Mathematics - joy and rigour

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MATHEMATICS – JOY AND RIGOUR

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Why did you become a subject leader in mathematics? Why mathematics?

Why and in what ways is mathematics important to you?

What was it that brought you into mathematics in the first place?

How do you yourself feel about mathematics?

As a former subject leader in mathematics in a secondary school, and in many other roles that I have played in mathematics education since then, I have no doubt as to why I am in mathematics education, and it boils down to two words – *joy* and *rigour*. I enjoy engaging in mathematics; I like its beauty and elegance, I appreciate the challenge of a mathematical problem and particularly the feeling of success when I contribute to its resolution. I value the ways in which we can express ideas succinctly and precisely in mathematics and appreciate the importance of a clearly defined language, its associated rules and the necessity for proof. The feelings of joy in stimulation, challenge and success are allied to recognition of the centrality of rigour and its manifestations in definitions, forms of expression and styles of proof.

I am going to say more about what I mean by joy and rigour, but let me start by saying that if we can enable our pupils to experience the joy and value the rigour of mathematics then we are doing a good job as educators.

What is it that you enjoy about mathematics? Do you know what teachers in your department enjoy about mathematics?

The authors of the Cockcroft report, produced in 1982, revolutionary in its time and still relevant in many respects today, wrote "Mathematics is a difficult subject both to teach and to learn" (para 228, p.67). Wide experience in learning mathematics and in working with learners of mathematics supports the statement that mathematics is found difficult by many pupils. This is fine – playing the piano is difficult, making a successful soufflé is difficult, building bridges that do not fall down is difficult. Many things that are worth doing are difficult but this does not mean we cannot enjoy the challenge and the engagement, and indeed our success in achievement. Mathematics is also difficult to teach. The challenge here is not *just* to present mathematics to learners in a way that is clear and easy to understand (where often the teacher is the one doing all the work), but really to *seek out the essence* of what we are trying to teach and to bring all our powers of motivational analysis to constructing a classroom environment that maximizes opportunity for pupils also to appreciate this essence. Most of the rest of this chapter is about what I understand this to mean.

In 2003, the Swedish National Agency for Education presented a report called *The Joy to Learn – Focus on Mathematics*.[1] One important finding in their report was that many pupils in Swedish schools experience mathematics as boring, and not challenging or joyful – probably not so different from many pupils in UK classrooms. In a study of disaffection in secondary mathematics classrooms in the UK, Elena Nardi and Susan Steward found that pupils on whom the study focused "apparently engage with mathematical tasks in the classroom mostly out of a sense of professional obligation and under parental pressure. They seem to have a minimal appreciation and gain little joy out of this engagement". Nardi and Steward go on to say,

Most pupils we observed and interviewed view mathematics as a tedious and irrelevant body of isolated, non-transferable skills, the learning of which offers little opportunity for activity. In addition to this perceived irrelevance, and in line with previous research that attributes pupil alienation from mathematics to its abstract and symbolic nature, pupils often found the use of symbolism alienating. (Nardi & Steward, 2003, p. 361. Emphasis in original.)

These authors point also to the school experience of the pupils they studied. Pupils resented what they perceived as *rote learning* activity, *rule-and-cue* following, and some saw mathematics as an "elitist subject that exposes the weakness of the intelligence of any individual who engages with it." Other researchers in the UK have looked closely at pupils' experience in the classroom and talk about underachievement in mathematics in relation to, for example, the effects of *setting* on pupils' opportunity and performance; to ethnic and social groups who have difficulty with school expectations, particularly in terms of language; to the kinds of tasks and activity presented in classrooms and the emphasis on testing (e.g. Boaler, 1997; Houssart, 2004; Lee, 2006; Watson, 2006). Thus we see pupils who are alienated due to the intrinsic nature of mathematics and pupils who underachieve in mathematics because of the practices and social norms within schools where they are taught.

If we focus on the need to share joy in mathematics as a solution, the question arises – what is needed for teachers to be able to awaken the 'joy to learn mathematics' in their pupils? I would claim that the answer lies, first and foremost, in how we, as educators, think of and promote mathematics itself. What areas of mathematics do we like and appreciate? Which ones stimulate us and give us pleasure? For example, we might think of the golden section, aspects of fractal geometry, ideas of infinity, algebraic conciseness in expressing an idea, a dissection proof for Pythagoras theorem, symmetry… Whatever it is that brings us personally to appreciation of mathematics, we need to exercise some of the passion from our own joy of mathematics in creating learning opportunities for our pupils.

Now for my second key word: *rigour*. The Cockcroft report, again, said "Mathematics provides a means of communication which is powerful, concise and unambiguous" (para 3, p. 1). The reason why mathematics is powerful, concise and unambiguous is to do with its logical consistency, its means of

1. See Straesser, Brandell, Grevholm, & Helenius (2004).

expressing generality and its use of abstraction to capture the essence of ideas. It uses elegant forms of expression to capture complex relationships and succinct pieces of logic to prove complicated propositions. G. H. Hardy is reported as saying "A mathematician, like a painter or a poet, is a master of pattern" (Davis & Hersh, 1981, p. 173). Recognition of pattern, expression of generality in pattern and abstraction from pattern are essential to being mathematical and doing mathematics. Rigour lies in the expression of generality and what is allowable as proof. Logical consistency demands rules, and rules have to be applied according to agreed systems of logic. Understanding and gaining fluency with rules is both central to mathematical success and part of what makes mathematics difficult to learn. Reducing mathematics to the rules, however, leaves the subject bereft of meaning, joy or passion. People who find mathematics hard and boring might do so because all they see is the rules: the rules lack connection to the exciting ideas for which they are invented, and tedious exercises to practice disembodied rules can be mind-numbing. However, practicing the rules, like arpeggios, is essential to bringing out the beauty of the music. So a central challenge for teaching mathematics is how to embody the rules of mathematical engagement in a way that is exciting and challenging and joyful.

A classroom story[2]

Look at Figure 1 here.

What is it?

What shape is it?

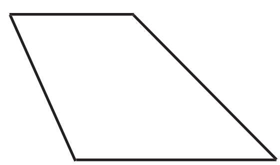

Figure 1: The teacher's drawing

What would be your reaction to someone who said it was a square?

A class of 12 year olds had been asked by their teacher to name the above shape, which he had drawn on the board. Someone said that it was a trapezium. Some pupils agreed with this, others disagreed.

The teacher said, 'If you think it's not a trapezium then what is it?' One boy said, tentatively, 'It's a square…'.

There were murmurings, giggles, 'a square?!'. But the boy went on '…sort of flat.'

The teacher looked puzzled, as if he could not see a square either. He invited the boy to come out to the board and explain his square. The boy did. He indicated that you had to be looking down on the square – as if it were on your book, only tilted. He moved his hand to illustrate.

'Oh' said the teacher. 'Oh, I think I see what you mean…does anyone else

2. Jaworski, 1988, pp. 287/8.

see what he means?' There were more murmurings, puzzled looks, tentative nods.

The teacher drew onto the shape, modifying it to produce Figure 2 below.

There were responses of oooooh yes (!) and nods around the class.

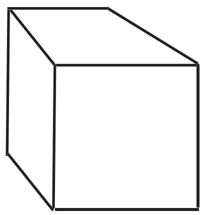

Figure 2: The revised drawing

What started out potentially as a right/wrong answer – trapezium or not trapezium – turned into an opportunity for the class to extend its visions of two dimensional space. Seeing the apparent two-dimensional figure in three-dimensions opened up other correct answers to the original question. The teacher, at first not seeing what the boy was getting at, nevertheless provided opportunity for explanation. The boy was encouraged to explain, possibly motivated and made to feel good about his contribution, and the class was encouraged to respect and value what was offered. Mathematically, perceptions of relations in shape and space were extended for this class as well perhaps as a shift from expectations of simple right/wrong answers. The teacher's intervention opened up the situation for the pupils, engaging their curiosity and offering challenge.

Curiosity and challenge

Have you noticed, when you travel, how many fellow travellers are engaging with puzzles of some sort? Whether crosswords, Sudoku, or other kinds of puzzles in puzzle books, there seems to be avid engagement in puzzling. It seems that we like to puzzle things out. For as long as I can remember, *The Guardian* on Saturdays has offered Chris Maslanka's puzzle corner, and one of the things I have liked about it is that some explanation is usually provided. Often a puzzle has a mathematical solution and I feel inordinately pleased if I have worked it out! The author has engaged my curiosity and offered me a challenge. I feel pleasure, joy in my success. How can I, as a teacher, bring the experience of such joy to pupils' engagement in mathematics in the classroom?

I remember observing a lesson in which the class of 12 year olds was invited to engage with a problem ostensibly about penning sheep. A farmer had 36 panels of fencing, each of length 1 m. to fashion a sheep pen in a field against a wall. What was the largest area of grass he could fence in? Most groups agreed that the best case was a pen of 9m by 18m. So then the teacher said, "Oh dear, before building the fence, one panel was destroyed by rain and wind. So in the end he had only 35 panels. What should he do then?" Two groups in the class were having an argument. One group said the best pen would be 9m by 17m. The other said there was a better answer, if you had 8.75m by 17.5m. The first group objected: "you can't have 8.75 of a panel"!! The teacher opened up a discussion: "what do the answers mean"? Pupils had the opportunity to contrast the mathematical with the practical: the latter giving the bigger area, the former providing a more workable solution. As pupils left the room at the end of the lesson, one said to the teacher, "That was brilliant miss!"

In this classroom activity, groups had worked on 'puzzling' in a mathematical problem related to a context in which their curiosity was engaged. Pupils had puzzled out their own solution and then defended it in the light of challenge from their peers. Their engagement was evident in their bright faces, excited voices and wide participation. It seemed here that for one group, the mathematical solution was what mattered, never mind the sheep. For the other group, the practical situation was what the problem was about. The difference in viewpoint, and each group's commitment to their own solution, gave the teacher an opportunity to encourage debate and point to the factors involved in different solutions.

In Table 1 below, I have related curiosity and challenge to joy and rigour as I see them.

	Joy	Rigour
Curiosity	Through having my curiosity stimulated, I experience joy in my motivation to engage. My curiosity leads me into the problem and fires my dealing with challenge. Curiosity is itself a powerful stimulant.	Curiosity provides an incentive to engage with the mathematics and make sense of the rigour needed to engage successfully. Because I see a need for the rigour, I do not turn away from it.
Challenge	Challenging me in *productive* ways – i.e., I can engage with the challenge and need to put my mental (and maybe social) skills into the task – gives me joy in worthwhile and possibly productive engagement and outcome.	Rigour is challenging. I need some real motivation to engage with the challenge of rigour and this can come from the joy that I experience from engagement and the desire for success that motivate dealing with rigour.

Table 1: Joy, Rigour, Curiosity and Challenge

One of the reviewers of this chapter responded as follows to the ideas above "I started to think about 'time' here. That allowing pupils to be curious and challenged may need time when they are apparently doing nothing because they are thinking or trying things out. The joy comes from spending that time and succeeding in meeting the challenge. Therefore it is time well spent, time when pupils 'grow mathematically.' This then links to your discussions of time later on".

The practical and the mathematical both in their own ways stimulate curiosity and offer challenge. Ultimately, it does not matter from which source the challenge and stimulus come, so long as the result is deeper, more serious engagement with mathematics involving joy and rigour. We get, here, into a meta-level of consideration – beyond the usual concerns of the teacher in planning for a lesson. This is why this discussion is especially important for the subject leader. When the SL appreciates the more general principles behind activity and links this to particularities of designing activity for classrooms, the result can be especially fruitful for mathematics within a school. I shall return to these ideas a bit later.

Mathematics 'versus' the real world

In *penning sheep*, we saw a task set in a (perhaps pseudo) real world context. The context there might be seen to have been fruitful in generating mathematical discussion. However, real world contexts do not always prove to be fruitful in generating *mathematical* thinking. A difference in perception – mathematics versus real world – is sometimes at the heart of issues in or about learning mathematics. In 1979 Margaret Brown (1979, p. 362) reported from research into 11-12 year old children's solutions to problems involving number operations. One question asked:

> A gardener has 391 daffodils. These are to be planted in 23 flowerbeds. Each flowerbed is to have the same number of daffodils. How do you work out how many daffodils will be planted in each flowerbed?

The following interview took place between a pupil YG and the interviewer MB:

YG You er… I know what to do but I can't say it…

MB Yes, well you do it then. Can you do it?

YG Those are daffodils and these are flowerbeds, large you see… Oh! They're being planted in different flowerbeds, you'd have to put them in groups…

MB Yes, how many would you have in each group? What would you do with 23 and 391, if you had to find out?

YG See if I had them, I'd count them up… say I had 20 of each… I'd put 20 in that one, 20 in that one…

MB Suppose you had some left over at the end when you've got to 23 flowerbeds?

YG I'd plant them in a pot (!!)

It seems clear that for YG the practical situation was predominant in his or her thinking and the strategy offered was quite reasonable in practical terms. Mathematically we might be looking for something different – for example 391÷23, from which the answer is 17. It is important to recognize that this answer is to the mathematical question 391÷23, not necessarily to the question asked above. In fact a mathematical modelling operation has to be undertaken to fit these two together. Often however, such an operation is treated implicitly as if the two situations are isomorphic. Offering the daffodil question in a classroom could open up opportunity for different solutions to emerge, so that issues like this could be discussed with the pupils. Otherwise, how can pupils become aware of relationships between the mathematical and the practical, the real world and the world of mathematics?

Barry Cooper and Máiréad Dunne report a striking outcome from their research into children's solutions to questions in the national mathematics tests: that is that children from working-class families do less well on average than children from middle-class families on items that involve everyday situations. They report on an item in an SCAA test in 1994. We see a copy of the item in the figure below.

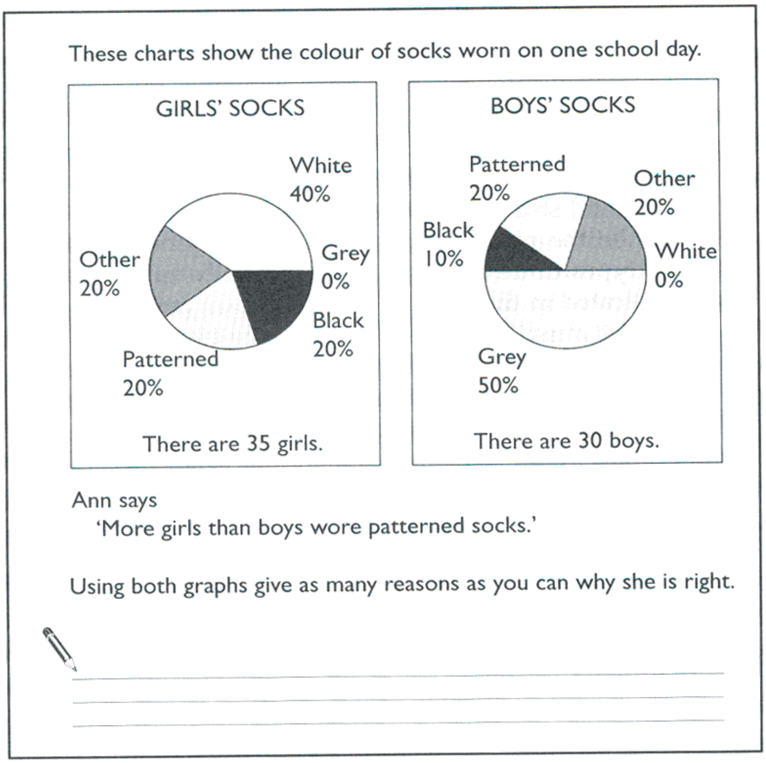

Figure 4.1 The socks item: interpreting statistical diagrams (SCAA 1994)

Cooper and Dunne report from responses of two children, Diane and Mike ("a girl of high measured 'ability' from a professional middle-class background and a boy of average measured ability from a working-class background", p. 43), to the question above. Although Diane considers the practicalities involved, she is able to cut through the everyday context and perceive that the required answer is that Ann is right "because although it's the same proportion, there are more girls" (p.47). Mike "fails to avoid this trap" (p. 47). I quote here from the interview with Mike in which he starts by reading the text of the test item.

Mike These charts show the colour of socks worn on one school day. [pause] That says that they, the girls, wore more patterned socks than the boys, but it says they both, they both had the same [sounding puzzled].

BC So, what do you think then? What do you want to say? [pause]

Mike Is it – I think, really, boys just wear, like, plain old sporty socks, white socks – unless they're, like, teachers' pets – with the socks up here, and things – socks all the way up to their knees. [pointing to his knees during this] But the girls, the girls seem to have more pattern on their socks – they're white and they've got patterns on all of them. The boys just have plain old sporty things with something like sport written down them. Not much of a pattern.

BC So you want – you don't agree with that then?

Mike No.

BC OK, What about this bit here? [Interviewer points to the 35/30 statements.] There are 35 girls. There are 30 boys. Does that make a difference?

Mike It might do [pause] in one way. Or another. But [pause] I mean, really, you've got five more girls that boys and, like, they're just going for it.

I have chosen this as an example, not to suggest that Mike's responses are typical of working-class children, but to exemplify the kind of response that is possible to such a question. How might a teacher deal with such a response? If the nature of the question draws a child into a practical situation which obscures the mathematical answer that is sought, how reasonable are the expectations that a child will give the mathematical answer?

If we want children to think mathematically in practical situations what can we do as teachers to prepare them to see the mathematics and perceive the question being asked?

Narratives

Above, I have offered narratives, stories from different sources – two from my own experience, two from the literature. I will refer to them respectively as "it's a square", "penning sheep", "daffodils" and "socks". Each narrative captures for me certain aspects of the classroom mathematical milieu that relates to the issue I highlighted above: *a central challenge for teaching mathematics is how to embody the rules in mathematical engagement* (rigour) *in a way that is exciting and challenging and joyful*.

Joy might come from a number of sources including pleasure in thinking and engagement with others and the challenge of argument and also the pleasure in engaging with mathematical concepts and seeing inside relationships. The four narratives seem to afford opportunity to notice aspects of these. Rigour lies in the relating of concepts and clear delineation in definition and argument. I challenged myself to try to say what aspects of joy and rigour I find encapsulated in these narratives – why are they important enough to include in this book?

In "It's a square", I see the teacher taking the opportunity, when it arises, to *open up* mathematics. In one stroke, he values the thinking of one pupil, encourages a respectful and collaborative ethos in his classroom and enables attention to, perhaps otherwise unconsidered, mathematical relationships. On the one hand, it is important for pupils to recognize particular geometrical figures, like a trapezium, and distinguish them from other figures, such as squares or parallelograms. This is delineation: mathematical thinking requires clear mathematical arguments for how a trapezium is different from a square – what are the properties that distinguish? This is going beyond visual perceptions and recognition to property articulation and definition – the rigour. However, a rehearsal or noting down of properties and definitions might not occasion much joy. The visual perception in seeing how the apparent trapezium (two dimensions) could be seen as a square (in three dimensions) affords a moment of joy – the rising "oooooh yes!" of the pupils' response. Such new perceptions open up new possibilities – ultimately perhaps an entry to different geometries such as projective geometry.

Of course, this narrative is only the beginning; there are many questions to follow. How would I follow up this situation if it happened in my class? What are our goals as teachers, with respect to our pupils, to the curriculum, to assessment scores and so on? How can we work in ways that foster serendipitous moments while ensuring pupils work on the rigour of mathematical concepts? How can we use class time most fruitfully in relation to all our objectives? I come back to these questions later in the chapter.

In the sheep penning narrative, I see a deep tackling of rigour alongside joyful engagement in mathematical argument that relates concepts and real world issues. The problem, which appears in many text books in this or some other form, is designed to address different combinations of factors of a number and relate these to area and perimeter of plane shapes. Its contextualization in penning sheep introduces a (pseudo) real world situation to enable thinking about the mathematical concepts. Here, we see some pupils engaging with the mathematics – perhaps forgetting the sheep – and others analyzing their possible results in terms of the situation in which the problem was posed. Both seem important, and the argument that ensued took the pupils deeply into mathematical properties (the rigour) and their relation to a real world situation. I saw the pupil's words, "That was brilliant miss", to capture his pleasure in the argument which I would like to think captured also some enjoyment in mathematical understanding that was secure enough to allow the depth of argument. We might also see opportunities here for opening up ideas of mathematical modelling and the difference between a pure mathematical solution and one that fits a given real world situation.

Issues relating to a real world situation and its relation to mathematical concepts can be seen in both *daffodils* and *socks*. It seems that VG in *daffodils* and Mike in *socks* were both caught up in the real world context to which the problem refers. The exclamation marks at the end of the quoted dialogue in *daffodils* perhaps recognizes that the pupil's answer was not what the interviewer was expecting or hoping for. It was not a mathematical answer. But perhaps many humans in the situation described would do what VG suggested – *put them in a pot*. How is VG to know that the problem is not actually about daffodils? Similarly, for Mike, in the socks problem; he seems to have trouble separating the mathematical question from everyday issues to do with choice of socks. Could those setting this question have anticipated such a response? Given the research of Cooper and Dunne, those in a responsible position for setting such questions should now perhaps be more aware of how the problem might appear for some pupils, rather than seeing it just from their own point of view.

There seem to be two sides to the issue. A goal of these problems can be seen to be to situate mathematics in some recognizable everyday context, so that the context might help pupils to understand what is required mathematically. However, for some pupils, the context is powerfully dominant, with mathematics taking a second place, leaving them little opportunity for mathematising the situation. I have been in many classrooms where well meaning tasks, situating mathematics in a context familiar to pupils, have resulted in activity that was only peripherally mathematical. In many cases, pupils were having a good time talking through the issues in the everyday situation, perhaps tackling meaningful and important social issues, but not really doing mathematics. Thus the situation provided fun in social terms, but the fun was not mathematically related and little rigour was evident. This, I think, contrasts with the fun and rigour in *penning sheep*.

Questions for a teacher

The questions I asked earlier are relevant for all teachers and especially for subject leaders who have a responsibility for the kind of mathematical ethos generated in classrooms in a school. I think therefore, that the questions are worth repeating.

- How would I follow up this situation (it's a square) if it happened in my class?
- What are our goals as teachers, with respect to our pupils, to the curriculum, to assessment scores and so on?
- How can we work in ways that foster serendipitous moments while ensuring pupils work on the rigour of mathematical concepts?
- How can we use class time most fruitfully in relation to all our objectives?

How would I follow up this situation (it's a square) if it happened in my class?

Of course, there are many ways of responding to a pupil who gives a surprising comment or answer. One thing I have learned to do is pause. Pausing allows thinking time for the teacher, and perhaps allows pupils also to think. Pausing allows the teacher to show that mathematics requires thinking about while a quick and ready answer might foster a belief that quick and ready answers are the norm in mathematics. Pausing allows me to consider alternative forms of action to the one which might seem most natural.

One of the problems in mathematics is the perception that questions have right or wrong answers and there is nothing in between. The answer "trapezium" is one right answer, but not the only one. For it to be the only one, the questioner would have had to be much more precise about what was required: e.g. "given a two-dimensional shape with just two sides parallel…". Definitions of plane shapes require such precise language, and pupils need to be able to appreciate such definitions and the constraints they impose on what is possible. Such awareness rules out the possibility of different interpretations. Perhaps a teacher can lead discussion in the classroom to distinguish such cases. We might see the opportunity presented here to afford discussion of the rigour of definition versus the perception of alternative perspectives.

What are our goals as teachers, with respect to our pupils, to the curriculum, to assessment scores and so on?

Teachers are under enormous pressure to conform to the curriculum, the strategy, the tests and the examinations. We owe it to our pupils to ensure they have the very best opportunity to achieve in all the different kinds of assessment with which they are faced. It is not surprising therefore if sometimes teachers lose sight of what brought them to mathematics in the first place. Perhaps their own joy in mathematics has abated over the years. Perhaps they would like more time to take diversions from the strict day to day following of routines and rules. Perhaps there is some wish that circumstances could be different; a recognition that the status quo is not conducive to teaching mathematics as they dream might be possible. However, perhaps again, it is too hard to go against imposed norms, whether they come from external forces or are part of socio-historical ways of being in school.

I took part in a research project recently in which mathematics educators from a university worked with teachers from a range of schools (lower primary to upper secondary) to develop mathematics teaching and learning in classrooms.[3] In workshops, we engaged in mathematics together through a variety of mathematical problems designed to provide an interesting and challenging environment for doing mathematics – an *investigative* ethos.[4] One upper secondary teacher spoke vehemently against the possibility of using such problems in his teaching. He claimed that the syllabus was too demanding, and there was just not enough time for investigative problems. Yet he acknowledged the *fun* of engaging with these problems in the workshops. A suggestion was that the investigative problems had no place in the regular syllabus-directed teaching; they would need extra time, time that was not available.

So, incorporating *fun* into teaching needs extra time? Certainly I would question this, since I have seen many teachers offer tasks in ways that engage pupils in mathematically challenging activities from which they gain enjoyment. Enjoyment comes from being drawn into the mathematics along with their peers and, although challenged, feeling accessibility and possibility for success. The challenges need to be seen by pupils as relating to the mainstream syllabus and of relevance to tests and examinations – not 'extras'

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3. See Bjuland & Jaworski (2009) for an account of this project and some of the issues it generated for teachers.
4. See Jaworski (1994) for a study of an investigative approach to teaching mathematics.

designed for social amelioration. Pupils are quick to see though the latter. So, the seeking-out or design of syllabus-related tasks might be seen as a crucial factor along with thoughtful response to pupils' questions and suggestions.[5]

How can we work in ways that foster serendipitous moments while ensuring pupils work on the rigour of mathematical concepts?

By serendipitous moments, I mean moments that arrive 'out of the blue' that offer opportunity to work on mathematical ideas in a fruitful, perhaps novel context that brings relevance to the mathematics. This context can be a mathematical context as well as an everyday context. In 'It's a square' above, we saw a pupil relating a mathematical context to the real world in terms of his own exercise book. The combination, seized on by the teacher, allowed further development of a mathematical idea. In "penning sheep", the argument that arose between two groups allowed the teacher to get pupils to examine their solution critically and develop their judgement on the suitability of a mathematical solution to the context to which it relates, mathematical or real world.

There seem to be two possibilities to address – one is planning for the serendipitous moments, and the other is seizing them when they arise. In the second case, *seizing the moment* is first a case of recognition: recognizing that there is a moment to be seized and being prepared to go with it, perhaps to abandon the carefully prepared tasks for a lesson. Recognition is something to cultivate. John Mason (e.g., 2002) talks about "noticing in the moment": when we are sufficiently aware of a concept or issue, we notice it when it arises and the noticing gives us the opportunity to act differently. This is where the pause is valuable: it allows us to register the noticing, and make a quick decision as to how to move on. It allows us to avoid always taking the well worn path, and only thinking afterwards what else might have been possible.

However, experience helps too. A teacher, planning a task for his lesson, said to me that he would offer the task in a particular way, expecting that pupils would ask a certain question – because in his experience, pupils always did ask that question. The question gave him opportunity to draw pupils' attention to mathematical choices in working on the task. True to his expectations, as I observed in the classroom, pupils did ask the question and the desired discussion took place.[6] Responding to serendipitous moments, seizing the moment, can lead to possibility to create such moments in the future.

Regarding rigour; seizing the moment can lead to an interlude in what has been planned. The interlude may, or may not provide opportunities to emphasise rigour. However, when experience from seizing the moment leads to pre-planning on a future occasion, the 'what is planned' around the moment can certainly have a rigorous foundation. When the serendipitous moment can be re-created in the future, it provides opportunity for rigour to be built in at an appropriate point.

5. The *Journal of Mathematics Teacher Education* published a special issue on design of tasks and issues relating to their purpose and use in classrooms (JMTE, 2007, Volume 10, issues 4/5/6).

6. For details see Jaworski, 1994, pp. 146-7.

How can we use class time most fruitfully in relation to all our objectives? This is not an easy question to answer. I believe it means challenging pupils to engage with mathematics so that they experience both joy and rigour. How we do this is something for every teacher and every department to work at in a serious way. Quite some years ago, Richard Skemp distinguished two kinds of mathematical understanding – relational and instrumental. He writes:

> By the former (relational) is meant what I … have always meant by understanding: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realizing that for many pupils and their teachers the possession of such a rule, and ability to use it, is what they mean by 'understanding'. (Skemp 1989, p.2)

Instrumental understanding enables pupils to do what is asked in the short term, and is very specifically focused. Outside the very local conditions of the particular understanding, pupils may be unable to apply or sometimes even recall what they have understood. Some times instrumental understanding is referred to as rote or *procedural* learning.[7] On the other hand relational understanding involves a conceptual appreciation of what is involved in and underpins the particularities addressed. This allows the learner to relate ideas within and across topics and apply them in different contexts and circumstances. Relational understanding is often linked to conceptual learning. It seems to me that to experience joy and appreciate rigour in mathematics learners need to experience some degree of relational understanding and conceptual learning. Just knowing what to do in certain circumstances can be frustrating and limiting. Many pupils see through this, and realise they are being denied what really matters. Often they put this down to their own deficiencies because this is how it is often presented – putting pupils in 'lower' sets if they cannot manage what is demanded in the 'higher' sets, for example. Clare Lee (2006, p. 6), reporting on research in which she focused on developing pupils' use of mathematical language, wrote:

> Many pupils come to the classroom with the idea that they have a predetermined and fixed level of ability. In mathematics they are often worried that this level is low. [This] may have been reinforced by 'setting' or 'grouping' procedures in schools, but in other ways as well. The approaches that I am advocating depend on the idea that everyone can become better able to use mathematical ideas by addressing the particular difficulties in learning that they have. This may be a new idea to the pupils. If in the past a pupil had tried but failed to learn mathematics, it is unsurprising if he or she gives up trying. In these circumstances the choice for pupils may seem to be between appearing to be lazy and not trying, or trying and giving the impression of being stupid. It seems, on balance, to be a sensible decision when pupils decide they would rather be thought lazy than stupid.

7. See for example Brown, 1979 p. 354.

If what pupils see as a result of their school mathematical experience is that their choice is between seeming lazy or stupid, then we are certainly failing them.

So, what is a teacher to do?

There is no prescription, but I offer some thoughts on how I have tried to tackle the issue.

I ask, what is the mathematics I have to teach (according to the curriculum)? I try to go beyond the text book presentation to really try to analyse what are the central concepts (the essence), and particularly where pupils might have difficulty. Of course, sometimes I have to work hard at the mathematics myself, to ensure I understand it.[8]

- I look at how others have addressed these concepts – the text book author, other writers, what I can find on the web. I think about what I need in order to understand and ways in which this mathematics excites or stimulates me. I look for interesting problems that can engage and challenge pupils. I think about possible questions or prompts I might use to get pupils thinking.9
- I try to think of my pupils and how they might respond – which ones will need more help, support, challenge? Any class, even in a system with finely divided sets, is mixed ability: I ask how I can respond to different learning needs and preferred approaches to learning.
- I devise a set of tasks resulting from my own thinking and analysis and drawing on the various resources I have used. I bring these to the classroom and use my own energy and personality in presenting them in an interesting and challenging way that stimulates pupils to engage.
- In the classroom I try to encourage collaborative working between pupils, support and respect for each other, and a classroom ethos of serious mathematical engagement, dealing with challenge together and providing support relevant to the needs.
- I reflect on outcomes related to pupils' engagement and understanding, and whether I need to modify tasks for further use or organise activity differently another time.

Of course, all of this is VERY demanding of the teacher. As teachers we too need support and challenge. The subject leader therefore has a very important role to play in encouraging and stimulating teachers of mathematics within a school.

In what ways can a subject leader approach this task?

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8. Some years ago the Centre for Mathematics Education at the Open University produced a series of booklets for teachers who wanted to update their own mathematics. These are still relevant today. The Project Update site provides free access to these materials. *http://labspace.open.ac.uk/course/view.php?id=4780*
9. See Watson and Mason, 1998.

Inquiry in mathematics, learning and teaching

Before fully discussing inquiry I will start by coming back to joy and rigour. I assume that even those teachers who have become disillusioned over the years once experienced joy and appreciated the rigour of mathematics. How is it possible to re-awaken this where necessary? Perhaps there are newer teachers in a school who are still excited and motivated who can be brought into the creating of an ethos of joy and rigour within a mathematics department. It seems to me that a collaborative, inclusive approach has most chance of success. And a key word or concept that I personally recommend is "inquiry".

Inquiry means to ask questions and seek answers; to recognise problems and seek solutions; to wonder, imagine, invent and explore. It can mean these things for ourselves as teachers in designing activity for the classroom and reflecting critically on the outcomes. It can mean these things also for our pupils as they engage with mathematics. A book that I find exceptionally valuable, written by Stephanie Prestage and Pat Perks (2001), is called *Adapting and Extending Secondary Mathematics Activities: New tasks for old*. In it, Prestage and Perks look at traditional tasks such as one finds in a text book, and suggest an alternative slant on the task so that it offers pupils something to think about or explore; engaging pupil in mathematical inquiry. An example is:

> Pythagoras Theorem
> What right angled triangles can you find with an hypotenuse of 17cm? (Page 25)

The authors make the point that such a task is different from traditional exercises which ask more direct questions with single right or wrong answers. They write:

> Solving the problem requires the algorithm to be used many times as a pupil makes decisions about the number and types of solutions. This is better than a worksheet any day, and requires little preparation. (Prestage and Perks, 2001, p. 25)

In my experience, when pupils engage with such tasks they experience joy in their engagement, as in penning sheep, and mathematics becomes more *real*, more accessible, something they believe they can engage with and have success. And this is also related to rigour: pupils start to see why it is important to do things in certain ways; they recognise inter-relationships between mathematical topics and gain insights to justification and proving.

I mentioned above "reflecting critically on the outcomes". This is an essential part of the inquiry process. We need to keep in mind what we are trying to achieve and review critically what we seem to have achieved. How have pupils engaged with the tasks we offered them? Mathematically speaking, are they able to do the mathematics involved? For example, can they apply Pythagoras' theorem suitably in finding lengths in triangles? Can they solve a quadratic equation? Can they work with the unitary method in ratio problems? Can they find the reflected image of a shape when the mirror line is not horizontal or vertical? And what about deeper levels of understanding: do pupils engage with the mathematical concepts that underlie what they are doing? To what extent? And how do we know? I am thinking here of Skemp's two kinds of understanding. To what extent are pupils developing relational understanding of the mathematics with which they engage in the classroom?

Reflecting critically leads to 'metaknowing' (Wells, 1999). *Meta*knowing is knowing about knowing, being more aware of what we know and what we need to know, and conscious of issues or tensions in our activity. Sometimes, issues and tension arise because we cannot achieve what we want to achieve due to the system within which we operate; the school system, the educational system, the social system more widely which includes parents and youth culture. Ways of doing and being within a school can be both empowering and constraining; the curriculum and examination structures which are often externally imposed can also constrain what is possible – or perhaps seem to do so. Parents can be both supportive and critically demanding. Youth culture influences how young people see themselves and what therefore they are prepared to do and engage with. Such factors and their associated empowerment/constraint affect all teachers within a system, although some may see it differently from others. It seems therefore worthwhile for teachers together to explore what is possible, how to engage in ways that seem fruitful while coping with or circumventing the constraints. This suggests having some *collaborative inquiry* in which teachers support each other in thinking about innovative ways of working with pupils and in reflecting on outcomes.

The NCETM (National Centre for Excellence in Teaching Mathematics) supports teachers in engaging in inquiry or small-scale research in classrooms. For example, a project with which I am currently involved includes 4 schools in which teachers are exploring how to challenge young people to be enthusiastic about mathematics in the GCSE-A Level interface. These teachers are each devising an innovative programme for a selected group of pupils and exploring the outcomes for the pupils involved and in terms of those going on to A Level mathematics and further mathematical studies.[10] As the teachers and the university team talk with each other about what is happening in the schools, what is planned and the outcomes that are being experienced, we intrinsically address joy and rigour. There is joy in the way the teachers express themselves about mathematics and their pupils, and their critical reflection addresses ways in which innovation is achieving its aims with respect to mathematical learning outcomes in which rigour is fundamental.

A key element here is collaboration across schools and between teachers and university academics. Together we bring diverse knowledge and expertise to the project. Such diversity is valuable in providing expertise and experience to deal with the different facets of the project. Teachers are the experts in their school environments, and in working with current systems in education. The university academics bring knowledge about doing research and of the wider educational literature that can inform practice. By discussing and reflecting together, each one can develop understanding of issues and we can support each other in seeking resolution. Undertaking inquiry within such a

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10. Further information can be obtained from the director of the project Dr Rod Bond at Loughborough University: R.M.Bond@lboro.ac.uk

supportive structure both motivates and helps sustain development. When several teachers within one school are involved, shared understandings of school practice and expectations and individual ideas for innovative practice, together with input and encouragement from university colleagues, can lead both to enhancements in thinking and practice and a strength to deal with issues. The initiative for such activity has to come from somewhere, and the subject leader is one obvious source.

What is needed to make this possible? Perhaps a first step is for the subject leader to start to engage personally in inquiry into learning and teaching and encourage pupils to inquire in mathematics in the classroom. In meetings with other mathematics teachers, opportunity can be taken to introduce some anecdote from the classroom to stimulate discussion. An inquiry task can be introduced for discussion between teachers. A short article, perhaps from the NCETM website,[11] or from a journal like *Mathematics Teaching*, or *Mathematics in Schools*, can be read and discussed between teachers. The subject leader might contact an academic in mathematics education at the local university or college teacher education programme to come and discuss possibilities for development. It seems essential for the subject leader to want to promote a positive environment for teaching and learning mathematics in the school; an environment in which both teachers and pupils express joy in mathematics and in which the rigour of mathematics is centrally addressed.

So joy and rigour, curiosity and challenge; how can these concepts start to be part of the ethos of a mathematics department? When I was head of mathematics in a secondary school, I persuaded my colleagues to join with me occasionally in a problem-solving session – sometimes in school, sometimes out of school in a more social environment. One or more of us found some interesting problem or problems to work on, and inevitably our joint engagement led to questions, discussion, sometimes to argument, usually to new insights, and overall to pleasure in mathematical engagement. Like the teachers in the workshops mentioned above, we found it fun to engage. Engagement brought us closer in spirit and philosophy, and we could discuss some of the serious issues of engaging pupils in our classrooms. This is just one way of developing an ethos in which joy and rigour, curiosity and challenge become familiar friends and clearly relate to doing mathematics and enjoying it.

11. Websites: NCETM: http://www.ncetm.org.uk/, ATM: http://www.atm.org.uk/, MA: http://www.m-a.org.uk/

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