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ARITHMETIC EQUALITY STATEMENTS: NUMERICAL BALANCE AND NOTATIONAL SUBSTITUTION

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Numerous studies have investigated the benefits of teaching young children that the equals sign means "is the same as" and presenting a variety of statement forms such as a+b=b+a and c=a+b. However, an important and overlooked aspect of equivalence relations is that of replacing one term with another, which implies a "can be substituted for" meaning of the equals sign. I report a trial with a pair of primary pupils working on a computer-based task that requires viewing equality statements in terms of both numerical balance and notational substitution. I present screenshots and transcript excerpts to illustrate how they articulated and coordinated balance and substitution in order to achieve the task goals.

INTRODUCTION

Young children tend to view the equals sign as a place-indicator for an arithmetic result, as in 2+3=5 (Behr, Erlwanger, & Nichols, 1976). This view can prove exclusive and stubborn (McNeil & Alibali, 2005), and can lead to later difficulties with equation solving (Knuth, Stephens, McNeil, & Alibali, 2006). Numerous studies have demonstrated that such difficulties can be reduced if young children are taught the equals sign means "is the same as" and are exposed to a variety of statement forms, such as 5=2+3, 2+3=3+2 and so on (e.g. Baroody & Ginsburg, 1983; Li, Ding, Capraro, & Capraro, 2008; Molina, Castro, & Castro, 2008). However, a full conception of equivalence relations involves not just numerical sameness but also the substitution of equivalent terms (Collis, 1975). In fact the notion of substitution underpins the transitivity, reflexivity and symmetry that define mathematical equivalence (Skemp, 1986). This paper reports a trial with a pair of primary children as part of a wider study into the pedagogic affordances of a computer-based task in which the equals sign means both "is the same as" and "can be substituted for".

STRUCTURAL APPROACHES TO NOTATING TASK DESIGN

Approaches to notating-task design can be referential or structural (Kirshner, 2001). Referential approaches, such as modelling, import symbol meaning from external objects. In structural approaches meaning instead arises from the relationships between symbols. In practice, however, symbols are often implicit referents to abstractions such as numbers and arithmetic principles (Dörfler, 2006). For example, to present 2+3=3+2 as a question of numerical balance is to ask for a comparison of the number referenced by each side. Pedagogically the statement can be considered a referent to the (abstract) principle of commutativity, by which a learner might satisfy herself of numerical balance without knowing the actual number on either side.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 257-264. Thessaloniki, Greece: PME.

Dörfler has argued this implicit role of symbols as referents to otherwise inaccessible abstractions is widespread but unacknowledged in mathematics education, and is a factor in many learners' sense of alienation.

> [they] often fail to get close to those genuine objects which mathematics purportedly is all about, they believe they lack the necessary abilities to think 'abstractly', and they are convinced that they do not understand what they are expected to understand. They want to reach through the representations to the abstract objects but without success. (p.100)

Dörfler suggests learners should be presented with notating tasks in which symbols and their transformations are themselves the "very objects" of mathematical study. Learning symbolic mathematics can then be seen as an exploratory, empirical activity in which learners make discoveries and test hypotheses. For example, if 2+3=3+2 is presented as a rule for making the substitutions 2+3 → 3+2 and 3+2 → 2+3 it becomes a reusable tool for transforming notation, rather than a closed question of balance that is discarded once assessed. The commutative property of 2+3=3+2 can be observed as an exchange of numerals when the statement is used to transform arithmetic notation.

TASK DESIGN

In this section I describe a computer-based task in which learners are presented with sets of inter-related arithmetic statements for making substitutions, akin to simultaneous equations in algebra. The software presents a sequence of "puzzles" comprising equality statements and a boxed term at the top of the screen (Figure 1). The goal is to use the statements to make substitutions in the boxed term (45+16) in order to transform it into the equivalent numeral (61). We might begin by selecting the statement 45=32+13 and use it to transform the boxed term 45+16 → 32+13+16; we might then use the statement 32+13=13+32 to transform 32+13+16 → 13+32+16; and so on until 61 appears in the box. (Note that substitutions are reversible due to the symmetry of equivalence relations).

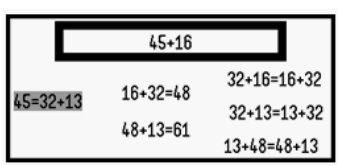

Figure 1: Screenshot of the computer-based task.

The task supports several novel pedagogic affordances which have been investigated in previous trials with pairs of 9 and 10 year old children (Jones, in press). For example, the task requires searching for multiple occurrences of numerals and terms in order to determine where substitutions can be made, and this is quite distinct from viewing statements as isolated questions of balance.

The affordance of interest here is that arithmetic principles can be observed as transformations of symbols rather than as relationships between the abstract numbers

they reference. The substitutive effect of 32+13=13+32 can be observed as the commutation of numerals played out on the computer screen. Likewise the substitutive effect of 45=32+13 can be observed as partitioning the numeral 45. (Note that the statement is partitional only if viewed in terms of replacing the numeral 45 and might as readily be written 32+13=45).

In previous trials all the children readily articulated and exploited commutative transformations ("swap", "switch round") when solving puzzles, and about half articulated and exploited partitional transformations ("split", "separate"). This is not to claim that a child who observes the transformation 32+13 → 13+32 on a computer screen and describes it as "swapping the numbers round" is necessarily drawing on the principle of commutativity; she may simply be indifferent to conservation of quantity (Baroody & Gannon, 1984). This was explicitly tested in one set of trials in which some puzzles contained false equalities, as in 77=11+33. It was found the children did not comment on the imbalance of such statements and their presence had no effect on how they worked with and talked about the puzzles (Jones, 2008). This is because the task promotes a "can be substituted for" meaning for the equals sign *instead of* an "is the same as" meaning. Analogous to algebraic symbol manipulation, there is simply no advantage to considering numerical balance or conservation of quantity when working towards the task goal of transforming the boxed term into a single numeral.

In the remainder of the paper I report a pilot trial of a variation on the task that seeks to overcome this disconnect between numerical balance and notational substitution. Instead of working through presented puzzles the children were challenged to make their own using provided keypad tools. This necessitates coordinating sameness and substitutive views of the equals sign because when inputting a statement learners must ensure it is both numerically balanced and capable of making substitutions. (The software allowed false equalities to be inputted but displayed them using the ≠ symbol, and they could not be used to make substitutions). Moreover, the children were given access to +, × and − operators (in previous trials the presented puzzles contained only + operators) and so needed to consider the conservation of quantity across transformations. For example, 2+3=3+2 could be used to make a substitution in 2+3+10 but not in 2+3×10.

FOCUS

The purpose of the pilot trial was to gain initial insights into how children might coordinate balance and substitutive views of equality statements when making puzzles. Any conclusions drawn from a single trial are necessarily limited but can provide novel insights in light of the predictions inferred from previous trials. Two forms of evidence for children coordinating numerical balance and notational substitution were sought: (i) the complexity and functionality of their puzzle designs; (ii) their discussion when working together on puzzle making. In previous trials substitution was articulated by comments such as "we can use that [a+b=b+a] to swap them [a and b] round", and so on. In this trial, assuming the children succeed at

making coherent puzzles, we should expect such comments to be constructively combined with discussion about the numerical balance of statements and, when the ordering of operations is a factor, discussion about the consistency of the boxed term's value across transformations.

METHOD

The method used was paired trialling and qualitative analysis for evidence of talking about mathematics in novel ways (Noss and Hoyles 1996). The participants were two talkative primary school children from the same class (Yusuf, male, 9 years; Sasha, female, 10 years). Both had been involved in a previous trial some one month prior and were familiar with the puzzle solving task but not the keypad gadgets or puzzle making task.

I introduced the keypad gadgets, challenged the children to make their own puzzles, and remained present throughout the trial to ask for verbal elaborations (e.g. "Why do you think that didn't work?") and offer encouragement. Initially I provided keypads with only + operators and later introduced × and then − operators. The total trial lasted 43 minutes. Data were captured as audiovisual movies of the children's discussion and on-screen interactions, and transcribed and analysed for articulations of the numerical balance and substitutive effects of equality statements.

FINDINGS

After being shown how to use the keypad gadgets, Yusuf and Sasha made eight solvable puzzles during the trial (Figure 2). The first puzzle (a) contains two statements both of which are substitutive with regard to the boxed term, but only one of which is required to solve it (i.e. produce a single numeral in the box). The second puzzle (b) contains four statements, one of which is redundant (30+2=2+30). The next three puzzles (c-e) contain between two and five statements, all of which are required to solve the puzzles. The final three puzzles (f-h) all have more than one type of operator in the boxed term and all contain one or three redundant statements that play no part in solving them. (The redundant statements are 5×5=25, 10×5=50 and 50=10×5 in puzzle f; 50+8=58 in puzzle g; and 8×6=48 in puzzle h).

The children therefore successfully made complex puzzles comprising multiple statements that were both balanced and substitutive with regard to the boxed term. The presence of redundant statements in all the puzzles where the boxed term includes more than one type of operator (f-h) also suggests the children experienced some challenges ensuring the conservation of quantity across transformations where the ordering of operations was a factor.

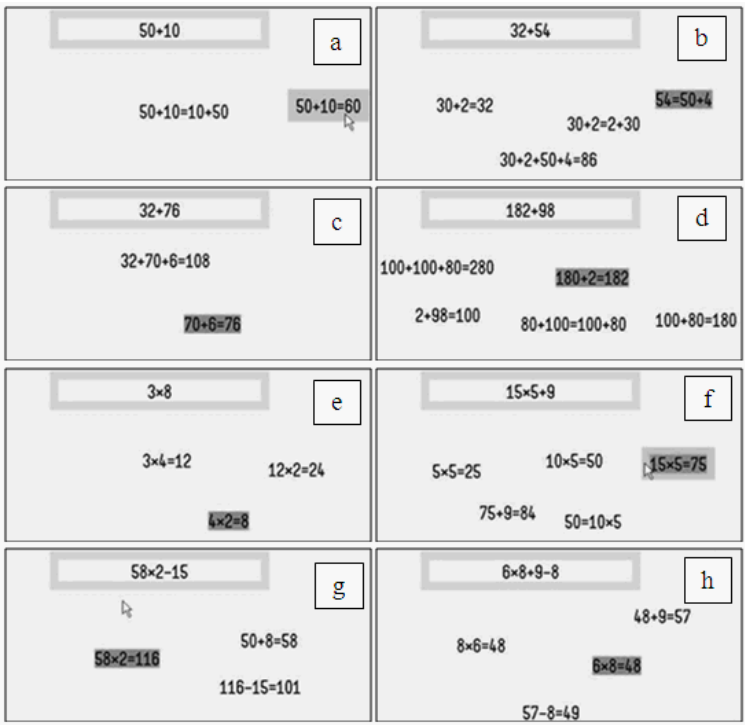

Figure 2: Puzzles made by the children.

The remainder of this section presents three transcript excerpts to illustrate how the children's discussion provides evidence that they considered and coordinated numerical balance and substitutive meanings for equality statements.

The first excerpt is from the start of the second puzzle (Figure 2b) and illustrates the children attempting to make a statement that is both balanced and substitutive. Yusuf had put 32+54 in the box and wanted to partition the numerals (turn 1). Sasha suggested 30+2 for one side of the statement (turn 2) and Yusuf inputted 30+2=2+30 without giving a clear reason (turn 5). He had entered a numerically balanced statement but, as Sasha pointed out (turns 6-8), it was not substitutive with regard to the boxed term. Following this Yusuf attempted to make the statement 32=23 (turn 10), which is substitutive but imbalanced. Sasha then suggested a statement that is both balanced and substitutive (i.e. 30+2=32; note that Sasha intended this statement to partition the numeral 32 suggesting a right to left substitutive reading). ("R" is the researcher).

1 Yusuf: I want to split them two *[the numerals in the boxed term]* apart.

2 Sasha: Let's do 30 add 2.

3 Yusuf: You can't do that bit because that's 2, that's 30 plus 2.

4 Sasha: Oh yeah.

5 Yusuf: *[unclear]* We can do two more sums, innit? *[pause]* 2 plus 30, so we want, and then we just place that there. *[enters 30+2=2+30]* Then...

6 Sasha: I don't think that one works.

7 R: Why not?

8 Sasha: Because it doesn't give 30 add 2 in the box.

9 R: Do you see what she means, Yusuf?

10 Yusuf: Mm-hm. Wait, 32 then put *[unclear]* over here, what's it called? 23, and then... *[inputs 32 and 23 at the keypad, giving 32≠23 on screen]*. I don't care if the equals sign's not there but I know how to do it. Does the equals sign have to be there? *[attempts to select 32≠23 to make a substitution but finds he can't]*

11 R: Sasha, have you any ideas how to help?

12 Sasha: Um, maybe you can make that sum and then put the answer on the other side, and then, make something that equals 32 and make the other one.

13 R: Okay. Yeah, I think I understand but show us what you mean.

14 Yusuf: 30 plus 2 you're saying, and what else? Okay, you get that but... *[inputs 30+2=32 and transforms the boxed term 32+54 → 30+2+54]* Yeah because we can split that *[32]* into that *[30+2]*.

The second excerpt is from the start of the sixth puzzle (Figure 2f) and illustrates the children's awareness of the need for conservation of quantity across transformations. They had entered 15×5+9 in the box and Yusuf suggested entering 5+9=14 to make a substitution but then decided against it (turns 15-17). When prompted both children indicated that 15×5+9 and 15×14 would give different results (turns 18-20).

15 Yusuf: 5 plus 9, no you can't do that. 5 plus 9, because that would make 14. 15 times 14. No. *[laughs]*

16 R: Sorry, what do you mean "no"?

17 Yusuf: Swap it round *[i.e. replace 5+9 with 14 in the boxed term]* makes 15 plus 14. *[he presumably meant 15* times *14]* So that won't work.

18 R: Why not?

19 Yusuf: It's going to be different.

20 Sasha: *[talking at same time as Yusuf...]* Because then you get another answer.

The third excerpt is from the start of the seventh puzzle (Figure 2g) and illustrates the children's attempts to substitute the term 2−15. Yusuf had entered 58×2−15 in the box and Sasha suggested the statement 15−2=13 (turns 21-25). Yusuf pointed out that it could not make a substitution in the boxed term because the 15 and 2 are commuted and attempted to input 2−15=15−2. He seemed doubtful of its validity and was unsurprised when it appeared on screen as 2−15≠15−2 (turn 26). He then suggested the × operator in the boxed term was the problem (probably alluding to conservation

of quantity across transformations – turn 29) but Sasha pointed out that the statement was imbalanced (turns 27-28 and 30-32).

21 Sasha: I think 15 takeaway 2.

22 Yusuf: 15 takeaway 2?

23 Sasha: Yeah, and then ...

24 Yusuf: Are you sure?

25 Sasha: ...do 13

26 Yusuf: No, first we got to switch that *[i.e. 15−2 in Sasha's suggested statement]* around. *[pause]* Let's do 2 takeaway 15. *[inputs 2−15=15−2]* Let's see if that equals, er, recognises it, first. *[the statement appears on screen as 2−15≠15−2]* No, the equals sign isn't there so we can't do it.

27 R: Why isn't the equals sign there?

28 Sasha: Because 2 takeaway 15 is minus 13.

29 Yusuf: *[talking over Sasha...]* Because the top *[boxed term]* has a times. So you can't do, because that one is...

30 R: Sasha, say that again I don't think Yusuf heard it.

31 Sasha: Um, because 2 takeaway 15 equals minus 13.

32 Yusuf: Yeah, that's true.

CONCLUSION AND FURTHER WORK

The task offered an exploratory structural approach to notating in which the statements made by the children were rules for further mathematical activity, rather than static expressions of arithmetic principles. Consistent with Dörfler's (2006) vision of symbolic mathematics learning, the children's focus was on patterns of actions within the puzzles, notably commuting and partitioning numerals. They made predictions about the effect of potential substitutions in an exploratory and reflective manner that took account of the *substitutivity* of statements with regard to the boxed term (turns 6-8 and 26), the *numerical balance* of statements (turns 27-32), and the *conservation of quantity* across transformations (turns 15-20 and 29). This enabled the children to confront and discuss issues such as the ordering of operations in 15×5+9 (turns 15-20) and the non-commutativity of 2−15 (turns 21-32) in a way that was purposeful and meaningful to them.

In this manner the children engaged with a duality of "is the same as" and "can be substituted for" meanings of the equals sign. This stands in contrast to the wider literature that reports young children engaging exclusively with the sameness meaning, which supports discerning statements by truthfulness, and previous trials from this study (Jones, 2008) in which they engaged exclusively with the substitutive meaning, which supports viewing statements in terms of potential transformations of notation, akin to algebraic equations. It is tentatively speculated that such a duality of meanings for the equals sign might help with the transition from arithmetic and algebra, although this conjecture requires further investigation.

The outcomes of this pilot trial have been used to design and conduct further trials of children making their own puzzles. Analysis so far supports and builds on the findings reported here.

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