# On Poisson structures of HYDRODYNAMIC TYPE AND THEIR DEFORMATIONS 

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A Doctoral Thesis

Submitted in partial fulfilment of the requirements
for the award of
Doctor of Philosophy of Loughborough University

March 2016
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A HUNDRED TIMES EVERY DAY I REMIND MYSELF THAT MY INNER AND OUTER LIFE DEPEND ON THE LABOURS OF OTHER MEN, LIVING AND DEAD, AND THAT I MUST EXERT MYSELF IN ORDER TO GIVE IN THE SAME MEASURE AS I HAVE RECEIVED AND AM STILL RECEIVING.

Albert Einstein - The World as I See It (1949)


#### Abstract

Systems of quasilinear partial differential equations of the first order, known as hydrodynamic type systems, are one of the most important classes of nonlinear partial differential equations in the modern theory of integrable systems. They naturally arise in continuum mechanics and in a wide range of applications, both in pure and applied mathematics.

Deep connections between the mathematical theory of hydrodynamic type systems with differential geometry, firstly revealed by Riemann in the nineteenth century, have been thoroughly investigated in the eighties by Dubrovin and Novikov. They introduced and studied a class of Poisson structures generated by a flat pseudo-Riemannian metric, called first-order Poisson brackets of hydrodynamic type. Subsequently, these structures have been generalised in a whole variety of different ways: degenerate, non-homogeneous, higher order, multi-dimensional, and non-local.

The first part of this thesis is devoted to the classification of such structures in two dimensions, both non-degenerate and degenerate. Complete lists of such structures are provided for a small number of components, as well as partial results in the multi-component non-degenerate case.

In the second part of the thesis we deal with deformations of Poisson structures of hydrodynamic type. The deformation theory of Poisson structures is of great interest in the theory of integrable systems, and also plays a key role in the theory of Frobenius manifolds. In particular, we investigate deformations of two classes of structures of hydrodynamic type: degenerate one-dimensional Poisson brackets and non-semisimple biHamiltonian structures associated with Balinskiǐ-Novikov algebras. Complete classification of second-order deformations are presented for two-component structures.


## Acknowledgements

First and foremost, I would like to express my deepest and sincere gratitude to my advisors Prof. Jenya Ferapontov and Prof. Paolo Lorenzoni, for their excellent supervision and continuous support throughout these three years, for their patience, motivation, enthusiasm and immense knowledge. I could not have imagined having better advisors and mentors for my PhD studies.

Besides my advisors, I would like to thank academic and support staff in the Department of Mathematical Sciences at Loughborough University, in particular Dr. Alexey Bolsinov and Dr. Vladimir Novikov for their insightful discussions and suggestions.

I am grateful to Dr. Antonio Moro and Prof. Raffaele Vitolo for all the support, encouragement, advice and fruitful discussions, and to Dr. Alberto Della Vedova for useful remarks and productive collaboration in our joint project.

My sincere thanks also goes to my fellow students, in particular Pierpaolo Calligaris, Matt Tranter, Dr. Adam Hughes and Dr. Mariano Galvagno for stimulating discussions, encouragement and friendship, and to all of those who supported me in any respect during the completion of this project.

I would like to dedicate a special thanks to Dr. Alessandro Pezzotta for his longstanding friendship and moral support, and to my family for supporting me spiritually throughout these three years and my life in general.

Last but not the least, in full gratitude I would like to acknowledge Michela (my future wife) who encouraged, inspired, supported, assisted me through this venture, and for her many sacrifices to help my pursuit of a higher education degree.

## Contents

Abstract ..... i
Acknowledgements ..... ii
Introduction ..... 1
Organisation of the thesis and summary of the main results ..... 6
1 Poisson geometry and Dubrovin-Novikov brackets ..... 16
1.1 Poisson brackets on finite-dimensional manifolds ..... 16
1.1.1 Poisson-Lichnerowicz cohomology ..... 19
1.2 Local Poisson structures on loop spaces ..... 20
1.2.1 Deformations of local Poisson structures ..... 24
1.2.2 Poisson-Lichnerowicz cohomology on loop spaces ..... 26
1.2.3 Dubrovin-Novikov brackets ..... 27
1.2.4 Deformations of Dubrovin-Novikov brackets ..... 30
1.3 Multi-dimensional Dubrovin-Novikov brackets ..... 31
1.3.1 Non-degenerate multi-dimensional brackets ..... 33
2 Classification of non-degenerate Hamiltonian operators in 2D ..... 37
2.1 Linear Killing tensors with zero Nijenhuis torsion ..... 38
2.2 The splitting lemma ..... 43
2.3 Classification results ..... 48
2.3.1 One-component case ..... 51
2.3.2 Two-component case ..... 51
2.3.3 Three-component case ..... 51
2.3.4 Four-component case ..... 53
2.4 The single Jordan block case ..... 59
2.4.1 Proof of Theorem 2.11 ..... 62
2.4.2 Relation to Frobenius manifolds ..... 69
2.5 Multi-dimensional Hamiltonian operators ..... 73
3 Degenerate Dubrovin-Novikov structures and integrable systems ..... 79
3.1 Degenerate one-dimensional Hamiltonian operators ..... 81
3.1.1 Two-component case ..... 82
3.1.2 Three-component case ..... 83
3.2 Degenerate two-dimensional Hamiltonian operators ..... 85
3.2.1 Two-component case ..... 85
3.2.2 Three-component case ..... 86
3.3 Hamiltonian systems of hydrodynamic type in $2+1$ dimensions ..... 88
3.3.1 The method of hydrodynamic reductions ..... 88
3.3.2 Generalised two-dimensional gas dynamics equations ..... 91
3.3.3 Three-component Hamiltonian systems with degenerate structure ..... 93
4 Deformations of degenerate Dubrovin-Novikov structures ..... 97
4.1 The action of Miura-type transformations of the second kind ..... 100
4.1.1 Deformation results ..... 101
4.2 The action of local changes of coordinates ..... 105
4.2.1 Two-component case ..... 105
4.2.2 Three-component case ..... 106
5 Deformations of non-semisimple Poisson pencils of hydrodynamic type ..... 109
5.1 The semisimple case ..... 110
5.2 Linear Poisson pencils of hydrodynamic type ..... 113
5.2.1 Invariant bilinear forms and bi-Hamiltonian structures ..... 113
5.3 Classification results ..... 115
5.4 Invariants of bi-Hamiltonian structures ..... 119
5.4.1 The cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$. ..... 119
5.4.2 The cases N 4 and N 6 with $\kappa=-2$ ..... 121
5.5 Truncated structures ..... 122
5.6 Example ..... 127
Concluding remarks ..... 128
Bibliography ..... 131

## Introduction

Hamiltonian formalism plays fundamental role in the field of integrable systems. One of the principal tools, considered the most natural basis for constructing a Hamiltonian formalism of finite- and infinite-dimensional dynamical systems, is represented by Poisson brackets. The theory of finite-dimensional Poisson brackets as a part of differential geometry was developed by a number of geometers beginning with Darboux and Lie (turn of the twentieth century). The approach to the infinite-dimensional case, which began as a natural consequence of this theory, has taken hold in the mathematics community since the early seventies. In particular, the theory of Poisson brackets for a special class of systems, namely the systems of hydrodynamic type (first-order quasi-linear systems of PDEs), introduced in 1983 by Dubrovin and Novikov [32], has been an area of intensive research in recent decades.

Systems of hydrodynamic type appear in a wide range of applications, namely hydrodynamics, chemical kinetics, fluid mechanics, gas dynamics, general relativity, the Whitham averaging method, the theory of Frobenius manifolds and so on (see the review papers [34, 96] for further details and references). These systems are given by equations of first-order

$$
\begin{equation*}
u_{t}^{i}=v_{j}^{i \alpha}(\mathbf{u}) \frac{\partial u^{j}}{\partial x^{\alpha}}, \quad \mathbf{u}=\mathbf{u}\left(x^{1}, \ldots, x^{d}, t\right), \quad i=1, \ldots, n, \quad \alpha=1, \ldots, d, \tag{1}
\end{equation*}
$$

where the standard summation rule over repeated indices is assumed. The field variables $\mathbf{u}=\left(u^{1}, \ldots, u^{n}\right)$ are usually the density of momentum, energy, mass, or others. In the one-dimensional situation $(d=1)$, a Poisson bracket on a space of fields $u^{1}(x), \ldots, u^{n}(x)$ is called a bracket of hydrodynamic type, or a Dubrovin-Novikov bracket, if it has the form

$$
\begin{equation*}
\left\{u^{i}(x), u^{j}(y)\right\}=g^{i j}(\mathbf{u}(x)) \delta^{\prime}(x-y)+b_{k}^{i j}(\mathbf{u}(x)) u_{x}^{k} \delta(x-y) . \tag{2}
\end{equation*}
$$

Therefore, for any pair of functionals $F=\int \psi\left(u, u_{x}, u_{x x}, \ldots\right) d x, G=\int \phi\left(u, u_{x}, u_{x x} \ldots\right) d x$ we have

$$
\begin{equation*}
\{F, G\}=\int \frac{\delta F}{\delta u^{i}(x)} P^{i j} \frac{\delta G}{\delta u^{j}(x)} d x, \quad P^{i j}=g^{i j}(\mathbf{u}(x)) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}(x)) u_{x}^{k} \tag{3}
\end{equation*}
$$

where the first-order differential operator $P^{i j}$ is called a Hamiltonian operator or a Poisson bivector of hydrodynamic type. Without any loss of generality, the term Poisson structure (of hydrodynamic type) can refer both to the Poisson bracket (of hydrodynamic type) and the Hamiltonian operator.

Remark. Note that when we consider Poisson structures (and not the systems associated with them), without any loss of generality we omit writing the dependence of local coordinates $\mathbf{u}$ on the independent variable $t$.

As already noted by Riemann in the case of hydrodynamic type systems (1), the theory of Poisson brackets of hydrodynamic type is a tensorial theory. Indeed, the class of Poisson brackets (2) is invariant under invertible smooth changes of the field variables of the form $u^{i} \rightarrow v^{i}$, where $u^{i}=u^{i}(\mathbf{v})$ : the coefficients $g^{i j}(\mathbf{u})$ are transformed like components of a tensor of type $(0,2)$, while assuming that the matrix $g^{i j}(\mathbf{u})$ is non-degenerate, the quantities $\Gamma_{j k}^{i}(\mathbf{u})$, defined by $b_{k}^{i j}(\mathbf{u})=-g^{i s}(\mathbf{u}) \Gamma_{s k}^{j}(\mathbf{u})$, are transformed like the Christoffel symbols of a differential-geometric connection.

Poisson brackets of hydrodynamic type for which $\operatorname{det}\left(g^{i j}(\mathbf{u})\right) \neq 0$ are called nondegenerate, and clearly, the non-degeneracy condition is invariant under local changes of the field variables. Dubrovin and Novikov proved that, in the non-degenerate case, expression (2) defines a Poisson bracket if and only if the tensor $g^{i j}(\mathbf{u})$ is symmetric (that is, it specifies a pseudo-Riemannian metric with upper indices) and the connection $\Gamma_{j k}^{i}(\mathbf{u})$ is compatible with the metric $g_{i j}(\mathbf{u})$ (with lower indices) and has no torsion, and the curvature tensor vanishes. In other words, the metric $g_{i j}(\mathbf{u})$ must be flat and $\Gamma_{j k}^{i}(\mathbf{u})$ is the associated Levi-Civita connection. This immediately establishes Darboux's theorem for such structures: in the flat coordinates of the metric $g_{i j}(\mathbf{u})$, the bracket (2) (or, equivalently, the Hamiltonian operator $P^{i j}$ defined in (3)) takes constant coefficient form.

Since the pioneering work of Dubrovin and Novikov, several authors have investigated generalisations of the theory of Poisson brackets of hydrodynamic type:

- Ferapontov and Mokhov [78] (see also [41, 73, 77]) found a non-local generalisa-
tion of these structures: they are defined by arbitrary pseudo-Riemannian metrics of constant Riemannian curvature (zero Riemannian curvature leads to DubrovinNovikov brackets) and play a very important role in the theory of systems of hydrodynamic type, for instance, in the theory of the Whitham equations.
- The study of non-homogeneous Poisson structures of hydrodynamic type, which have applications in the theories of Heisenberg magnets, the Korteweg-de Vries equation, N -wave equations, and other non-homogeneous systems of hydrodynamic type, leads to the theory of Killing-Poisson bivectors on manifolds of constant Riemannian curvature, constructed in [77].
- The problem of classification of general local homogeneous Poisson structures, proposed by Dubrovin and Novikov in [35], is far from being completely solved: only homogeneous Poisson structures of zero-order (Darboux), first-order (Dubrovin and Novikov [32]), and second-order (Potëmin [85, 84], Doyle [24]) are completely classified, while partial results are available for third-order structures (Potëmin [85, 83], Doyle [24], Baladin and Potëmin [6], Ferapontov, Pavlov and Vitolo [51, 52]). Third-order homogeneous Poisson structures arise in the theory of Monge-Ampere equations (Ferapontov and Mokhov [47]) and equations of associativity in twodimensional topological field theory (Ferapontov, Galvão, Mokhov and Nutku [42]).
- In the Hamiltonian theory of multi-dimensional systems of hydrodynamic type (1), local multi-dimensional Poisson structures cannot be reduced in general to constant coefficients form. The obstruction can be expressed in terms of tensorial relations [35, 74] and the classification of such structures, proposed again by Dubrovin and Novikov in [35], reduces to the classification of algebras of certain type [35, 74]. This problem was firstly addressed in [74] resulting in a complete description of one- and two-component structures. In the author's joint work with Ferapontov and Lorenzoni [46], the two-dimensional case is discussed. Adopting a differentialgeometric point of view, the classification up to four components is obtained, as well as a complete classification for any number of components in a special case (details will be given in Chapter 2). Important examples of hydrodynamic type systems in two spatial dimensions occur in gas dynamics, shallow water theory, combustion theory, general relativity, nonlinear elasticity and magneto-fluid dynamics [66].
- Equations of gas dynamics are Hamiltonian with respect to Poisson brackets of hydrodynamic type with degenerate metrics. Some results about degenerate Poisson brackets were announced for the first time by Grinberg in 1985, in a short communication [56], and later investigated by Bogoyavlenskij [9, 10]. In the author's recent works [87], a complete classification of one- and two-dimensional degenerate structures is obtained up to the three-component case. Unfortunately, a fully geometric interpretation of this class of structures is not yet clear.

Another interesting problem in the modern theory of integrable systems is the classification of systems of the form

$$
\begin{equation*}
u_{t}^{i}=F^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \mathbf{u}_{x x}, \ldots, \mathbf{u}_{(m)}, \ldots\right), \quad i=1, \ldots, n \tag{4}
\end{equation*}
$$

Clearly, hydrodynamic type systems are a special subclass of equations of this kind, where the functions $F^{i}$ depend arbitrarily on the local field $\mathbf{u}=\left(u^{1}, \ldots, u^{n}\right)$ and linearly on the first derivatives $\mathbf{u}_{x}=\left(u_{x}^{1}, \ldots, u_{x}^{n}\right)$. Among integrable systems of such form, the most interesting are bi-Hamiltonian systems introduced for the first time by Magri in [65]. The importance of this class is due to the fact that the bi-Hamiltonian structure captures all integrability properties. A system is called bi-Hamiltonian if it can be written as a Hamiltonian system with respect to two compatible Poisson brackets, where compatible means that each linear combination of the two Poisson brackets, defining the bi-Hamiltonian structure, is still a Poisson bracket [65]. Compatible Poisson brackets of hydrodynamic type arise in the theory of Frobenius manifolds [25], and are strictly related to the theory of compatible and almost compatible metrics introduced by Mokhov [70, 71]. Dubrovin proved that the compatibility of two Poisson structures of hydrodynamic type is equivalent to the fact that the correspondent flat metrics form a flat pencil (precise definition will be given in Chapter 1). Therefore, the theory of bi-Hamiltonian structures of hydrodynamic type has a strongly geometric nature. Since the bi-Hamiltonian structure encodes all the characteristics of an integrable system, Dubrovin and Zhang [38] proposed to study integrable perturbations of systems of the form (4), by studying perturbations (deformations) of their associated bi-Hamiltonian structure, and classifying them modulo Miura transformations. For instance, the famous Korteweg-de Vries equation,

$$
\begin{equation*}
u_{t}=u u_{x}+\epsilon^{2} u_{x x x}, \tag{5}
\end{equation*}
$$

can be seen as a deformation of the less renowned, but equally important, Hopf equation $u_{t}=u u_{x}$. The KdV equation (5) is Hamiltonian with respect to two different Poisson structures: the Gardner-Zakharov-Faddeev bracket [53, 100], given by $\{u(x), u(y)\}=$ $\delta^{\prime}(x-y)$, and the Magri bracket [65],

$$
\begin{equation*}
\{u(x), u(y)\}=u \delta^{\prime}(x-y)+\frac{1}{2} u_{x} \delta(x-y)-\epsilon^{2} \delta^{\prime \prime \prime}(x-y), \tag{6}
\end{equation*}
$$

which are compatible. Therefore, the KdV equation is also an example of a bi-Hamiltonian equation. Note that the bracket (6) is not a first-order Poisson bracket of hydrodynamic type since it contains the term $\delta^{\prime \prime \prime}(x-y)$. However, it can be interpreted as a deformation of the dispersionless limit (corresponding to the case $\epsilon \rightarrow 0$ ), which clearly is a Poisson bracket of Dubrovin-Novikov type.

The study of deformations of non-degenerate bi-Hamiltonian structures of hydrodynamic type was originally motivated by questions arising in the theory of Frobenius manifolds, Gromov-Witten invariants and topological field theory [36, 37, 25, 26, 38, 55]. In this setting, the deformations satisfy some additional constraints ( $\tau$-structure, Virasoro constraints) and the undeformed structure is related to a Frobenius manifold [25] (see [27, $28,57,67]$ for further details on the theory of Frobenius manifolds). In the Dubrovin and Zhang approach [38], the pencil of metrics [25, 40], defining the dispersionless limit of the bi-Hamiltonian structure, is assumed to be semisimple, meaning that there exists a special set of coordinates such that both metrics of the pencil take diagonal form. In the scalar case, a complete classification of such deformations has been obtained, see Lorenzoni [64], Liu and Zhang [63, 60], Arsie and Lorenzoni [2], Carlet, Posthuma and Shadrin [16, 18]. The semisimple case has been thoroughly investigated by many authors (Barakat [8], Dubrovin, Liu and Zhang [31], Dubrovin and Zhang [38], Liu and Zhang [62], Carlet, Posthuma and Shadrin [17]), leading to a wide understanding and a complete classification of such structures. Only recently, in the author's joint work with Della Vedova and Lorenzoni [22], the non-semisimple case has been discussed for the first time.

On the other hand, deformations of a single non-degenerate Dubrovin-Novikov bracket are completely understood: Getzler [54] and independently Degiovanni, Sciacca and Magri [21] proved that any such deformation is trivial, that is, can be obtained via Miura transformation. This is not the case for deformations of degenerate structures, as recently shown in the author's paper [87]: these deformations depend on arbitrary func-
tions, which cannot be eliminated by Miura transformations. A similar behaviour (nontriviality) has recently been observed also in the case of deformations of multi-dimensional structures of hydrodynamic type [19, 15].

Remark. In the whole thesis, unless otherwise specified, according to the Einstein notation, the summation over repeating upper and lower indices is assumed.

## Organisation of the thesis and summary of the main results

As we have seen, the study of Poisson brackets of hydrodynamic type and their generalisations has been an area of intensive research over the last three decades. Although some branches have been fully understood, this theory still offers interesting aspects to be investigated. In this thesis we are mainly interested in the classification of two-dimensional Hamiltonian operators, both degenerate and non-degenerate, as well as the deformation theory for two classes of structures, never discussed before: one-dimensional degenerate Dubrovin-Novikov brackets and non-semisimple bi-Hamiltonian structures of hydrodynamic type.

A general overview concerning the theory of Poisson brackets for finite- and infinitedimensional systems, and in particular the theory developed in the framework of hydrodynamic type systems, is given in Chapter 1. The aim of this introductory chapter is two-fold. On one hand, we introduce the basics needed for a better understanding of the thesis: Poisson structures and their deformations, Poisson-Lichnerowicz cohomology, multi-dimensional Hamiltonian operators. On the other hand, we recall the main results obtained in these areas mainly due to Dubrovin, Novikov and Mokhov. Other more specific results will be presented in the appropriate chapters.

The new results presented in Chapters 2, 3, 4 and 5, are published in the author's joint works with Ferapontov and Lorenzoni [46], Della Vedova and Lorenzoni [22], and in the author's own works [87, 86]. Let us summarise the main results obtained, chapter by chapter.

We point out that, due to the technical nature of some cumbersome computations, proofs of some theorems in Chapters 3, 4 and 5 are omitted. Detailed proofs can be found in the original papers or on the arXiv version.

## Chapter 2: Classification of non-degenerate Hamiltonian operators in 2D

Any non-degenerate 2D Hamiltonian operator,

$$
\begin{equation*}
P^{i j}=g^{i j}(\mathbf{u}) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}) u_{x}^{k}+\tilde{g}^{i j}(\mathbf{u}) \frac{d}{d y}+\tilde{b}_{k}^{i j}(\mathbf{u}) u_{y}^{k}, \tag{7}
\end{equation*}
$$

is defined by a pair of contravariant metrics $g$ and $\tilde{g}$. It was demonstrated by Dubrovin and Novikov [35] that in the flat coordinates of the first metric $g$, the second one, that is $\tilde{g}$, must be linear. Mokhov [70] showed that these two metrics must be almost-compatible, that is the Nijenhuis torsion of $L=\tilde{g} g^{-1}$ must be identically zero. Moreover, in [76, 74] he gives general relations for the coefficients of the Hamiltonian operator of the form (7), which follow from the Jacobi identity. Our first result establishes a link between 2D Hamiltonian operators (7) and the theory of Killing tensors:

Theorem. Let $g$ and $\tilde{g}$ be two flat metrics. Formula (7) defines a Hamiltonian operator if and only if the following conditions are satisfied:

1. Linearity of $\tilde{g}^{i j}$ in the flat coordinates of $g$. Invariantly, this means $\nabla^{2} \tilde{g}=0$ where $\nabla$ denotes covariant differentiation in the Levi-Civita connection of $g$.
2. The vanishing of the Nijenhuis torsion of the affinor $L_{j}^{i}=\tilde{g}^{i l} g_{l j}$.
3. The Killing condition for the bivector $\tilde{g}: \nabla^{i} \tilde{g}^{k j}+\nabla^{k} \tilde{g}^{i j}+\nabla^{j} \tilde{g}^{i k}=0$.

Furthermore, the flatness of $g$ and the above three conditions imply the flatness of $\tilde{g}$.
Thus, the classification of Hamiltonian operators of the form (7) is reduced to the classification of linear Killing bivectors with zero Nijenhuis torsion in flat pseudo-Euclidean spaces. Using the fact that any Killing bivector in flat space is the sum of symmetrised tensor products of Killing vectors, we obtain a complete classification of 2D Hamiltonian operators with $n \leq 4$ components.

The Killing condition also plays a key role in the proof of the splitting property for Hamiltonian operators, which can be seen as an analogue of the splitting lemma for affinors with zero Nijenhuis torsion proved by Bolsinov and Matveev [11] in the context of projectively equivalent metrics. First of all we recall their result. Let $M$ be an $n$-dimensional manifold, and let $L$ be an affinor on $M$ with zero Nijenhuis torsion. Suppose that there exists a frame (not necessarily holonomic) in which $L$ takes block diagonal
form,

$$
L=\left(\begin{array}{ll}
A & 0  \tag{8}\\
0 & B
\end{array}\right)
$$

where $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)=\emptyset$. Then there exists a coordinate system $(\mathbf{u}, \mathbf{v})=\left(u^{1}, \ldots, u^{m}\right.$, $\left.v^{m+1}, \ldots, v^{n}\right)$ such that $A$ depends on $\mathbf{u}$ and $B$ depends on $\mathbf{v}$ only, that is, $L$ is a direct sum of two affinors (both with vanishing Nijenhuis torsion). Adding the Killing condition, we show how to extend this splitting structure to the metrics, namely we prove that, in the same coordinate system, the two metrics $g$ and $\tilde{g}$ also assume block diagonal forms,

$$
g=\left(\begin{array}{cc}
g_{1}(\mathbf{u}) & 0 \\
0 & g_{2}(\mathbf{v})
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{cc}
\tilde{g}_{1}(\mathbf{u}) & 0 \\
0 & \tilde{g_{2}}(\mathbf{v})
\end{array}\right) .
$$

This suggests the definition of reducible operators: given an $m$-component operator $P_{1}$ with the dependent variables $u^{1}, \ldots, u^{m}$, and an $(n-m)$-component operator $P_{2}$ with the dependent variables $v^{m+1}, \ldots, v^{n}$, their direct sum is the $n$-component operator $P$ defined by the formula

$$
P=\left(\begin{array}{cc}
P_{1} & 0 \\
0 & P_{2}
\end{array}\right)
$$

on the combined set of variables $\left(u^{1}, \ldots, u^{m}, v^{m+1}, \ldots, v^{n}\right)$. The corresponding metrics $g$, $\tilde{g}$ will be direct sums of the metrics defining $P_{1}$ and $P_{2}$. Operators of this type will be called reducible. Thus, our second result can be formulated as follows:

The Splitting Lemma. Let $P$ be a Hamiltonian operator such that the corresponding affinor $L=\tilde{g} g^{-1}$ can be represented in the block-diagonal form (8) in some (non-holonomic) frame, and let $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)=\emptyset$. Then $P$ decouples into a direct sum of two Hamiltonian operators, with the corresponding affinors $A$ and $B$.

Thus, any Hamiltonian operator (7) can be represented as a direct sum of irreducible operators $P_{\alpha}$ (each generated by a pair of flat metrics $g_{\alpha}, \tilde{g}_{\alpha}$, defined on a manifold of dimension $n_{\alpha}$ ) such that the corresponding affinor $L_{\alpha}=\tilde{g}_{\alpha} g_{\alpha}^{-1}$ either has a unique real eigenvalue of multiplicity $n_{\alpha}$, or a pair of complex conjugate eigenvalues of the same multiplicity (in the last case $n_{\alpha}$ must be even).

As a consequence of the splitting lemma we will prove that, if the affinor $L$ is diagonal, then the Hamiltonian operator can be brought to constant coefficient form. This
generalises the analogous result of [69] obtained under the additional assumption of the simplicity of the spectrum of $L$. In what follows, we will be interested in Hamiltonian operators which are not reducible, and not transformable to constant coefficient form.

Our approach to the classification of Hamiltonian operators in 2D is based on the Killing property, and allows us to obtain a full classification up to four components. In the three-component case, the main result is as follows.

Theorem. Any irreducible non-constant three-component Hamiltonian operator in 2D can be brought (by a change of the dependent variables $u^{i}$ ) to the form $\pm P$ where $P$ can have one of the two following canonical forms (in both cases the affinor $L$ is a single $3 \times 3$ Jordan block):

1. Jordan block with constant eigenvalue

$$
P=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
-2 u^{2} & u^{3} & \lambda \\
u^{3} & \lambda & 0 \\
\lambda & 0 & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{ccc}
-u_{y}^{2} & 2 u_{y}^{3} & 0 \\
-u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

2. Jordan block with non-constant eigenvalue

$$
P=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{ccc}
-u_{y}^{1} & \frac{1}{2} u_{y}^{2} & 2 u_{y}^{3} \\
-u_{y}^{2} & \frac{1}{2} u_{y}^{3} & 0 \\
-u_{y}^{3} & 0 & 0
\end{array}\right) .
$$

In the four-component situation calculations become more complicated, and we get several canonical forms labelled by Segre types of the affinor $L$.

Although our approach works for any number of components $n$, for $n>4$ computations become rather cumbersome. The main difficulty is when the affinor $L$ consists of several Jordan blocks with the same eigenvalue. In the case of a single $n \times n$ Jordan block we obtain the following result:

Theorem. Let P be a Hamiltonian operator (7) such that the affinor $L=\tilde{g} g^{-1}$ is a single $n \times n$ Jordan block with non-constant eigenvalue. Then there exists a coordinate system in which $g$ and
$\tilde{g}$ can be reduced to the following canonical forms:

$$
g= \pm\left(\right), \quad \tilde{g}= \pm \begin{cases}\mu^{(n ; 0)} & \text { if } n \not \equiv 1 \bmod 3, \\ \mu^{(n ; 0)}+\kappa_{1} \mu^{\left(n ; \frac{n-1}{3}\right)} & \text { if } n \equiv 1 \bmod 3, n \neq 4, \\ \mu^{(4 ; 0)}+\kappa_{1} \mu^{(4 ; 1)}+\mu_{0} & \text { if } n=4 .\end{cases}
$$

Here $\kappa_{1}$ is an arbitrary constant, the symmetric bivector $\mu^{(n ; k)}$ is defined as

$$
\mu^{(n ; k) i j}=[3(i+j)-2(n+2-k)] u^{i+j-1+k},
$$

and $\mu_{0}$ is the constant symmetric matrix $\mu_{0}^{i j}=\delta^{i, 4-j}+\lambda \delta^{i, 5-j}, \lambda=$ const.
An analogous statement can be proved for the constant eigenvalue case. This result, combined with the splitting lemma, provides a complete classification of 2D operators of Dubrovin-Novikov type in the case of a direct sum of Jordan blocks with distinct eigenvalues. We also show that the case of a single $n \times n$ Jordan block with non-constant eigenvalue gives rise to the trivial non-semisimple Frobenius manifold whose underlying Frobenius algebra corresponds to the cohomology ring of $\mathbf{C P}^{n-1}$.

Finally, we extend our approach to Hamiltonian operators in dimensions higher than two. This leads to a complete description of three-component operators which are essentially three-dimensional, and cannot be transformed to constant coefficients:

Theorem. Any non-degenerate three-component Hamiltonian operator in 3D, which is not transformable to constant coefficients, can be brought by a local change of the dependent variables $u^{i}$ to

$$
P^{i j}=g^{i j} \frac{d}{d x}+\tilde{g}^{i j} \frac{d}{d y}+\left(c_{1} g^{i j}+c_{2} \tilde{g}^{i j}+h_{0}^{i j}\right) \frac{d}{d z}+\tilde{b}_{k}^{i j}\left(u_{y}^{k}+c_{2} u_{z}^{k}\right),
$$

where $c_{1}, c_{2}$ are constants, $\tilde{b}_{k}^{i j}$ are the contravariant Christoffel symbols of $\tilde{g}$, and the contravariant metrics $g, \tilde{g}, h_{0}$ assume one of the two following canonical forms

- form 1 :

$$
g=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right), \quad h_{0}=\left(\begin{array}{lll}
\nu & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

where $\nu=$ const;

- form 2 :

$$
g=\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & u^{2} & 0 \\
u^{2} & 0 & 0 \\
0 & 0 & \lambda
\end{array}\right), \quad h_{0}=\left(\begin{array}{lll}
0 & \mu & 0 \\
\mu & 0 & 0 \\
0 & 0 & \nu
\end{array}\right)
$$

where $\lambda, \mu, \nu$ are constants.

## Chapter 3: Degenerate Dubrovin-Novikov structures and integrable systems

In this chapter, we give a complete list of two- and three-component Poisson structures of hydrodynamic type with degenerate metrics in 1D and 2D. For instance, in the twocomponent case, any one-dimensional structure can be brought by a local change of coordinates to one of the following canonical forms,

$$
P=\left(\begin{array}{cc}
\partial_{x} & 0  \tag{9}\\
0 & 0
\end{array}\right), \quad P=\left(\begin{array}{cc}
\partial_{x} & -\frac{u_{x}^{2}}{u^{1}} \\
\frac{u_{x}^{2}}{u^{1}} & 0
\end{array}\right),
$$

where $\partial_{x}=\frac{d}{d x}$.
Degenerate two-dimensional structures of hydrodynamic type, defined through a pair of metrics $g$ and $\tilde{g}$, are classified according to the rank of the pencil $g-\lambda \tilde{g}$. In the twodimensional case, rank 0 structures are trivial while for rank 1 we obtain

$$
P=\left(\begin{array}{cc}
\partial_{x}+u^{2} \partial_{y}+\frac{1}{2} u_{y}^{2} & -\epsilon \frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}} \\
\epsilon \frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}} & 0
\end{array}\right),
$$

where $\epsilon$ can be either 0 or 1 .
Furthermore, we discuss the integrability by the method of hydrodynamic reductions for Hamiltonian systems arising from degenerate two-dimensional three-component structures. Our analysis leads to the following statment.

Theorem. The method of hydrodynamic reductions imposes additional differential constraints under which Hamiltonian equations arising from three-component two-dimensional Hamiltonian operators reduce to known classes of systems considered before:

- rank-zero structures lead to trivial systems

$$
u_{t}^{1}=u_{t}^{2}=u_{t}^{3}=0
$$

- rank-one structures lead to one dimensional systems of the form

$$
u_{t}^{1}+f\left(u^{1}\right) u_{x}^{1}=0, \quad u_{t}^{2}=u_{t}^{3}=0,
$$

- rank-two structures lead either to one dimensional systems of the form

$$
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{3}=0,
$$

or two-component non-degenerate Hamiltonian systems

$$
\begin{gathered}
u_{t}^{1}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{2}}\right)_{y}=0, \\
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}+\left(h_{u^{2}}\right)_{y}=0, \\
u_{t}^{1}+\left(2 u^{1} h_{u^{1}}+u^{2} h_{u^{2}}-h\right)_{x}+\left(u^{1} h_{u^{2}}\right)_{y}=0, \quad u_{t}^{2}+\left(u^{2} h_{u^{1}}\right)_{x}+\left(2 u^{2} h_{u^{2}}+u^{1} h_{u^{1}}-h\right)_{y}=0,
\end{gathered}
$$

plus the trivial equation $u_{t}^{3}=0$, or to the systems

$$
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}+\left(h_{u^{3}}\right)_{y}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{3}+\left(h_{u^{1}}\right)_{y}=0 .
$$

## Chapter 4: Deformations of degenerate Dubrovin-Novikov structures

Given a Poisson structure $P$ defined on a manifold $M$, a deformation of order $k$ of $P$ is a formal series

$$
P^{\epsilon}=P+\epsilon P_{1}+\epsilon^{2} P_{2}+\ldots
$$

satisfying the condition $\left[P^{\epsilon}, P^{\epsilon}\right]=\mathcal{O}\left(\epsilon^{k+1}\right)$ for any value of the parameter $\epsilon$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket. Deformation theory for non-degenerate Poisson structures of hydrodynamic type in 1D is completely understood (under the assumption of homogeneity, further details will be given later). It has been proved that any such deformation is trivial, that is $P^{\epsilon}$ can be reduced to $P$ by the action of the Miura group. In this chapter we show that in the two-component case, first- and second-order deformations of
degenerate structures are not trivial, that is, they cannot be eliminated by Miura transformations, and we prove that they depend on a certain number of arbitrary functions of the variable $u^{2}$.

Theorem. Up to Miura-type transformations, the following holds:

- first-order deformations of $(9)_{1}$ depend on 2 functions of $u^{2}$, and second-order deformations on 6 functions of $u^{2}$;
- first-order deformations of $(9)_{2}$ depend on 1 function of $u^{2}$, and second-order deformations on 2 functions of $u^{2}$.

In the three-component case, we provide some examples of non-trivial first-order deformations (as we will see, in this case second-order deformations involve too many unknown functions, and computations become very hard), focusing on the Poisson structures given by

$$
P=\left(\begin{array}{ccc}
0 & u_{x}^{3} & 0 \\
-u_{x}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & \partial_{x} & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

In particular, our results imply that the first homogeneous component of the second Poisson-Lichnerowicz cohomology group for all the structures we have considered, does not vanish. This implies that the second cohomology group for such degenerate structures is not trivial, contrary to what happens in the non-degenerate case [54, 21].

## Chapter 5: Deformations of non-semisimple bi-Hamiltonian structures of hydrodynamic type

The last chapter is devoted to the study of deformations of non-semisimple two-component bi-Hamiltonian structures related to two-dimensional Balinskiir-Novikov algebras [7] and the associated invariant bilinear forms. The undeformed structures can be reduced to the form

$$
P_{2}^{i j}-\lambda P_{1}^{i j}=g^{i j} \frac{d}{d x}+b_{k}^{i j} u_{x}^{k}-\lambda \eta^{i j} \frac{d}{d x}
$$

where $g^{i j}=\left(b_{k}^{i j}+b_{k}^{j i}\right) u^{k}$ and the coefficients $b_{k}^{i j}$ and $\eta^{i j}$ are constant. We focus on the following cases:

| Type | $P_{2}^{i j}$ |
| :---: | :---: |
| (T3) $P_{1}^{i j}$ |  |
| (N5) | $\left(\begin{array}{cc}0 & -u^{1} \partial_{x} \\ -u^{1} \partial_{x}-u_{x}^{1} & 0\end{array}\right)$ |
| $\left(\begin{array}{cc}0 & u^{1} \partial_{x}+u_{x}^{1} \\ u^{1} \partial_{x} & 2\left(u^{1}+u^{2}\right) \partial_{x}+u_{x}^{2}+u_{x}^{1}\end{array}\right)$ | $\left(\begin{array}{cc}0 & \eta^{12} \partial_{x} \\ \eta^{12} \partial_{x} & \eta^{22} \partial_{x}\end{array}\right)$ |
| $\left(\begin{array}{cc}0 & (1+\kappa) u^{1} \partial_{x}+u_{x}^{1} \\ (1+\kappa) u^{1} \partial_{x}+\kappa u_{x}^{1} & 2 u^{2} \partial_{x}+u_{x}^{2}\end{array}\right)$ | $\left(\begin{array}{cc}0 & \eta^{12} \partial_{x} \\ \eta^{12} \partial_{x} & \eta^{22} \partial_{x}\end{array}\right)$ |

According to Bai-Meng's list [4], N3 corresponds to $\kappa=1$, while N4 to $\kappa=0$.
We show that in most cases the second-order deformations are parametrised by two functions of a single variable, while in two exceptional cases the second-order deformations are parametrised by four functions.

Theorem. Second-order deformations can be reduced by Miura transformations to the form

$$
\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)
$$

where

- in the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2, X=P_{1} \delta H-P_{2} \delta K$ with

$$
H[u]=\int \sum_{i, j}\left(h_{i j} u_{x}^{i} \log u_{x}^{j}\right) d x, \quad K[u]=\int \sum_{i, j}\left(k_{i j} u_{x}^{i} \log u_{x}^{j}\right) d x,
$$

and the functions $h_{i j}$ and $k_{i j}$ are uniquely determined in terms of two arbitrary functions $F_{1}, F_{2}$ depending only on the eigenvalue of the tensor $L=g \eta^{-1}$;

- in the cases N4 and N6 with $\kappa=-2, X$ depends on four functions $F_{1}, F_{2}, F_{3}, F_{4}$ of eigenvalue of the tensor $L=g \eta^{-1}$.

In particular, in the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, second-order deformations are quasi-trivial, that is, they can be reduced to the dispersionless limit by a quasiMiura transformation.

It turns out that in the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, one function is invariant with respect to the subgroup of Miura transformations preserving the dispersionless limit, and another function is related to a one-parameter family of truncated
structures. In the two exceptional cases N 4 and N 6 with $\kappa=-2$, two functions are invariants, and two are related to a two-parameter family of truncated structures.

We finally provide an example corresponding to the lift of deformations of the biHamiltonian structure associated with the KdV equation. This example suggests that deformations of non-semisimple pencils corresponding to the lifted invariant parameters are unobstructed.

# Poisson geometry and Dubrovin-Novikov brackets 

In this chapter we summarise some of the main notions we need from the general theory of Poisson brackets for finite- and infinite-dimensional systems. The overview of finitedimensional Poisson structures can be found in several text books on classical mechanics, for instance $[1,80]$, while to describe the Poisson structures on loop spaces we follow [33, 38]. Furthermore, we also recall the main aspects of Dubrovin-Zhang approach to the deformation theory of bi-Hamiltonian structures of hydrodynamic type [38] (following the detailed overview given in [2]), the notion of Poisson-Lichnerowicz cohomology [59], as well as the known results in the framework of multi-dimensional Dubrovin-Novikov brackets [35, 74, 69].

### 1.1 Poisson brackets on finite-dimensional manifolds

Let $M$ be a (smooth) finite-dimensional manifold, $n=\operatorname{dim} M$, and let $x^{1}, \ldots, x^{n}$ be a system of local coordinates on $M$.

Definition 1.1. A Poisson bracket is a bilinear operation $\{\}:, C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$ which satisfies, for all $f, g, h \in C^{\infty}(M)$ :
(i) skew-symmetry: $\{f, g\}=-\{g, f\}$;
(ii) Leibniz identity: $\{f g, h\}=f\{g, h\}+g\{f, h\}$;
(iii) Jacobi identity: $\{\{f, g\}, h\}+\{\{g, h\}, f\}+\{\{h, f\}, g\}=0$.

A symplectic structure on $M$ is a non-degenerate closed 2-form $\Omega=\omega_{\alpha \beta} d x^{\alpha} \wedge d x^{\beta}$. Non-degenerate means that the skew-symmetric matrix $\omega_{\alpha \beta}$ is non-singular for all points $\mathbf{x} \in M$, that is $\operatorname{det} \omega_{\alpha \beta}(\mathbf{x}) \neq 0$. Thus, since skew-symmetric matrix in odd dimensions is
necessarily singular, a symplectic structure is defined only if the dimension of $M$ is even. The following general statement holds.

Darboux's theorem (Symplectic form). Let $\Omega$ be a closed non-degenerate differential 2-form in a neighbourhood of a point $\mathbf{x}$ in the space $\mathbb{R}^{2 n}$. Then in some neighbourhood of $\mathbf{x}$ one can choose a coordinate system $\left(p_{1}, \ldots, p_{n}, q^{1}, \ldots, q^{n}\right)$ such that the 2-form has the standard form $\Omega=\sum_{i=1}^{n} d p_{i} \wedge d q^{i}$.

Let $P$ denote the inverse matrix of $\omega$, that is $P^{\alpha \beta} \omega_{\beta \gamma}=\delta_{\gamma}^{\alpha}$. We will see that the matrix $P^{\alpha \beta}$ determines everything important for the theory of Hamiltonian systems. Given $P^{\alpha \beta}$, we can define a natural operation on the functions $f, g \in C^{\infty}(M)$, that is

$$
\begin{equation*}
\{f, g\}=P^{\alpha \beta} \frac{\partial f}{\partial x^{\alpha}} \frac{\partial g}{\partial x^{\beta}} . \tag{1.1}
\end{equation*}
$$

It is easy to check that (1.1) satisfies bilinearity, skew-symmetry and Leibniz rule, while the Jacobi identity is non-obvious. In local coordinates we have that

$$
\left\{x^{\alpha}, x^{\beta}\right\}=P^{\alpha \beta}, \quad\left\{\left\{x^{\alpha}, x^{\beta}\right\}, x^{\gamma}\right\}=\frac{\partial P^{\alpha \beta}}{\partial x^{k}} \frac{\partial x^{\gamma}}{\partial x^{p}} P^{k p}=\frac{\partial P^{\alpha \beta}}{\partial x^{k}} P^{k \gamma},
$$

so the Jacobi identity is given by

$$
\begin{equation*}
\frac{\partial P^{\alpha \beta}}{\partial x^{k}} P^{k \gamma}+\frac{\partial P^{\gamma \alpha}}{\partial x^{k}} P^{k \beta}+\frac{\partial P^{\beta \gamma}}{\partial x^{k}} P^{k \alpha}=0, \quad \forall \alpha, \beta, \gamma . \tag{1.2}
\end{equation*}
$$

Definition 1.2. A skew-symmetric $C^{\infty}(M)$-tensor field $P^{\alpha \beta}$ satisfying (1.2) is called Poisson structure on $M$.

In the scientific literature, the tensor field $P$ is also called a Poisson bivector or, equivalently, a Hamiltonian operator.

Definition 1.3. A manifold $M$ endowed with a Poisson structure is called a Poisson manifold.
Remark. If $P^{\alpha \beta}$ is non-singular and $\omega_{\alpha \beta}$ denotes the inverse matrix, (1.2) is also equivalent to

$$
\frac{\partial \omega_{\alpha \beta}}{\partial x^{k}}+\frac{\partial \omega_{\gamma \alpha}}{\partial x^{k}}+\frac{\partial \omega_{\beta \gamma}}{\partial x^{k}}=0, \quad \forall \alpha, \beta, \gamma,
$$

that is

$$
d\left(\sum_{\alpha<\beta} \omega_{\alpha \beta} d x^{\alpha} \wedge d x^{\beta}\right)=0,
$$

i.e. the closedness of the 2 -form $\omega_{\alpha \beta} d x^{\alpha} \wedge d x^{\beta}$.

If we restrict our attention to the domain where the rank of the Poisson structure is constant (in particular, on the open subset where it achieves its maximum) the geometric picture underlying the symplectic foliation simplifies considerably. In fact, we can introduce local coordinates which make the foliation of particularly simple canonical form. This is the content of Darboux's theorem.

Darboux's theorem (Poisson form). Let $M$ be an $n$-dimensional Poisson manifold of constant rank $2 m \leq n$ everywhere. At each $x_{0} \in M$ there exist canonical local coordinates $(p, q, z)=$ $\left(p^{1}, \ldots, p^{m}, q^{1}, \ldots, q^{m}, z^{1}, \ldots, z^{l}\right), 2 m+l=n$, in terms of which the Poisson bracket takes the form

$$
\{f, g\}=\sum_{i=1}^{m}\left(\frac{\partial f}{\partial q^{i}} \frac{\partial g}{\partial p^{i}}-\frac{\partial g}{\partial p^{i}} \frac{\partial f}{\partial q^{i}}\right)
$$

The leaves of the symplectic foliation intersect the coordinate chart in the slices $\left\{z^{1}=c_{1}, \ldots, z^{l}=\right.$ $\left.c_{l}\right\}$ determined by the distinguished coordinates $z$.

A Poisson bracket defines an anti-homomorphism between the space of the smooth functions $C^{\infty}(M)$ and the space of vector fields on $M$ by

$$
H \mapsto X_{H}:=\{\cdot, H\}, \quad\left[X_{H_{1}}, X_{H_{2}}\right]=-X_{\left\{H_{1}, H_{2}\right\}} .
$$

The object $X_{H}$ is called a Hamiltonian vector field, and the corresponding dynamical system

$$
\begin{equation*}
\frac{d}{d t} x^{\alpha}=\left\{x^{\alpha}, H\right\}, \quad \alpha=1, \ldots, n, \tag{1.3}
\end{equation*}
$$

is called a Hamiltonian system with the Hamiltonian $H(u)$. Clearly, (1.3) is a symmetry of the Poisson bracket, namely, $\operatorname{Lie}_{X_{H}}\{\}=$,0 .

Definition 1.4. A function $f \in C^{\infty}(M)$ is called a Casimir for the given Poisson bracket if it belongs to the kernel of the Poisson bracket, i.e. iffor any function $g \in C^{\infty}(M)$ we have $\{f, g\}=0$. A function $f$ which is a Casimir for the given Poisson bracket is automatically an integral of the Hamiltonian system (1.3)

### 1.1.1 Poisson-Lichnerowicz cohomology

The notion of Poisson cohomology of a Poisson manifold $(M,\{\}$,$) was introduced for the$ first time by Lichnerowicz [59]. It can be described introducing the Schouten-Nijenhuis bracket [88, 79]. Let us denote the space of multivectors on $M$ by $\Lambda^{*}$ (the space of $k$-vectors is then denoted by $\Lambda^{k}$ ). The Schouten-Nijenhuis bracket is a bilinear extension of the Liederivative of vector fields to the space of multivectors, given by the map $[]:, \Lambda^{k} \times \Lambda^{l} \rightarrow$ $\Lambda^{k+l-1}$, uniquely determined by the following properties:

- supersymmetry: $[b, a]=(-1)^{k+l}[a, b], \quad a \in \Lambda^{k}, b \in \Lambda^{l}$;
- graded Leibniz: $[c, a \wedge b]=[c, a] \wedge b+(-1)^{l k+k} a \wedge[c, b], \quad a \in \Lambda^{k}, c \in \Lambda^{l}$;
- $[f, g]=0$ for $f, g \in \Lambda^{0},[v, f]=v^{i} \frac{\partial f}{\partial x^{i}}$ for $v \in \Lambda^{1}, f \in \Lambda^{0}$ and for $v_{1}, v_{2} \in \Lambda^{1},\left[v_{1}, v_{2}\right]$ is the usual commutator of vector fields.

In particular, for a vector field $v$ and a multivector $a$ we have $[v, a]=\operatorname{Lie}_{v} a$. For a pair of bivectors $h=\left(h^{i j}\right)$ and $f=\left(f^{i j}\right)$, their Schouten-Nijenhuis brackets is given by the following trivector

$$
[h, f]^{i j k}=\frac{\partial h^{i j}}{\partial x^{s}} f^{s k}+\frac{\partial f^{i j}}{\partial x^{s}} h^{s k}+\frac{\partial h^{k i}}{\partial x^{s}} f^{s j}+\frac{\partial f^{k i}}{\partial x^{s}} h^{s j}+\frac{\partial h^{j k}}{\partial x^{s}} f^{s i}+\frac{\partial f^{j k}}{\partial x^{s}} h^{s i}
$$

Let us point out that the Jacobi condition (1.2) is equivalent to $[P, P]^{i j k}=0$.
The Schouten-Nijenhuis bracket satisfies the so called graded Jacobi identity [79]

$$
(-1)^{k m}[[a, b], c]+(-1)^{m l}[[c, a], b]+(-1)^{l k}[[b, c], a]=0, \quad a \in \Lambda^{k}, b \in \Lambda^{l}, c \in \Lambda^{m}
$$

It follows that, for a Poisson bivector $P$, the map $d: \Lambda^{k} \rightarrow \Lambda^{k+1}$, such that for $Q \in \Lambda^{k}$ we have $d Q=[P, Q]$, is a differential, that is $d^{2}=0$.

Definition 1.5. The cohomology of the complex $\left(\Lambda^{*}, d\right)$ is called Poisson-Lichnerowicz cohomology of $(M,\{\}$,$) .$

Usually, it is denoted by $H^{*}(M,\{\})=,\oplus_{k \geq 0} H^{k}(M,\{\}$,$) , or just by H P^{*}$, where $P$ is the Poisson bivector related to the Poisson bracket $\{$,$\} . In particular:$

- $H^{0}(M,\{\}$,$) corresponds to the ring of Casimirs of the Poisson bracket;$
- $H^{1}(M,\{\}$,$) is the quotient of the Lie algebra of infinitesimal symmetries over the$ subalgebra of Hamiltonian vector fields;
- $H^{2}(M,\{\}$,$) coincides with the space of infinitesimal deformations of the Poisson$ bracket modulo those that can be obtained by infinitesimal changes of coordinates.


### 1.2 Local Poisson structures on loop spaces

In this section we want to describe an appropriate class of Poisson brackets on the loop space of a (smooth) $n$-dimensional manifold $M$, that is the space $\mathcal{L}(M)$ of all smooth maps $S^{1} \rightarrow M$. Here $\mathcal{L}(M)$ is treated formally in the spirit of formal variational calculus (see [23,20]) and it is defined in terms of the ring of functions on it. Let $U \subset M$ be a chart on $M$ with the coordinates $u^{1}(x), \ldots u^{n}(x)$. We denote by $\mathcal{A}=\mathcal{A}(U)$ the ring of differential polynomials in the independent variables $u_{(s)}^{i} i=1, \ldots, n, s=1,2, \ldots$,

$$
f\left(x, \mathbf{u}, \mathbf{u}_{x}, \ldots\right)=\sum_{i_{1}, s_{1}, \ldots, i_{m}, s_{m}} f_{i_{1}, s_{1} ; \ldots ; i_{m}, s_{m}}(x, \mathbf{u}) u_{\left(s_{1}\right)}^{i_{1}} \ldots u_{\left(s_{m}\right)}^{i_{m}},
$$

where $\mathbf{u}=\left(u^{1}, \ldots, u^{n}\right), \mathbf{u}_{(s)}=\left(u_{(s)}^{1}, \ldots, u_{(s)}^{n}\right)$ with $u_{(s)}^{i}=\frac{d^{s}}{d x^{s}} u^{i}(x)$. We also require that the coefficients $f_{i_{1}, s_{1} ; \ldots ; i_{m}, s_{m}}(x, \mathbf{u})$ of these differential polynomials are smooth functions on $S^{1} \times M$. Denote by $\mathcal{A}_{0}=\mathcal{A} / \mathbb{R}$ the space of differential polynomials modulo constants, and $\mathcal{A}_{1}=\mathcal{A}_{0} d x$. Then we have a well-defined map $d: \mathcal{A}_{0} \rightarrow \mathcal{A}_{1}$ such that

$$
f \mapsto d f:=\left(\frac{\partial f}{\partial x}+\sum_{i, s} \frac{\partial f}{\partial u_{(s)}^{i}} u_{(s+1)}^{i}\right) d x .
$$

The quotient space $\Lambda=\mathcal{A}_{1} / d \mathcal{A}_{0}$ is called the space of local functionals on $\mathcal{L}(M)$, whose elements are expressed as integrals over $S^{1}$ of a representative differential polynomial, namely

$$
\Lambda \ni \lambda=\int_{S^{1}} f\left(x, \mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(m)}\right) d x
$$

Since we are considering suitable boundary conditions, two elements $\lambda_{1}$ and $\lambda_{2}$ in $\Lambda$ are specified by a differential polynomial up to a total derivative.

In order to study Poisson bivectors on the loop space, we need to introduce the notion of local multivectors ( $k$-vectors).

Definition 1.6. A local $k$-vector $\alpha$ on the loop space $\mathcal{L}(M)$ is a formal infinite sum of the form

$$
\alpha=\sum \frac{1}{k!} \partial_{x_{1}}^{s_{1}} \ldots \partial_{x_{k}}^{s_{k}} A^{i_{1}, \ldots, i_{k}} \frac{\partial}{\partial u_{\left(s_{1}\right)}^{i_{1}}\left(x_{1}\right)} \wedge \cdots \wedge \frac{\partial}{\partial u_{\left(s_{k}\right)}^{i_{k}}\left(x_{k}\right)},
$$

with coefficients

$$
A^{i_{1}, \ldots, i_{k}}=\sum_{p_{2}, \ldots, p_{k} \geq 0} B_{p_{2}, \ldots, p_{k}}^{i_{1}, \ldots, i_{k}}\left(\mathbf{u}\left(x_{1}\right), \mathbf{u}_{x}\left(x_{1}\right), \ldots\right) \delta^{\left(p_{2}\right)}\left(x_{1}-x_{2}\right) \ldots, \delta^{\left(p_{k}\right)}\left(x_{1}-x_{k}\right)
$$

where $B_{p_{2}, \ldots, p_{k}}^{i_{1}, \ldots, i_{k}}\left(\mathbf{u}\left(x_{1}\right), \mathbf{u}_{x}\left(x_{1}\right), \ldots\right) \in \mathcal{A}$ (the ring of differential polynomials) and the coefficients $A^{i_{1}, \ldots, i_{k}}$, called the components of the $k$-vector $\alpha$, satisfy the skew-symmetry condition with respect to simultaneous exchange ( $i_{r}, x_{r}$ ) with $\left(i_{s}, x_{s}\right)$.
$\delta$-functions, as well as their derivatives and products, are defined by the formulae

$$
\begin{gathered}
\int f(y) \delta(x-y) d y=f(x), \quad \int f(y) \delta^{(m)}(x-y) d y=\frac{d^{m} f(x)}{d x^{m}} \\
\int f\left(x_{1}, \ldots, x_{k}\right) \delta^{\left(m_{2}\right)}\left(x_{1}-x_{2}\right) \ldots \delta^{\left(m_{k}\right)}\left(x_{1}-x_{k}\right) d x_{2} \ldots d x_{k} \\
=\left.\partial_{x_{2}}^{m_{2}} \ldots \partial_{x_{f}}^{m_{k}} f\left(x_{1}, \ldots, x_{k}\right)\right|_{x_{1}=x_{2}=\ldots=x_{k}} .
\end{gathered}
$$

We denote the space of local $k$-vectors as $\Lambda_{\text {loc }}^{k}$. For example, for $k=0$, the subspace $\Lambda_{\mathrm{loc}}^{0} \subset \Lambda_{\mathrm{loc}}^{*}$ is identified with the space of local functionals of the form

$$
\begin{equation*}
F=\int_{S^{1}} f\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) d x, \quad f\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) \in \mathcal{A}_{0} . \tag{1.4}
\end{equation*}
$$

In the case where $k=1$, the subspace $\Lambda_{\mathrm{loc}}^{1}$ represents the space of local vector fields on $\mathcal{L}(M)$, and its generic element is expressed by the following formula

$$
\begin{equation*}
\xi=\sum_{i=1}^{n} \sum_{s \geq 0} \partial_{x}^{s} X^{i}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) \frac{\partial}{\partial u_{(s)}^{i}} \tag{1.5}
\end{equation*}
$$

The components $X^{i}$ in (1.5) do not depend explicitly on the variable $x$ and for this reason they are called translationally invariant evolutionary vector fields.

Let us consider the space of local bivectors ( $k=2$ ). A generic element $P \in \Lambda_{\mathrm{loc}}^{2}$ has the
form

$$
\begin{equation*}
P=\frac{1}{2} \sum \partial_{x}^{r} \partial_{y}^{s} A^{i j} \frac{\partial}{\partial u_{(r)}^{i}(x)} \wedge \frac{\partial}{\partial u_{(s)}^{j}(y)}, \tag{1.6}
\end{equation*}
$$

where

$$
A^{i j}=A^{i j}\left(x-y ; \mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right)=\sum_{t \geq 0} A_{t}^{i j}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) \delta^{(t)}(x-y) .
$$

In order to characterise which local bivector $P \in \Lambda_{\mathrm{loc}}^{2}$ corresponds to a Poisson bivector we need to introduce a criterion for the Jacobi identity. We have seen that in the finitedimensional case the Jacobi identity can be written in terms of the Schouten-Nijenhuis bracket (see Section 1.1). An infinite-dimensional version of this bracket can be defined on the space of local multivectors with its natural gradation $\Lambda_{\mathrm{loc}}^{*}=\Lambda_{\mathrm{loc}}^{0} \oplus \Lambda_{\mathrm{loc}}^{1} \oplus \Lambda_{\mathrm{loc}}^{2} \oplus \ldots$, through a bilinear operation

$$
[,]: \Lambda_{\mathrm{loc}}^{r} \times \Lambda_{\mathrm{loc}}^{s} \rightarrow \Lambda_{\mathrm{loc}}^{r+s-1}, \quad r, s \geq 0 .
$$

Let us describe how the Schouten-Nijenhuis bracket operates on certain pairs of local multi-vectors:

- For any $F, G \in \Lambda_{\mathrm{loc}}^{0}$ we have $[F, G]=0$ identically.
- If $\xi$ is a local vector (1.5) and $F$ is a local functional (1.4), then

$$
[\xi, F]=\int_{S^{1}} \sum_{t \geq 0} \sum_{i=1}^{n}\left(\partial_{x}^{t} X^{i}\right) \frac{\partial f}{\partial u_{(t)}^{i}} d x=\int_{S^{1}} \sum_{i=1}^{n} X^{i} \frac{\delta F}{\delta u^{i}(x)} d x,
$$

where

$$
\frac{\delta F}{\delta u^{i}(x)}=\sum_{t \geq 0}(-1)^{t} \partial_{x}^{t}\left(\frac{\partial f}{\partial u_{(t)}^{i}}\right)
$$

is the variational derivative of the local functional $F$. Observe that $[\xi, F]$ is indeed an element of $\Lambda_{\mathrm{loc}}^{0}$.

- The Schouten-Nijenhuis bracket of two local vector fields $\xi, \eta$ of the form (1.5), with components $X^{i}, Y^{i}$ respectively, is again a vector field $\mu$ given by

$$
\mu=[\xi, \eta]=\sum_{s, i, j, t} \partial_{x}^{s}\left(X_{(t)}^{j} \frac{\partial Y^{i}}{\partial u_{(t)}^{j}}-Y_{(t)}^{j} \frac{\partial X^{i}}{\partial u_{(t)}^{j}}\right) \frac{\partial}{\partial u_{(s)}^{i}} .
$$

- The Schouten-Nijenhuis bracket of a local bivector $P$ of the form (1.6) and a local functional $F$ gives rise to a local vector field whose components are

$$
[P, F]^{i}=\sum_{j, k} A_{k}^{i j} \partial_{x}^{k} \frac{\delta F}{\delta u^{j}(x)}
$$

- Analogously the Schouten-Nijenhuis bracket of a local bivector $P$ (1.6) and a local vector field $\xi$ (1.5) is again a local bivector whose components are given by

$$
\begin{equation*}
[P, \xi]^{i j}=\sum_{k, s}\left(\partial_{x}^{s} X^{k}(\mathbf{u}(x), \ldots) \frac{\partial A^{i j}}{\partial u_{(s)}^{k}(x)}-\frac{\partial X^{i}(\mathbf{u}(x), \ldots)}{\partial u_{(s)}^{k}(x)} \partial_{x}^{s} A^{k j}-\frac{\partial X^{j}(\mathbf{u}(y), \ldots)}{\partial u_{(s)}^{k}(y)} \partial_{y}^{s} A^{i k}\right) \tag{1.7}
\end{equation*}
$$

We refer to formula (1.7) as Lie derivative of a translationally invariant bivector $P$ along a translationally invariant vector field $\xi$, that is, $\left(\operatorname{Lie}_{\xi} P\right)^{i j}=[P, \xi]^{i j}$.

- If $P$ and $Q$ are two translationally invariant bivectors with the components $A^{i j}(x-$ $\left.y ; \mathbf{u}(x), \mathbf{u}(y) ; \mathbf{u}_{x}(x), \mathbf{u}_{x}(y), \ldots\right)$ and $B^{i j}\left(x-y ; \mathbf{u}(x), \mathbf{u}(y) ; \mathbf{u}_{x}(x), \mathbf{u}_{x}(y) ; \ldots\right)$, that we denote respectively by $A_{x, y}^{i j}$ and $B_{x, y}^{i j}$, then the Schouten-Nijenhuis bracket $[P, Q]$ is a translation invariant trivector with the components

$$
\begin{gathered}
{[P, Q]_{x, y, z}^{i j k}=} \\
\frac{\partial A_{x, y}^{i j}}{\partial u_{(s)}^{l}(x)} \partial_{x}^{s} B_{x, z}^{l k}+\frac{\partial B_{x, y}^{i j}}{\partial u_{(s)}^{l}(x)} \partial_{x}^{s} A_{x, z}^{l k}+\frac{\partial A_{x, y}^{i j}}{\partial u_{(s)}^{l}(y)} \partial_{y}^{s} B_{y, z}^{l k}+\frac{\partial B_{x, y}^{i j}}{\partial u_{(s)}^{l}(y)} \partial_{y}^{s} A_{y, z}^{l k} \\
+\frac{\partial A_{z, x}^{k i}}{\partial u_{(s)}^{l}(z)} \partial_{z}^{s} B_{z, y}^{l j}+\frac{\partial B_{z, x}^{k i}}{\partial u_{(s)}^{l}(z)} \partial_{z}^{s} A_{z, y}^{l j}+\frac{\partial A_{z, x}^{k i}}{\partial u_{(s)}^{l}(x)} \partial_{x}^{s} B_{x, y}^{l j}+\frac{\partial B_{z, x}^{k i}}{\partial u_{(s)}^{l}(x)} \partial_{x}^{s} A_{x, y}^{l j} \\
+\frac{\partial A_{y, z}^{j k}}{\partial u_{(s)}^{l}(y)} \partial_{y}^{s} B_{y, x}^{l i}+\frac{\partial B_{y, z}^{j k}}{\partial u_{(s)}^{l}(y)} \partial_{y}^{s} A_{y, x}^{l i}+\frac{\partial A_{y, z}^{j k}}{\partial u_{(s)}^{l}(z)} \partial_{z}^{s} B_{z, x}^{l i}+\frac{\partial B_{y, z}^{j k}}{\partial u_{(s)}^{l}(z)} \partial_{z}^{s} A_{z, x}^{l i} .
\end{gathered}
$$

As in the finite-dimensional case, the Schouten-Nijenhuis bracket satisfies the following properties for every $a \in \Lambda_{\mathrm{loc}}^{k}, b \in \Lambda_{\mathrm{loc}}^{l}, c \in \Lambda_{\mathrm{loc}}^{m}$ :

- supersymmetry: $[a, b]=(-1)^{k l}[b, a]$;
- graded Jacobi: $(-1)^{k m}[[a, b], c]+(-1)^{m l}[[c, a], b]+(-1)^{l k}[[b, c], a]=0$.

We can now introduce the notion of local Poisson structure.
Definition 1.7. A local bivector $P \in \Lambda_{\mathrm{loc}}^{2}$ of the form (1.6) is called a local Poisson structure on $\mathcal{L}(M)$ if $[P, P]=0$.

### 1.2.1 Deformations of local Poisson structures

A local Poisson structure $P \in \Lambda_{\mathrm{loc}}^{2}$ gives rise to a Poisson bracket on the space of local functionals by the formula

$$
\begin{equation*}
\{F, G\}=\int_{S^{1}} \sum_{k \geq 0} \frac{\delta F}{\delta u^{i}(x)} A_{k}^{i j}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots\right) \partial_{x}^{k} \frac{\delta G}{\delta u^{j}(x)} d x \tag{1.8}
\end{equation*}
$$

which, using a special choice of local functionals, namely, $F=\int u^{i}(w) \delta(w-x) d w, G=$ $\int u^{j}(w) \delta(w-y) d w$, leads to the usual representation of a Poisson structure:

$$
\left\{u^{i}(x), u^{j}(y)\right\}=\sum_{k \geq 0} A_{k}^{i j}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) \delta^{(k)}(x-y) .
$$

It has been observed that the rescaling of the independent variable $\psi_{\epsilon}: x \mapsto x \epsilon$ induces a natural gradation on the space $\Lambda_{\text {loc }}^{k}$. Now we describe how the various ingredients rescale under $\psi_{\epsilon}$. First of all we define $\left(\psi_{\epsilon} u^{i}\right)(x)=u^{i}(\epsilon x)$, which immediately implies

$$
\frac{d\left(\psi_{\epsilon} u^{i}\right)(x)}{d x^{s}}=\epsilon^{s} u_{(s)}^{i}(x), \quad \frac{\partial}{\partial u_{(s)}^{i}(\epsilon x)}=\frac{1}{\epsilon^{s}} \frac{\partial}{\partial u_{(s)}^{i}(x)} .
$$

Concerning the scaling of $\delta$ distribution and its derivatives, let us consider

$$
\begin{gathered}
\int f(x) \delta^{(s)}(\epsilon x) d x=\int f\left(\frac{z}{\epsilon}\right) \delta^{(s)}(z) \frac{d z}{\epsilon}=(-1)^{s} \int \frac{d^{s} f\left(\frac{z}{\epsilon}\right)}{d z^{s}} \delta(z) \frac{d z}{\epsilon}= \\
=(-1)^{s} f_{(s)}(0) \frac{1}{\epsilon^{s+1}}=\int f(x) \delta^{(s)}(x) \frac{1}{\epsilon^{s+1}} d x
\end{gathered}
$$

from which we get

$$
\delta^{(s)}(\epsilon x)=\delta^{(s)}(x) \frac{1}{\epsilon^{s+1}}, \quad \delta^{(s)}(x)=\epsilon^{s+1}\left(\psi_{\epsilon}\left(\delta^{(s)}\right)\right)(x) .
$$

With this information, one can show that the rescaling $\psi_{\epsilon}$ induces a decomposition on $\Lambda_{\mathrm{loc}}^{k}$ into monomials of different degrees.

For simplicity, we focus on the cases of $\Lambda_{\mathrm{loc}}^{1}$ and $\Lambda_{\mathrm{loc}}^{2}$. As seen above, any local vector field $\xi$ has the form (1.5). Since the changes of $\partial_{x}^{s}$ and $\frac{\partial}{\partial u_{(s)}^{i}}$ induced by the rescaling $\psi_{\epsilon}$ are reciprocal to each other, the splitting of the vector field $\xi$ into homogeneous monomials depends only on its components $X^{i}$. In general, the components $X^{i}$ split under the
rescaling into homogeneous monomials as follows:

$$
X^{i}=a^{i}(\mathbf{u})+\epsilon \sum_{j=1}^{n} b_{j}^{i}(\mathbf{u}) u_{x}^{j}+\epsilon^{2}\left(\sum_{j=1}^{n} e_{j}^{i}(\mathbf{u}) u_{x x}^{j}+\sum_{j, l=1}^{n} h_{j l}^{i}(\mathbf{u}) u_{x}^{j} u_{x}^{l}\right)+\ldots
$$

and this gives rise to an analogous decomposition as

$$
\Lambda_{\mathrm{loc}}^{1}=\bigoplus_{k=0}^{\infty} \Lambda_{k, \mathrm{loc}}^{1},
$$

where $\Lambda_{k, \text { loc }}^{1} \subseteq \Lambda_{\text {loc }}^{1}$ is the space of local vector fields $\xi$ whose components $X^{i}$ are homogeneous differential polynomials of degree $k$. Following the same procedure, let us consider the space of local bivectors $\Lambda_{\text {loc }}^{2}$. A local bivector $P$ with components $A^{i j}$ is given by (1.6). Again, since the terms $\partial_{x}^{r} \partial_{y}^{s}$ and $\frac{\partial}{\partial u_{(r)}^{i}(x)} \wedge \frac{\partial}{\partial u_{(s)}^{j}(y)}$ have reciprocal scaling factors, the decomposition of the local bivector $P$ into homogeneous monomials under the action of $\psi_{\epsilon}$ is completely controlled by the way in which its components $A^{i j}$ decompose. Thus, rewriting $A^{i j}$ as

$$
\sum_{t \geq 0} \sum_{l \geq 0}\left(A_{t}^{i j}\right)_{l} \delta^{(t)}(x-y),
$$

where $\left(A_{t}^{i j}\right)_{l}$ is the homogeneous component of degree $l$ of the differential polynomial $A_{t}^{i j}$, and applying $\psi_{\epsilon}$, we obtain

$$
\sum_{t \geq 0} \sum_{l \geq 0}\left(A_{t}^{i j}\right)_{l} \epsilon^{l+t+1} \delta^{(t)}(x-y) .
$$

Setting $k=l+t+1$, it reads

$$
\begin{equation*}
\sum_{k=1}^{\infty} \epsilon^{k} \sum_{t=0}^{k-1}\left(A_{t}^{i j}\right)_{k-1-t} \delta^{(t)}(x-y) . \tag{1.9}
\end{equation*}
$$

In this way the components $A^{i j}$ of the bivector $P$ decompose in homogeneous terms $\left[A^{i j}\right]_{k}$ of the form

$$
\left[A^{i j}\right]_{k}=\sum_{t=0}^{k-1}\left(A_{t}^{i j}\right)_{k-1-t} \delta^{(t)}(x-y) .
$$

As above, we have an induced decomposition of $\Lambda_{\mathrm{loc}}^{2}$ as

$$
\Lambda_{\mathrm{loc}}^{2}=\bigoplus_{k \geq 1} \Lambda_{k, \mathrm{loc}}^{2}
$$

where $P \in \Lambda_{k, \text { loc }}^{2}$ if its components $A^{i j}$ are of the form (1.9). We can now define the notion of a deformation of a Poisson bivector $P \in \Lambda_{k, l \mathrm{loc}}^{2}$.

Definition 1.8. Any Poisson structure of the form

$$
\begin{equation*}
(P+Q) \in \Lambda_{\mathrm{loc}}^{2} \tag{1.10}
\end{equation*}
$$

where $Q=\sum_{s \geq 1} Q_{s}$ and $Q_{s} \in \Lambda_{s+k, \text { loc }}^{2}$ is called a deformation of $P \in \Lambda_{k, \text { loc }}^{2}$.
Finally, under the rescaling $\psi_{\epsilon}$, the deformation (1.10) transforms into $\epsilon^{2}\left(P+\sum_{s \geq 1} \epsilon^{s} Q_{s}\right)$, so (1.10) can be rewritten as

$$
P+\sum_{s \geq 1} \epsilon^{s} Q_{s}
$$

In the general case, it turns out that the space of $j$-multivectors $\Lambda_{\mathrm{loc}}^{j}$ can be decomposed into terms which are homogeneous under rescaling, that is $\Lambda_{\mathrm{loc}}^{j}=\oplus_{k} \Lambda_{k, \text { loc }}^{j}$, where any element $P \in \Lambda_{k, \text { loc }}^{j}$ is transformed to $\epsilon^{k} P$ under rescaling.

### 1.2.2 Poisson-Lichnerowicz cohomology on loop spaces

Let $P \in \Lambda_{2, \text { loc }}^{2}$ be a Poisson bivector (the choice of the space $\Lambda_{2, \text { loc }}^{2}$ will be clear in the next section) and let $d_{P}$ be a map defined as follows

$$
d_{P}: \Lambda_{\mathrm{loc}}^{j} \rightarrow \Lambda_{\mathrm{loc}}^{j+1}, \quad d_{P}(a)=[P, a] .
$$

Clearly, $d_{P}$ maps $\Lambda_{k, \text { loc }}^{j}$ to $\Lambda_{k+2, \text { loc }}^{j+1}$. Moreover, using the graded Jacobi identity satisfied by the Schouten-Nijenhuis bracket, and the fact that $P$ is a Poisson structure, one can easily prove that $d_{P}$ is a differential, namely $d_{P}^{2}=0$ identically. This property allows us to define the analogues of the cohomology groups, defined in the finite-dimensional framework by Lichnerowicz (see Section 1.1.1), through

$$
H^{j}(\mathcal{L}(M), P)=\frac{\operatorname{Ker}\left\{d_{P}: \Lambda_{\mathrm{loc}}^{j} \rightarrow \Lambda_{\mathrm{loc}}^{j+1}\right\}}{\operatorname{Im}\left\{d_{P}: \Lambda_{\mathrm{loc}}^{j-1} \rightarrow \Lambda_{\mathrm{loc}}^{j}\right\}} .
$$

Remembering that $d_{P}$ preserves the natural decomposition of the space of $j$-multivectors $\Lambda^{j}$ into components homogeneous under rescaling, each cohomology group inherits a natural decomposition into homogeneous parts. Thus, we can introduce

$$
H_{k}^{j}(\mathcal{L}(M), P)=\frac{\operatorname{Ker}\left\{d_{P}: \Lambda_{k, \text { loc }}^{j} \rightarrow \Lambda_{k+2, \text { loc }}^{j+1}\right\}}{\operatorname{Im}\left\{d_{P}: \Lambda_{k-2, \text { loc }}^{j-1} \rightarrow \Lambda_{k, \text { loc }}^{j}\right\}},
$$

where a class $[\alpha] \in H_{k}^{j}(\mathcal{L}(M), P)$ if any of its representative can be chosen in $\Lambda_{k, \text { loc }}^{j}$. This leads to the following decomposition $H^{j}(\mathcal{L}(M), P)=\oplus_{k} H_{k}^{j}(\mathcal{L}(M), P)$, which is typical of the infinite dimensional situation. Indeed, there is no analogous correspondence in the finite dimensional case.

### 1.2.3 Dubrovin-Novikov brackets

We have seen that a local Poisson structure $P \in \Lambda_{\mathrm{loc}}^{2}$ of the form (1.6) gives rise to a Poisson bracket on the space of local functionals by formula (1.8). Without any loss of generality, we can identify the Poisson structure $P$ by its components,

$$
\begin{equation*}
\left\{u^{i}(x), u^{j}(y)\right\}=P^{i j}=\sum_{k=0}^{N} A_{k}^{i j}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots\right) \delta^{(k)}(x-y), \tag{1.11}
\end{equation*}
$$

for a certain number $N \in \mathbb{N}$.
Remark. The bivector $P^{i j}$ can be represented as

$$
A^{i j}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots ; \frac{d}{d x}\right) \delta(x-y)
$$

where the differential operators $A^{i j}$ are given by

$$
A^{i j}\left(\mathbf{u}(x), \mathbf{u}_{x}(x), \ldots ; \frac{d}{d x}\right)=\sum_{s} A_{s}^{i j} \frac{d^{s}}{d x^{s}} .
$$

For multivectors of higher rank the language of differential operator was used by Olver [81]. In what follows, we will denote by $P$ the Poisson structure (equivalently, Poisson bivector or Hamiltonian operator) written in terms of differential operators.

It is natural to require that the Poisson structure is independent on the choice of local coordinates on $M$. Thus, the coefficients $A_{k}^{i j}$ have to transform in an suitable way when
we apply a change of coordinates $\mathbf{v}=\mathbf{v}(\mathbf{u})$. The transformation law of these coefficients is determined by the Leibniz identity

$$
\left\{v^{p}(x), v^{q}(y)\right\}=\frac{\partial v^{p}}{\partial u^{i}}(x) \frac{\partial v^{q}}{\partial u^{j}}(y)\left\{u^{i}(x), v^{j}(y)\right\},
$$

together with the following identities for the derivatives of $\delta$-function

$$
f(y) \delta^{(k)}(x-y)=\sum_{l=0}^{k}\binom{k}{l} f^{(l)}(x) \delta^{(k-l)}(x-y) .
$$

Let us assign degrees to the derivatives putting

$$
\operatorname{deg} \frac{d^{s} u^{i}}{d x^{s}}=s, \quad s=1,2, \ldots
$$

and $\operatorname{deg} f(\mathbf{u})=0$ if the function $f$ is independent of the derivatives.
Definition 1.9. A Poisson structure (1.11) is graded homogeneous of the degree $D$ if the coefficients are graded homogeneous polynomials in the derivatives of the degrees $\operatorname{deg} A_{k}^{i j}=D-k$, for $k=0,1, \ldots, N$.

Clearly, the order $N$ of (1.11) cannot be greater than the degree $D$. Using the transformation property described before one can easily prove that the degree $D$ does not depend on the choice of local coordinates $u^{1}, \ldots, u^{n}$.

For instance, the graded homogeneous Poisson structure of degree 0 has the form

$$
\left\{u^{i}(x), u^{j}(y)\right\}=h^{i j}(\mathbf{u}(x)) \delta(x-y),
$$

where $h^{i j}(\mathbf{u})$ is a usual (i.e. finite-dimensional) Poisson structure on the manifold $M$.
The main example, which will be the starting point of our analysis, is the graded homogeneous Poisson structure of degree 1, namely

$$
\begin{equation*}
\left\{u^{i}(x), u^{j}(y)\right\}=g^{i j}(\mathbf{u}(x)) \delta^{\prime}(x-y)+b_{k}^{i j}(\mathbf{u}(x)) u_{x}^{k} \delta(x-y), \tag{1.12}
\end{equation*}
$$

where $g^{i j}(\mathbf{u})$ and $b_{k}^{i j}(\mathbf{u})$ are some functions on $M$ depending on the choice of local coordinates. In other words, for arbitrary functionals $F[\mathbf{u}]$ and $G[\mathbf{u}]$ a Poisson bracket of the
form

$$
\begin{equation*}
\{F, G\}=\int_{S^{1}} \frac{\delta F}{\delta u^{i}(x)}\left(g^{i j}(\mathbf{u}(x)) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}(x)) u_{x}^{k}\right) \frac{\delta G}{\delta u^{j}(x)} d x, \tag{1.13}
\end{equation*}
$$

is defined. Thus, the differential operator $P$ is given by

$$
\begin{equation*}
P^{i j}(\mathbf{u})=g^{i j}(\mathbf{u}(x)) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}(x)) u_{x}^{k} . \tag{1.14}
\end{equation*}
$$

This class of Poisson brackets is invariant under transformations of the field variables $v^{i}=$ $v^{i}\left(u^{1}, \ldots, u^{n}\right)$. More precisely, the coefficients $g^{i j}$ and $b_{k}^{i j}$ transform as object of differential geometry, according to the rules

$$
\begin{gathered}
\tilde{g}^{m l}(\mathbf{v})=\frac{\partial v^{l}}{\partial u^{i}} \frac{\partial v^{m}}{\partial u^{j}} g^{i j}(\mathbf{u}), \\
\tilde{b}_{s}^{m l}(\mathbf{v})=\frac{\partial v^{l}}{\partial u^{i}} \frac{\partial^{2} v^{m}}{\partial u^{j} \partial u^{k}} \frac{\partial u^{k}}{\partial v^{s}} g^{i j}(\mathbf{u})+\frac{\partial v^{l}}{\partial u^{i}} \frac{\partial u^{k}}{\partial v^{s}} \frac{\partial v^{m}}{\partial u^{j}} b_{k}^{i j}(\mathbf{u}) .
\end{gathered}
$$

Poisson structures of the form (1.12) (or, equivalently, (1.13)) were introduced and studied by Dubrovin and Novikov [32, 35, 34], and they are called Poisson brackets of hydrodynamic type, or Dubrovin-Novikov brackets. If $\operatorname{det}\left(g^{i j}\right) \neq 0$, we say that the Poisson bracket is non-degenerate. Dubrovin and Novikov provided a complete description of non-degenerate Poisson brackets in terms of Riemannian geometry.

Theorem 1.1 (Dubrovin-Novikov, [32]). Let $g^{i j}$ be non-degenerate. Formula (1.12) defines a Poisson brackets if and only if the tensor $g^{i j}$ is symmetric, i.e it specifies a pseudo-Riemannian metric (with upper indices), and the connection $\Gamma_{j k}^{i}=-g_{j s} b_{k}^{s i}$ is compatible with the metric $g^{i j}$ and has zero curvature and torsion.

As a direct consequence, doing a change of the dependent variables $u^{i}(x) \mapsto w^{\alpha}(\mathbf{u}(x))$, we can rewrite the Poisson structure (1.12) in the following constant form

$$
\left\{w^{\alpha}\left(x_{1}\right), w^{\beta}\left(x_{2}\right)\right\}=\eta^{\alpha \beta} \delta^{\prime}\left(x_{1}-x_{2}\right), \quad\left(\text { or, equivalently, } \quad P^{\alpha \beta}=\eta^{\alpha \beta} \frac{d}{d x}\right),
$$

where the constant coefficients $\eta^{\alpha \beta}$ are the entries of the matrix of the metric in the flat coordinates $t$.

Remark. If $P \in \Lambda_{\text {loc }}^{2}$ is a bivector of hydrodynamic type, that is of the form (1.12), then $P \in \Lambda_{2, \text { loc }}^{2}$. Furthermore, any element of $\Lambda_{2, \text { loc }}^{2}$ is indeed a bivector of hydrodynamic type.

### 1.2.4 Deformations of Dubrovin-Novikov brackets

In the framework of the theory of Frobenius manifolds [27,36,38], Dubrovin conjectured the triviality of homogeneous formal deformations of structures (1.14). The problem formulated by Dubrovin can be stated as follows. Let us consider a Poisson manifold $M$ endowed with a Poisson structure of hydrodynamic type (bivector) $P_{0}$ which satisfies the Jacobi condition written in terms of the Schouten bracket on the algebra of multivector fields on $M$, that is $\left[P_{0}, P_{0}\right]=0$.

Definition 1.10. A deformation of order $k$ of a Poisson bivector $P_{0}$ is a formal series

$$
P^{\epsilon}=P_{0}+\epsilon P_{1}+\epsilon^{2} P_{2}+\ldots
$$

in the space of bivector fields on $M$ satisfying the condition $\left[P^{\epsilon}, P^{\epsilon}\right]=\mathcal{O}\left(\epsilon^{k+1}\right)$ for any value of the parameter $\epsilon$.

In particular, if $\left[P^{\epsilon}, P^{\epsilon}\right]=0$, we say that $P^{\epsilon}$ is a deformation of $P_{0}$. A deformation (of order $k$ ) is trivial if there exists a Miura transformation $\phi_{\epsilon}: M \rightarrow M$,

$$
\phi_{\epsilon}=\sum_{m=0}^{\infty} \epsilon^{m} \phi_{m},
$$

which pulls back $P^{\epsilon}$ to $P_{0}$, that is $P^{\epsilon}=\phi_{\epsilon *} P_{0}$. Therefore, the allowed deformations $P^{\epsilon}$ are formal series of the form

$$
P_{k}=\sum_{s=0}^{k+1} A_{s}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{k+1}\right) \frac{d^{k+1-s}}{d x^{k+1-s}},
$$

where the entries of the $n \times n$ matrices $A_{s}$ are homogenous polynomials of degree $s$ in the $x$-derivative, namely $\operatorname{deg}\left(A_{s}\right)=s$.

Problem. Does there exists a Miura transformation that brings the homogeneous deformation $P^{\epsilon}$ to $P_{0}$ ?

This problem can be reformulated in cohomological terms. Indeed, as we have seen in Section 1.2.2, triviality of deformations is equivalent to the vanishing of the second cohomology group in the Poisson-Lichnerowicz cohomology. Getzler [54] and independently Degiovanni, Sciacca and Magri [21] solved Dubrovin's conjecture proving that this cohomology group is trivial (in particular, Getzler proved that all positive integer cohomology
groups are trivial, that is, $H^{k}(\mathcal{L}(M), Q)=0$ for $Q \in \Lambda_{2, \text { loc }}^{2}$ and $\left.k=1,2, \ldots\right)$. This result immediately implies that any deformation of $\omega$ of the form

$$
P^{\epsilon}=Q+\sum_{k=1}^{\infty} \epsilon^{k} P_{k}, \quad P_{k} \in \Lambda_{k+2, \mathrm{loc}}^{2}
$$

can be obtained from $Q$ by performing a Miura transformation, namely a change of dependent variables $u^{1}, \ldots, u^{n}$ of the form

$$
\begin{equation*}
\tilde{u}^{i}=F_{0}^{i}(\mathbf{u})+\sum_{k} \epsilon^{k} F_{k}^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \mathbf{u}_{x x}, \ldots\right) \tag{1.15}
\end{equation*}
$$

where $F_{k}^{i}$ are differential polynomials in the derivatives of $u^{i}$ of degree $k$ and $\operatorname{det} \frac{\partial F_{0}}{\partial \mathbf{u}} \neq 0$. Indeed, the Poisson condition $\left[P^{\epsilon}, P^{\epsilon}\right]=0$, implies that $P_{1}$ is a cocycle of $\omega$ and therefore a coboundary

$$
P_{1}=\operatorname{Lie}_{X_{1}} Q
$$

for a suitable vector field $X_{1}$. This means that, performing a transformation of the form (1.15) generated by the vector field $-X_{1}$, we can eliminate the term in $\epsilon$, obtaining a local Poisson bivector of the form

$$
\tilde{P}^{\epsilon}=Q+\sum_{k=2}^{\infty} \epsilon^{k} \tilde{P}_{k}
$$

Analogously, there exists a vector field $X_{2}$ such that $\tilde{P}_{2}=\operatorname{Lie}_{X_{2}} Q$ and the transformation (1.15) generated by $-X_{2}$ allows us to eliminate $\tilde{P}_{2}$. Following this procedure, step by step, we can reduce $P^{\epsilon}$ to $Q$. Finally, the reducing transformation of the form (1.15) is the composition of the infinite sequence of transformations (1.15) generated by the vector fields $-X_{1},-X_{2},-X_{3}$ and so on.

### 1.3 Multi-dimensional Dubrovin-Novikov brackets

The multi-dimensional analogue of the Drubrovin-Novikov brackets (1.12) has the form

$$
\begin{equation*}
\left\{u^{i}(\mathbf{x}), u^{j}(\mathbf{y})\right\}=g^{i j \alpha}(\mathbf{u}(\mathbf{x})) \delta_{\alpha}(\mathbf{x}-\mathbf{y})+b_{k}^{i j \alpha}(\mathbf{u}(\mathbf{x})) u_{\alpha}^{k} \delta(\mathbf{x}-\mathbf{y}) \tag{1.16}
\end{equation*}
$$

where $\mathbf{u}=\left(u^{1}, \ldots, u^{n}\right)$ are local coordinates on a smooth $n$-dimensional manifold $M$, $\mathbf{x}=\left(x^{1}, \ldots, x^{N}\right)$ and $\mathbf{x}=\left(y^{1}, \ldots, y^{N}\right)$ are independent variables, $\delta_{\alpha}(\mathbf{x}-\mathbf{y})=\frac{\partial \delta(\mathbf{x}-\mathbf{y})}{\partial x^{\alpha}}$,
$u_{\alpha}^{k}=\frac{\partial u^{k}}{\partial x^{\alpha}}, \alpha=1, \ldots, N$. The differential operator $P$ associated with (1.16) is defined by

$$
\begin{equation*}
P^{i j}(\mathbf{u})=g^{i j \alpha}(\mathbf{u}) \frac{d}{d x^{\alpha}}+b_{k}^{i j \alpha}(\mathbf{u}) u_{\alpha}^{k} . \tag{1.17}
\end{equation*}
$$

The operator (1.17) is called non-degenerate if $\operatorname{det} g^{i j \alpha} \neq 0$ for any $\alpha$. However, since under unimodular changes of the spatial variable $x^{\alpha}=c_{\beta}^{\alpha} \tilde{x}^{\beta}, c_{\beta}^{\alpha}=\operatorname{const}$, $\operatorname{det}\left(c_{\beta}^{\alpha}\right)=1$, for fixed $i, j, k$, the objects $g^{i j \alpha}$ and $b_{k}^{i j \alpha}$ transform like vectors, that is

$$
\begin{equation*}
g^{i j \alpha}=c_{\beta}^{\alpha} \tilde{g}^{i j \beta}, \quad b_{k}^{i j \alpha}=c_{\beta}^{\alpha} b_{k}^{i j \beta}, \tag{1.18}
\end{equation*}
$$

it is sufficient to assume that $g^{i j \alpha}$ is non-degenerate for one $\alpha$, or similarly, that there exists a linear combination of $g^{i j \alpha}$ such that $\operatorname{det}\left(\lambda_{\alpha} g^{i j \alpha}\right) \neq 0, \lambda_{\alpha} \in \mathbb{R}$.

As in the one-dimensional case, the form of the Poisson bracket (1.16) is invariant under local transformations of coordinates, $\mathbf{v}=\mathbf{v}(\mathbf{u})$. In particular, for each $\alpha$ the coefficients $g^{i j \alpha}$ transform as components of a contravariant tensor of rank 2, and the coefficients $b_{k}^{i j \alpha}$ are transformed as components of the contravariant Levi-Civita connection.

Although the bilinearity property and Leibniz identity are fulfilled, the condition of skew-symmetry and the Jacobi identity for a Poisson bracket (1.16) impose very severe restrictions on the coefficients $g^{i j \alpha}(\mathbf{u})$ and $b_{k}^{i j \alpha}(\mathbf{u})$.

Theorem 1.2 ([76]). A bracket of the form (1.16) is a Poisson bracket, (or, equivalently, an operator of the form (1.17) is a Hamiltonian operator) i.e. it is skew-symmetric and satisfies the Jacobi identity, if and only if the following relations for the coefficients of the operator are fulfilled:

$$
\begin{gather*}
g^{i j \alpha}=g^{j i \alpha},  \tag{1.19}\\
\frac{\partial g^{i j \alpha}}{\partial u^{k}}=b_{k}^{i j \alpha}+b_{k}^{j i \alpha},  \tag{1.20}\\
\sum_{(\alpha, \beta)}\left(g^{s i \alpha} b_{s}^{j r \beta}-g^{s j \beta} b_{s}^{i r \alpha}\right)=0,  \tag{1.21}\\
\sum_{(i, j, r)}\left(g^{s i \alpha} b_{s}^{j r \beta}-g^{s j \beta} b_{s}^{i r \alpha}\right)=0,  \tag{1.22}\\
\sum_{(\alpha, \beta)}\left[g^{s i \alpha}\left(\frac{\partial b_{s}^{j r \beta}}{\partial u^{q}}-\frac{\partial b_{q}^{j r \beta}}{\partial u^{s}}\right)+b_{s}^{i j \alpha} b_{q}^{s r \beta}-b_{s}^{i r \alpha} b_{q}^{s j \beta}\right]=0, \tag{1.23}
\end{gather*}
$$

$$
\begin{gather*}
g^{s i \beta} \frac{\partial b_{q}^{j r \alpha}}{\partial u^{s}}-b_{s}^{i j \beta} b_{q}^{s r \alpha}-b_{s}^{i r \beta} b_{q}^{j s \alpha}=g^{s j \alpha} \frac{\partial b_{q}^{i r \beta}}{\partial u^{s}}-b_{s}^{j i \alpha} b_{q}^{s r \beta}-b_{q}^{i s \beta} b_{s}^{j r \alpha},  \tag{1.24}\\
\frac{\partial}{\partial u^{k}}\left[g^{s i \alpha}\left(\frac{\partial b_{s}^{j r \beta}}{\partial u^{q}}-\frac{\partial b_{q}^{j r \beta}}{\partial u^{s}}\right)+b_{s}^{i j \alpha} b_{q}^{s r \beta}-b_{s}^{i r \alpha} b_{q}^{s j \beta}\right]+\sum_{(i, j, r)}\left[b_{q}^{s i \beta}\left(\frac{\partial b_{k}^{j r \alpha}}{\partial u^{s}}-\frac{\partial b_{s}^{j r \alpha}}{\partial u^{k}}\right)\right] \\
+\frac{\partial}{\partial u^{q}}\left[g^{s i \beta}\left(\frac{\partial b_{s}^{j r \alpha}}{\partial u^{k}}-\frac{\partial b_{k}^{j r \alpha}}{\partial u^{s}}\right)+b_{s}^{i j \beta} b_{k}^{s r \alpha}-b_{s}^{i r \beta} b_{k}^{s j \alpha}\right]+\sum_{(i, j, r)}\left[b_{k}^{s i \alpha}\left(\frac{\partial b_{q}^{j r \beta}}{\partial u^{s}}-\frac{\partial b_{s}^{j r \beta}}{\partial u^{q}}\right)\right]=0 . \tag{1.25}
\end{gather*}
$$

The signs $\sum_{(\alpha, \beta)}$ and $\sum_{(i, j, k)}$ mean cyclic summation in the indicated indices. Note that in one-dimensional case these conditions reduce to Grinberg's conditions [56]. These relations imply the following important property.

Lemma 1.3 ([35, 69]). Every multi-dimensional Poisson bracket of the form (1.16) is always the sum of one-dimensional Poisson brackets with respect to each of the independent variables $x^{\alpha}$.

In other words, every summand on the right-hand side of the formula (1.17), namely

$$
g^{i j \alpha}(\mathbf{u}) \frac{d}{d x^{\alpha}}+b_{k}^{i j \alpha}(\mathbf{u}) u_{\alpha}^{k}
$$

defines a one-dimensional Hamiltonian operator with respect to $x^{\alpha}$.

### 1.3.1 Non-degenerate multi-dimensional brackets

Multi-dimensional Poisson brackets of hydrodynamic type, introduced by Dubrovin and Novikov in [35], have been thoroughly investigated by Mokhov [74, 76, 69] in the nondegenerate case. Under this assumption, in virtue of Dubrovin-Novikov theorem (see Theorem 1.1), Lemma 1.3 implies that all tensors $g^{i j \alpha}$ must be flat contravariant metrics, and each affine connection $\Gamma_{j k}^{i \alpha}=-g_{j m}^{\alpha} b_{k}^{m i \alpha}$ must be compatible with the respective metric $g^{i j \alpha}$, and has zero torsion and zero Riemann curvature, i.e. $\Gamma_{j k}^{i \alpha}$ is the Levi-Civita connection [35]. Hence, each non-degenerate multi-dimensional Poisson brackets of the form (1.16) is uniquely determined by the flat metrics $g^{i j \alpha}$, which must satisfy further restrictions. Thus, the classification problem of non-degenerate multi-dimensional Poisson brackets of hydrodynamic type, proposed by Dubrovin and Novikov in [35], can be reduced to a classification of admissible set of flat metrics $g^{i j \alpha}$.

The main difference with respect to the one-dimensional case is that, although all the
metrics $g^{i j \alpha}$ must be flat, they can no longer be reduced to a constant coefficient form simultaneously: an obstruction is given by the tensors $T_{j k}^{i \alpha \beta}(\mathbf{u})=\Gamma_{j k}^{i \beta}(\mathbf{u})-\Gamma_{j k}^{i \alpha}(\mathbf{u})$. Dubrovin and Novikov proved that the vanishing of the obstruction tensors is a necessary and sufficient condition for the existence of coordinates where the bracket (1.16) takes constant coefficient forms [35]. Under the assumption of non-degeneracy, Theorem 1.2 can be reformulated in term of the obstruction tensor.

Theorem 1.4 (Mokhov [74]). Flat non-degenerate metrics $g^{i j \alpha}(\mathbf{u})$ define a multi-dimensional Hamiltonian operator of the form (1.17) if and only if the following relations are fulfilled:

$$
\begin{gather*}
T^{i j k \alpha \beta}(\mathbf{u})=T^{k j i \alpha \beta}(\mathbf{u}),  \tag{1.26}\\
\sum_{(i, j, k)} T^{i j k \alpha \beta}(\mathbf{u})=0,  \tag{1.27}\\
T^{i j s \alpha \beta}(\mathbf{u}) T_{s t}^{r \alpha \beta}(\mathbf{u})=T^{i r s \alpha \beta}(\mathbf{u}) T_{s t}^{j \alpha \beta}(\mathbf{u}),  \tag{1.28}\\
\nabla_{r}^{\alpha} T^{i j k \alpha \beta}(\mathbf{u})=0, \tag{1.29}
\end{gather*}
$$

where $T_{j k}^{i \alpha \beta}(\mathbf{u})=\Gamma_{j k}^{i \beta}(\mathbf{u})-\Gamma_{j k}^{i \alpha}(\mathbf{u}), T^{i j k \alpha \beta}(\mathbf{u})=g^{k s \beta}(\mathbf{u}) g^{i r \alpha}(\mathbf{u}) T_{r s}^{j \alpha \beta}(\mathbf{u})$, the sign $\sum_{(i, j, k)}$ means summation over all cyclic permutations of indices $(i, j, k), \nabla_{r}^{\alpha}$ is the covariant derivative given by the connection $\Gamma_{j k}^{i \alpha}(\mathbf{u})$, and $\Gamma_{j k}^{i \alpha}(\mathbf{u})$ is the Levi-Civita connection generated by the metric $g^{i j \alpha}(\mathbf{u})$.

These tensorial conditions imply another important property of non-degenerate multidimensional brackets, as stated in the following theorem.

Theorem 1.5 ([35, 74]). A non-degenerate multi-dimensional Hamiltonian operator (1.17) for $n=1$ can be reduced to constant form, and for $n \geq 2$ can be reduced to a linear form, that is the metrics $g^{i j \alpha}(\mathbf{u})$ assume the form $g^{i j \alpha}(\mathbf{u})=c_{k}^{i j \alpha} u^{k}+g_{0}^{i j \alpha}$, where $c_{k}^{i j \alpha}=b_{k}^{i j \alpha}+b_{k}^{j i \alpha}, b_{k}^{i j \alpha}$ and $g_{0}^{i j \alpha}$ are constants.

In particular, since all the metrics are flat, we can always choose a system of coordinates where one metric, say $g^{i j \alpha}$ with $\alpha$ fixed, is reduced to constant coefficient form, and all the remaining metrics, $g^{i j \beta}$ with $\beta \neq \alpha$, are linear. Such coordinates are called flat coordinates for the metric $g^{i j \alpha}$.

This is a first step towards a geometric interpretation of the tensorial conditions (1.26)(1.29). The second step is related to the theory of compatible metrics constructed by Mokhov
in [70].

Definition 1.11. Two Riemannian or pseudo-Riemannian contravariant metrics $g^{i j 1}(\mathbf{u})$ and $g^{i j 2}(\mathbf{u})$ are called compatible if for any linear combination of these metrics

$$
g^{i j}(\mathbf{u})=\lambda_{1} g^{i j 1}(\mathbf{u})+\lambda_{2} g^{i j 2}(\mathbf{u})
$$

where $\lambda_{1}$ and $\lambda_{2}$ are arbitrary constants such that $\operatorname{det}\left(g^{i j}(\mathbf{u})\right) \not \equiv 0$, the coefficients of the corresponding Levi-Civita connections and the components of the corresponding tensors of Riemannian curvature are related by the same linear formula:

$$
\begin{align*}
b_{k}^{i j}(\mathbf{u}) & =\lambda_{1} b_{k}^{i j 1}(\mathbf{u})+\lambda_{2} b_{k}^{i j 2}(\mathbf{u})  \tag{1.30}\\
R_{k l}^{i j}(\mathbf{u}) & =\lambda_{1} R_{k l}^{i j 1}(\mathbf{u})+\lambda_{2} R_{k l}^{i j 2}(\mathbf{u}) \tag{1.31}
\end{align*}
$$

We shall also say in this case that the metrics $g^{i j 1}(\mathbf{u})$ and $g^{i j 2}(\mathbf{u})$ form a pencil of metrics.

If for any linear combination of metrics only (1.30) is fulfilled, then the metrics are called almost compatible. Almost-compatibility is also equivalent to the vanishing of the Nijenhuis tensor of the affinor (that is, $(1,1)$-tensor) $L_{j}^{i}(\mathbf{u})=g^{i k 2}(\mathbf{u}) g_{k j}^{1}(\mathbf{u})$, where $g^{i k 1}(\mathbf{u}) g_{k j}^{1}(\mathbf{u})=\delta_{j}^{i}$, which is defined by

$$
\begin{equation*}
\mathcal{N}_{j k}^{i}(\mathbf{u})=L_{j}^{s}(\mathbf{u}) \frac{\partial L_{k}^{i}(\mathbf{u})}{\partial u^{s}}-L_{k}^{s}(\mathbf{u}) \frac{\partial L_{j}^{i}(\mathbf{u})}{\partial u^{s}}-L_{s}^{i}(\mathbf{u}) \frac{\partial L_{k}^{s}(\mathbf{u})}{\partial u^{j}}+L_{s}^{i}(\mathbf{u}) \frac{\partial L_{j}^{s}(\mathbf{u})}{\partial u^{k}} \tag{1.32}
\end{equation*}
$$

see $[71,40]$ for further details.
Concerning Mokhov's conditions, by straightforward computations one can easily see that condition (1.26) is equivalent to the almost-compatibility of each pair of metrics $\left(g^{i j \alpha}, g^{i j \beta}\right)$, while adding (1.28) we get the compatibility.

Theorem 1.6 ([75]). All metrics $g^{i j \alpha}(\mathbf{u}), 1 \leq \alpha \leq N$, defining a multi-dimensional Poisson bracket of the form (1.16) are mutually compatible.

A pair of compatible flat metric defines a flat pencil of metrics. We emphasize that this notion plays an important role in the theory of Frobenius manifolds introduced by Dubrovin [29, 27], and in the theory of compatible Dubrovin-Novikov brackets. According to Magri [65], two Poisson brackets are called compatible if each their linear combination is a Poisson bracket. In the framework of hydrodynamic Poisson brackets, two
non-degenerate Dubrovin-Novikov brackets are compatible if and only if the correspondent flat metrics form a flat pencil [27, 25]. Therefore, Theorem 1.6 says that all onedimensional Dubrovin-Novikov brackets forming a multi-dimensional Poisson brackets of hydrodynamic type are also mutually compatible. This means that the study of nondegenerate multi-dimensional Poisson brackets of hydrodynamic type corresponds to the study of a subclass of compatible one-dimensional Dubrovin-Novikov brackets, identified by the additional conditions (1.27) and (1.29).

## 2

## Classification of non-degenerate Hamiltonian operators in 2D

In this chapter, based on the author's joint work with E.V. Ferapontov and P. Lorenzoni [46], we address the classification of non-degenerate Hamiltonian operators of DubrovinNovikov type in two dimensions. As we mentioned in Section 1.3.1, such operators are generated by two flat metrics, that we now denote with $g, \tilde{g}$, which can be assumed nondegenerate without any loss of generality. In this notation, the operator (1.17) takes the form

$$
\begin{equation*}
P^{i j}=g^{i j}(\mathbf{u}) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}) u_{x}^{k}+\tilde{g}^{i j}(\mathbf{u}) \frac{d}{d y}+\tilde{b}_{k}^{i j}(\mathbf{u}) u_{y}^{k} \tag{2.1}
\end{equation*}
$$

with $b_{k}^{i j}=-g^{i s} \Gamma_{s k}^{j}$ and $\tilde{b}_{k}^{i j}=-\tilde{g}^{i s} \tilde{\Gamma}_{s k}^{j}$, where $\Gamma$ and $\tilde{\Gamma}$ are the Levi-Civita connections of $g$ and $\tilde{g}$, the obstruction tensor is given by $T_{j k}^{i}=\tilde{\Gamma}_{j k}^{i}-\Gamma_{j k^{\prime}}^{i}$, and Theorem 1.4 reads

Theorem 2.1. Let $g$ and $\tilde{g}$ be two flat metrics. Formula (2.1) defines a Hamiltonian operator if and only if the obstruction tensor satisfies the relations

$$
\begin{gather*}
T^{i j k}=T^{k j i},  \tag{2.2}\\
\sum_{(i, j, k)} T^{i j k}=0  \tag{2.3}\\
T^{i j s} T_{s t}^{r}=T^{i r s} T_{s t}^{j},  \tag{2.4}\\
\nabla T^{i j k}=0  \tag{2.5}\\
\tilde{\nabla} T^{i j k}=0 \tag{2.6}
\end{gather*}
$$

Here $T^{i j k}=g^{i r} \tilde{g}^{k s} T_{r s}^{j}$ and $\nabla, \tilde{\nabla}$ are covariant derivatives given by the Levi-Civita connections of the metrics $g, \tilde{g}$.

As we have seen in Section 1.3.1, these conditions imply that, in the flat coordinates of
$g$, the second metric $\tilde{g}$ becomes linear, so that the classification of such operators reduces to the classification of algebras of certain type [35, 74]. This problem was addressed in [74], resulting in a complete description of one- and two-component operators of the form (2.1). Here we adopt a differential-geometric point of view, starting from the vanishing of the Nijenhuis torsion of the affinor $L_{j}^{i}=\tilde{g}^{i k} g_{k j}$.

In the case when $L$ has simple spectrum, (Hamiltonian operators of this type are known as non-singular or semisimple), the results of [69] imply the existence of coordinates where the Hamiltonian operator $P$ takes constant coefficient form. It turns out that all interesting (non-constant) examples correspond to the case when $L$ has non-trivial Jordan block structure. The simplest known example of this kind is provided by the twocomponent Mokhov's Hamiltonian operator

$$
P=\left(\begin{array}{ll}
0 & 1  \tag{2.7}\\
1 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
-2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{cc}
-u_{y}^{1} & 2 u_{y}^{2} \\
-u_{y}^{2} & 0
\end{array}\right)
$$

which is related to the Lie algebra of vector fields on the plane [35, 74]. It is generated by the flat contravariant metrics

$$
g=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{cc}
-2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right)
$$

One can easily see that, for generic values of $u^{1}, u^{2}$, the corresponding affinor $L=\tilde{g} g^{-1}$ is a single $2 \times 2$ Jordan block.

### 2.1 Linear Killing tensors with zero Nijenhuis torsion

In this section we rewrite Mokhov's conditions (2.2)-(2.6) in a form which is more suitable for our purposes, making link with the theory of Killing tensors.

Theorem 2.2. Let $g$ and $\tilde{g}$ be two flat metrics which define the Hamiltonian operator (2.1). The Mokhov conditions (2.2)-(2.6) are equivalent to the following:

1. Linearity of the bivector $\tilde{g}^{i j}$ in flat coordinates of $g$. Invariantly, this means $\nabla^{2} \tilde{g}=0$ where $\nabla$ denotes covariant differentiation in the Levi-Civita connection of $g$.
2. The vanishing of the Nijenhuis torsion of the affinor $L_{j}^{i}=\tilde{g}^{i l} g_{l j}$.
3. The Killing condition for the bivector $\tilde{g}$ :

$$
\begin{equation*}
\nabla^{i} \tilde{g}^{k j}+\nabla^{k} \tilde{g}^{i j}+\nabla^{j} \tilde{g}^{i k}=0 . \tag{2.8}
\end{equation*}
$$

Moreover, the flatness of $g$ and the above three conditions imply the flatness of the second metric $\tilde{g}$.

## Proof:

(a). Condition (2.2) is equivalent to the vanishing of the Nijenuis torsion of $L$.

This was proved by Mokhov [71, 72], here we briefly recall the proof. Let $\tilde{b}_{k}^{i j}=-\tilde{g}^{i s} \tilde{\Gamma}_{s k}^{j}$ be contravariant Christoffel symbols of the second metric,

$$
\tilde{b}_{k}^{i j}=-\tilde{g}^{i s} \tilde{\Gamma}_{s k}^{j}=-\frac{1}{2} \tilde{g}^{i s} \tilde{g}^{j t}\left(\partial_{s} \tilde{g}_{t k}+\partial_{k} \tilde{g}_{s t}-\partial_{t} \tilde{g}_{s k}\right)
$$

By definition they satisfy the conditions

$$
\begin{aligned}
\partial_{k} \tilde{g}^{i j} & =\tilde{b}_{k}^{i j}+\tilde{b}_{k}^{j i} \\
\tilde{g}^{i l} \tilde{b}_{l}^{j k} & =\tilde{g}^{j l} \tilde{b}_{l}^{i k} .
\end{aligned}
$$

Written in flat coordinates of $g$, the condition (2.2) reads

$$
g^{i l} \tilde{b}_{l}^{j k}=g^{j l} \tilde{b}_{l}^{i k} .
$$

This means that the contravariant Christoffel symbols of the pencil $g_{\lambda}=\tilde{g}-\lambda g$ are equal to the pencil of the Christoffel symbols of $\tilde{g}$ and $g$. Thus, the metrics $\tilde{g}$ and $g$ are almost compatible, and this is known to be equivalent to the vanishing of the Nijenhuis torsion [70].
(b). Condition (2.3) is equivalent to the Killing property.

Using (2.2) we can rewrite (2.3) as

$$
\sum_{(i, j, k)}\left[T^{i j k}+T^{k j i}\right]=0
$$

In flat coordinates of $g$ we have

$$
\sum_{(i, j, k)}\left[T^{i j k}+T^{k j i}\right]=
$$

$$
\begin{aligned}
& =\tilde{g}^{k s} g^{i r} \tilde{\Gamma}_{r s}^{j}+\tilde{g}^{i s} g^{k r} \tilde{\Gamma}_{r s}^{j}+\tilde{g}^{i s} g^{j r} \tilde{\Gamma}_{r s}^{k}+\tilde{g}^{j s} g^{i r} \tilde{\Gamma}_{r s}^{k}+\tilde{g}^{j s} g^{k r} \tilde{\Gamma}_{r s}^{i}+\tilde{g}^{k s} g^{j r} \tilde{\Gamma}_{r s}^{i} \\
& =-\left[g^{i r} \tilde{b}_{r}^{k j}+g^{k r} \tilde{b}_{r}^{i j}+g^{j r} \tilde{b}_{r s}^{i k}+g^{i r} \tilde{b}_{s}^{j k}+g^{k r} \tilde{b}_{s}^{j i}+g^{j r} \tilde{b}_{r}^{k i}\right] \\
& =-\left[g^{i r} \partial_{r} \tilde{g}^{k j}+g^{k r} \partial_{r} \tilde{g}^{i j}+g^{j r} \partial_{r} \tilde{g}^{i k}\right] \\
& =-\left[\partial^{i} \tilde{g}^{k j}+\partial^{k} \tilde{g}^{i j}+\partial^{j} \tilde{g}^{i k}\right]=0 .
\end{aligned}
$$

In invariant notation, this gives the Killing condition,

$$
\nabla^{i} \tilde{g}^{k j}+\nabla^{k} \tilde{g}^{i j}+\nabla^{j} \tilde{g}^{i k}=0,
$$

here $\nabla$ is the Levi-Civita connection of $g$.
(c). Condition (2.5) is equivalent to the linearity of $\tilde{g}$ in flat coordinates of $g$.

In flat coordinates of $g$,(2.5) implies

$$
\partial_{r}\left(T^{i j k}+T^{i k j}\right)=\partial_{r}\left[g^{i t}\left(\tilde{b}_{t}^{k j}+\tilde{b}_{t}^{j k}\right)\right]=\partial_{r} \partial^{i} \tilde{g}^{j k}=0 .
$$

This means that $\tilde{g}$ is linear. Conversely, assuming that $\tilde{g}$ is linear in flat coordinates of $g$, and using (2.2) and (2.3), we obtain (2.5):

$$
0=\partial_{r}\left(T^{i j k}+T^{i k j}\right)=\partial_{r}\left(T^{i j k}+T^{j k i}\right)=-\partial_{r} T^{k i j} .
$$

(d). Conditions (2.4) and (2.5) are equivalent to the flatness of $\tilde{g}$.

Condition (2.5) means that, in flat coordinates of $g$, the contravariant Christoffel symbols $\tilde{b}_{k}^{i j}$ are constant. This follows from the identity

$$
-\partial_{r} T^{k i j}=\partial_{r}\left(g^{k m} \tilde{b}_{m}^{i j}\right)=g^{k m} \partial_{r} \tilde{b}_{m}^{i j}=0 .
$$

Similarly, condition (2.6) means that, in flat coordinates of $\tilde{g}$, the contravariant Christoffel symbols $b_{k}^{i j}$ are constant. Written in flat coordinates of $g$, the condition (2.4) reads

$$
\tilde{g}^{s q} g^{i p} \tilde{\Gamma}_{p q}^{j} \tilde{\Gamma}_{s t}^{r}=\tilde{g}^{s q} g^{i p} \tilde{\Gamma}_{p q}^{r} \tilde{\Gamma}_{s t}^{j},
$$

or

$$
g^{i p}\left(\tilde{b}_{p}^{s j} \tilde{\Gamma}_{s t}^{r}-\tilde{b}_{p}^{s s} \tilde{\Gamma}_{s t}^{j}\right)=0,
$$

that is

$$
\tilde{b}_{p}^{s j} \tilde{\Gamma}_{s t}^{r}-\tilde{b}_{p}^{s r} \tilde{\Gamma}_{s t}^{j}=0,
$$

which is equivalent to

$$
\tilde{b}_{p}^{s j} \tilde{b}_{s}^{i r}-\tilde{b}_{p}^{s \tau_{b}^{i j}}=0
$$

Due to (2.5), the vanishing of the curvature of $\tilde{\nabla}$, written in flat coordinates of $g$, reads

$$
g^{i s}\left(\partial_{s} \tilde{b}_{p}^{j r}-\partial_{p} \tilde{b}_{s}^{j r}\right)-\tilde{b}_{s}^{i j} \tilde{b}_{p}^{s r}+\tilde{b}_{s}^{i r} \tilde{b}_{p}^{s j}=-\tilde{b}_{s}^{i j} \tilde{b}_{p}^{s r}+\tilde{b}_{s}^{i r} \tilde{b}_{p}^{s j}=0 .
$$

(e). Condition (2.6) can be dropped.

Recall that, in flat coordinates of $g$, we have $T^{l k j}=-g^{l m} \tilde{b}_{m}^{k j}$ and $T_{j k}^{i}=\tilde{\Gamma}_{j k}^{i}=\tilde{\Gamma}_{k j}^{i}=T_{k j}^{i}$ (by the symmetry of $\tilde{\nabla}$ ). Thus,

$$
\begin{aligned}
\tilde{\nabla}_{r} T^{i j k} & =\tilde{\nabla}_{r}\left(-g^{i m} \tilde{b}_{m}^{k j}\right)=-\left(\tilde{\nabla}_{r} g^{i m} \tilde{b}_{m}^{k j}+g^{i m} \tilde{\nabla}_{r} \tilde{b}_{m}^{k j}\right) \\
& =-\tilde{b}_{m}^{k j}\left(\partial_{r} g^{i m}+\tilde{\Gamma}_{r l}^{i} g^{l m}+\tilde{\Gamma}_{r l}^{m} g^{l i}\right)-g^{i m}\left(\partial_{r} \tilde{b}_{m}^{k j}+\tilde{\Gamma}_{r l}^{k} \tilde{b}_{m}^{l j}+\tilde{\Gamma}_{r l}^{j} \tilde{b}_{m}^{k l}-\tilde{\Gamma}_{r m}^{l} \tilde{b}_{l}^{k j}\right) \\
& =-\tilde{b}_{m}^{k j} \tilde{\Gamma}_{r l}^{i} g^{l m}-\tilde{b}_{m}^{k j} \tilde{P}_{r l}^{m} g^{k i}-g^{i m} \tilde{\Gamma}_{r l}^{k} \tilde{b}_{m}^{l j}-g^{i m} \tilde{\Gamma}_{r l}^{j} \tilde{b}_{m}^{k l}+g^{i m} \tilde{\Gamma}_{r m}^{l} \tilde{b}_{l}^{k j} \\
& =-\left(\tilde{\Gamma}_{r l}^{i} g^{l m} \tilde{b}_{m}^{k j}+\tilde{\Gamma}_{r l}^{k} g^{i m} \tilde{b}_{m}^{l j}+\tilde{\Gamma}_{r l}^{j} g^{i m} \tilde{b}_{m}^{k l}\right) \\
& =-\left(T_{r l}^{i} T^{l k j}+T_{r l}^{k} T^{i l j}+T_{r l}^{j} T^{i k l}\right) .
\end{aligned}
$$

Using conditions (2.2), (2.3) and (2.4), this term vanishes. Indeed, by (2.2) it reads

$$
\tilde{\nabla}_{r} T^{i j k}=-\left(\underline{T_{r l}^{i}} T^{j k l}+T_{r l}^{k} T^{i l j}+\underline{T_{r l}^{j} T^{i k l}}\right) .
$$

Using (2.4) for the underlined terms we obtain

$$
\tilde{\nabla}_{r} T^{i j k}=-T_{r l}^{k}\left(T^{j i l}+T^{i l j}+T^{i j l}\right),
$$

and, by (2.3)

$$
\tilde{\nabla}_{r} T^{i j k}=-T_{r l}^{k}\left(-T^{l j i}+T^{i j l}\right) .
$$

The last term vanishes by (2.2).
(f). The flatness of $\tilde{g}$ follows from the flatness of $g$, linearity of $\tilde{g}$, the Killing condition, and the vanishing of the Nijenhuis torsion.

Since $\tilde{b}_{j k}^{i}$ are constant in flat coordinates of $g$ (see step (c)), from the condition $\partial_{k} \tilde{g}^{i j}=$ $\tilde{b}_{k}^{i j}+\tilde{b}_{k}^{j i}$, it follows immediately

$$
\begin{equation*}
\tilde{g}^{i j}=\left(\tilde{b}_{l}^{i j}+\tilde{b}_{l}^{j i}\right) u^{l}+g_{0}^{i j} . \tag{2.9}
\end{equation*}
$$

Thus in flat coordinates of $g$ the condition $\tilde{g}^{i l} \tilde{b}_{l}^{j k}=\tilde{g}^{j} l \tilde{b}_{l}^{i k}$ implies

$$
\begin{equation*}
\left(\tilde{b}_{m}^{i l}+\tilde{b}_{m}^{l i}\right) \tilde{b}_{l}^{j k}=\left(\tilde{b}_{m}^{j l}+\tilde{b}_{m}^{l j}\right) \tilde{\tilde{b}}_{l}^{i k} . \tag{2.10}
\end{equation*}
$$

Moreover, the Killing condition

$$
g^{i s}\left(\underline{\tilde{b}_{s}^{k j}}+\underline{\tilde{b}_{s}^{j k}}\right)+g^{k s}\left(\tilde{b}_{s}^{i j}+\underline{\tilde{b}_{s}^{j i}}\right)+g^{j s}\left(\tilde{b}_{s}^{i k}+\tilde{b}_{s}^{k i}\right)=0,
$$

can be rewritten, using (2.2) for the underlined terms, as

$$
\begin{aligned}
0=g^{k s} \tilde{b}_{s}^{i j} & +g^{j s} \tilde{b}_{s}^{i k}+g^{k s} \tilde{b}_{s}^{i j}+g^{j s} \tilde{b}_{s}^{k i}+g^{j s}\left(\tilde{b}_{s}^{i k}+\tilde{b}_{s}^{k i}\right) \\
& =2 g^{k s} \tilde{b}_{s}^{i j}+2 g^{j s} \tilde{b}_{s}^{i k}+2 g^{j s} \tilde{b}_{s}^{k i} \\
& =2\left(g^{k s} \tilde{b}_{s}^{i j}+g^{j s}\left(\tilde{b}_{s}^{i k}+\tilde{b}_{s}^{k i}\right)\right),
\end{aligned}
$$

that is

$$
\begin{equation*}
\tilde{b}_{s}^{i j} g^{s k}+\left(\tilde{b}_{s}^{k i}+\tilde{b}_{s}^{i k}\right) g^{s j}=0 \tag{2.11}
\end{equation*}
$$

Taking into account the above condition (2.11), the equation (2.10) becomes

$$
g_{m r}\left(\underline{g^{s l} \tilde{b}_{l}^{i r} \tilde{b}_{s}^{j k}}-g^{j s} \tilde{b}_{s}^{l r} \tilde{b}_{l}^{i k}\right)=0 .
$$

Using (2.2) for the underlined terms, we finally get

$$
g_{m r} g^{j s}\left(\tilde{b}_{s}^{l k} \tilde{b}_{l}^{i r}-\tilde{b}_{s}^{l r} \tilde{b}_{l}^{i k}\right)=0
$$

In what follows we will need an alternative form of the Killing condition (2.8), namely

$$
\begin{equation*}
g^{i s} \partial_{s} \tilde{g}^{k j}+g^{k s} \partial_{s} \tilde{g}^{i j}+g^{j s} \partial_{s} \tilde{g}^{i k}-\tilde{g}^{i s} \partial_{s} g^{k j}-\tilde{g}^{k s} \partial_{s} g^{i j}-\tilde{g}^{j s} \partial_{s} g^{i k}=0 . \tag{2.12}
\end{equation*}
$$

This can be easily obtained: computing the covariant derivative of $\tilde{g}^{i j}$ we get

$$
\begin{gather*}
g^{k s} \nabla_{s} \tilde{g}^{i j}=g^{k s}\left(\partial_{s} \tilde{g}^{i j}+\Gamma_{s m}^{i} \tilde{g}^{m j}+\Gamma_{s m}^{j} \tilde{g}^{i m}\right) \\
=g^{k s} \partial_{s} \tilde{g}^{i j}-b_{m}^{k i} \tilde{g}^{m j}-b_{m}^{k j} \tilde{g}^{i m} \tag{2.13}
\end{gather*}
$$

Using $\partial_{s} g^{i j}=b_{s}^{i j}+b_{s}^{j i}$ and substituting (2.13) into the Killing condition (2.8), one arrives at (2.12).

### 2.2 The splitting lemma

The Killing condition plays a key role in the proof of the splitting property for Hamiltonian operators. First of all, let us give a definition.

Definition 2.1. Given an m-component operator $P_{1}$ with the dependent variables $u^{1}, \ldots, u^{m}$, and an $(n-m)$-component operator $P_{2}$ with the dependent variables $v^{m+1}, \ldots, v^{n}$, their direct sum is the $n$-component operator $P$ defined by the formula

$$
P=\left(\begin{array}{cc}
P_{1} & 0 \\
0 & P_{2}
\end{array}\right)
$$

on the combined set of variables $\left(u^{1}, \ldots, u^{m}, v^{m+1}, \ldots, v^{n}\right)$. The corresponding metrics $g, \tilde{g}$ will be direct sums of the metrics defining $P_{1}$ and $P_{2}$. Operators of this type will be called reducible. The main result of this section can be stated as follows.

Lemma 2.3 (Splitting Lemma for Hamiltonian operators). Let $P$ be a Hamiltonian operator such that the corresponding affinor $L=\tilde{g} g^{-1}$ can be represented in the block-diagonal form

$$
L=\left(\begin{array}{ll}
A & 0  \tag{2.14}\\
0 & B
\end{array}\right)
$$

in some (non-holonomic) frame, and let $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)=\emptyset$. Then $P$ decouples into a direct sum of two Hamiltonian operators, with the corresponding affinors $A$ and $B$.

Thus, any Hamiltonian operator (2.1) can be represented as a direct sum of irreducible operators $P_{\alpha}$ (each generated by a pair of flat metrics $g_{\alpha}, \tilde{g}_{\alpha}$, defined on a manifold of dimension $n_{\alpha}$ ) such that the corresponding affinor $L_{\alpha}=\tilde{g}_{\alpha} g_{\alpha}^{-1}$ either has a unique real
eigenvalue of multiplicity $n_{\alpha}$, or a pair of complex conjugate eigenvalues of the same multiplicity (in the last case $n_{\alpha}$ must be even).

Lemma 2.3 can be seen as an analogue of the splitting lemma for affinors with zero Nijenhuis torsion proved by Bolsinov and Matveev [12] in the context of projectively equivalent metrics.

Lemma 2.4 (Splitting Lemma, [12]). Let L be an affinor with zero Nijenhuis torsion on a manifold $M, \operatorname{dim} M=n$. Suppose there exists a (non-holonomic) frame in which $L$ takes block diagonal form,

$$
L=\left(\begin{array}{ll}
A & 0 \\
0 & B
\end{array}\right)
$$

where $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)=\emptyset$. Then there exists a local coordinate system $\left(u^{1}, \ldots, u^{m}, v^{m+1}, \ldots\right.$, $\left.v^{n}\right)$ such that

$$
L=\left(\begin{array}{cc}
A(\mathbf{u}) & 0 \\
0 & B(\mathbf{v})
\end{array}\right) .
$$

Using the Killing condition, one can extend the splitting structure to the metrics. First of all we recall two well-known facts from linear algebra.

1. In the hypothesis of the above lemma, if $g$ and $\tilde{g}$ are two non-degenerate symmetric bivectors related by the affinor $L$, that is $\tilde{g}^{i j}=L_{k}^{j} g^{k i}$, then $g$ and $\tilde{g}$ assume the form

$$
g=\left(\begin{array}{cc}
\sigma & 0  \tag{2.15}\\
0 & \eta
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{cc}
\tilde{\sigma} & 0 \\
0 & \tilde{\eta}
\end{array}\right) .
$$

2. Let $A$ and $B$ be two square matrices (not necessarily of the same size), such that $\operatorname{Spec}(A) \cap \operatorname{Spec}(B)=\emptyset$. Suppose that $A C=C B$ for a certain matrix $C$. Then $C=0$. For convenience of the reader, let us briefly prove this last statement. Let us bring $A$ to upper triangular form, that is $A=X \Lambda X^{-1}$ where

$$
\Lambda=\left(\begin{array}{ccc}
\lambda^{1} & & * \\
& \ddots & \\
0 & & \lambda^{m}
\end{array}\right)
$$

and $\lambda^{i}$ are the eigenvalues of $A$. Then

$$
X \Lambda X^{-1} C=C B \quad \Rightarrow \quad \Lambda X^{-1} C=X^{-1} C B
$$

Let $X^{-1} C=\tilde{C}$. Thus

$$
\begin{equation*}
\Lambda \tilde{C}=\tilde{C} B \tag{2.16}
\end{equation*}
$$

where $\Lambda$ is upper triangular and $\lambda^{i} \in \operatorname{Spec}(A)$. Let $\tilde{C}_{1}, \ldots, \tilde{C}_{m}$ be the rows of $\tilde{C}$. Comparing the $m^{\prime}$ th rows in (2.16) we get

$$
\lambda^{m} \tilde{C}_{m}=\tilde{C}_{m} B .
$$

Since $\lambda^{m} \notin \operatorname{Spec}(B)$, then $\tilde{C}_{m} \equiv 0$. Comparing the ( $m-1$ )'th rows in (2.16) we get (since $\tilde{C}_{m}=0$ )

$$
\lambda^{m-1} \tilde{C}_{m-1}=\tilde{C}_{m-1} B,
$$

so that $\tilde{C}_{m-1} \equiv 0$. By induction, this implies $\tilde{C} \equiv 0$, so that $C \equiv 0$ as $X$ is non-degenerate.
Lemma 2.5. In the hypothesis of Lemma 2.4, let $g$ and $\tilde{g}$ be two non-degenerate symmetric bivectors (2.15) such that $\tilde{g}^{i j}=L_{k}^{j} g^{k i}$. If the Killing condition (2.8) holds, then $\sigma, \tilde{\sigma}$ must depend only on $\mathbf{u}=\left(u^{1}, \ldots, u^{m}\right)$, and $\eta, \tilde{\eta}$ must depend only on $\mathbf{v}=\left(v^{m+1}, \ldots, v^{n}\right)$.

## Proof:

By Lemma 2.4, $A=A(\mathbf{u})$ is an $m \times m$ matrix, and $B=B(\mathbf{v})$ is an $(n-m) \times(n-m)$ matrix. Let $I=\{1, \ldots, m\}$ and $J=\{m+1, \ldots, n\}$. We know that if $i \in I$ and $j \in J$, then $g^{i j}=0$. Then, for $i \in I$ and $j, k \in J$, the condition (2.12) leads to

$$
g^{i s} \partial_{s} \tilde{g}^{k j}-\tilde{g}^{i s} \partial_{s} g^{k j}=0
$$

in particular,

$$
\sigma^{i s} \partial_{s} \tilde{\eta}^{k j}-\tilde{\sigma}^{i s} \partial_{s} \eta^{k j}=0
$$

By hypothesis $g$ is non-degenerate, thus $\sigma$ is non-degenerate, then multiplying by the inverse matrix $\sigma_{l i}$ we obtain

$$
\partial_{l}\left(B_{p}^{k} \eta^{p j}\right)-A_{l}^{s} \partial_{s} \eta^{k j}=0,
$$

as $\tilde{\sigma}^{s i} \sigma_{i l}=A_{l}^{s}$. Since $l \in I$ and the elements of $B$ depend on $\mathbf{v}$ only, our relation becomes

$$
B_{p}^{k} \partial_{l} \eta^{p j}-A_{l}^{s} \partial_{s} \eta^{k j}=0 .
$$

Fixing $j$, let $C_{i}^{k}=\partial_{i} \eta^{k j}$. Thus, we get $B_{p}^{k} C_{l}^{p}=C_{s}^{k} A_{l}^{s}$, that is $B C=C A$. As $\operatorname{Spec}(A) \cap$ $\operatorname{Spec}(B)=\emptyset$, we can conclude that $C \equiv 0$. Thus

$$
\partial_{i} \eta^{j k}=0, \forall i \in I, \forall j, k \in J .
$$

If we now take $i \in J$ and $j, k \in I$, following the same method we get

$$
\partial_{i} \sigma^{j k}=0, \forall i \in J, \forall j, k \in I .
$$

This establishes Lemma 2.3 (Splitting Lemma for Hamiltonian operators). It allows us to focus on affinors with one single eigenvalue, otherwise we can split them and consider each block separately.

As a simple application of Lemma 2.3 we can establish Darboux's theorem for Hamiltonian operators whose affinor $L$ is diagonal (has no non-trivial Jordan blocks: note that we allow coinciding eigenvalues). It is based on the following result:

Proposition 2.6. Let $L$ be a diagonal affinor, $g$ be a flat contravariant metric, and $\tilde{g}=L g$. Suppose that the Nijenhuis torsion of L vanishes, and the Killing condition holds. Then there exists a coordinate system where $L$ and $g$ take constant coefficient form.

## Proof:

Since the Nijenhuis torsion of $L$ vanishes, using Lemma 2.4 we can bring $L$ to block diagonal form,

$$
L=\left(\begin{array}{llll}
L_{1} & & & \\
& L_{2} & & \\
& & \ddots & \\
& & & L_{k}
\end{array}\right)
$$

Here each $L_{i}$ is a scalar operator with the same eigenvalue,

$$
L_{i}=\left(\begin{array}{lll}
\lambda^{i} & & \\
& \ddots & \\
& & \lambda^{i}
\end{array}\right)
$$

$\lambda^{i} \neq \lambda^{j}$ for $i \neq j$, and $\lambda^{i}$ depends on coordinates of its own block only. By Lemma 2.5, we have

$$
g=\left(\begin{array}{ccc}
g_{\lambda^{1}} & & \\
& \ddots & \\
& & g_{\lambda^{k}}
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{lll}
\lambda^{1} g_{\lambda^{1}} & & \\
& \ddots & \\
& & \lambda^{k} g_{\lambda^{k}}
\end{array}\right)
$$

where $g_{\lambda^{i}}$ depends on coordinates of its own block only. Thus we can consider each block separately. For instance, suppose $L_{1}$ is an $m \times m$ scalar operator with the eigenvalue $\lambda^{1}$. Let us set $\lambda=\lambda^{1}$ and $h=g_{\lambda^{1}}$. We know that $\lambda$ and $h$ depend on $u^{1}, \ldots, u^{m}$ only, and no other block depends on these coordinates. Condition (2.12) leads to

$$
h^{k j} h^{i s} \partial_{s} \lambda+h^{j i} h^{k s} \partial_{s} \lambda+h^{i k} h^{j s} \partial_{s} \lambda=0 .
$$

Since $h$ is non-degenerate, contracting with $h_{q i} h_{p j}$ we get

$$
\delta_{p}^{k} \partial_{q} \lambda+h_{p q} h^{k s} \partial_{s} \lambda+\delta_{q}^{k} \partial_{p} \lambda=0 .
$$

Setting $q=k$ and summing over $k$ we obtain

$$
\partial_{p} \lambda+h_{p k} h^{k s} \partial_{s} \lambda+m \partial_{p} \lambda=0 \quad \Rightarrow \quad(m+2) \partial_{p} \lambda=0 .
$$

Thus $\lambda$ must be constant, as $m>0$. Since $g$ is flat, we can find a change of coordinates which brings $h$ to constant form. As $L_{1}$ is a constant scalar operator, it retains its form in any coordinate system. Similarly $\lambda^{i}$ and $g_{\lambda^{i}}$ can be reduced to constant form.

This leads to the following
Theorem 2.7. Consider a non-degenerate Hamiltonian operator (2.1) such that the affinor $L_{j}^{i}=$ $\tilde{g}^{i k} g_{k j}$ has (pointwise) diagonal Jordan normal form. Then this operator can be reduced to constant coefficient form by a local change of coordinates.

This extends the analogous result of Mokhov [69] obtained under the additional as-
sumption of simplicity of the spectrum of $L$.
Suppose that $g$ has Euclidean signature (or, more generally, there exists a non-degenerate Euclidean combination of the form $\lambda g+\mu \tilde{g})$. Then the affinor $L$ can be brought to diagonal form. By Theorem 2.7 we have

Corollary 2.8. If one of the contravariant metrics which define a $2 D$ Hamiltonian operator is Euclidean, then the operator can be reduced to constant coefficient form.

This shows that the most interesting case is when each representative of the pencil $\lambda g+\mu \tilde{g}$ is essentially pseudo-Euclidean, and the affinor $L$ has non-trivial Jordan block structure.

### 2.3 Classification results

In this section we classify Hamiltonian operators of type (2.1) with the number of components $n \leq 4$. This will be done up to arbitrary transformations of the dependent variables $u^{i}$. Our approach is based on the following two fundamental facts:

1. Any Killing bivector in flat space is the sum of symmetrized tensor products of Killing vectors (see, e.g. [94, 95, 68]);
2. A pair of symmetric bivectors can be brought to the Segre normal form [89] (see, e.g. [58] for a modern description).

We recall that (see Theorem 1.5) the first metric $g$ can always be reduced to constant form, and the second one must be linear, that is

$$
\begin{equation*}
\tilde{g}=c_{k}^{i j} u^{k}+g_{0}^{i j} \tag{2.17}
\end{equation*}
$$

here $g$ and $g_{0}$ are constant symmetric matrices, and $c_{k}^{i j}$ are constant coefficients. Taking 'generic' values $u_{0}^{k}$ of the variables $u^{k}$ and applying the shift of variables,

$$
u^{k} \rightarrow u_{0}^{k}+v^{k},
$$

we obtain the transformed metric,

$$
\begin{equation*}
\tilde{g}=c_{k}^{i j} v^{k}+\tilde{g}_{0}^{i j} \tag{2.18}
\end{equation*}
$$

The genericity of $u_{0}^{k}$ allows us to assume that the Segre type of the pair $(g, \tilde{g})$ is the same as that of $\left(g, \tilde{g}_{0}\right)$. Recall that the Segre type of a pair of symmetric forms can be read off the Jordan normal form of the corresponding affinor $L$, see below. Bringing $g$ and $\tilde{g}_{0}$ to the Segre normal form leads to a considerable simplification of calculations. Furthermore, the splitting lemma allows us to consider irreducible cases only, where the affinor $L$ either has one real eigenvalue, or two complex conjugate eigenvalues.

The theory of normal forms of pairs of symmetric bilinear forms is based on the following result, see e.g. [58]:

Theorem 2.9. Suppose $L$ is a $g$-selfadjoint operator on a real vector space $V$. There exist a canonical basis $e_{1}, \ldots, e_{n} \in V$ in which $L$ and $g$ can be simultaneously reduced to the following block diagonal canonical forms:

$$
L_{c a n}=\left(\begin{array}{llll}
L_{1} & & & \\
& L_{2} & & \\
& & \ddots & \\
& & & L_{s}
\end{array}\right), \quad g_{c a n}=\left(\begin{array}{llll}
g_{1} & & & \\
& g_{2} & & \\
& & \ddots & \\
& & & g_{s}
\end{array}\right) \text {, }
$$

where

$$
g_{j}= \pm\left(\begin{array}{llll} 
& & & 1 \\
& & 1 & \\
& . & & \\
1 & & &
\end{array}\right)
$$

and

$$
L_{j}=\left(\begin{array}{cccc}
\lambda^{j} & 1 & & \\
& \lambda^{j} & \ddots & \\
& & \ddots & 1 \\
& & & \lambda^{j}
\end{array}\right)
$$

in the case of real eigenvalues $\lambda^{j} \in \mathbb{R}$ (real Jordan block), or

$$
L_{j}=\left(\begin{array}{ccccccccc}
a & b & 1 & 0 & & & & & \\
-b & a & 0 & 1 & & & & & \\
& & a & b & & & & & \\
& & -b & a & & \ddots & & & \\
& & & & \ddots & & & & \\
& & & & & a & b & 1 & 0 \\
& & & & & & -b & a & 0 \\
& & & & & & \\
& & & & & & & a & b \\
& & & & & & & -b & a
\end{array}\right)
$$

in the case of complex conjugate eigenvalues $\lambda_{1,2}^{j}=a \pm i b$ (complex Jordan block). It is assumed that for each $j$ the blocks $g_{j}$ and $L_{j}$ are of the same size.

Remark. Let us briefly comment on what we mean by Segre type. Suppose $n=4$ and let us consider the affinor $L=\tilde{g} g^{-1}$. In the case of two complex conjugate eigenvalues $\nu+i \lambda$ and $\nu-i \lambda$, the canonical form of $L$ reads

$$
\left(\begin{array}{cccc}
\nu & -\lambda & 1 & 0 \\
\lambda & \nu & 0 & 1 \\
0 & 0 & \nu & -\lambda \\
0 & 0 & \lambda & \nu
\end{array}\right)
$$

In the case of a single real eigenvalue we have the following four canonical forms:

$$
\begin{aligned}
& \left(\begin{array}{llll}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda
\end{array}\right),\left(\begin{array}{llll}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{array}\right),\left(\begin{array}{llll}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda
\end{array}\right),\left(\begin{array}{llll}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{array}\right) . \\
& \text { Segre type }[4] \quad \text { Segre type }[(3,1)] \quad \text { Segre type }[(2,2)] \quad \text { Segre type }[(2,1,1)]
\end{aligned}
$$

Segre type indicates the number and sizes of Jordan blocks with the same eigenvalue $\lambda$.

### 2.3.1 One-component case

It was shown in $[32,74]$ that any one-component operator can be reduced to constant coefficient form, $P=\lambda \partial_{x}+\mu \partial_{y}$, here $\lambda$ and $\mu$ are arbitrary constants.

### 2.3.2 Two-component case

The two-component situation is also understood completely [35, 74]: we have only one non-constant Hamiltonian operator (2.7), the corresponding affinor $L$ is a single Jordan block with non-constant eigenvalue:

$$
P=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
-2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{cc}
-u_{y}^{1} & 2 u_{y}^{2} \\
-u_{y}^{2} & 0
\end{array}\right)
$$

Let us give an alternative proof of this result based on the Killing condition. First we reduce $g$ to flat coordinates,

$$
g=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

recall that, by Corollary 2.8, $g$ must be Lorentzian. Since $\tilde{g}$ is a Killing tensor of $g$, it is a quadratic expression in the isometries $u^{1} \partial_{1}-u^{2} \partial_{2}, \partial_{1}, \partial_{2}$. Since $\tilde{g}$ is linear, the first isometry can only enter linearly, so that

$$
\tilde{g}=\left(u^{1} \partial_{1}-u^{2} \partial_{2}\right)\left(\alpha \partial_{1}+\beta \partial_{2}\right)+\gamma \partial_{1}^{2}+2 \delta \partial_{1} \partial_{2}+\epsilon \partial_{2}^{2}
$$

here $\alpha, \beta, \gamma, \delta, \epsilon$ are arbitrary constants. The vanishing of the Nijenhuis torsion of the corresponding affinor $L$ gives

$$
\left(\alpha u^{1}+\gamma\right) \beta=0, \quad\left(\beta u^{2}-\epsilon\right) \alpha=0
$$

Without any loss of generality one can take $\beta=0$. In this case $\alpha$ must be nonzero, otherwise $\tilde{g}$ will have constant coefficients. Then $\epsilon=0$, and modulo translations of $u^{1}, u^{2}$ we arrive at the required expression (2.7).

### 2.3.3 Three-component case

Our main result can be summarised as follows.

Theorem 2.10. Any irreducible non-constant three-component Hamiltonian operator in 2D can be brought (by a change of the dependent variables $u^{i}$ ) to the form $\pm P$ where $P$ can have one of the two following canonical forms (in both cases the affinor $L$ is a single $3 \times 3$ Jordan block):

1. Jordan block with constant eigenvalue

$$
P=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
-2 u^{2} & u^{3} & \lambda \\
u^{3} & \lambda & 0 \\
\lambda & 0 & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{ccc}
-u_{y}^{2} & 2 u_{y}^{3} & 0 \\
-u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

2. Jordan block with non-constant eigenvalue

$$
P=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{ccc}
-u_{y}^{1} & \frac{1}{2} u_{y}^{2} & 2 u_{y}^{3} \\
-u_{y}^{2} & \frac{1}{2} u_{y}^{3} & 0 \\
-u_{y}^{3} & 0 & 0
\end{array}\right) .
$$

## Proof:

Since the complex conjugate case cannot occur (it requires an even number of components), we only need to consider the cases where the affinor $L$ has one triple eigenvalue, and has Segre type $[3]$ or $[(2,1)]$. Since the case $[(2,1)]$ gives no non-constant irreducible examples, we will concentrate on Segre type [3]. Then there exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{ccc}
0 & 1 & \lambda \\
1 & \lambda & 0 \\
\lambda & 0 & 0
\end{array}\right) .
$$

The general solution of Mokhov's conditions is given by the two-parameter family $\tilde{g}=$ $\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{2}+\tilde{g}_{0}$, where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\tilde{g}_{1}=\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{ccc}
-2 u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

In the non-constant eigenvalue case, $\kappa_{1} \neq 0$, using the following transformations which
preserve both $g$ and $\tilde{g}_{0}$,

$$
\begin{aligned}
u^{1} & \rightarrow \frac{1}{\kappa_{1}} u^{1}+\frac{2 \kappa_{2}}{\kappa_{1}^{2}} u^{2}-\frac{2 \kappa_{2}^{2}}{\kappa_{1}^{3}} u^{3}-\frac{2 \kappa_{2}}{\kappa_{1}^{3}} \\
u^{2} & \rightarrow u^{2}-\frac{2 \kappa_{2}}{\kappa_{1}} u^{3}+\frac{2 \kappa_{1}-2}{\kappa_{1}} \\
u^{3} & \rightarrow \kappa_{1} u^{3}
\end{aligned}
$$

we can reduce the above family to $\tilde{g}=\tilde{g}_{1}+\tilde{g}_{0}$ (that is, we can set $\kappa_{1}=1, \kappa_{2}=0$ ). After that we can eliminate $\tilde{g}_{0}$ by appropriate translations of $u^{2}$ and $u^{3}$, arriving at the final answer $\tilde{g}=\tilde{g}_{1}$. Similarly, in the constant eigenvalue case, $\kappa_{1}=0$, we can set $\kappa_{2}=1$, and use an appropriate translations of $u^{3}$ to arrive at the normal form above. In detail, if $\kappa_{2}>0$, the transformation given by

$$
u^{1} \rightarrow \alpha u^{1}, \quad u^{2} \rightarrow u^{2}, \quad u^{3} \rightarrow \alpha^{-1} u^{3}+\alpha-1, \quad \alpha=\kappa_{2}^{-\frac{1}{2}}
$$

preserves $g$ and $\tilde{g}$ and implies $\tilde{g}=\tilde{g}_{1}+\tilde{g}_{0}$. Using a shift of $u^{3}$, we finally get the normal form above. If $\kappa_{2}<0$, it is sufficient to choose $\alpha=\left(-\kappa_{2}\right)^{-\frac{1}{2}}$ in the above transformation, and then applying a shift of $u^{3}$ we get the same result.

### 2.3.4 Four-component case

The four-component situation is more complicated since we have more Segre types. In this section we present the results of classification of four-component Hamiltonian operators of the form (2.1) with one real eigenvalue, as well as with two complex conjugate eigenvalues (the latter turn out to be complexifications of the $2 \times 2$ operator (2.7)). We will only give canonical forms for the contravariant metrics $g$, $\tilde{g}$ : the symbols $\tilde{b}_{k}^{i j}$ of the second metric can be computed directly. We skip the details of calculations: these follow the procedure outlined at the beginning of Section 2.3, and are essentially the same as in the proof of Theorem 2.10.

## Segre type [(2,1,1)]

One can show that this case leads to constant coefficient operators.

## Segre type [(2,2)]

By Theorem 2.9, we have to consider two different cases.
Case 1: There exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
1 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & 1 & \lambda \\
0 & 0 & \lambda & 0
\end{array}\right)
$$

The general solution of Mokhov's conditions is given by $\tilde{g}=\sum_{i=1}^{4} \kappa_{i} \tilde{g}_{i}+\tilde{g}_{0}$, where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\begin{gathered}
\tilde{g}_{1}=\left(\begin{array}{cccc}
u^{1} & -\frac{1}{2} u^{2} & \frac{1}{2} u^{3} & 0 \\
-\frac{1}{2} u^{2} & 0 & 0 & 0 \\
\frac{1}{2} u^{3} & 0 & 0 & -\frac{1}{2} u^{2} \\
0 & 0 & -\frac{1}{2} u^{2} & 0
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
u^{4} & 0 & -\frac{1}{2} u^{2} & 0 \\
0 & 0 & 0 & 0 \\
-\frac{1}{2} u^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \\
\tilde{g}_{3}=\left(\begin{array}{cccc}
0 & \frac{1}{2} u^{4} & -\frac{1}{2} u^{1} & 0 \\
\frac{1}{2} u^{4} & 0 & 0 & 0 \\
-\frac{1}{2} u^{1} & 0 & -u^{3} & \frac{1}{2} u^{4} \\
0 & 0 & \frac{1}{2} u^{4} & 0
\end{array}\right), \quad \tilde{g}_{4}=\left(\begin{array}{cccc}
0 & 0 & \frac{1}{2} u^{4} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} u^{4} & 0 & -u^{2} & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
\end{gathered}
$$

The eigenvalue of the corresponding affinor $L$ is $\frac{1}{2}\left(\kappa_{3} u^{4}-\kappa_{1} u^{2}\right)+\lambda$. Using symmetries which preserve $g$ and $\tilde{g}_{0}$ one can set the coefficients $\kappa_{3}$ and $\kappa_{4}$ equal to zero, arriving at the normal form

$$
\tilde{g}=\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{2}+\tilde{g}_{0}
$$

Case 2: There exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
1 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & -1 & -\lambda \\
0 & 0 & -\lambda & 0
\end{array}\right) .
$$

The eigenvalue of the corresponding affinor $L$ is $\frac{1}{2}\left(\kappa_{3} u^{4}-\kappa_{1} u^{2}\right)+\lambda$. The general solution
of Mokhov's conditions is given by $\tilde{g}=\sum_{i=1}^{4} \kappa_{i} \tilde{g}_{i}+\tilde{g}_{0}$, where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\begin{aligned}
& \tilde{g}_{1}=\left(\begin{array}{cccc}
u^{1} & -\frac{1}{2} u^{2} & \frac{1}{2} u^{3} & 0 \\
-\frac{1}{2} u^{2} & 0 & 0 & 0 \\
\frac{1}{2} u^{3} & 0 & 0 & \frac{1}{2} u^{2} \\
0 & 0 & \frac{1}{2} u^{2} & 0
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
u^{4} & 0 & \frac{1}{2} u^{2} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} u^{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \\
& \tilde{g}_{3}=\left(\begin{array}{cccc}
0 & \frac{1}{2} u^{4} & \frac{1}{2} u^{1} & 0 \\
\frac{1}{2} u^{4} & 0 & 0 & 0 \\
\frac{1}{2} u^{1} & 0 & u^{3} & -\frac{1}{2} u^{4} \\
0 & 0 & -\frac{1}{2} u^{4} & 0
\end{array}\right), \quad \tilde{g}_{4}=\left(\begin{array}{cccc}
0 & 0 & \frac{1}{2} u^{4} & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} u^{4} & 0 & u^{2} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{aligned}
$$

Using symmetries which preserve $g$ and $\tilde{g}_{0}$ one can reduce the above four-parameter family to one of the following normal forms:

$$
\tilde{g}=\tilde{g}_{0}+\left\{\begin{array}{l}
\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{4} \\
\kappa_{1} \tilde{g}_{2}+\kappa_{2} \tilde{g}_{3} \\
\tilde{g}_{2} \pm \tilde{g}_{4} \\
\tilde{g}_{1} \pm \tilde{g}_{3}+\kappa_{1} \tilde{g}_{4}
\end{array} \quad \kappa_{1}, \kappa_{2}=\right.\text { const }
$$

## Segre type [(3,1)]

Here we also have two different cases.
Case 1: There exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
0 & 1 & \lambda & 0 \\
1 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda
\end{array}\right)
$$

The general solution of Mokhov's conditions is given by $\tilde{g}=\sum_{i=1}^{4} \kappa_{i} \tilde{g}_{i}+\tilde{g}_{0}$, where $\kappa_{i}$ are
arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\begin{aligned}
& \tilde{g}_{1}=\left(\begin{array}{cccc}
2 u^{1} & \frac{1}{2} u^{2} & -u^{3} & \frac{1}{2} u^{4} \\
\frac{1}{2} u^{2} & -u^{3} & 0 & 0 \\
-u^{3} & 0 & 0 & 0 \\
\frac{1}{2} u^{4} & 0 & 0 & -u^{3}
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
u^{2} & -\frac{1}{2} u^{3} & 0 & 0 \\
-\frac{1}{2} u^{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \\
& \tilde{g}_{3}=\left(\begin{array}{cccc}
u^{4} & 0 & 0 & -\frac{1}{2} u^{3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{1}{2} u^{3} & 0 & 0 & 0
\end{array}\right), \quad \tilde{g}_{4}=\left(\begin{array}{cccc}
0 & \frac{1}{2} u^{4} & 0 & -\frac{1}{2} u^{2} \\
\frac{1}{2} u^{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{1}{2} u^{2} & 0 & 0 & 0
\end{array}\right) .
\end{aligned}
$$

The eigenvalue of the corresponding affinor $L$ is $\lambda-\kappa_{1} u^{3}$. Using symmetries which preserve $g$ and $\tilde{g}_{0}$ one can bring the above four-parameter family to one of the following canonical forms:

$$
\tilde{g}=\tilde{g}_{0}+\left\{\begin{array}{l}
\kappa_{1} \tilde{g}_{2}+\kappa_{2} \tilde{g}_{3} \\
\kappa_{1} \tilde{g}_{3}+\kappa_{2} \tilde{g}_{4} \\
\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{2}+\kappa_{3} \tilde{g}_{4}
\end{array} \quad \kappa_{1}, \kappa_{2}, \kappa_{3}=\right.\text { const }
$$

Case 2: There exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
0 & 1 & \lambda & 0 \\
1 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda
\end{array}\right)
$$

The general solution of Mokhov's conditions is given by $\tilde{g}=\sum_{i=1}^{4} \kappa_{i} \tilde{g}_{i}+\tilde{g}_{0}$, where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\tilde{g}_{1}=\left(\begin{array}{cccc}
2 u^{1} & \frac{1}{2} u^{2} & -u^{3} & \frac{1}{2} u^{4} \\
\frac{1}{2} u^{2} & -u^{3} & 0 & 0 \\
-u^{3} & 0 & 0 & 0 \\
\frac{1}{2} u^{4} & 0 & 0 & u^{3}
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
u^{2} & -\frac{1}{2} u^{3} & 0 & 0 \\
-\frac{1}{2} u^{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

$$
\tilde{g}_{3}=\left(\begin{array}{cccc}
u^{4} & 0 & 0 & \frac{1}{2} u^{3} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} u^{3} & 0 & 0 & 0
\end{array}\right), \quad \tilde{g}_{4}=\left(\begin{array}{cccc}
0 & \frac{1}{2} u^{4} & 0 & \frac{1}{2} u^{2} \\
\frac{1}{2} u^{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{2} u^{2} & 0 & 0 & 0
\end{array}\right) .
$$

The eigenvalue of the corresponding affinor $L$ is $\lambda-\kappa_{1} u^{3}$. Using symmetries which preserve $g$ and $\tilde{g}_{0}$ one can bring the above four-parameter family to one of the following normal forms:

$$
\tilde{g}=\tilde{g}_{0}+\left\{\begin{array}{l}
\kappa_{1} \tilde{g}_{2}+\kappa_{2} \tilde{g}_{3} \\
\kappa_{1} \tilde{g}_{3}+\kappa_{2} \tilde{g}_{4} \\
\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{2}+\kappa_{3} \tilde{g}_{4}
\end{array} \quad \kappa_{1}, \kappa_{2}, \kappa_{3}=\right.\text { const }
$$

## Segre type [4]

This is the case where the corresponding affinor $L$ is a single Jordan block (see Section 2.4 for the general theory). There exists a coordinate system where $g$ and $\tilde{g}_{0}$ take the form

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
0 & 0 & 1 & \lambda \\
0 & 1 & \lambda & 0 \\
1 & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0
\end{array}\right)
$$

It turns out that the general solution of Mokhov's conditions is $\tilde{g}=\sum_{i=1}^{3} \kappa_{i} \tilde{g}^{i}+\tilde{g}_{0}$ where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\begin{gathered}
\tilde{g}_{1}=\left(\begin{array}{cccc}
-u^{1} & -\frac{1}{2} u^{2} & 0 & \frac{1}{2} u^{4} \\
-\frac{1}{2} u^{2} & 0 & \frac{1}{2} u^{4} & 0 \\
0 & \frac{1}{2} u^{4} & 0 & 0 \\
\frac{1}{2} u^{4} & 0 & 0 & 0
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
2 u^{2} & \frac{1}{2} u^{3} & -u^{4} & 0 \\
\frac{1}{2} u^{3} & -u^{4} & 0 & 0 \\
-u^{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \\
\tilde{g}_{3}=\left(\begin{array}{cccc}
u^{3} & -\frac{1}{2} u^{4} & 0 & 0 \\
-\frac{1}{2} u^{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{gathered}
$$

Here the eigenvalue of the affinor $L$ is $\frac{1}{2} \kappa_{1} u^{4}+\lambda$. Using symmetries which preserve $g$ and $\tilde{g}_{0}$ one can bring the above three-parameter family to one of the following normal forms: in the non-constant eigenvalue case

$$
\tilde{g}=\tilde{g}_{0}+\tilde{g}_{1}+\kappa_{1} \tilde{g}_{2}, \quad \kappa_{1}=\text { const }
$$

while in the constant eigenvalue case

$$
\tilde{g}=\tilde{g}_{0}+\left\{\begin{array}{l}
\tilde{g}_{2} \\
k_{1} \tilde{g}_{3}
\end{array} \quad \kappa_{1}=\text { const } .\right.
$$

## Complex conjugate case

In the case of two pairs of complex conjugate eigenvalues $\nu+i \lambda$ and $\nu-i \lambda$, there exists a coordinate system such that

$$
g^{i j}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right), \quad \tilde{g}_{0}^{i j}=\left(\begin{array}{cccc}
0 & 1 & -\lambda & \nu \\
1 & 0 & \nu & \lambda \\
-\lambda & \nu & 0 & 0 \\
\nu & \lambda & 0 & 0
\end{array}\right)
$$

The general solution of Mokhov's conditions is $\tilde{g}=\kappa_{1} \tilde{g}_{1}+\kappa_{2} \tilde{g}_{2}+\tilde{g}_{0}$ where $\kappa_{i}$ are arbitrary constants, and the bivectors $\tilde{g}_{i}$ are as follows:

$$
\tilde{g}_{1}=\left(\begin{array}{cccc}
2 u^{2} & -2 u^{1} & -u^{4} & u^{3} \\
-2 u^{1} & -2 u^{2} & u^{3} & u^{4} \\
-u^{4} & u^{3} & 0 & 0 \\
u^{3} & u^{4} & 0 & 0
\end{array}\right), \quad \tilde{g}_{2}=\left(\begin{array}{cccc}
2 u^{1} & 2 u^{2} & -u^{3} & -u^{4} \\
2 u^{2} & -2 u^{1} & -u^{4} & u^{3} \\
-u^{3} & -u^{4} & 0 & 0 \\
-u^{4} & u^{3} & 0 & 0
\end{array}\right)
$$

Using symmetries which preserve the form of $g$ one can eliminate $\tilde{g}_{0}$, and bring $\tilde{g}$ to the normal form

$$
\tilde{g}^{i j}=\left(\begin{array}{cccc}
2 u^{2} & -2 u^{1} & -u^{4} & u^{3} \\
-2 u^{1} & -2 u^{2} & u^{3} & u^{4} \\
-u^{4} & u^{3} & 0 & 0 \\
u^{3} & u^{4} & 0 & 0
\end{array}\right)
$$

The eigenvalues of the corresponding affinor $L$ are $u^{3} \pm i u^{4}$. Note that this case is a complexification of the two-component operator (2.7), which can be achieved via the following recipe (see [11] for more details): each complex entry $a+i b$ of $g^{\mathbb{C}}$ and $\tilde{g}^{\mathbb{C}}$ is replaced by the $2 \times 2$ block

$$
\left(\begin{array}{cc}
-b & a \\
a & b
\end{array}\right),
$$

where $g^{\mathbb{C}}$ and $\tilde{g}^{\mathbb{C}}$ are the complexified bivectors of the operator (2.7):

$$
g^{\mathbb{C}}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \tilde{g}^{\mathbb{C}}=\left(\begin{array}{cc}
-2 z^{1} & z^{2} \\
z^{2} & 0
\end{array}\right),
$$

$z^{1}=u^{1}+i u^{2}, z^{2}=u^{3}+i u^{4}$.

### 2.4 The single Jordan block case

Let us begin with examples of $n$-component Hamiltonian operators of the single Jordan block type. The Hamiltonian property of these examples will be proved later in this section.

Example 1. One of the most important examples was discovered by Mokhov [74]. Here the first $n \times n$ contravariant metric is constant and anti-diagonal,

$$
g=\left(\begin{array}{lll} 
& & 1 \\
& . & \\
1 & &
\end{array}\right)
$$

while the second contravariant metric $\tilde{g}$ is defined as follows:

$$
\tilde{g}^{i j}=\left(b_{k}^{i j}+b_{k}^{j i}\right) u^{k}, \quad \begin{cases}b_{k}^{i j}=0 & \text { if } k \neq i+j-1, \\ b_{i+j-1}^{i j}=3 j-n-2 & \text { otherwise }\end{cases}
$$

One can verify that the Jordan normal form for the corresponding affinor $L$ is a single Jordan block with non-constant eigenvalue (for any $n \neq 4$ : in the exceptional case $n=4$ the affinor $L$ is the sum of two $2 \times 2$ Jordan blocks). We will refer to this case as the Mokhov operator. The affinor $L$ is given by $L_{j}^{i}=[3(i-j)+n-1] u^{n+i-j}$. The equivalent form for $\tilde{g}$
is

$$
\tilde{g}^{i j}=[3(i+j)-2(n+2)] u^{i+j-1}
$$

for $i+j-1 \leq n$, and 0 otherwise (in what follows, we use the following convention: if $\alpha>n$ then $\left.u^{\alpha} \equiv 0\right)$. For $n=2,3,4$ the explicit form of $\tilde{g}$ is as follows:

$$
\tilde{g}=\left(\begin{array}{cc}
-2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-4 u^{1} & -u^{2} & 2 u^{3} \\
-u^{2} & 2 u^{3} & 0 \\
2 u^{3} & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{cccc}
-6 u^{1} & -3 u^{2} & 0 & 3 u^{3} \\
-3 u^{2} & 0 & 3 u^{3} & 0 \\
0 & 3 u^{3} & 0 & 0 \\
3 u^{3} & 0 & 0 & 0
\end{array}\right) .
$$

Example 2. Another $n$-component example has $g$ the same as in Example 1, while the second contravariant metric is given by

$$
\tilde{g}^{i j}=\left(b_{k}^{i j}+b_{k}^{j i}\right) u^{k}+\lambda g^{i j}, \quad \begin{cases}b_{k}^{i j}=0 & \text { if } k \neq i+j \\ b_{i+j}^{i j}=3 j-n-1 & \text { otherwise }\end{cases}
$$

with $\lambda=$ const. One can verify that this pair of contravariant metrics defines a Hamiltonian operator for any $n \geq 3$ (the case $n=2$ is trivial since all $b_{k}^{i j}$ vanish). The corresponding affinor $L$ is a single Jordan block with constant eigenvalue $\lambda$. For instance, for $n=3,4$ the second contravariant metric reads

$$
\tilde{g}=\left(\begin{array}{ccc}
-2 u^{2} & u^{3} & \lambda \\
u^{3} & \lambda & 0 \\
\lambda & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{cccc}
-4 u^{2} & -u^{3} & 2 u^{4} & \lambda \\
-u^{3} & 2 u^{4} & \lambda & 0 \\
2 u^{4} & \lambda & 0 & 0 \\
\lambda & 0 & 0 & 0
\end{array}\right) .
$$

The aim of this section is to give a complete description of the case where the affinor $L$ is a single Jordan block. We will see that the Mokhov example plays fundamental role in this picture. To formulate our main result, let us introduce symmetric bivectors $\mu^{(n ; k)}$ as follows:

$$
\begin{equation*}
\mu^{(n ; k) i j}=[3(i+j)-2(n+2-k)] u^{i+j-1+k} \tag{2.19}
\end{equation*}
$$

In particular, $\mu^{(n ; 0)}$ coincides with the second contravariant metric $\tilde{g}$ of the Mokhov operator from Example 1. Note also that $\mu^{(n ; k)}=0$ for $k>n-2$. Let us present the explicit
form for some $\mu^{(n ; k)}$ :
$\mu^{(3 ; 1)}=\left(\begin{array}{ccc}-2 u^{2} & u^{3} & 0 \\ u^{3} & 0 & 0 \\ 0 & 0 & 0\end{array}\right), \mu^{(4 ; 1)}=\left(\begin{array}{cccc}-4 u^{2} & -u^{3} & 2 u^{4} & 0 \\ -u^{3} & 2 u^{4} & 0 & 0 \\ 2 u^{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right), \mu^{(4 ; 2)}=\left(\begin{array}{cccc}-2 u^{3} & u^{4} & 0 & 0 \\ u^{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{array}\right)$.
We will show that in the case when the affinor $L$ is a single Jordan block, the general solution of Mokhov's conditions reads

$$
\begin{equation*}
\tilde{g}=\tilde{g}_{0}+\sum_{m=0}^{n-2} \xi_{m} \mu^{(n ; m)}, \tag{2.20}
\end{equation*}
$$

where $\xi_{m}$ are arbitrary constants, and

$$
g= \pm\left(\begin{array}{lllll} 
& & & 1 \\
& & & & \\
& . & & \\
1 & & & &
\end{array}\right), \quad \tilde{g}_{0}= \pm\left(\begin{array}{llll} 
& & & \\
& & & \\
& . & & \\
& & & \\
1 & . & & \\
\lambda & & & \\
& & &
\end{array}\right)
$$

Here the eigenvalue of $L$ equals $\xi_{0}(n-1) u^{n}+\lambda$. In the non-constant eigenvalue case, $\xi_{0} \neq 0$, we have the following result:

Theorem 2.11. Let $P$ be a Hamiltonian operator (2.1) such that the affinor $L=\tilde{g} g^{-1}$ is a single $n \times n$ Jordan block with non-constant eigenvalue. Then there exists a coordinate system in which $g$ and $\tilde{g}$ can be reduced to the following canonical forms:

$$
g= \pm\left(\begin{array}{ll} 
& . \\
& . \\
1 &
\end{array}\right), \quad \text { if } n \neq 1 \bmod 3, \begin{array}{ll}
\mu^{(n ; 0)} & \text { if } n \equiv 1 \bmod 3, n \neq 4 \\
\mu^{(n ; 0)}+\kappa \mu^{\left(n ; \frac{n-1}{3}\right)} & \\
\mu^{(4 ; 0)}+\kappa \mu^{(4 ; 1)}+\tilde{g}_{0} & \text { if } n=4
\end{array}
$$

Here $\kappa$ is an arbitrary constant.
In the constant eigenvalue case, $\xi_{0}=0$, we have several canonical forms depending on how many coefficients among $\xi_{i}$ are equal to zero:

Theorem 2.12 (Constant eigenvalue case). Suppose $\xi_{i}=0$ for $i=0, \ldots, \alpha-1$. Then the
family (2.20) can be reduced to

$$
\tilde{g}=\mu^{(n ; \alpha)}+\kappa \mu^{(n ; \alpha+m)}+\tilde{g}_{0}, \quad m=\frac{n-1+2 \alpha}{3}
$$

if $m \in \mathbb{N}$, otherwise to

$$
\tilde{g}=\mu^{(n ; \alpha)}+\tilde{g}_{0}
$$

### 2.4.1 Proof of Theorem 2.11

The idea of the proof is as follows: first, we find the general solution of Mokhov's equations. It turns out (Proposition 2.14) that this solution depends on $n-1$ parameters plus the constant $\lambda$ appearing in $\tilde{g}_{0}$. Using orthogonal transformations, we then reduce this family of solutions to various normal forms (Lemma 2.16 and Proposition 2.17). We work in coordinates where $g$ and $\tilde{g}_{0}$ take canonical form

$$
g= \pm\left(\begin{array}{llll} 
& & & 1 \\
& & 1 & \\
& . & & \\
1 & & &
\end{array}\right), \quad \tilde{g}_{0}= \pm\left(\begin{array}{llll} 
& & & \\
& & & \\
& . & & \lambda \\
& & & \\
1 & . & & \\
\lambda & & &
\end{array}\right)
$$

For definiteness, we consider the + sign. In what follows we will need the following result:

Proposition 2.13. The Killing vectors of $g$ are the following $\frac{1}{2} n(n-1)$ vector fields:

$$
X_{(\alpha, \beta)}=u^{\alpha} \partial_{\beta}-u^{n+1-\beta} \partial_{n+1-\alpha}, \quad X_{\gamma}=\partial_{\gamma}
$$

here $\alpha+\beta<n+1$, and $\partial_{\alpha}=\frac{\partial}{\partial u^{\alpha}}$.

The affinor $L$ and the metric $\tilde{g}$ are given by

$$
\begin{aligned}
L_{j}^{i} & =c_{j k}^{i} u^{k}+\tilde{g}_{0}^{i l} g_{l j}, \\
\tilde{g}^{i j} & =L_{l}^{i} g^{l j}=c_{n+1-j, k}^{i} u^{k}+\tilde{g}_{0}^{i j} .
\end{aligned}
$$

These have to satisfy a set of constraints (note that the vanishing of the Nijenhuis torsion, $\mathcal{N}(L)=0$, gives two types of relations: linear and quadratic in $c_{j k}^{i}$ ):

- Linear part of the condition $\mathcal{N}(L)=0$ reads

$$
\begin{equation*}
c_{j, i-1}^{k}-c_{i, j-1}^{k}+c_{i j}^{k+1}-c_{j i}^{k+1}=0 ; \tag{2.21}
\end{equation*}
$$

- Quadratic part of the condition $\mathcal{N}(L)=0$ reads

$$
c_{i l}^{s} c_{j s}^{m}-c_{j l}^{s} c_{i s}^{m}+c_{s l}^{m} c_{i j}^{s}-c_{s l}^{m} c_{j i}^{s}=0 ;
$$

- Symmetry of $\tilde{g}$ gives

$$
\begin{equation*}
c_{j k}^{n+1-i}=c_{i k}^{n+1-j} ; \tag{2.22}
\end{equation*}
$$

- The Killing condition gives

$$
\begin{equation*}
c_{j k}^{n+1-i}+c_{i j}^{n+1-k}+c_{k i}^{n+1-j}=0 . \tag{2.23}
\end{equation*}
$$

Remarkably, the linear system (2.21)-(2.23) can be solved explicitly:
Proposition 2.14. The general solution of the linear system (2.21)-(2.23) is given by (2.20),

$$
\tilde{g}=\tilde{g}_{0}+\sum_{m=0}^{n-2} \xi_{m} \mu^{(n ; m)}
$$

where $\xi_{m}$ are arbitrary constants. The eigenvalue of the corresponding affinor $L$ is $\xi_{0}(n-1) u^{n}+\lambda$.

## Proof:

The key observation allowing one to prove Proposition 2.14 by induction is as follows. Suppose $c_{j i}^{n}=0$. In this case it is easy to see that $c_{1 i}^{k}$ and $c_{j 1}^{k}$ must also vanish, indeed, from (2.22) we have

$$
c_{1 k}^{n+1-j}=c_{j k}^{n},
$$

and from (2.23) and (2.22) we obtain

$$
c_{1 k}^{n+1-j}+c_{1 j}^{n+1-k}+c_{k 1}^{n+1-j}=0 .
$$

Then the remaining equations for $c_{i j}^{k}$, with $i, j=2, \ldots, n$ and $k=1, \ldots, n-1$, coincide with the system one obtains in the $(n-1)$-component case with $\tilde{c}_{i j}^{k}=c_{i-1, j-1}^{k}$, allowing one to
use inductive assumption.
Our strategy will be the following: first we show that the above equations imply $c_{j i}^{n}=$ $0, c_{1 i}^{k}=0$ and $c_{j 1}^{k}=0$ apart from $c_{n n}^{n}=c, c_{1 n}^{1}=c$ and $c_{n 1}^{1}=-2 c$. We already know a solution with $c \neq 0$, which comes from Mokhov's example. The generic solution can be written as a linear combination of Mokhov's solution and a solution of the system with $c_{j i}^{n}=c_{1 i}^{k}=c_{j 1}^{k}=0$ (in which case we can use inductive assumption as outlined above).

From (2.22) and (2.21) we have (for $j \neq 1$ )

$$
c_{1, j-1}^{k}=c_{n+1-k, j-1}^{n}=c_{j, n-k}^{n} .
$$

Using this identity we can write (2.21) as

$$
c_{j, n-k}^{n}-c_{1 j}^{k+1}+c_{j 1}^{k+1}=0 .
$$

Similarly, from (2.23) we obtain

$$
c_{j, n-k}^{n}+c_{j 1}^{k+1}+c_{1 j}^{k+1}=0 .
$$

Combining these two conditions we get $c_{1 j}^{i}=0$, for any $i \neq 1, j \neq 1$. Writing out (2.23) with $j=k=1$ we get

$$
c_{11}^{n+1-i}+c_{1 i}^{n}+c_{i 1}^{n}=2 c_{11}^{n+1-i}+c_{1 i}^{n}=0,
$$

which implies $c_{11}^{k}=0$. Summarizing, we have

$$
c_{1 j}^{i}=0, \quad \forall i \neq 1,
$$

which, for symmetry reasons, implies

$$
c_{j k}^{n}=0, \quad \forall j \neq n,
$$

and

$$
c_{j 1}^{i}=0, \quad \forall i \neq 1, j \neq n .
$$

Our next remark is that $c_{1 j}^{1}=0$ for $j=1, \ldots, n-2$. This follows from (2.21) evaluated at
$k=i=1$,

$$
c_{1, j-1}^{1}-c_{1 j}^{2}+c_{j 1}^{2}=0
$$

This readily implies $c_{n, k}^{n}=0$ and $c_{k, 1}^{1}=0$ for $k=1, \ldots, n-2$, as well as $c_{n, 1}^{k}=0$ for $k=3, \ldots, n$. It is also easy to see that the three non-vanishing coefficients $c_{n n}^{n}, c_{1 n}^{1}, c_{n 1}^{1}$ are related by

$$
c_{n n}^{n}=c_{1 n}^{1}, \quad c_{n 1}^{1}=-2 c_{1 n}^{1}
$$

We still need to prove that $c_{1, n-1}^{1}=0$. Due to the above computations the first column of the affinor $L$ has the form $(\nu, 0, \ldots, 0)^{t}$ where $\nu=c_{1, n-1}^{1} u^{n-1}+c_{1, n}^{1} u^{n}+\lambda$ is the (unique) eigenvalue of $L$. Similarly, the last row of $L$ is given by $(0, \ldots, 0, \nu)$. Let us denote by $\left(e_{(1)}, \ldots, e_{(n)}\right)$ the canonical frame of the pair $(L, g)$. Thus,

$$
\begin{equation*}
L_{k}^{i} e_{(p)}^{k}=\nu e_{(p)}^{i}+e_{(p-1)}^{i} \tag{2.24}
\end{equation*}
$$

It follows from the vanishing of the Nijenhuis torsion of $L$ that $e_{(i)}(\nu)=0$ for $i=1, \ldots, n-$ 1, where $e_{(i)}(\nu)$ denotes the Lie derivative of $\nu$ in direction $e_{(i)}$, see [11]. Due to the form of the affinor we have (set $i=n$ in (2.24)):

$$
e_{(p)}^{n}=0, \quad p=1, \ldots, n-1
$$

This means that $e_{(1)}, \ldots, e_{(n-1)}$ do not contain $\frac{\partial}{\partial u^{n}}$, and thus $\nu$ must depend on $u^{n}$ only, so that $c_{1, n-1}^{1}=0$.

This proves that the general solution is given by (2.20). A direct computation shows that (2.20) also satisfies the quadratic conditions coming from $\mathcal{N}(L)=0$.

Thus, the general solution depends on $n-1$ parameters plus $\lambda$ (see (2.20)). At this point one might wonder whether this number can be reduced. The answer is yes, the list of normal forms is presented below. In order to proceed, we need the following statement.

Lemma 2.15. The $n-2$ vector fields $X_{(k)}=\sum_{i=1}^{n-k}(n-k+1-2 i) u^{i+k} \partial_{i}$, where $k=1, \ldots, n-2$, satisfy the relations

1. $\operatorname{Lie}_{X_{(k)}} g=0$ (thus, they are isometries of $g$ ),
2. $\operatorname{Lie}_{X_{(k)}} \mu^{(n ; \alpha)}=p_{[n, k, \alpha]} \mu^{(n ; \alpha+k)}$,
3. $\operatorname{Lie}_{X_{(k)}}^{m} \mu^{(n ; \alpha)}=\left(\prod_{s=0}^{m-1}\left(p_{[n, k, \alpha]}-2 k s\right)\right) \mu^{(n ; \alpha+m k)}$, where the coefficients $p_{[n, k, \alpha]}$ are defined as $p_{[n, k, \alpha]}=3 k+1-n-2 \alpha$.

## Proof:

The condition 1 follows immediately by the proposition 2.13 , since $X_{(k)}$ are linear combinations of the vector fields $X_{(\alpha, \beta)}$.

A straightforward computation shows that

$$
\operatorname{Lie}_{X_{(k)}} \mu^{(n ; \alpha)}=(3 k+1-n-2 \alpha)(3(i+j)-2(n+2-\alpha-k)) u^{i+j+\alpha-1+k} .
$$

Thus condition 2 holds, which is equivalent to condition 3 for $m=1$. By induction, let us assume that it is true for $m$, thus

$$
\begin{gathered}
\operatorname{Lie}_{X_{(k)}}^{m+1} \mu^{(n ; \alpha)}=\operatorname{Lie}_{X_{(k)}}\left(\operatorname{Lie}_{X_{(k)}}^{m} \mu^{(n ; \alpha)}\right)=\operatorname{Lie}_{X_{(k)}}\left(\prod_{s=0}^{m-1}\left(p_{[n, k, \alpha]}-2 k s\right) \mu^{(n ; \alpha+m k)}\right)= \\
=\left(\prod_{s=0}^{m-1}\left(p_{[n, k, \alpha]}-2 k s\right)\right) \operatorname{Lie}_{X_{(k)}} \mu^{(n ; \alpha+m k)} .
\end{gathered}
$$

Since $\mu^{(n ; \alpha+m k) i j}=[3(i+j)-2(n+2-\alpha-m k)] u^{i+j-1+\alpha+m k}$, it is easy to see that its Lie derivative reads $(3 k+1-n-2 \alpha-2 k m) \mu^{(n ; \alpha+(m+1) k)}$, that is,

$$
\begin{gathered}
\operatorname{Lie}_{X_{(k)}}^{m+1} \mu^{(n ; \alpha)}=\left(\prod_{s=0}^{m-1}\left(p_{[n, k, \alpha]}-2 k s\right)\right)(3 k+1-n-2 \alpha-2 k m) \mu^{(n ; \alpha+(m+1) k)}, \\
=\left(\prod_{s=0}^{m}\left(p_{[n, k, \alpha]}-2 k s\right)\right) \mu^{(n ; \alpha+(m+1) k)} .
\end{gathered}
$$

Then, 2 is fulfilled.

Consider now the general solution (2.20). Note that in the non-constant eigenvalue case, $\xi_{0} \neq 0$, one can eliminate the constant term $\tilde{g}_{0}$ by a translation of variables $u^{i}$. Let $S_{0}$ be the resulting $n-1$ parameter family of solutions,

$$
\begin{equation*}
S_{0}=\sum_{i=0}^{n-2} \xi_{i} \mu^{(n ; i)}, \tag{2.25}
\end{equation*}
$$

and let $\mathbf{L}_{[k]}$ be the Lie series

$$
\mathbf{L}_{[k]}=\exp \left(t_{k} \operatorname{Lie}_{X_{(k)}}\right)=\sum_{s \geq 0} \frac{t_{k}^{s}}{s!} \operatorname{Lie}_{X_{(k)}}^{s},
$$

where $X_{(k)}$ are as in Lemma 2.15. We point out that, when applied to $\mu^{(n ; k)}$ for $n$ fixed, $\mathbf{L}_{[k]}$ consists of a finite number of terms: recall that $\mu^{(n ; i)}=0$ for $i>n-2$.

Lemma 2.16. If $\xi_{0} \neq 0$, then it can be set equal to one.

## Proof:

Let us consider the scaling transformation

$$
v^{i}=\gamma^{\frac{n+1}{2}-i} u^{i},
$$

where $\gamma \neq 0$ is an arbitrary constant. It is easy to see that this preserves the form of $g$. Indeed, we have

$$
g=\delta^{i, n+1-j} \partial_{u^{i}} \partial_{u^{j}}=\delta^{i, n+1-j} \gamma^{n+1-i-j} \partial_{v^{i}} \partial_{v^{j}}=\delta^{i, n+1-j} \gamma^{0} \partial_{v^{i}} \partial_{v^{j}}=\delta^{i, n+1-j} \partial_{v^{i}} \partial_{v^{j}},
$$

where $\partial_{u^{i}}=\frac{\partial}{\partial u^{i}}$ and $\partial_{v^{i}}=\frac{\partial}{\partial v^{2}}$. Taking a metric of the form (2.19) and applying the above transformation, we get

$$
\begin{gathered}
\mu^{(n ; k) i j}(\mathbf{u}) \partial_{u^{i}} \partial_{u^{j}}=[3(i+j)-2(n+2-k)] \gamma^{i+j-1+k-\frac{n+1}{2}} v^{i+j-1+k} \gamma^{n+1-i-j} \partial_{v^{i}} \partial_{v^{j}} \\
=[3(i+j)-2(n+2-k)] v^{i+j-1+k} \gamma^{\frac{n-1}{2}+k} \partial_{v^{i}} \partial_{v^{j}} \\
=\gamma^{\frac{n-1}{2}+k} \mu^{(n ; k) i j}(\mathbf{v}) \partial_{v^{i}} \partial_{v^{j}} .
\end{gathered}
$$

Thus, setting $\gamma=\xi_{0}^{-\frac{2}{n-1}}$, we can reduce the coefficient of $\mu^{(n ; 0)}$ to 1 .
To finish the proof of Theorem 2.11 we need the following
Proposition 2.17. Suppose $\xi_{0} \neq 0$. Then

1. if $n \not \equiv 1 \bmod 3$, there exists an orthogonal transformation which brings the ( $n-1$ )-parameter solution $S_{0}$ to $\mu^{(n ; 0)}$;
2. if $n \equiv 1 \bmod 3, n \neq 4$, there exists an orthogonal transformation which brings $S_{0}$ to the one-parameter family $\mu^{(n ; 0)}+\kappa \mu^{\left(n ; \frac{n-1}{3}\right)}$,
where $\kappa$ is an arbitrary constant.

## Proof:

By Lemma 2.16 we can consider the family $S_{0}$ in the form

$$
S_{0}=\mu^{(n ; 0)}+\sum_{i=1}^{n-2} \kappa_{i} \mu^{(n ; i)}
$$

where $\kappa_{i}$ are arbitrary constant coefficients. Suppose $n \not \equiv 1 \bmod 3$, then the coefficients $p_{[k, n, 0]}$ defined in Lemma 2.15 do not vanish. Let us apply $\mathbf{L}_{[1]}$ to $S_{0}$ and look at the coefficient of $\mu^{(n ; 1)}$ :

$$
\begin{gathered}
\mathbf{L}_{[1]} S_{0}=S_{0}+t_{1} \operatorname{Lie}_{X_{(1)}} S_{0}+\frac{t_{1}^{2}}{2} \operatorname{Lie}_{X_{(1)}}^{2} S_{0}+\ldots+\frac{t_{1}^{n-2}}{(n-2)!} \operatorname{Lie}_{X_{(1)}}^{n-2} S_{0} \\
=\mu^{(n ; 0)}+\left(\kappa_{1}+t_{1} p_{[n, 1,0]}\right) \mu^{(n ; 1)}+\ldots
\end{gathered}
$$

We can always choose $t_{1}$ such that the coefficient of $\mu^{(n ; 1)}$ is zero. Let us call $S_{1}$ the resulting $(n-2)$-parameter family:

$$
S_{1}=\left.\mathbf{L}_{[1]} S_{0}\right|_{t_{1}=-\frac{\kappa_{1}}{p_{[n, 1,0]}}}=\mu^{(n ; 0)}+\sum_{i=2}^{n-2} \tilde{\kappa}_{i} \mu^{(n ; i)}
$$

Applying $\mathbf{L}_{[2]}$ and looking at the coefficient of $\mu^{(n ; 2)}$ we obtain

$$
\mathbf{L}_{[2]} S_{1}=\mu^{(n ; 0)}+\left(\tilde{\kappa}_{2}+t_{2} p_{[n, 2,0]}\right) \mu^{(n ; 2)}+\ldots
$$

Again, we can choose $t_{2}$ such that the coefficient of $\mu^{(n ; 2)}$ vanishes, and so on. Ultimately, we get

$$
\mathbf{L}_{[n-2]} \mathbf{L}_{[n-3]} \cdots \mathbf{L}_{[1]} S_{0}=\mu^{(n ; 0)}
$$

as required.
To prove the second part of the proposition, let us set $n=3 m+1$. It is easy to see that $\operatorname{Lie}_{X_{(m)}} \mu^{(n ; 0)}=0$, since the coefficient $p_{[n, m, 0]}$ vanishes. Note that in this case $p_{[n, k, 0]} \neq 0$ for $k \neq m$. For fixed $m$, until $k=m-1$ we can apply the same procedure as above, obtaining

$$
S_{m-1}=\mu^{(n ; 0)}+\sum_{i=m}^{n-2} \tilde{\kappa}_{i} \mu^{(n ; i)}
$$

At this point, applying $\operatorname{Lie}_{X_{(m)}}$ to $S_{m-1}$, we cannot eliminate the coefficient of $\mu^{(n ; m)}$, since $\operatorname{Lie}_{X_{(m)}} \mu^{(n ; 0)}=0$. However, applying $\operatorname{Lie}_{X_{(m+1)}}$ and looking at the coefficient of $\mu^{(n ; m+1)}$,

$$
\mathbf{L}_{[m+1]} S_{m-1}=\mu^{(n ; 0)}+\tilde{\kappa}_{m} \mu^{(n ; m)}+\left(\tilde{\kappa}_{m+1}+t_{m+1} p_{[n, m+1,0]}\right) \mu^{(n ; m+1)}+\ldots
$$

we can eliminate it. Following the same method, we arrive at the canonical form

$$
S_{c a n}=\mu^{(n ; 0)}+\tilde{\kappa}_{m} \mu^{(n ; m)} .
$$

The case $n=4$ is special. Indeed, the $4 \times 4$ Mokhov metric does not correspond to the single Jordan block case. In the $n=4$ Jordan block case, normal forms are presented in Section 2.3.4. This concludes the proof of Theorem 2.11. In the constant eigenvalue case, $\xi_{0}=0$, we cannot eliminate $\tilde{g}_{0}$ by a shift. Finally, Theorem 2.12 can be established by following the same procedure as above.

### 2.4.2 Relation to Frobenius manifolds

In this section we demonstrate the relation between Mokhov's Hamiltonian operator and the trivial Frobenius manifold associated with the cohomology ring of projective space. For this purpose let us briefly recall the definition of a Frobenius manifolds [29, 27].

Definition 2.2. A Frobenius manifold ( $M, g, o, e, E$ ) is a manifold $M$ endowed with a (pseudo)Riemannian metric $g$, a product $\circ$ on the tangent spaces $T_{u} M$ and a pair of vector fields e and $E$ such that

- the product $\circ$ is commutative and associative:

$$
c_{j k}^{i}=c_{k j}^{i}, \quad c_{j l}^{i} c_{k h}^{l}=c_{k l}^{i} c_{j h}^{l} .
$$

- $g$ is flat, invariant with respect to $\circ$,

$$
g_{i k} c_{j l}^{k}=g_{j k} c_{i l}^{k},
$$

and the associated Levi-Civita connection $\nabla$ is compatible with $\circ$ :

$$
\begin{equation*}
\nabla_{l} c_{j k}^{i}=\nabla_{j} c_{l k}^{i} . \tag{2.26}
\end{equation*}
$$

This implies that there exists a function F, called the Frobenius potential, such that, in flat coordinates for $g$,

$$
c_{i j k}=g_{i l} c_{j k}^{l}=\partial_{i} \partial_{j} \partial_{k} F .
$$

- The product o has a unity e which is flat: $\nabla e=0$.
- The Euler vector field $E$ satisfies:

$$
\nabla \nabla E=0, \quad \operatorname{Lie}_{E} e=-e, \quad \operatorname{Lie}_{E} \circ=\circ, \quad \operatorname{Lie}_{E} g=(2-d) g,
$$

for some constant $d$. The existence of the Euler vector field is related to the existence of a flat contravariant metric called the intersection form. In local coordinates it is defined by the formula

$$
\tilde{g}^{i j}=g^{i l} c_{l k}^{j} E^{k} .
$$

If, in the flat coordinates for $g$, the functions $c_{j k}^{i}$ are constant, the Frobenius manifold is called trivial [27]. In this case, the Frobenius potential is a cubic polynomial, $F=\frac{1}{6} c_{i j k} u^{i} u^{j} u^{k}$.

It is known that flat pencils of metrics associated with a Frobenius manifold are exact [25]. Let us recall what is the exactness of a flat pencil of metrics.

Definition 2.3. Let $(g, \tilde{g})$ a be pair of flat contravariant metrics. We say that they define an exact flat pencil of metrics if there exists a vector fields $X$ such that the conditions

$$
\operatorname{Lie}_{X} \tilde{g}=g, \quad \operatorname{Lie}_{X} g=0
$$

are satisfied.
It is easy to check that the associated Poisson bivectors $P$ and $\tilde{P}$ satisfy the relations $\operatorname{Lie}_{X} \tilde{P}=P$ and $\operatorname{Lie}_{X} P=0$. A straightforward computation shows that in the case of a single Jordan block with non-constant eigenvalue, the flat pencil given by the pair of flat contravariant metrics $\left(g, \mu^{(n ; 0)}\right)$ is exact. Indeed, let us consider the vector field
$X=(n-1)^{-1} \partial_{u^{n}}$. Clearly, $\operatorname{Lie}_{x} g=0$. Computing the Lie derivative of $\mu^{(n ; 0)}$ we get

$$
\begin{gathered}
\left(\operatorname{Lie}_{X} \mu^{(n ; 0)}\right)^{i j}=X^{m} \partial_{u^{m}} \mu^{(n ; 0) i, j}=(n-1)^{-1} \partial_{u^{n}}\left((3(i+j)-2(n+2)) u^{i+j-1}\right) \\
=(n-1)^{-1}(3(i+j)-2(n+2)) \partial_{u^{n}} u^{i+j-1}=(n-1)^{-1}(3(i+j)-2(n+2)) \delta^{i, n+1-j} \\
=(n-1)^{-1}(n-1) \delta^{i, n+1-j}=g^{i j} .
\end{gathered}
$$

Remark. In general, using Mokhov's conditions it is easy to prove that if the pair ( $g, \tilde{g}$ ) defines a 2D Hamiltonian operator, then in flat coordinates of $g, g$ itself and the homogeneous linear part of $\tilde{g}$ define an exact flat pencil of metrics. More precisely, we have $\operatorname{Lie}_{X} g^{i j}=\tilde{g}_{\text {hom }}^{i j}$ and $\operatorname{Lie}_{X} \tilde{g}_{\text {hom }}^{i j}=0$, where $\tilde{g}_{\text {hom }}^{i j}=\left(\tilde{b}_{l}^{i j}+\tilde{b}_{l}^{j i}\right) u^{l}$ and $X^{i}=-\tilde{g}_{\text {hom }}^{i s} g_{s l} u^{l}$. Moreover, $X$ is constant in flat coordinates of $\tilde{g}_{\mathrm{hom}}\left(\tilde{\nabla}_{\mathrm{hom}} X=0\right)$.

Thus, this fact suggest the existence of a relation between Hamiltonian operators and the theory of Frobenius manifolds. In the case of Mokhov's Hamiltonian operator, we can easily define a trivial Frobenius manifold associated with it.

Theorem 2.18. The metric

$$
g_{i j}= \begin{cases}1 & \text { if } i+j=n+1, \\ 0 & \text { otherwise }\end{cases}
$$

the structure constants

$$
c_{j k}^{i}=g^{i l} c_{l j k}= \begin{cases}1 & \text { if } l=2 n+1-j-k=n+1-i \text { that is } j+k-i=n, \\ 0 & \text { otherwise },\end{cases}
$$

the unity $e=\frac{\partial}{\partial u^{n}}$, and the Euler vector field $E=\sum_{k=1}^{n}(3 k-2 n-1) u^{k} \frac{\partial}{\partial u^{k}}$ define a trivial Frobenius manifold with $d=3$. Moreover, the intersection form,

$$
\tilde{g}^{i j}=g^{i l} c_{l k}^{j} E^{k}= \begin{cases}{[3(i+j)-2 n-4] u^{i+j-1}} & \text { if } i+j-1 \leq n, \\ 0 & \text { otherwise },\end{cases}
$$

coincides with the second metric of Mokhov's operator.

## Proof:

Commutativity of the product is trivial. Associativity

$$
\begin{equation*}
c_{j l}^{i} c_{k m}^{l}=c_{k l}^{i} c_{j m}^{l} \tag{2.27}
\end{equation*}
$$

is also satisfied since the left hand side of (2.27) is different from 0 if $l=i-j=k+m$ and the right hand side of (2.27) is different from 0 if $l=i-k=j+m$. Finally, applying the Euler vector field to the Frobenius potential we get
$\sum_{l=1}^{n}(3 l-2 n-1) u^{l} \frac{\partial F}{\partial u^{l}}=\sum_{i, j=1}^{n}[3 i-2 n-1+3 j-2 n-1+3(2 n+1-i-j)-2 n-1]=0$.
In order to prove that Christoffel symbols $\tilde{b}_{k}^{i j}$ coincide with Christoffel symbols of Mokhov examples we have to prove that

$$
\begin{align*}
\partial_{k} \tilde{g}^{i j} & =\tilde{b}_{k}^{i j}+\tilde{b}_{k}^{j i}  \tag{2.28}\\
\tilde{g}^{j l} \tilde{b}_{l}^{i k} & =\tilde{g}^{i l} \tilde{b}_{l}^{j k} \tag{2.29}
\end{align*}
$$

Condition (2.28) is trivially satisfied if $k \neq i+j-1$, while for $k=i+j-1$ we have we get

$$
\partial_{k} \tilde{g}^{i j}=3(i+j)+2 n-4=3 j-n-2+3 i-n-2=\tilde{b}_{k}^{i j}+\tilde{b}_{k}^{j i} .
$$

Moreover,
$\tilde{g}^{j} \tilde{b}_{l}^{i k}=g^{j, i+k-1} \tilde{b}_{i+k-1}^{i k}=[3(i+j+k-1)-2 n-4](3 k-n-2)=g^{j, i+k-1} \tilde{b}_{i+k-1}^{i k}=\tilde{g}^{i l} \tilde{b}_{l}^{j k}$.

To conclude this section we compare the Frobenius algebra underlying Mokhov's example with the Frobenius algebra structure on the full cohomology ring of projective space $H^{*}\left(\mathbf{C P}^{d}\right)$. This can be defined with respect to the natural basis $e_{1}=1, e_{2}=$ $\omega, \ldots, e_{d+1}=\omega^{d}$, generated by powers of the standard Kahler form normalized as $\int_{\mathbf{C P}^{d}} \omega^{d}=1$. The contravariant components of the metric $g$ and the structure constants $c_{j k}^{i}$ are defined respectively by

$$
g_{i j}= \begin{cases}1 & \text { if } i+j=n+1, \\ 0 & \text { otherwise }\end{cases}
$$

and $e_{j} \wedge e_{k}=c_{j k}^{i} e_{i}=e_{j+k-1}$, that is,

$$
c_{j k}^{i}= \begin{cases}1 & \text { if } j+k-i=1 \\ 0 & \text { otherwise }\end{cases}
$$

Putting $i^{\prime}=n+1-i$ we obtain the Frobenius algebra of Mokhov's example.

### 2.5 Multi-dimensional Hamiltonian operators

The theory developed by Mokhov holds for a generic $N$-dimensional Hamiltonian operator

$$
\begin{equation*}
P^{i j}=\sum_{\alpha=1}^{N}\left(g^{i j \alpha}(\mathbf{u}) \frac{d}{d x^{\alpha}}+b_{k}^{i j \alpha}(\mathbf{u}) u_{\alpha}^{k}\right) \tag{2.30}
\end{equation*}
$$

here $\mathbf{u}=\mathbf{u}(\mathbf{x})=\left(u^{1}(\mathbf{x}), \ldots, u^{n}(\mathbf{x})\right)$ are local coordinate on a certain smooth $n$-dimensional manifolds $M, \mathbf{x}=\left(x^{1}, \ldots, x^{N}\right)$ are independent variables and $u_{\alpha}^{k}=\frac{\partial u^{k}}{\partial x^{\alpha}}$. In the nondegenerate situation, that is when a generic linear combination of $g^{\alpha}$ is non-degenerate (without any loss of generality we will assume that each $g^{\alpha}$ is non-degenerate: this can always be achieved by a suitable linear transformation of the independent variables $x^{\alpha}$ ), the Mokhov's conditions involving the obstruction tensor must be satisfied by each pair of metrics appearing in the differential operator (see Section 1.3). This implies that each pair of metrics $\left(g^{\beta}, g^{\gamma}\right)$ for $\beta \neq \gamma$ must define a 2D Hamiltonian operator. Moreover, we have seen that all metrics $g^{\alpha}$ must be flat and all the coefficients $b_{k}^{i j \alpha}$ define Levi-Civita connection with respect to the metrics $g^{\alpha}$ corresponding to them (see Lemma 1.3). Therefore, Mokhov's theorem for a generic $N$-dimensional Hamiltonian operator (Theorem 1.4) can be rewritten as follows.

Theorem 2.19. Suppose $g^{\alpha}$ are flat contravariant metrics. An operator of the form (2.30) defines a $N$-dimensional Hamiltonian operator if and only if the following conditions are fulfilled for all $\beta, \gamma$ such that $\beta \neq \gamma$ :

1. Linearity of $g^{\beta}$ in flat coordinates of $g^{\gamma}$.
2. Vanishing of the Nijenhuis torsion of the affinors $L^{(\beta \gamma)}=g^{\beta}\left(g^{\gamma}\right)^{-1}$.
3. The Killing condition between each pair of metrics:

$$
\nabla^{i} g^{k j \gamma}+\nabla^{k} g^{i j \gamma}+\nabla^{j} g^{i k \gamma}=0
$$

where $\nabla$ corresponds to $g^{\beta}$.

Remark. It is sufficient to require the flatness of only one of the metrics $g^{\alpha}$. Indeed, let us suppose that the metric $g^{\alpha}$ is flat. Then, since the pair $\left(g^{\alpha}, g^{\beta}\right)$ must define a 2D Hamiltonian operator for all $\beta$, the linearity, Nijenhuis and Killing conditions imply the flatness of $g^{\beta}$ (see Theorem 2.2).

It was demonstrated by Mokhov [69] that there exist no non-constant 3D Hamiltonian operator with one or two components. Here we show that there exist only two non trivial three-components 3D Hamiltonian operators.

Theorem 2.20. Every three-components 3D Hamiltonian operator either can be reduced to constant form, or can be reduced to

$$
P=\left(\begin{array}{ccc}
\partial_{z} & 0 & \partial_{x}  \tag{2.31}\\
0 & \partial_{x} & 0 \\
\partial_{x} & 0 & 0
\end{array}\right)+\left(\begin{array}{ccc}
-2 u^{2} \partial_{y}-u_{y}^{2} & u^{3} \partial_{y}+2 u_{y}^{3} & 0 \\
u^{3} \partial_{y}-u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

or

$$
P=\left(\begin{array}{ccc}
0 & \partial_{x} & 0  \tag{2.32}\\
\partial_{x} & 0 & 0 \\
0 & 0 & \partial_{x}+\partial_{z}
\end{array}\right)+\left(\begin{array}{ccc}
-2 u^{1} \partial_{y}-u_{y}^{1} & u^{2} \partial_{y}+2 u_{y}^{2} & 0 \\
u^{2} \partial_{y}-u_{y}^{2} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

by a local change of coordinates $u^{i}$ and a linear change of the independent space variables $x, y, z$.

## Proof:

Since we are interested in the non-constant case, we will consider 3D operators as deformation of 2D non-constant operators which have been classified already (Theorem 2.10). There exist only three such operators, defined by the following pairs of contravariant metrics:

$$
\begin{gather*}
g=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right),  \tag{2.33}\\
g=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{2} & u^{3} & \lambda \\
u^{3} & \lambda & 0 \\
\lambda & 0 & 0
\end{array}\right), \tag{2.34}
\end{gather*}
$$

and (reducible case)

$$
g=\left(\begin{array}{lll}
0 & 1 & 0  \tag{2.35}\\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & u^{2} & 0 \\
u^{2} & 0 & 0 \\
0 & 0 & \lambda
\end{array}\right) .
$$

Fixing one of the above pairs, let us add a third contravariant metric $h$. Since we are in the flat coordinates of the first metric $g$, the bivector $h$ must be linear. Since the pair $(g, h)$ satisfies the Killing condition, we can represent $h$ as a sum of symmetrized tensor products of infinitesimal isometries of $g$. Assuming this, let us consider the above three cases separately.

Case (2.33): Checking the Killing condition for the pair $(\tilde{g}, h)$ we obtain that $h$ must be a linear combination of $g$ and $\tilde{g}$. This means that our operator is essentially twodimensional.

Case (2.34): Checking the Killing condition for the pair $(\tilde{g}, h)$ we obtain $h=c_{1} g+c_{2} \tilde{g}+h_{0}$ where

$$
h_{0}=\left(\begin{array}{lll}
\nu & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

One can verify that this ansatz for $h$ satisfies all other conditions.
Case (2.35): In this case, it is no longer sufficient to consider the Killing condition alone: we also need the linearity of $h$ with respect to $\tilde{g}$, that is $\nabla^{2} h=0$, where $\nabla$ correponds to $\tilde{g}$. These conditions imply $h=c_{1} g+c_{2} \tilde{g}+h_{0}$, where

$$
h_{0}=\left(\begin{array}{lll}
0 & \mu & 0 \\
\mu & 0 & 0 \\
0 & 0 & \nu
\end{array}\right)
$$

One can verify that this ansatz for $h$ satisfy all other conditions.
It remains to note that all constants appearing in the classification can be eliminated by linear transformations of $x, y, z$ leading to the normal forms (2.31) and (2.32).

If one does not allow linear changes of the independent variables $x, y, z$, the analogue of Theorem 2.20 reads as follows

Theorem 2.21. Any non-degenerate three-component Hamiltonian operator in $3 D$, which is not
transformable to constant coefficients, can be brought by a transformation of the dependent variables $u^{i}$ to

$$
P^{i j}=g^{i j} \frac{d}{d x}+\tilde{g}^{i j} \frac{d}{d y}+\left(c_{1} g^{i j}+c_{2} \tilde{g}^{i j}+h_{0}^{i j}\right) \frac{d}{d z}+\tilde{b}_{k}^{i j}\left(u_{y}^{k}+c_{2} u_{z}^{k}\right)
$$

where $c_{1}, c_{2}$ are constants, $\tilde{b}_{k}^{i j}$ are the contravariant Christoffel symbols of $\tilde{g}$, and the contravariant metrics $g, \tilde{g}, h_{0}$ assume one of the two canonical forms

- form 1 :

$$
g=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & -\frac{1}{2} u^{2} & u^{3} \\
-\frac{1}{2} u^{2} & u^{3} & 0 \\
u^{3} & 0 & 0
\end{array}\right), \quad h_{0}=\left(\begin{array}{ccc}
\nu & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

here $\nu=$ const;

- form 2 :

$$
g=\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \tilde{g}=\left(\begin{array}{ccc}
-2 u^{1} & u^{2} & 0 \\
u^{2} & 0 & 0 \\
0 & 0 & \lambda
\end{array}\right), \quad h_{0}=\left(\begin{array}{ccc}
0 & \mu & 0 \\
\mu & 0 & 0 \\
0 & 0 & \nu
\end{array}\right)
$$

here $\lambda, \mu, \nu$ are constants.

It follows from the proof of Theorem 2.20 that any non-degenerate three-component Hamiltonian operator in 4D is essentially 3D, or can be transformed to constant coefficient form. We point out that there exists non-trivial examples of Hamiltonian operators in any dimension.

Example 3. The following expression provides the example of a non-constant irreducible $N$-component Hamiltonian operator in $N$ dimensions:

$$
P^{i j}=\eta^{i j} \frac{d}{d x^{1}}+g^{i j} \frac{d}{d x^{2}}+b_{k}^{i j} u_{x^{2}}^{k}+\sum_{m=1}^{N-2} h^{i j m} \frac{d}{d x^{m+2}}
$$

where

- $\eta$ is the constant $N \times N$ anti-diagonal metric $\eta^{i j}=\delta^{i, N+1-j}$;
- the metric $g$ in components is $g^{i j}=\mu^{(N ; N-2)}+g_{0}^{i j}$, where $\mu^{(N ; N-2)}$ is defined by (2.19), namely

$$
\mu^{(N ; N-2) i j}=[3(i+j)-8] u^{i+j+N-3},
$$

and $g_{0}^{i j}=\delta^{i, N-j}+\lambda \delta^{i, N+1-j} ;$

- $b_{k}^{i j}$ are the contravariant Christoffel symbols of $g$, namely

$$
b_{N-1}^{11}=-1, \quad b_{N}^{12}=2, \quad b_{N}^{21}=-1,
$$

and all the remaining coefficients equal to 0 ;

- the $N-2$ constant metrics $h^{m}$ are defined by $h^{i j m}=\delta^{i m} \delta_{m}^{j}$.

For $N=3$, this example corresponds to (2.31).
Let us demonstrate that this example is actually a $N$-dimensional Hamiltonian operator. We consider the metrics $h$ as a unique metric $\tilde{h}$ defined by $N-2$ arbitrary constants $\alpha_{m}$, that is $\tilde{h}^{i j}=\alpha_{i+2} \delta^{i j}$, for $i=1, \ldots, N-2$. For clearness, let us write the metrics in matrix form, namely

$$
g=\left(\begin{array}{ccccccc}
-2 u^{N-1} & u^{N} & 0 & \cdots & 0 & 1 & \lambda \\
u^{N} & 0 & \cdots & 0 & 1 & \lambda & \\
0 & \cdots & 0 & 1 & \lambda & & \\
\vdots & . & . & . & . \cdot & & \\
0 & 1 & \lambda & & & \\
1 & \lambda & & & & \\
\lambda & & & & &
\end{array}\right), \quad \tilde{h}=\left(\begin{array}{cccccc}
\alpha_{3} & & & & & \\
& \alpha_{4} & & & & \\
& & \ddots & & & \\
& & & \alpha_{N-1} & & \\
& & & & \alpha_{N} & \\
& & & & & 0 \\
& & & & & \\
& & & & & \\
& & &
\end{array}\right) .
$$

We already know that the pair $(\eta, g)$ is a 2D Hamiltonian operator. Since $\tilde{h}$ is constant, then the pair $(\eta, \tilde{h})$ also defines a 2D Hamiltonian operator. We want to show that the pair $(\tilde{h}, g)$ also satisfies all Mokhov's condition. Linearity is obvious. The Killing condition reads

$$
\tilde{h}^{i s} \partial_{s} g^{j k}+\tilde{h}^{j s} \partial_{s} g^{k i}+\tilde{h}^{k s} \partial_{s} g^{i j}=0
$$

Since $g$ depends on $u^{N-1}, u^{N}$ and $\tilde{h}^{i j}=0$ for $i, j \in\{N-1, N\}$, then the Killing condition is easily fulfilled. It remains to prove that the Nijenhuis torsion vanishes. We will show
that $L=\tilde{h} g^{-1}$ is constant, and then its Nijenhuis torsion is zero.
Let us compute $g^{-1}$. Since $g$ is upper triangular, the inverse is lower triangular, in particular of the form

$$
g^{-1}=\left(\begin{array}{ccccc} 
& & & & \\
& & & & * \\
& & & \cdot & \vdots \\
& & * & * & * \\
& . & * & * & * \\
* & \ldots & * & * & *
\end{array}\right) .
$$

It is not difficult to see that in this particular case the coordinates $u^{N-1}, u^{N}$ appear only in the $2 \times 2$ block identified by the square. Thus, the product of $h$ and $g^{-1}$ reads

$$
L=\left(\begin{array}{lll}
* & \ldots & * \\
& \ddots & \vdots \\
& & * \\
& &
\end{array}\right)
$$

where all the non zero coefficients * are constant. Therefore the Nijenhuis tensor vanishes. This proves that the triple $(\eta, g, \tilde{h})$ defines a 3D Hamiltonian operator, depending on the $N-2$ parameters $\alpha_{i}$. Choosing $h^{i j m}=\delta^{i m} \delta_{m}^{j} \alpha_{m+2}$, up to linear change of the independent variables $x^{3}, \ldots, x^{N}$, that is $x^{i} \rightarrow \frac{x^{i}}{\alpha_{i}}$, we finally get the starting $N-2$ metrics $h^{m}$. By construction, the metrics $\left(\eta, g, h^{1}, \ldots, h^{N-2}\right)$ define a $N$-component Hamiltonian operator in $N$ dimensions.

## Degenerate Dubrovin-Novikov structures and integrable systems

In the previous chapter we discussed first-order Hamiltonian operators of hydrodynamic type under the assumption of non-degeneracy. One might wonder what happens if we remove this requirement. Let us recall that a 2D Hamiltonian operator

$$
\begin{equation*}
P^{i j}=g^{i j}(\mathbf{u}) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}) u_{x}^{k}+\tilde{g}^{i j}(\mathbf{u}) \frac{d}{d y}+\tilde{b}_{k}^{i j}(\mathbf{u}) u_{y}^{k}, \tag{3.1}
\end{equation*}
$$

is degenerate if $\operatorname{det}(g+\lambda \tilde{g})=0$ for all $\lambda \in \mathbb{R}$. The study of such structures is motivated by the existence of systems of hydrodynamic type which admit a Hamiltonian formulation with a degenerate operator. An example is given by two-dimensional isentropic gas dynamics equations (see, for instance [44])

$$
\begin{equation*}
\rho_{t}+(\rho u)_{x}+(\rho v)_{y}=0, \quad u_{t}+u u_{x}+v u_{y}+\frac{p_{x}}{\rho}=0, \quad v_{t}+u v_{x}+v v_{y}+\frac{p_{y}}{\rho}=0, \tag{3.2}
\end{equation*}
$$

where $p=p(\rho)$ is the equation of state. One can easily see that this system can be written in Hamiltonian form as $\mathbf{u}_{t}+P h_{\mathbf{u}}=0$, where the operator $P$ is given by

$$
P^{i j}=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{3.3}\\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \frac{u_{y}-v_{x}}{\rho} \\
0 & \frac{v_{x}-u_{y}}{\rho} & 0
\end{array}\right),
$$

the Hamiltonian density $h$ is $h(\rho, u, v)=\frac{1}{2} \rho\left(u^{2}+v^{2}\right)+k(\rho)$, and the equation of state $p$ and the function $k$ are related by $p_{\rho}=\rho k_{\rho \rho}$. Clearly, in this case there is no linear combination $g+\lambda \tilde{g}$ such that $\operatorname{det}(g+\lambda \tilde{g}) \neq 0$.

According to Lemma 1.3, in the degenerate situation the $x$-part and the $y$-part of the operator (3.1), which we call respectively $P_{(x)}$ and $P_{(y)}$, must both define one-dimensional

Hamiltonian operators of hydrodynamic type. Unfortunately, since both metrics $g^{i j}$ and $\tilde{g}^{i j}$ are degenerate, Dubrovin-Novikov theorem (Theorem 1.1) does not hold. This leads to two main problems:

1. The symbols $b_{k}^{i j}$ and $\tilde{b}_{k}^{i j}$ are not uniquely determined by the respective metrics, that is, they are no longer the contravariant Christoffel symbols of $g^{i j}$ and $\tilde{g}^{i j}$.
2. The analogues of Darboux theorem for Hamiltonian operators (which is a straightforward consequence of Dubrovin-Novikov theorem) does not hold, that is, we cannot reduce a degenerate one-dimensional first-order Hamiltonian operator of hydrodynamic type to constant coefficient form.

Therefore, in order to discuss the classification of degenerate Hamiltonian operators in 2D, we firstly need to deal with the one-dimensional case.

In $2+1$ dimensions, a quasilinear system is said to be integrable if it can be decoupled in infinitely many ways into a pair of compatible $m$-component 1D systems in Riemann invariants [43]. Ferapontov and Khusnutdinova proved that the requirement of the existence of sufficiently many $m$-component reductions provides an effective classification criterion. The method of hydrodynamic reductions, which is a natural analogue of the generalised hodograph transform in higher dimensions, leads to finite-dimensional moduli spaces of integrable Hamiltonians.

The purpose of this chapter is two-fold. Starting from the classification of degenerate brackets in 1D, we want to describe degenerate Hamiltonian operators of hydrodynamic type in 2D. Our analysis leads to a complete classification of two- and three-component degenerate structures. Secondly, we study the integrability, by the method of hydrodynamic reductions, of Hamiltonian systems arising from three-component structures we classified.

The one-dimensional discussion is based on the author's paper [87]. The classification of 2D structures and the analysis of integrability is based on the author's paper [86].

### 3.1 Degenerate one-dimensional Hamiltonian operators

One-dimensional Poisson structures (Hamiltonian operators) of hydrodynamic type, already mentioned in Section 1.2.3 formula (1.14), namely

$$
\begin{equation*}
P^{i j}(\mathbf{u})=g^{i j}(\mathbf{u}) \frac{d}{d x}+b_{k}^{i j}(\mathbf{u}) u_{x}^{k}, \tag{3.4}
\end{equation*}
$$

with degenerate metric, that is $\operatorname{det}\left(g^{i j}\right)=0$, where first studied by Grinberg [56] in 1985, and later investigated by Bogoyavlenskij [9, 10]. The requirement that such structures satisfy skew-symmetry and Jacobi conditions implies constraints on the differentialgeometric objects $g^{i j}$ and $b_{k}^{i j}$.

Theorem 3.1 ([56]). Operator (3.4) defines a Poisson structure if and only if the pair $(g, b)$ satisfies the following conditions

$$
\begin{gather*}
g^{i j}=g^{j i},  \tag{3.5}\\
\frac{\partial g^{i j}}{\partial u^{k}}=b_{k}^{i j}+b_{k}^{j i},  \tag{3.6}\\
g^{t k} b_{t}^{j i}=g^{t j} b_{t}^{k i},  \tag{3.7}\\
b_{t}^{i j} b_{r}^{t k}-b_{t}^{i k} b_{r}^{t j}=g^{t i}\left(\frac{\partial b_{r}^{j k}}{\partial u^{t}}-\frac{\partial b_{t}^{j k}}{\partial u^{r}}\right),  \tag{3.8}\\
\sum_{(i, j, k)}\left[\left(\frac{\partial b_{t}^{i j}}{\partial u^{q}}-\frac{\partial b_{q}^{i j}}{\partial u^{t}}\right) b_{r}^{t k}+\left(\frac{\partial b_{t}^{i j}}{\partial u^{r}}-\frac{\partial b_{r}^{i j}}{\partial u^{t}}\right) b_{q}^{t k}\right]=0, \tag{3.9}
\end{gather*}
$$

where $\sum_{(i, j, k)}$ means cyclic summation over $i, j, k$.
As mentioned in Section 1.3, this result has been later generalised to the multi-dimensional case by Mokhov, see Theorem 1.2. To the best of our knowledge, up to now a fully geometric interpretation of these equations (under the assumption of degeneracy) is not clear. Moreover, there is no classification of such structures in the literature. Our first aim is to obtain this classification up to three-component case.

In the non-degenerate situation, there always exists a system of coordinates where the pair ( $g, b$ ) assumes constant form. In the degenerate case, this is not true, but a weaker result holds:

Theorem 3.2 ([56]). Suppose that the bivector (3.4) defines a $n$-component Hamiltonian operator, and $\operatorname{rank}\left(g^{i j}\right)=m \leq n$. Then $g^{i j}$ can be reduced to a constant form.

Although we can easily classify all possible canonical forms for degenerate constant metrics, the symbols $b_{k}^{i j}$ are no longer defined through $g^{i j}$. Fixing $g^{i j}$, the coefficients $b_{k}^{i j}$ can be found solving equations (3.5)-(3.9).

In her paper [56], Grinberg gives a description of two- and three-component degenerate Hamiltonian operators (one-component case is trivial), without explicitly writing out the canonical forms. Here, starting from her results, we list all possible canonical forms, up to arbitrary changes of dependent variables.

Remark. Once the metric is fixed and Grinberg's conditions are solved, in order to reduce them to canonical forms we need a change of coordinates which preserves the form of the metric. Following [56], this class of transformations is called admissible. Unfortunately, in general, under admissible change of coordinates, the symbols $b_{k}^{i j}$ do not transform like components of a ( 2,1 )-tensor.

Lemma 3.3 ([87]). Suppose that $0<\operatorname{rank}\left(g^{i j}\right)=m<n$. Among all transformations which preserve the form of the constant metric $g^{i j}$, those which transform the symbols $b_{k}^{i j}$ as components of a $(2,1)$-tensor must be of the form

$$
\begin{equation*}
v^{r}\left(u^{1}, \ldots, u^{n}\right)=c_{1}^{r} u^{1}+\ldots c_{m}^{r} u^{m}+F^{r}\left(u^{m+1}, \ldots, u^{n}\right), \quad r=1, \ldots, n, \tag{3.10}
\end{equation*}
$$

where $c_{s}^{i}$ are constants and $c_{s}^{r}=0$ for $r \in\{m+1, \ldots, n\}$.
Of course, the requirement of admissibility imposes some further restrictions on the coefficients $c_{s}^{r}$.

### 3.1.1 Two-component case

Fixing the number of components, degenerate metrics can be characterised by their rank. In particular, for $n=2$, we have to investigate metrics with $\operatorname{rank}\left(g^{i j}\right)=0,1$. In twocomponent case, we have only two canonical forms.

Theorem 3.4. Any non-trivial degenerate two-component Hamiltonian operator of DubrovinNovikov type in 1D can be brought, by a change of the dependent variables, to one of the following two canonical forms:

1. Constant form

$$
P=\left(\begin{array}{cc}
\partial_{x} & 0  \tag{3.11}\\
0 & 0
\end{array}\right),
$$

2. Non-constant form

$$
P=\left(\begin{array}{cc}
\partial_{x} & -\frac{u_{x}^{2}}{u^{1}}  \tag{3.12}\\
\frac{u_{x}^{2}}{u^{1}} & 0
\end{array}\right) .
$$

## Proof:

If $\operatorname{rank}\left(g^{i j}\right)=0$ then the Hamiltonian operator is identically zero [56]. Suppose $\operatorname{rank}\left(g^{i j}\right)=$ 1 , thus the metric can be reduced by local changes to constant form. Without any loss of generality we can assume

$$
g^{i j}=\left(\begin{array}{ll}
1 & 0  \tag{3.13}\\
0 & 0
\end{array}\right)
$$

By a straightforward computation we obtain that all $b_{k}^{i j}$ vanish except $b_{2}^{12}=-b_{2}^{21}$, which has to satisfy the condition

$$
\partial_{1} b_{2}^{12}=\left(b_{2}^{12}\right)^{2} .
$$

If $b_{2}^{12}=0$, all the coefficients $b_{k}^{i j}$ vanish and we have the constant solution (3.11). Otherwise, for $b_{2}^{12} \neq 0$ we get

$$
b_{2}^{12}=\frac{1}{f\left(u^{2}\right)-u^{1}}
$$

for an arbitrary $f\left(u^{2}\right)$. Applying the admissible transformation

$$
\begin{equation*}
v^{1}=u^{1}-f\left(u^{2}\right), \quad v^{2}=u^{2}, \tag{3.14}
\end{equation*}
$$

we can reduce $b_{2}^{12}$ to $-\frac{1}{v^{1}}$ obtaining (3.12).
Remark. A generic admissible transformation for the metric (3.13) is given by

$$
\begin{equation*}
v^{1}=u^{1}+F\left(u^{2}\right), \quad v^{2}=G\left(u^{2}\right), \tag{3.15}
\end{equation*}
$$

and therefore it is of the form (3.10). This implies that the symbols $b_{k}^{i j}$ transform as tensors so that the structures (3.11) and (3.12) cannot be equivalent. Indeed, in the first case the coefficients $b_{k}^{i j}$ vanish, while in (3.12) they are non-zero.

### 3.1.2 Three-component case

In the three-component case there are three distinct possibilities: $\operatorname{rank}\left(g^{i j}\right)=0,1,2$.

Theorem 3.5. Any non-trivial degenerate three-component Hamiltonian operator of DubrovinNovikov type in 1D can be brought, by a change of the dependent variables, to one of the following canonical forms:

- $\operatorname{rank}(g)=0:$

$$
P=\left(\begin{array}{ccc}
0 & u_{x}^{3} & 0  \tag{3.16}\\
-u_{x}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

- $\operatorname{rank}(g)=1:$

$$
\begin{align*}
& P=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{x} & u_{x}^{3} & 0 \\
-u_{x}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{x} & 0 & -\frac{u_{x}^{3}}{u^{1}} \\
0 & 0 & 0 \\
\frac{u_{x}^{3}}{u^{1}} & 0 & 0
\end{array}\right),  \tag{3.17}\\
& P=\left(\begin{array}{ccc}
\partial_{x} & -\frac{u_{x}^{2}}{u^{1}} & -\frac{u_{x}^{3}}{u^{1}} \\
\frac{u_{x}^{x}}{u^{1}} & 0 & 0 \\
\frac{u_{x}^{3}}{u^{1}} & 0 & 0
\end{array}\right),
\end{align*}
$$

- $\operatorname{rank}(g)=2$ :

$$
\begin{align*}
& P=\left(\begin{array}{ccc}
0 & \partial_{x} & 0 \\
\partial_{x} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), P=\left(\begin{array}{ccc}
0 & \partial_{x} & -\frac{u_{x}^{3}}{u^{2}} \\
\partial_{x} & 0 & 0 \\
\frac{u_{x}^{3}}{u^{2}} & 0 & 0
\end{array}\right), P=\left(\begin{array}{ccc}
0 & \partial_{x} & \frac{u_{x}^{3}}{u^{3} u^{1}-u^{2}} \\
\partial_{x} & 0 & \frac{-u^{3} u_{x}^{3}}{u^{3} u^{1}-u^{2}} \\
\frac{-u_{x}^{3}}{u^{3} u^{1}-u^{2}} & \frac{u^{3} u_{x}^{3}}{u^{3} u^{1}-u^{2}} & 0
\end{array}\right),  \tag{3.18}\\
& P=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & \partial_{x} & 0 \\
0 & 0 & 0
\end{array}\right), P=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & \partial_{x} & -\frac{u_{x}^{3}}{u^{2}} \\
0 & \frac{u_{x}^{3}}{u^{2}} & 0
\end{array}\right), P=\left(\begin{array}{ccc}
\partial_{x} & 0 & \frac{-u^{3} u_{x}^{3}}{u^{3} u^{1}-u^{2}} \\
0 & \partial_{x} & \frac{u_{x}^{3}}{u^{3} u^{x}-u^{2}} \\
\frac{u^{3} u_{x}^{3}}{u^{3} u^{2}-u^{2}} & \frac{-u_{x}^{3}}{u^{3} u^{2}-u^{2}} & 0
\end{array}\right) . \tag{3.19}
\end{align*}
$$

Furthermore, the canonical forms (3.18) and (3.19) are equivalent under complex transformations.
The proof of this theorem can be obtained by a straightforward computation, and can be found in the appendix of [87].

### 3.2 Degenerate two-dimensional Hamiltonian operators

Classification of degenerate operators of hydrodynamic type (3.1) in 2D can be obtained solving general Mokhov's conditions (1.19)-(1.25) directly, but the analysis of these conditions is not straightforward. In order to deal with it, we firstly fix the pair $(g, b)$ given by the classification of 1D Hamiltonian operators described in the previous section, and then we find $(\tilde{g}, \tilde{b})$ solving (1.19)-(1.25). After that, we look for canonical forms of 2D structures using transformations which preserve the form of the first structure given by $(g, b)$. As we will see, in some cases these transformations are not enough to eliminate all the functional parameters appearing in the 2D structure. Let us firstly introduce a definition.

Definition 3.1. A degenerate Hamiltonian operator of the form (3.1) is called trivial if it is identically zero, or if it can be reduced to the form

$$
\begin{equation*}
\tilde{g}^{i j}=\xi g^{i j}, \quad \tilde{b}_{k}^{i j}=\xi b_{k}^{i j} \tag{3.20}
\end{equation*}
$$

for $\xi$ constant.

Notice that allowing linear changes of the independent variables $x, y$, an operator satisfying (3.20) is essentially 1 D .

### 3.2.1 Two-component case

Here we provide a full description of the two-component case.

Theorem 3.6. Any non-trivial degenerate two-component Hamiltonian operator of DubrovinNovikov type in 2D can be brought, by a change of the dependent variables, to the following form

$$
P=\left(\begin{array}{cc}
\partial_{x}+u^{2} \partial_{y}+\frac{1}{2} u_{y}^{2} & -\epsilon \frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}}  \tag{3.21}\\
\epsilon \frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}} & 0
\end{array}\right),
$$

where $\epsilon$ can be either 0 or 1 .

## Proof:

First of all, the case $g=\tilde{g}=0$ gives no non-trivial solutions. In the case where the rank of the pencil $g^{i j}+\lambda \tilde{g}^{i j}$ is constantly equal to one, there exists a coordinates system $\left(u^{1}, u^{2}\right)$
where

$$
g^{i j}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right), \quad \tilde{g}^{i j}=\left(\begin{array}{ll}
f & 0 \\
0 & 0
\end{array}\right),
$$

here $f=f\left(u^{1}, u^{2}\right)$ is some function. Let us fix the $P_{(x)}$ structure.
Case (3.11). If $h_{k}^{i j}$ are all identically zero, conditions (1.19)-(1.25) imply

$$
f=f\left(u^{2}\right), \quad \tilde{b}_{2}^{11}=\frac{f^{\prime}}{2},
$$

and all other $\tilde{b}_{k}^{i j}$ equal to zero. If $f=\xi$ is constant, than $\tilde{g}=\xi g, \tilde{b}=\xi b$. Otherwise, using a transformation which preserves $P_{(x)}$, that is of the form (3.15), we can easily reduce $f$ to $v^{2}$, obtaining (3.21) with $\epsilon=0$.

Case (3.12). Suppose $b_{2}^{21}=-b_{2}^{12}=\frac{1}{u^{1}}$. Conditions (1.19)-(1.25) imply

$$
f=f\left(u^{2}\right), \quad \tilde{b}_{2}^{11}=\frac{f^{\prime}}{2}, \quad \tilde{b}_{2}^{21}=-\tilde{b}_{2}^{12}=\frac{f}{u^{1}} .
$$

If $f=\xi$ is constant, than $\tilde{g}=\xi g, \tilde{b}=\xi b$. Otherwise, let us assume $f$ non-constant. Transformations which preserve $P_{(x)}$ are given by $u^{1}=v^{1}, u^{2}=F\left(v^{2}\right)$, then we can always choose $F$ such that $f$ reduces to $v^{2}$ in the new coordinate system, obtaining (3.21) with $\epsilon=1$.

### 3.2.2 Three-component case

The analysis of the three-component situation is more complicated. Let us consider separately the cases according to the rank of the pencil $g_{\lambda}=g-\lambda \tilde{g}$. The results can be stated as follows.

Theorem 3.7. $\operatorname{Rank}\left(g_{\lambda}\right)=0$. Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables, to one of the following forms:

$$
P=\left(\begin{array}{ccc}
0 & u_{x}^{3}+u^{1} u_{y}^{3} & 0  \tag{3.22}\\
-u_{x}^{3}-u^{1} u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
0 & u_{x}^{3}+u^{3} u_{y}^{3} & 0 \\
-u_{x}^{3}-u^{3} u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

In this case, we do not need any linear change of the independent variables $x, y$.

Theorem 3.8. $\operatorname{Rank}\left(g_{\lambda}\right)=1$. Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables and linear change of $x$ and $y$, to one of the following forms:

$$
\begin{gather*}
P=\left(\begin{array}{ccc}
\partial_{x}+\epsilon\left(u^{2} \partial_{y}+\frac{u_{y}^{2}}{2}\right) & 0 & h u_{y}^{2} \\
0 & 0 & 0 \\
-h u_{y}^{2} & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{x}+f \partial_{y}+\frac{\partial_{2} f u_{y}^{2}+\partial_{3} f u_{y}^{3}}{2} & u_{x}^{3}+h u_{y}^{3} & 0 \\
-u_{x}^{3}-h u_{y}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \\
P=\left(\begin{array}{ccc}
\partial_{x}+f \partial_{y}+\frac{\partial_{2} f u_{y}^{2}+\partial_{3} f u_{y}^{3}}{2} & 0 & -\frac{u_{x}^{3}-h u_{y}^{2}+f u_{y}^{3}}{u^{1}} \\
0 & 0 & 0 \\
\frac{u_{x}^{3}-h u_{y}^{2}+f u_{y}^{3}}{u^{1}} & 0 & 0
\end{array}\right),  \tag{3.23}\\
P=\left(\begin{array}{ccc}
\partial_{x}+u^{2} \partial_{y}+\frac{u_{y}^{2}}{2} & -\frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}} & -\frac{u_{x}^{3}+u^{2} u_{y}^{3}}{u^{1}} \\
\frac{u_{x}^{2}+u^{2} u_{y}^{2}}{u^{1}} & 0 & 0 \\
\frac{u_{x}^{3}+u^{2} u_{y}^{3}}{u^{1}} & 0 & 0
\end{array}\right), \tag{3.25}
\end{gather*}
$$

where $f=f\left(u^{2}, u^{3}\right), h=h\left(u^{2}, u^{3}\right)$ are arbitrary functions and $\epsilon$ can be either 0 or 1 .

Theorem 3.9. $\operatorname{Rank}\left(g_{\lambda}\right)=2$. Any non-trivial degenerate three-component Hamiltonian operator of Dubrovin-Novikov type in 2D can be brought, by a change of the dependent variables and linear change of $x$ and $y$, to one of the following forms:

$$
\begin{gather*}
P=\left(\begin{array}{ccc}
-2 u^{1} \partial_{y}-u_{y}^{1} & \partial_{x}+u^{2} \partial_{y}+2 u_{y}^{2} & \epsilon u_{y}^{3} \\
\partial_{x}+u^{2} \partial_{y}-u_{y}^{2} & 0 & 0 \\
-\epsilon u_{y}^{3} & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
0 & \partial_{x} & \partial_{y} \\
\partial_{x} & 0 & 0 \\
\partial_{y} & 0 & 0
\end{array}\right),  \tag{3.26}\\
P=\left(\begin{array}{ccc}
p \partial_{y}+\frac{p^{\prime} u_{y}^{3}}{2} & \partial_{x}+q \partial_{y}+\epsilon u_{y}^{3} & 0 \\
\partial_{x}+\partial_{y} q-\epsilon u_{y}^{3} & r \partial_{y}+\frac{r^{\prime} u_{y}^{3}}{2} & 0 \\
0 & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
\partial_{y} & \partial_{x} & -\frac{u_{x}^{3}}{u^{2}} \\
\partial_{x} & 0 & 0 \\
\frac{u_{x}^{3}}{u^{2}} & 0 & 0
\end{array}\right),  \tag{3.27}\\
P=\left(\begin{array}{ccc}
\epsilon \partial_{y} & \partial_{x}+u^{3} \partial_{y} & -\frac{u_{x}^{3}+u^{3} u_{y}^{3}}{u^{2}} \\
\partial_{x}+\partial_{y} u^{3} & 0 & 0 \\
\frac{u_{x}^{3}+u^{3} u_{y}^{3}}{u^{2}} & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
0 & \partial_{x} & -\frac{u_{x}^{3}-u_{y}^{1}}{u^{2}} \\
\partial_{x} & 0 & \partial_{y} \\
\frac{u_{x}^{3}-u_{y}^{1}}{u^{2}} & \partial_{y} & 0
\end{array}\right), \tag{3.28}
\end{gather*}
$$

$$
\begin{gather*}
P=\left(\begin{array}{ccc}
0 & \partial_{x} & \partial_{y}-\frac{u_{x}^{3}-u_{y}^{2}}{u^{2}} \\
\partial_{x} & 0 & 0 \\
\partial_{y}+\frac{u_{x}^{3}-u_{y}^{2}}{u^{2}} & 0 & 0
\end{array}\right), \quad P=\left(\begin{array}{ccc}
u^{1} \partial_{y}+\frac{u_{y}^{1}}{2} & \partial_{x}-\frac{u^{2}}{2} \partial_{y}-u_{y}^{2} & -\frac{u_{x}^{3}}{u^{2}} \\
\partial_{x}-\frac{u^{2}}{2} \partial_{y}+\frac{u_{y}^{2}}{2} & 0 & 0 \\
\frac{u_{x}^{3}}{u^{2}} & 0 & 0
\end{array}\right),  \tag{3.29}\\
P=\left(\begin{array}{ccc}
\partial_{y} & \partial_{x}-u^{3} \partial_{y} & \frac{u_{x}^{3}-2 u^{3} u_{y}^{3}}{u^{3} u^{1}} \\
\partial_{x}-\partial_{y} u^{3} & u^{3} \partial_{y} u^{3} & -\frac{\left.u^{3} u_{x}^{3}-2 u^{3}\right)^{3} u_{y}^{3}}{u^{3} u^{1}-u^{2}} \\
-\frac{u_{x}^{3}-2 u^{3} u_{y}^{3}}{u^{3} u^{1}-u^{2}} & \frac{u^{3} u_{x}^{3}-2\left(u^{3}\right)^{2} u_{y}^{3}}{u^{3} u^{1}-u^{2}} & 0
\end{array}\right),  \tag{3.30}\\
P=\left(\begin{array}{ccc}
\kappa \partial_{y} \frac{1}{u^{3}} & \partial_{x}-\kappa \partial_{y}+\frac{\kappa u_{y}^{3}}{2 u^{3}} & \frac{u_{x}^{3}-2 \kappa u_{y}^{3}}{u^{3} u^{1} u^{2}} \\
\partial_{x}-\kappa \partial_{y}-\frac{\kappa u_{y}^{3}}{2 u^{3}} & \kappa u^{3} \partial_{y}+\frac{u_{y}^{3}}{2} & -\frac{u^{3} u_{x}^{3}-2 \kappa u^{3} u_{y}^{3}}{u^{3} u^{1}-u^{2}} \\
-\frac{u_{x}^{3}-2 \kappa u_{y}^{3}}{u^{3} u^{1}-u^{2}} & \frac{u^{3} u_{x}^{3}-2 \kappa u^{3} u_{y}^{3}}{u^{3} u^{1}-u^{2}} & 0
\end{array}\right), \tag{3.31}
\end{gather*}
$$

where $p, q, r$ are arbitrary functions on $u^{3}, \kappa$ is constant and $\epsilon$ can be either 0 or 1 .
The proofs of these theorems can be found in the appendix of [86]. Let us point out that, after the change of coordinates $u^{1}=u, u^{2}=\rho, u^{3}=v$, the operator $(3.28)_{2}$ corresponds to the Hamiltonian operator for the 2D equations of gas dynamics (3.3).

### 3.3 Hamiltonian systems of hydrodynamic type in 2+1 dimensions

In this section we discuss (2+1)-dimensional Hamiltonian systems of hydrodynamic type,

$$
\begin{equation*}
\mathbf{u}_{t}+A(\mathbf{u}) \mathbf{u}_{x}+B(\mathbf{u}) \mathbf{u}_{y}=0, \tag{3.32}
\end{equation*}
$$

which are representable in the form $\mathbf{u}_{t}+P h_{\mathbf{u}}=0$, where $h(\mathbf{u})$ is a Hamiltonian density and $P$ is a two-dimensional Hamiltonian operator of differential-geometric type (3.1). A $(2+1)$-dimensional quasi-linear system is said to be integrable if it can be decoupled in infinitely many ways into a pair of compatible $m$-component one-dimensional systems in Riemann invariants. This definition of integrability follows from the method of hydrodynamic reduction introduced by Ferapontov and Khusnutdinova in [43].

### 3.3.1 The method of hydrodynamic reductions

The method of hydrodynamic reductions is based on the existence of exact solutions of (2+1)-dimensional system (3.32) of the form $\mathbf{u}=\mathbf{u}\left(R^{1}, \ldots, R^{m}\right)$, where the Riemann in-
variants $\mathbf{R}=\left(R^{1}, \ldots, R^{m}\right)$ solve a pair of commuting diagonal systems

$$
\begin{equation*}
R_{t}^{i}=\lambda^{i}(\mathbf{R}) R_{x}^{i}, \quad R_{y}^{i}=\mu^{i}(\mathbf{R}) R_{x}^{i} . \tag{3.33}
\end{equation*}
$$

Let us point out that we do not impose any constraint on the number of Riemann invariants: $m$ is arbitrary. Therefore, the ( $2+1$ )-dimensional system (3.32), is decoupled into a pair of diagonal (1+1)-dimensional systems given by (3.33). These solutions are known as nonlinear interactions of $m$ planar simple waves.

It turns out that the commutativity of the flows (3.33) is equivalent to the following constraints on the characteristic speeds $\lambda^{i}, \mu^{i}$ [96]:

$$
\begin{equation*}
\frac{\partial_{j} \lambda^{i}}{\lambda^{j}-\lambda^{i}}=\frac{\partial_{j} \mu^{i}}{\mu^{j}-\mu^{i}}, \quad i \neq j, \quad \partial_{j}=\frac{\partial}{\partial R^{j}}, \tag{3.34}
\end{equation*}
$$

(no summation). Imposing these restrictions, the general solution of systems (3.33) is given by the implicit generalised hodograph formula [96]

$$
\begin{equation*}
v^{i}(\mathbf{R})=x+\lambda^{i}(\mathbf{R}) t+\mu^{i}(\mathbf{R}) y, \quad i=1, \ldots, m . \tag{3.35}
\end{equation*}
$$

Here the functions $v^{i}(\mathbf{R})$ are characteristic speeds of the general flow commuting with (3.33), namely, the general solution of the linear system

$$
\begin{equation*}
\frac{\partial_{j} v^{i}}{v^{j}-v^{i}}=\frac{\partial_{j} \lambda^{i}}{\lambda^{j}-\lambda^{i}}=\frac{\partial_{j} \mu^{i}}{\mu^{j}-\mu^{i}}, \quad i \neq j . \tag{3.36}
\end{equation*}
$$

By straightforward computation, the substitution of $\mathbf{u}\left(R^{1}, \ldots, R^{m}\right)$ into (3.32), using (3.33), leads to

$$
\begin{equation*}
\left(E+\lambda^{i} A+\mu^{i} B\right) \partial_{i} \mathbf{u}=0, \quad i=1, \ldots, m \tag{3.37}
\end{equation*}
$$

where $E$ is the $n \times n$ identity matrix. This means that both $\lambda^{i}$ and $\mu^{i}$ have to satisfy the dispersion relation

$$
\begin{equation*}
\operatorname{det}\left(E+\lambda^{i} A+\mu^{i} B\right)=0 \tag{3.38}
\end{equation*}
$$

The construction of nonlinear interactions of $m$ planar simple waves can be summarised as follows. First of all, we have to decouple the initial (2+1)-dimensional system (3.32) into a pair of commuting flows (3.33), by solving the equations (3.34), (3.37) for $\mathbf{u}(\mathbf{R}), \lambda^{i}(\mathbf{R})$, $\mu^{i}(\mathbf{R})$ as functions depending on the Riemann invariants $R^{1}, \ldots, R^{m}$. It is not difficult to
see that for $m \geq 3$ the system given by these equations is overdetermined. Thus, in general this system does not possess solutions. However, if we are able to construct a particular reduction of the form (3.33), the second step is quite straightforward: we have to solve the linear system given by (3.36) for the functions $v^{i}(\mathbf{R})$, and then obtain $R^{1}, \ldots, R^{m}$ as functions of $t, x, y$ from the implicit hodograph formula (3.35).

What can we say about the number of $m$-component reductions that a ( $2+1$ )-dimensional system (3.32) may admit? Analysing equations (3.34) and (3.37), one can prove that this number is parametrised, up to changes of variables of the form $R^{i} \rightarrow f^{i}\left(R^{i}\right)$, by $m$ arbitrary functions of a single variable. Remarkably, this number does not depend on $n$. This leads to the following definition.

Definition 3.2 ([43]). A (2+1)-dimensional quasi-linear system is said to be integrable if it possesses $m$-component reductions of the form (3.33) parametrised by marbitrary functions of a single argument.

Looking at the structure of equations (3.34) and (3.37), one can see that their consistency conditions involve only triples of indices $i \neq j \neq k$. Moreover, all these conditions are completely symmetric in $i, j$ and $k$, and then it is enough to verify them setting, for instance, $i=1, j=2, k=3$. This means that the existence of non-trivial three-component reductions implies the existence of $m$-component reductions for arbitrary $m$.

Remark. We require that $\lambda^{i}$ and $\mu^{i}$ do not satisfy any linear relation, otherwise we would not have sufficiently many arbitrary functions of a single argument. Indeed, let us suppose that $\mu^{i}=a \lambda^{i}+b$. Condition (3.34) reads $\partial_{j} a \lambda^{i}+\partial_{j} b=0$, which implies $a$ and $b$ constant. Thus, solutions of the system (3.33),

$$
R_{t}^{i}=\lambda^{i} R_{x}^{i}, \quad R_{y}^{i}=\left(a \lambda^{i}+b\right) R_{x}^{i},
$$

are of the form $R^{i}=R^{i}(x+b y, t+a y)$. These solutions correspond to travelling wave reductions, and they clearly do not contain enough arbitrary functions.

### 3.3.2 Generalised two-dimensional gas dynamics equations

The equations of two-dimensional isentropic gas dynamics (3.2) can be written in the matrix form (3.32) where $\mathbf{u}=(\rho, u, v)^{t}$ and

$$
A=\left(\begin{array}{ccc}
u & \rho & 0 \\
\frac{c^{2}}{\rho} & u & 0 \\
0 & 0 & u
\end{array}\right), \quad B=\left(\begin{array}{ccc}
v & 0 & \rho \\
0 & v & 0 \\
\frac{c^{2}}{\rho} & 0 & v
\end{array}\right),
$$

here $c^{2}=p^{\prime}(\rho)$ is the sound speed. As demonstrated in [90], there exist potential flows describing nonlinear interaction of two sound waves which are locally parametrised by four arbitrary functions of a single argument.

Furthermore, as we have already mentioned, the system (3.2) admits the Hamiltonian formulation $\mathbf{u}_{t}+P h_{\mathbf{u}}=0$ where $P$ is given by (3.3) and the Hamiltonian density $h$ is $h(\rho, u, v)=\frac{1}{2} \rho\left(u^{2}+v^{2}\right)+k(\rho)$. Let us now assume $h=h(\rho, u, v)$ generic, hence the system $\mathbf{u}_{t}+P h_{\mathbf{u}}=0$ reads

$$
\begin{equation*}
\rho_{t}+\left(h_{u}\right)_{x}+\left(h_{v}\right)_{y}=0, \quad u_{t}+\left(h_{\rho}\right)_{x}+\frac{u_{y}-v_{x}}{\rho} h_{v}=0, \quad v_{t}+\left(h_{\rho}\right)_{y}+\frac{v_{x}-u_{y}}{\rho} h_{u}=0 . \tag{3.39}
\end{equation*}
$$

Let us consider the Riemann invariants $R^{1}, \ldots, R^{m}$ solving

$$
R_{x}^{i}=\lambda^{i}(\mathbf{R}) R_{t}^{i}, \quad R_{y}^{i}=\mu^{i}(\mathbf{R}) R_{t}^{i}, \quad i=1, \ldots, m .
$$

By straightforward computation, the substitution $\rho=\rho(\mathbf{R}), u=u(\mathbf{R}), v=v(\mathbf{R})$ into (3.39) implies

$$
\begin{gather*}
\left(1+\lambda^{i} h_{\rho u}+\mu^{i} h_{\rho v}\right) \partial_{i} u+h_{\rho \rho} \lambda^{i} \partial_{i} \rho=0,  \tag{3.40}\\
\left(1+\lambda^{i} h_{\rho u}+\mu^{i} h_{\rho v}\right) \partial_{i} v+h_{\rho \rho} \mu^{i} \partial_{i} \rho=0,  \tag{3.41}\\
\left(1+\lambda^{i} h_{\rho u}+\mu^{i} h_{\rho v}\right) \partial_{i} \rho+\left(\lambda^{i} h_{u u}+\mu^{i} h_{u v}\right) \partial_{i} u+\left(\lambda^{i} h_{u v}+\mu^{i} h_{v v}\right) \partial_{i} v=0, \tag{3.42}
\end{gather*}
$$

here $i=1, \ldots, m, \partial_{i}=\frac{\partial}{\partial R^{i}}$. Note that since $\mu^{i} \partial_{i} u=\lambda^{i} \partial_{i} v$ (this easy follows from (3.40) and (3.41), assuming $1+\lambda^{i} h_{\rho u}+\mu^{i} h_{\rho v} \neq 0$ ), one has $u_{y}=v_{x}$. Thus, solutions are necessarily
potential. Then, setting $u=\varphi_{x}$ and $v=\varphi_{y}$, system (3.39) reads

$$
\begin{equation*}
\rho_{t}+\left(h_{u}\right)_{x}+\left(h_{v}\right)_{y}=0, \quad \varphi_{x t}+\left(h_{\rho}\right)_{x}=0, \quad \varphi_{y t}+\left(h_{\rho}\right)_{y}=0 . \tag{3.43}
\end{equation*}
$$

The last two equations give $\varphi_{t}+h_{\rho}=0$, so we finally have the following system

$$
\begin{equation*}
\rho_{t}+\left(h_{u}\right)_{x}+\left(h_{v}\right)_{y}=0, \quad \varphi_{t}+h_{\rho}=0 . \tag{3.44}
\end{equation*}
$$

If we consider the partial Legendre transform

$$
\begin{equation*}
\tilde{\rho}=h_{\rho}, \quad \tilde{u}=u, \quad \tilde{v}=v, \quad \tilde{h}=h-\rho h_{\rho}, \tag{3.45}
\end{equation*}
$$

the derivatives with respect to the new variables are

$$
\begin{equation*}
\tilde{h}_{\tilde{\rho}}=-\rho, \quad \tilde{h}_{\tilde{u}}=h_{u}, \quad \tilde{h}_{\tilde{v}}=h_{v}, \tag{3.46}
\end{equation*}
$$

and we can rewrite system (3.44) in the form

$$
\left(\tilde{h}_{\tilde{\rho}}\right)_{t}+\left(\tilde{h}_{\tilde{u}}\right)_{x}+\left(\tilde{h}_{\tilde{v}}\right)_{y}=0, \quad \varphi_{t}=\tilde{\rho}, \quad \varphi_{x}=\tilde{u}, \quad \varphi_{y}=\tilde{v}
$$

The function $\tilde{h}$ depends only on $\varphi_{x}, \varphi_{y}, \varphi_{t}$ and thus we obtain a three-dimensional EulerLagrange equation (setting $\tilde{h}=f$ )

$$
\begin{equation*}
\left(f_{\varphi_{x}}\right)_{x}+\left(f_{\varphi_{y}}\right)_{y}+\left(f_{\varphi_{t}}\right)_{t}=0 \tag{3.47}
\end{equation*}
$$

corresponding to a Lagrangian density of the form $f\left(\varphi_{x}, \varphi_{y}, \varphi_{t}\right)$. For example, the Lagrangian density $f=u_{x}^{2}+u_{y}^{2}-2 e^{u_{t}}$ leads to the Boyer-Finley equation $u_{x x}+u_{y y}=e^{u_{t}} u_{t t}$ [13].

In [45] Ferapontov, Khusnutdinova and Tsarev derived a system of partial differential equations for the Lagrangian density $f\left(\varphi_{x}, \varphi_{y}, \varphi_{t}\right)$ which are necessary and sufficient for the integrability of the equation (3.47) by the method of hydrodynamic reductions (see also [49] for further details). Setting $a=\varphi_{x}, b=\varphi_{y}, c=\varphi_{t}$, these conditions can be represented in a remarkable compact form:

Theorem 3.10 ([45]). For a non-degenerate Lagrangian, the Euler-Lagrange equation (3.47) is
integrable by the method of hydrodynamic reductions if and only if the density $f$ satisfies the relation

$$
\begin{equation*}
d^{4} f=d^{3} f \frac{d H}{H}+\frac{3}{H} \operatorname{det}(d M) ; \tag{3.48}
\end{equation*}
$$

here $d^{3} f$ and $d^{4} f$ are the symmetric differentials of $f$. The Hessian $H$ and the $4 \times 4$ matrix $M$ are defined as follows:

$$
H=\operatorname{det}\left(\begin{array}{ccc}
f_{a a} & f_{a b} & f_{a c}  \tag{3.49}\\
f_{a b} & f_{b b} & f_{b c} \\
f_{a c} & f_{b c} & f_{c c}
\end{array}\right), \quad M=\left(\begin{array}{cccc}
0 & f_{a} & f_{b} & f_{c} \\
f_{a} & f_{a a} & f_{a b} & f_{a c} \\
f_{b} & f_{a b} & f_{b b} & f_{b c} \\
f_{c} & f_{a c} & f_{b c} & f_{c c}
\end{array}\right) .
$$

The differential $d M=M_{a} d a+M_{b} d b+M_{c} d c$ is a matrix-valued form

$$
\begin{aligned}
d M= & \left(\begin{array}{cccc}
0 & f_{a a} & f_{a b} & f_{a c} \\
f_{a a} & f_{a a a} & f_{a a b} & f_{a a c} \\
f_{a b} & f_{a a b} & f_{a b b} & f_{a b c} \\
f_{a c} & f_{a a c} & f_{a b c} & f_{a c c}
\end{array}\right) d a+\left(\begin{array}{cccc}
0 & f_{a b} & f_{b b} & f_{b c} \\
f_{a b} & f_{a a b} & f_{a b b} & f_{a b c} \\
f_{b b} & f_{a b b} & f_{b b b} & f_{b b c} \\
f_{b c} & f_{a b c} & f_{b b c} & f_{b c c}
\end{array}\right) d b \\
& +\left(\begin{array}{cccc}
0 & f_{a c} & f_{b c} & f_{c c} \\
f_{a c} & f_{a a c} & f_{a b c} & f_{a c c} \\
f_{b c} & f_{a b c} & f_{b b c} & f_{b c c} \\
f_{c c} & f_{a c c} & f_{b c c} & f_{c c c}
\end{array}\right) d c .
\end{aligned}
$$

Finally, we recall that the equations of gas dynamics possess double waves only, and are not integrable by the method of hydrodynamic reductions [44]. On the other hand, the generalised equations (3.39) define a (2+1)-dimensional integrable system when the Lagrangian density $f\left(\varphi_{x}, \varphi_{y}, \varphi_{t}\right)$, obtained from the Hamiltonian density $h(\rho, u, v)$ performing a partial Legendre transform (3.45), satisfies the conditions given by Theorem 3.10.

### 3.3.3 Three-component Hamiltonian systems with degenerate structure

We have seen that the degenerate Hamiltonian operator $(3.28)_{2}$ leads to a class of integrable systems related to Lagrangian densities of the form $f\left(\varphi_{x}, \varphi_{y}, \varphi_{t}\right)$. Here we are going to discuss all three-component cases arising from our classification.

The aim of this section is to apply the method of hydrodynamic reductions to threecomponent Hamiltonian systems given by $\mathbf{u}_{t}+P h_{\mathbf{u}}=0$, where $P$ is a Hamiltonian structure appearing in Theorems 3.7, 3.8 and 3.9. Let us identify the Hamiltonian operators we obtained with the rank of the pencil $g_{\lambda}$. For instance, we call rank-zero structures the Hamiltonian operators listed in Theorem 3.7.

Theorem 3.11. The method of hydrodynamic reductions imposes additional differential constraints under which equations under study reduce to known classes of systems considered before:

- rank-zero structures lead to trivial systems

$$
u_{t}^{1}=u_{t}^{2}=u_{t}^{3}=0,
$$

- rank-one structures lead to one dimensional systems of the form

$$
u_{t}^{1}+f\left(u^{1}\right) u_{x}^{1}=0, \quad u_{t}^{2}=u_{t}^{3}=0
$$

- rank-two structures lead either to one dimensional systems to the form

$$
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{3}=0
$$

or two-component non-degenerate Hamiltonian systems

$$
\begin{gather*}
u_{t}^{1}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{2}}\right)_{y}=0  \tag{3.50}\\
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}+\left(h_{u^{2}}\right)_{y}=0,  \tag{3.51}\\
u_{t}^{1}+\left(2 u^{1} h_{u^{1}}+u^{2} h_{u^{2}}-h\right)_{x}+\left(u^{1} h_{u^{2}}\right)_{y}=0, \quad u_{t}^{2}+\left(u^{2} h_{u^{1}}\right)_{x}+\left(2 u^{2} h_{u^{2}}+u^{1} h_{u^{1}}-h\right)_{y}=0, \tag{3.52}
\end{gather*}
$$

plus the trivial equation $u_{t}^{3}=0$, or to the systems

$$
\begin{equation*}
u_{t}^{1}+\left(h_{u^{2}}\right)_{x}+\left(h_{u^{3}}\right)_{y}=0, \quad u_{t}^{2}+\left(h_{u^{1}}\right)_{x}=0, \quad u_{t}^{3}+\left(h_{u^{1}}\right)_{y}=0 . \tag{3.53}
\end{equation*}
$$

We point out that the integrability of two-component non-degenerate Hamiltonian
systems (3.50), (3.51) and (3.52), generated respectively by the Hamiltonian operators

$$
P=\left(\begin{array}{cc}
\partial_{x} & 0 \\
0 & \partial_{y}
\end{array}\right), P=\left(\begin{array}{cc}
0 & \partial_{x} \\
\partial_{x} & \partial_{y}
\end{array}\right), P=\left(\begin{array}{cc}
2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & u^{1} \\
u^{1} & 2 u^{2}
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{cc}
u_{x}^{1} & u_{y}^{1} \\
u_{x}^{2} & u_{y}^{2}
\end{array}\right)
$$

is completely understood, see [50] for further details. Furthermore, as we showed above, system (3.53) reduces to the three-dimensional Euler-Lagrange equations (3.47) after performing a partial Legendre transformation of the form (3.45).

## Proof of Theorem 3.11:

First of all, let us remark that if $u_{t}^{i}=0$, for some $i$, the method of hydrodynamic reductions necessarily implies $u^{i}=$ const. Secondly, if one of the equations of the system is of the form $u_{t}^{i}+\phi(\mathbf{u}) u_{x}^{i}+\psi(\mathbf{u}) u_{y}^{i}=0$, the method of hydrodynamic reductions implies $\left(\lambda^{j}+\phi+\psi \mu^{j}\right) \partial_{j} u^{i}=0$, which leads to $u^{i}=$ const, since we are imposing that $\lambda^{j}$ and $\mu^{j}$ do not satisfy any linear relation. Furthermore, in these cases we can replace $u^{i}$ with a constant, and then the Hamiltonian will depend on $u^{j}$ for $j \neq i$.

Using these observations, the proof is straightforward. Rank-zero structures easily lead to trivial systems. For the rank-one structures we always have $u^{2}$ and $u^{3}$ constant, which leads to an operator of the form

$$
P=\left(\begin{array}{ccc}
\partial_{x}+\kappa \partial_{y} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \kappa=\text { const }
$$

which is essentially one-dimensional (up to linear changes of the independent variables $x$ and $y$ ).

The analysis of rank-two structures is a bit more complicated. In the cases (3.26) ${ }_{1}$ and $(3.29)_{2}$, the method of hydrodynamic reductions implies $u^{3}=$ const. Thus, up to a change of local coordinates $u^{1}, u^{2}$, the $3 \times 3$ degenerate Hamiltonian operator reduces to a direct sum of the $2 \times 2$ two-component non-degenerate Mokhov's Hamiltonian operator [74, 69]

$$
P=\left(\begin{array}{cc}
2 u^{1} & u^{2} \\
u^{2} & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & u^{1} \\
u^{1} & 2 u^{2}
\end{array}\right) \frac{d}{d y}+\left(\begin{array}{cc}
u_{x}^{1} & u_{y}^{1} \\
u_{x}^{2} & u_{y}^{2}
\end{array}\right)
$$

and the trivial $1 \times 1$ operator $P=0$.

In the cases $(3.27)_{1,2},(3.28)_{1}$, (3.30) and (3.31) the method of hydrodynamic reductions implies again $u^{3}=$ const. These structures reduce to direct sum of constant $2 \times 2$ twocomponent non-degenerate Hamiltonian operator, and the trivial $1 \times 1$ operator $P=0$. Constant $2 \times 2$ non-degenerate Hamiltonian operators are known [50]: if they do not reduce to one-dimensional operator

$$
P=\left(\begin{array}{cc}
0 & \partial_{x} \\
\partial_{x} & 0
\end{array}\right),
$$

(for instance, $(3.28)_{1}$ for $\epsilon=1$ ), they can be brought to one of the following two forms

$$
P=\left(\begin{array}{cc}
\partial_{x} & 0 \\
0 & \partial_{y}
\end{array}\right), \quad P=\left(\begin{array}{cc}
0 & \partial_{x} \\
\partial_{x} & \partial_{y}
\end{array}\right),
$$

by a change of local coordinates $u^{1}, u^{2}$ and a linear change of the independent variables $x, y$.

It remains to consider the cases $(3.26)_{2}$ and (3.29) ${ }_{1}$. It is not difficult to see that in both cases we get $u_{y}^{2}=u_{x}^{3}$. Therefore, solutions are necessarily potential. Then, setting $u^{2}=\varphi_{x}$ and $u^{3}=\varphi_{y}$, the system leads to (3.43).

Remark. We have seen that most of the Hamiltonian systems, arising from three-component degenerate Poisson structures of hydrodynamic type in 2D, present an equation of the form $u_{t}^{i}=0$, or $u_{t}^{i}+\phi(\mathbf{u}) u_{x}^{i}+\psi(\mathbf{u}) u_{y}^{i}=0$, which, on application of the method of hydrodynamic reductions, leads to $u^{i}=$ const. It is remarkable that only Poisson structures $(3.26)_{2},(3.28)_{2},(3.29)_{1}$ generate "non-trivial" three-component Hamiltonian systems, and that, by imposing the method of hydrodynamic reductions, these systems reduce to the generalisation of the gas dynamics equations, given by (3.53).

## Deformations of degenerate Dubrovin-Novikov structures

In this chapter we discuss second-order deformations of two-component Poisson structures in 1D we have classified in the previous chapter, and investigate which of those deformations can be obtained by Miura transformations. The Miura group coincides with the semidirect product of the subgroup of diffeomorphisms (local change of coordinates) on the manifold $M$ and the subgroup of Miura-type transformations

$$
\begin{equation*}
u^{i} \rightarrow u^{i}+\epsilon A_{j}^{i}(u) u_{x}^{j}+\epsilon^{2}\left(B_{j}^{i}(u) u_{x x}^{j}+\frac{1}{2} C_{j k}^{i}(u) u_{x}^{j} u_{x}^{k}\right)+\ldots \tag{4.1}
\end{equation*}
$$

see [38] for further details. Following [63], we call the transformations of these two subgroups the Miura-type transformations of the first and second kind respectively.

As we will see, the action of the subgroup of diffeomorphisms is not straightforward: it leads to several branches. Thus, for simplicity, we firstly discuss the action of the subgroup of Miura-type transformations of the second kind, and only at a later time we analyse the action of local changes of coordinates.

Even though higher-order deformations can be obtained following the same procedure, the computations become much more complicated. Furthermore, we also analyse some examples of first-order deformations for three-component structures.

In the non-degenerate case any such deformation is trivial, that is, can be obtained via Miura transformation. In this chapter we demonstrate that in the degenerate case this class of deformations is non-trivial, and depends on a certain number of arbitrary functions.

As defined in the introduction, a deformation of order $k$ of a $n$-component Poisson
bivector $P_{0}$ is a formal series

$$
P^{\epsilon}=P_{0}+\epsilon P_{1}+\epsilon^{2} P_{2}+\ldots
$$

satisfying the condition $\left[P^{\epsilon}, P^{\epsilon}\right]=\mathcal{O}\left(\epsilon^{k+1}\right)$, where each coefficient $P_{k}$ has degree $k+1$, and is given by

$$
P_{k}=\sum_{s=0}^{k+1} A_{s}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{k+1}\right) \frac{d^{k+1-s}}{d x^{k+1-s}}, \quad \operatorname{deg}\left(A_{s}\right)=s
$$

The form of the operator $P_{k}$ depends on an increasing number of arbitrary functions of the coordinates $u^{i}, i=1, \ldots, n$. Furthermore, these functions must be chosen in such a way that $P_{k}$ is skew-symmetric, namely $P_{k}^{*}=-P_{k}$.

In particular, the first two coefficients, $P_{1}$ and $P_{2}$, have the form

$$
\begin{aligned}
P_{1}^{i j}= & A^{i j}(\mathbf{u}) \frac{d^{2}}{d x^{2}}+\sum_{k} B_{k}^{i j}(\mathbf{u}) u_{x}^{k} \frac{d}{d x}+\sum_{k} C_{k}^{i j}(\mathbf{u}) u_{x x}^{k}+\sum_{r \leq k} D_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}, \\
P_{2}^{i j}= & E^{i j}(\mathbf{u}) \frac{d^{3}}{d x^{3}}+\sum_{k} F_{k}^{i j}(\mathbf{u}) u_{x}^{k} \frac{d^{2}}{d x^{2}}+\left(\sum_{k} G_{k}^{i j}(\mathbf{u}) u_{x x}^{k}+\sum_{r \leq k} H_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}\right) \frac{d}{d x} \\
& +\sum_{k} L_{k}^{i j}(\mathbf{u}) u_{x x x}^{k}+\sum_{k, r} M_{k r}^{i j}(\mathbf{u}) u_{x x}^{k} u_{x}^{r} \sum_{s \leq r \leq k}+N_{s r k}^{i j}(\mathbf{u}) u_{x}^{s} u_{s}^{r} u_{x}^{k} .
\end{aligned}
$$

This means that the operator $P_{1}$ is defined by $\frac{n^{2}\left(n^{2}+5 n+2\right)}{2}$ functions depending on the variables $u^{1}, \ldots, u^{n}$, while the operator $P_{2}$ is given by $\frac{n^{2}(n+2)\left(n^{2}+10 n+3\right)}{6}$ functions in the variables $u^{1}, \ldots, u^{n}$. Thus, one can see that the number of unknown functions is quite high already for a low number of components: for $n=2$ we have 104 unknowns, while for $n=3$ they are 432. Of course, imposing the skew-symmetry condition, this number falls.

Remark. In order to simplify the computations, it is convenient to substitute the coefficients $D, H, N$ with $\tilde{D}, \tilde{H}, \tilde{N}$ such that

$$
\begin{array}{llll}
\tilde{D}_{r k}^{i j}=\tilde{D}_{k r}^{i j}=\frac{1}{2} D_{r k}^{i j} \quad \text { if } \quad r<k, & \text { otherwise } & \tilde{D}_{k k}^{i j}=D_{k k}^{i j}, \\
\tilde{H}_{r k}^{i j}=\tilde{H}_{k r}^{i j}=\frac{1}{2} H_{r k}^{i j} \quad \text { if } \quad r<k, & \text { otherwise } & \tilde{H}_{k k}^{i j}=H_{k k}^{i j},
\end{array}
$$

$$
\begin{gathered}
\tilde{N}_{s r k}^{i j}=\tilde{N}_{k s r}^{i j}=\tilde{N}_{r k s}^{i j}=\tilde{N}_{s k r}^{i j}=\tilde{N}_{k r s}^{i j}=\tilde{N}_{r s k}^{i j}=\frac{1}{6} N_{s r k}^{i j} \text { if } s<r<k, \\
\tilde{N}_{r r s}^{i j}=\tilde{N}_{r s r}^{i j}=\tilde{N}_{s r r}^{i j}=\frac{1}{3} N_{r r s}^{i j} \text { if } r<s, \\
\tilde{N}_{r r s}^{i j}=\tilde{N}_{r s r}^{i j}=\tilde{N}_{s r r}^{i j}=\frac{1}{3} N_{s r r}^{i j} \text { if } r>s, \\
\tilde{N}_{r r r}^{i j}=N_{r r r}^{i j} .
\end{gathered}
$$

In this way, the summations involving these coefficients become

$$
\begin{gathered}
\sum_{r \leq k} D_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}=\sum_{r, k} \tilde{D}_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}, \quad \sum_{r \leq k} H_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}=\sum_{r, k} \tilde{H}_{r k}^{i j}(\mathbf{u}) u_{x}^{r} u_{x}^{k}, \\
\sum_{s \leq r \leq k} N_{s r k}^{i j}(\mathbf{u}) u_{x}^{s} u_{s}^{r} u_{x}^{k}=\sum_{s, r, k} \tilde{N}_{s r k}^{i j}(\mathbf{u}) u_{x}^{s} u_{x}^{r} u_{x}^{k} .
\end{gathered}
$$

Lemma 4.1. A first-order deformation is skew-symmetric if and only if the following conditions hold

$$
\begin{gather*}
A^{i j}=-A^{j i},  \tag{4.2}\\
B_{k}^{i j}=-2 \partial_{k} A^{j i}+B_{k}^{j i},  \tag{4.3}\\
C_{k}^{i j}=-\partial_{k} A^{j i}+B_{k}^{j i}-C_{k}^{j i},  \tag{4.4}\\
\tilde{D}_{r k}^{i j}=-\partial_{r} \partial_{k} A^{j i}+\frac{1}{2}\left(\partial_{r} B_{k}^{j i}+\partial_{k} B_{r}^{j i}\right)-\tilde{D}_{r k}^{j i} . \tag{4.5}
\end{gather*}
$$

Provided that the above conditions are satisfied, a second-order deformation is skew-symmetric if and only if the following conditions hold

$$
\begin{gather*}
E^{i j}=E^{j i},  \tag{4.6}\\
F_{k}^{i j}=3 \partial_{k} E^{j i}-F_{k}^{j i},  \tag{4.7}\\
G_{k}^{i j}=3 \partial_{k} E^{j i}-2 F_{k}^{j i}+G_{k}^{j i},  \tag{4.8}\\
\tilde{H}_{r k}^{i j}=3 \partial_{r} \partial_{k} E^{j i}-\partial_{r} F_{k}^{j i}-\partial_{k} F_{r}^{j i}+\tilde{H}_{r k}^{j i},  \tag{4.9}\\
L_{k}^{i j}=\partial_{k} E^{j i}-F_{k}^{j i}+G_{k}^{j i}-L_{k}^{j i},  \tag{4.10}\\
M_{r k}^{i j}=3 \partial_{r} \partial_{k} E^{j i}-2 \partial_{k} F_{r}^{j i}-\partial_{r} F_{k}^{j i}+\partial_{k} G_{r}^{j i}+2 \tilde{H}_{r k}^{j i}-M_{r k}^{j i}, \tag{4.11}
\end{gather*}
$$

$$
\begin{align*}
\tilde{N}_{s r k}^{i j}= & \partial_{s} \partial_{r} \partial_{k} E^{j i}-\frac{1}{3}\left(\partial_{s} \partial_{r} F_{k}^{j i}+\partial_{r} \partial_{k} F_{s}^{j i}+\partial_{k} \partial_{s} F_{r}^{j i}\right)  \tag{4.12}\\
& +\frac{1}{3}\left(\partial_{s} \tilde{H}_{r k}^{j i}+\partial_{r} \tilde{H}_{k s}^{j i}+\partial_{k} \tilde{H}_{s r}^{j i}\right)-\tilde{N}_{s r k}^{j i} .
\end{align*}
$$

The proof is a straightforward computation (see [87] for further details). For instance, for $n=2$ the number of unknown functions falls to 42 .

### 4.1 The action of Miura-type transformations of the second kind

Let us start with deformations of order 1. These deformations have to satisfy the Jacobi condition $\left[P_{0}, P_{1}\right]=0$. We want to eliminate deformations that can be obtained by an Miura-type transformations of the second kind, that is, those that can be written as $\operatorname{Lie}_{X} P_{0}$, where $X$ is a suitable vector field of degree 1 . In the non-degenerate case, it has been proved that all deformations of order 1 can be written in this way, but we will show that in the degenerate case this is not true.

Secondly, concerning deformations of order 2, namely

$$
P^{\epsilon}=P_{0}+\epsilon P_{1}+\epsilon^{2} P_{2}+\mathcal{O}\left(\epsilon^{3}\right),
$$

we have to consider the Jacobi condition $2\left[P_{0}, P_{2}\right]+\left[P_{1}, P_{1}\right]=0$. In our cases, first-order deformations $P_{\epsilon}$ cannot be reduced to $P_{0}$. In order to simplify the form of second-order deformations without changing lower order terms, we have to use Miura-type transformations of the second kind like

$$
\begin{equation*}
\operatorname{Lie}_{Y} P_{1}+\operatorname{Lie}_{Z} P_{0} \tag{4.13}
\end{equation*}
$$

where $Z$ is an arbitrary vector field of degree 2 and $Y$ is a vector field of degree 1 which is a symmetry for $P_{0}$, namely $\operatorname{Lie}_{Y} P_{0}=0$.

To better understand this formula, let us consider the Lie series given by the vector field $\epsilon Y+\epsilon^{2} Z$, we have

$$
\mathcal{L}_{\epsilon Y+\epsilon^{2} Z}\left(P^{\epsilon}\right)=P_{0}+\epsilon\left(P_{1}+\operatorname{Lie}_{Y} P_{0}\right)+\epsilon^{2}\left(P_{2}+\operatorname{Lie}_{Y} P_{1}+\frac{1}{2} \operatorname{Lie}_{Y}^{2} P_{0}+\operatorname{Lie}_{Z} P_{0}\right)+\mathcal{O}\left(\epsilon^{3}\right)
$$

Since $\operatorname{Lie}_{Y} P_{0}$ is assumed to vanish, we obtain exactly (4.13).

### 4.1.1 Deformation results

In two-component case, we have two different Poisson structures with degenerate metric (Theorem 3.4), one constant and one non-constant, which we call $P_{0}^{(1)}$ and $P_{0}^{(2)}$ respectively:

$$
P_{0}^{(1)}=\left(\begin{array}{cc}
\partial_{x} & 0 \\
0 & 0
\end{array}\right), \quad P_{0}^{(2)}=\left(\begin{array}{cc}
\partial_{x} & -\frac{u_{x}^{2}}{u^{1}} \\
\frac{u_{x}^{2}}{u^{L}} & 0
\end{array}\right) .
$$

Theorem 4.2. • All first-order deformations of $P_{0}^{(1)}$ can be reduced by Miura-type transformations of the second kind to $P=P_{0}^{(1)}+\epsilon P_{1}+\mathcal{O}\left(\epsilon^{2}\right)$ where

$$
P_{1}=\left(\begin{array}{cc}
0 & -p u_{x x}^{2}-q\left(u_{x}^{2}\right)^{2}  \tag{4.14}\\
p u_{x x}^{2}+q\left(u_{x}^{2}\right)^{2} & r u_{x}^{2} \partial_{x}+\frac{1}{2}\left(r u_{x}^{2}\right)_{x}
\end{array}\right),
$$

here $p, q, r$ are arbitrary functions of $u^{2}$.

- All second-order deformations of $P_{0}^{(1)}$ can be reduced by Miura-type transformations of the second kind to $P=P_{0}^{(1)}+\epsilon P_{1}+\epsilon^{2} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)$, where $P_{1}$ is given by (4.14) with the constraint $r=0$, and

$$
P_{2}=\left(\begin{array}{cc}
0 & 0  \tag{4.15}\\
0 & \alpha^{22}
\end{array}\right) \frac{d^{3}}{d x^{3}}+\left(\begin{array}{cc}
0 & 0 \\
0 & \beta^{22}
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{cc}
0 & 0 \\
0 & \gamma^{22}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & \eta^{12} \\
-\eta^{12} & \eta^{22}
\end{array}\right),
$$

with

$$
\begin{gathered}
\alpha^{22}=e, \quad \beta^{22}=\frac{3 e^{\prime}}{2} u_{x}^{2}, \quad \gamma^{22}=g u_{x x}^{2}+h\left(u_{x}^{2}\right)^{2}, \\
\eta^{12}=\left(2 p^{2} u^{1}-l\right) u_{x x x}^{2}+p q u_{x}^{1}\left(u_{x}^{2}\right)^{2}+p^{2} u_{x}^{1} u_{x x}^{2}+\left(2 u^{1}\left(p q^{\prime}+q^{2}\right)-n\right)\left(u_{x}^{2}\right)^{3} \\
+\left(2 p u^{1}\left(3 q+p^{\prime}\right)-m\right) u_{x}^{2} u_{x x}^{2}, \\
\eta_{2}= \\
\frac{1}{2}\left(g u_{x x}^{2}+h\left(u_{x}^{2}\right)^{2}\right)_{x}-\frac{1}{4}\left(e^{\prime} u_{x}^{2}\right)_{x x},
\end{gathered}
$$

where $p, q, e, g, h, l, m, n$ are arbitrary functions of $u^{2}$, and ' denotes the derivative with respect to $u^{2}$. Furthermore, it is always possible to reduce to zero one of the two functions $m$ or $n$.

Theorem 4.3. - All first-order deformations of $P_{0}^{(2)}$ can be reduced by Miura-type transforma-
tions of the second kind to $P=P_{0}^{(2)}+\epsilon P_{1}+\mathcal{O}\left(\epsilon^{2}\right)$, where

$$
P_{1}=\left(\begin{array}{cc}
0 & 0  \tag{4.16}\\
0 & \frac{r}{\left(u^{1}\right)^{3}} u_{x}^{2}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & -\frac{s}{\left(u^{1}\right)^{3}}\left(u_{x}^{2}\right)^{2} \\
\frac{s}{\left(u^{1}\right)^{3}}\left(u_{x}^{2}\right)^{2} & \frac{1}{2}\left(\frac{r}{\left(u^{1}\right)^{3}} u_{x}^{2}\right)_{x}
\end{array}\right)
$$

here $r, s$ are arbitrary functions of $u^{2}$.

- All second-order deformations of $P_{0}^{(2)}$ can be reduced by Miura-type transformations of the second kind to $P=P_{0}^{(2)}+\epsilon P_{1}+\epsilon^{2} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)$, where $P_{1}$ is given by (4.16) and

$$
P_{2}=\left(\begin{array}{cc}
0 & 0  \tag{4.17}\\
0 & \alpha^{22}
\end{array}\right) \frac{d^{3}}{d x^{3}}+\left(\begin{array}{cc}
0 & 0 \\
0 & \beta^{22}
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{cc}
0 & \gamma^{12} \\
\gamma^{12} & \gamma^{22}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{21} & \eta^{22}
\end{array}\right)
$$

with

$$
\begin{aligned}
\alpha^{22}= & \frac{r^{2}}{2\left(u^{1}\right)^{4}}, \quad \beta^{22}=\frac{3 r r^{\prime}}{2\left(u^{1}\right)^{4}} u_{x}^{2}-\frac{3 r^{2}}{\left(u^{1}\right)^{5}} u_{x}^{1}, \quad \gamma^{12}=\frac{19 s r}{6\left(u^{1}\right)^{5}}\left(u^{1} u_{x x}^{2}-u_{x}^{1} u_{x}^{2}\right) \\
\gamma^{22}= & \frac{15 r^{2}}{2\left(u^{1}\right)^{6}}\left(u_{x}^{1}\right)^{2}-\frac{2 r^{2}}{\left(u^{1}\right)^{5}} u_{x x}^{1}-\frac{1}{\left(u^{1}\right)^{5}}\left(\frac{9 r r^{\prime}}{2}+p\right) u_{x}^{1} u_{x}^{2}+\frac{p}{\left(u^{1}\right)^{4}} u_{x x}^{2}, \\
\eta^{12}= & \frac{5 s r}{2\left(u^{1}\right)^{4}} u_{x x x}^{2}-\frac{5 s r}{2\left(u^{1}\right)^{5}} u_{x x}^{1} u_{x}^{2}-\frac{32 s r}{3\left(u^{1}\right)^{5}} u_{x}^{1} u_{x x}^{2}+\frac{3 s r^{\prime}+s^{\prime} r}{\left(u^{1}\right)^{4}} u_{x}^{2} u_{x x}^{2} \\
& +\frac{32 s r}{3\left(u^{1}\right)^{6}}\left(u_{x}^{1}\right)^{2} u_{x}^{2}-\frac{3 s r^{\prime}+s^{\prime} r}{\left(u^{1}\right)^{5}} u_{x}^{1}\left(u_{x}^{2}\right)^{2}-\frac{2 s^{2}}{\left(u^{1}\right)^{5}}\left(u_{x}^{2}\right)^{3} \\
\eta^{21}= & \frac{2 s r}{3\left(u^{1}\right)^{4}} u_{x x x}^{2}-\frac{2 s r}{3\left(u^{1}\right)^{5}} u_{x x}^{1} u_{x}^{2}-\frac{31 s r}{6\left(u^{1}\right)^{5}} u_{x}^{1} u_{x x}^{2}+\frac{13 s^{\prime} r+s r^{\prime}}{6\left(u^{1}\right)^{4}} u_{x}^{2} u_{x x}^{2} \\
& +\frac{31 s r}{6\left(u^{1}\right)^{6}}\left(u_{x}^{1}\right)^{2} u_{x}^{2}-\frac{13 s^{\prime} r+s r^{\prime}}{6\left(u^{1}\right)^{5}} u_{x}^{1}\left(u_{x}^{2}\right)^{2}+\frac{2 s^{2}}{\left(u^{1}\right)^{5}}\left(u_{x}^{2}\right)^{3}, \\
\eta^{22}= & \frac{1}{2\left(u^{1}\right)^{5}}\left(\frac{3 r r^{\prime}}{2}-5 p\right) u_{x}^{1} u_{x x}^{2}-\frac{15 r^{2}}{2\left(u^{1}\right)^{7}}\left(u_{x}^{1}\right)^{3}-\frac{1}{2\left(u^{1}\right)^{5}}\left(\frac{5 r r^{\prime}}{2}+p\right) u_{x x}^{1} u_{x}^{2} \\
& +\frac{1}{2\left(u^{1}\right)^{4}}\left(p^{\prime}-\frac{3}{2}\left(\left(r^{\prime}\right)^{2}+r r^{\prime \prime}\right)\right) u_{x}^{2} u_{x x}^{2}+\frac{5}{2\left(u^{1}\right)^{6}}\left(\frac{3 r r^{\prime}}{2}+p\right)\left(u_{x}^{1}\right)^{2} u_{x}^{2} \\
& +\frac{1}{2\left(u^{1}\right)^{4}}\left(p-\frac{r r^{\prime}}{2}\right) u_{x x x}^{2}-\frac{r^{2}}{2\left(u^{1}\right)^{5}} u_{x x x}^{1}-\frac{1}{4\left(u^{1}\right)^{4}}\left(3 r^{\prime} r^{\prime \prime}+r r^{\prime \prime \prime}\right)\left(u_{x}^{2}\right)^{3} \\
& +\frac{5 r^{2}}{\left(u^{1}\right)^{6}} u_{x}^{1} u_{x x}^{1}-\frac{1}{2\left(u^{1}\right)^{5}}\left(\frac{3}{2}\left(p^{\prime}-\left(r^{\prime}\right)^{2}+r r^{\prime \prime}\right)\right) u_{x}^{1}\left(u_{x}^{2}\right)^{2}
\end{aligned}
$$

where $r, s, p$ are arbitrary functions of $u^{2}$ and 'denote the derivative with respect to $u^{2}$.
These results can be obtained performing a cumbersome computation. Details of the proof are given in the author's paper [87].

The classification of three-component Poisson structures with degenerate metric is quite extensive (Theorem 3.5), so we have decided to study only some of them. Especially, we describe first-order deformations for the following operators:

$$
P_{0}^{(3)}=\left(\begin{array}{ccc}
0 & u_{x}^{3} & 0 \\
-u_{x}^{3} & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P_{0}^{(4)}=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad P_{0}^{(5)}=\left(\begin{array}{ccc}
\partial_{x} & 0 & 0 \\
0 & \partial_{x} & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

We refer again to [87] for the proofs of the following results.
Theorem 4.4. All first-order deformations of $P_{0}^{(3)}$ can be reduced by Miura-type transformations of the second kind to $P=P_{0}^{(3)}+\epsilon P_{1}+\mathcal{O}\left(\epsilon^{2}\right)$, where

$$
P_{1}=\left(\begin{array}{ccc}
0 & -\alpha^{21} & 0  \tag{4.18}\\
\alpha^{21} & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{ccc}
\beta^{11} & \beta^{12} & \beta^{13} \\
\beta^{21} & \beta^{22} & \beta^{23} \\
\beta^{13} & \beta^{23} & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
\gamma^{11} & \gamma^{12} & \gamma^{13} \\
\gamma^{21} & \gamma^{22} & \gamma^{23} \\
\gamma^{31} & \gamma^{32} & 0
\end{array}\right)
$$

with

$$
\begin{aligned}
\alpha^{21}= & a, \\
\beta^{11}= & \left(2 \partial_{2} a-b_{2}^{21}-\partial_{2} s-\partial_{2} r\right) u_{x}^{1}+b_{2}^{11} u_{x}^{2}, \\
\beta^{12}= & \left(\partial_{1} s-2 \partial_{1} a\right) u_{x}^{1}+\left(b_{2}^{21}-2 \partial_{2} a\right) u_{x}^{2}-2 \partial_{3} a u_{x}^{3}, \\
\beta^{13}= & \frac{b_{2}^{21}+\partial_{2} s+\partial_{2} r}{2} u_{x}^{3}, \\
\beta^{21}= & \partial_{1} s u_{x}^{1}+b_{2}^{21} u_{x}^{2}, \\
\beta^{22}= & b_{1}^{22} u_{x}^{1}+\partial_{1} r u_{x}^{2}, \\
\beta^{23}= & -\left(\partial_{1} s+\frac{\partial_{1} r}{2}\right) u_{x}^{3}, \\
\gamma^{11}= & \left(\partial_{2} a-\frac{b_{2}^{21}-\partial_{2} s-\partial_{2} r}{2}\right) u_{x x}^{1}+\left(\partial_{1} \partial_{2} a-\frac{\partial_{1} b_{2}^{21}-\partial_{1} \partial_{2} s-\partial_{1} \partial_{2} r}{2}\right)\left(u_{x}^{1}\right)^{2} \\
& +\left(\partial_{2} \partial_{3} a-\frac{\partial_{3} b_{2}^{21}-\partial_{2} \partial_{3} s-\partial_{2} \partial_{3} r}{2}\right) u_{x}^{1} u_{x}^{3}+\frac{\partial_{3} b_{2}^{11}}{2} u_{x}^{2} u_{x}^{3}+\frac{b_{2}^{11}}{2} u_{x x}^{2} \\
& +\left(\partial_{2}^{2} a-\frac{\partial_{2} b_{2}^{21}+\partial_{2}^{2} r+\partial_{2}^{2} s-\partial_{1} b_{2}^{11}}{2}\right) u_{x}^{1} u_{x}^{2}+\frac{\partial_{2} b_{2}^{11}}{2}\left(u_{x}^{2}\right)^{2}, \\
= & \left(\frac{\partial_{2}^{2} s+\partial_{2}^{2} r+3 \partial_{2} b_{2}^{21}-\partial_{1} b_{2}^{11}}{4}-\frac{\partial_{2}^{2} a}{2}\right)\left(u_{x}^{2}\right)^{2}+\left(\partial_{1} \partial_{3} s-2 \partial_{1} \partial_{3} a\right) u_{x}^{1} u_{x}^{3} \\
& +\left(\frac{3 \partial_{1} b_{2}^{21}+3 \partial_{1} \partial_{2} s}{2}+\partial_{1} \partial_{2} r-\partial_{2}^{2} a\right) u_{x}^{1} u_{x}^{2}+\left(\partial_{3} b_{2}^{21}-\partial_{2} \partial_{3} a\right) u_{x}^{2} u_{x}^{3}
\end{aligned}
$$

$$
\begin{aligned}
& +\left(\frac{3 \partial_{1}^{2} s}{2}+\frac{\partial_{2} b_{1}^{22}+\partial_{1}^{2} r}{4}-\partial_{1}^{2} a\right)\left(u_{x}^{1}\right)^{2}+\left(b_{2}^{21}-\partial_{2} a\right) u_{x x}^{2}-\partial_{3} a u_{x x}^{3} \\
& +\left(\partial_{1} s-\partial_{1} a\right) u_{x x}^{1}-\partial_{3}^{2} a\left(u_{x}^{3}\right)^{2}, \\
\gamma^{13}= & \left(\partial_{2} s+\partial_{2} r\right) u_{x x}^{3}+\frac{\partial_{1} b_{2}^{21}-\partial_{1} \partial_{2} s}{2} u_{x}^{1} u_{x}^{3}+\frac{\partial_{3} b_{2}^{21}+\partial_{2} \partial_{3} s+\partial_{2} \partial_{3} r}{2}\left(u_{x}^{3}\right)^{2}, \\
\gamma^{21}= & \left(\frac{\partial_{2} b_{2}^{21}+\partial_{1} b_{2}^{11}-\partial_{2}^{2} s-\partial_{2}^{3} r}{4}-\frac{\partial_{2}^{2} a}{2}\right)\left(u_{x}^{2}\right)^{2}-\left(\frac{\partial_{2} b_{1}^{22}+\partial_{1}^{2} r}{4}+\frac{\partial_{1}^{2} s}{2}\right)\left(u_{x}^{1}\right)^{2} \\
& -\left(\frac{\partial_{1} b_{2}^{21}+\partial_{1} \partial_{2} s}{2}+\partial_{1} \partial_{2} r\right) u_{x}^{1} u_{x}^{2}-\partial_{2} \partial_{3} a u_{x}^{2} u_{x}^{3}, \\
\gamma^{22}= & \frac{b_{1}^{22}}{2} u_{x x}^{1}+\frac{\partial_{1} r}{2} u_{x x}^{2}+\frac{\partial_{3} b_{1}^{22}}{2} u_{x}^{1} u_{x}^{3}+\frac{\partial_{2} b_{1}^{22}+\partial_{1}^{2} r}{2} u_{x}^{1} u_{x}^{2}+\frac{\partial_{1} \partial_{3} r}{2} u_{x}^{2} u_{x}^{3} \\
& +\frac{\partial_{1} b_{1}^{22}}{2}\left(u_{x}^{1}\right)^{2}+\frac{\partial_{1} \partial_{2} r}{2}\left(u_{x}^{2}\right)^{2}, \\
\gamma^{23}= & \frac{\partial_{1} b_{2}^{21}-\partial_{1} \partial_{2} s}{2} u_{x}^{2} u_{x}^{3}-\left(\partial_{1} \partial_{3} s+\frac{\partial_{1} \partial_{3} r}{2}\right)\left(u_{x}^{3}\right)^{2}-\left(\partial_{1} s+\partial_{1} r\right) u_{x x}^{3}, \\
\gamma^{31}= & \frac{b_{2}^{21}-\partial_{2} s-\partial_{2} r}{2} u_{x x}^{3}+\left(\partial_{1} \partial_{2} s+\frac{\partial_{1} \partial_{2} r}{2}\right) u_{x}^{1} u_{x}^{3}+\frac{\partial_{2} b_{2}^{21}+\partial_{2}^{2} s+\partial_{2}^{2} r}{2} u_{x}^{2} u_{x}^{3}, \\
\gamma^{32}= & \frac{\partial_{1} r}{2} u_{x x}^{3}-\left(\partial_{1}^{2} s+\frac{\partial_{1}^{2} r}{2}\right) u_{x}^{1} u_{x}^{3}-\frac{\partial_{1} b_{2}^{21}+\partial_{1} \partial_{2} s+\partial_{1} \partial_{2} r}{2} u_{x}^{2} u_{x}^{3},
\end{aligned}
$$

where $a, r, s, b_{2}^{11}, b_{2}^{21}, b_{1}^{22}$ are arbitrary functions of $u^{1}, u^{2}, u^{3}$, and $\partial_{k}$ means partial derivative with respect to $u^{k}$, for $k=1,2,3$.

Theorem 4.5. All first-order deformations of $P_{0}^{(4)}$ can be reduced by Miura-type transformations of the second kind to $P=P_{0}^{(4)}+\epsilon P_{1}+\mathcal{O}\left(\epsilon^{2}\right)$, where

$$
P_{1}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{4.19}\\
0 & 0 & -\alpha^{32} \\
0 & \alpha^{32} & 0
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \beta^{22} & \beta^{23} \\
0 & \beta^{32} & \beta^{33}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
0 & -\gamma^{21} & -\gamma^{31} \\
\gamma^{21} & \gamma^{22} & \gamma^{23} \\
\gamma^{31} & \gamma^{32} & \gamma^{33}
\end{array}\right)
$$

with

$$
\begin{gathered}
\alpha^{32}=a, \quad \beta^{i j}=b_{2}^{i j} u_{x}^{2}+b_{3}^{i j} u_{x}^{3} \quad(i \geq j), \quad \beta^{23}=\beta^{32}-2 a_{x} \\
\gamma^{i j}=c_{2}^{i j} u_{x x}^{2}+c_{3}^{i j} u_{x x}^{3}+e_{22}^{i j}\left(u_{x}^{2}\right)^{2}+e_{23}^{i j} u_{x}^{2} u_{x}^{3}+e_{33}^{i j}\left(u_{x}^{3}\right)^{2} \quad(i>j) \\
\gamma^{23}=\beta_{x}^{32}-a_{x x}-\gamma^{32}, \quad \gamma^{i i}=\frac{1}{2} \beta_{x}^{i i}
\end{gathered}
$$

where $a, b_{k}^{r s}, c_{k}^{i j}, e_{m k}^{i j}(f o r r \geq s$ and $k \geq m$ and $i>j$, where $i, j=1,2,3$ and $r, s, m, k=2,3$ ) are arbitrary functions of $u^{2}, u^{3}$.

Theorem 4.6. All first-order deformations of $P_{0}^{(5)}$ can be reduced by Miura-type transformations
of the second kind to $P=P_{0}^{(5)}+\epsilon P_{1}+\mathcal{O}\left(\epsilon^{2}\right)$, where

$$
P_{1}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{4.20}\\
0 & 0 & 0 \\
0 & 0 & \beta^{33}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{ccc}
0 & -\gamma^{21} & -\gamma^{31} \\
\gamma^{21} & 0 & -\gamma^{32} \\
\gamma^{31} & \gamma^{32} & \gamma^{33}
\end{array}\right)
$$

with

$$
\beta^{33}=b u_{x}^{3}, \quad \gamma^{33}=\frac{1}{2}\left(b u_{x}^{3}\right)_{x}, \quad \gamma^{i j}=e^{i j}\left(u_{x}^{3}\right)^{2}+c^{i j} u_{x x}^{3} \quad(i>j),
$$

where $b, c^{i j}, e^{i j}$, for $i>j$, are arbitrary functions of $u^{3}$. Furthermore, it is always possible to reduce to zero one of the functions $e^{21}$ or $c^{21}$.

### 4.2 The action of local changes of coordinates

The classification provided in the previous section has been obtained using Miura-type transformations of the second kind. As we pointed out above, the whole Miura group contains also local changes of coordinates which preserve the dispersionless limit of our structures.

### 4.2.1 Two-component case

Let us consider deformations of the structure $P_{0}^{(1)}$, Theorem 4.2. Local changes of coordinates which preserve the form of the dispersionless term $P_{0}^{(1)}(u)$ are of the form $u^{1}=v^{1}+\omega_{1}\left(v^{2}\right), u^{2}=\omega_{2}\left(v^{2}\right)$. Let us apply this transformation to the bivector $P_{1}$ given by (4.14), using the transformation rule $P(v)=J P(u) J^{t}$, where ${ }^{t}$ means the transpose and $J_{j}^{i}=\frac{\partial v^{i}}{\partial u^{j}}$. We have

$$
J=\left(\begin{array}{cc}
1 & -\frac{\omega_{1}^{\prime}}{\omega_{2}^{2}} \\
0 & \frac{1}{\omega_{2}^{\prime}}
\end{array}\right)
$$

where prime denotes the derivative with respect to $v^{2}$. Looking at the coefficient of $\partial_{x}$ in (4.14), it transforms as

$$
\left(\begin{array}{cc}
0 & 0 \\
0 & r\left(u^{2}\right) u_{x}^{2}
\end{array}\right) \mapsto\left(\begin{array}{cc}
-\frac{\left(\omega_{1}^{\prime}\right)^{2}}{\omega_{2}^{\prime}} r\left(\omega_{2}\right) v_{x}^{2} & \frac{\omega_{1}^{\prime}}{\omega_{2}^{\prime}} r\left(\omega_{2}\right) v_{x}^{2} \\
\frac{\omega_{1}^{\prime}}{\omega_{2}^{\prime}} r\left(\omega_{2}\right) v_{x}^{2} & \frac{1}{\omega_{2}^{\prime}} r\left(\omega_{2}\right) v_{x}^{2}
\end{array}\right) .
$$

In the general case where $r \neq 0$, this transformation suggests to set $\omega_{1}^{\prime}=0$, otherwise we would have a new arbitrary function in the coefficient of $\partial_{x}$ in $P_{1}$. Without any loss of generality, at this stage we can consider $\omega_{1}=0$. Looking at the whole bivector $P^{1}$, by straightforward computation, we get the following rule for the arbitrary functions $r, p, q$ :

$$
r\left(u^{2}\right) \mapsto \frac{r\left(\omega_{2}\right)}{\omega_{2}^{\prime}}, \quad p\left(u^{2}\right) \mapsto p\left(\omega_{2}\right), \quad q\left(u^{2}\right) \mapsto p\left(\omega_{2}\right) \frac{\omega_{2}^{\prime \prime}}{\omega_{2}^{\prime}}+q\left(\omega_{2}\right) \omega_{2}^{\prime},
$$

(if $r=0$, the action of local changes of coordinates is still the same, namely, the function $\omega_{1}$ is not involved in the transformation of $p$ and $q$ ). Thus, with a suitable choice of $\omega_{2}$, one can eliminate the function $q$.

Looking at the deformations of order two (4.15), since $r=0$, we still have the freedom of one arbitrary function due to $\omega_{1}$. Suppose we have used $\omega_{2}$ to simplify $p$ or $q$. Then, the coefficient of $\partial_{x}^{3}$ transforms as

$$
\left(\begin{array}{cc}
0 & 0 \\
0 & e\left(u^{2}\right)
\end{array}\right) \quad \mapsto \quad\left(\begin{array}{cc}
-\left(\omega_{1}^{\prime}\right)^{2} e\left(v^{2}\right) & \omega_{1}^{\prime} e\left(v^{2}\right) \\
\omega_{1}^{\prime} e\left(v^{2}\right) & e\left(v^{2}\right)
\end{array}\right) .
$$

Once again, this means that $\omega_{1}^{\prime}=0$, otherwise we would have an extra function. Summarising, up to diffeomorphisms, we are able to simplify at most one arbitrary function in the first and second deformation of $P_{0}^{(1)}$.

Considering $P_{0}^{(2)}$, a generic change of coordinates which preserves its form is given by $u^{1}=v^{1}, u^{2}=\omega\left(v^{2}\right)$. Here, looking at first-order deformations, Theorem 4.3, the two arbitrary functions $r$ and $s$ appearing in $P_{1}$ transform as

$$
s\left(u^{2}\right) \mapsto s\left(\omega_{2}\right) \omega_{2}^{\prime}, \quad r\left(u^{2}\right) \mapsto \frac{r\left(\omega_{2}\right)}{\omega_{2}^{\prime}} .
$$

Therefore, in this case we can also simplify at most one single function.

### 4.2.2 Three-component case

Although the analysis of the three-component case can be performed in the same way, computations become much more complicated. Therefore, it is not always possible to provide a complete description of the action of local changes of coordinates on the structures we studied. In this subsection, we are going to describe the action of the group of diffeomorphisms on second-order deformations of $P_{0}^{(5)}$, since this is the only case where
we can provide a detailed analysis.
Up to Miura-type transformations of the second kind, a first-order deformation of $P_{0}^{(5)}$ reduces to the one described in Theorem 4.6. Diffeomorphisms which preserve the form of $P_{0}^{(5)}$ are

$$
u^{1}=v^{1} \cos \kappa+v^{2} \sin \kappa+\varphi_{1}\left(v^{3}\right), \quad u^{2}=v^{1} \sin \kappa-v^{2} \cos \kappa+\varphi_{1}\left(v^{3}\right), \quad u^{3}=\varphi_{3}\left(v^{3}\right),
$$

Without any loss of generality, we can set $\kappa=0$. The coefficient of $\partial_{x}$ in (4.20) transforms as

$$
\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & b\left(u^{3}\right)
\end{array}\right) \mapsto\left(\begin{array}{ccc}
\frac{\left(\varphi_{1}^{\prime}\right)^{2} b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{x}^{\prime}} & \frac{\varphi_{1}^{\prime} \varphi_{2}^{\prime} b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} & \frac{-\varphi_{1}^{\prime} b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} \\
\frac{\varphi_{1}^{\prime} \varphi_{2}^{\prime} b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} & \frac{\left(\varphi_{2}^{\prime}\right)^{2} b^{\prime}\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} & \frac{\left.-\varphi_{2}^{\prime} b \varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} \\
\frac{\left.-\varphi_{1}^{\prime} b \varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} & \frac{-\varphi_{2}^{\prime} b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}} & \frac{b\left(\varphi_{3}\right) v_{x}^{3}}{\varphi_{3}^{\prime}}
\end{array}\right),
$$

here ' denote the derivative with respect to $v^{3}$. Therefore, when $b \neq 0$, we have to impose $\varphi_{i}^{\prime}=0$, for $i=1,2$, otherwise two new functions would appear in the coefficient of $\partial_{x}$. Setting $\varphi_{i}=\xi_{i}$, where $\xi_{i}=$ const, $i=1,2$, the functions appearing in (4.20) transform as

$$
b \mapsto \frac{b}{\varphi_{3}^{\prime}}, \quad e^{21} \mapsto e^{21}\left(\varphi_{3}^{\prime}\right)^{2}+c^{21} \varphi_{3}^{\prime \prime}, \quad c^{21} \mapsto c^{21} \varphi_{3}^{\prime}, \quad e^{3 j} \mapsto \frac{e^{3 j}\left(\varphi_{3}^{\prime}\right)^{2}+c^{3 j} \varphi_{3}^{\prime \prime}}{\varphi_{3}^{\prime}}, \quad c^{3 j} \mapsto c^{3 j},
$$

for $j=1,2$ (here $e^{i j}, c^{i j}$ on the left hans side depend on $u^{3}$, while on the right hand side they depend on $\varphi_{3}\left(v^{3}\right)$ ). Thus, in the most general case, namely $b \neq 0$, local changes of coordinates allow to reduce by one the number of arbitrary functions appearing in the deformation. For instance, we can choose to reduce $b$ to 1 . Let us recall that Miura-type transformations of the second kind allow to reduce to 0 one function among $e^{21}$ and $c^{21}$. Thus, we have the following

Theorem 4.7. Up to Miura transformations, a generic second-order deformation of $P_{0}^{(5)}$ depends on 5 functions of $u^{3}$.

At this point, one could ask: if $b=0$, how does the group of diffeomorphisms act on the structure? Although this is a reasonable question, a deeper analysis of this case does not provide any further information about the general form of the deformation we are studying. However, it is remarkable that under this strong assumption ( $b=0$ ), we still have the freedom of three arbitrary functions $\varphi_{1}, \varphi_{2}, \varphi_{3}$. Let us discuss this sub-case. Clearly, the number of arbitrary functions appearing in the deformation is already re-
duced by one, since $b$ is assumed to be zero. The functions $e^{i j}, c^{i j}$ transform as

$$
\begin{gathered}
e^{21} \mapsto e^{21}\left(\varphi_{3}^{\prime}\right)^{2}+c^{21} \varphi_{3}^{\prime \prime}-\frac{\varphi_{2}^{\prime}\left(e^{31}\left(\varphi_{3}^{\prime}\right)^{2}+c^{31} \varphi_{3}^{\prime \prime}\right)-\varphi_{1}^{\prime}\left(e^{32}\left(\varphi_{3}^{\prime}\right)^{2}+c^{32} \varphi_{3}^{\prime \prime}\right)}{\varphi_{3}^{\prime}}, \\
c^{21} \mapsto c^{21} \varphi_{3}^{\prime}-c^{31} \varphi_{2}^{\prime}+c^{32} \varphi_{1}^{\prime}, \quad e^{3 j} \mapsto \frac{e^{3 j}\left(\varphi_{3}^{\prime}\right)^{2}+c^{3 j} \varphi_{3}^{\prime \prime}}{\varphi_{3}^{\prime}}, \quad c^{3 j} \mapsto c^{3 j},
\end{gathered}
$$

for $j=1,2$. Let us assume for simplicity that all $e^{i j}, c^{i j}$ are non-zero (otherwise, we should discuss case by case). Therefore, both $e^{21}$ and $c^{21}$ can be brought to 0 , using $\varphi_{1}, \varphi_{2}$. Finally, the freedom of $\varphi_{3}$ allows to simplify other functions among $e^{31}, e^{32}, c^{31}$ and $c^{32}$.

Corollary 4.8. Let $b=0$ in (4.20). Up to Miura transformations, second-order deformations of $P_{0}^{(5)}$ depend on 3 functions of $u^{3}$.

Changes of local coordinates which preserve the form of the undeformed Poisson structure $P_{0}^{(3)}$ and $P_{0}^{(4)}$ are quite easy to compute. For $P_{0}^{(3)}$ these transformations are given by

$$
u^{1}=\varphi_{1}\left(v^{1}, v^{2}, v^{3}\right), \quad u^{2}=\varphi_{2}\left(v^{1}, v^{2}, v^{3}\right), \quad u^{3}=\varphi_{3}\left(v^{3}\right),
$$

with the constraint

$$
\partial_{1} \varphi_{1} \partial_{2} \varphi_{2}-\partial_{2} \varphi_{1} \partial_{1} \varphi_{2}=\partial_{3} \varphi_{3}, \quad \partial_{i}=\frac{\partial}{\partial v^{i}},
$$

while for $P_{0}^{(4)}$ we have

$$
u^{1}=v^{1}+\varphi_{1}\left(v^{2}, v^{3}\right), \quad u^{2}=\varphi_{2}\left(v^{2}, v^{3}\right), \quad u^{3}=\varphi_{3}\left(v^{2}, v^{3}\right) .
$$

Unfortunately, the action of these transformations on the respective deformed structures are very cumbersome, and we are not going to describe it.

Summarising, as we have seen, the action of the subgroup of diffeomorphisms leads to several branches for each case, depending wherever the functional parameters are constant, zero or arbitrary. Furthermore, the number of additional arbitrary functions appearing in these transformations, is always lower than the number of functional parameters appearing in the deformations. This implies that, in each cases we have studied, we cannot reduce the deformation to its dispersionless term, and therefore the deformation is not trivial.

## Deformations of non-semisimple Poisson pencils of hydrodynamic type

Two Poisson brackets are called compatible if any their linear combination is still a Poisson bracket [65]. Therefore, a pair of compatible Poisson structures $P_{1}, P_{2}$ implicitly defines a one-parameter family of Poisson structures $\Pi_{\lambda}=P_{2}-\lambda P_{1}(\lambda \in \mathbb{R})$ called Poisson pencil or bi-Hamiltonian structure (note that here and in the rest of this chapter $P_{1}$ and $P_{2}$ denote two Poisson structures and they do not necessarily represent first- and second-order deformations as in the previous chapter). Poisson pencils of hydrodynamic type and their deformations play an important role in the modern theory of integrable PDEs. Originally the study of such structures was motivated by questions arising in the theory of Frobenius manifolds, Gromov-Witten invariants and topological field theory [25, 38]. In this setting, deformations satisfy some additional constraints ( $\tau$-structure, Virasoro constraints) and the undeformed pencil is related to a Frobenius manifold [25].

A perturbative approach to the study of these deformations was developed by Dubrovin and Zhang in [38]. In their approach, the full pencil

$$
\begin{equation*}
\Pi_{\lambda}^{i j}=P_{2}^{i j}+\sum_{k \geq 1} \epsilon^{k} \sum_{l=0}^{k+1} A_{2 ; k, l}^{i j} \frac{d^{k-l+1}}{d x^{k-l+1}}-\lambda\left(P_{1}^{i j}+\sum_{k \geq 1} \epsilon^{k} \sum_{l=0}^{k+1} A_{1 ; k, l}^{i j} \frac{d^{k-l+1}}{d x^{k-l+1}}\right) \tag{5.1}
\end{equation*}
$$

where $A_{\theta ; k, l}^{i j}=A_{\theta ; k, l}^{i j}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(l)}\right)$ are homogeneous differential polynomials of degree $l$, is obtained via a bi-Hamiltonian deformation procedure from the dispersionless limit $\epsilon \rightarrow 0$ :

$$
\begin{equation*}
P_{2}^{i j}-\lambda P_{1}^{i j}=g_{2}^{i j} \frac{d}{d x}+b_{2 ; k}^{i j} u_{x}^{k}-\lambda\left(g_{1}^{i j} \frac{d}{d x}+b_{1 ; k}^{i j} u_{x}^{k}\right) . \tag{5.2}
\end{equation*}
$$

The pencil of metrics $g_{\lambda}=g_{2}-\lambda g_{1}$ defining this limit (5.2) is assumed to be semisimple, meaning that there exists a special set of coordinates, the roots $\left(r^{1}, \ldots, r^{n}\right)$ of the equation $\operatorname{det} g_{\lambda}=0$, such that both metrics of the pencil $g_{\lambda}$ take diagonal form. It can be shown (see
[40]) that in this canonical coordinates $\mathbf{r}=\left(r^{1}, \ldots, r^{n}\right)$ the metric can be represented in the form $g_{1}^{i j}=f^{i}(\mathbf{r}) \delta_{i j}, g_{2}^{i j}=r^{i} f^{i}(\mathbf{r}) \delta_{i j}$.

Whereas the semisimple case is fairly well understood, the non-semisimple case is largely open. Besides computational difficulties, the lack of canonical coordinates, or at least of a normal form theorem for non-semisimple pencils, makes a unified approach to the problem very hard. For this reason, in the joint work with A. Della Vedova and P. Lorenzoni [22], we studied the general case focusing on two special subcases where computations are feasible:

- Deformations of Poisson pencils related to two-dimensional Balinskiǐ-Novikov algebras [7] and the associated invariant bilinear forms.
- Lifts of deformations of semisimple structures.

In order not to stretch the thesis, in this chapter we mainly discuss the results obtained concerning the first class of structures. The second part, related to the lift of deformations of semisimple structures, will only be mentioned. Details can be found in [22].

### 5.1 The semisimple case

The deformation theory of Poisson structures defined on a loop space has been already introduced in Section 1.2.1. In the case of Dubrovin-Novikov brackets, we have seen that any such deformation is trivial, that is, can be obtained via Miura transformation (see Section 1.2.4). In the bi-Hamiltonian setting, we have an analogous definition of $k$-order deformation: pencil (5.1) defines a deformation of order $k$ if $\left[\Pi_{\lambda}, \Pi_{\lambda}\right]=\mathcal{O}\left(\epsilon^{k+1}\right)$, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket (see Section 1.2). Moreover, two deformations $\Pi_{\lambda}$ and $\tilde{\Pi}_{\lambda}$ of the same pencil are considered equivalent if they are related by a Miura transformation of the form

$$
\begin{equation*}
\tilde{u}^{i}=u^{i}+\sum_{k \geq 1} \epsilon^{k} F_{k}^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(k)}\right) \tag{5.3}
\end{equation*}
$$

where $F_{k}^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(k)}\right)$ are differential polynomials of degree $k$. This means that two pencils belonging to the same class are related by

$$
\tilde{\Pi}_{\lambda}^{i j}=L_{k}^{* i} \Pi_{\lambda}^{k l} L_{l}^{j},
$$

where

$$
L_{k}^{i}=\sum_{s}\left(-\partial_{x}\right)^{s} \frac{\partial \tilde{u}^{i}}{\partial u_{(s)}^{k}}, \quad L_{k}^{* i}=\sum_{s} \frac{\partial \tilde{u}^{i}}{\partial u_{(s)}^{k}} \partial_{x}^{s}
$$

Under the assumption of semisemplicity, Dubrovin, Liu and Zhang proved that this equivalence classes are labelled by $n$ functions $c^{i}\left(r^{i}\right)$ called central invariants [61, 31], obtained expanding the roots $\lambda^{i}$ of the equation

$$
\begin{equation*}
\operatorname{det}\left(g_{2}^{i j}-\lambda g_{1}^{i j}+\sum_{k \geq 1}\left(A_{2 ; k, 0}^{i j}(u)-\lambda A_{1 ; k, 0}^{i j}(u)\right) p^{k}\right)=0, \tag{5.4}
\end{equation*}
$$

near $\lambda^{i}=r^{i}$ :

$$
\begin{equation*}
\lambda^{i}=r^{i}+\sum_{k=1}^{\infty} \lambda_{2 k}^{i} p^{2 k}, \tag{5.5}
\end{equation*}
$$

and selecting the coefficient of $p^{2}$. The central invariants are then defined as [30, 61]:

$$
c^{i}=\frac{1}{3} \frac{\lambda_{2}^{i}}{g_{1}^{i i}}=\frac{1}{\left(f^{i}\right)^{2}}\left(S_{2}^{i i}-r^{i} S_{1}^{i i}+\sum_{k \neq i} \frac{\left(R_{2}^{k i}-r^{i} R_{1}^{k i}\right)^{2}}{f^{k}\left(r^{k}-r^{i}\right)}\right), \quad i=1, \ldots, n,
$$

where $f^{i}$ are the diagonal components of the contravariant metric $g_{1}$ in canonical coordinates, and

$$
R_{\theta}^{i j}(\mathbf{u})=A_{\theta ; 1,2}^{i j}(\mathbf{u}), \quad S_{\theta}^{i j}(\mathbf{u})=A_{\theta ; 2,3}^{i j}(\mathbf{u}), \quad i, j=1, \ldots, n, \quad \theta=1,2 .
$$

They can also be defined by (see [39])

$$
c_{i}=-\frac{1}{3 f^{i}} \operatorname{Res}_{\lambda=r^{i}} \operatorname{Tr}\left[g_{\lambda}^{-1}\left(S_{\lambda}^{i j}+\left(g_{\lambda}^{-1}\right)_{l k} R_{\lambda}^{l i} R_{\lambda}^{k j}\right)\right],
$$

where $S_{\lambda}^{i j}=S_{2}^{i j}-\lambda S_{1}^{i j}$ and $R_{\lambda}^{i j}=R_{2}^{i j}-\lambda R_{1}^{i j}$.
In this framework the following facts should be mentioned:

- Each function $c^{i}$ depends only on the corresponding canonical coordinate $r^{i}$, and it is invariant with respect to Miura transformations (5.3) [61].
- Two deformations (of the same pencil) belong to the same equivalence class if and only if they have the same central invariants [31].
- For any choice of the dispersionless limit and of the central invariants the equiv-
alence classes are not empty. This fact, suggested by some computations (for the scalar case see [64, 2]), has been proved only recently: by Liu and Zhang in the scalar case [60] and by Carlet, Posthuma and Shadrin in the general semisimple case [17]. The proof is based on the vanishing of certain cohomology groups introduced in [61].
- Given the dispersionless limit $\omega_{\lambda}$ and the central invariants $c^{i}\left(r^{i}\right)$, there exists a Miura transformation (5.3) reducing the pencil to the standard form [61]

$$
\begin{aligned}
\Pi_{\lambda} & =P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X_{\left(c_{1}, \ldots, c_{n}\right)}} P_{1}+\epsilon^{4} \Pi_{4}+\epsilon^{6} \Pi_{6}+\ldots \\
& =P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{Y_{\left(c_{1}, \ldots, c_{n}\right)}} P_{2}+\epsilon^{4} \Pi_{4}+\epsilon^{6} \Pi_{6}+\ldots
\end{aligned}
$$

where the polynomial vector fields $X_{\left(c_{1}, \ldots, c_{n}\right)}$ and $Y_{\left(c_{1}, \ldots, c_{n}\right)}$ can be written as the difference of two Hamiltonian vector fields

$$
X_{\left(c_{1}, \ldots, c_{n}\right)}=P_{2} \delta H-P_{1} \delta K, \quad Y_{\left(c_{1}, \ldots, c_{n}\right)}=P_{2} \delta \tilde{H}-P_{1} \delta \tilde{K}
$$

with non polynomial Hamiltonian densities:

$$
\begin{align*}
& H[\mathbf{r}]=\sum_{i=1}^{n} \int c^{i}\left(r^{i}\right) r_{x}^{i} \log r_{x}^{i} d x, \quad K[r]=\sum_{i=1}^{n} \int r^{i} c^{i}\left(r^{i}\right) r_{x}^{i} \log r_{x}^{i} d x .  \tag{5.6}\\
& \tilde{H}[\mathbf{r}]=\sum_{i=1}^{n} \int \frac{c^{i}\left(r^{i}\right)}{r^{i}} r_{x}^{i} \log r_{x}^{i} d x, \quad \tilde{K}[r]=\sum_{i=1}^{n} \int c^{i}\left(r^{i}\right) r_{x}^{i} \log r_{x}^{i} d x . \tag{5.7}
\end{align*}
$$

- The coefficients $F_{k}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(k)}\right)$ of the Miura transformation (5.3) are assumed to depend polynomially on the derivatives of $u^{i}$. Removing this assumption the classification problem becomes "trivial": all deformations turn out to be equivalent to their dispersionless limit. This remarkable property was discovered in [31] and is called quasi-triviality. For instance, it is easy to check that the canonical quasi-Miura transformation generated by the Hamiltonian $H$ defined in the formula (5.6) reduces the pencil $\Pi_{\lambda}^{i j}$ to the form $P_{2}^{i j}-\lambda P_{1}^{i j}+\mathcal{O}\left(\epsilon^{4}\right)$.


### 5.2 Linear Poisson pencils of hydrodynamic type

In this section we introduce Poisson pencils of hydrodynamic type related to BalinskiǐNovikov algebras [7] and the associated invariant bilinear forms [3]. These are Poisson pencils that can be reduced to the form

$$
P_{2}^{i j}-\lambda P_{1}^{i j}=g^{i j} \frac{d}{d x}+b_{k}^{i j} u_{x}^{k}-\lambda \eta^{i j} \frac{d}{d x},
$$

where $g^{i j}$ depends linearly on the variables $\left(u^{1}, \ldots, u^{n}\right)$ and the coefficients $b_{k}^{i j}$ and $\eta^{i j}$ are constant. Therefore, $P_{2}^{i j}$ is a linear Hamiltonian operator of hydrodynamic type. As proved by Balinskiǐ and Novikov in [7] these operators have the form

$$
P^{i j}=\left(b_{k}^{i j}+b_{k}^{j i}\right) u^{k} \frac{d}{d x}+b_{k}^{i j} u_{x}^{k},
$$

where the numbers $b_{k}^{i j}$ are the structure constants of an algebra $B$ satisfying the following properties

$$
\begin{aligned}
a \cdot(b \cdot c) & =b \cdot(a \cdot c) \\
(a \cdot b) \cdot c-a \cdot(b \cdot c) & =(a \cdot c) \cdot b-a \cdot(c \cdot b)
\end{aligned}
$$

We refer to them as Balinskiǐ-Novikov algebras, even if in the literature they are often called Novikov algebras (following [82]).

First approach to the study of such algebras was made by Zelmanov [101]. In low dimensions, the problem of classification was addressed by Bai and Meng [4,5] and recently by Burde and de Graaf [14], resulting in a complete description of one-, two- and three-dimensional Balinskiir-Novikov algebras. Unfortunately, a full classification of these structures of dimension four and higher is far from being complete.

### 5.2.1 Invariant bilinear forms and bi-Hamiltonian structures

Given a Balinskiǐ-Novikov algebra $B$, as observed in [93], any invariant bilinear symmetric form on it gives rise to a bi-Hamiltonian structure in a canonical way. For convenience of the reader let us briefly recall how they are defined. Let $e^{1}, \ldots, e^{n}$ be a basis of $B$, and let $b_{k}^{i j}$ be the corresponding structure constants. A bilinear form $\eta: B \times B \rightarrow F$ is called
invariant if and only if

$$
\eta\left(e^{i} \cdot e^{j}, e^{k}\right)=\eta\left(e^{i}, e^{k} \cdot e^{j}\right)
$$

Bai and Meng classified invariant bilinear forms on two- and three-dimensional Balin-skiǐ-Novikov algebras in [4, 3]. For future reference we recall the two-dimensional classification in the following table.

Table 5.1: Two-dimensional Balinskiǐ-Novikov algebras and invariant bilinear forms.

| Type | Characteristic matrix $e^{i} \cdot e^{j}$ | Linear Poisson structure | Invariant bilinear forms |
| :---: | :---: | :---: | :---: |
| (T1) | $\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{ll}\eta^{11} & \eta^{12} \\ \eta^{21} & \eta^{22}\end{array}\right)$ |
| (T2) | $\left(\begin{array}{ll}e^{2} & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{cc}2 u^{2} \partial_{x}+u_{x}^{2} & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{cc}\eta^{11} & \eta^{12} \\ \eta^{12} & 0\end{array}\right)$ |
| (T3) | $\left(\begin{array}{cc}0 & 0 \\ -e^{1} & 0\end{array}\right)$ | $\left(\begin{array}{cc}0 & -u^{1} \partial_{x} \\ -u^{1} \partial_{x}-u_{x}^{1} & 0\end{array}\right)$ | $\left(\begin{array}{cc}0 & \eta^{12} \\ \eta^{12} & \eta^{22}\end{array}\right)$ |
| (N1) | $\left(\begin{array}{cc}e^{1} & 0 \\ 0 & e^{2}\end{array}\right)$ | $\left(\begin{array}{cc}2 u^{1} \partial_{x}+u_{x}^{1} & 0 \\ 0 & 2 u^{2} \partial_{x}+u_{x}^{2}\end{array}\right)$ | $\left(\begin{array}{cc}\eta^{11} & 0 \\ 0 & \eta^{22}\end{array}\right)$ |
| (N2) | $\left(\begin{array}{ll}e^{1} & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{cc}2 u^{1} \partial_{x}+u_{x}^{1} & 0 \\ 0 & 0\end{array}\right)$ | $\left(\begin{array}{cc}\eta^{11} & 0 \\ 0 & \eta^{22}\end{array}\right)$ |
| (N3) | $\left(\begin{array}{ll}e^{1} & e^{2} \\ e^{2} & 0\end{array}\right)$ | $\left(\begin{array}{cc}2 u^{1} \partial_{x}+u_{x}^{1} & 2 u^{2} \partial_{x}+u_{x}^{2} \\ 2 u^{2} \partial_{x}+u_{x}^{2} & 0\end{array}\right)$ | $\left(\begin{array}{cc}\eta^{11} & \eta^{12} \\ \eta^{12} & 0\end{array}\right)$ |
| (N4) | $\left(\begin{array}{ll}0 & e^{1} \\ 0 & e^{2}\end{array}\right)$ | $\left(\begin{array}{cc}0 & u^{1} \partial_{x}+u_{x}^{1} \\ u^{1} \partial_{x} & 2 u^{2} \partial_{x}+u_{x}^{2}\end{array}\right)$ | $\left(\begin{array}{ll}\eta^{11} & \eta^{12} \\ \eta^{21} & \eta^{22}\end{array}\right)$ |
| (N5) | $\left(\begin{array}{cc}0 & e^{1} \\ 0 & e^{1}+e^{2}\end{array}\right)$ | $\left(\begin{array}{cc}0 & u^{1} \partial_{x}+u_{x}^{1} \\ u^{1} \partial_{x} & 2\left(u^{1}+u^{2}\right) \partial_{x}+u_{x}^{2}+u_{x}^{1}\end{array}\right)$ | $\left(\begin{array}{cc}0 & \eta^{12} \\ \eta^{12} & \eta^{22}\end{array}\right)$ |
| (N6) | $\begin{gathered} \left(\begin{array}{cc} 0 & e^{1} \\ \kappa e^{1} & e^{2} \end{array}\right) \\ \kappa \neq 0,1 \end{gathered}$ | $\left(\begin{array}{cc}0 & (1+\kappa) u^{1} \partial_{x}+u_{x}^{1} \\ (1+\kappa) u^{1} \partial_{x}+\kappa u_{x}^{1} & 2 u^{2} \partial_{x}+u_{x}^{2}\end{array}\right)$ | $\left(\begin{array}{cc}0 & \eta^{12} \\ \eta^{12} & \eta^{22}\end{array}\right)$ |

Remark. Notice that the case N4 with $\eta^{11} \neq 0$ is semisimple. For this reason we will consider only the case $\eta^{11}=0$. The cases N3 and N4 can be considered as subcases of N6, if we remove the constraints $\kappa \neq 0,1$. Indeed, for $\kappa=0$ we easily get N 4 ( with $\eta^{11}=0$ ) while N 3 is equivalent to the case $\kappa=1$, up to swapping the local coordinates $u^{1}, u^{2}$. According to [4], this distinction is due to different algebraic properties: the cases N3 and N4 are characterized by the associativity of the algebra, while this is not the case for N6
with $\kappa \neq 0,1$. However, for our purposes, we do not need to distinguish these cases.
Let us point out that adding the constraint $\eta^{21}=\eta^{12}$ in T 1 and N 4 , the bilinear invariant forms associated with two-dimensional Balinskiǐ-Novikov algebra become symmetric. As observed by Strachan and Szablikowski in [93] the associated Hamiltonian operator $\eta^{i j} \partial_{x}$ is compatible with the linear Hamiltonian operator defining the Balinskiǐ-Novikov algebra.

Remark. A pair of compatible flat metrics defines a (2+1)-Poisson structure of hydrodynamic type under some additional conditions. Among the structures coming from two component Balinskiǐ-Novikov algebras, such additional conditions are satisfied just by $N 6$ with $\kappa=-2[35,74,69,46]$.

### 5.3 Classification results

In this section we provide a classification of second-order deformations of Poisson pencils coming from two-dimensional Balinskiǐ-Novikov algebras. We have to distinguish two cases:

1. The cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, where second-order deformed structures depend on two functions.
2. The remaining cases N 4 (which corresponds to $\kappa=0$ ) and N 6 with $\kappa=-2$, namely

$$
g_{1}=\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{12} & \eta^{22}
\end{array}\right), \quad g_{2}=\left(\begin{array}{cc}
0 & \pm u^{1} \\
\pm u^{1} & 2 u^{2}
\end{array}\right)
$$

where second-order deformed structures depend on four functions.
Theorem 5.1. • In the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, second-order deformations can be reduced by a Miura transformation to the form

$$
\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X_{\left(F_{1}, F_{2}\right)}} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)
$$

with $X_{\left(F_{1}, F_{2}\right)}=P_{1} \delta H-P_{2} \delta K$, where

$$
H[\mathbf{u}]=\int \sum_{i, j}\left(h_{i j} u_{x}^{i} \log u_{x}^{j}\right) d x, \quad K[\mathbf{u}]=\int \sum_{i, j}\left(k_{i j} u_{x}^{i} \log u_{x}^{j}\right) d x
$$

and the functions $h_{i j}$ and $k_{i j}$ are uniquely determined in terms of two arbitrary functions $F_{1}, F_{2}$ depending only on the (unique) eigenvalue of the affinor $L=g_{2} g_{1}^{-1}$. Calling $\mathcal{K}=\left(k_{i j}\right)$ and $\mathcal{H}=\left(h_{i j}\right)$, we have $\mathcal{K}=L^{T} \mathcal{H}$, where $L^{T}$ means the transpose of $L$, and $\mathcal{H}$ is given respectively for each case by

- T3: $h_{12}=h_{22}=0$ and

$$
h_{11}=\frac{e^{-\frac{\eta^{12} u^{2}}{\eta^{2} u^{1}}}}{3 \eta^{12}}\left(\eta^{22} u^{1} F_{2}^{\prime}+\frac{\eta^{12} u^{2}+\eta^{22} u^{1}}{u^{1}} F_{2}\right)-F_{1}, \quad h_{21}=-\frac{e^{-\frac{\eta^{12} u^{2}}{\eta^{2} u^{1}}}}{3} F_{2} ;
$$

- N5: $h_{12}=h_{22}=0$ and

$$
\begin{gathered}
h_{11}=\frac{\sqrt{2 \eta^{12}\left(u^{1}+u^{2}\right)-\eta^{22} u^{1}} F_{2}^{\prime}}{3 \eta^{12}}+\frac{\left(2 \eta^{12}-\eta^{22}\right) F_{2}}{6 \eta^{12} \sqrt{2 \eta^{12}\left(u^{1}+u^{2}\right)-\eta^{22} u^{1}}}+\frac{F_{1}}{2 \eta^{12}} \\
h_{21}=\frac{1}{3 \sqrt{2 \eta^{12}\left(u^{1}+u^{2}\right)-\eta^{22} u^{1}}} F_{2} ;
\end{gathered}
$$

- N3, N6 $(\kappa \neq 0,-1,-2): h_{12}=h_{22}=0$ and

$$
\begin{gathered}
h_{11}=\frac{\left(2 \eta^{12} u^{2}-(\kappa+1) \eta^{22} u^{1}\right)^{\frac{\kappa+1}{2}} F_{2}^{\prime}-\frac{\eta^{22}\left(2 \eta^{12} u^{2}-(\kappa+1) \eta^{22} u^{1}\right)^{\frac{\kappa-1}{2}} F_{2}}{3(\kappa+1)^{2} \eta^{12}}}{6 \eta^{12}} \\
+\frac{F_{1}}{\eta^{12} \kappa(\kappa+2)}, \\
h_{21}=\frac{\left(2 \eta^{12} u^{2}-(\kappa+1) \eta^{22} u^{1}\right)^{\frac{k-1}{2}}}{3(\kappa+1)} F_{2} ;
\end{gathered}
$$

where $F_{i}=F_{i}\left(u^{1}\right), i=1,2$.

- In the case N4, namely

$$
g_{2}=\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{12} & \eta^{22}
\end{array}\right), \quad g_{1}=\left(\begin{array}{cc}
0 & u^{1} \\
u^{1} & 2 u^{2}
\end{array}\right),
$$

second-order deformations can be reduced by a Miura transformation to the form

$$
\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)
$$

where

$$
X^{i}=X_{1}^{i} u_{x x}^{1}+X_{2}^{i}\left(u_{x}^{1}\right)^{2}+X_{3}^{i} u_{x}^{1} u_{x}^{2}+X_{4}^{i}\left(u_{x}^{2}\right)^{2}+X_{5}^{i} u_{x x}^{2},
$$

with

$$
\begin{aligned}
& X_{1}^{1}=0 \\
& X_{2}^{1}=\theta F_{1} \\
& X_{3}^{1}=\partial_{1}\left(\theta F_{2}\right) \\
& X_{4}^{1}=\partial_{2}\left(\theta F_{2}\right), \\
& X_{5}^{1}=\theta F_{2}, \\
& X_{1}^{2}=0 \\
& X_{2}^{2}=\theta F_{3} \\
& X_{3}^{2}=\partial_{1}\left(\theta^{\frac{1}{2}} F_{4}-\frac{\partial_{1} F_{2}}{\eta^{12}}\right), \\
& X_{4}^{2}=\partial_{2}\left(\theta^{\frac{1}{2}} F_{4}-\frac{\partial_{1} F_{2}}{\eta^{12}}\right), \\
& X_{5}^{2}=\theta^{\frac{1}{2}} F_{4}-\frac{\partial_{1} F_{2}}{\eta^{12}}
\end{aligned}
$$

In the above formulae, $F_{i}$ are 4 arbitrary functions of $u^{1}$ and $\theta=\left(\eta^{22} u^{1}-2 \eta^{12} u^{2}\right)^{-1}$.

- In the case N6 with $\kappa=-2$, namely

$$
g_{1}=\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{12} & \eta^{22}
\end{array}\right), \quad g_{2}=\left(\begin{array}{cc}
0 & -u^{1} \\
-u^{1} & 2 u^{2}
\end{array}\right),
$$

second-order deformations can be reduced by a Miura transformation to the form

$$
\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X} P_{2}+\mathcal{O}\left(\epsilon^{3}\right),
$$

where

$$
X^{i}=X_{1}^{i} u_{x x}^{1}+X_{2}^{i}\left(u_{x}^{1}\right)^{2}+X_{3}^{i} u_{x}^{1} u_{x}^{2}+X_{4}^{i}\left(u_{x}^{2}\right)^{2}+X_{5}^{i} u_{x x}^{2},
$$

with

$$
\begin{aligned}
& X_{1}^{1}=0, \\
& X_{2}^{1}=2 \eta^{22} \theta\left(\theta^{\frac{3}{2}} F_{4}-\frac{\partial_{1}\left(\theta^{2} F_{2}\right)}{\eta^{12}}\right)+\theta F_{1}, \\
& X_{3}^{1}=2 \eta^{12} \theta^{\frac{5}{2}} F_{4}-\partial_{1}\left(\theta^{3} F_{2}\right), \\
& X_{4}^{1}=-4 \eta^{12} \theta^{4} F_{2},
\end{aligned}
$$

$$
\begin{aligned}
& X_{5}^{1}=\theta^{3} F_{2}, \\
& X_{1}^{2}=0, \\
& X_{2}^{2}=F_{3}, \\
& X_{3}^{2}=\partial_{1}\left(\theta^{\frac{3}{2}} F_{4}\right)-\frac{\partial_{1}^{2}\left(\theta^{2} F_{2}\right)}{\eta^{12}}, \\
& X_{4}^{2}=4 \partial_{1}\left(\theta^{3} F_{2}\right)+\partial_{2}\left(\theta^{\frac{3}{2}} F_{4}\right), \\
& X_{5}^{2}=\theta^{\frac{3}{2}} F_{4}-\frac{\partial_{1}\left(\theta^{2} F_{2}\right)}{\eta^{12}} .
\end{aligned}
$$

In the above formulae, $F_{i}$ are 4 arbitrary functions of $u^{1}$ and $\theta=\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)^{-1}$.

Due to its technical nature, and cumbersome computations, we refer to [22] for the proof of the previous theorem.

Corollary 5.2. In the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, all second-order deformations are quasi-trivial.

## Proof:

By construction, the canonical quasi-Miura transformation generated by $H[\mathbf{u}]$ reduces the pencil to its dispersionless limit up to terms of order $\mathcal{O}\left(\epsilon^{3}\right)$.

Remark. General Miura transformations have the form

$$
u^{i} \rightarrow \tilde{u}^{i}=f^{i}(\mathbf{u})+\sum_{k \geq 1} \epsilon^{k} F_{k}^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(k)}\right)
$$

where $\operatorname{det} \frac{\partial f^{i}}{\partial u^{j}} \neq 0$. In this thesis we are interested in Miura transformations preserving the disperionless limit and for this reason we consider the subgroup

$$
u^{i} \rightarrow \tilde{u}^{i}=u^{i}+\sum_{k \geq 1} \epsilon^{k} F_{k}^{i}\left(\mathbf{u}, \mathbf{u}_{x}, \ldots, \mathbf{u}_{(k)}\right)
$$

Indeed, the only diffeomorphism preserving both metrics of the pencil is the identity.

### 5.4 Invariants of bi-Hamiltonian structures

As already mentioned in Section 5.1, the central invariants for deformations of semisimple Poisson pencils of hydrodynamic type (5.1) are related to the roots of the equation

$$
\operatorname{det}\left(g_{2}^{i j}-\lambda g_{1}^{i j}+\sum_{k \geq 1}\left(A_{2 ; k, 0}^{i j}(\mathbf{u})-\lambda A_{1 ; k, 0}^{i j}(\mathbf{u})\right) p^{k}\right)=0
$$

Expanding these roots near $\lambda^{i}=r^{i}$ one obtains a series:

$$
\begin{equation*}
\lambda^{i}=r^{i}+\sum_{k=1}^{\infty} \lambda_{k}^{i} p^{k} \tag{5.8}
\end{equation*}
$$

whose coefficients are invariants (up to permutations) with respect to Miura transformations as shown by Dubrovin, Liu and Zhang in [30]. Due to the skew-symmetry of the pencil, the sum and product of the roots contain only even powers of $p$. In the semisimple case, expansions (5.8) contain only even powers of $p$, while in the non-semisimple case, odd powers may appear. For instance, in the case of deformations of non semisimple pencils associated with two-dimensional Balinskiǐ-Novikov algebras one obtains the expansions

$$
\begin{equation*}
\lambda^{1}=u^{1}+\sum_{k=1}^{\infty} \lambda_{k}^{1} p^{k}, \quad \lambda^{2}=u^{1}+\sum_{k=1}^{\infty} \lambda_{k}^{2} p^{k} \tag{5.9}
\end{equation*}
$$

where, due to skew-symmetry,

$$
\begin{equation*}
\lambda_{2 k+1}^{1}+\lambda_{2 k+1}^{2}=0, \quad \lambda_{2 k}^{1}-\lambda_{2 k}^{2}=0 . \tag{5.10}
\end{equation*}
$$

Thus it is natural to divide Poisson pencils associated with Balinskiǐ-Novikov algebras in two classes: those admitting as invariants $\lambda_{1}^{1}=-\lambda_{1}^{2}$ and $\lambda_{2}^{1}=\lambda_{2}^{2}$ (and eventually higher order coefficients of the expansions (5.9)) and those admitting as invariants only $\lambda_{2}^{1}=\lambda_{2}^{2}$ (and eventually higher order coefficients of the expansions (5.9)).

### 5.4.1 The cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$.

In the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, the expansions of $\lambda^{i}$ do not contain the linear term in $p$, and coefficients of the quadratic terms $\lambda_{2}^{1}=\lambda_{2}^{2}$ are related to the
functional parameter $F_{2}$.
Theorem 5.3. Let $P_{\lambda}=P_{2}-\lambda P_{1}$ bi-Hamiltonian structure corresponding to one of the BalinskiǐNovikov algebras T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$ and the associated symmetric bilinear invariant form $\eta$. Let us consider a bi-Hamiltonian structures $\Pi_{\lambda}$ of the form (5.1) with leading term $P_{\lambda}^{i j}$. Then the coefficients $\lambda_{2}^{1}$ and $\lambda_{2}^{2}$ of the expansion (5.8) coincide, and are related to the functional parameter $F_{2}$ by the formulae:

- T3: $\lambda_{2}^{i}=\frac{u^{1}}{\eta^{12}} e^{-\frac{\eta^{12} u^{2}}{\eta^{22} u^{1}}} F_{2}\left(u^{1}\right) ;$
- N5: $\lambda_{2}^{i}=-\frac{u^{1} F_{2}\left(u^{1}\right)}{\eta^{12} \sqrt{2 \eta^{12}\left(u^{1}+u^{2}\right)-\eta^{22} u^{1}}}$;
- N3, N6 with $\kappa \neq 0,-1,-2: \lambda_{2}^{i}=-\frac{(\kappa+1) u^{1}\left(2 \eta^{12} u^{2}-(\kappa+1) \eta^{22} u^{1}\right)^{\frac{\kappa-1}{2}}}{\eta^{12}} F_{2}\left(u^{1}\right)$.


## Proof:

We are going to prove this statement in the case T3 with $\eta^{22} \neq 0$. In this case the dispersionless limit is given by

$$
P_{1}^{i j}=\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{12} & \eta^{22}
\end{array}\right) \frac{d}{d x}, \quad P_{2}^{i j}=\left(\begin{array}{cc}
0 & -u^{1} \\
-u^{1} & 0
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & 0 \\
-u_{x}^{1} & 0
\end{array}\right)
$$

If we write the pencil in the standard form

$$
\Pi_{\lambda}^{i j}=P_{\lambda}^{i j}+\sum_{k=1}^{2} \epsilon^{k} \sum_{l=0}^{k+1}\left(A_{2 ; k, l}^{i j}\left(\mathbf{u}, \ldots, \mathbf{u}_{(l)}\right)-\lambda A_{1 ; k, l}^{i j}\left(\mathbf{u}, \ldots, \mathbf{u}_{(l)}\right)\right) \frac{d^{k-l+1}}{d x^{k-l+1}}+\mathcal{O}\left(\epsilon^{3}\right)
$$

the first two terms of the expansion (5.8) are

$$
\begin{align*}
\lambda_{1}^{i} & =0  \tag{5.11}\\
\lambda_{2}^{i} & =\frac{1}{\eta^{12}}\left(S_{2}^{12}+\frac{\left(R_{2}^{12}\right)^{2}}{u^{1}}+\frac{\eta^{22} S_{2}^{11}}{2 \eta^{12}}+\frac{u^{1} S_{1}^{12}+R_{1}^{12} R_{2}^{12}}{\eta^{12}}\right) \tag{5.12}
\end{align*}
$$

where

$$
R_{\theta}^{i j}(\mathbf{u})=A_{\theta ; 1,2}^{i j}(\mathbf{u}), \quad S_{\theta}^{i j}(\mathbf{u})=A_{\theta ; 2,3}^{i j}(\mathbf{u}), \quad i, j=1, \ldots, n, \quad \theta=1,2
$$

We know from general theory that these coefficients are invariant up to permutations. The condition $\lambda_{2 n}^{1}=\lambda_{2 n}^{2}$ implies that they are genuine invariants.

Using this, the proof is a straightforward computation: substituting the relations

$$
R_{1}=R_{2}=S_{1}=\left(\begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}\right), \quad S_{2}=\left(\begin{array}{cc}
0 & u^{1} e^{-\frac{\eta^{12} u^{2}}{\eta^{2} u^{1}}} F_{2}\left(u^{1}\right) \\
u^{1} e^{-\frac{\eta^{12} u^{2}}{\eta^{2} u^{1}}} F_{2}\left(u^{1}\right) & *
\end{array}\right)
$$

in the formula (5.12) we get the result. Remaining cases can be proved following the same procedure.

Remark. The invariant $\lambda_{2}^{i}$ can be also written as

$$
\lambda_{2}^{i}=-\frac{1}{2} \operatorname{Res}_{\lambda=\hat{\lambda}} \operatorname{Tr}\left(g_{\lambda}^{-1} \Lambda_{\lambda}\right)
$$

where $\hat{\lambda}$ is the eigenvalue of the affinor $L=g_{2} g_{1}^{-1}$ and $\Lambda_{\lambda}^{i j}=S_{\lambda}^{i j}+\frac{1}{2}\left(g_{\lambda}^{-1}\right)_{l k} R_{\lambda}^{l i} R_{\lambda}^{k j}$.

### 5.4.2 The cases N4 and N6 with $\kappa=-2$

In the remaining cases the expansion of $\lambda^{i}$ contains also the linear term in $p$, and the invariants $\lambda_{1}^{1}=-\lambda_{1}^{2}$ and $\lambda_{2}^{1}=\lambda_{2}^{2}$ are related to the functional parameters $F_{2}$ and $F_{4}$ respectively.

Theorem 5.4. Let $P_{\lambda}=P_{2}-\lambda P_{1}$ bi-Hamiltonian structure corresponding to one of the BalinskiǐNovikov algebras N4 and N6 with $\kappa=-2$ and the associated symmetric bilinear invariant form $\eta$. Let us consider bi-Hamiltonian structures $\Pi_{\lambda}$ of the form (5.1) with the leading term $P_{\lambda}^{i j}$. Then, the invariants $\left(\lambda_{1}^{i}\right)^{2}$ and $\lambda_{2}^{i}$ are related to the functional parameters $F_{2}$ and $F_{4}$ through the formulae:

- N4:

$$
\begin{aligned}
\left(\lambda_{1}^{i}\right)^{2} & =\frac{2 u^{1} F_{2}}{\left(\eta^{12}\right)^{3}} \\
\lambda_{2}^{i} & =\frac{\partial_{1}\left(u^{1} F_{2}\right)}{\left(\eta^{12}\right)^{2}}-\frac{u^{1} F_{4}}{\eta^{12} \sqrt{-2 \eta^{12} u^{2}+\eta^{22} u^{1}}}
\end{aligned}
$$

- $N 6, \kappa=-2$ :

$$
\begin{aligned}
\left(\lambda_{1}^{i}\right)^{2} & =\frac{2 u^{1} F_{2}}{\left(\eta^{12}\right)^{3}\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)^{2}}, \\
\lambda_{2}^{i} & =\frac{u^{1} F_{4}}{\eta^{12}\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)^{3 / 2}}-\frac{\left(2 \eta^{12} u^{2}-\eta^{22} u^{1}\right) F_{2}+u^{1} F_{2}^{\prime}}{\left(\eta^{12}\right)^{2}\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)^{3}} .
\end{aligned}
$$

## Proof:

We outline the proof in the case N 4 (corresponding to $\kappa=0$ ). In this case, the standard form of the pencil is

$$
\tilde{\Pi}_{\lambda}^{i j}=P_{\lambda}^{i j}+\epsilon^{2} \Theta^{i j}+\mathcal{O}\left(\epsilon^{3}\right)=P_{\lambda}^{i j}+\epsilon^{2}\left(\Theta_{(3)}^{i j} \frac{d^{3}}{d x^{3}}+\Theta_{(2)}^{i j} \frac{d^{2}}{d x^{2}}+\Theta_{(1)}^{i j} \frac{d}{d x}+\Theta_{(0)}^{i j}\right)+\mathcal{O}\left(\epsilon^{3}\right)
$$

where

$$
P_{\lambda}^{i j}=\left(\begin{array}{cc}
0 & u^{1} \\
u^{1} & 2 u^{2}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & u_{x}^{1} \\
0 & u_{x}^{2}
\end{array}\right)-\lambda\left(\begin{array}{cc}
0 & \eta^{12} \\
\eta^{12} & \eta^{22}
\end{array}\right) \frac{d}{d x}
$$

and

$$
\Theta_{(3)}=\left(\begin{array}{cc}
\frac{2 u^{1} F_{2}}{2 \eta^{12} u^{2}-\eta^{22} u^{1}} & \frac{u^{1} F_{2}^{\prime}}{\eta^{12}}-\frac{u^{1} F_{4}}{\sqrt{-2 \eta^{12} u^{2}+\eta^{22} u^{1}}}+\frac{2 u^{2} F_{2}}{2 \eta^{12} u^{2}-\eta^{22} u^{1}} \\
\frac{u^{1} F_{2}^{\prime}}{\eta^{12}}-\frac{u^{1} F_{4}}{\sqrt{-2 \eta^{12} u^{2}+\eta^{22} u^{1}}}+\frac{2 u^{2} F_{2}}{2 \eta^{12} u^{2}-\eta^{22} u^{1}} & \frac{4 u^{2} F_{2}^{\prime}}{\eta^{12}}-\frac{4 u^{2} F_{4}}{\sqrt{-2 \eta^{12} u^{2}+\eta^{22} u^{1}}}
\end{array}\right) .
$$

From the general theory and relations (5.10) we know that $\left(\lambda_{1}^{i}\right)^{2}$ and $\lambda_{2}^{i}$ are invariants. Using the invariance, the proof is a straightforward computation. The case N6 with $\kappa=$ -2 can be treated in a similar way.

Remark. The function $\Theta_{(3)}^{12}$ can be also written as

$$
\Theta_{(3)}^{12}=-\frac{\eta^{12}}{2} \operatorname{Res}_{\lambda=\hat{\lambda}} \operatorname{Tr}\left(g_{\lambda}^{-1} \Lambda_{\lambda}\right)
$$

where $\hat{\lambda}$ is the eigenvalue of the affinor $L=g_{2} g_{1}^{-1}$, and $\Lambda_{\lambda}^{i j}=S_{\lambda}^{i j}+\frac{1}{2}\left(g_{\lambda}^{-1}\right)_{l k} R_{\lambda}^{l i} R_{\lambda}^{k j}$.

### 5.5 Truncated structures

In Theorems 5.3, 5.4 we proved the invariant nature of some functional parameters appearing in deformations. In this section we prove that the remaining parameters are related to truncated structures. These are Poisson pencils of the form (5.1) depending polynomially on the parameter $\epsilon$ (that is, the sum in (5.1) contains finitely many terms). We show that setting to zero the invariant parameters we obtain deformations that are Miura equivalent to truncated pencils up to the order three. More precisely we prove that in the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$ the additional parameter provides a one-parameter family of truncated structures, while in the cases N 4 and N6 with $\kappa=-2$
the two additional parameters provide a two-parameter family of truncated structures.
Theorem 5.5. In the cases T3, N3, N5 and N6 with $\kappa \neq 0,-1,-2$, second-order deformations with $F_{2}=0$ can be reduced by a Miura transformation to the form $\Pi_{\lambda}=P_{\lambda}+\epsilon^{2} \Theta+\mathcal{O}\left(\epsilon^{3}\right)$, where

$$
\Theta=\left(\begin{array}{cc}
0 & 0  \tag{5.13}\\
0 & 2 f
\end{array}\right) \frac{d^{3}}{d x^{3}}+\left(\begin{array}{cc}
0 & 0 \\
0 & 3 f_{x}
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{cc}
0 & 0 \\
0 & f_{x x}
\end{array}\right) \frac{d}{d x},
$$

with $f=f\left(u^{1}\right)$. Moreover, the truncated pencil $P_{\lambda}+\epsilon^{2} \Theta$ is a Poisson pencil.

## Proof:

The form (5.13) can be easily obtained from the results of Theorem 5.1 rescaling the function $F_{1}$. In particular, we have to set:

- $F_{1}\left(u^{1}\right)=\frac{f\left(u^{1}\right)}{u^{1}}$, for T3;
- $F_{1}\left(u^{1}\right)=-\frac{\eta^{12} f\left(u^{1}\right)}{u^{1}}$, for N5;
- $F_{1}\left(u^{1}\right)=-\frac{\eta^{12} \kappa f\left(u^{1}\right)}{(1+\kappa) u^{1}}$, for N3, N6 with $\kappa \neq 0,-1,-2$.

To prove that $P_{\lambda}+\epsilon^{2} \Theta$ is a Poisson pencil, we have to show that

$$
\begin{aligned}
& \frac{1}{2}[\Theta, \Theta]^{i j k}(x, y, z)= \\
& \quad \frac{\partial \Theta^{i j}(x, y)}{\partial u_{(s)}^{l}(x)} \partial_{x}^{s} \Theta^{l k}(x, z)+\frac{\partial \Theta^{k i}(z, x)}{\partial u_{(s)}^{l}(z)} \partial_{z}^{s} \Theta^{l j}(z, y)+\frac{\partial \Theta^{j k}(y, z)}{\partial u_{(s)}^{l}(y)} \partial_{y}^{s} \Theta^{l i}(y, x)=0 .
\end{aligned}
$$

Taking into account that $\Theta^{11}=\Theta^{12}=\Theta^{21}=0$ and $\frac{\partial \Theta^{22}}{\partial u_{(s)}^{2}}=0$, we obtain the result.

Theorem 5.6. In the case N6 with $\kappa=-2$, second-order deformations with $F_{2}=F_{4}=0$ can be reduced by a Miura transformation to the form $\Pi_{\lambda}=P_{\lambda}+\epsilon^{2} \Theta+\mathcal{O}\left(\epsilon^{3}\right)$, where

$$
\Theta=\left(\begin{array}{cc}
0 & 0  \tag{5.14}\\
0 & 2 f
\end{array}\right) \frac{d^{3}}{d x^{3}}+\left(\begin{array}{cc}
0 & 0 \\
0 & 3 f_{x}
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{cc}
0 & 0 \\
0 & f_{x x}+2 g
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
0 & 0 \\
0 & g_{x}
\end{array}\right),
$$

with $f=f\left(u^{1}\right)$ and $g=\left(h\left(u^{1}\right) u_{x}^{1}\right)_{x}+h\left(u^{1}\right) u_{x x}^{1}$. Moreover, the truncated pencil $P_{\lambda}+\epsilon^{2} \Theta$ is a Poisson pencil.

## Proof:

Here we prove only the first part of the theorem. The second part can be obtained as above by straightforward computation.

By Theorem 5.1 we have

$$
\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)
$$

where components of the vector field $X$ are given by

$$
X^{1}=\theta F_{1}\left(u_{x}^{1}\right)^{2}, \quad X^{2}=F_{3}\left(u_{x}^{1}\right)^{2}
$$

with $\theta=\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)^{-1}$. The Miura transformation

$$
u^{i} \rightarrow \exp (-\epsilon Y) u^{i}, \quad i=1,2
$$

generated by the vector field $Y$ with components

$$
\begin{aligned}
Y^{1} & =-\eta^{12} R u_{x x}^{1}-\eta^{12} \partial_{1} R\left(u_{x}^{1}\right)^{2}-\eta^{12} \partial_{2} R u_{x}^{1} u_{x}^{2} \\
Y^{2} & =-\eta^{22} R u_{x x}^{1}-\eta^{22} \partial_{1} R\left(u_{x}^{1}\right)^{2}+\left(\eta^{12} \partial_{1} R-\eta^{22} \partial_{2} R\right) u_{x}^{1} u_{x}^{2}+\eta^{12} \partial_{2} R\left(u_{x}^{2}\right)^{2}+\eta^{12} R u_{x x}^{2}
\end{aligned}
$$

where $R=\frac{u^{1} F_{1}}{2 \eta^{12}\left(2 \eta^{12} u^{2}+\eta^{22} u^{1}\right)}$, reduces the pencil to the form $P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{\tilde{X}} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)$, where

$$
\begin{aligned}
\tilde{X}^{1}= & -\frac{\theta u^{1} F_{1} u_{x x}^{1}}{2}-\left(\frac{\theta u^{1} F_{1}^{\prime}}{2}-\theta^{2}\left(\eta^{12} u^{2}+\eta^{22} u^{1}\right) F_{1}\right)\left(u_{x}^{1}\right)^{2}+\theta^{2} \eta^{12} u^{1} F_{1} u_{x}^{1} u_{x}^{2} \\
\tilde{X}^{2}= & -\frac{\theta \eta^{22} u^{1} F_{1} u_{x x}^{1}}{2 \eta^{12}}+\frac{\theta u^{1} F_{1} u_{x x}^{2}}{2}+\left(\frac{\theta u^{1} F_{1}^{\prime}}{2}+\theta^{2}\left(\eta^{12} u^{2}+\eta^{22} u^{2}\right) F_{1}\right) u_{x}^{1} u_{x}^{2} \\
& -\left(\frac{\theta \eta^{22} u^{1} F_{1}^{\prime}}{2 \eta^{12}}+\theta^{2} \eta^{22} u^{2} F_{1}-F_{3}\right)\left(u_{x}^{1}\right)^{2}-\theta^{2} \eta^{12} u^{1} F_{1}\left(u_{x}^{2}\right)^{2}
\end{aligned}
$$

To conclude it is easy to check that $\operatorname{Lie}_{\tilde{X}} P_{2}$ coincides with (5.14), setting $F_{1}=-\frac{2 \eta^{12} f}{u^{1}}$ and $F_{3}=-\frac{h}{u^{1}}$.

Theorem 5.7. In the case N4 with $F_{2}=F_{4}=0$, second-order deformations can be reduced by a

Miura transformation to the form $\Pi_{\lambda}=P_{\lambda}+\epsilon^{2} \Theta+\mathcal{O}\left(\epsilon^{3}\right)$, where

$$
\Theta=\left(\begin{array}{cc}
0 & 0  \tag{5.15}\\
0 & q_{3}^{22}
\end{array}\right) \frac{d^{3}}{d x^{3}}+\left(\begin{array}{cc}
0 & q_{2}^{12} \\
-q_{2}^{12} & q_{2}^{22}
\end{array}\right) \frac{d^{2}}{d x^{2}}+\left(\begin{array}{cc}
q_{1}^{11} & q_{1}^{12} \\
q_{1}^{21} & q_{1}^{22}
\end{array}\right) \frac{d}{d x}+\left(\begin{array}{cc}
q_{0}^{11} & q_{0}^{12} \\
q_{0}^{21} & q_{0}^{22}
\end{array}\right),
$$

with

$$
\begin{aligned}
q_{3}^{22}= & 2 f, \\
q_{2}^{12}= & 4 \theta \eta^{12} f u_{x}^{1}, \\
q_{2}^{22}= & 3 f^{\prime} u_{x}^{1}, \\
q_{1}^{11}= & -8\left(\theta \eta^{12}\right)^{2} f\left(u_{x}^{1}\right)^{2}, \\
q_{1}^{12}= & \left(2 \theta \eta^{12} f^{\prime}-2 \theta^{2} \eta^{12} \eta^{22} f+2 \theta^{2} h\right)\left(u_{x}^{1}\right)^{2}, \\
q_{1}^{21}= & \left(-6 \theta \eta^{12} f^{\prime}-10 \theta^{2} \eta^{12} \eta^{22} f+2 \theta^{2} h\right)\left(u_{x}^{1}\right)^{2}+16\left(\theta \eta^{12}\right)^{2} f u_{x}^{1} u_{x}^{2}-8 \theta \eta^{12} f u_{x x}^{1}, \\
q_{1}^{22}= & \left(f^{\prime \prime}+2 \theta\left(\eta^{12}\right)^{-1} h^{\prime}+6 \theta^{2}\left(\eta^{12}\right)^{-1} \eta^{22} h\right)\left(u_{x}^{1}\right)^{2}-8 \theta^{2} h u_{x}^{1} u_{x}^{2}+\left(f^{\prime}+4 \theta\left(\eta^{12}\right)^{-1} h\right) u_{x x}^{1}, \\
q_{0}^{11}= & -\left(4\left(\theta \eta^{12}\right)^{2} f^{\prime}+8 \theta^{3}\left(\eta^{12}\right)^{2} \eta^{22} f\right)\left(u_{x}^{1}\right)^{3}+16\left(\theta \eta^{12}\right)^{3} f\left(u_{x}^{1}\right)^{2} u_{x}^{2}-8\left(\theta \eta^{12}\right)^{2} f u_{x}^{1} u_{x x}^{1}, \\
q_{0}^{12}= & \left(2 \theta^{2} h^{\prime}+4 \theta^{3} \eta^{22} h\right)\left(u_{x}^{1}\right)^{3}-8 \theta^{3} \eta^{12} h\left(u_{x}^{1}\right)^{2} u_{x}^{2}+4 \theta^{2} h u_{x}^{1} u_{x x}^{1}, \\
q_{0}^{21}= & \left(-2 \theta \eta^{12} f^{\prime \prime}-8 \theta^{2} \eta^{12} \eta^{22} f^{\prime}-12 \theta^{3} \eta^{12}\left(\eta^{22}\right)^{2} f\right)\left(u_{x}^{1}\right)^{3} \\
& +\left(12\left(\theta \eta^{12}\right)^{2} f^{\prime}+40 \theta^{3}\left(\eta^{12}\right)^{2} \eta^{22} f\right)\left(u_{x}^{1}\right)^{2} u_{x}^{2}+\left(-8 \theta \eta^{12} f^{\prime}-16 \theta^{2} \eta^{12} \eta^{22} f\right) u_{x}^{1} u_{x x}^{1} \\
& -32\left(\theta \eta^{12}\right)^{3} f u_{x}^{1}\left(u_{x}^{2}\right)^{2}+8\left(\theta \eta^{12}\right)^{2} f u_{x}^{1} u_{x x}^{2}+16\left(\theta \eta^{12}\right)^{2} f u_{x x}^{1} u_{x}^{2}-4 \theta \eta^{12} f u_{x x x}^{1}, \\
q_{0}^{22}= & \left(\theta\left(\eta^{12}\right)^{-1} h^{\prime \prime}+4 \theta^{2}\left(\eta^{12}\right)^{-1} \eta^{22} h^{\prime}+6 \theta^{3}\left(\eta^{12}\right)^{-1}\left(\eta^{22}\right)^{2} h\right)\left(u_{x}^{1}\right)^{3} \\
& +\left(-6 \theta^{2} h-20 \theta^{3} \eta^{22} h\right)\left(u_{x}^{1}\right)^{2} u_{x}^{2}+\left(4 \theta\left(\eta^{12}\right)^{-1} h^{\prime}+8 \theta^{2}\left(\eta^{12}\right)^{-1} \eta^{22} h\right) u_{x}^{1} u_{x x}^{1} \\
& +16 \theta^{3} \eta^{12} h u_{x}^{1}\left(u_{x}^{2}\right)^{2}-2 \theta^{2} h u_{x}^{1} u_{x x}^{2}-4 \theta^{2} h u_{x x}^{1} u_{x}^{2}+\theta\left(\eta^{12}\right)^{-1} h u_{x x x}^{1},
\end{aligned}
$$

where $f=f\left(u^{1}\right), h=h\left(u^{1}\right)$ and $\theta=\left(2 \eta^{12} u^{2}-\eta^{22} u^{1}\right)^{-1}$. Moreover, the truncated pencil $P_{\lambda}+\epsilon^{2} \Theta$ is a Poisson pencil.

## Proof:

By Theorem 5.1 we have $\Pi_{\lambda}=P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{X} P_{2}+\mathcal{O}\left(\epsilon^{3}\right)$, where the components of the vector field $X$ are given by

$$
X^{1}=-\theta F_{1}\left(u_{x}^{1}\right)^{2}, \quad X^{2}=-\theta F_{3}\left(u_{x}^{1}\right)^{2}
$$

with $\theta=\left(2 \eta^{12} u^{2}-\eta^{22} u^{1}\right)^{-1}$. The Miura transformation

$$
u^{i} \rightarrow \exp (-\epsilon Y) u^{i}, \quad i=1,2,
$$

generated by the vector field $Y$ with components

$$
\begin{aligned}
& Y^{1}=-\eta^{12} R u_{x x}^{1}-\eta^{12} \partial_{1} R\left(u_{x}^{1}\right)^{2}-\eta^{12} \partial_{2} R u_{x}^{1} u_{x}^{2}, \\
& Y^{2}=-\eta^{22} R u_{x x}^{1}-\eta^{22} \partial_{1} R\left(u_{x}^{1}\right)^{2}+\left(\eta^{12} \partial_{1} R-\eta^{22} \partial_{2} R\right) u_{x}^{1} u_{x}^{2}+\eta^{12} \partial_{2} R\left(u_{x}^{2}\right)^{2}+\eta^{12} R u_{x x}^{2},
\end{aligned}
$$

where $R=-\frac{u^{1} F_{1}}{2 \eta^{12}\left(2 \eta^{12} u^{2}-\eta^{22} u^{1}\right)}$, reduces the pencil to the form

$$
P_{2}-\lambda P_{1}+\epsilon^{2} \operatorname{Lie}_{\tilde{X}} P_{2}+\mathcal{O}\left(\epsilon^{3}\right),
$$

where

$$
\begin{aligned}
X^{1}= & \frac{\theta u^{1} F_{1} u_{x x}^{1}}{2}+\left(\frac{\theta u^{1} F_{1}^{\prime}}{2}-\theta^{2}\left(\eta^{12} u^{2}-\eta^{22} u^{2}\right) F_{1}\right)\left(u_{x}^{1}\right)^{2}-\theta^{2} \eta^{12} u^{1} F_{1} u_{x}^{1} u_{x}^{2}, \\
X^{2}= & \frac{\theta \eta^{22} u^{1} F_{1} u_{x x}^{1}}{2 \eta^{12}}-\frac{\theta u^{1} F_{1} u_{x x}^{2}}{2}-\left(\frac{\theta u^{1} F_{1}^{\prime}}{2}+\theta^{2}\left(\eta^{12} u^{2}+\eta^{22} u^{2}\right) F_{1}\right) u_{x}^{1} u_{x}^{2} \\
& +\left(\frac{\theta \eta^{22} u^{1} F_{1}^{\prime}}{2 \eta^{12}}+\theta^{2} \eta^{22} u^{2} F_{1}-\theta F_{3}\right)\left(u_{x}^{1}\right)^{2}+\theta^{2} \eta^{12} u^{1} F_{1}\left(u_{x}^{2}\right)^{2} .
\end{aligned}
$$

To conclude the first part of the theorem we observe that $\operatorname{Lie}_{\tilde{X}} P_{2}=\Theta\left(F_{1}=\frac{2 \eta^{12} f}{u^{1}}\right.$ and $F_{3}=-\frac{h}{\eta^{12} u^{1}}$. The second part is a cumbersome computation.

Remark. Truncated Poisson pencils of the form

$$
\begin{equation*}
\Pi_{\lambda}^{i j}=P_{\lambda}^{i j}+\epsilon \sum_{l=0}^{2}\left(A_{2 ; 1, l}^{i j}-\lambda A_{1 ; 1, l}^{i j}\right) \frac{d^{2-l}}{d x^{2-l}}+\epsilon^{2} \sum_{l=0}^{3}\left(A_{2 ; 2, l}^{i j}-\lambda A_{1 ; 2, l}^{i j}\right) \frac{d^{3-l}}{d x^{3-l}}, \tag{5.16}
\end{equation*}
$$

where $P_{\lambda}$ is a Poisson pencil of hydrodynamic type associated with a Balinskiǐ-Novikov algebra, appear in [93]. In this case, the coefficients

$$
A_{2 ; 1,0}^{i j}, A_{1 ; 1,0}^{i j}, A_{2 ; 2,0}^{i j}, A_{1 ; 2,0}^{i j}
$$

are related to second and third order cocycles of the Balinskiǐ-Novikov algebra. In order to reduce deformations of the form (5.16) to the canonical form $\Pi_{\lambda}=P_{\lambda}+\epsilon^{2} \Theta+\mathcal{O}\left(\epsilon^{3}\right)$,
one has to peform a Miura transformation producing (in general) infinitely many terms in the right hand side of (5.16). For this reason, Strachan-Szablikowski truncated pencils correspond in our framework to non truncated pencils.

### 5.6 Example

Let us consider second-order deformations of N3 obtained in Theorem 5.1, and set $\eta^{22}=$ $0, \eta^{12}=1, F_{1}\left(u^{1}\right)=0$ and $F_{2}\left(u^{1}\right)=-\frac{f\left(u^{1}\right)}{u^{1}}$. The Miura transformation $u^{i} \rightarrow \exp (-\epsilon Y) u^{i}$, $i=1,2$, generated by the vector field $Y$ with components

$$
Y^{1}=\frac{f^{\prime}}{3} u_{x x}^{1}+\frac{f^{\prime \prime}}{3}\left(u_{x}^{1}\right)^{2}, \quad Y^{2}=-\frac{f^{\prime \prime}}{3} u_{x}^{1} u_{x}^{2}-\frac{f^{\prime}}{3} u_{x x}^{2},
$$

reduces the pencil to the form

$$
\hat{\Pi}_{\lambda}=\left(\begin{array}{cc}
0 & \Pi_{\lambda}  \tag{5.17}\\
\Pi_{\lambda} & \sum_{t} v_{(t)} \frac{\partial \Pi_{\lambda}}{\partial u_{(t)}}
\end{array}\right)
$$

where $\Pi_{\lambda}$ coincides with

$$
\begin{equation*}
\Pi_{\lambda}=2 u^{1} \frac{d}{d x}+u_{x}^{1}-\lambda \frac{d}{d x}+\epsilon^{2}\left(2 s \frac{d^{3}}{d x^{3}}+3 s_{x} \frac{d^{2}}{d x^{2}}+s_{x x} \frac{d}{d x}\right)+\mathcal{O}\left(\epsilon^{3}\right), \tag{5.18}
\end{equation*}
$$

with $s=s\left(u^{1}\right)$. The structure (5.17) is, up to terms of order $\epsilon^{3}$, the complete lift of deformations of the dispersionless scalar structure

$$
\begin{equation*}
P=2 u \frac{d}{d x}+u_{x}-\lambda \frac{d}{d x}, \tag{5.19}
\end{equation*}
$$

here $u^{1}=u$. This complete lift can be viewed as an infinite-dimensional analogue of the complete lift introduced by Yano and Kobayashi [97, 98, 99]. Further details can be found in [22].

It is well known that the scalar structure (5.19) admits deformations to any order. This suggests that deformations of non-semisimple pencils corresponding to the invariant parameter are unobstructed.

Remark. One can prove that the complete lift of a semisimple structure leads to a nonsemisimple pencil [22].

## Concluding remarks

Poisson structures of hydrodynamic type and their deformations have been the subject of extensive research in recent years, and this area is still offering challenging problems. In this framework, we restricted ourselves to three main topics: classification of twodimensional Poisson structures, both degenerate and non-degenerate (for a small number of components), deformations of degenerate one-dimensional structures, and deformations of non-semisimple Poisson pencils.

Firstly, using a novel approach to the study of multi-dimensional brackets, based on Riemannian geometry, we obtained a complete list of two-dimensional non-degenerate Dubrovin-Novikov structures with three and four components, as well as the classification of multi-component non-degenerate structures in the case where the corresponding affinor consists of Jordan blocks with distinct eigenvalues.

- Our calculations demonstrate that the most challenging case is the one where the Jordan normal form of the affinor $L=\tilde{g} g^{-1}$ consists of several Jordan blocks with the same eigenvalue. To complete the classification, one needs to understand the structure of such operators: due to the splitting lemma, the general Hamiltonian operator would be representable as their direct sum.
- Given any 2D Poisson structures from our list, it would be interesting to classify Hamiltonians which generate integrable $2+1$ dimensional systems of hydrodynamic type. The existing results suggest that integrable Hamiltonians form finitedimensional moduli spaces, and are quite non-trivial even for constant-coefficient operators, see [43, 48, 50] for the first steps in this direction.
- It would be interesting to develop a deformation theory of 2D Hamiltonian operators in the spirit of $[54,21,38]$, and to investigate triviality of Poisson cohomology in 2D. Some results in this direction were recently obtained in [19, 15].

Then, we obtained the classification (up to three components) of degenerate two-dimensional structures, analysing also the integrability by the method of hydrodynamic reduction for all of the three-component structures we classified.

- Obtaining a complete classification of degenerate structures (at least in four components) is still an open problem. The main obstacle is the lack of a full description of 1D degenerate Poisson brackets. Indeed, already for four-component 1D degenerate structures, the computation of Jacobi conditions is quite complicated [56, 87].
- Any 2D Hamiltonian operator gives rise to a pair of 1D compatible brackets of Dubrovin-Novikov type, and therefore a bi-Hamiltonian structure. This property is still true in the case of degenerate structures [76, 69]. Degenerate bi-Hamiltonian structures of hydrodynamic type were firstly investigated by Strachan [92, 91], revealing a nice relation with the theory of Frobenius manifolds with degenerate metric. It turns out that some of the degenerate bi-Hamiltonian structures arising from our classification are not of the kind investigated by Strachan. It would be interesting to analyse these structures and to study a possible correspondence with the theory of Frobenius manifolds.

In the framework of deformation theory for Poisson brackets of hydrodynamic type with degenerate metric, our main contributions include the proof that in the two-component case, first- and second-order deformations are not trivial, as well as examples of nontrivial first-order deformations for some three-component structures.

- Our results suggest the following conjecture.

Conjecture. The $k$-order deformations of two-component Poisson structures with degenerate metric are characterised by functions depending on the single variable $u^{2}$.

Unfortunately, the number of unknowns in this problem grows rapidly with the increase of the order of deformations, and computations become more and more complicated. Thus, it seems necessary to find a different approach in order to prove the conjecture.

- A deeper analysis of the three-component case would be an important step to better understand what happens in a more general context, in order to generalise our results:

Conjecture. If a matrix $g$ which defines a $n$-component Poisson structure of hydrodynamic type $P$ has rank $m<n$, deformations of $P$ are characterised by arbitrary functions depending only on the set of variables $\left(u^{m+1}, \ldots, u^{n}\right)$.

Finally, we analysed deformations of two-component non-semisimple Poisson pencils of hydrodynamic type associated with Balinskiir-Novikov algebras. We proved that in most cases second-order deformations are parametrised by two functions of a single variable: one function is invariant with respect to the subgroup of Miura transformations preserving the dispersionless limit, and another function is related to a one-parameter family of truncated structures. In two exceptional cases, second-order deformations are parametrised by four functions: two are invariants and two are related to a two-parameter family of truncated structures.

- Our computations provide the first step towards the study of deformations of nonsemisimple Poisson pencils of hydrodynamic type. It would be interesting to investigate higher order deformations and to increase the number of components. Unfortunately, as in the case of deformations of a single degenerate structure, increasing the order, or the number of components, leads to very complicated computations.
- The undeformed structures we considered are non-semisimple and therefore they are related to non-semisimple Frobenius manifolds. Furthermore, as observed in [22], the lift of a semisimple Frobenius manifold leads to a non-semisimple Frobenius manifold. Since this class of structures has not been deeply studied yet, this could be a starting point for further investigations.


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