# THE ROLE OF SUBORDINATION AND FADING IN LEARNING FORMAL ALGEBRAIC NOTATION AND SOLVING EQUATIONS: THE CASE OF YEAR 5 STUDENTS

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*A group of 9-10 year olds, who previously had not met letters or formal algebraic notation, were taught over three lessons which led up to solving linear equations using software which produced formal algebraic notation as a consequence of making movements round a grid. Tasks using the software were understandable in terms of movements but the only information provided to carry out tasks was the formal notation. This feature of subordinating the notation to the required task was examined along with the way in which the strong visual support offered by the software was faded in activities. Students gained some success in solving linear equations but the greatest success came in the way students became confident with reading and writing formal notation.*

## INTRODUCTION

There is a recognition that students experience many difficulties when learning algebra (e.g. Herscovics, 1989, MacGregor and Stacey, 1997). The learning of formal notation can also pose problems even though students may be able to express their algebraic ideas verbally (Zazkis and Liljedahl, 2002). Sometimes classroom activities only involve writing something in correct notation as an end point in the activity, such as finding rules for geometric patterns and this can lead to only a few using formal notation (Ma, 2009). Teacher actions can help some students with this (Warren, 2006) but there is little purpose to writing in formal notation except for following a teacher's wishes. There are situations where there is a need to use notation in order to achieve something. For example, when notation is required to work with technology, such as with spreadsheets (Ainley, 1999). However invariably that notation is particular to the software and the way expressions are entered is not as one would write those expressions in formal algebraic notation on paper. Ball and Stacey (2005) highlighted the dilemma of students knowing whether or not the notation they were using with their Computer Algebra System (CAS) was specific to that particular CAS or was part of a standard formal notational system. Coles and Brown (2001) talked about the need to use algebra and here I explore a situation where there is a need to use standard notation. Vygotsky (1962) suggested that for a concept formation to begin there needs to be a problem which cannot be solved other than through the formation of new concepts. This idea that a task cannot be solved unless something new is learned raises difficulties such as whether someone could get anywhere if the new is not learned first. Also the issue that the task might not be understood in the first place if it involves something which is not yet known.



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However, this idea lies at the heart of the notion of subordination (Hewitt, 1996). With subordination, what is to be learned is subordinate to the successful completion of a task which, in itself, can be understood without the need to know in advance that which is yet to be learned. So, in trying to learn A (in this case the reading and writing of formal algebraic notation), a task is designed where A is required to be used in order to be successful at B, yet B can be understood independently of A. Also, significantly, there is feedback from the desired end task, B (which is understood by students), on the appropriate use or otherwise of A. Thus someone can be learning about A through seeing the effect of actions with A on the desired task B.

A computer program was developed which had such features in relation to formal algebraic notation and one aspect of this paper looks at the role subordination played in this learning. A feature of the software was that it offered visual and kinaesthetic support for work with notation and another aspect of this paper is to consider not only the 'scaffolding' this provided for learning formal notation and solving linear equations but also ways in which that support faded (so that students worked directly with the formal notation and where no longer dependent upon the visual support).

## THE SOFTWARE

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 2 | 4 | 6 | 8 | 10 |
| 3 | 3 | 6 | 9 | 12 | 15 |

**Figure 1: the grid structure**

The software is called *Grid Algebra*[1] and is based upon a grid of numbers arranged in multiplication tables, with row 1 being the one times table, row 3 being the three times table, etc. (see Figure 1). The software allows the user to decide the number of rows in view and most of the lessons with this group of students involved working with just the first two rows. A key aspect of the software is that the focus can shift from numbers onto relationships between numbers on the grid through movement. Each movement is associated with a mathematical operation: subtraction to the left; addition to the right; multiplication down; and division up.

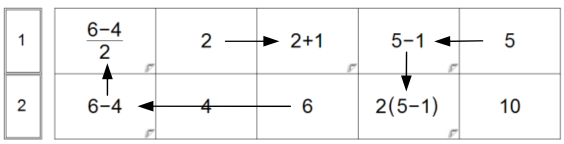


**Figure 2: examples of movements on the grid**

Any number, or indeed letter (as letters can be placed in cells as well), can be picked up and dragged to another cell and the associated mathematical operation shown (see

[1] *Grid Algebra* is available from the Association of Teachers of Mathematics (ATM) http://www.atm.org.uk/shop/products/sof071.html

Figure 2).The expressions formed through such movements are shown in formal notation. Each expression represents both a historical artefact of a journey which has been carried out and also numerically represents the value associated with the particular cell in which that expression sits. Thus $\frac{6-4}{2}$ in Figure 2 represents a journey starting with the number 6, moving to the left and then up; and it also represents the number 1 which is the numerical value of that cell. Quite complex expressions can be created through dragging a number on a long journey involving several separate movements. The design of the software helps students use what they already intuitively know about movement (e.g. order – this way followed by that way; inverse – reverse the journey) to support new learning within mathematics (e.g. order of operations within an expression, inverse operations). The fact that the software was used with an Interactive Whiteboard (IWB) means that students came up to the board and physically moved expressions around the grid and so there was a strong kinaesthetic and visual aspect to using the software. It is important for me to be clear that I was involved in developing the software so that this fact is transparent.

## THE STUDY

The students were a mixed ability group of 9-10 year olds chosen from a two-form entry primary school. The school has a multi-cultural intake with a larger number than average of students having free school meals and achieves below average in terms of National Curriculum results at Key Stage 2. There were 21 students and these were chosen on the basis of parental agreement to take part in the study. The UK National Curriculum (NC) levels of the students, based upon teacher assessments, ranged from level 2 to level 5. The group was taught by myself over three consecutive days, one hour on the first day and one and a half hours on the second and third days. Prior to these lessons the students had not met formal notation before within school and this included the use of a division line rather than the ÷ symbol. Neither had they met letters standing for numbers. The mathematical focus during the three lessons included familiarity with the grid, coming to read formal notation, introducing letters, multiplying out brackets, inverse operations and solving linear equations. A key aspect of the way in which the lessons were run was that nothing was explained to the students, instead there was use of questioning and an expectation that the students would notice and abstract rules for how the software presented notation, and also for them to engage in mathematical tasks based upon that notation.

## FRAMEWORKS

The pedagogic approach to algebra was based upon the framework of arbitrary/necessary divide (Hewitt, 1999) where mathematics is viewed in terms of those things which are arbitrary (names and conventions) and those things which are necessary (properties and relationships). As the arbitrary are socially agreed names and conventions, these need to be provided for a learner, whereas those things which are necessary can be noticed and deduced by a learner (see Hewitt, 1999 for more detail). The software provided what was arbitrary: the way in which formal notation was written. However, the properties and relationships were not provided either by the software nor myself and the focus was on the students noticing and accounting for why certain things happened. The lessons were video recorded and written work was collected. These were analysed within the framework of the Discipline of Noticing (Mason, 2002) where the viewing of such things is acknowledged never to be neutral, and that our own experiences and interests are significant in what we stress and ignore and hence notice and mark to be of significance. Links were made between particular incidents and themes developed from these links. Once themes were developed the videos and written work were viewed again with these themes in mind. A number of themes were identified and here the themes of scaffolding/fading of the visual support offered by the grid, and subordination, were of particular relevance.

## LEARNING NOTATION

When movements were first shown on the grid, a period of 15 minutes was spent with students noticing how expressions were written, and predicting how they might be written ahead of movements being made. Then an activity was introduced where a number was taken on a journey and the middle stages of the journey were deleted so that the grid only had the start number in its original cell and the final expression in the cell in which the journey was finished. The task was for them to re-create the expression through dragging the start number. In this case number 15 was taken on a journey was indicated in Figure 3 which resulted in the final expression of $\frac{2(15+2)-6}{2}+2$. Note that the arrows in Figure 3 did not appear on the screen.

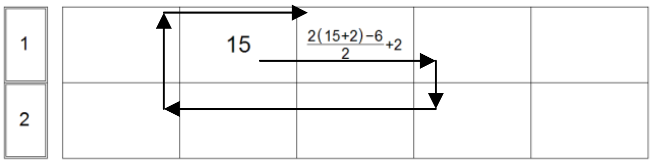


**Figure 3: the number 15 is taken on a journey to produce an expression**

In trying to re-create this expression the class were calling out directions such as “Go across”, “Up”, “Go that way”, “Down”. So the reading of the expression was generally articulated in terms of movement around the visual support of the grid rather than mathematical operations. Later on students worked individually on computer generated tasks which presented the final expression (rather than seeing it being created) and this expression was placed outside the grid so that the students did not know which cell they would end up in. This reduced the visual support which had been offered. When working on this task students were forced to interpret the formal notation as this was the only clue available on the screen. Thus interpretation of the notation was subordinate to the main aim of the task, and this informed their movement on the grid. This movement resulted in new expressions being created and they could be informed as to their success or otherwise by seeing the consequences of their movements on the expression created. This not only informed students whether their movements were correct but also whether their interpretation of the notation of the target expression was correct. Some mistakes in movements were due to learning about the particularities of the software, for example that multiplication is a movement down rather than a movement up. Other mistakes were about the mathematics rather than the software. One pair, Paulette and Sofia (both NC level 3), were trying to re-create the expression $2\left(\frac{10-4}{2}+1\right)$ and started by dragging the 10 (which was in a cell in row 2) upwards and this produced $\frac{10}{2}$. Seeing that this did not look the same as what appeared in the final expression they decided instead to move the 10 to the left to produce 10-4 and then they continued successfully to create the desired expression. Their initial reading of the notation was that dividing by two was the first operation. However, the feedback from trying this informed them that it was not correct and so they tried the subtraction first instead. So they were learning about order of operations through gaining feedback from seeing the consequences of their actions. In order to judge whether they were correct or not they only had to notice what was the same or different about the expression they were creating and the expression they were trying to re-produce.

During the second lesson, students carried on with similar activities on the IWB which involved the notation being subordinated. Whilst this was happening the language being used in the classroom was encouraged to shift from descriptions of movements to mathematical operations and this change gradually began to happen. For example, rather than saying "go down", the students began saying "multiply by" whilst working with increasingly complex expressions, such as $2\left(\frac{2\left(\frac{2(33-2)+2}{2}+3\right)-2}{2}-1\right)-4$. Significantly letters were introduced and I will not go into detail here about the way in which this was done except to say that after several activities of finding journeys, the emphasis was on the mathematical operations and not on the start number (which was, in fact, irrelevant to the tasks). So when a letter was introduced to the grid and a journey made with this, little reaction was made as the letter was not significant to the task.

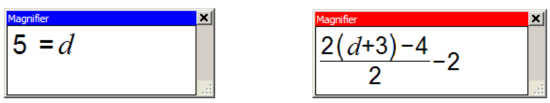


Figure 4: support of the grid taken away

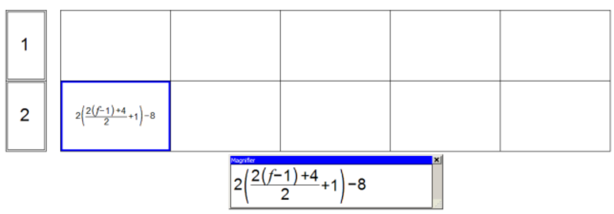


Figure 5: where is the letter *f*?

As activities continued the visual support was gradually reduced further. For example, the letter d was placed into a cell and taken on a journey to produce the expression $\frac{2(d+3)-4}{2}-2$. Then the number 5 was placed into the original cell containing the letter d. The software had a magnifier which was placed in that cell and which brought up in a separate window showing 5=d. Another magnifier was placed into the cell with the final expression producing another separate window. Finally the grid was hidden to leave only those two windows in view (see Figure 4).

The task was to find the number which would be accepted into the window of the final expression. Now the support of movements around the grid had been withdrawn and there was a need to work directly with the formal notation. Later towards the end of lesson two when they were again working individually, one student, Abbas (NC level 3), explained how he worked out the value of $5\left(\frac{b}{5}+1\right)$ given $b = 36$ .

> "It tells you $b$ equals thirty-six so you know that 36 (points to the $b$), this line stands for division (points to division line). So you do thirty-six divided by six, which is… six, and then you do six add one which is seven. You know the plus one goes first and not the five because the plus one is inside the brackets so you do that first. So six plus one equals seven. So these brackets stand for times and whatever is on the left hand side of it is the number that you times it by. So I do five times seven, which is… thirty-five, so thirty-five should be the answer."

This gives an example of the general confidence with formal notation which many of the students had gained by this time.

## INVERSE JOURNEYS

In the third lesson, tasks concerned trying to make inverse journeys, starting with a final expression and trying to return to the letter it started from. Initially a route was marked on the grid so that there was strong visual support offered to see what the inverse journey would be. Also the students knew intuitively what was involved in reversing a journey and that supported what was needed to be done mathematically (opposite movement relates to inverse operation, and reverse order of movements relates to undoing the last operation first, etc). Later no route was marked and the task was to find where the start letter would have to be (see Figure 5, where the expression in the final cell is magnified in a separate window below the grid). The software had a feature where if a movement was made which was the inverse of the last operation within the expression, then that operation would be cancelled in the expression, thus simplifying it. This offered visual feedback in that if an incorrect movement was made students would see that an operation within the expression had not been cancelled. One student was unsure whether to subtract four or divide by two when he had got down to the expression $\frac{2(f-1)+4}{2}$ . In beginning to drag this expression to the left to subtract four he noticed that moving it one cell to the left resulted in the expression $\frac{2(f-1)+4}{2} - 1$. Seeing this he changed his mind and moved it down in order to multiply instead, and this resulted in $2(f-1)+4$ (cancelling the division). He then continued until he was left with just the letter *f*. This was typical of a number of students when they were unsure of which operation to undo next, a movement was tried and they saw the consequences of that movement upon the expression. This is a feature of subordination where the consequences of actions can be seen and understood even if they are unsure as to the correctness of their interpretation of the order of operations within the expression. In this way they gradually made less incorrect moves as they became more sure of the order in which operations needed to be inversed. Although I will not go into detail here, similar tasks were then carried out where a journey was inversed in order to solve equations. The level of support offered for these activities started with the route drawn on the grid, then no route was drawn and only the final equation and starting letter was on view within the grid; and finally the grid was hidden completely and only the final equation was on view, similar to a standard pen and paper question.

## CONCLUSIONS

The visual support seemed to help students engage with confidence in algebraic work which would not normally be introduced to them until they were much older. This was done within a relatively short period of time of three lessons. The design of the software and the lessons attempted to bring in the 'scaffolding' of visual movements within the grid and gradually 'fade' that support away so that students were working directly with the formal notation. At the end of the third lesson the students were given a choice of two sheets to work. These sheets had linear equations to solve: one sheet gave some visual support and the other did not. The students only had about 10 minutes to work on these sheets (due to that being what was left of the final lesson) and so there was not time to see whether those that chose the sheets with visual support would have answered the more traditional equation sheet just as well. There was much success, for example I offer one of the more able students in Figure 8 and perhaps the weakest student in Figure 9.

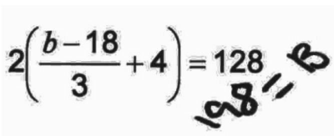


**Figure 8: one of more able students**

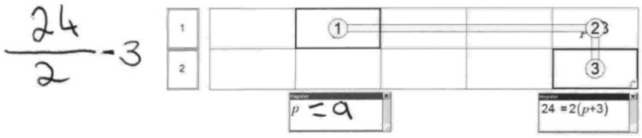


**Figure 9: one of weakest students**

What was of significance for me was although there were mistakes made by some students with solving equations, there seemed to be a confidence with reading and writing formal notation from everyone. For example, writing expressions formally takes longer to achieve than reading such expressions yet the work of possibly the weakest student (Figure 9) included use of a division line and correct placing of the subtraction following that division. I felt that a little more time was needed to see whether all students would be able to solve equations without the visual support and as such I was unsure as to the success of the fading process. However, they all appeared to be comfortable with reading and writing formal notation. The role of subordination seemed significant as the students' focus of attention was with formal notation initially but then this shifted onto other things such as substituting and solving equations whilst formal notation was still subordinate to the carrying out of the tasks. The students were never explicitly asked to remember the notation, nor was the notation explained or justified by myself, yet it was used with confidence by all students. This meant that whatever success there was with solving equations can be viewed as a bonus to the learning of formal notation.

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