solving', and 'balancing'. A student is told that each of these operations has an inverse. Presumably a student would have to believe this as a fact and so this asks the student to use their *memory*. The examples of operations and of inverses are few and consequently there is little information for a student to *abstract* a sound understanding of each of these words. This is equally true of 'equation', where no definition is given and there are only three examples from which to *abstract* what is common to them. There may be a number of possible common elements, such as they each have a letter, and so there is no guarantee that a student will form an understanding which will fit '3+2=5' as being an equation as well. There is an assumption that each of the equations may be interpreted as involving operations. For example, the equations given as examples do not have a multiplication sign shown whereas a few lines above there is an expression with a multiplication sign. Thus, without previous mathematical knowledge, a student might have difficulty in progressing further.

Earlier, the notion of inverse was shown as a diagram and as an expression using symbols. A student may be able to form an *association* between these so that with one form might come another. However, no help is given to assist the establishment of that association.

The student reading this page is told that the method to solve equations involves the ideas of *inverse operations* and *balancing*. A student is *not being asked to be creative* to develop ways in which an equation might be solved, instead they are being presented with a process to remember. A student is also told that doing the same thing to both sides of the equation means that the two sides remain equal. Once again, like most of this page, a student is presented with a fact and so is in a position of using *memory* in order to hold that information. They are presented with little other material to make use of their other powers.

7.3 Conclusions

> Most grown children no longer practice the powerful learning skills they once utilized so well, the very skills that enable them to be on top of things.
> Gattegno (1986, p128)

The three selections from these text books are typical of much prepared work for mathematics classrooms, and give the impression of students being asked to make extensive use of their memory and to practise within the same subordinate level as that of the mathematical content they have been asked to remember. There is occasionally reliance on previous mathematical knowledge, without which continuation through the pages might prove difficult for a student. Students are encouraged to follow instructions of what to do and how to do it, thus creativity is rarely called upon. Their powers of abstraction, extraction and association are also rarely called upon, and when they are, there is little material for the students to work on. Thus, I assume that there is no expectation for these powers to be used since there is little material on the page to stimulate their productive use. For this reason, I am little surprised by Gattegno's observation quoted above. Dewey called for children's powers to be enhanced, not ignored, by the subject matter:

> What is needed is not an inventory of personal motives which we suppose children to have, but a consideration of their *powers*, their tendencies in action, and the ways in which these can be carried forward by a given subject-matter. (1975, p62)

Ways of presenting mathematics to students, such as are in these text books, do not encourage or help those students to use the very powers which have proved so productive to them in their early learning. Instead, there is a reliance on memory, which I have already indicated is an expensive way to hold information in terms of personal effort and time. Gattegno says that:

> First, we need to stop detouring people and forcing on them the wholly inadequate techniques of memorization and routine employment. Second, we need to develop ways of working that keep individuals in contact with their creative powers. In other words, we need to create the educational methods that give everyone the chance to extend freely the very successful spontaneous learning of early childhood. (1986, p128)

There needs to be a different balance in the powers students are asked to employ in their learning. Gattegno suggests that:

> We cannot save time by being instructed, but perhaps we can save time by recognizing and utilizing more effectively the various roles of the various powers of our self made evident in our various learnings. (1973, p58)

Having recognised such powers in the last chapter, I will discuss the utilisation of those powers in the coming chapters such that personal time can be saved in the learning process. Effort needs to be saved as well since, even if the approaches offered from these text books are successful in terms of students holding the required information, it will have been with an unnecessary expenditure of personal effort due to it being achieved through such great use of memorisation. It is also possible that the approach will not be successful for a number of students since reliance on memory gives the opportunity to forget.

Vygotsky talks of developmental growth with regard to the zone of proximal development:

> What is the zone of proximal development today will be the actual developmental level tomorrow - that is, what a child can do with assistance today she will be able to do by herself tomorrow. (1978, p87)

I propose a development of that theme by saying that *what a student can do with conscious effort today, they can employ effortlessly tomorrow*. This transition requires the student to develop meaning for this something, since there are decisions to be made as to when it is appropriate for this something to be employed. Also, there is a need to make what was a conscious act into an automatic act. Brownell has used the phrase *meaningful habitation*, which encompasses these ideas:

> I have adopted the phrase "meaningful habitation." "Habitation" describes the almost automatic way in which the required response is invariably made; "meaningful" implies that the seemingly simple behavior has a firm basis in understanding. The particular word or phrase for this last step in meaningful learning is unimportant; but the *idea*, and its difference from "memorization," *are* important. (1956, p133)

The difference between memorisation and meaningful habitation is significant on two counts. Memorisation fails to assist something becoming automatic since that something is being retained as a conscious activity when automatism requires the avoidance of conscious activity. Secondly, memorisation treats something as if it was arbitrary and hence does not assist the development of meaning. Thus, methods such as those used in these text books fail to assist the desired progression of learning by focusing on the use of memory. In the coming chapters I will show how the effective use of powers of children and the employment of functionalisation and imagery can help develop progression in learning and meaningful habitation of skills.

Hughes finishes his book, *Children and Number. Difficulties in Learning Mathematics*, by discussing the difficulties for teachers to find sufficient time and resources to give new and important ideas to children:

> Clearly there is a challenge facing us. We have on our side, however, a strength which is often underestimated: the immense capacity of young children to grasp difficult ideas if they are presented in ways which interest them and make sense to them. It is not always easy to design situations which meet these criteria but, as I have tried to show in this book, the attempt to do so is usually worthwhile. If we can redesign our educational environments in the same way so that, instead of nullifying and

> ignoring young children's strengths, we are able to bring them into play and build on them, then I am confident that we will be able to meet the challenge currently facing us. (1990, p184)

I suggest that such approaches as are offered by these texts books are *nullifying and ignoring young children's strengths* - namely the powers of children, of which they made such good use before entering a classroom. In the next three chapters I will outline a set of principles which can guide a teacher to work with students such that the learning environment can bring into play and build on the strengths they had as young children and not put such emphasis on the use of memory. Hughes calls for the need to change educational environments but does not detail how this can be done. I will add more substance and detail to this aim by producing principles by which a teacher can assist students to make greater use of their powers.

7.4 Backwards and forwards

I have analysed some pages of three text books in terms of the powers of children which are called upon. I have found that few powers are called upon and that there is an emphasis on the use of memory, along with repeated practice at the same subordinate level, as a way of holding the information contained on the page. These are, of course, only three text books of the many which are available. Furthermore, I have taken only one page from each of the texts. There are many text books and other pages I could have used as examples. However, I decided that to do more than this was not necessary since this is not a thesis primarily concerned with the analysis of current text books in the UK, but rather with the powers of children and the way students are asked to hold information. Consequently, the purpose of this chapter has been to demonstrate that there is room to improve and extend the ways powers of children are called upon in mathematics lessons, and that other ways of working are required to adjust an over reliance on memory.

In the following three chapters I will develop a theory of how imagery and functionalisation can be used in learning and teaching mathematics, and how an environment can be created which enables students to make extended use of their powers as they learn mathematics in terms of awarenesses rather than facts to be memorised. As I progress through the chapters I will develop principles which assist the above, which I call the principles of economy. To do this, I describe three areas of practice, one in each chapter, which are examples containing many of the principles of economy. I wish to stress at this point that I have deliberately kept the examples of practice to a number which is small but still sufficient to demonstrate the principles involved. Thus, these three are representative of a number of other examples of practice which I could have included. The only reason I have not included more is that they would not add further to the development of the theory and thus would unnecessarily lengthen the thesis.

8 Towards economy: practice and theory 1

8.1 Introduction

For me, the development of theory has gone alongside the development of my practice as a teacher. At times, I have found myself doing something, in the moment during a lesson, which appeared to be particularly productive in terms of helping a student learn. My action was not one I had considered beforehand. On reflection I have sometimes been able to interpret what happened in a way which has informed my theory. At other times, I have used theory to plan considered actions in the classroom. This inter-relationship between myself as a practitioner and as a theorist has been important to me as each has informed the other. In this chapter, I attempt to demonstrate this by, firstly, describing a way in which I have developed a particular well known image so as to use children's powers; and, secondly, to continue with the development of my theory heading towards some general principles of economy. Several of the summary statements from chapters 3, 4 and 5 will feed into this chapter and, when reference is made to them, the statements will appear as footnotes with an indication of which section they came from.

This pattern of practice and theory will continue over the next three chapters. The three examples of practice, one in each chapter, are chosen to illustrate both a development in my practice as a teacher, and to give an example from which the theory can be developed further. Only a few examples of practice are given since I believe that these are sufficient to construct my theory which formulates principles of economy. Different or more examples could have been given but these have not been included for the sake of economy. Thus, each example of practice is given as an exemplar rather than a particular. In this first example of practice, I follow on from the previous chapter by contrasting how a text book presents a well known geometric image with some ways I have developed the image to provide more opportunities for students to make extensive use of their powers.

The theory is developed across the next three chapters and so the theory sections can be considered as a whole whilst each section will relate to the example of practice presented in that chapter.

8.2 Practice 1: Parallel lines

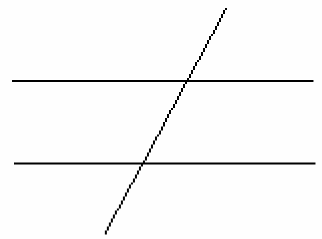


Figure 18: Drawing of two parallel lines and a transversal line.

In the early 1980's, I heard from Laurinda Brown about a starting point for a lesson. This consisted of a drawing of two parallel lines with one transversal across them (see Figure 18). She related to me that Dick Tahta had used this starting point with a class and worked with the students where they were asked to justify statements they made about this drawing. I heard that some of the students had, amongst many other things, talked about one angle rotating on top of another, or translating from one point to another. This came at a time when I was beginning to become aware of my own imagery and the fact that I could imagine movement within static drawings (see Hewitt (1986)). The drawing itself is a standard one which is contained in several text books.

I began to consider what movements I would make to help me see connections between angles in certain situations. I produced a short 8mm film containing a number of movements of lines and angles, and during 1984, I developed some of these movements into a computer film (unpublished) with the help of a programmer attached to the Resources for Learning Development Unit in Bristol. Here, I wish to concentrate on one particular movement related to the drawing above.

Consider one of the parallel lines moving up and down but always remaining parallel to its original position. Now imagine an angle indicated between this line and the transversal line. As the line moves, the angle goes with it (see Figure 19).

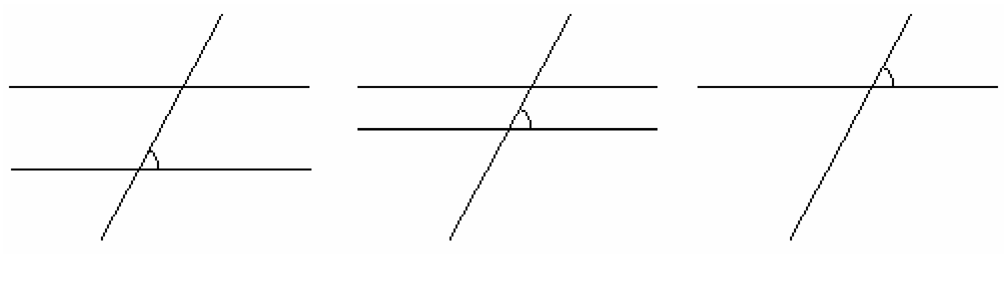
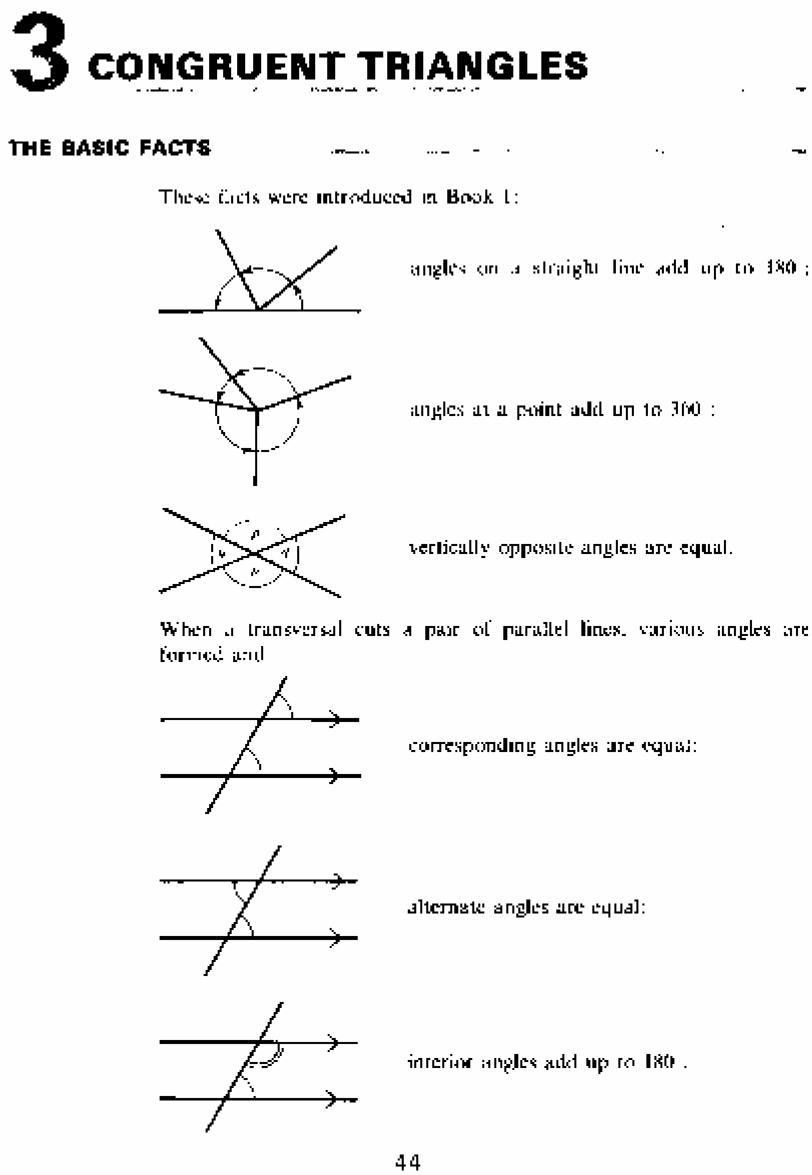


Figure 19: Drawings representing the movement of one of the parallel lines.

At some time during this movement, the line becomes coincident with the second parallel line.

When I have shown this movement to students, it has been as part of the above film (or computer program) which involves many different movements and situations. Thus, when students talked about what they saw in the film, they were *extracting* parts from the complexity with which they were presented. The fact that a part has been extracted means that it is positioned within a situation which extends beyond the particularity of that part. This gives the possibility for that student to be aware of a more general situation whilst considering the particular. Compare this with Figure 20 showing a page from *ST(P) Mathematics 3A* (Bostock, et al (1985, p44)):



3 CONGRUENT TRIANGLES

THE BASIC FACTS

These facts were introduced in Book 1:

angles on a straight line add up to 180°;

angles at a point add up to 360°;

vertically opposite angles are equal.

When a transversal cuts a pair of parallel lines, various angles are formed and

corresponding angles are equal;

alternate angles are equal;

interior angles add up to 180°.

44

Figure 20: *ST(P) Mathematics 3A*, p44.

Here, a student is presented with a particular drawing and a general statement. In order to relate the two, the student has to consider what is the generality of the drawing. The fact that they have not extracted this drawing from a dynamic situation, where the movement offers the generality, a student is stuck with needing to generalise from the particular. Even if a student is successful with this, Mason (1993a, p3) points out that the generalisation they consider may not be the generalisation the teacher, or author of the text book, intended.

Students use their *sense of truth* within the mathematical essence of *same/different* in order to state assuredly that the angle remains constant whilst the line moves in the film, and from the different positions of the moving line, use *abstraction* to state that the certain angles are equal. I have found that students make these statements with ease and certainty when presented with the film[21] . In this way a geometric property becomes known because students use images where they establish the property for themselves. Thus, there is less reliance on the need for this property to be remembered. If it is forgotten, the images can be recalled and worked on anew. In this way knowledge is generated rather than memorised[22] . In contrast, the text book presents the general statement as if it is something to memorise rather than a mathematical awareness.

An additional consideration is that the film presents the moving dynamics of a mathematical situation through the same sense as that which a student uses in manipulating their own visual images. Thus, there is no requirement for a student to translate from one sense to another. Again, this contrasts with the statement in the text book *When a transversal cuts a pair of parallel lines* which requires the reader to translate the written words into what they mean in terms of the picture presented. The film offers no need for translation.

8.3 Theory (1)

The ability to work with images enables a learner to explore a situation by observing the consequences of certain freedoms and constraints. In the case of the parallel lines above, the freedoms include the movement of lines, the constraints include keeping a line parallel to its original position. By exploring the freedoms within the constraints through movement, a learner can see a continuum of particular cases which gives a sense of the generality of those particular freedoms and constraints. Consequently, one drawing can be viewed as a frozen frame from the movement encapsulating the generality and as such a particular drawing becomes an exemplar of the generality. It is

[21] **{Movement allows all possibilities to be considered and properties noted which might otherwise be difficult within a few particular examples}** - 4.2.2; **{The given examples can be positioned within the noted properties}** - 4.2.2
[22] **{Generate knowledge rather than memorise it}** - 4.2.3

important to have the generality first since a different set of freedoms and constraints will produce a different generality. For example, a new set of freedoms and constraints may allow the 'transversal' to be the arc of a circle of any radius and of any centre. In which case Figure 18 is a case where the circle has infinite radius. With this new sense of generality (which I suggest the reader gains from exploring their own image for a while), the angles where the circular arc crosses the two parallel lines will not necessarily be the same, in fact the arc may not even cross one or even both of the lines. Thus, awarenesses can only be gained within a particular sense of generality and each set of freedoms and constraints produce their own particular generality. Movement and the use of images allows the particular generality to be construed which means that awarenesses will be placed within the context of an agreed generality. The use of imagery along with an understanding of a particular generality means that a learner can generate movements and look at them afresh rather than having to memorise a particular drawing. Other generalities can be explored through *what if...* exercises where the learner can be in control of defining their own set of freedoms and constraints.

PRINCIPLE OF ECONOMY: *use imagery and movement so that awarenesses can be gained within a context of generality.*

The careful construction of a dynamic, moving image can lead to asking students to use their powers and mathematical essences to become aware of mathematical content, and to see it, not as a mathematical fact to be memorised, but as an awareness of something which can be noticed within the context of generality which they are capable of generating themselves. In order to have the opportunity to gain an awareness, there needs to be sufficient material available for the learner to use their powers of extraction, abstraction and association. The dynamic movement of an image can offer such quantities of material, or a static image may also hold sufficient complexity for children's powers to be called upon. Learners are used to dealing with a complex environment in many aspects of their life. Examples include growing up surrounded by people using sound as part of an, as yet unknown, language; playing computer games; and learning to drive. Wheeler pointed out that all of the skills we have learned involved meeting complexity at the outset, and said that *when we want to teach a child a skill, we must offer enough of the complexity to work on from the beginning;* (1970, p28). Yet complexity is often avoided in classrooms with this resulting in students not being encouraged to use their powers.

For example, explanations are offered, or statements made, rather than students being encouraged to use their power to abstract meaning. The properties listed in the text book can all be abstracted from awarenesses gained from observing the film or computer program mentioned in the last section. The text book gives separate drawings for each property, thus the properties are presented in isolation rather than coming from a whole, such as a film, where each property can be extracted if required. Working with the whole and extracting whenever there is a desire to attend to a particular aspect, means that the properties remain linked and connected with each other. The desire to itemise and reduce the mathematics curriculum to 'manageable' units, denies students the opportunity to use their powers, the very things they have used to gain such impressive learning throughout their life.

PRINCIPLE OF ECONOMY: *offer sufficient complexity for learners to have the material necessary to make use of their powers*.

Towards the end of section 8.2, I made use of the word *translation*. I do so in a way which is similar to Janvier who states:

> By a translation process, we mean the psychological processes involved in going from one mode of representation to another, for example, from an equation to a graph. (1987, p27)

However, I add the possibility of moving from one sense to another, such as the processes involved to change an instruction or example received through one sense, into meaningful physical or mental action in a different sense. Lesh uses *translation* in a similar way to Janvier whilst also considering possible changes in senses:

> ... a child who has difficulty translating from real situations to written symbols may find it helpful to begin by translating from real situations to spoken words and then translate from spoken words to written symbols; (1987, p36)

However, I will also consider the possibility of the mode of representation remaining the same, as with the physical act of playing a particular shot in squash[23] - the learner is attempting to repeat the same physical action. However, the visual impact of watching a demonstration or hearing instructions have to be changed into appropriate timing and activation of muscles. Thus, I would describe such a demonstration or instruction as requiring a translation from the visual sense to the kinesthetic domain of activating muscles, even though the same mode of representation is involved.

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[23] **{Reduce the need for a learner to translate from one medium to another}** - 3.2.4

An example, of the need to translate within mathematics, is of a teacher who tells a student that a square is *a four sided plane shape which has all its sides and angles equal*. A student would have to translate the words they heard into some personal image of a shape which conforms to such criteria. Here the translation is from the hearing of someone else's words to the creation of a personal picture. In this case, the representation has changed from words to a mental picture as well as from the sense of hearing to that of seeing. Bruner gives an example from his experience where some children were unable to make the translation from the sense of hearing to the activation of muscles. In this case, as what often happens in a classroom, the teacher repeats what they said, only louder! There can be a lack of appreciation that the job of translating is not a trivial one:

> I heard a sailing instructor a few years ago engage with two children in a shouting match about "getting the luff out of the main"; the children understood every single word, but the sentence made no contact with their muscles. It was a shocking performance, like so much that goes on in school. (1966, p10).

These examples, along with Janvier's, demonstrate my usage of *translation* as involving either going from one mode of representation to another, or going from one sense to another, or both.

This notion of translation is linked to the dynamics of the Neutral Zone[24] . Anything a teacher does or says[25] , does not go straight into a student's understanding, it becomes material[26] for which a student will have actively to develop meaning in order to have an understanding of what is being asked of them. Even if this is achieved, the student will have to work on translating that into their own thoughts or actions.

PRINCIPLE OF ECONOMY: *reduce the need for translation*.

If a teacher wants to help a student to do something of which they have no knowledge at present, one way is to tell them through explanation or demonstration which leads to the need for translation as mentioned above. An alternative is to work with what a student can do[27] and help them move towards a desired goal by attempting to focus their attention on certain aspects of what they can do and not on other aspects of what

[24] **{Neutral Zone}** - 4.2.5
[25] **{Offerings}** - 4.2.5
[26] **{Material}** - 4.2.5
[27] **{Pay attention to what you can do rather than what you cannot do}** - 3.2.4

they can do[28] . A teacher can act as an editor and/or amplifier of certain aspects of what the student can do[29] and, because the teacher is working with what the student does, this reduces the amount of translation the student needs to do. In this way a teacher can attempt to help a student become aware of certain aspects of what they can already do.

A teacher can ask a group of students to share some of the images they recall from seeing a film such as the one I described concerning parallel lines. What the students share are their own images, which may or may not be similar to sequences in the film, and these constitute the material with which the teacher has to work. The teacher does not work on the film images but works on the students' images. Once shown, the film has performed its function of stimulating the students to have mathematical images which can then be worked on. When a student does share their image either in terms of a verbal statement or a drawing, then they are revealing some awarenesses they have. After all, they cannot state what they are not aware of. Thus, it is these awarenesses a teacher can work with, and with pedagogic skills, the teacher can work with the awarenesses, which are contained within the offerings given, to help students gain new awarenesses.

I recall a colleague, Adrian Underhill, saying at a meeting in Bristol in 1983, something similar to *My task as a teacher is to attend to the learner. The learner's task is to attend to the mathematics*. This relates to Dewey who said:

> The problem of the pupils is found in *subject matter*; the problem of teachers is *what the minds of pupils are doing with this subject matter*. (1933, p275)

Also, Wheeler said:

> He [the teacher] must use every means he can find to focus the attention of the children on the problem, and this means that he *must* efface himself from their attention. On the other hand, the children will be at the centre of *his* attention because he must study them to know how to help them keep to their task. (1970, p27)

PRINCIPLE OF ECONOMY: *use pedagogical awarenesses to work with the learner's mathematical awarenesses*.

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[28] **{Affecting attention is possible}** - 4.2.6
[29] **{Teacher as amplifier}** - 4.2.8; **{Teacher as editor}** - 4.2.8

In order to work in this way, a teacher needs to pay attention to the awarenesses and abilities students demonstrate at that moment in time. Since the teacher is working with what the students are doing or saying, they will not know what material they have to work with unless they pay attention to the students[30] . There is an immediate problem here for the writers of text books since they will not know in advance what a particular set of students may be saying or doing. The medium of the written page cannot respond to the thoughts and actions of the reader. Thus, written material is of limited value.

One role written material can play is as an initiator of an activity or conversation. Even then there may be the question *what level do I pitch it at?* What abilities can someone assume students have? One set of abilities are the powers of children which students still possess. Another set of abilities are children's awareness of mathematical essences which they have come to know through their everyday learning. Whether particular mathematical content can be assumed is another matter. The possibility of students being unable to engage, because they have forgotten some mathematical content, can be avoided by approaching the desired mathematical content through evoking children's powers and relating to mathematical essences[31] . The use of film offers the possibility for students to gain an awareness about corresponding and alternate angles directly from working on the film without the need for knowing about previous mathematical knowledge such as the number of degrees in a triangle or a straight line, or having to be able to measure accurately using a protractor. Thus, the film offers a direct entry to angle properties, and it is a *pedagogic* awareness that leads to the consideration of using a film. I wish to contrast this with an awareness of other mathematical content which can form the foundation for learning these angle properties (such as number of degrees in a straight line,...). This latter approach I refer to as approaching teaching as a *mathematician* since reference is constantly being made to other items of mathematics which will need to be known in order to learn new mathematical content. Approaching mathematical content as a pedagogue can mean avoiding the need to remember previous mathematical knowledge and offering a direct entry to the desired mathematics using the powers of children and mathematical essences[32] .

In a recently published text book, *Mathematics Level 8* (Holderness (1992, p14)), ratio is approached following a section on fractions (see Figure 21):

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[30] **{A teacher is in a position to make pedagogic decisions when they become aware of what their student is aware of}** - 5.2.2

[31] **{Approach mathematics through powers used in early learning rather than previous mathematical content}** - 3.2.5

[32] **{Questioning my understanding of an area of mathematics as a pedagogue and not as a mathematician}** - 4.2.9; **{What are the pedagogical roots of a piece of mathematics?}** - 4.2.9

14 FRACTIONS AND RATIOS

[redacted]

A ratio is a way of comparing the sizes of two quantities.
e.g. A quantity divided in the ratio 2 : 3 (read as 2 to 3)
means that the 1st share is $\frac{2}{3}$ of the 2nd share,
and the 2nd share is $\frac{3}{2}$ times the 1st share

Ratios have no units: they are just numbers

Examples

1 Express the amounts of £1.40 : 60p as a ratio in its simplest form

First express them as a fraction and then write that fraction as a ratio.

$$\frac{£1.40}{60p} = \frac{140p}{60p} = \frac{\cancel{140}^{7}}{\cancel{60}_{3}} = \frac{7}{3}. \quad \text{The ratio is } 7 : 3$$

2 Express 30 cm, 1.5 m, 2.5 m as a ratio in its simplest form

Since there are 3 quantities we cannot write them as a fraction.

30 cm : 1.5 m : 2.5 m = 30 : 150 : 250 = 3 : 15 : 25

3 Divide 60 kg in the ratio 2 : 3

First divide 60 kg into 5 equal parts (since 2 + 3 = 5)
60 kg ÷ 5 = 12 kg
The 1st share is 2 × 12 kg = 24 kg
The 2nd share is 3 × 12 kg = 36 kg

4 Increase 2.5 m in the ratio 11 : 10.

The new amount is $\frac{11}{10}$ of 2.5 m $= \frac{11}{1\cancel{0}} \times 25\cancel{0}$ cm = 275 cm = 2.75 m.

5 Decrease 3.6 kg in the ratio 4 : 9.

The new amount is $\frac{4}{9}$ of 3.6 kg $= \frac{4}{\cancel{9}} \times \cancel{3.6}^{0.4}$ kg = 1.6 kg.

Figure 21: *Mathematics Level 8*, p14

Ratio is explained by using the mathematical content previously mentioned in the earlier pages of the book. Thus, for a student to be successful at ratio on this page, they would have to remember the previous work on fractions. This I describe as approaching the mathematical content of ratio as a mathematician, in the sense that it is based on the idea of mathematical content being built up from previous mathematical content.

In contrast, I developed the following activity (see Hewitt, et al (1987b)) which approaches ratio requiring minimal previous mathematical knowledge but which calls upon the mathematical essences of order, inverse and same/difference. I attempted to approach the teaching of ratio by creating an activity which gives direct access to operational activity in the area of ratio without the need for students to remember previously encountered mathematical content. Thus, I approached this task as a pedagogue. This fictitious account of a lesson is based on my experience of giving several similar lessons to students of different ages and is included for the reader to contrast the approach to ratio with that given in the text book above.

I begin by asking for someone to help. Howard offers. I ask him what table he is really good at and he replies the 5 times table.

DH: Let us practise: 3.
Howard: I don't understand. What am I meant to do?
DH: You said you were good at the 5 times table. So let's practise it. I'm saying 3.
Howard: 15. (Said with some uncertainty)

DH: 8.
Howard: 40. (More confident)

DH: 5.
Howard: 25.

This continues for some time. I ask the whole class what is happening going from me to Howard. They reply *times 5*.

DH: Now let us try it the other way round. You start (speaking to Howard).
Howard: 7.
DH: Ow! Now you have made it difficult for me. Say a number that makes it easy.

Howard looks confused. Others in the class help by telling him he has to say something in the 5 times table.

Howard: 20.
DH: 4. And another.

Howard: 45.
DH: 9. Is this right? (To the whole class.)

This continues for a while. I then ask what is happening going back from Howard to me. They reply *dividing by 5*.

DH: What happened from me to Howard?
Class: Times 5.

DH: What happened from Howard to me?
Class: Divide by 5.

I then ask for someone else to help me. John volunteers and says he's good at the 7 times table.

DH: Right. Let us practice this: 8.
John: 56.

DH: 2.
John: 14.

DH: 1.
John: 7.

I ask the class if this is right and what is happening going from me to John. They reply *times 7*. We then practice coming back from John to me and the class tell me that it is dividing by 7.

Now the work really begins, we start by carrying on a bit longer:

John: 70.
DH: 10.

John: 49.
DH: 7. (Now I turn and look expectantly at Howard.)
Howard: (Pause)...What? (He has a slightly confused look.)
DH: I said 7.
Howard: Oh, 35.

I look expectantly at John.

John: 63.
DH: 9. (Looking at Howard)
Howard: 45.

Looking again at John.

John: 14.
DH: 2.
Howard: 10.

John: 21. [I write '21' on the board]
DH: 3.
Howard: 15. [I complete on the board '21:15']

This continues with the notation going on the board as they say their numbers, then...

John: 42.
DH: (To the whole class) Don't say anything but do you know what I will say?
DH: What is Howard going to say?
Class: 30. ['42:30' is written on the board]

I can now withdraw from the central role I played although the class will still make reference to the number I would say. They can make the journey from Howard to John or John to Howard by maintaining the image of going through me. My freedom allows me to watch individuals closely and work on those that may not look comfortable or confident.

After a while I bring in a third person (usually someone who looks the least confident, if they are willing). Kath chooses the 9 times table, and I work with her in the same way I did with the others so that she is comfortable with journeying from me to her and back again. Then on the board I write:

Howard	:	John	:	Kath
5	:	7	:	9

Kath starts:

Kath: 45. -- : -- : 45

I point to me but put a finger over my mouth so that everyone thinks what I will say but does not say it. I point to John.

John: 35.	--	: 35	: 45

I point back to me and then to Howard.

Howard: 25.	25	: 35	: 45

We continue playing, with different people starting and varying the order. More people are brought in, different tables chosen. Calculators are given out so that we are not restricted to easy numbers. Since the attention has always been on the inherent algebra of the situation (i.e. the process), there is no difference whether we start with 45 or 617.9074, we still do the same. Likewise we are no longer restricted to whole number tables. Gradually less is said, the tables become less connected to particular people and questions are asked through the notation (which has been learned as a consequence of the activity). For example:

7	: 2	: 3.6	: 5.13
	: 23.7	:	: ?

One power of this activity is that it is devoid of context. Because of this fact, I can now put it in any context I wish:

Simple change of units including foreign currency:

Francs	: Pounds	: Dollars
9.77	: 1	: 1.62
?	: 17.50	:

Or, percentage:

Original amount	: V.A.T.
100%	: 17.5%
312	: ?

Or, scales:

Map	:	Real life
1	:	50 000
5.2cm	:	?

PRINCIPLE OF ECONOMY: *view mathematical content through the eyes of a pedagogue rather than a mathematician. This means considering mathematical content in terms of children's powers, mathematical essences, and awarenesses*.

With both of the two situations described above, the film concerning parallel lines, and the ratio activity, students can come to know the desired mathematical content through awarenesses from the situations rather than having to be told by a teacher. Thus, the above principle involves creating an activity whereby the content can be obtained directly from awarenesses obtained from engaging in the activity with little or no reliance on previously encountered mathematical content. Mathematical essences can be called upon since students have experiences of these within their daily lives. In this way, mathematical content can be approached in a number of possible orders since each item of mathematical content on a syllabus need require little or no previous mathematical content[33] .

PRINCIPLE OF ECONOMY: *reduce the need for a learner to remember previously encountered mathematical content*.

8.4 Backwards and forwards

In this chapter I took a well known image and considered how the use of that image could be developed so as to make use of children's powers.

In the next chapter, I take another well known idea but describe in detail the way in which I have developed a precise and calculated way of working with students which not only makes use of the powers of children but also drives some of the students' learning into functionalisation.

[33] **{Mathematical sophistication does not imply learning difficulty}** - 3.2.5; **{Without attention to the powers of children, we may be seduced into basing the order, approach and progression of out teaching on mathematical sophistication}** - 3.2.5

9 Towards economy: practice and theory 2

9.1 Introduction

This activity is based on a well known and used idea. However, I have adapted and developed the idea, in terms of the way I work with students, as a consequence of my theoretical considerations on the principles of economy. This way of working uses many of the principles mentioned in the last chapter and also develops strategies to drive some of the learning students do into becoming functionings. Parts of a lesson are described in detail and analysed in terms of children's powers called upon and the way in which I work towards functionalisation. The precise and calculated way in which I work in this lesson is the result of many lessons of using this idea with many students of different ages and abilities.

As with the previous chapter, this particular lesson is used as an exemplar in order to demonstrate some of the principles of economy in a practical classroom activity, and also as a vehicle to develop my theory further. Thus, section 9.3, *Theory (2)*, is a continuation of the development of my theory from the last chapter, as well as analysing aspects of this particular lesson. Several of the summary statements from chapters 3, 4 and 5 will feed into this chapter and, when reference is made to them, the statements will appear as footnotes with an indication of which section they came from.

9.2 Practice 2: Think of a number

Something is done to an unknown number and the result is stated. The task for the students is for them to find the original number. I began considering the potential of this activity after hearing Gattegno (at a seminar in London, 1982) say that he would expect students to be able to articulate how they arrived at finding the original number if the case presented was arithmetically accessible and involved about two operations (see Fyfe (1957) for someone else who has been influenced by Gattegno with regard to this activity). I tried this out on a number of classes and found this to be true. Since then I have been developing and refining ways in which I can make use of this fact to help students develop algebraic awarenesses and gain confidence with formal notation in as little time as possible. The following quotes are taken from a lesson in 1990 with a middle set year 9 (13-14 year olds), sections of the lesson form part of the Open University video *Working mathematically on symbols in key stage 3* (OU (1991)).

During the lesson, I tried to draw attention to the processes involved in finding the unknown number.

1	DH:	*I'm thinking of another number. OK? I add three, then I times by two, and I get 14... Think of a number, I add three, I times by two, and get 14... Naome?*
2	Naome:	Four. (DH looks at several individuals and each says four)
3	DH:	*I think of a number...* (hand placed by ear).
4	Class:	Four.
5	DH:	*I add three...*
6	Class:	Seven.
7	DH:	*Times by two...*
8	Class:	14.
9	DH:	*So does it work?* (straight face).
10	Class:	Yes.
11	DH:	*Wonderful. But how can you get my number from what I said, because I said... what did I say?... I said... add three, times by two, and get 14. How can you find out my number from what I just said? Sam?*
12	Sam:	You divide 14 by two and take three from the end.
13	DH:	*Say it just once more.*
14	Sam:	You divide 14 by two and you take three off the number you get when you've done that.
15	DH:	*OK. We'll do that. Say it again.*
16	Sam:	You divide 14 by two...
17	DH:	(Clicks fingers) *Which is?* (to class)
18	Class:	Seven. (DH points to Sam)
19	Sam:	And you take three off that.
20	DH:	*Which is?*
21	Class:	Four.
22	DH:	*Does that work?*
23	Class:	Yes.
24	DH:	*OK. Right. And what did I say? What did I end up saying again? I said I'm thinking of a number and... Go on Shona.*
25	Shona:	Times by two.
26	DH:	*Say it again.*
27	Shona:	Add three, times two... equals 14.
28	DH:	*And what are you going to do to find out my number?*
29	Naome:	14 divided by two, take three.
30	DH:	*And what did I say?*
31	Shona:	Think of a number, add three, times by two, equals 14. (DH points to Naome)
32	Naome:	14 take... divide by two and then take three.
33	Shona:	Think of a number, add three, times by 2, equals 14.

34	Naome:	14...
35	DH:	*Now, whilst this is going on, I'm going to ask the rest of you to tell me what is different about what I said and what you did to work it out. OK?*
36	Shona:	Think of a number, add three, times two, equals 14.
37	Naome:	14 divided by two, take three, is four.
38	DH:	*Ah. Shh! Don't give away the answer. Hang on. Right. What's different? Say it again* (to Shona). *Sorry you're going to get really bored by this. Right, go on.*
39	Shona:	Think of a number, add three, times by two, equals 14.
40	Naome:	14 divided by two, take three.
41	Shona:	Think of a number, add three, times by two, equals 14.
42	Naome:	14 divided by two, take three.
43	DH:	*Go on Ben.*
44	Ben:	Four.
45	DH:	*Sorry?*
46	Ben:	Four.
47	DH:	*Right, the number's four. OK. What is different about what I said and what Naome is saying about how to work it out?*
48	Ben:	Turning it around the other way.
49	DH:	*Turning what around?*
50	Ben:	The numbers.
51	DH:	*OK. Can you just say the numbers? Shona.*
52	Shona:	Think of a number, add three.
53	DH:	*Right, just the numbers.*
54	Shona:	Three... two... 14.
55	Naome:	14... two... three.
56	DH:	*So that's right is it? OK. Right. Is that right? Uha. And what else is different? The whole lot again* (to Shona).
57	Shona:	Think of a number, add three, times two, equals 14.
58	Naome:	14 divided by two, take three.
59	DH:	*What else is different? What else is different? Jo.*
60	Jo:	Instead of... you got divided and take away instead of add and times.
61	DH:	*Right, so can you just say the... which one did Shona say?*
62	Jo:	Add three, times two.
63	DH:	*OK. So just say the three bit.*
64	Shona:	Add three.
65	DH:	*Just say the three bit* (to Naome).
66	Naome:	Take three.
67	Jo:	So it is the opposite.
68	DH:	*Uha. And what other number... say the other* (to Shona).

69 Shona: Times two.
70 Naome: Divide by two.
71 Jo: It's the opposite again.

Before considering some of the detail, there are some general observations I wish to make. The questions I asked focused attention on the processes involved in finding my number. By asking such questions I wanted to affect the students so that their attention would become focussed on things which were significant to these processes. Throughout the lesson, my questions generated responses from the students and it was these responses, or offerings, I worked with. The offerings also became the material which others in the class could work with[34]. Each offering gave me some indication of what that individual was aware of, and this informed decisions I made as to what I might or might not do next[35]. It also meant that I was working with their awarenesses by using my own awarenesses[36].

Here, I begin by presenting the problem and attention is on the answer (line 2). I check this answer out with the class (lines 3-10). Then I shift attention from performing the arithmetic to the operations themselves (line 11). Here, I appeal to someone's notion of *inverse*. I have an expectation that someone in the class will be able to articulate the inverse operations, as Gattegno had suggested and as my experience has borne out. This articulation (line 12) concerns the particular situation I presented. The fact that Sam could do this successfully does not mean that she is aware of what she has done[37] and can use those generalisations in more complex situations. Thus, my task is to help students reflect on that doing to develop new awarenesses which can be applied to more complex situations[38].

A checking process is gone through (lines 15-23) again which gives an opportunity for other students in the class to be sure that this process does successfully arrive at my 'unknown' answer (which is now known to be four).

I set up Naome and Shona to repeat the process I said when presenting the problem and the process Sam did in getting to the unknown number. This provides material for the class to work on when engaging in the task I set in line 35. This task requires students to use their notion of *difference* with this material and to do this they will need to *extract* parts of what Naome and Shona are saying and compare them.
Ben articulates an awareness that the numbers are turned around (lines 48 and 50). This

[34] **{Offerings}** - 4.2.5; **{Material}** - 4.2.5
[35] **{A teacher is in a position to make pedagogic decisions when they become aware of what their student is aware of}** - 5.2.2
[36] **{Working with awareness}** - 5.2.2
[37] **{Our knowledge consists of more than that which we might be able to articulate}** - 3.2.4
[38] **{Successfully do, followed by reflecting on that doing}** - 3.2.4

indicates a *difference* and also indicates a *sameness* since Ben refers to 'the' numbers, the same numbers.

I used the material Naome and Shona provided by acting as an editor[39] in order to shift attention onto what Ben had just talked about (lines 51-53). I used the technique again in lines 63, 65 and 68 to place attention on Jo's awareness of the operations changing.

Ben's awareness is concerned with *order* and Jo's is concerned with *inverse* operations. Thus, they are working with mathematical essences.

I continued the lesson by making use of Jo's awareness by applying it to other particular numbers. In this way I am *amplifying* the attention that has been placed in the changing of the operations[40] . This amplification of attention in the operations has the consequence of keeping attention away from the number involved.

72. DH: *So, if I do times two, you're going to...?*
73. Jo: Divide by two.
74. DH: *If I were to add seven, you are going to?* (to class) *I'm going to add seven. What are you going to do?*
75. Girl3: Take seven.
76. DH: *If I add 49?*
77. Boy2: Take 49.
78. DH: *If I add 73?*
79. Girl4: Take 73.
80. DH: *If I add 59.2?*
81. Girl5: Take 59.2.
82. DH: *If I add 73.9?*
83. Several: Take 73.9.
84. DH: *If I were to add 0.00003?*
85. Class: Take 0.00003.
86. DH: *If I were to add 59 million?*
87. Class: Take 59 million.
88. DH: *If I add 73.*
89. Class: Take 73.
90. DH: *If I add 59.2?*
91. Class: Take 59.2.
92. DH: *If I add beta?*
93. Class: Take (away) beta.

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[39] **{Teacher as editor}** - 4.2.8
[40] **{Teacher as amplifier}** - 4.2.8

94. DH: *If I add gamma?*
95. Class: Take gamma.
96. DH: *If I times by 72?*
97. Class: Divide by 72.
98. DH: *If I times by delta?*
99. Class: Divide by delta.

Attention was on what is different - the changing of the operations, and not on what stays the same - the number. Thus, the number became just some word or words which were repeated after the operation had been changed. This placement of attention meant that I could say 'beta' or any other word and it would be repeated. The generalisation which was established was that the number does not matter in terms of what you do. At this stage I could get the students to 'do' the inverse correctly when Greek letters appeared since their attention stayed with an awareness they had of the operations.

A few minutes later the lesson continued as below. During this part of the lesson the following equation was gradually created on the blackboard:

$$6\left(\frac{2(x+3)-5}{3}+72\right)=100$$

142. DH: *I'm thinking of a number. Oh dear, what am I going to do with this one? Oh yes, I'm going to add three, times by two, take away five... divide by three... add 72... Got a problem with this? Do you want me to write down what I am doing?*
143. Class: Yes.
144. DH: *OK. So,... let me see... I am thinking of a number* (writes 'x' on the board)... *I add three* (writes '+3')... *then I'm going to... multiply by* (writes brackets round the expression so far)... *two* (writes '2' in front of the brackets)... *then I am going to take away five* (writes '-5')... *then I am going to divide by* (writes a line underneath the expression so far)... *three* (writes '3' below the line)... *then I'm going to...* (makes a noise whilst going along the division line from left to right, writes '+' following on from the division line and makes a different double noise whilst the addition sign is being written) *add... 72* (writes '72' after the addition sign)... *then I'm going to multiply by* (writes brackets round the expression so far)... *ummm... six* (writes '6' in front of the brackets)... *and I get* (writes '=' to the right of the expression so far)... *umm... 100* (writes '100' to the right of the equals sign)... *So you are going to?...*

145. Shona: Think of a number, add three, times by two, take...
146. Girl5: Five.
147. Shona: ...five, divided by three, times six, add 72, equals 100.
148. DH: *I think I said add 72 and then times by six.*
149. Shona: Oh yeh... add 72, times by six, equals 100.

As the lesson continued, the following equation was gradually written on the board below the original one:

$$\frac{3\left(\frac{100}{6} - 72\right) + 5}{2} - 3 = x$$

150. DH: *OK. So, how am I going to work out my number? Gemma.*
151. Gemma: 100 divided by six.
152. DH: (Writes '100') *100.* (Points to '100' in the original equation)
153. Gemma: Divide by six.
154. DH: *Aha. And you're dividing by six because...* (writes a line underneath the '100' and then '6' underneath that line)
155. Gemma: Because it was times six. So you... (unclear)... divide by six.
156. DH: *So that's that done.* (Points to '6' in original equation)
157. Gemma: Yeh. Then take 72.
158. DH: *Aha.* (Writes '-72' to the right of the division line) *OK. And after that, I've done that.* (Covers '72' in original equation)
159. Gemma: Times three.
160. DH: *Because I...*
161. Girl8: Divided...
162. Gemma: Divided three...
163. DH: *So you're going to...*
164. Gemma: ... so it's times three.
165. DH: *Times by...* (Writes brackets round the expression so far and writes '3' to the left of the brackets) *Right.* (Covers '3' in the original equation) *We are left with this.* (Hugs '2(x + 3) - 5' in the original equation)
166. Gemma: Divide it by two... oh no...
167. Girl9: That's what I thought.
168. Gemma: Add five.
169. DH: *Go on.*
170. Gemma: Add five. Divide it by two.
171. DH: *Sorry... add...* (Writes '+' to the right of the brackets)

172. Girl9: Five.
173. Gemma: Five. (DH writes '5' to the right of the addition sign)
174. DH: *Right. So that's that done.*
175. Gemma: Divide by two.
176. DH: *Aha.* (Makes a noise whilst drawing a line underneath the expression so far) *Divide by two.* (Writes '2' underneath the division line)
177. Gemma: Take three.
178. DH: *Right and...* (Makes noise whilst going along the division line from left to right, and a different noise whilst drawing '-') *take...*
179. Gemma: Three.
180. DH: *Three* (Writes '3' to the right of the subtraction sign) *And I end up with...*
181. Girl10: A number.
182. DH: (writes '=' to the right of the expression so far) *What do I end up with?*
183. Several: Four.
184. DH: *Gosh you worked that out carefully. I hadn't worked that out... Well I end up with the number I was thinking of, whatever that is. And when I said "I'm thinking of a number", what did I write down? When I first started off saying "I'm thinking of a number", what did I write down?*
185. Student2: 'x'.
186. DH: *Right.* (Writes 'x' to the right of the equals sign) *So I end up with whatever that number is.* (Bangs on the 'x' in the equation just written) *And we could work it out and find out what it is. Has anyone got a calculator?*
187. Several: Yes
188. DH: *Does anyone need a calculator?*
189. Several: Yes.

By this stage, the task for many of the students had shifted to describe how to get my number rather than attempting to state what the number was. For some students there was still some attention on the latter (line 183). I help to shift their attention away from this by choosing to ignore such attempts (lines 184-186). The fact that the suggestion of *four* was incorrect, and possibly a carry over from the previous activities, was not a factor in my choice to ignore it. I would have ignored a correct answer as well.

In line 142 I began giving a list of operations I did to my number. In my mind, this particular list of operations was an example of a general case. I indicated this implicitly by my comment *Oh dear, what am I going to do with this one?* and by the hesitations I made between saying what I was doing to my number in line 144. I was indicating that I was making this up as I was going on. This is such that there is a sense of the general in this particular example. The example is exemplary, or paradigmatic (Freudenthal

(1980)), containing a sense of the general, rather than a particular case. Freudenthal (ibid, p198) criticised times when children where given *... a shower of exercises instead of one that hits the mark*. However, by gaining a sense of the arbitrariness of the operations mentioned, the example in the above lesson contains a sense of the general and so becomes paradigmatic. An additional point is that the availability of calculators (lines 186-189) means that I do not have to be concerned with potentially 'difficult' calculations even when the students turn their attention to carrying out the operations. This allows me the freedom to make up operations and not have to be concerned whether the students will have the ability to perform these calculations either mentally or with paper and pencil.

The ability to describe the process was something which several members of the class had shown confidence with. In line 142, I deliberately gave a list of operations which were too long for the class to remember (at the beginning of the lesson I had stated that they wouldn't need to do any writing so they could pack things away). This created a need to have a way of recording the list of operations. By not allowing them to write anything, I provided the one record of what I said with the equation written on the board. Thus, this formal notation became the only means they had to hold on to what had been said. Vygotsky (1978, p38-39) identifies two types of memory. He calls *natural memory* the retention of actual experiences, and *mediated memory* where there is use of signs to assist remembering. The notation provided on the board acts as a mediating agent for them to recall the operations and their order.

The task for the class remained the same - to say what they would do to get my number. Although many people in the class knew what to do, they needed to be aware of what had been said and in which order. Thus, they had to use the formal notation in order successfully to carry out the task. The notation became subordinate to the task. The attention remained with the task of saying what they would do in order to find my number. Although the focus remained with the task, the focus went through the subordinated notation on the way to the task. Thus, although there was attention on the notation, that attention was still focussed on the task at a higher effectual level. There were never questions of the sort *what does that line mean?* or *why didn't you write the multiplication sign?* which would have indicated that the focus of the attention was on the notation.

As the lesson progressed, new tasks were set which still required the formal notation to be subordinated. With the use of calculators, the students carried out the calculations to find out my number. This involved them interpreting the notation to know what calculations had to be done and in which order. Still later in the lesson, I stopped saying what I was doing to my number and just wrote the notation on the board instead. As I wrote, I asked them to say what I was doing to my number. At each stage the notation was playing a subordinate role.

Introducing the notation as something which was subordinate to a task at a higher effectual level, and which always remained so, meant that interpreting formal notation was practised whilst attention was elsewhere on the main task[41] . This helped drive the interpretation of formal notation into becoming a functioning. Thus, this was where the learning for life was taking place. The really productive learning during this lesson was concerned with formal notation, not the solving of linear equations. The solving of equations was not being driven into functionalisation because it was not being made subordinate to a task at a higher effectual level.

In 1985, I taught a mixed ability year 8 (12-13 year olds) class. For about two weeks we spent time working on solving linear equations. The first lesson had been similar to the one detailed above. The next academic year, the year group were split into sets and one boy, Paul, had gone into set three out of four and was being taught by another member of the department. In the summer term of that year I met Paul when he had been sent out of his class. He had no work to do so I asked him whether he had done any algebra this year. He said that he hadn't. So I thought I would find out how many of the ideas from the previous year had remained with him. I gave Paul the following information written on a piece of paper, asked him to solve it for me, and left the room.

$$\frac{3\left(2\left(\frac{3x+6}{7}\right)-6\right)}{7}+8=y \qquad \mathrm{y} = 20.8$$

On returning I found that Paul had written:

$$7\left(\frac{\left(\frac{\left(7\left(\frac{\mathrm{y}}{3}\right)-6\right)}{2}\right)+6}{3}\right)-8=\mathrm{x}$$

The following conversation took place:

DH: *If I double and add one, what are you doing?*
Paul: Halving and taking one.

[41] **{Attention is placed at a higher level to that which is to be learned}** - 5.2.3

DH: *I've got a number. I times it by two, add one and get five. Do you know what my number is?*

Paul: I take one and divide by two.

DH: *That's not what you said before.*

Paul: Ah... I know...

Paul began writing again and I left the room. When I returned I found that Paul had now written:

$$\left(\frac{7\left(\frac{\left(\frac{7y - 8}{3}\right) + 6}{2}\right) - 6}{3}\right) = x$$

One year on from doing such work, Paul was having partial success at the perceived task of solving the equation, but what he was quite confident with was interpreting and writing what he wanted within formal notation.

The *think of a number* lesson is not isolated. I develop the work to consider what might happen if I had not started with 'x' in an equation such as:

$$\frac{3(x + 4) - 7}{5} + 2 = 14$$

but had started with '4', or '3', or '2', or '(x + 4)'. I consider equations which contain a variety of letters, sometimes with no numbers at all. Another development involves simultaneous equations, where there are two numbers I am thinking of and the students are trying to find them out. The details of such a lesson are included in appendix 2, but here I will only mention the fact that some of the students were trying to solve the simultaneous equations algebraically. In doing so, there was sometimes the need to re-arrange equations before they arrived at one which was in a form which they could solve. Thus, they were having to subordinate the manipulation of the notation in order to be in a position to carry out the desired task of finding my numbers. This subordination drives the manipulation of equations into becoming a functioning, and the manipulation of equations subordinates the reading and interpreting of the notation. Thus, the latter becomes two subordinate levels away from the main focus which helps drive that further into functionalisation.

A final comment about this lesson concerns the use of sound and movement. At times I wished to draw attention to the fact that something has been written on the board in a particular way. For example, in line 144, I made one sound whilst I was going along a division line from left to right, and another double sound as I wrote an addition sign at the end of the line. The movement is to draw attention to the placing of the addition sign and the use of sound is offering an aural image associated with this placement. Although it did not happen in this lesson, there have been several lessons I have taken where some students have ended up making equivalent noises as they write parts of an equation. The recalling of the aural image can bring with it the association of the movement and placement of symbols. I will develop this aspect further in chapter 10.

9.3 Theory (2)

Throughout this lesson, care was taken as to the placement of the students' attention. This is not something which a teacher has control[42] , however, a teacher can attempt to affect where the students place their attention by using techniques such as editing (lines 51-71) and amplifying (lines 72-99). Attention can also be affected by questioning. If someone is attempting to answer a question, then their attention may become placed in what they perceive as relevant to that question. Thus, the placement of the students' attention can be affected by the choice of questions which are asked. For example by asking the question *how can you get my number from what I said...?* I encouraged attention to be shifted away from finding my original number, and towards the operations involved in getting to the number[43] . The placement of attention is crucial to many of the principles of economy of learning and teaching mathematics[44] .

There were a number of times during this part of the lesson when something new was being introduced to the students. The two occasions I will consider here are (a) the introduction of 'beta' and (b) the introduction of formal written algebraic notation. Both of these are arbitrary and so need to be given to students in some way[45] . When both of these were introduced, I had attempted to have the students' attention somewhere else.

(a) With respect to 'beta', I was amplifying the awareness that the operation was changing, and with attention on the operation that meant that attention was away from the numbers themselves. The sense that the number stays the same and it is only the operation which needs attention, means that any number-name could be said and they would know what to do. Attention was taken away from the number-name to such an

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[42] **{Affecting attention cannot be guaranteed}** - 4.2.6
[43] **{Affecting attention is possible}** - 4.2.6
[44] **{Where attention is placed is crucial to what learning may take place}** - 4.2.7
[45] **{Provide that which is arbitrary}** - 4.2.3

extent that any word could be said, such as 'beta', and the students would have their attention on the operation, change that, and repeat the word, 'beta', afterwards. Thus, the introduction of the letter came after there had already been established an awareness that the name stays the same, and this was partly achieved by shifting the students' attention away from the number-names and onto the operations. In this way, there was nothing special about 'beta', the generality had already been established and any word could have been said at that time.

The build-up to this part of the lesson (lines 72-99) involved accepting 'correct' replies from those that offered them, and gradually having more of the class feeling confident that they could join in with replies which would be accepted. At this stage all that can be established is that the vast majority of the class, if not all of them, could respond in a way which was acceptable. I had taken their attention away from the number-names and introduced 'beta' such that they were likely to respond 'correctly'. Thus, the aim for me was to get the students to be able to respond 'correctly' when arbitrary numbers and letters were involved. This is a process of enculturation, with students coming to know what is accepted within their social environment. In this case I play a role of accepting some things and ignoring others. Once students are responding 'correctly', there is the possibility of them reflecting on what they are now able to do, and it is this reflection which offers the opportunity for them to become aware of, for example, how letters may be used, or how the mathematical essence of inverse can be applied to arithmetic operations. A new awareness can come from reflecting on what they can already do; they cannot reflect on what they cannot do. Thus, there is a need to be able to do something first in order to be in a position to reflect on it afterwards[46] . Whatever understanding someone has may be in terms of knowing what needs to be done and how to do it. Meaning and understanding can be developed through the process of reflection on this doing. Dewey suggests that some understanding and meaning is required before reflection, as an act of thinking, is possible:

> We reflect in order that we may get hold of the full and adequate significance of what happens. Nevertheless, *something* must be already understood, the mind must be in possession of some meaning that it has mastered, or else thinking is impossible. (1933, p139)

I suggest that I can know something in terms of the physical movements it involves, or the sounds required, and in this sense I have meaning for this something. However, I may not be aware of how it relates to other things, or why there are certain responses from others when I do this something. However, I am still in a position to reflect on

[46] **{Successfully do, followed by reflecting on that doing}** - 3.2.4

these experiences once I am able to do this something. Thus, I am in a position of being able to develop meaning because my ability to do this something enables me to explore, to generate experiences which can be reflected on.

(b) The second introduction of something new, which I mentioned, was formal notation. This occurred first at lines 142-144. The notation was not introduced until the students appeared to be confident with the process involved in solving such linear equations. Thus, the students knew what to do once they were given a set of operations on the original number. I kept the attention on the task of finding my number by the fact that this followed on from similar activities and I asked such questions as *So you are going to?* (line 144) and *So, how am I going to work out my number?* (line 150). By keeping the attention on the task of finding my number, it was kept away from the notation itself. Yet, the notation was required in order to carry out this task. The list of operations was deliberately too long for anyone to be able to remember them, this forces[47] the students to subordinate the notation in order to carry out the task of saying how they would find my number. The notation was all that was available for them to know what the list of operations had been. Thus, it might be seen as an apparent paradox that I am claiming to keep the attention away from the notation whilst the notation needs to be attended to in order for the main task to be completed. However, the apparent paradox disappears if you consider attention to involve more than merely looking at something. The notation needs to be looked at and studied sufficiently in order to gain the information required for the main task. The attention remains with the main task and informs which aspect of the notation is looked at and studied. In this way students *look through* (Mason (1993b)) the notation to work on their main task. Dewey comments that

> ... a question to be answered, an ambiguity to be resolved, sets up an end and holds the current of ideas to a definite channel... *The nature of the problem fixes the end of thought*, and *the end controls the process of thinking*. (1933, p15)

Attention on the task drives the reading and interpretation of the notation. Thus, the successful reading and interpreting of the notation is assisted by the fact that the person's attention is not focussed on the notation per se but is needing the notation for what they are attending to[48] . Dewey gives an example of sensory development where attention is best placed on tasks to subordinate the use of senses:

> Sense perception does not occur for its own sake or for purposes

[47] **{Forcing awareness}** - 5.2.3

[48] **{Learning is not always taking place where the observable attention is}** - 3.2.3

> of training, but because it is an indispensable factor of success in doing what one is trying to do. Although it is not designed for sense-training, this method effects sense-training in the most economical and thoroughgoing way. Various schemes have been devised by teachers for cultivating sharp and prompt observation of forms, as by writing words (even in an unknown language), making arrangements of figures and geometrical forms, and having pupils reproduce them after a momentary glance. Children often attain great skill in quick seeing and full reproducing of even complicated meaningless combinations. But such methods, however valuable as occasional games and diversions, compare very unfavorably with the training of eye and hand that comes as an incident of work with tools in wood or metals, or such activities as gardening, cooking, or the care of animals. Training by isolated exercises leaves no deposit, leads nowhere; (1933, p249-250)

The fact that something is being subordinated means that it is also being practised. However, the practice is carried out whilst progress is being made. Thus the practice is not practice per se but is more productive for two reasons. Firstly, progress is being made at the same time and secondly, practice through progress will drive that which is being practised into becoming a functioning.

PRINCIPLE OF ECONOMY: *place attention in an activity which subordinates the desired learning. This means that something is practised whilst progress is made at a higher subordinate level (practice through progress)*.

A series of lessons can develop a progression of subordinations such that something which is introduced in one lesson can become subordinated to a task, then subordinated to something which is itself subordinated to another task,... Such a situation was indicated in the simultaneous equation lesson mentioned in the last section, where the reading and interpretation of notation is subordinated to the manipulation of equations which is subordinated to the task of finding my original two numbers. This progression of subordinations drives the reading and interpreting of notation into a functioning - something which a learner knows so well that they hardly have to give it any conscious thought.

PRINCIPLE OF ECONOMY: *use successive levels of subordination to drive functionalisation*.

When the formal notation is introduced to students, their interpretation of the symbols and positioning of the symbols will involve coming to know how the operations are represented. Additionally, for the task of finding my number, they will have to be aware of the order of the operations as well. As I am speaking to the students, telling them what I am doing to my original number, there is an inherent order which comes through the fact that speech is said in time. Thus, one operation is said before another. However, with an expression such as:

$$6\left(\frac{2(x + 3) - 5}{3} + 72\right) = 100$$

all the symbols exist within our field of vision at the same time. So, there is a choice about what we might attend to first. One possible order could come from our experience of reading. So, for someone in the UK, they might place an order of left to right on the expression. However, the order of the operations on my original number is not the order of the symbols as read from left to right. Thus, students have the task of placing an order on this expression as well as reading the operations themselves.

The fact that time places an inherent order on things can be made use of. In line 144, I repeated the list of operations and wrote the expression on the board in the same order. Thus, the expression was being written in the order of the operations and not from left to right. This I will say more about in the next chapter with regard to the construction of the computer program *GRID Algebra* (Hewitt (1992)). Another use of time within this line is the fact that I wrote the symbols at the same time as they were being said. Thus, what is already known (the words describing the operations) occurs at the same time as what is new (the written notation on the board). The use of simultaneity in this way helps establish an association between the two[49] . Thus, students are being asked to make use of this power as part of the activity.

PRINCIPLE OF ECONOMY: *use simultaneity to help establish desired associations.*

9.4 Backwards and forwards

In this chapter I have described how a well known idea can be developed in the detail of how a teacher works with the students during the activity. I have described how the way of working with the students can not only ask them to make use of their powers but also mathematical essences can be called upon, in this case *order*, *inverse*, and *sameness/difference*. I have considered the placement of attention and its role in helping

[49] **{Simultaneity}** - 5.2.1; **{Association}** - 5.2.1

something to become introduced and used immediately, thus subordinating the new, which in turn helps drive the learning of the new into becoming a functioning. Within this scenario, something is learned by firstly achieving the doing of it, followed by reflection on that doing.

In the next section, I will describe a computer program which I developed to include, within its structure, several of the principles of economy.

10 Towards economy: practice and theory 3

10.1 Introduction

In the last two chapters a number of principles of economy have been developed and a final part to my work in this study has been the practical application of writing a computer program, *GRID Algebra* (Hewitt (1992)), which incorporates principles of economy within its structure. In this section I will describe how several of the principles have been incorporated. In particular I will relate aspects of the program to some of the teaching techniques I discussed in the previous chapter regarding *Think of a Number*. One challenge for me in developing the program was the fact that the program could be used by anyone and so I had to try to incorporate as many of the principles of economy as I could into the structure of the program so that these principles would still exist no matter how the program was used. As with *Think of a number*, I decided to tackle the area of algebra, with which I had observed students having difficulty and which others have described as a difficult area for students to learn (for example, see Tall and Thomas (1991)). In particular, I had observed lessons where students had been working with confidence and success at tasks such as finding the area of specified rectangles. Towards the end of the examples set, there would be a rectangle that had 'x' as the length of one of the sides, instead of a number. When students arrived at this rectangle, they became confused and wanted to know what 'x' was. The teacher would attempt to give an explanation but this often did not appear to help. As Thom has stated *The 'meaning' of an algebraic symbol is established with difficulty or is non-existent* (1973, p207). And Freudenthal commented:

> With the upmost patience teachers have tried to engrave in their pupils' minds that letters in algebra mean something, that no formula is meaningful unless the meaning of its components is told, and that algebra is not a meaningless game with 26 letters. It was to no avail. (1973, p290)

After repeated failed attempts at trying to explain the meaning of an 'x', I have seen students' confidence diminish to such an extent that some students no longer felt sure about what they were doing with examples, which had all numbers and no letters present. It appears to be important how the notion of using a letter is introduced. Thus, my first interest was in the introduction of letters to arithmetic.

My second interest was with formal notation. From my experience in the classroom I know that students are capable of creating their own notation. Yet there is a standard notation which is used within the mathematical community. Thus, there can be a tension between the valid and productive activity of students developing notation, and the shift to using a socially endorsed notation so that they can share in the mathematical communications of the world at large.

Within *Think of a number* I gradually developed a way in which I worked with the students so as to make such a lesson as productive as I could. As I did so, my theories about learning and teaching developed along with my practice. In contrast, *GRID Algebra* was an application of those theories. *Think of a number* relies on the teacher acting in certain ways in order for the notation to become functionalised, so the challenge I gave myself with *GRID Algebra* was to incorporate several principles of economy into the structure of a computer program. I developed the program between April 1990 and December 1991.

I suggest that one difficulty students have with formal notation is that there is no apparent reason for why things are written as they are. I say no reason because an equally valid but different notation could have developed. It is a convention rather than something to be discovered as an obvious way of representing arithmetic. Thus, formal notation needs to be given to students in some way. Just as it is not something to be discovered, it is also not something to be explained because there are no reasons for why this choice of notation should be preferred over all other options. Thus, I decided that notation had to be given and not explained. This is similar to the situation where a young child is faced with adults using a particular language. It is given in the sense that it is part of the environment the child grows up in; and it is not explained but used. Thus, what is of significance is the environment offered by the computer program and the way that environment behaves when someone is engaging with it.

The structure in this chapter will be based on *Practice* and *Theory* as with the last two chapters. The *Practice* will be concerned with the design of the computer program *GRID Algebra*. It will begin with a description of the program and will then have sub-headings which refer to each of the principles of economy developed in the *Theory* sections of the last two chapters. In each sub-heading I will discuss the design of the computer program in relation to that particular principle of economy. The *Theory* section will continue the development of my theory from the last two chapters in the light of my experiences of designing the software. This development will bring to light additional principles which have not been mentioned so far. Several of the summary statements from chapters 3, 4 and 5 will feed into this chapter and, when reference is made to them, the statements will appear as footnotes with an indication of which section they came from.

10.2 Practice 3: GRID Algebra

10.2.1 Brief description of *GRID Algebra*

In this section I will describe some basic aspects of the computer program. Many other facilities exist and a photocopy of the accompanying booklet is included in appendix 3.

GRID Algebra is based on a rectangular grid of cells (see Figure 22). In each cell a number, letter or algebraic expression can be placed. The contents of the cells is based on the times tables with:

the 1st row containing the 1 times table;
the 2nd row containing the 2 times table;
the 3rd row containing the 3 times table;...etc...

1
2
3
4
5
6

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25
6	12	18	24	30

Figure 22: *GRID Algebra* with 6 row grid and numbers added.

Initially, the cells appear blank. All the numbers can be placed in if required. Also, there is a numbers drawer where single numbers can be collected and placed into a cell. Likewise, single letters can be collected from a letter drawer and placed in a cell. Additionally, expressions can be entered into cells with the use of a 'calculator'.

Expressions (when using this word, I will include single letters or numbers as much as expressions using letters and numbers) can be picked up and dragged by the mouse either to the right, left, up or down. Each direction is associated with an arithmetic operation:

right is concerned with addition
left is concerned with subtraction
down is concerned with multiplication
up is concerned with division

The expression changes automatically as it is moved according to the direction and the number of cells moved (see Figure 23). Since the fourth row is the 4 times table, each movement to the right in that row is associated with an addition of 4. In row 2, it would be an addition of 2. Going from the 5 times table to the 1 times table involves division by 5.

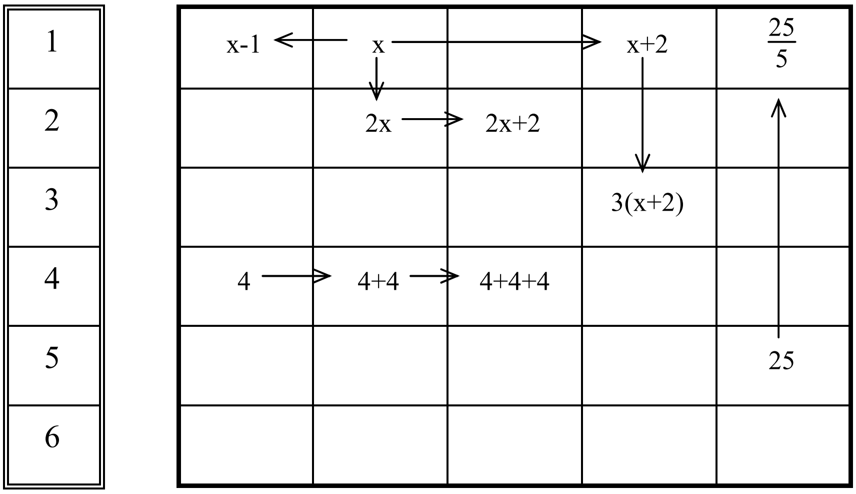

Figure 23: *GRID Algebra*: some examples of expressions obtained through movements on the grid.

In this way, considerable work can be done on the arithmetic connections between numbers, the inter-relationship between algebraic expressions, as well as links between algebra and number.

10.2.2 Use imagery and movement so that awarenesses can be gained within a context of generality (see 8.3)

Algebraic expressions are created in *GRID Algebra* by the learner using the mouse to pick up numbers or letters and moving them horizontally or vertically about a grid. This involves the learner physically moving their hand in a way which relates to the movements on the grid. In this way, both physical (through the movement of the mouse) and visual (through the movements on the screen) dynamics are involved in the creation of algebraic expressions. So, I have offered an image to which students could relate (moving horizontally and vertically) whilst still engaging students directly with formal algebraic notation in the form of expressions which appear on the screen.

The expressions remain on the screen after the movements have been made. Thus, they can represent the physical and visual dynamics that took place at one time. Equally, expressions which were created by someone else in a prepared grid, or expressions entered by the calculator, have the possibility for a learner to enter into and reconstruct the potential physical and visual movements they represent. Equally, a learner can imagine journeys taking place on the screen and consider the expressions which would appear if such journeys took place. Thus, there are three elements in the dynamics between the written algebraic expression and the imagery of physical movement and visual perception. The first is when both occur simultaneously when a learner is moving expressions about the grid; the second is when a learner is given an expression and considers the associated movement; and the third is when they imagine possible movements and consider the associated expressions. Particular imagery is developed through the first of these, and is used by the second and third through challenges in which a learner is involved within the grid structure.

A learner, through exploring movement within the grid, can gain a sense of the generality of the grid in terms of relating movements they make with the expressions they see. Thus awarenesses gained are placed in the context of possible movements on the grid.

10.2.3 Offer sufficient complexity for learners to have the material necessary to make use of their powers (see 8.3)

A learner can drag expressions around the grid in any way they choose through a combination of horizontal and vertical movements. Each time an expression is dragged and dumped, the expression changes according to the movement performed and the new expression appears in the appropriate cell. This new expression can then be dragged and dumped in another cell with the expression changing once more. This can continue with each new movement, either vertically or horizontally, representing an arithmetic operation which is expressed within the notation of the new expression. Thus, three such movements will produce expressions with three arithmetic operations such as:

$$3+4-2-1 \qquad \text{(see Figure 24)}$$

or

$$\frac{3x+6}{2} \qquad \text{(see Figure 25)}$$

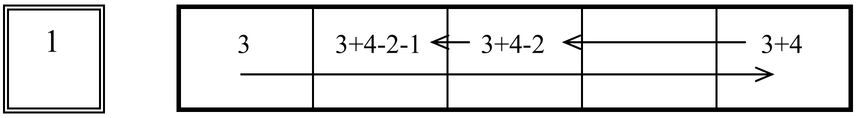

Figure 24: *GRID Algebra*: example of three movements in row 1.

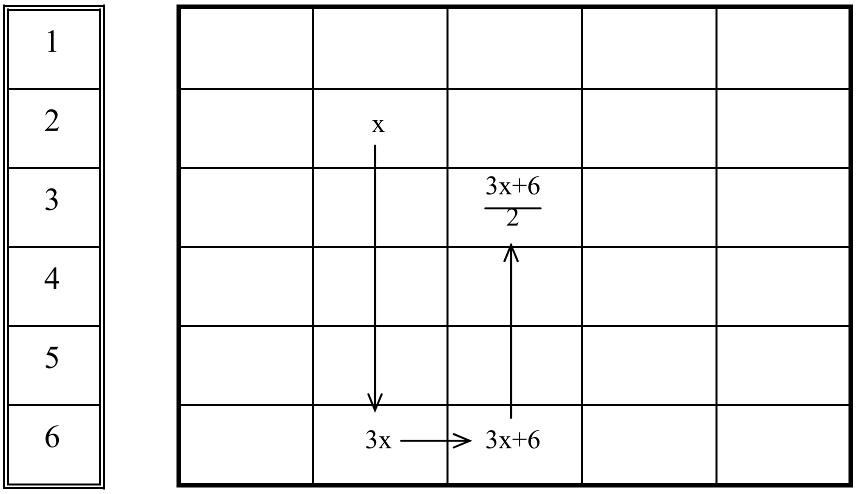

Figure 25: *GRID Algebra*: example of three movements on 6 row grid.

When making a journey from cell x to cell y in the following example (see Figure 26), some expressions can be generated from making journeys which involve only two movements:

$2(x+3)$ (see Figure 27) or $2x+6$ (see Figure 28)

Figure 26: *GRID Algebra*: grid presenting task of going from x to y.

Figure 27: *GRID Algebra*: one possible journey involving two movements.

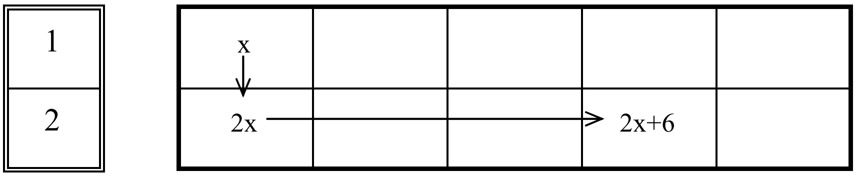

Figure 28: *GRID Algebra*: another possible journey involving two movements.

Other journeys, which begin and end in the same positions as those above, may involve several more movements:

$$2\left(\frac{2x+2}{2}+1\right)+2 \qquad \text{(see Figure 29)}$$

or

$$2\left(\frac{2\left(\frac{2(x+4)-6}{2}+1\right)+4}{2}-1\right) \qquad \text{(see Figure 30)}$$

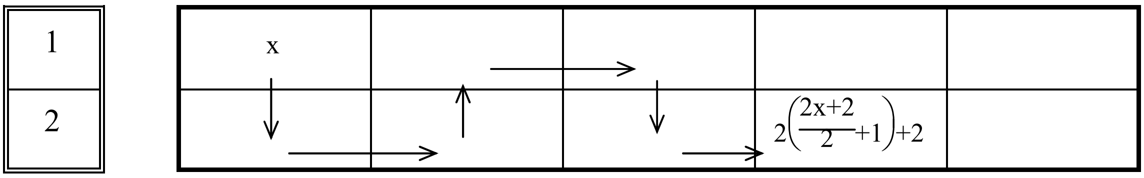

Figure 29: *GRID Algebra*: a journey involving six movements (the intermediate expressions are not included for the sake of clarity).

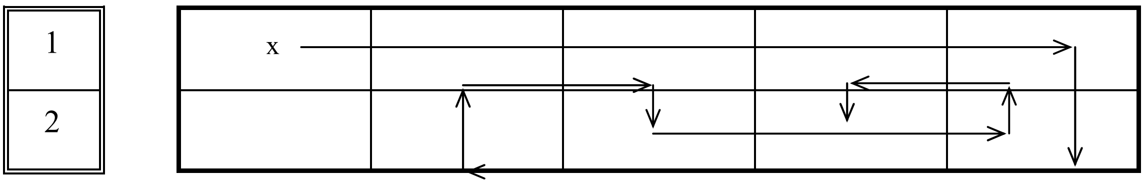

Figure 30. *GRID Algebra*: another journey, this time involving ten movements (intermediate and final expressions are not included for the sake of clarity).

Complex expressions can be generated simply as a result of movements, without the requirement of understanding such expressions. Thus, a complex environment can be created through a learner exploring what can be done on the grid. The material which is created through this exploration is the algebraic expressions appearing in the cells along with the accumulation of imagery which occurs as a consequence of their creation.

A situation can develop in which a learner becomes involved in a world of complexity where they see the consequences of their actions in the algebraic expressions created. They can generate as much material as they require since the generation of expressions only requires them to make physical movements with the mouse. They are then in a situation where they can make use of their powers in order to make sense of the complexity within the computer environment. I am not suggesting by this that a learner be left to their own devices on first meeting the program, but that elements exist within the program to allow for complexity to be generated easily. There is a significant role for a teacher to present challenges or appropriately prepared grids which offer particular structures for learners to explore.

10.2.4 Reducing need for a learner to remember previously encountered mathematical content (see 8.3)

GRID Algebra does not require the learner to remember previously encountered mathematical content since it is an environment which does not require the learner to enter information in order to proceed. If a learner cannot remember addition facts, *GRID Algebra* is a program a teacher could use to help the learner to become aware of the symbol '+' in relation to the movements made with the mouse and the numbers appearing in the grid. In other words, knowledge not remembered can be worked on anew through awarenesses gained whilst exploring the grid environment within the program. It is an environment designed to respond to what a learner does by informing them, through algebraic notation, of the movements they have made with the mouse or the calculations they have entered in the calculator. Thus, it is always possible for someone to proceed and generate material by making movements with the mouse. This material is then available for them to work with by using their powers. It is a program which is designed for awareness rather than memory. There is no request for a learner to remember but there are several opportunities for a learner to gain awarenesses.

10.2.5 Reducing the need for translation (see 8.3)

One aim for the computer program is for learners to become familiar and confident with understanding and interpreting formal notation. Whilst moving around the grid, the expressions which appear in the cells are written in standard formal notation. Thus the operations which take place through the movements are presented immediately in a desired form of formal notation. This can help reduce the need to translate an understanding of the operations into formal notation at another time. Work related to the teaching of algebra through other computer media, notably LOGO and spreadsheets (see Pozzi (1991), Sutherland (1992)), all use a notation which is different from that used by mathematicians and scientists in the community. Translation of the understandings learners have developed through the use of these other computer media, into the use and interpretation of formal notation, will be necessary.

There is, however, another translation which is involved in *GRID Algebra*, and that is the translation between the three elements of movement, operations and notation. A learner can make movements over the grid and know what they have done in terms of the movements themselves. They can also know what they have done in terms of the operations which are associated with those movements. Additionally they can know what they have done in terms of the notation which appears on the screen.

As an example I relate a time (20th February 1993) I was working with two year 11 students who were in a hospital school. Their teacher had reported that both of them had found work concerned with algebra difficult and, before my visit, the teacher had spent some time with them working with *GRID Algebra*. At one point, I had a grid with numbers in each cell and had made a journey from one of the numbers and finished in a cell in row 6. Afterwards, I rubbed out the expression *3(2)* which appeared as an intermediate step in one of the cells. Thus, there was no evidence of the journey I had made. Whilst I was doing this, the two students, whom I will call Nigel and Ben, were looking away from the screen.

1
2
3
4
5
6

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25
6	3(2)+6	18	24	30

Figure 31: *GRID Algebra*: problem grid for Nigel and Ben.

They were presented with the grid in Figure 31, and the task for them was to say what I had done.

N: That's times. The two...
B: *You've picked the two down and brought it down to the six and then you've moved it across to the twelve. So what you've done is you've pulled it down to there and went across to there.*
N: Yes, so three times two equals six.
B: *Three two's are six. Six and six...*
N: Is twelve.
B: *... is twelve.*

Both of them only have the notation, *3(2)+6*, and its position on the grid to look at. Thus, their statements indicate the interpretation they had of the written notation with regard to the grid and the other numbers on the grid. Nigel states immediately *That's times*. A question with regard to this statement is <u>what</u> is *times*? There is no multiplication sign within the notation, yet he recognised that there was 'timesing' going on. His interpretation of the notation is concerned with the arithmetic operations. Ben, however, makes statements which are concerned with the movements which may have taken place and does not mention arithmetic operations in his initial sentences. This is an example of the dynamic which exists between the movements, arithmetic operations and the formal notation.

There is the possibility of a translation being required between comments about the movements on the grid and the arithmetic operations, and vice versa. This is a consequence of offering the image of the grid and the structure within it. Thus, I have not succeeded completely in avoiding translation in this respect. However, I have avoided other possible demands on translation by having the learner work directly with notation. There could have been demands to translate from other forms of representation, for example other computer notation systems as mentioned earlier, or practical materials such as Cuisenaire rods.

<u>10.2.6 Use pedagogical awarenesses to work with the learner's mathematical awarenesses (see 8.3)</u>

There is nothing that can be written into a computer program which can control the awareness of a teacher. Indeed, nothing can control the learner's awareness either. At best, awarenesses might be influenced by what does or does not happen on the screen and what is emphasised within the software. Thus, the program can act as an editor and amplifier. One decision I made was not to have the computer accept anything inconsistent with the structure of the grid. If something is entered into a cell, either by the calculator or from one of the drawers, which is not consistent with the rest of the current grid, then the colour of the expression is different to accepted expressions and there appears a little 'dustbin' by the side of it. There are choices about how the learner might be informed that their expression is not going to be accepted in this cell. For example, a loud noise might come from the machine. Or, some words could appear such as ***WRONG!*** These two options would act as an amplifier. However, I decided to have the program act as much as possible like an editor instead by having the expression disappear as soon as the learner begins to do anything else. Thus, I want the learner to be informed that the expression is not going to be accepted, however I do not want to make a big fuss over it.

I want the learner's awareness to be concerned with operations rather than calculations. Thus, the program never carries out a calculation, and the fact that the program is concerned with movements encourages a learner's awareness to be with their associated operations. Attention with the calculations would divert attention to the arithmetic rather than the algebra of operations. Additionally, as with *Think of a Number* in the last chapter, attention taken away from the numbers and onto the operations can encourage general statements to be made which are independent of the starting number. The awareness of the irrelevance of the starting number allows the possibility of a letter being used to start with and so can help with the introduction of 'x'.

As for the awareness of the teacher, this is not something the program makes any attempt to influence except for the fact that the program is deliberately designed so that there is no explicit set of challenges with which a learner is presented when they sit at the machine. Thus, there is an implied requirement for a teacher to make pedagogic decisions about how they will use the program with their class.

10.2.7 View mathematical content through the eyes of a pedagogue rather than a mathematician (see 8.3)

Students already demonstrate that if they move their hand to the right and want to return it to the position it was before, then they move their hand to the left. Furthermore, they move it to the left the same distance as they had moved it to the right. It is only this mathematical essence, as an awareness, which is required to be related naturally to the grid environment in order for the students to be engaging in inverse in a mathematical context. As they perform these actions, they can use their powers to note the association of right with addition and left with subtraction. This, along with the use of sameness/difference, can lead to the articulation of what inverse is about with respect to addition and subtraction. No explanations are required, only an awareness coming from reflecting on what they can already do and what happens on the screen as they do it.

The program is designed so that it is only these mathematical essences and powers of children which are required to relate to the inverse of more complex journeys involving all four arithmetic operations. Although the expressions may appear complex, they are associated with a journey which can be visualised and, as a consequence, the inverse journey can also be visualised. Once associations are formed between movements and operations, the inverse journey brings with it the inverse collection of operations in their appropriate order.

The mathematical essence of sameness/difference is used by associating the idea of same position and different position with the notion of equivalent expressions and different expressions depending upon whether they appear in the same cell or not.

The mathematical essence of order is used by making it only possible to move in one direction at a time. In this way, one operation is carried out before another can begin. This sets up a natural order in time which can assist in the reading of the expressions. A learner can observe the fact that as an expression is created through a succession of movements, the different operations are not written in order from left to right. However, through the image the learner obtains by physically carrying out the journey and watching what happens on the screen, they can re-create their journey and read the final algebraic expression with the knowledge and direct experience of order. For example, the generation of the two expressions:

$$\frac{2(x+6) - 4}{4} \quad \text{and} \quad \frac{2(x+6)}{4} - 4$$

are carried out by the same physical movements for 2(x+6) (see Figure 32), however, the generation of the rest of each expression is generated by the carrying out of movements in a different order. The first expression will involve moving to the left first and then moving up (see Figure 33). The second expression will require moving up first before moving to the left (see Figure 34). The expressions end up in different cells and consequently are not equivalent.

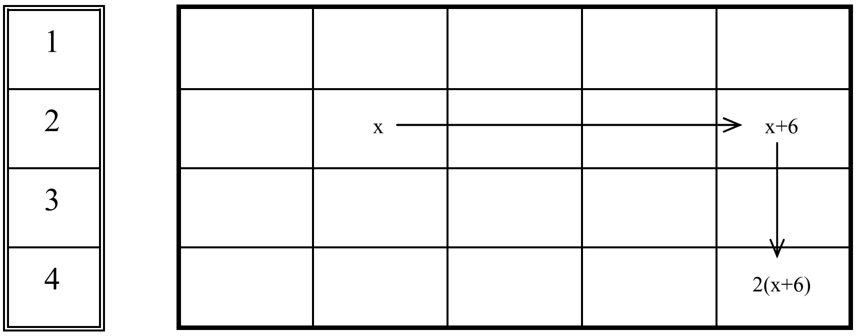

Figure 32: *GRID Algebra*: initial movements which are common to two different expressions.

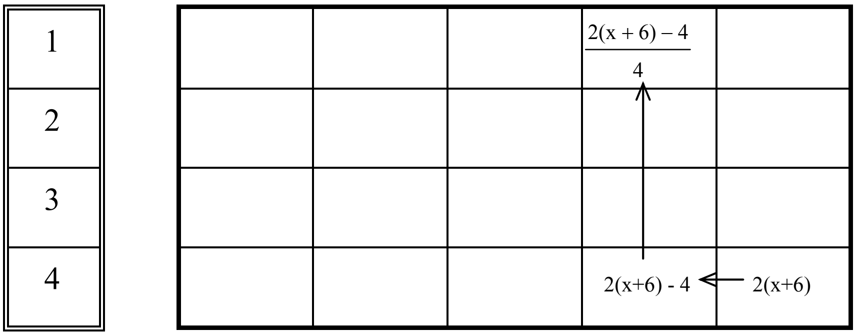

Figure 33: *GRID Algebra*: remaining movements for one of the expressions.

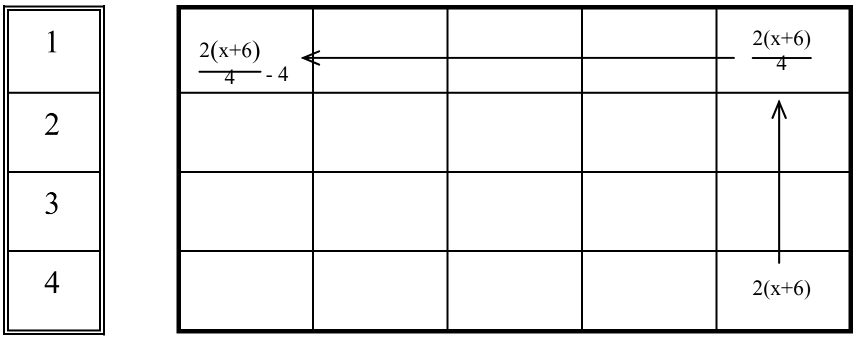

Figure 34: *GRID Algebra*: remaining movements for the other expression.

The program only allows movement in one direction at a time and so a learner is forced into making one movement before another. Thus, time places a natural order on the carrying out of the physical movements, and this natural order is associated with the algebraic expressions appearing on the screen. In this way, *GRID Algebra* uses a natural awareness of order in time to place an order in the operations involved in algebraic expressions.

Thus, the program is designed to call upon powers of children and mathematical essences and as such has been created from pedagogic reasoning rather than consideration of levels of mathematical knowledge.

10.2.8 The placement of attention (see 9.3)

Although this heading is not a principle, the placement of attention is a fundamental issue with regard to the principles of economy. Thus, I have decided to give this issue a section of its own.

I have mentioned that I have observed many students have difficulties when a letter such as 'x' is introduced in a mathematics lesson. The appearance of 'x' sometimes results in students being confused when up to that point they were clear and confident about what they were doing. Reflecting on such situations, I have become aware of where the students' attention is being asked to be placed. In the lesson I discussed about finding areas of rectangles, the task has been to calculate the answer, the area of the rectangle. Yet, what is being asked with the introduction of 'x' is for the students not to place their attention into the answer but to place their attention in the process of getting the answer. This sudden demand for the student to shift attention is one element of the difficulty they experience. The desired algebraic statement is concerned with the generality of the process involved in finding areas, it is not involved in the generality of the answers. Thus, I ask the question, if you are interested in one, why place attention in the other?

In designing *GRID Algebra* I tried to keep attention in the processes rather than answers. Thus, there are no answers given and no answers required by the program. Furthermore, the active part of the program for a learner is the moving of the mouse as expressions are dragged from one cell to another. These movements are associated with operations rather than their results. Thus attention is in the processes which can lead to students gaining awarenesses within the domain of operations rather than answers.

The introduction of 'x' has been of interest to me due to the difficulties many students have. In *Think of a number* in 9.3, the fact that a student's attention is taken away from the particularity of numbers and into the generality of operations gives the opportunity for a letter to be introduced. In the case of *Think of a number*, the numbers and operations were being said. With attention in the operations in finding the inverse, the particularity of the number-names became unimportant since all that had to be done with them was to repeat them. It was at this time that the word 'beta' could be said, rather than a number-name, and this appeared to be accepted without difficulty.

With *GRID Algebra*, the numbers and operations are not being said but are being expressed in formal notation. However, as with *Think of a number*, I wanted to place attention away from the particularity of the numbers. Attention cannot be diverted on its own.

For the next 5 minutes, do not think about eggs.

This is an example of me drawing your attention to something which I do not want you to pay attention to. People to whom I have given this exercise, report thinking of eggs straight away.

As soon as you said it, I thought about eggs.

For the next ten minutes... not thinking about eggs came to my mind.

For a lot of that period I was reading and in the first 15 minutes I was reading it didn't make any difference I was... about twice every minute or so... I'd think about not thinking about eggs.

They seemed to be successful not so much by attempting to suppress their thoughts but by putting their attention into something else. For one person it was having a conversation about his mother, for another it was listening to jazz and reading a book.

...but I wasn't trying not to think about eggs. I was reading... and had some jazz on the tape deck...

Thus, with *GRID Algebra* I encourage attention to be placed elsewhere in the area of movement and operations. This serves two functions. This first is to place attention where algebraic awarenesses can be found. The second is to take attention away from the particularities of the starting number or answer. It is the moving around the grid which is the focus of attention, not <u>what</u> you move around the grid. There are two 'drawers' available in the screen, one containing numbers and the other containing letters, where the symbol can be picked up and placed in a cell on the grid. Once the attention is in aspects of moving <u>something</u> around the screen, then that something could come from the letters drawer as much as the numbers drawer. Just as in *Think of a number* when the number-name becomes just a word that is said and you could say 'beta' just as well, so with *GRID Algebra* the written number becomes just a symbol and so 'x' would do just as well.

Valerie Walkerdine said:

> Formal academic mathematics, as an axiomatic system, is built precisely on a bounded discourse, in which the practice operates by means of suppression of all aspects of multiple signification. The forms are stripped of meaning, and the mathematical signifiers become empty. (1990, p96)

It is when the number-names and the written numbers are stripped of their meaning and these signifiers become empty that 'beta', or 'x', or whatever, can be introduced.

In April 1992, I ran a session at the joint conference of the Association of Teachers of Mathematics and the Mathematical Association entitled *Towards fluency with algebra* which involved using *GRID Algebra*. One of the participants, Gill, commented that she wanted to take a number and drag it round the grid and look at the expressions obtained. However, she did not want the original number to become confused with the operations performed on it. She decided to use '7' since she was not going to add, subtract, multiply or divide by seven. Her group ended up with various expressions on the grid involving '7'.

It appears that Gill used '7' as if it was an 'x'. Gill talked of the symbol '7' no longer representing the number seven but as a symbol which would not be confused with the other symbols which would be generated as the '7' was dragged round the grid.

Here, the '7' as used by Gill is closer to the formal academic mathematics Walkerdine talks of, than the 'x' as viewed by the students mentioned above. The symbol '7' may signify the number seven, seven things, etc, but such significations have been suppressed in Gill's case, and it is this suppression which enables Gill to work with '7' in a powerful way. The students, from the area of rectangles lesson, were obsessed with the question of what 'x' might signify. Whilst this obsession remains and they are in the process of developing meaning for 'x' rather than 'stripping it of meaning', they are unable to use 'x' as part of a formal system.

For the students, it is not a matter of suppressing meanings, since they have not yet developed meanings. However, the 'x' remains the focus of attention as they try to develop their meaning. For Gill, along with the suppression of meanings came an attention elsewhere. She wanted to attend to the operations taking place with the dragging of the '7'. She chose '7' in order that the symbol would not be confused with other symbols generated from these operations. It was these other symbols that were the focus of her attention, along with the imagery of the journeys over the grid. She was attending to the meaning of the other symbols, the journeys made on the grid and the process carried out on the object '7'.

The placement of attention has been a crucial aspect of the designing of this computer program.

10.2.9 Place attention in an activity which subordinates the desired learning (see 9.3)

There are two intentions of the program I wish to discuss under this heading.

(a) The first intention of the program is to help students become comfortable with using and interpreting formal algebraic notation. With this in mind I made notation the only thing which appears on the screen within the grid. Thus, it is the notation which has within it the story of what has happened in terms of journeys and operations. In fact the notation provides the link between the movements on the grid and the arithmetic operations.

Movements on grid <--------> **Symbolic expression** <------> Arithmetic operations

There is a need for something to provide this link since a grid which only contains the numbers (see Figure 22) does not contain sufficient information to establish this link.

Addition could have been a movement down as much as a movement to the right. Likewise, multiplication could have been a movement to the right as much as a movement down. The connection is an arbitrary one, it is only the notation which informs you of which structure is imposed on the grid. Thus, all conversation about the operations comes from the notation which appears in the cells.

The activities I suggest in the accompanying booklet to the program (see Appendix 3) focus attention on various tasks to do with the movements and operations on the grid. Throughout many of the activities, the notation is the only information which appears on the screen to help the students with their task. So formal notation can be met for the first time and immediately subordinated in consideration of questions related to the grid and movements round it. In this way, students practice interpreting the notation whilst their attention is on higher order tasks which require the subordination of the notation.

Thus, written into the program is the notion that formal notation will be practised through progress on other tasks. This practice through progress will help the functionalisation of notation.

(b) The second intention for the program was to help with the introduction of letters. In the program letters are available to be taken from the letter 'drawer' and placed on the grid. The only thing which can be done to these letters is to move them round the grid. Thus, the letters, and indeed the numbers, are subordinated to the task of moving something round the grid. With the attention in the moving rather than what is being moved, the use of letters is subordinate and so helps the functionalisation of their use in expressions.

Order of operations is another aspect which is subordinated since the creation of expressions depends upon either making movements or entering the expression via the calculator. With regard to the movements, these are accepted one direction at a time and thus one operation at a time. Consequently, the written expression is created in the order of the operations rather than an order of left to right. Entering expressions in the calculator also requires the operations to be entered in the order they are performed. In fact the calculator is designed to assume that the entries made are in the order that they are carried out, and they are displayed with this assumption. Thus, if the expression

$$2(8-9)+6$$

were to be entered into the calculator, it would have to be entered in the following order:

$\boxed{8}$ -> $\boxed{-}$ -> $\boxed{9}$ -> $\boxed{x}$ -> $\boxed{2}$ -> $\boxed{+}$ -> $\boxed{6}$

In fact, it is usual with using *GRID Algebra* to be thinking in terms of the order of the operations, rather than the order of left to right, because the activity of movements round the grid are carried out, and define, the order of the operations. Entry of expressions in the calculator is required to be done in the order of the operations, which means that the order of operations becomes subordinate to the requirement of entering expressions. This subordination helps students to be able to look at algebraic expressions and interpret them in terms of order. This helps students to overcome what Tall and Thomas call the *parsing obstacle*:

> ... there is considerable cognitive conflict between the deeply

> ingrained implicit understanding of natural language and the symbolism of algebra. In most Western civilisations, both algebra and natural language are spoken, written and read sequentially from left to right. There are exceptions to this, for instance, numbers in some languages may exhibit a reversal... However, this is nothing compared with the subtle rules of precedence which occur in algebra. For instance, the expression 3x+2 is both read and processed from left to right, however, 2+3x is read from left to right as "two plus three x", but computed from right to left, with the product of 3 and x calculated before the sum. This difficulty of unravelling the sequence in which the algebra must be processed, conflicting with the sequence of natural language, we term the *parsing obstacle*. (1991, p125-126)

I have noticed that the demand for operational order (the order of calculation) within *GRID Algebra* does affect the way in which some people write expressions on paper after using the program for a while. I have observed expressions such as the one above being written on paper in the order 8, -, 9, brackets, 2, +, 6. Thus, this also helps with the parsing order by the fact that someone may write on paper, as well as read, in the order of calculation rather than a left to right order.

10.2.10 Use successive levels of subordination to drive functionalisation (see 9.3)

Activities using *GRID Algebra* can be devised such that the notation is subordinate to those activities. As an example, consider an activity where students are asked to take different routes from one cell, 'x', to another cell, 'y', and look at the expressions which have ended up in the 'y' cell. After several journeys have been made, the students are then asked to use the calculator to enter other expressions which would be accepted in this cell. Consideration of 'simplest' routes can lead to discussion of the distributive law when two expressions such as:

$$2(x+3) \qquad 2x+6$$

are the result of two 'simple' routes. Likewise, the reverse journeys can lead to considering:

$$\frac{y}{2} - 3 \qquad \frac{y-6}{2}$$

This new awareness of distribution could then be subordinated into the following activity:

$$\frac{12(x - 6)+8}{2}$$

A cell has the above expression in it. The task is to reduce the number of numbers above the division line by one, whilst obeying the following rules:

1 - Only numbers are allowed to change
2 - All operations must remain the same

This new expression is to be entered into the same cell and be accepted.

Repeat the process, having one number fewer above the division line each time. Continue until there is only 'x' above the division line.

$$3\left(\frac{x - 12}{3}+4\right)$$

The above expression is in another cell. This time the task is to enter an expression in the same cell with one less number within the bracket. Same rules as before. Repeat the process until there is only 'x' inside the bracket.

This is an example of how something, such as order, can be subordinated to a task which can bring a new awareness, such as distributivity, which in turn can be subordinated to a new task. Thus, the original awareness of order can become driven into becoming functionalised through successive levels of subordination. Such a series of activities and lessons will obviously be dependant upon the teacher and the students involved, but this as an example of how *GRID Algebra* lends itself to such levels of subordination.

10.2.11 Use simultaneity to help establish desired associations (see 9.3)

There are associations between the possible movement round the grid and arithmetic operations. Moving to the right is associated with addition; to the left is associated with subtraction; downwards with multiplication; and upwards with division. As I mentioned in section 10.2.9, these are arbitrary associations and it is only the notation which informs you of them[50] . I could have attempted to inform the user of the program by having a statement appear on the screen to say what the associations are. However, this would have required memory and translation. Instead, I have used simultaneity by having the notation changing appropriately at the same time as movements are taking place. Furthermore, the notation appears in the cell corresponding to the movement which is taking place. Thus, there is simultaneity in time and space between movement and the notation. This helps establish the associations between the movements and the notation.

Additional to this association, there is the association between the notation and the arithmetic operations. For example, the fact that '2(5+3)' means adding five and three and then multiplying by two. These associations can come from relating the notation with the numbers that appear in the grid (if the option NUMBERS is chosen). I made the decision to help establish an association between brackets and multiplication by having an option NOTATION where every time an expression is dragged down, brackets appear round it and a number appears in front. This is standard when expressions such as 'x+4' are multiplied. However, it is not standard when the expression is either a number or a single letter. I felt that this association is generally worth establishing even though there are exceptions, since the exceptions can be dealt with separately. I will discuss further the idea of structure first, exceptions later in section 10.3 below. The other notation option will result in the exceptions being written as they usually appear (e.g. '2y' rather than '2(y)', and '3x5' rather than '3(5)').

In *GRID Algebra* you can only move in one direction, horizontally or vertically, each time. This means that, if you consider one stage in a journey to involve picking up an expression, dragging it, and dumping it in another cell, then each stage will involve exactly one operation. Thus, there is an association between the number of stages and the number of operations.

The calculator is designed to make use of simultaneity by showing the form of the notation with each key which is clicked. For example, not only does '3' appear when that number is clicked on, but also brackets appear when the multiplication sign is clicked (depending upon the NOTATION option chosen) and a line will appear under

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[50] **{Experiencing knowledge of the arbitrary in the course of my actions}** - 5.2.3

the expression entered so far when the division button is clicked. The principle of simultaneity is also respected with regard to order. The order of entering an expression into the calculator must be the order of the operations within the expression.

10.3 Theory (3)

Formal algebraic notation of arithmetic operations is something which is established within a culture. It existed long before any of us were born and is still used in written communication of mathematics. In this respect it has similarities to established languages. In contrast to convention, young children demonstrate the ability to invent sounds and combinations of sounds in an attempt to communicate. I recall holding a child, of about one and a half years of age, who said a collection of sounds and noises whilst pointing out the window and turning his head to me as if to see whether I had understood. I perceived considerable emotion within what he said. I also felt totally inadequate as to how I might reply. Children are perfectly capable of using sound to communicate, the trouble is that within the community there are a particular set of sounds which are used in communication. If the child is to succeed in communicating then they not only have to have something which they want to communicate but they also need to learn an established language through which to communicate. Likewise, students are quite capable of inventing their own notation (Billington and Evans (1987), Brown (1990)) but there is a need to learn formal notation if they wish to read and communicate with an established community. Since both language and notation are arbitrary, it would be complete luck if a single person happened to invent those particular forms without being informed of them in some way. A young child is informed of language by it being spoken in the environment within which they are engaged. If it wasn't provided then that particular language would not have been developed by the child. This is demonstrated by the fact that children do develop the language within the environment they grow up. Thus, if it is desirable for students to use formal algebraic notation, then it must be provided for them. Everything which is arbitrary will need to be provided[51] since otherwise the fact that something is arbitrary means there is choice, and a student may well choose an alternative. This can be useful in some respects, namely helping someone become aware that something is only a convention, or if the focus of a lesson is more concerned with someone developing their ideas. However, if there is a desire for something arbitrary, such as formal algebraic notation, to be learned, then a student making other choices does not help towards the learning of this particular notation. In this sense it is not a productive use of time or effort. Also, Ginsburg makes a comment about a gap between formal written methods and a child's informal knowledge of arithmetic:

[51] **{Provide that which is arbitrary}** - 4.2.3

> They [children] usually count on their fingers to get a sum. Methods like this work easily and well. Next children are taught various written procedures for accomplishing the same purposes. Unfortunately, they often fail to understand the necessity or rationale for written methods. Nevertheless, they are imposed on them and in school they are required to use them. The result is not only a bizarre written arithmetic, but a gap between it and children's informal knowledge. (1977, p125)

The fact that *GRID Algebra* provides the written notation, and it is this formal notation which is the material with which students develop their informal methods of operation and manipulation, means that an equivalent gap within algebra is not created in the first place.

PRINCIPLE OF ECONOMY: *provide that which needs to be given, namely the arbitrary*.

The particular symbols '2', '3', and '5' are arbitrary and thus need to be provided. However, the statement '2+3=5' is not arbitrary within the structure of our number system. Students can become aware of this fact which connects the numbers together and so it is not a necessity for this fact to be provided for them. An awareness which results in '2+3=5' can be applied to other situations which can bring a collection of other statements connecting numbers together. However, if '2+3=5' is presented as a fact to be memorised without an awareness, then there is nothing else to be gained. Memory is a means by which certain information can be stored, however, without awareness there can be no application to other situations. This is one reason why memory has limited value in terms of economy. The other reason is that memory is expensive in terms of personal effort and time[52] . It is for these two reasons that memory needs to be kept to where it is needed and used no more than that.

PRINCIPLE OF ECONOMY: *restrict the use of memory to that which is arbitrary*.

Formal algebraic notation is something which is arbitrary and thus something which is a candidate for the use of memory. However, I have indicated in 10.2.9 how, within *GRID Algebra*, the notation is subordinated to challenges and activities since it is the only information which appears on the screen. Hence memory can be assisted by the use of subordination to help drive something which is arbitrary into becoming a functioning. This means that it is possible to reduce the demands on personal time and

[52] **{Memory is expensive}** - 4.2.3

energy further by subordinating the arbitrary at a time when it is first provided, and thus using functionalisation to assist memory.

In 1989, I observed a Spanish lesson at my school. During part of the lesson, the class were asked to stand up and move in particular ways according to the Spanish words the teacher called out. I did not know any of these words before this activity, but I still joined in. A single word was said and I noticed the class turn their heads to the right. I turned my head the same way. As we continued, there was little time between one word and the next. This meant that there was not sufficient time for me to reflect for a length of time on the word I had heard. I found myself following what the students did, initially by copying them since I did not know the words. Then, as some words were repeated a second and third time, I began to move at the same time as the students. The speed at which the words were said meant that I had to act, there was no time to ponder. After a while, I found myself knowing the words in a way which did not require translation into English. I heard and I moved. I could not say that I ever made a conscious effort to remember the words, yet quite quickly they became words I knew sufficiently to respond appropriately within the social setting I was placed. I had a feeling of learning the words without the demand of memory[53] . I cannot say what the words are now but this is not surprising considering I have had no further practice since that time, but it does not alter the fact of how quickly and efficiently I was able to respond to the words appropriately at the time.

The words were introduced and simultaneously subordinated to the task of moving in an accepted way. For me, the accepted way was defined by doing the same as the rest of the class. The task was to make these movements at the same time as the students or even quicker. This is an example of something which is arbitrary but to which I was able to respond appropriately without having to consciously memorise[54] .

Imagery can also offer help to memorisation when what is to be retained is arbitrary. As an example, consider part of the *Think of a Number* lesson discussed in chapter 9. In line 144 of the transcript of that lesson, I was gradually writing algebraic notation which represented what I was doing to my number:

144. DH: *OK. So,... let me see... I am thinking of a number* (writes 'x' on the board)... *I add three* (writes '+3')... *then I'm going to... multiply by* (writes brackets round the expression so far)... *two* (writes '2' in front of the brackets)... *then I am going to take away five* (writes '-5')... *then I am going to divide by* (writes a line underneath the expression so far)... *three*

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[53] **{Learning without the need to remember}** - 3.2.3

[54] **{Attention remains with the activity rather than the need to remember arbitrary knowledge}** - 5.2.3

> (writes '3' below the line)... *then I'm going to...* (makes a noise whilst going along the division line from left to right, writes '+' following on from the division line and DH makes a different double noise whilst the addition sign is being written) *add... 72* (writes '72' after the addition sign)... *then I'm going to multiply by* (writes brackets round the expression so far)... *ummm... six* (writes '6' in front of the brackets)... *and I get* (writes '=' to the right of the expression so far)... *umm... 100* (writes '100' to the right of the equals sign)... *So you are going to?...*

Having written the dividing by three, I am about to continue writing the expression by adding on 72. What I do is to make a noise as I slowly go along the division line and then make a double noise as I write the '+' sign. The noises are offered as aural images, and they are said at the same time as I am writing the '+' sign. Thus, I use simultaneity to help form an association between the sounds and the particular aspect of written notation. If students recall the aural image, then the association can draw attention to the particular way in which that aspect of notation was written. That is why I make three sounds, each one is associated with one line - a long sound for travelling along the division line, and two short sounds for the addition sign. In several lessons where I have offered such images, I have observed that students do recall the image since I hear them making sounds themselves whilst they are working on their own. Furthermore, these sounds are not made at random times but are made at the same time as they carry out the associated actions or writing, in their book.

The positioning of the '+' sign is a convention and is thus arbitrary. However, the offering of an image combined with an association to this convention, gives the opportunity for memory not to be relied upon. I do not ask students to memorise the position of the '+' sign. Instead, the recalling of the image brings with it the association with the convention.

The use of imagery with association has been used by a number of people who have developed impressive memory. Luria (1969) researched the techniques used by the Russian mnemonist Sheresheveskii. The primary techniques involved turning the arbitrary data to be memorised into a story-line involving vivid images. Each piece of data had an associated image. Sheresheveskii used to be able to recall the images, and along with the images came the associated data. Techniques using imagery along with association have been suggested in various books concerned with improving memory, see Finkel (1993) for example. It is not uncommon for images to be used by teachers to help students memorise something. One example I have seen used is shown in Figure 35.

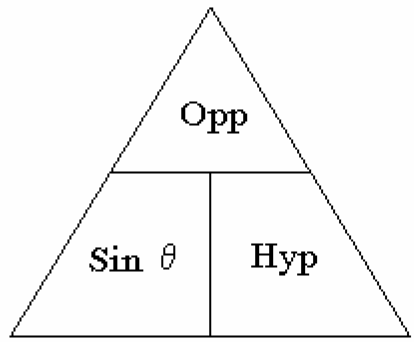

Figure 35: Image used for remembering variations of the sine ratio.

I have seen this used by a number of teachers to assist students who have difficulty manipulating algebraic expressions. This is designed to help with the trigonometric equation:

$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}}$$

If I wish to remember what sin θ is equal to, then I can look at the triangle and see that the other two (Opp and Hyp) are positioned such that Opp is above Hyp and so the above equation is as it is, with the Opp above the Hyp. There are times when the Opposite side may be unknown and the equation needs to be re-arranged. I can avoid that task of re-arranging by looking at the triangle and seeing that the other two (sin θ and Hyp) are next to one another on the same line, so the appropriate equation has them like this:

$$\text{Opp} = \sin\theta \text{ x Hyp}$$

Likewise, the equation to find the Hypotenuse has Opp above the sin θ:

$$\text{Hyp} = \frac{\text{Opp}}{\sin\theta}$$

The image is used as an aid to memory where the position in the triangle is associated with the position within an algebraic equation. However, there are some severe limitations to the use of this particular image. The image does not attempt to help an understanding of the algebraic processes involved in the manipulation of one form of the equation to another. Indeed, there is a sense of there being three separate equations which can be obtained from attending to each of the three sections in turn. This is different to the idea of there being one equation which can be manipulated into different forms. Each form of the equation is not arbitrary, an original form is an expression of a mathematical property, and subsequent manipulations are the result of an algebraic awareness of inverse. Thus, these equations are expressions of awareness whereas the image offered ignores the possibility of them being an expression of awareness but put them at the level of something which is arbitrary - something to be memorised. This runs against a principle of economy of only using memory for that which is arbitrary. The original image of the triangle can be recalled but there is a need either for a mathematical awareness to place the correct expressions in each of the sections of the triangle, or there is a reliance on memorisation for this as well.

Imagery combined with association can reduce the need for memorisation *of the arbitrary*. Those things which are not arbitrary can be known through awareness by employing powers of children. However, awareness can only bring a learner to the decision that something is arbitrary, not what that arbitrary thing is. It is at such times that the use of memory is to be considered. Since memory is expensive in terms of personal time and effort, I have shown that there are ways in which the potential demands on time and effort can be reduced by using either functionalisation, or imagery combined with association instead.

PRINCIPLE OF ECONOMY: *reduce the need for memorising the arbitrary by using functionalisation, or by using imagery combined with association.*

Formal algebraic notation is arbitrary and so there are no reasons why the notation must be like it is. Explaining the form of notation to a student is not appropriate. Explanation of an arbitrary decision might give a student the impression that there are reasons for why something must be how it is, when this is clearly not so since it was an arbitrary decision. Explanations in such a situation can lead to a waste of a student's time and effort by shifting their attention onto possible reasons which do not exist. A student can become confused by the apparent contradiction of someone giving explanations for what appears to be arbitrary. A response to this confusion is to ask the question *why?* This question can obstruct their ability to accept and use the arbitrary. For example, attention being placed on why a door is called *door* can obstruct the acceptance and use of the word. Examples of such arbitrary things in mathematics are:

Cartesian co-ordinates;
number of degrees in one complete turn;

notation;
number-names;
'answers' to arithmetic problems being written on the right of the '=' sign;
...

I am not saying that there were not reasons at the time for the particular choice made. For example there were reasons why the unit of turn divided one revolution into 360. However, there is no reason why there must be 360 units in a complete turn. In today's environment of decimalisation, students may well consider that 100 would be a more sensible choice. The question *why 360?* is interesting, and I am not suggesting that it is not a valuable question to pursue. However, I am stating that it is not a question which helps someone's ability to use and interpret degrees. To this aim, pursuit of the question *why 360?* is uneconomic at the time when the desire is for students to learn to use and interpret degrees. Thus, an attempted explanation, which shifts attention onto the question is also uneconomic to this aim[55] .

If explanations are given with regard to things which are not arbitrary, then this is an attempt to transfer meaning to a student which is not possible due to the dynamics of the Neutral Zone[56] . The development of meaning can only be the responsibility of the students themselves[57] . Furthermore, such attempts to explain can lead to the 'explanation trap'[58] where students become passive and stop using their own powers.

PRINCIPLE OF ECONOMY: *avoid explanations*.

I have included number-names in my list of those things which are arbitrary. Not only are the names, such as *six* and *two*, arbitrary but also there is a arbitrary structure placed on these words to generate *sixty two*, for example. I said in 3.2.5 that the order in which the names are learned is significant in terms of economy of learning. For example, having learned the names from *one* to *nine*, learning *ten* will only give a child the possibility of knowing how to say one more number. However, learning *-ty* means that they can say *six-ty two* or *nine-ty seven* or... Thus, there becomes an issue of economy in how the number-names may be learned.

In section 10.2, I mentioned that *GRID Algebra* offers the option NOTATION to have brackets appear every time something is multiplied even though there are exceptions to this rule. Here, I develop further the argument for structure first, exceptions later with

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55 **{No explanations}** - 3.2.4
56 **{Neutral Zone}** - 4.2.5
57 **{It is the learner who develops meaning} - 3.2.3.**
58 **{The explanation trap}** - 4.2.1

regard to the structure in the number-names.

Karen Fuson gives the structure of the English sequence of number words from one to one hundred as:

> (a) a rote list of twelve words;
> (b) words 'thirteen' through 'nineteen' repeat the early number words 'three, four,..., nine' but the irregular 'thir-' and 'fif-' obfuscate this pattern;
> (c) a decade pattern of *x*-ty, *x*-ty one, *x*-ty two,..., *x*-ty nine in which the *x* words are regular repetitions of the first nine words for 'four' and 'six' through 'nine' but are not regular for two, three or five (i.e. for 'twenty', 'thirty', and 'fifty').
> (1991, p28)

From this list, we can count the number of words which the children will need to remember in order to say the number-names up to one hundred.

(a) gives us 12 words;
(b) involves repeating some of the words in (a) and so are not new words to be remembered. However, the ending '-teen' does count as one, and so will each of the irregularities 'thir-' and 'fif-'. This gives a further 3 words.
(c) involves some of the words in (a) again. The new words are '-ty', along with the irregularities 'twenty', 'thirty', and 'fifty'. The spelling difference of 'forty' to 'four-ty' is not counted as an exception since Fuson is concerned with the saying of the words rather than the writing of them. Thus, there are an additional 4 words here.

This gives a total of 19 words. Including 'hundred' would give 20 words.

This agrees with Caleb Gattegno who states in a footnote, with reference to the following table:

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

> We can count exactly the "burden to the memory" represented by

> what actually needs to be remembered for the full sequence to be generated. On the first line, "one", "two", "three", "four", "five", "six", "seven", "eight", "nine", or *nine* signs; on the second line, "ten", "twenty", "thirty", "fifty", "-ty" of *five* signs, plus "eleven", "twelve", "thirteen", "fifteen", "-teen", or another *five*; and on the third line, "a hundred" - or *twenty* sign-sound combinations altogether. (1971, p111)

However, in Gattegno (1988, p10), he had reduced this to 18 words by becoming aware that the number-names for 13 and 30 both use 'thir-'. Thus 'thirteen' and 'thirty' only require the one word 'thir-' along with '-teen' and '-ty'. Similarly for 15 and 50. Gattegno does not consider the word 'and' at this stage. This may be due to the fact that numbers can still be said without the use of the word. It may also be considered that 'and' will be learned within the acquisition of the English language and, when that is achieved, it can be incorporated within the number-names. For the moment I will stay with the 18 words, although it will make no difference to my argument if 'and' is counted. The main point is that all the number-names up to 'nine hundred (and) ninety nine' can be achieved through remembering only 18 words.

The way in which this is achieved is to generate number-names from the 18 words. Such generations involve saying one name after another (e.g. 'six' and '-ty'). After this generation has been done once to achieve 'sixty', the ability to say one after the other can be used again to generate 'sixty-five', and another twice for 'two hundred sixty-five'. This ability is one which children already possess and demonstrate as they engage in exploring the sounds they can make. Thus, it is not something which requires further cost in terms of what the learner has to remember. It is only the 18 words which are required to be remembered, although there is work involved in gaining the structure behind the number-names.

In appendix 1, I give an account of some work I did with Idris (3 years 10 months) on 21st April 1990 for 15 minutes. This was only 13 days after my encounter with Helen and Mark (see 3.2.5) and I had been thinking about the idea of working on rules before dealing with exceptions[59] . Idris could say the number-names with confidence up to the mid-twenties. However, although he could say 21, 22, 23, he was not able to say 31, 32, 33, or 41, 42, 43. By working on the structure of the number-names using the computer program *Counter*[60] , and ignoring exceptions, Idris was able to say all the names from 'one' to 'ninety-nine' within the 15 minutes.

Idris's mother, Linda, had not been present when I had worked with Idris and only knew

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[59] **{Rules first, exceptions later}** - 3.2.5

[60] *Counter* is available as part of *Some Lessons In Mathematics With A Microcomputer 2 (SLIMWAM2)* from the Association of Teachers of Mathematics, 7 Shaftesbury Street, Derby.

that I had done some number work with him. A few weeks later, she reported an incident she had with Idris whilst waiting at the cheese counter in a supermarket. The system at the counter was for a customer to take a ticket with a number on it and wait for their number to be displayed. Linda asked Idris what number was being displayed. He replied *fivety-eight*. Linda told him that it was said *fifty-eight*. Idris would not accept this and they had an argument as to how 58 was said. Linda reported to me that although he had not accepted *fifty-eight* at the supermarket, he was soon saying all the numbers up to 99 correctly without her having had any further discussions with him about the number-names. You may recall, from your own experiences, children applying rules within language such as the past tenses of irregular verbs. There is a period of time when the rule dominates and is applied to irregular as well as regular verbs despite adult correction. Then, suddenly, the exceptions of the irregular verbs are said in their conventional, irregular form. It appears that the rule is established before exceptions are dealt with.

The meeting of exceptions is a disturbance and a cost has to be paid in order to remember these exceptions. The disturbance Idris felt reveals that the saying of 58 was viewed as an exception and thus the overall structure was present for Idris in the majority of the number-names. Idris had managed to discover that there were exceptions and what those exceptions were by listening to the way other people say numbers and by finding out the reactions others have to the way he says the number-names (as with the cheese counter incident).

Exceptions will always have to be learned independently because of the fact that they are exceptions. However, the rest can be learned within a structure which will make greater use of what children can do and make less demands on memory. The exceptions interfere with the structure as a whole and thus it is the structure which needs to be engaged with first.

If the number-names are presented in numerical order for children to learn then there are so many irregularities in the first 20 number-names, that children have little choice but to employ memory for each name. Furthermore, the irregularities are such during the 'teens' that there is little assistance to help children link the number-names with the written symbols. For example, even though *nineteen* obeys a *teen*-rule, the *nine* is said <u>before</u> the *teen*, yet '9' is written <u>after</u> the '1'. The lack of a structure within the number-names which relates to the structure of written symbols is in contrast to some other languages, such as Chinese, where they are closely related. Fuson and Kwon (1991) have indicated that this could account for observations that English speaking children have extra difficulty with some of their number work compared to Chinese speaking children. Approaching the English number-names initially through overall structures may be economic in terms of later learning as well, when number names are related to work with written symbols.

PRINCIPLE OF ECONOMY: *work on overall structures first, leaving exceptions until later*.

I designed *GRID Algebra* so that students can generate material without the requirement for them to remember any previously encountered mathematical knowledge. At the same time I wanted the program to be a mathematical environment in which students could engage. One way in which I did this was to have mathematical notation appear whenever students performed movements with expressions on the grid. Thus, a physical movement produces a mathematical expression. At first a student may not have an understanding of what appears on the screen. Yet, what appears on the screen is only a consequence of the decisions and actions they make[61] .

Hawkins commented that *The adult's function, in the child's learning, is to provide a selective feedback from the child's own choice and action* (1969, p23). The computer program takes on this function and provides feedback which is selective in the sense that feedback is in terms of formal algebraic expressions and where only 'correct' expressions are allowed on the grid. Furthermore, all that appears on the grid is a consequence of a student's own choice and action.

The expressions which appear are not random (although they may appear so initially to a student), they conform to a strict structure which is consistent with some algebraic properties of arithmetic. A desire to gain control[62] in this environment means to gain control of the algebraic expressions which appear on the screen. To gain control of the algebraic expressions means to gain control of some algebraic properties of arithmetic.

In order to be in a position to engage in the challenge of gaining control of an environment then that environment must have inherent within it the possibility for it to be controlled. In other words the environment should not be randomly governed so that it behaves largely independent of what a student may do. Furthermore, feedback of actions or decisions taken are required since control requires informed decisions and actions.

In a classroom this feedback may be in the form of marked exercises that a student has completed. When I have taken some time before marking a piece of work, I have noticed that the student requires some time and effort to remind themself of what the work was about and what they had done. The feedback I gave them on such occasions

[61] **{Experiencing the consequences of their actions}** - 3.2.1; **{Seeing the consequences of their decisions}** - 5.2.2
[62] **{Gaining control}** - 3.2.1

has not appeared to be as useful to them as other occasions when I have given feedback sooner. It is another indication of the effectiveness of simultaneity that feedback is more productive the sooner it is given after the actions themselves. In fact, one positive aspect of computers is that they are capable of giving immediate feedback. For example, LOGO is a computer environment which gives immediate feedback to the commands given by students.

One type of feedback can be informing a student whether what they have done is right or wrong. This gives the student limited information with which to re-consider their decision or action. At present, I am trying to juggle three balls when one ball is occasionally thrown from behind my back, passing over my head and caught in front. As I attempt to throw the ball from behind my back I get much more information than whether it was right or wrong. I see exactly the flight of the ball and whether it arrives too much to the right or left, or whether it arrives too far forward for me to catch and continue juggling. I get this information through being able to observe the reality of what the consequence of my throw was. I have set my own challenge of what I want to do with that ball, that of catching it in front to join the two other balls to continue the juggle. Whether the throw was right or wrong depended upon the challenge I had set myself. Thus, as long as I am clear about my challenge, and I am informed of the consequences of my actions, then I am capable of deciding myself whether the throw was right or wrong. More importantly, I have far more information than right or wrong with which to improve my later actions. Papert (1980, p22-23) talked of the fact that learning involves making mistakes and going through a process of *de-bugging*. If LOGO did not reveal on the screen the consequences of students' decisions then de-bugging would not be possible.

Dewey talks of the important role consequences play in the development of meaning:

> We may sum up by stating that things gain meaning when they are used as *means to bring about consequences* (or as means to prevent the occurrence of undesired consequences), or as standing for *consequences* for which we have to discover *means*. The relation of *means - consequence* is the center and heart of all understanding. (1933, p146)

By informing someone of the consequences of their actions immediately, the consequences can drive improvements in actions (or means), and actions (or means) can be explored to learn consequences. Either way the dynamic goes, accounting for the relationship between actions (or means) and consequences is at the heart of developing meaning.

PRINCIPLE OF ECONOMY: *enable students to see immediately the consequences of their actions and decisions*.

10.4 Backwards and forwards

I have described how the development of the computer program *GRID Algebra* has incorporated within its structure the principles of economy identified in chapters 8 and 9. Further development of my theory has also identified some additional principles to those already identified. The following chapter will present the conclusions from this study, which will include a list of these principles as a summary.

11 Conclusions

11.1 Introduction

In this chapter I present the conclusions to my thesis starting with a list of the principles of economy in learning and teaching mathematics which have been developed over the last three chapters. It is important to realise that these principles, presented without the examples from which they were generated, could be read as over general statements and easily interpreted as bland injunctions. The force of this thesis is to provide:

(a) accounts of situations to which the reader can relate in terms of their own experiences; and
(b) exercises which help the reader to awarenesses.

It is from both of these that the principles were derived. Thus, although a bare list is provided as a form of summary, the principles should not be read in isolation from the work which has generated them. As Elliott stated: *General rules are guides to reflection distilled from experience and not substitutes for it* (1991, p50). In this case the experience is both my own which is described throughout this thesis, and also the reader's, whose experiences I have called upon to relate to aspects of this work.

I have written, in brackets after each principle, the section where the principle was stated.

11.2 Principles of economy

- View mathematical content through the eyes of a pedagogue rather than a mathematician. This means considering mathematical content in terms of children's powers, mathematical essences, and awarenesses (8.3);

- Use pedagogical awarenesses to work with the learner's mathematical awarenesses (8.3);

- Reduce the need for a learner to remember previously encountered mathematical content (8.3);

- Work on overall structures first, leaving exceptions until later (10.3);

- Offer sufficient complexity for learners to have the material necessary to make use of their powers (8.3);

- Avoid explanations (10.3);

- Reduce the need for translation (8.3);

- Enable students to see immediately the consequences of their actions and decisions (10.3);

- Use simultaneity to help establish desired associations (9.3);

- Use imagery and movement so that awarenesses can be gained within a context of generality (8.3);

- Place attention in an activity which subordinates the desired learning. This means that something is practised whilst progress is made at a higher subordinate level (practice through progress) (9.3);

- Use successive levels of subordination to drive functionalisation (9.3);

- Provide that which needs to be given, namely the arbitrary (10.3);

- Restrict the use of memory to that which is arbitrary (10.3);

- Reduce the need for memorising the arbitrary by using functionalisation, or by using imagery combined with association (10.3).

11.3 Conclusions

This thesis has shown that learning and teaching mathematics can be viewed in terms of the personal effort and time a student is required to provide in the learning process. This establishes the principle of *economy*, to minimise the amount of time and personal effort given by a student, and to maximise the amount of their learning[63] . Economic teaching is that teaching which assists economic learning. The principles of economy which are listed above are principles for a teacher in order that their students can have a learning environment which supports the productive way students worked when they were very young.

[63] **{Economy is concerned with time and effort spent for the learning gained}** - 4.2.2

I have analysed several events which have been significant in the development of my theory, in order to identify powers children possess and use as part of daily human activity in coming to know and gain some control over the world that surrounds them. Additionally, I have identified three ways in which information is held and demonstrated that memory is the most costly of these in terms of economy. Yet it is memory on which text books often rely as a way to support learning. Due to the relatively large amounts of personal time and effort required to memorise something, one principle is for memory to be restricted to that which is arbitrary. Those things which are not arbitrary are the concern of awareness. They can become known to a student through the use of the student's powers and the student's sense of truth. The task for a teacher is to devise an activity of sufficient complexity that provides the material with which a student can use their powers to gain this awareness.

Teachers cannot give their students awareness. All a teacher can do is to make offerings. These offerings can (a) provide material which a student can use their powers to work on, and (b) attempt to affect a student's attention. None of these two can be guaranteed since both require the active involvement of the students themselves. I have developed the idea of a Neutral Zone, a fictitious zone in which all offerings, from students and teachers, are placed. These offerings become potential material for other people, however, the potentiality only becomes an actuality through the active involvement of someone choosing to place their attention appropriately. Thus, the dynamic of the zone is such that it requires a positive action *from* someone to contribute to the zone, and a positive placement of attention *from* someone to relate to another's contribution (or, indeed, to reflect on their own contribution). All aspects of communication require someone to actively do something, whether it is talking or hearing, showing or seeing. The arrows do not represent the physical dynamics of the photons of light or sound waves, but represent the personal energy in the form of attention which is required for communication to take place. This attention can result in talking, listening, etc. There is nothing passive about the process of communication. As a consequence, my image of the Neutral Zone has arrows coming *from* each person to the zone (see Figure 37). This contrasts with a traditional image of a student *receiving* information from a teacher (see Figure 36). Figure 37 shows several arrows going into the Neutral Zone, since there are likely to be contributions made by other people as well as a teacher, even if those people are not particularly directing their contributions towards the student. For example, someone in the room may be turning round to someone behind them, or another person may be tapping their pen on a desk. Some contributions may be potential distractions for the student. However, the student may decide to place their attention in these distractions rather than the teacher's contributions.

Figure 36: Traditional image of student receiving information.

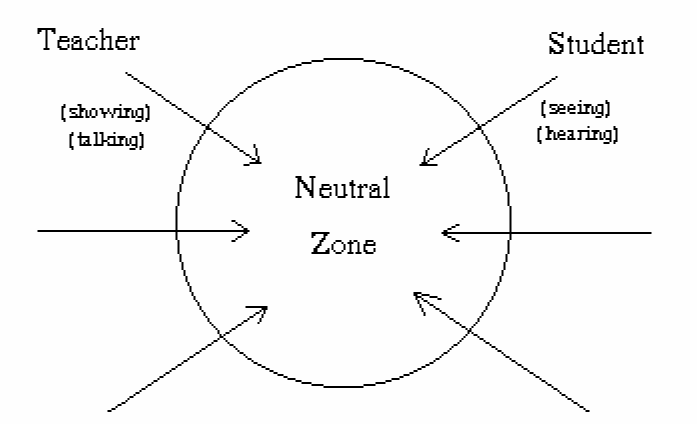

Figure 37: Dynamics involved in communication through the Neutral Zone.

A traditional style of teaching involves a teacher explaining something they want the students to do. The idea being that the students will know what they have to do by listening to what the teacher is saying. In such a situation, there is often a need for the student to translate from listening to an explanation to carrying out the doing. This translation is due to either a difference in the mode of representation, or the need to go from one of the student's senses to another, or both (see Figure 38). I have described a way to avoid the student making such translations by the teacher working with the material the students offer, rather than the teacher producing their own material, via explanations, which the students would then have to translate into their own actions. By acting as an editor or an amplifier, the teacher is only asking the students to do a subset or an elaboration of what they have already done. Thus, there is no requirement to translate (see Figure 39).

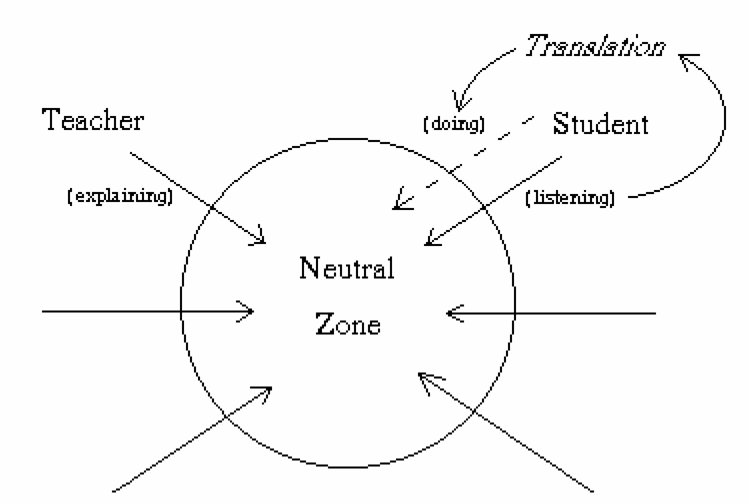

Figure 38: Diagram representing the need for translation with a traditional style of teaching. (The dotted line happens at a later time than the solid lines).

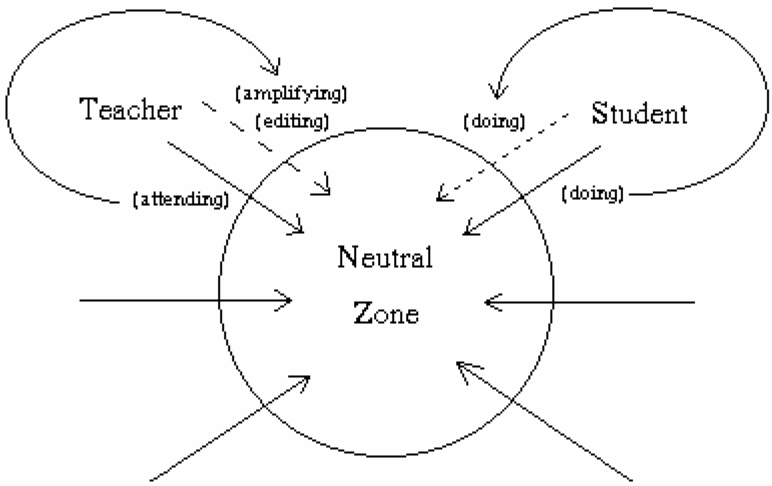

Figure 39: Avoiding the need for translation through the use of editing and amplifying. (The dotted lines happen at a later time to the solid lines, and the student's dotted line happens after the teacher's dotted line).

Gattegno's (1971) list of powers of children was unstructured. I have modified his list and provided a structure which relates these together (see Figure 14 in chapter 6). In this structure, *will*, *a sense of truth*, and *creativity* are used to direct *attention*, which produces *stressing and ignoring* as a consequence. The result of this stressing and ignoring can be either *extraction*, *abstraction*, or *association* depending upon the material which was stressed and ignored. It is the last three of these which I have chosen to describe as children's powers. As young children go about their everyday learning, these powers are employed continually. I have linked children's powers with the mathematics syllabus of schools through the notion of mathematical essences, which are the manifestations of these powers in the domain of mathematics. *Sameness/difference*, *order*, and *inverse* are employed within daily learning as a young child, and are also employed within all areas of mathematics. These essences, along with the powers, are available for students to use in their mathematics lessons. They have been employed through the student's life in their learning as a human being, and so are functionings. The main problem which prevents students using these powers in mathematics lessons is the fact that the students have learned through years of attending lessons that these powers are rarely called upon. Instead, students have experienced countless hours of being told things which they are then expected to memorise. As a consequence, students have learned to adapt to the expected discourse within a lesson. Thus, although these powers and mathematical essences are always available, students may take a while to become used to a different set of expectations and discourse within a mathematics lesson.

Mathematics is often described in a structured way similar to building walls with bricks. A brick of a certain level is placed on bricks of lower levels. Likewise, a certain level of mathematical content is described in terms of previous levels of mathematical content. This metaphor is one which has been adopted by some people within mathematics education by assuming that because mathematics is described in such a way, that implies it is to be learned in a similar way. Consequently, the learning of new mathematics relies on the previously learned mathematics of lower levels to be remembered and understood. This creates so many problems in the classroom with students having forgotten the mathematics (since it is often asked to be memorised) of the lower levels on which the new mathematics is to be built. This means that the lesson cannot progress since the old mathematics has to be re-visited and built again before progression can take place. In fact, it is sometimes the case that a succession of layers of mathematical content need to be re-visited. In such cases a teacher might describe the lesson as one where they had to go right *back to basics*. The throwing away of this building bricks metaphor would help the advancement of economic learning and teaching of mathematics. Each new area of mathematics can be considered in terms of an environment which will call upon children's powers of extraction, abstraction, and association (and the manifestation of these powers as mathematical essences), instead of remembering previously encountered mathematical facts. Then, new mathematics is encountered as an awareness within this environment gained through the use of children's powers. In this way, Bruner's claim that:

> ...any subject can be taught effectively in some intellectually honest form to any child at any stage of development. (1960, p33)

can be viewed as quite possible within mathematics since a new area of mathematics does not need a set of previous mathematics lessons for students to have learned all the mathematics of the lower levels. Instead an environment can be constructed such that the new mathematics can be gained through an awareness which only requires the use of children's powers.

I have demonstrated that memorisation of the arbitrary can be assisted or avoided by the use of functionalisation, or by the use of imagery combined with association. Indeed, it is functionalisation which appears so strong in the dynamics of early learning. Without the ability to drive something into becoming a functioning, there would not be sufficient spare energy in terms of attention for new challenges to be engaged in. Growth in learning requires attention to be freed from the old to be available to meet the new. Functionalisation is the processes of driving something into unconscious mastery. I repeat the quote from Whitehead *Civilisation advances by extending the number of important operations which we can perform without thinking about them* (1925, p42). Functionalisation drives this process and is a powerful way in which young children learn. The old dictum of *practice makes perfect* does not describe adequately the way in which I observe young children practice. Rather, they continue to progress whilst practising by finding challenges which subordinate a newly acquired skill. The writers of text books, and the planners of many mathematics lessons, have not learned from this as most of the exercises they provide ask students to practice at the same subordinate level, basing their hopes for learning on the idea of practice makes perfect.

It is through functionalisation above all else that I see a way forward for the learning of mathematics to become more efficient in that more mathematics can be learned more thoroughly in less time and with less effort. I have produced a theoretical framework to account for the transition from meeting something new into that something becoming a functioning. This process of functionalisation is described through the notion of levels of subordination and practice through progress. The placing of attention in a task at a higher subordinate level to the desired learning can drive that learning into becoming a functioning. To assist functionalisation further, successive levels of subordination can take place with each one driving the original learning increasingly into something which no longer requires conscious consideration.

The viewing of learning and teaching mathematics in terms of economy is not just an academic exercise of observation. I have demonstrated that creating resources, planning lessons, and ways of working with students, can be based on the idea of economy. The establishment of the *principle* of economy which appears in the title of this thesis has led to the production of a set of *principles* to assist the development of economy in the learning and teaching of mathematics.

Levi-Strauss commented:

> What makes a steel ax (sic) superior to a stone ax is not that the first one is better made than the second. They are equally well made, but steel is quite different from stone. In the same way we may be able to show that the same logical processes operate in myth as in science, and that man has always been thinking equally well; the improvement lies, not in the alleged progress of man's mind, but in the discovery of new areas to which it may apply its unchanged and unchanging powers. (1963, p230)

Students of all ages have the powers they used so effectively as young children, yet many teachers organise their lessons in such a way that these powers are not called upon. The principles I have produced are guide-lines to apply our 'unchanged and unchanging powers' to mathematics classrooms. The same powers used on the 'stone' of our early learning such as walking and talking, can be applied to the 'steel' of mathematical content such as multiplication tables and calculus. I have shown that many commercially prepared materials do not apply the powers of children to the school mathematical content.

Many of the principles are concerned with the placement of attention, either as a statement of where it would be desirable for attention to be placed or where it is not desirable for attention to be placed. The importance of attention has been recognised by Mason and Davis (1988) (also, see Mason (1989)), who have been concerned with the shifting of attention. Here, I have been more concerned with the placement of attention. The significance of attention means that an entry for mathematics educators into the issues of economy can be made through an attention to the issue of attention. Where is the teacher's attention placed? Where is a student's attention placed? How can attention be shifted? Where might it be shifted to?

In the light of increasing pressures and demands on teachers to improve the level of students' achievements in mathematics, I have established *economy* as a powerful perspective through which to view the dynamics of learning and teaching mathematics. It can lead to more productive activity in mathematics classrooms through students working in ways which have proved so productive in their early pre-school learning.

12 Further questions relating to study

In the light of the work I have carried out in this study, there are questions which have arisen for me which did not exist when I began my work. It is the nature of learning that increased awareness brings with it as many questions as answers. I will consider here some questions which are relevant to the work I have done in this study but which I have not explicitly tackled. Many of the questions have come as a consequence of the learning I have gained through doing this work. Other questions are ones of which I was aware before engaging in this study, but I have not directly addressed them in this work.

One issue I have had in mind throughout much of this work is that of motivation. I decided not to make explicit reference to motivation as I felt that would have directed me away from the particular task of this study. However, one observation I wish to express now is that when students are in an environment where their powers are being used and challenged, I notice that they are fully involved and engaged in the activity. It appears that motivation is not an issue in such circumstances. Thus, I have a conjecture that, when their powers are called upon, students are sufficiently engaged and no additional or external motivation is required. By external I mean motivation which is generated separately to the activity itself. Examples of such external motivation include giving marks or grades for work, rewards, and punishments. The relationship between motivation and the demands placed on students in terms of their powers is an area of possible future work.

I chose a particular classification of children's powers which identified *extraction*, *abstraction*, and *association*. I said at the time that there might be other classifications which could have been made. I have found these particular classifications to be helpful when considering what principles might be involved in teaching economically. Are there other choices which would be more beneficial to future development of teaching economically?

I have used the word *awareness* throughout this thesis without going into explicit detail as to the precise role of awareness in the learning process. Gattegno states that *only awareness is educable*, which places awareness at the centre of education. My work has been mainly concerned with creating situations whereby there is opportunity for students to gain new awarenesses rather than being placed in situations where they are only being asked to memorise. I have described how these new awarenesses can be forced into becoming functionings and I have briefly mentioned that during the functionalisation process there are opportunities for reflection on what is known, which can lead to further awarenesses. This role of reflection is one which raises further questions. There has been some attention given to reflection (see 9.3), however, the question I raise here is concerned with when it is appropriate to ask a learner to reflect. A possible implication from my work is that the primary focus might be on helping someone to be able to do something first, through such methods as functionalisation, and that reflection at this stage may not be of benefit. It is a secondary matter to ask that person to reflect on what they can now do. This can sound traditional, however, I believe that it is clear from this thesis that the methods involved in helping someone to do something are far removed from such notions as following instructions. As I have mentioned earlier, something can be understood in terms of an action or a process, something which is *done*, the significance of which can be gained through reflection. I believe that the role of reflection in the process of functionalisation is an important area for further work.

As part of my work I have developed the computer software *GRID Algebra* as an example of a resource which is based on principles of economy. The issue for me within the context of this study was what could be written into the software itself. I have indicated that effective use of the software will depend upon the teacher and students and that this is intentional. Thus, the actual use made of *GRID Algebra* in the classroom has not been tackled here. Further work relating to the use of this program, and incorporating principles of economy, would be welcome.

There is the important question of how a teacher educator might work with students if the educator considers the principle of economy to be important. Thus, how might the principles of economy be applied to the situation of working with students on the principle of economy?

Finally, this thesis has been concerned with the learning and teaching of mathematics, and most of my examples and research have come from mathematics classrooms. However, I strongly believe that there are implications for my study in other areas of the curriculum. I am calling for the teaching of mathematics to be based on powers which students have used in their everyday learning since being a young child, and that the use of these powers produce economic learning. Students have these same powers in other curriculum classrooms as well. Thus, there is the question of how the principles of economy can be applied to other subject areas, and to what extent the principles might change according to the particular nature of those subject areas.

I identified mathematical essences as those root elements of mathematics which students have also met through the use of their powers in everyday learning. Are there essences in other subject areas and, if so, what are they?

This thesis has built on much of the work Gattegno, in particular, has done. However, I am aware that it is only a stepping stone towards more productive ways of teaching in classrooms. For me, this study has both refined my practice as a teacher and enabled me to articulate the theoretical underpinning of principles leading towards economic learning and teaching.

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Appendix 1 - Teaching Idris the number-names from 1 to 99

On 21st April 1990, I worked with Idris, then aged 3 years 10 months, for 15 minutes. I had been thinking about the incidents with Helen and Mark (see 3.2.5) and considered what I might do to help Idris learn the number-names up to 100 in an economic way, in order that he might have a productive learning experience in the time available. I knew that I only needed Idris to know the number-names from *one* to *nine*, and these he knew well.

I used the computer program *Counter*[1]This program presents an initial starting number on the screen. This starting number can be chosen by the user. The number then steps up, by a constant amount, in regular time intervals. This constant, known as the step number, can also be chosen by the user. I started with a start number of '1' displayed on the screen which stepped up in ones. Idris called out the number-names as the numbers appeared on the screen. I stopped when the number arrived at '9'.

There was one new name I wanted Idris to know, and that was *ty*. I asked Idris to say *ty*, and then set the program to start with '10' on the screen and to step up in tens. As the counter started going, I pointed to the tens digit as I was saying:

10	*one*-ty
20	*two*-ty
30	*three*-ty

Idris joined in... *four*-ty, *five*-ty, *six*-ty, *seven*-ty, *eight*-ty, and *nine*-ty. We practised this a few more times and stopped the counter on '60'. Changing the step to one, I continued, saying:

60	sixty
61	sixty-one
62	sixty-two
63	sixty-three
.	.
.	.

[1] *Counter* is available as part of *Some Lessons In Mathematics With A Microcomputer 2 (SLIMWAM2)* from the Association of Teachers of Mathematics, 7 Shaftesbury Street, Derby.

Each time, I pointed to the units digit. I pointed in an attempt to affect Idris's attention; to indicate that it was the symbol I pointed at that related to the changes in what were are saying. I varied the speed of the counter so that Idris and I had a demanding but manageable rhythm as we said the number-names.

Next, I changed the start number to 40 and stepped through in ones. We repeated this, starting at different multiples of ten. Idris was saying these number-names on his own now. After a while, I started at one and stepped through in ones up to 99. Idris made the odd mistake but was able to say the vast majority of the number-names without further help from me. I finished off by putting random two-digit numbers on the screen for him to say. This he did successfully.

In all I had spent 15 minutes with Idris. In this time, he learned to say to least 70 more number-names than he could say before. Obviously, there was more work to do with 10, 20, 30, and 50 being exceptions, and 11 to 19 having their own rule with exceptions. However, much had been achieved with relatively little effort over a short period of time and only exceptions were left and the *-teen* rule (which is in itself an exception to the main structure). By their very nature, the exceptions would have to be learned individually, at least they could be worked on within a larger framework of which Idris now had command.

Appendix 2 - Simultaneous equations lesson

This account is based on tape recorded lessons with a mixed ability year 8 (12-13 year olds) class which took place in September 1989.

Beginning

Previously, the class had done some work that had made them comfortable with algebraic notation (*Think of a number*).

I'm thinking of two numbers.

I write on the board:

x   y

Do you know what they are?

There are some guesses.

Do you <u>*know*</u> *what they are*?

Now, they say that they don't know.

Well, I'll give you some information:

$$x + y = 10$$

As suggestions are called out, I write them on the board:

5	5
3	7
6	4
.	.
.	.

Someone calls out '7,3'. Someone else says that we have already got that. However, I choose to write it up on the board. Following this, several more suggestions are made. These are put up on the board as well.

After a while, the question arises as to whether we have all the possibilities. Ways of checking are discussed. Once they are sure, I write:

2.5 ?

This releases a number of other possibilities. We spend some time exploring decimals and fractions. This is an opportunity to do some work and practice in these areas. I continue until there is a strong feeling of continuing for ever. If possible, I want students to gain a feeling for the infinite. Once I feel this has been achieved, I write:

-1 ?

This unleashes even more possibilities (more than infinite?). Along with gaining a sense of the infinite, we are also practising and working with negative numbers.

By now, we have several blackboards of possibilities. I ask them to put these possibilities on a graph. Scales are discussed, I check that each student labels their axes in an appropriate way and plot points correctly.

Issues arise as to how accurately points can be plotted, as some of the decimals and fractions are a little awkward. The negative numbers create a need to extend axes.

Not every student works on all these things. It is a mixed ability class. Some students are trying to carefully draw axes and plot a few points successfully, others have redrawn their axes and are considering all four quadrants.

The points appear to lie on a line. This is observed and shared, leading to an awareness by the whole class. Some students find the odd points that do not fit this; these are checked.

There are a number of points on your graph. What about the spaces inbetween?

Gradually, inbetween points are added as students find that they, too, conform to the rule $x + y = 10$. As more points are added, more spaces appear. Eventually a 'line' is formed through the saturation of points. Infinity is once again present.

Do you know my two numbers?

They reply that the numbers could be any of them.

With the *Think of a number* activity, I had one unknown number. They were able to find it by me giving them just one piece of information in the form of an equation. Now that there are two unknown numbers, we find that one equation is not sufficient.

In that case, I will tell you some more information:

$$3x + y = 15$$

There are some suggestions, but they do not find it so easy to generate possibilities with this equation.

What if x was 4?

They are already familiar with the notation, due to the work we had previously done. They say that 3x must be 12 and so y is 3. By thinking up a number for x, we can determine what y is. This technique means they can generate more possibilities for this equation. With these being written on the board, we find that none of them also satisfy my first equation, $x + y = 10$. So, they still do not know my two numbers. However, the situation has changed. Previously, there were an infinite number of possibilities, now, we cannot find one.

I ask them to plot the possibilities for the second equation on the same graph as the first. Once again, they appear to lie on a line. Spaces are filled and checked until the saturation of points forms a line.

By now, some students have already found what my two numbers are. For the others, I draw their attention to a special point that happens to be on both the lines.

Development

Pairs of equations are given for the class to solve. They vary in the way they are written in order that there is practice at working with algebraic notation. For example:

$y = 4(x + 6) \quad x + y = 9$

or

$3x = 5 + y \quad y - x + 6 = 0$

One of the examples I put on the board is

$y = 2(x + 3) \quad y = 2(x + 5)$

A number of the class tell me, as they come to do this, that the two lines don't cross. I reply that perhaps they might meet up somewhere off the page of their book. One girl, Sarah, asks for larger graph paper. However, when she extends her graph, she still finds that the lines do not meet. Stuart and Gavin are discussing what will happen if they continue the lines in either direction. I hear railway lines mentioned. Rowena says, concerning the tables she made of the possibilities for each equation, *this is always four more than that, so they can never be the same*. Other students overhear this and look at their tables of values. They agree that the second equation is always four more than the first.

I ask (pointing to the appropriate places in the equations), *why four more, when this says 3 and that is only two more, 5?*

Because you are doubling it, several reply. Here, I find that the previous work I did (1), which developed understanding of the notation, is paying off.

So, is $y = 2x + 6$ another way of writing the first equation?

They seem happy with this and tell me that the second equation can be written as $y = 2x + 10$. We practise expanding brackets and factorising for a while.

I return to the two equations and ask what my two numbers were. I want to spend a while dwelling on the fact that two equations do not always produce a solution. I ask if they can tell me two other equations like this. We draw them to check whether their graphs meet.

Another pair of equations I have up on the board is

$$2x + y = 8 \qquad 4x + 2y = 16$$

Joseph comes up to me and says *It's the same graph. It still could be any of them*.

Sometimes two equations leave me no more knowledgeable than when I knew only one of them.

We explore other equations that will also give us the same graph.

Is there anything else that can happen with two lines?

Algebraic solution

Whilst the class are solving sets of simultaneous equations by graphical methods, I mention to them that anyone who wishes to work on finding my numbers using algebra should come round the front table. I say that they might not find the work particularly easy but it's a nice challenge. I ensure that particular students do come to the table whilst not discouraging others. Some of the students that come will find it hard to cope with what we will do. This does not bother me, they will either surprise me or choose how long to stay with us.

I write on the board

$$y = 3x + 4 \qquad y = 2x - 3$$

I point to the relevant places in the first equation as I say *y is the same as 3x + 4*. I ask one student, Fiona, to say *y*, another, William, to say *3x + 4*. As they repeat these, I say that the first equation tells me that these are the same.

I ask Sarah to read out the second equation, *y = 2x - 3*. I ask her to read it out again, only this time, when she sees (I point to Fiona) *y*, she says (I point to William) *3x + 4*, as they are the same. Now Sarah says *3x + 4 = 2x - 3*. I write this on the board. I hesitate at this point to see if they seem comfortable with this.

Now I am left with only one number to find instead of two.

I write on the board

$$5 = 6 - 1$$

saying that I want to move the 1 across to the other side of the equals sign. Nicola says *Oh, it's plus 1. It changes*.

$$5 + 1 = 6$$

There are some comments indicating that they note a connection between this and the previous work we had done on algebra. They are now able to tell me how I can move the 3 across:

$$3x + 4 = 2x - 3$$
$$3x + 4 + 3 = 2x$$

At this point we spend a while messing around with this equation, pushing the 3 and 4 from one side of the equation to the other and back again.

We then looked at the 2x. Sarah says *2x, that's times, so you divide by 2*. Again, this stems from our previous work. I write on the board:

$$\frac{3x + 4 + 3}{2} = x$$

I say that this is true but that I am going to take 2x across and not just the 2. Sarah asks me why, I reply by saying that her question may be answered in a while. I ask them to think of the equation as

$$3x + 4 + 3 = 0 + 2x$$

They accept this. *Now move the 2x across*:

$$3x + 4 + 3 - 2x = 0$$

I cover up everything but the *3x* and ask how many x's there are. They say 3. I then cover up everything except the - *2x* and ask how many there are here. Some say 2, so I cover up the - and say that this would be 2. I uncover it again and they say -2. Now, I reveal the whole equation and they tell me there is just one x. I write

$$x +$$

saying that, when there is only one of them, it is usual to just write *x*. Continuing, they want me to write *4 + 3*, but I state that they can work that out.

$$x + 7 = 0$$

From this, they know what x is, and, by substituting this in one of the original equations, they can find out what y is.

Rowena, Sarah, William, Joseph and Clare try to solve another pair of equations, working on the blackboard. They work well together and spot each other's mistakes. Between them, they are successful. Fiona is able to tackle quite complex ones on her own. For example:

$$3x + 2y = 13 \qquad\qquad 5y = \frac{3(x + 7) - 4}{6}$$

Although they are working on solving simultaneous equations, they are mainly practising manipulating and solving linear algebraic equations, multiplying out brackets, collecting like terms, using negative numbers and fractions. Simultaneous equations are the vehicle through which these are practised.

Appendix 3 - *GRID Algebra* booklet