A spin-coefficient approach to space–times with torsion

A SPIN-COEFFICIENT APPROACH TO SPACE-TIMES WITH TORSION

by

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ABSTRACT

The Newman-Penrose formalism, which has been extremely useful in general relativity, is extended to include the possibility of space-times with torsion. Initially, Riemann-Cartan geometry is discussed and the torsion and contortion tensors are defined. The possible alternative but equivalent approaches in developing the formalism are given in Chapters 3 and 4. These involve the use of tetrads and spinor dyads. In the definition of the spin coefficients, the components of the contortion tensor appear as correction terms to the spin coefficients defined in the associated Riemannian space-time. The algebraic structure of the curvature is analysed in both its tensor and spinor forms and distinctive labels are given to the additional 16 independent components that vanish in Riemannian space-time. The generalised Newman-Penrose identities are obtained in Chapter 5 and consist of 4 lengthy sets of equations. Of these, the Bianchi identities for the torsion are a new feature and consist of essential integrability conditions on the contortion tensor.

In Chapter 6, it is shown how the formalism may be used to generate exact solutions. Two different forms for the contortion tensor are considered and simple plane wave geometries are obtained.

The formalism is applicable to any theory of gravitation that includes torsion. Of particular interest however is the Einstein-Cartan theory which is introduced in Chapter 7. Classical Neutrino fields and semi-classical spin fields are considered in this theory in Chapters 8 and 9. A number of general properties and exact solutions are obtained for these sources. For example it is shown that

the only possible ghost neutrino solutions are those with a plane wave metric. The exact solutions for the semi-classical spin fluid have in general non-zero acceleration and so provide the first "tilted" cosmological models in the Einstein-Cartan theory. However, these models are very special and have an equation of state corresponding to "stiff matter".

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NOTATION

The conventions for tensor indices are:

$\alpha, \beta, \gamma, \ldots$ range over the values 1, 2, 3, 4.

The conventions for tetrad indices are:

$\ell, m, n, \ldots$ range over the values 1, 2, 3, 4.

The conventions for spinor indices are:

$A, B, C, \ldots$ range over the values 0, 1.

$\ldots, \dot{X}, \dot{Y}, \dot{Z}$ range over the values $\dot{0}, \dot{1}$.

The conventions for dyad indices are:

$a, b, c, \ldots$ range over the values 0, 1.

$\ldots, \dot{x}, \dot{y}, \dot{z}$ range over the values $\dot{0}, \dot{1}$.

(Note that when different conventions are used for indices, they will be specified in the thesis).

Symmetrization is denoted by round brackets, e.g.

$$B_{(\alpha\beta\gamma)} = \frac{1}{3!}\left(B_{\alpha\beta\gamma} + B_{\beta\gamma\alpha} + B_{\gamma\alpha\beta} + B_{\alpha\gamma\beta} + B_{\gamma\beta\alpha} + B_{\beta\alpha\gamma}\right)$$

and anti-symmetrization by square brackets, e.g.

$$B_{[\alpha\beta\gamma]} = \frac{1}{3!}\left(B_{\alpha\beta\gamma} + B_{\beta\gamma\alpha} + B_{\gamma\alpha\beta} - B_{\alpha\gamma\beta} - B_{\gamma\beta\alpha} - B_{\beta\alpha\gamma}\right)$$

The symbol "c.c" denotes complex conjugate.

The signature of space-time is chosen to be $---+$.

Cartan's torsion tensor and the contorsion tensor are respectively denoted by $S_{\mu\nu}{}^{\kappa}$ and $K_{\mu\nu}{}^{\kappa}$.

The sign of the Riemann-Christoffel curvature tensor is specified by the Ricci identities

$$2\,\nabla_{[\mu}\nabla_{\nu]}A^{\lambda} = -\,R_{\mu\nu\kappa}{}^{\lambda}\,A^{\kappa} + 2\,K_{[\mu\nu]}{}^{\kappa}\,\nabla_{\kappa}\,A^{\lambda}$$

where "∇" denotes covariant differentiation with respect to the affine connection.

The Ricci tensor and curvature scalar are defined by

$$R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha}, \quad R = R_{\alpha}{}^{\alpha}$$

Finally, the permutation tensor is defined by

$$\varepsilon_{\mu\nu\kappa\lambda} = \varepsilon_{[\mu\nu\kappa\lambda]}$$

and

$$\varepsilon_{1234} = \sqrt{-g}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$.

CHAPTER 1

INTRODUCTION

The spin coefficient approach to general relativity was used in a classic paper by Newman and Penrose (1962) to develop a notation which has now become a familiar tool to numerous relativists. In the paper they were concerned primarily with gravitational radiation in the context of Einstein's theory. However their notation, also known as the Newman-Penrose (N-P) formalism, has since been successfully applied to a wider class of problems and is now a well established method in relativity theory [see for example the general reviews in Alekseev and Khlebnikov (1978), Carmeli (1977), Frolov (1979) and Kramer et al (1980)]. It is particularly useful for the analysis of gravitational fields and for obtaining algebraically special exact solutions of the field equations.

Because of its tremendous success, a mathematical extension of the formalism from Riemannian space-times (V_4) to the more general Riemann-Cartan space-times (U_4) may be of use to gravitational theories expressed in terms of the latter. Such an extension is presented in the first half of this thesis. The U_4 affine structure is more general than that of the associated V_4. The connection in U_4 is taken to be generally asymmetric and its antisymmetric part is called Cartan's torsion tensor. Therefore if this tensor vanishes then U_4 reduces to V_4. The "metric postulate", which is the compatability condition between the symmetric metric and the asymmetric connection, is also imposed so that local Minskowskian structure can be ensured. The space-times, considered in this thesis, are then restricted to the class of Riemann-Cartan manifolds. The essential geometry of these manifolds is given in Chapter 2.

A N-P type formalism for space-times with torsion is presented in Chapters 3, 4 and 5. Two equivalent methods may be considered in developing the formalism which culminates in the generalised N-P identities of Chapter 5. These involve the use of tetrads and spinor dyads. Emphasis will be placed throughout this thesis on the tetrad approach given in Chapter 3. However the spinor dyad approach given in Chapter 4 will be useful when dealing with Neutrino fields.

In Chapter 3, the spin coefficients are defined in terms of the Ricci rotation coefficients by taking the same linear combinations and using the same labels as Newman and Penrose. It is found that the contortion tensor, defined in Chapter 2, adapts to this scheme a little more naturally than the torsion tensor. Its components are treated as correction terms to the values of the spin coefficients in a V_4.

A new decomposition of the U_4 Riemann-Christoffel curvature tensor is presented in Section 3.2. This tensor now has 36 independent components which are defined explicitly with respect to the tetrad basis. Of these the 16 components of the Ricci tensor are represented by the familiar components of its symmetric part, Φ_{AB} and Λ, and 3 new complex components Φ_A which represent its anti-symmetric part. The remaining 20 independent components belong to the trace free part of the curvature tensor $C_{\kappa\lambda\mu\nu}$. This part is decomposed into the generalised Weyl tensor $A_{\kappa\lambda\mu\nu}$, a tensor $B_{\kappa\lambda\mu\nu}$ and a scalar D. The components of $A_{\kappa\lambda\mu\nu}$ are expressed in terms of the five familiar complex components Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4. The remaining new components belonging to the tensor $B_{\kappa\lambda\mu\nu}$ and the scalar D are represented respectively by the components of a 3 x 3 hermitian matrix Θ_{AB} and a real parameter Σ.

The generalised N-P identities, given in Chapter 5, are considerably longer and more numerous than the original ones. They also include a new set of identities, introduced in Section 5.3, which are essential integrability conditions on the torsion. These equations are called the Bianchi identities for the torsion and originate from the generalisation of a symmetry property of the curvature tensor defined in a V_4. In the application of this formalism, the Bianchi identities for the torsion have proven to be an important new feature and a starting point in the simplification of the full set of equations.

At first sight the generalised N-P identities appear to be unmanageable. However it should be noted that any gravitational theory involving torsion is mathematically complicated in any approach. Thus extending the N-P formalism to such a complicated form is not out of proportion and in some applications the form of the contortion tensor drastically reduces the number and the length of the equations to a point at which they can be handled. This is seen in Chapter 6 where two different forms for the contortion tensor have been chosen in order to obtain simple plane-wave geometries for space-times with torsion. Although this exercise is not physically important, it is instructive and shows how the formalism can be used to generate metrics.

The motivation for the development of this formalism has not been purely mathematical. Riemann-Cartan manifolds are the space-times for the so-called torsion theories of gravity that extend Einstein's theory to include the concept of spin angular momentum. The formalism developed here is applicable to any torsion theory which considers such an extension.

The mathematician Élie Cartan introduced torsion and appreciated that it might be connected with the intrinsic angular momentum of matter (Cartan 1922, 1923, 1924, 1925). For historical reasons however his work was largely forgotten until Sciama (1962) and Kibble (1961) took an interest in the problem. They rediscovered the field equations for a gravitational theory incorporating spin and torsion. This theory will be called the Einstein-Cartan theory in this thesis although some authors refer to it as the Einstein-Cartan-Sciama-Kibble (ECSK) theory. The essential features of the theory are presented in Chapter 7.

It should be emphasised that the Einstein-Cartan theory is only one of many gravitational theories involving torsion that are being considered at present. However it is the most aesthetically pleasing and takes spin into account more naturally.

I have attempted in this thesis not only to present a spin-coefficient formalism for space-times with torsion, but also to show how such a formalism can be used to obtain general results and exact solutions in the context of the Einstein-Cartan theory. The physical sources considered are the Neutrino field and classical spin fluids. Exact solutions are presented in Chapters 8 and 9. Chapter 10 concludes the thesis with suggestions for further work.

Some of the material included in the first half of this thesis has been published in a joint paper with Dr J B Griffiths (1980a)*. Another joint paper containing the material of Chapter 9 has been submitted for publication (Griffiths and Jogia 1981). Some parts of this thesis have been summarised in an abstract published jointly with Dr J B Griffiths (1980b).

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* It should be noticed that the definitions of the components of the curvature tensor in spinor form are, at two points, inconsistent with their use in the remainder of this publication. This may be corrected by replacing, in equation (4.24) of the paper, the following:

$$\chi = \frac{i}{4}(\Omega - \Omega), \qquad \Psi_2 - \frac{2}{3}i\chi = \Psi_{ABCD}\ o^A o^B \iota^C \iota^D$$

However it has been found to be more appropriate to redefine these components in this thesis so that the original definitions given in the spinor section of the publication are retained (with χ being relabelled Σ).

CHAPTER 2

RIEMANN-CARTAN GEOMETRY

Torsion theories of gravity use a Riemann-Cartan geometry for their description of space-time. It is the purpose of this chapter to present the essential features of this geometry. Detailed descriptions are given in Schroedinger (1963) and Lovelock and Rund (1975). Riemann-Cartan geometry is also summarised with reference to the Einstein-Cartan theory by Hehl (1973, 1974) and Kuchowicz (1975).

2.1 Affine Geometry

In order to decide upon an invariant method of varying vectors from point to point, a concept of parallel displacement must be introduced. Namely, a vector A^{α}, parallely displaced from the points x^{μ} to $x^{\mu} + dx^{\mu}$ changes according to the affinity postulate

$$\delta A^{\alpha} = - \Gamma_{\mu\nu}{}^{\alpha} A^{\nu} dx^{\mu}$$

in which δA^{α} is assumed to be bilinear in A^{ν} and dx^{μ}. The affine connection $\Gamma_{\mu\nu}{}^{\lambda}$ is assumed to be generally asymmetric in its lower indices and therefore the order of these indices is important when stating the affinity postulate. In the terminology of Schrödinger (1963), a manifold equipped with a $\Gamma_{\mu\nu}{}^{\lambda}$ is called a linearly connected space.

The antisymmetric part of the connection transforms as a tensor and is called Cartan's torsion tensor:

$$S_{\mu\nu}{}^{\lambda} := \Gamma_{[\mu\nu]}{}^{\lambda} \tag{2.1}$$

This tensor is a purely affine quantity and has in general 24 independent components. It vanishes in standard Riemannian geometry.

The covariant derivative with respect to $\Gamma_{\mu\nu}{}^{\lambda}$ of a contravariant vector field A^{α} is defined by

$$\nabla_{\mu} A^{\alpha} = \partial_{\mu} A^{\alpha} + \Gamma_{\mu\nu}{}^{\alpha} A^{\nu}. \tag{2.2}$$

Covariant differentiation of an arbitrary rank tensor field can be defined in the usual manner.

It is possible at this stage to introduce repeated covariant differentiation. This yields the Ricci identity

$$(\nabla_{\mu}\nabla_{\nu}-\nabla_{\nu}\nabla_{\mu})A^{\lambda} = - R_{\mu\nu\kappa}{}^{\lambda} A^{\kappa} - 2 S_{\mu\nu}{}^{\kappa} \nabla_{\kappa}A^{\lambda} \tag{2.3}$$

where the Riemann-Christoffel curvature tensor is defined by

$$R_{\mu\nu\kappa}{}^{\lambda} = -\partial_{\mu}\Gamma_{\nu\kappa}{}^{\lambda}+\partial_{\nu}\Gamma_{\mu\kappa}{}^{\lambda} - \Gamma_{\mu\alpha}{}^{\lambda}\Gamma_{\nu\kappa}{}^{\alpha} + \Gamma_{\nu\alpha}{}^{\lambda}\Gamma_{\mu\kappa}{}^{\alpha} \tag{2.4}$$

2.2 Metric Geometry

For a reasonable space-time, the theory of special relativity must be valid locally. This is accomplished, just as in V_4, by introducing a symmetric metric $g_{\mu\nu}$ which is locally taken to be the Lorentz metric of flat space-time. In addition, the "metric postulate"

$$\nabla_\lambda \, g_{\mu\nu} = 0 \tag{2.5}$$

is imposed so that units of length and time are preserved under parallel displacement. A manifold equipped with this metric and connection is called a Riemann-Cartan space-time (U_4). Equation (2.5) is seen to be a constraining equation with respect to the connection. Solving it we obtain

$$\Gamma_{\mu\nu}{}^{\lambda} = \left\{ {\lambda \atop \mu\nu} \right\} - K_{\mu\nu}{}^{\lambda} \tag{2.6}$$

where $\left\{ {\lambda \atop \mu\nu} \right\}$ is the Christoffel symbol of the 2nd kind and

$$K_{\mu\nu}{}^{\lambda} := - S_{\mu\nu}{}^{\lambda} + S_{\nu}{}^{\lambda}{}_{\mu} - S^{\lambda}{}_{\mu\nu} \tag{2.7}$$

is the contortion tensor. $K_{\mu\nu\kappa} = g_{\kappa\lambda} \, K_{\mu\nu}{}^{\lambda}$ has the symmetry property

$$K_{\mu(\nu\lambda)} = 0 \tag{2.8}$$

indicating that it has 24 independent components. The torsion tensor is given in terms of the contortion tensor by the useful equation

$$S_{\mu\nu}{}^{\lambda} = - K_{[\mu\nu]}{}^{\lambda} \qquad (2.9)$$

Although one is at liberty to work with either the torsion or the contortion tensors, preference is given to the contortion tensor because it describes, more directly, (non-Riemannian) rotational degrees of freedom of the space-time (Hehl 1973, 1974). This choice is particularly appropriate for the spin-coefficient formalism developed later.

It is convenient at this stage to define the preferred paths of particles in a U_4. One must distinguish between two classes of curves, both of which reduce to the "geodesics" of Riemannian geometry. Namely

Extremal curves (shortest or longest lines) are those curves which are of extremal length with respect to the metric of the manifold. Since the length between two given points depends only on the metric, the extremal equation is derivable from a variation of the interval:

$$\delta\int ds = \delta\int \sqrt{\left(g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds}\right)}\, ds = 0$$

The definition is independent of the torsion field. We obtain

$$\frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \mu\nu} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \qquad (2.10)$$

Autoparallel curves (straightest lines) are those curves over which a vector U^α is transported parallel to itself with respect to the total connection $\Gamma_{\mu\nu}{}^\lambda$ of the manifold. The defining equation is

$$U^\mu \nabla_\mu U^\alpha = 0 \qquad (2.11)$$

or, upon choosing a suitable affine parameter s,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma_{\mu\nu}{}^\alpha \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \qquad (2.12)$$

Theorem 2.1 Autoparallels and extremals coincide if, and only if, the contortion tensor is totally antisymmetric, i.e. $K_{\mu\nu\lambda} = K_{[\mu\nu\lambda]}$.

This theorem follows simply by noticing that only the symmetric part of the connection enters (2.12). Note that extremal curves have the same form as geodesics defined in the associated V_4, over which a vector is transported parallel to itself with respect to the Christoffel symbols.

Schouten (1954) and most other mathematicians use the term "geodesic" instead of autoparallel. In fact the path of a particle need not be extremal or autoparallel. In such cases the curve must be calculated from the field equations or conservation laws of some physical theory.

2.3 Properties of the Curvature Tensor

In terms of the contortion tensor, the Ricci identity (2.3) can be written

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\lambda = - R_{\mu\nu\kappa}{}^\lambda A^\kappa + (K_{\mu\nu}{}^\kappa - K_{\nu\mu}{}^\kappa) \nabla_\kappa A^\lambda \tag{2.13}$$

and the curvature tensor (2.4) can be seen to have the symmetry property

$$R_{(\mu\nu)\kappa}{}^\lambda = 0 \tag{2.14}$$

Upon introducing the metric, it also has the symmetry property

$$R_{\mu\nu(\kappa\lambda)} = 0 \tag{2.15}$$

(2.14) and (2.15) are the only symmetry properties of the curvature tensor which therefore has 36 independent components. The other identity of interest is the cyclic identity of V_4. The generalisation of this to U_4 yields the following:

$$\nabla_{[\lambda} K_{\mu\nu]}{}^\kappa = \tfrac{1}{2} R_{[\lambda\mu\nu]}{}^\kappa - K_{[\lambda\mu}{}^\alpha K_{\nu]\alpha}{}^\kappa + K_{\alpha[\nu}{}^\kappa K_{\lambda\mu]}{}^\alpha \tag{2.16}$$

These equations are integrability conditions on the contortion tensor and will be regarded here as the Bianchi identities for the torsion.

The integrability conditions on the curvature are given by the Bianchi identities for the curvature which take the form

$$\nabla_{[\lambda}R_{\mu\nu]\alpha}{}^{\beta} = -2\,K_{[\mu\nu}{}^{\kappa}R_{\lambda]\kappa\alpha}{}^{\beta} \qquad (2.17)$$

It is often useful to identify the terms that are responsible for the deviation of a U_4 quantity from its value in the associated V_4. Therefore a notation will now be introduced by which a degree sign is used to denote the V_4 value of a quantity. This is the value which depends only on the metric structure in U_4 and is obtained by replacing the connection with the Christoffel symbol alone. For example the V_4 value of the curvature tensor is denoted $R^{o}{}_{\mu\nu\kappa}{}^{\lambda}$ and the deviation of the U_4 curvature tensor from $R^{o}{}_{\mu\nu\kappa}{}^{\lambda}$ is given by the expansion

$$R_{\mu\nu\kappa}{}^{\lambda} = R^{o}{}_{\mu\nu\kappa}{}^{\lambda} + 2\nabla_{[\mu}K_{\nu]\kappa}{}^{\lambda} - 2K_{[\mu\nu]}{}^{\alpha}K_{\alpha\kappa}{}^{\lambda} - K_{\mu\kappa}{}^{\alpha}K_{\nu\alpha}{}^{\lambda} + K_{\nu\kappa}{}^{\alpha}K_{\mu\alpha}{}^{\lambda} \qquad (2.18)$$

CHAPTER 3

THE TETRAD APPROACH

Following the notation of Newman and Penrose (1962), we adopt a set of four null basis vectors (ℓ^μ, n^μ, m^μ, $\bar{m}^\mu$), defined at each point in space-time, of which ℓ^μ and n^μ are real and future pointing. The null tetrad is denoted by

$$e_m{}^\mu = (\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu) \qquad m = 1,2,3,4.$$

and satisfies the orthogonality conditions

$$\ell^\mu n_\mu = - m^\mu \bar{m}_\mu = 1, \ \ell^\mu m_\mu = n^\mu m_\mu = 0 \tag{3.1}$$

Components of the metric tensor in the tetrad frame are therefore given by:

$$\eta_{mn} = e_{m\mu}\, e_{n\nu}\, g^{\mu\nu} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \tag{3.2}$$

where the matrix $\eta_{mn} = \eta^{mn}$ is used to raise and lower tetrad indices. The metric is consistent with the contractions (3.1) if, and only if, it satisfies the "completeness relation"

$$g_{\mu\nu} = e_{m\mu} e_{n\nu} \eta^{mn} = \ell_\mu n_\mu + n_\mu \ell_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \tag{3.3}$$

Components of a tensor in the tetrad frame will now be referred to as its tetrad components. For example, the tetrad components of the contortion tensor are given by

$$K_{\ell mn} = K_{\lambda\mu\nu} \, e_{\ell}{}^{\lambda} \, e_{m}{}^{\mu} \, e_{n}{}^{\nu}$$

for which

$$K_{\ell(mn)} = 0$$

Tensor indices may be regained by inverting this operation.

3.1 <u>The Spin Coefficients and Contortion Components</u>

Complex Ricci rotation coefficients are defined by:

$$\gamma_{\ell mn} = e_{n}{}^{\nu} \, e_{m}{}^{\mu} \, \nabla_{\nu} \, e_{\ell\mu} \tag{3.4}$$

for which

$$\gamma_{(\ell m)n} = 0 \tag{3.5}$$

The latter identity is easily derived by covariantly differentiating equation (3.2). Also, by expanding (3.4), with the aid of (2.6), we obtain

$$\begin{aligned} \gamma_{\ell mn} &= e_{n}{}^{\nu} \, e_{m}{}^{\mu} \, [\delta_{\mu}{}^{\alpha} \, \partial_{\nu} - \left\{ {\alpha \atop \nu\mu} \right\} + K_{\nu\mu}{}^{\alpha}] \, e_{\ell\alpha} \\ &= \gamma^{o}{}_{\ell mn} + K_{nm\ell} \end{aligned} \tag{3.6}$$

Equation (3.6) shows that the tetrad components of the contortion tensor and the rotation coefficients are related. More precisely, the contortion components are the quantities by which the rotation coefficients differ from their values in the associated V_4. The unfortunate reverse order in the subscripts of these components is a result of maintaining the notations of Hehl and of Newman and Penrose. However, this does not present itself as a major problem in developing the formalism.

In the N-P formalism, the twelve spin coefficients are defined as linear combinations of $\gamma_{\ell mn}$. These definitions are carried over here identically. In addition it is also convenient to take the same combinations of the $K_{nm\ell}$ and use the same symbols for these combinations but with a subscript "1" to denote contortion components. Accordingly, we make the definitions.

$$\kappa = \ell^{\nu} m^{\mu} \nabla_{\nu} \ell_{\mu} \qquad \kappa_1 = K_{\lambda\mu\nu} \ell^{\lambda} m^{\mu} \ell^{\nu}$$

$$\rho = \bar{m}^{\nu} m^{\mu} \nabla_{\nu} \ell_{\mu} \qquad \rho_1 = K_{\lambda\mu\nu} \bar{m}^{\lambda} m^{\mu} \ell^{\nu}$$

$$\sigma = m^{\nu} m^{\mu} \nabla_{\nu} \ell_{\mu} \qquad \sigma_1 = K_{\lambda\mu\nu} m^{\lambda} m^{\mu} \ell^{\nu}$$

$$\tau = n^{\nu} m^{\mu} \nabla_{\nu} \ell_{\mu} \qquad \tau_1 = K_{\lambda\mu\nu} n^{\lambda} m^{\mu} \ell^{\nu}$$

$$\varepsilon = \tfrac{1}{2} \ell^{\nu} (n^{\mu} \nabla_{\nu} \ell_{\mu} - \bar{m}^{\mu} \nabla_{\nu} m_{\mu}) \qquad \varepsilon_1 = \tfrac{1}{2} K_{\lambda\mu\nu} \ell^{\lambda} (n^{\mu} \ell^{\nu} - \bar{m}^{\mu} m^{\nu})$$

$$\alpha = \tfrac{1}{2} \bar{m}^{\nu} (n^{\mu} \nabla_{\nu} \ell_{\mu} - \bar{m}^{\mu} \nabla_{\nu} m_{\mu}) \qquad \alpha_1 = \tfrac{1}{2} K_{\lambda\mu\nu} \bar{m}^{\lambda} (n^{\mu} \ell^{\nu} - \bar{m}^{\mu} m^{\nu})$$

$$\beta = \tfrac{1}{2} m^{\nu} (n^{\mu} \nabla_{\nu} \ell_{\mu} - \bar{m}^{\mu} \nabla_{\nu} m_{\mu}) \qquad \beta_1 = \tfrac{1}{2} K_{\lambda\mu\nu} m^{\lambda} (n^{\mu} \ell^{\nu} - \bar{m}^{\mu} m^{\nu})$$

$$\gamma = \tfrac{1}{2}n^{\nu}(n^{\mu}\nabla_{\nu}\ell_{\mu} - \overline{m}^{\mu}\nabla_{\nu}m_{\mu}) \qquad \gamma_1 = \tfrac{1}{2}K_{\lambda\mu\nu}n^{\lambda}(n^{\mu}\ell^{\nu} - \overline{m}^{\mu}m^{\nu})$$

$$\pi = -\ell^{\nu}\overline{m}^{\mu}\nabla_{\nu}n_{\mu} \qquad \pi_1 = -K_{\lambda\mu\nu}\ell^{\lambda}\overline{m}^{\mu}n^{\nu}$$

$$\lambda = -\overline{m}^{\nu}\overline{m}^{\mu}\nabla_{\nu}n_{\mu} \qquad \lambda_1 = -K_{\lambda\mu\nu}\overline{m}^{\lambda}\overline{m}^{\mu}n^{\nu}$$

$$\mu = -m^{\nu}\overline{m}^{\mu}\nabla_{\nu}n_{\mu} \qquad \mu_1 = -K_{\lambda\mu\nu}m^{\lambda}\overline{m}^{\mu}n^{\nu}$$

$$\nu = -n^{\nu}\overline{m}^{\mu}\nabla_{\nu}n_{\mu} \qquad \nu_1 = -K_{\lambda\mu\nu}n^{\lambda}\overline{m}^{\mu}n^{\nu} \qquad (3.7)$$

The spin coefficients may now be expressed in terms of their values in a V_4 and the contortion components, i.e.

$$\kappa = \kappa^{0} + \kappa_1, \qquad \rho = \rho^{0} + \rho_1 \text{ etc.} \qquad (3.8)$$

The definitions (3.7) can be given in the alternative forms:

$$\nabla_{\mu}\ell_{\nu} = (\gamma+\overline{\gamma})\ell_{\mu}\ell_{\nu} + (\varepsilon+\overline{\varepsilon})n_{\mu}\ell_{\nu} - (\alpha+\overline{\beta})m_{\mu}\ell_{\nu} - (\overline{\alpha}+\beta)\overline{m}_{\mu}\ell_{\nu} \qquad (3.9)(a)$$

$$- \overline{\tau}\,\ell_{\mu}m_{\nu} - \overline{\kappa}\,n_{\mu}m_{\nu} + \overline{\sigma}\,m_{\mu}m_{\nu} + \overline{\rho}\,\overline{m}_{\mu}m_{\nu}$$

$$- \tau\,\ell_{\mu}\overline{m}_{\nu} - \kappa\,n_{\mu}\overline{m}_{\nu} + \rho\,m_{\mu}\overline{m}_{\nu} + \sigma\,\overline{m}_{\mu}\overline{m}_{\nu}$$

$$\nabla_{\mu}n_{\nu} = -(\gamma+\overline{\gamma})\ell_{\mu}n_{\nu} - (\varepsilon+\overline{\varepsilon})n_{\mu}n_{\nu} + (\alpha+\overline{\beta})m_{\mu}n_{\nu} + (\overline{\alpha}+\beta)\overline{m}_{\mu}n_{\nu} \qquad (b)$$

$$+ \nu\,\ell_{\mu}m_{\nu} + \pi\,n_{\mu}m_{\nu} - \lambda\,m_{\mu}m_{\nu} - \mu\,\overline{m}_{\mu}m_{\nu}$$

$$+ \overline{\nu}\,\ell_{\mu}\overline{m}_{\nu} + \overline{\pi}\,n_{\mu}\overline{m}_{\nu} - \overline{\mu}\,m_{\mu}\overline{m}_{\nu} - \overline{\lambda}\,\overline{m}_{\mu}\overline{m}_{\nu}$$

$$\nabla_\mu m_\nu = \bar{\nu}\, \ell_\mu \ell_\nu + \bar{\pi}\, n_\mu \ell_\nu - \bar{\mu}\, m_\mu \ell_\nu - \bar{\lambda}\, \bar{m}_\mu \ell_\nu \tag{c}$$

$$- \tau\, \ell_\mu n_\nu - \kappa\, n_\mu n_\nu + \rho\, m_\mu n_\nu + \sigma\, \bar{m}_\mu n_\nu$$

$$+ (\gamma-\bar{\gamma})\ell_\mu m_\nu + (\varepsilon-\bar{\varepsilon})n_\mu m_\nu - (\alpha-\bar{\beta})m_\mu m_\nu - (\beta-\bar{\alpha})\bar{m}_\mu m_\nu$$

$$K_{\lambda\mu\nu} = (\tau_1 \ell_\lambda + \kappa_1 n_\lambda - \rho_1 m_\lambda - \sigma_1 \bar{m}_\lambda)(n_\mu \bar{m}_\nu - \bar{m}_\mu n_\nu) \tag{d}$$

$$+ (\gamma_1 \ell_\lambda + \varepsilon_1 n_\lambda - \alpha_1 m_\lambda - \beta_1 \bar{m}_\lambda)(\ell_\mu n_\nu - n_\mu \ell_\nu - m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu)$$

$$+ (\nu_1 \ell_\lambda + \pi_1 n_\lambda - \lambda_1 m_\lambda - \mu_1 \bar{m}_\lambda)(m_\mu \ell_\nu - \ell_\mu m_\nu) + \text{C.C.}$$

In general terms, the spin coefficients describe how the null tetrad varies from point to point. Some of them relate more directly to the geometric properties of the null congruences to which ℓ^μ and n^μ are tangent. (A null congruence in a region of space-time is a 3-parameter family of null curves such that exactly one curve passes through each point of the region (Pirani, 1964)). In particular, ℓ_μ is tangent to a null autoparallel congruence if, and only if, $\kappa = 0$. In addition, the null autoparallel congruence is affinely parameterised if, and only if, $\varepsilon+\bar{\varepsilon} = 0$. Similarly, ℓ_μ is tangent to a null extremal congruence if, and only if, $\kappa^0 = 0$ and this is affinely parameterised if, and only if, $\varepsilon^0+\bar{\varepsilon}^0 = 0$.

If ℓ^μ is tangent to an affinely parameterised null autoparallel congruence then the "optical scalars" (Sachs, 1961), associated with such a congruence are defined in terms of the spin coefficients by

$$\underline{\text{expansion}} \quad \theta = -\tfrac{1}{2}(\rho+\bar{\rho}) = \theta^{0} - \tfrac{1}{2}(\rho_1+\bar{\rho}_1) \tag{3.10(a)}$$

$$\underline{\text{shear}} \quad |\sigma| = \sigma\bar{\sigma} = [|\sigma|^{02} + \sigma^{0}\bar{\sigma}_1 + \bar{\sigma}^{0}\sigma_1 + \sigma_1\bar{\sigma}_1]^{\frac{1}{2}} \tag{b}$$

$$\underline{\text{twist}} \quad \omega = \frac{i}{2}(\rho-\bar{\rho}) = \omega^{0} + \frac{i}{2}(\rho_1 - \bar{\rho}_1) \tag{c}$$

where their V_4 values are defined with respect to an affinely parameterised extremal null congruence.

If ℓ^{μ} is tangent to both affinely parameterised autoparallels and affinely parameterised extremals then the optical scalars associated with the former need not be equal to their V_4 values defined with respect to the latter. The deviations of the scalars associated with autoparallels from their V_4 values are given in equations (3.10).

For a scalar field ϕ, it is easily shown that

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\phi = (K_{\mu\nu}{}^{\alpha} - K_{\nu\mu}{}^{\alpha})\nabla_{\alpha}\phi \tag{3.11}$$

Therefore ℓ^{μ} is proportional to the gradient of a scalar field ϕ if, and only if, $\kappa^{0} = 0$ and $\bar{\rho}^{0} = \rho^{0}$; if ℓ^{μ} is in fact equal to the gradient of ϕ then, in addition, $\varepsilon^{0}+\bar{\varepsilon}^{0} = 0$ and $\tau^{0} = \bar{\alpha}^{0}+\beta^{0}$.

Finally, the spin coefficients $-\nu$, $-\lambda$, $-\mu$ describe geometric properties for n^{μ} analogously to those determined by κ, σ, ρ for ℓ^{μ}.

3.2 The Curvature Tensor

The curvature tensor in U_4 has 36 independent components. In order to give these components distinctive labels with respect to the tetrad basis, it is necessary to suitably decompose the curvature tensor. We begin with the trace free part of the curvature tensor. This is given by

$$C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - R_{\lambda[\mu}\, g_{\nu]\kappa} + R_{\kappa[\mu}\, g_{\nu]\lambda} + \frac{R}{3}\, g_{\nu[\kappa}\, g_{\lambda]\mu} \tag{3.12}$$

where the Ricci tensor and curvature scalar are defined by

$$R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha}\,, \qquad R = R_{\alpha}{}^{\alpha} \tag{3.13}$$

In U_4 the Ricci tensor is not necessarily symmetric and so has 16 independent components. The remaining 20 independent components of the curvature tensor are given by $C_{\kappa\lambda\mu\nu}$ which may be regarded as a generalisation of the Weyl tensor defined in V_4. However, an alternative and more convenient generalisation of the V_4 Weyl tensor is given by

$$A_{\kappa\lambda\mu\nu} = \tfrac{1}{2}\left(C_{\kappa\lambda\mu\nu} + C_{\mu\nu\kappa\lambda}\right) - C_{[\kappa\lambda\mu\nu]} \tag{3.14}$$

This tensor has the symmetry properties

$$A_{(\kappa\lambda)(\mu\nu)} = A_{(\mu\nu)(\kappa\lambda)} = A_{[\kappa\lambda\mu\nu]} = 0 \tag{3.15}$$

which indicate that it has 10 independent components. It will be referred to here as the <u>generalised Weyl tensor</u>. The remaining 10 independent components of $C_{\kappa\lambda\mu\nu}$ are given by the tensor

$$B_{\kappa\lambda\mu\nu} = \tfrac{1}{2}\,(C_{\kappa\lambda\mu\nu} - C_{\mu\nu\kappa\lambda}), \tag{3.16}$$

which has 9 independent components, and the scalar

$$D = -\,\tfrac{1}{2}\,\varepsilon^{\kappa\lambda\mu\nu}\,R_{\kappa\lambda\mu\nu} \tag{3.17}$$

The decomposition of the curvature tensor in U_4 can therefore be given by

$$\begin{aligned} R_{\kappa\lambda\mu\nu} = {} & A_{\kappa\lambda\mu\nu} + \tfrac{1}{2}\,(R_{\lambda\mu}g_{\nu\kappa} - R_{\lambda\nu}g_{\mu\kappa} - R_{\kappa\mu}g_{\nu\lambda} + R_{\kappa\nu}g_{\mu\lambda}) \\ & - \frac{R}{6}\,(g_{\nu\kappa}g_{\lambda\mu} - g_{\nu\lambda}g_{\kappa\mu}) \\ & + B_{\kappa\lambda\mu\nu} + \frac{D}{12}\,\varepsilon_{\kappa\lambda\mu\nu} \end{aligned} \tag{3.18}$$

The tensors $B_{\kappa\lambda\mu\nu}$, $R_{[\mu\nu]}$ and the scalar D, vanish in the limit when torsion vanishes.

The authors Gambini and Herrera (1980) have recently obtained an alternative decomposition. They have found it convenient to express the components of the curvature described by $B_{\kappa\lambda\mu\nu}$, $R_{[\mu\nu]}$ and D, in terms of the tensor

$$D_{\mu\nu} = \tfrac{1}{2}\, \varepsilon_{\mu}{}^{\alpha\beta\gamma}\, R_{\alpha\beta\gamma\nu} \tag{3.19}$$

With this definition, it is easily shown that

$$R_{[\mu\nu]} = -\tfrac{1}{2}\, \varepsilon_{\mu\nu}{}^{\alpha\beta}\, D_{\alpha\beta} \tag{3.20}$$

The decomposition in terms of $D_{\mu\nu}$ is given by

$$R_{\kappa\lambda\mu\nu} = A_{\kappa\lambda\mu\nu} + \tfrac{1}{2}[R_{(\lambda\mu)}g_{\nu\kappa} - R_{(\lambda\nu)}g_{\mu\kappa} - R_{(\kappa\mu)}g_{\nu\lambda} + R_{(\kappa\nu)}g_{\mu\lambda}] \tag{3.21}$$

$$- \frac{R}{6}(g_{\nu\kappa}g_{\lambda\mu} - g_{\nu\lambda}g_{\kappa\mu}) + \frac{D}{12}\varepsilon_{\kappa\lambda\mu\nu}$$

$$- \frac{1}{4}[\varepsilon_{\kappa\mu\lambda}{}^{\alpha} D_{\alpha\nu} + \varepsilon_{\mu\lambda\nu}{}^{\alpha} D_{\alpha\kappa} + \varepsilon_{\lambda\nu\kappa}{}^{\alpha} D_{\alpha\mu} + \varepsilon_{\nu\kappa\mu}{}^{\alpha} D_{\alpha\lambda}]$$

where

$$D = D_{\alpha}{}^{\alpha}. \tag{3.22}$$

The components of the Ricci tensor are now expressed in terms of the familiar 9 components of a hermitian 3x3 matrix Φ_{AB}, the real parameter Λ and 3 new complex components Φ_A $(A,B = 0,1,2)$. These are defined by

$$\Phi_{00} = -\tfrac{1}{2} R_{\mu\nu}\ell^{\mu}\ell^{\nu}$$

$$\Phi_{01} = -\tfrac{1}{2} R_{(\mu\nu)}\ell^{\mu}m^{\nu}$$

$$\Phi_{02} = -\tfrac{1}{2} R_{\mu\nu}m^{\mu}m^{\nu}$$

$$\Phi_{11} = -\tfrac{1}{4} R_{(\mu\nu)} (\ell^\mu n^\nu + m^\mu \bar{m}^\nu)$$

$$\Phi_{12} = -\tfrac{1}{2} R_{(\mu\nu)} n^\mu m^\nu$$

$$\Phi_{22} = -\tfrac{1}{2} R_{\mu\nu} n^\mu n^\nu$$

$$\Lambda = \frac{1}{24} R$$

$$\Phi_0 = -\tfrac{1}{2} R_{[\mu\nu]} \ell^\mu m^\nu = -\frac{i}{2} D_{[\mu\nu]} \ell^\mu m^\nu$$

$$\Phi_1 = -\tfrac{1}{2} R_{[\mu\nu]} (\ell^\mu n^\nu - m^\mu \bar{m}^\nu) = -\frac{i}{2} D_{[\mu\nu]} (\ell^\mu n^\nu - m^\mu \bar{m}^\nu)$$

$$\Phi_2 = -\tfrac{1}{2} R_{[\mu\nu]} \bar{m}^\mu n^\nu = -\frac{i}{2} D_{[\mu\nu]} \bar{m}^\mu n^\nu \qquad (3.23)$$

The generalised Weyl tensor $A_{\kappa\lambda\mu\nu}$ is expressed in terms of the familiar 5 components

$$\Psi_0 = -A_{\kappa\lambda\mu\nu} \ell^\kappa m^\lambda \ell^\mu m^\nu$$

$$\Psi_1 = -A_{\kappa\lambda\mu\nu} \ell^\kappa n^\lambda \ell^\mu m^\nu$$

$$\Psi_2 = -\tfrac{1}{2} A_{\kappa\lambda\mu\nu} (\ell^\kappa n^\lambda \ell^\mu n^\nu - \ell^\kappa n^\lambda m^\mu \bar{m}^\nu) \qquad (3.24)$$

$$\Psi_3 = -A_{\kappa\lambda\mu\nu} \ell^\kappa n^\lambda \bar{m}^\mu n^\nu$$

$$\Psi_4 = -A_{\kappa\lambda\mu\nu} \bar{m}^\kappa n^\lambda \bar{m}^\mu n^\nu$$

It is convenient to express the tensor $B_{\kappa\lambda\mu\nu}$ in terms of the 9 components of a hermitian matrix Θ_{AB} (A, B = 0,1,2) as follows

$$\Theta_{00} = -i\, B_{\kappa\lambda\mu\nu} \ell^\kappa m^\lambda \ell^\mu \bar{m}^\nu = +\tfrac{1}{2} D_{\mu\nu} \ell^\mu \ell^\nu$$

$$\Theta_{01} = -\frac{i}{2} B_{\kappa\lambda\mu\nu} \ell^\kappa m^\lambda (\ell^\mu n^\nu - \bar{m}^\mu m^\nu) = +\tfrac{1}{2} D_{(\mu\nu)} \ell^\mu m^\nu$$

$$\Theta_{02} = i\, B_{\kappa\lambda\mu\nu}\, \ell^{\kappa} m^{\lambda} n^{\mu} m^{\nu} = + \tfrac{1}{2}\, D_{\mu\nu}\, m^{\mu} m^{\nu}$$

$$\Theta_{11} = \frac{i}{4}\, B_{\kappa\lambda\mu\nu} (\ell^{\kappa} n^{\lambda} - \bar{m}^{\kappa} m^{\lambda})(\ell^{\mu} n^{\nu} + \bar{m}^{\mu} m^{\nu}) = + \tfrac{1}{4}\, D_{(\mu\nu)} (\ell^{\mu} n^{\nu} + m^{\mu} \bar{m}^{\nu})$$

$$\Theta_{12} = - \frac{i}{2}\, B_{\kappa\lambda\mu\nu}\, n^{\kappa} m^{\lambda}\, (\ell^{\mu} n^{\nu} + \bar{m}^{\mu} m^{\nu}) = + \tfrac{1}{2}\, D_{(\mu\nu)}\, n^{\mu} m^{\nu}$$

$$\Theta_{22} = - i\, B_{\kappa\lambda\mu\nu}\, n^{\kappa} \bar{m}^{\lambda} n^{\mu} m^{\nu} = + \tfrac{1}{2}\, D_{\mu\nu}\, n^{\mu} n^{\nu} \qquad (3.25)$$

It should be noted that these components, when expressed in terms of $D_{\mu\nu}$, are analogous to the components Φ_{AB} of the trace free Ricci tensor. This analogy will be illustrated more clearly in Chapter 4. The remaining scalar component D is given the label

$$\Sigma = + \frac{1}{24} D \qquad (3.26)$$

The tensors $A_{\kappa\lambda\mu\nu}$ and $B_{\kappa\lambda\mu\nu}$ are given in terms of the components defined in (3.24) and (3.25) by

$$\begin{aligned} A_{\kappa\lambda\mu\nu} = & - \Psi_0\, U_{\kappa\lambda} U_{\mu\nu} + 2\, \Psi_1\, (U_{\kappa\lambda} M_{\mu\nu} + M_{\kappa\lambda} U_{\mu\nu}) \\ & - \Psi_2\, (U_{\kappa\lambda} U_{\mu\nu} + 4\, M_{\kappa\lambda} M_{\mu\nu} + V_{\kappa\lambda} V_{\mu\nu}) \\ & + 2\, \Psi_3\, (V_{\kappa\lambda} M_{\mu\nu} + M_{\kappa\lambda} V_{\mu\nu}) - \Psi_4\, V_{\kappa\lambda}\, V_{\mu\nu} + \text{C.C.} \end{aligned} \qquad (3.27)$$

$$B_{\kappa\lambda\mu\nu} = -\, i\; \Theta_{00}\overline{U}_{\kappa\lambda}U_{\mu\nu} + 2i\;\Theta_{01}\;\overline{M}_{\kappa\lambda}U_{\mu\nu} - i\;\Theta_{02}\;\overline{V}_{\kappa\lambda}U_{\mu\nu}$$

$$+ 2i\;\Theta_{10}\;\overline{U}_{\kappa\lambda}M_{\mu\nu} - 4i\;\Theta_{11}\;\overline{M}_{\kappa\lambda}M_{\mu\nu} + 2i\;\Theta_{12}\;\overline{V}_{\kappa\lambda}M_{\mu\nu}$$

$$- i\;\Theta_{20}\;\overline{U}_{\kappa\lambda}V_{\mu\nu} + 2i\;\Theta_{21}\;\overline{M}_{\kappa\lambda}V_{\mu\nu} - i\;\Theta_{22}\;\overline{V}_{\kappa\lambda}V_{\mu\nu} \qquad (3.28)$$

where 3 complex self dual tensors have been defined by

$$U_{\mu\nu} = 2\,\overline{m}_{[\mu}n_{\nu]},\; V_{\mu\nu} = 2\,\ell_{[\mu}m_{\nu]},\;\; M_{\mu\nu} = \ell_{[\mu}n_{\nu]} - m_{[\mu}\overline{m}_{\nu]}.$$

The trace free Ricci tensor is given in terms of the tetrad basis by

$$R_{\mu\nu} - \frac{R}{4}\,g_{\mu\nu} = -\Phi_{22}\ell_{\mu}\ell_{\nu} + 4\Phi_{12}\ell_{(\mu}\overline{m}_{\nu)} - 2\Phi_{11}(\ell_{(\mu}n_{\nu)} + m_{(\mu}\overline{m}_{\nu)})$$

$$- 2\Phi_{02}\,\overline{m}_{\mu}\overline{m}_{\nu} + 4\Phi_{01}\,n_{(\mu}\overline{m}_{\nu)} - \Phi_{00}\,n_{\mu}n_{\nu}$$

$$- 4\Phi_{2}\,\ell_{[\mu}m_{\nu]} + 2\Phi_{1}\,(\ell_{[\mu}n_{\nu]} - m_{[\mu}\overline{m}_{\nu]})$$

$$- 4\Phi_{0}\,\overline{m}_{[\mu}n_{\nu]} + \text{C.C.} \qquad (3.29)$$

Finally, the tensor $D_{\mu\nu}$ is given in terms of the tetrad basis by

$$D_{\mu\nu} = \Theta_{22}\ell_\mu\ell_\nu - 4\Theta_{12}\,\ell_{(\mu}\bar{m}_{\nu)} + 2\Theta_{11}(\ell_{(\mu}n_{\nu)} + \bar{m}_{(\mu}m_{\nu)})$$

$$+ 2\Theta_{02}\,\bar{m}_\mu\bar{m}_\nu - 4\Theta_{01}\,\bar{m}_{(\mu}n_{\nu)} + \Theta_{00}\,n_\mu n_\nu$$

$$+ 4i\,\Phi_2\,\ell_{[\mu}m_{\nu]} - 2i\,\Phi_1\,(\ell_{[\mu}n_{\nu]} + \bar{m}_{[\mu}m_{\nu]}) + 4i\,\Phi_0\,\bar{m}_{[\mu}n_{\nu]}$$

$$+ 3\,\Sigma\,g_{\mu\nu} + \text{C.C.} \qquad (3.30)$$

Variations of the spin coefficients and the tetrad components of the curvature tensor under a number of tetrad transformations are given in Appendix A.

CHAPTER 4

THE SPINOR APPROACH

4.1 Spinor Algebra and Analysis

Following the notations of Pirani (1965) and Bade and Jehle (1953), 2-component spinors are introduced at each point of space-time in a tangent two dimensional complex space. To each tensor, one may then associate a spinor. The correspondence between tensors and spinors is achieved by means of a set of four 2 x 2 Hermitian matrices $\sigma^{\mu}{}_{A\dot{X}}$ $(A = 0,1$ and $\dot{X} = \dot{0},\dot{1})$ satisfying

$$\sigma_{\mu A\dot{X}}\, \sigma_{\nu}{}^{B\dot{X}} + \sigma_{\nu A\dot{X}}\, \sigma_{\mu}{}^{B\dot{X}} = g_{\mu\nu}\, \delta_A{}^B \qquad (4.1)$$

Spinor indices are raised and lowered according to the rule

$$\xi^A = \varepsilon^{AB}\, \xi_B, \quad \xi_A = \xi^B\, \varepsilon_{BA} \qquad (4.2)$$

where

$$(\varepsilon_{AB}) = (\varepsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In Minkowski space-time, $\sigma_{\mu}{}^{A\dot{X}}$ may be taken to be the set of Pauli matrices

$$\sigma_1{}^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2{}^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3{}^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_4{}^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These satisfy the useful relations

$$\sigma^{\alpha}{}_{A\dot{X}}\, \sigma_{\alpha}{}^{B\dot{Y}} = \delta_{A}{}^{B}\, \delta_{\dot{X}}{}^{\dot{Y}} \tag{4.3}$$

$$\sigma_{\mu A\dot{X}}\, \sigma_{\nu}{}^{B\dot{X}} - \sigma_{\nu A\dot{X}}\, \sigma_{\mu}{}^{B\dot{X}} = i\varepsilon_{\mu\nu\alpha\beta}\, \sigma^{\alpha}{}_{A\dot{X}}\, \sigma^{\beta B\dot{X}} \tag{4.4}$$

$$\sigma^{\alpha}{}_{A\dot{X}}\, \sigma^{\beta B\dot{X}}\, \sigma^{\gamma}{}_{B\dot{Y}} = \tfrac{1}{2}(g^{\alpha\beta}\, \sigma^{\gamma}{}_{A\dot{Y}} - g^{\alpha\gamma}\, \sigma^{\beta}{}_{A\dot{Y}} + g^{\beta\gamma}\, \sigma^{\alpha}{}_{A\dot{Y}})$$

$$- \frac{i}{2}\, \varepsilon^{\alpha\beta\gamma\delta}\, \sigma_{\delta A\dot{Y}} \tag{4.5}$$

The spinor equivalent of a tensor is defined by associating with each tensor index a pair of spinor indices as follows

$$T^{\dots A\dot{W} \dots\dots}{}_{\dots\dots B\dot{X} \dots} = \sigma_{\mu}{}^{A\dot{W}}\, \sigma^{\nu}{}_{B\dot{X}}\, T^{\dots \mu \dots\dots}{}_{\dots\dots \nu \dots}$$

and inversely

$$T^{\dots \mu \dots\dots}{}_{\dots\dots \nu \dots} = \sigma^{\mu}{}_{A\dot{W}}\, \sigma_{\nu}{}^{B\dot{X}}\, T^{\dots A\dot{W} \dots\dots}{}_{\dots\dots B\dot{X} \dots}$$

Therefore a spinor associated with a real tensor is Hermitian (where indices that are not dotted become dotted and vice versa under complex conjugation).

It is easily shown using the identity

$$\varepsilon_{A[B}\varepsilon_{CD]} = 0,$$

that any spinor $\xi_{\cdots A \cdots B \cdots}$ must satisfy

$$\xi_{\cdots A \cdots B \cdots} - \xi_{\cdots B \cdots A \cdots} = \epsilon_{AB}\, \xi_{\cdots C \cdots}{}^{C}{}_{\cdots} \qquad (4.6)$$

Equation (4.6) is essential for simplifying the spinor equivalent of tensors with pairs of anti-symmetric indices. (See Pirani (1965)). The spinor equivalent of a real tensor which is anti-symmetric in a pair of indices $T_{\cdot\cdot\mu\nu\cdot\cdot} = T_{\cdot\cdot[\mu\nu]\cdot\cdot}$ satisfies

$$T_{\cdot\cdot A\dot{X}B\dot{Y}\cdot\cdot} = \tfrac{1}{2}[T_{\cdot\cdot A\dot{X}B\dot{Y}\cdot\cdot} - T_{\cdot\cdot B\dot{X}A\dot{Y}\cdot\cdot} + T_{\cdot\cdot B\dot{X}A\dot{Y}\cdot\cdot} - T_{\cdot\cdot B\dot{Y}A\dot{X}\cdot\cdot}]$$

$$= \tfrac{1}{2}\epsilon_{AB}\, T_{\cdot\cdot C\dot{X}}{}^{C}{}_{\dot{Y}\cdot\cdot} + \tfrac{1}{2}\epsilon_{\dot{X}\dot{Y}}\, T_{\cdot\cdot B\dot{Z}A}{}^{\dot{Z}} \qquad (4.7a)$$

Now let

$$T_{\cdot\cdot BA\cdot\cdot} = \tfrac{1}{2} T_{\cdot\cdot B\dot{Z}A}{}^{\dot{Z}}{}_{\cdot\cdot} = T_{\cdot\cdot AB\cdot\cdot} \qquad (4.7b)$$

Then

$$T_{\cdot\cdot A\dot{X}B\dot{Y}\cdot\cdot} = T_{\cdot\cdot AB\cdot\cdot}\,\epsilon_{\dot{X}\dot{Y}} + \bar{T}_{\cdot\cdot\dot{X}\dot{Y}\cdot\cdot}\,\epsilon_{AB} \qquad (4.7c)$$

Note that $T_{\cdot\cdot A\dot{X}B\dot{Y}\cdot\cdot}$ has been decomposed into parts symmetric and anti-symmetric in AB and $\dot{X}\dot{Y}$ alternately.

The covariant derivative of a spinor quantity is defined in terms of a <u>spinor affine connection</u> as follows

$$\nabla_\mu \xi^A = \partial_\mu \xi^A + \Gamma_{\mu B}{}^A \xi^B$$

In U_4, the requirements $\nabla_\mu \epsilon_{AB} = 0$ and $\nabla_\mu \sigma_\nu{}^{A\dot{X}} = 0$ respectively imply that

$$\Gamma_{\mu AB} = \Gamma_{\mu BA}$$

$$\Gamma_{\mu A}{}^B = \tfrac{1}{2} \sigma_\alpha{}^{B\dot{X}} \left(\partial_\mu \sigma^\alpha{}_{A\dot{X}} + \left\{ {\alpha \atop \mu\ \beta} \right\} \sigma^\beta{}_{A\dot{X}} - K_{\mu\beta}{}^\alpha \sigma^\beta{}_{A\dot{X}}\right)$$

and therefore determine the form for $\Gamma_{\mu A}{}^B$.

4.2 The Spin Coefficients and Contortion Components

The spinor approach to the spin coefficient formalism is based on a dyad of rank 1 spinors

$$\zeta_a{}^A = (o^A,\ \iota^A) \qquad\qquad a = (0,\ 1)$$

which are normalised such that

$$o_A \iota^A = -\iota_A o^A = 1 \tag{4.8}$$

and satisfy the completeness relation

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B \tag{4.9}$$

The null tetrad in vector space can be constructed from the basis spinors by

$$\ell^{\mu} = \sigma^{\mu}{}_{A\dot{X}}\, o^{A}\, \bar{o}^{\dot{X}}\ ,\ n^{\mu} = \sigma^{\mu}{}_{A\dot{X}}\, \iota^{A}\, \bar{\iota}^{\dot{X}},\ m^{\mu} = \sigma^{\mu}{}_{A\dot{X}}\, o^{A}\, \bar{\iota}^{\dot{X}} \tag{4.10}$$

If we represent the covariant derivative in the spinor form

$$\nabla_{A\dot{X}} = \sigma^{\mu}{}_{A\dot{X}}\ \nabla_{\mu}$$

then the analog of the rotation coefficients can be given in dyad components* by

$$\Gamma_{abc\dot{x}} = \zeta_{b}{}^{A}\, \zeta_{c}{}^{C}\, \bar{\zeta}_{\dot{x}}{}^{\dot{X}}\, \nabla_{C\dot{X}}\, \zeta_{aA} \tag{4.11}$$

where

$$\Gamma_{abc\dot{x}} = \Gamma_{bac\dot{x}} \tag{4.12}$$

With the aid of equation (4.7c) (noting the symmetry (2.8)) the spinor equivalent of the contortion tensor can be written in the form

$$K_{A\dot{X}B\dot{Y}C\dot{Z}} = K_{A\dot{X}BC}\, \epsilon_{\dot{Y}\dot{Z}} + \bar{K}_{\dot{X}A\dot{Y}\dot{Z}}\, \epsilon_{BC} \tag{4.13}$$

* Care must be taken when defining dyad components of odd rank spinors. The definition $\phi^{a} = \zeta^{a}{}_{A}\, \phi^{A}$ for a rank 1 spinor is inconsistent with the above notation. I will consistently define dyad components with respect to lower case spinors as in (4.11).

where

$$K_{A\dot{X}BC} = K_{A\dot{X}CB} \tag{4.14}$$

The derivative of a spinor can be expanded about its V_4 value by

$$\nabla_{A\dot{X}}\, \xi_B = \nabla^{o}{}_{A\dot{X}}\, \xi_B + K_{A\dot{X}B}{}^{C}\, \xi_C \tag{4.15}$$

Thus

$$\Gamma_{abc\dot{x}} = \Gamma^{o}{}_{abc\dot{x}} - K_{c\dot{x}ba} \tag{4.16}$$

where $\Gamma^{o}{}_{abc\dot{x}}$ are the dyad components of the spinor affine connection defined in the associated V_4 and $K_{c\dot{x}ba} = \zeta_c{}^{C}\, \bar{\zeta}_{\dot{x}}{}^{\dot{X}}\, \zeta_b{}^{B}\, \zeta_a{}^{A}\, K_{C\dot{X}BA}$. The distinct components of $\Gamma_{abc\dot{x}}$ and $K_{c\dot{x}ba}$ are equal to the spin coefficients and the contortion components respectively:

$\Gamma_{abc\dot{x}} =$

$c\dot{x}$ \ ab	00	01 or 10	11
$0\dot{0}$	κ	ϵ	π
$1\dot{0}$	ρ	α	λ
$0\dot{1}$	σ	β	μ
$1\dot{1}$	τ	γ	ν

$K_{c\dot{x}ab} =$

$c\dot{x}$ \ ab	00	01 or 10	11
$0\dot{0}$	$-\kappa_1$	$-\epsilon_1$	$-\pi_1$
$1\dot{0}$	$-\rho_1$	$-\alpha_1$	$-\lambda_1$
$0\dot{1}$	$-\sigma_1$	$-\beta_1$	$-\mu_1$
$1\dot{1}$	$-\tau_1$	$-\gamma_1$	$-\nu_1$

These are given by the expansions (c.f. (3.9))

$$\begin{aligned}\nabla_{B\dot{X}}\, o_A = {} & \gamma\, o_A o_B \bar{o}_{\dot{X}} - \alpha\, o_A o_B \bar{\iota}_{\dot{X}} - \beta\, o_A \iota_B \bar{o}_{\dot{X}} + \epsilon\, o_A \iota_B \bar{\iota}_{\dot{X}} \\ & - \tau\, \iota_A o_B \bar{o}_{\dot{X}} + \rho\, \iota_A o_B \bar{\iota}_{\dot{X}} + \sigma\, \iota_A \iota_B \bar{o}_{\dot{X}} - \kappa\, \iota_A \iota_B \bar{\iota}_{\dot{X}}\end{aligned} \tag{4.17a}$$

$$\nabla_{B\dot{X}}\,\iota_A = \nu\, o_A o_B \bar{o}_{\dot{X}} - \lambda\, o_A o_B \bar{\iota}_{\dot{X}} - \mu\, o_A \iota_B \bar{o}_{\dot{X}} + \pi\, o_A \iota_B \bar{\iota}_{\dot{X}}$$

$$- \gamma \iota_A o_B \bar{o}_{\dot{X}} + \alpha\, \iota_A o_B \bar{\iota}_{\dot{X}} + \beta\, \iota_A \iota_B \bar{o}_{\dot{X}} - \varepsilon\, \iota_A \iota_B \bar{\iota}_{\dot{X}} \qquad (4.17b)$$

$$K_{A\dot{X}BC} = (-\nu_1\, o_A \bar{o}_{\dot{X}} + \lambda_1 o_A \bar{\iota}_{\dot{X}} + \mu_1 \iota_A \bar{o}_{\dot{X}} - \pi_1 \iota_A \bar{\iota}_{\dot{X}})\, o_B o_C$$

$$+ (\gamma_1\, o_A \bar{o}_{\dot{X}} - \alpha_1 o_A \bar{\iota}_{\dot{X}} - \beta_1 \iota_A \bar{o}_{\dot{X}} + \varepsilon_1 \iota_A \bar{\iota}_{\dot{X}})(o_B \iota_C + \iota_B o_C) \qquad (4.17c)$$

$$+ (-\tau_1\, o_A \bar{o}_{\dot{X}} + \rho_1\, o_A \bar{\iota}_{\dot{X}} + \sigma_1\, \iota_A \bar{o}_{\dot{X}} - \kappa_1 \iota_A \bar{\iota}_{\dot{X}})\, \iota_B \iota_C$$

The spinor equivalent of the commutation relation, equation (3.11), is given by

$$(\nabla_{A\dot{W}}\,\nabla_{B\dot{X}} - \nabla_{B\dot{X}}\,\nabla_{A\dot{W}})\phi = (K_{A\dot{W}B\dot{X}}{}^{C\dot{Y}} - K_{B\dot{X}A\dot{W}}{}^{C\dot{Y}})\nabla_{C\dot{Y}}\,\phi \qquad (4.18)$$

Comparing the left hand side of this equation with equations (4.7) (a) and (b) we may write

$$\nabla_{A\dot{W}}\,\nabla_{B\dot{X}} - \nabla_{B\dot{X}}\,\nabla_{A\dot{W}} = \varepsilon_{AB}\,\nabla_{H(\dot{W}}\,\nabla^{H}{}_{\dot{X})} - \varepsilon_{\dot{W}\dot{X}}\,\nabla_{(A}{}^{\dot{P}}\nabla_{B)\dot{P}} \qquad (4.19)$$

The right hand side of (4.18) can be decomposed with the aid of (4.13). We obtain

$$(\nabla_{A\dot{W}}\nabla_{B\dot{X}} - \nabla_{B\dot{X}}\nabla_{A\dot{W}})\phi = [K_{A\dot{W}B}{}^{H}\nabla_{H\dot{X}} - K_{B\dot{X}A}{}^{H}\nabla_{H\dot{W}} + \bar{K}_{\dot{W}A\dot{X}}{}^{\dot{P}}\nabla_{B\dot{P}} - \bar{K}_{\dot{X}B\dot{W}}{}^{\dot{P}}\nabla_{A\dot{P}}]\phi$$

Equating the latter with (4.19) and contracting with $\varepsilon^{\dot{W}\dot{X}}$ gives the simplified spinor equivalent of the commutation relations:

$$\nabla_{(A}{}^{\dot P}\nabla_{B)\dot P}\,\phi = [K_{(A}{}^{\dot P}{}_{B)}{}^{H}\nabla_{H\dot P} - \bar{K}_{\dot P(A}{}^{\dot P\dot Q}\nabla_{B)\dot Q}]\phi \tag{4.20}$$

4.3 Decomposition of the Curvature Spinor

The symmetries (2.14) and (2.15) may be used, together with equation (4.7c), to decompose the spinor equivalent of the curvature tensor as follows:

$$\begin{aligned} R_{A\dot W B\dot X C\dot Y D\dot Z} &= Q_{ABCD}\,\epsilon_{\dot W\dot X}\,\epsilon_{\dot Y\dot Z} + \bar{Q}_{\dot W\dot X\dot Y\dot Z}\,\epsilon_{AB}\,\epsilon_{CD} \\ &\quad + P_{AB\dot Y\dot Z}\,\epsilon_{CD}\,\epsilon_{\dot W\dot X} + \bar{P}_{\dot W\dot X CD}\,\epsilon_{AB}\,\epsilon_{\dot Y\dot Z} \end{aligned} \tag{4.21}$$

where

$$Q_{ABCD} = Q_{(AB)(CD)} \tag{4.22}$$

$$P_{AB\dot Y\dot Z} = P_{(AB)(\dot Y\dot Z)} \tag{4.23}$$

The spinors (4.22) and (4.23) may be further decomposed. Q_{ABCD} written in terms of its totally symmetric part is given by

$$\begin{aligned} Q_{ABCD} &= Q_{(ABCD)} + \tfrac{1}{6}\,(Q_{ABCD} - Q_{ACDB}) + \tfrac{1}{6}\,(Q_{ABCD} - Q_{ADBC}) \\ &\quad + \tfrac{1}{6}(Q_{ABCD} - Q_{BCDA}) + \tfrac{1}{6}(Q_{ABCD} - Q_{CDBA}) + \tfrac{1}{6}(Q_{ABCD} - Q_{DBCA}) \end{aligned}$$

It is easily shown using this relation, with the aid of equation (4.6), that

$$Q_{ABCD} = -\Psi_{ABCD} + \frac{1}{4}\,[\Sigma_{AC}\varepsilon_{BD} + \Sigma_{BC}\varepsilon_{AD} + \Sigma_{AD}\varepsilon_{BC} + \Sigma_{BD}\varepsilon_{AC}]$$

$$+ \frac{1}{6}\,\Omega[\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{BC}\varepsilon_{AD}] \tag{4.24}$$

where

$$\Psi_{ABCD} = -Q_{(ABCD)} \tag{4.25}$$

$$\Sigma_{AB} = \varepsilon^{CD}\, Q_{C(AB)D} \tag{4.26}$$

$$\Omega = \varepsilon^{AC}\,\varepsilon^{BD}\, Q_{ABCD} \tag{4.27}$$

The spinors Ψ_{ABCD} and Σ_{AB} are the spinor equivalents of the generalised Weyl tensor $A_{\kappa\lambda\mu\nu}$, defined in (3.14), and the anti-symmetric part of the Ricci tensor $R_{[\mu\nu]}$ respectively.

The spinor $P_{AB\dot{Y}\dot{Z}}$ can be given in terms of its Hermitian and anti-Hermitian parts, of which the Hermitian part is the spinor equivalent of the symmetric part of the trace free Ricci tensor, as follows

$$P_{AB\dot{Y}\dot{Z}} = -\,\Phi_{AB\dot{Y}\dot{Z}} + i\;\;\Theta_{AB\dot{Y}\dot{Z}} \tag{4.28}$$

where

$$\Phi_{AB\dot{Y}\dot{Z}} = -\,\frac{1}{2}\,(P_{AB\dot{Y}\dot{Z}} + \bar{P}_{\dot{Y}\dot{Z}AB}) \tag{4.29}$$

$$\theta_{AB\dot{Y}\dot{Z}} = -\frac{1}{2}(P_{AB\dot{Y}\dot{Z}} - \overline{P}_{\dot{Y}\dot{Z}AB}) \tag{4.30}$$

Hence, the irreducible decomposition of the curvature spinor is given by

$$\begin{aligned} R_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}} = & - \Psi_{ABCD}\, \varepsilon_{\dot{W}\dot{X}}\, \varepsilon_{\dot{Y}\dot{Z}} - \varepsilon_{AB}\, \varepsilon_{CD}\, \overline{\Psi}_{\dot{W}\dot{X}\dot{Y}\dot{Z}} \\ & + \frac{1}{4}(\Sigma_{AC}\, \varepsilon_{BD} + \Sigma_{BC}\, \varepsilon_{AD} + \Sigma_{AD}\, \varepsilon_{BC} + \Sigma_{BD}\, \varepsilon_{AC})\varepsilon_{\dot{W}\dot{X}}\, \varepsilon_{\dot{Y}\dot{Z}} \\ & + \frac{1}{4}\varepsilon_{AB}\, \varepsilon_{CD}\, (\overline{\Sigma}_{\dot{W}\dot{Y}}\, \varepsilon_{\dot{X}\dot{Z}} + \overline{\Sigma}_{\dot{X}\dot{Y}}\, \varepsilon_{\dot{W}\dot{Z}} + \overline{\Sigma}_{\dot{W}\dot{Z}}\, \varepsilon_{\dot{X}\dot{Y}} + \overline{\Sigma}_{\dot{X}\dot{Z}}\, \varepsilon_{\dot{W}\dot{Y}}) \\ & + \frac{1}{6}\Omega(\varepsilon_{AC}\, \varepsilon_{BD} + \varepsilon_{BC}\, \varepsilon_{AD})\, \varepsilon_{\dot{W}\dot{X}}\, \varepsilon_{\dot{Y}\dot{Z}} \\ & + \frac{1}{6}\overline{\Omega}\, \varepsilon_{AB}\, \varepsilon_{CD}\, (\varepsilon_{\dot{W}\dot{Y}}\, \varepsilon_{\dot{X}\dot{Z}} + \varepsilon_{\dot{X}\dot{Y}}\, \varepsilon_{\dot{W}\dot{Z}}) \\ & - (\Phi_{AB\dot{Y}\dot{Z}} - i\, \Theta_{AB\dot{Y}\dot{Z}})\, \varepsilon_{CD}\, \varepsilon_{\dot{W}\dot{X}} \\ & - (\overline{\Phi}_{\dot{W}\dot{X}CD} + i\, \overline{\Theta}_{\dot{W}\dot{X}CD})\, \varepsilon_{AB}\, \varepsilon_{\dot{Y}\dot{Z}}. \end{aligned} \tag{4.31}$$

The components of the curvature tensor defined in § 3.2 using the tetrad approach can equivalently be obtained from the dyad components of the above spinors as follows:

$$
\begin{array}{lll}
\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D & , & \Lambda = -\frac{1}{12}\,(\Omega + \bar{\Omega}) \\[1em]
\Psi_1 = \Psi_{ABCD}\, o^A o^B o^C \iota^D & , & \Sigma = \frac{i}{12}\,(\Omega - \bar{\Omega}) \\[1em]
\Psi_2 = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D & , & \Phi_0 = \tfrac{1}{2}\, \Sigma_{AB} o^A o^B \\[1em]
\Psi_3 = \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D & , & \Phi_1 = \Sigma_{AB} o^A \iota^B \\[1em]
\Psi_4 = \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D & , & \Phi_2 = \tfrac{1}{2}\, \Sigma_{AB} \iota^A \iota^B \\[1em]
\Phi_{00} = \Phi_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{o}^{\dot{Y}} \bar{o}^{\dot{Z}} & , & \theta_{00} = \theta_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{o}^{\dot{Y}} \bar{o}^{\dot{Z}} \\[1em]
\Phi_{01} = \Phi_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{o}^{\dot{Y}} \bar{\iota}^{\dot{Z}} & , & \theta_{01} = \theta_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{o}^{\dot{Y}} \bar{\iota}^{\dot{Z}} \\[1em]
\Phi_{02} = \Phi_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}} & , & \theta_{02} = \theta_{AB\dot{Y}\dot{Z}}\, o^A o^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}} \\[1em]
\Phi_{11} = \Phi_{AB\dot{Y}\dot{Z}}\, o^A \iota^B \bar{o}^{\dot{Y}} \bar{\iota}^{\dot{Z}} & , & \theta_{11} = \theta_{AB\dot{Y}\dot{Z}}\, o^A \iota^B \bar{o}^{\dot{Y}} \bar{\iota}^{\dot{Z}} \\[1em]
\Phi_{12} = \Phi_{AB\dot{Y}\dot{Z}}\, o^A \iota^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}} & , & \theta_{12} = \theta_{AB\dot{Y}\dot{Z}}\, o^A \iota^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}} \\[1em]
\Phi_{22} = \Phi_{AB\dot{Y}\dot{Z}}\, \iota^A \iota^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}} & , & \theta_{22} = \theta_{AB\dot{Y}\dot{Z}}\, \iota^A \iota^B \bar{\iota}^{\dot{Y}} \bar{\iota}^{\dot{Z}}
\end{array}
\tag{4.32}
$$

4.4 Properties of the Curvature Spinor

In this section I will trace the steps of Penrose (1960) (given explicitly in Pirani (1965) § 3.8) in order to generalise the spinor equivalents of the V_4 Ricci identities and Bianchi identities for the curvature to U_4. The spinor equivalents of the Bianchi identities for the torsion are also obtained.

The Ricci identity (2.13) implies that, for a bivector $T_{\kappa\lambda} = T_{[\kappa\lambda]}$,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, T_{\kappa\lambda} = R_{\mu\nu\kappa}{}^{\alpha}\, T_{\alpha\lambda} + R_{\mu\nu\lambda}{}^{\alpha}\, T_{\kappa\alpha}$$

$$+ (K_{\mu\nu}{}^{\alpha} - K_{\nu\mu}{}^{\alpha})\, \nabla_\alpha T_{\kappa\lambda}$$

The spinor equivalent of this equation is given by

$$(\nabla_{A\dot{W}} \nabla_{B\dot{X}} - \nabla_{B\dot{X}} \nabla_{A\dot{W}})\, T_{C\dot{Y}D\dot{Z}} = R_{A\dot{W}B\dot{X}C\dot{Y}}{}^{E\dot{Q}}\, T_{E\dot{Q}D\dot{Z}} + R_{A\dot{W}B\dot{X}D\dot{Z}}{}^{E\dot{Q}}\, T_{C\dot{Y}E\dot{Q}}$$

$$+ (K_{A\dot{W}B\dot{X}}{}^{E\dot{Q}} - K_{B\dot{X}A\dot{W}}{}^{E\dot{Q}})\, \nabla_{E\dot{Q}}\, T_{C\dot{Y}D\dot{Z}} \qquad (4.33)$$

where we may choose

$$T_{C\dot{Y}D\dot{Z}} = \xi_C\, \xi_D\, \epsilon_{\dot{Y}\dot{Z}} \qquad (4.34)$$

We may substitute the equation (4.19) into the left hand side of the Ricci identity (4.33) and then consider separately the two equations obtained by contracting first with $\epsilon^{\dot{W}\dot{X}}$ and second with ϵ^{AB}. With the aid of the decomposition (4.31), the first contraction yields

$$-\xi_C \nabla_{(A}{}^{\dot{P}} \nabla_{B)\dot{P}} \xi_D - \xi_D \nabla_{(A}{}^{\dot{P}} \nabla_{B)\dot{P}} \xi_C = + 2\Psi_{ABE(C}\xi_{D)}\xi^E - \frac{2}{3}\Omega\, \xi_{(C}\epsilon_{D)(A}\xi_{B)}$$

$$+ \xi_{(A}\Sigma_{B)(C}\xi_{D)} - \xi^E\, \Sigma_{E(A}\epsilon_{B)(C}\xi_{D)}$$

$$- K_{(A}{}^{\dot{P}}{}_{B)\dot{P}}{}^{E\dot{Q}}\, \nabla_{E\dot{Q}}(\xi_C\, \xi_D) \tag{4.35}$$

We now multiply by $\eta^C \eta^D$, where η^C is an arbitrary spinor, and then remove the common factor $\xi_C \eta^C$. η^D now appears as a common factor and may also be removed to give

$$\nabla_{(A}{}^{\dot{P}} \nabla_{B)\dot{P}}\, \xi_C = -\Psi_{ABCD}\, \xi^D - \frac{1}{3}\Omega\, \xi_{(A}\epsilon_{B)C}$$

$$- \frac{1}{2}\, \xi_{(A}\Sigma_{B)C} + \frac{1}{2}\, \xi^D\, \Sigma_{D(A}\epsilon_{B)C} \tag{4.36}$$

$$+ K_{(A}{}^{\dot{Q}}{}_{B)}{}^{E} \nabla_{E\dot{Q}}\, \xi_C - \overline{K}_{\dot{P}(A}{}^{\dot{P}\dot{Q}} \nabla_{B)\dot{Q}}\, \xi_C$$

This equation is one of the two Ricci identities in spinor form.

The other is obtained by contracting initially with ϵ^{AB}. Then, following analogously the steps of the first contraction, the second Ricci identity is given by

$$\nabla_{H(\dot{W}}\nabla^{H}{}_{\dot{X})}\ \xi_D = + [\Phi_{DE\dot{W}\dot{X}} + i\theta_{DE\dot{W}\dot{X}}]\ \xi^E$$

$$+ K_{A(\dot{W}}{}^{AE}\nabla_{|E|\dot{X})}\ \xi_D - \overline{K}_{(\dot{W}}{}^{A}{}_{\dot{X})}{}^{\dot{Q}}\nabla_{A\dot{Q}}\ \xi_D \qquad (4.37)$$

Note that equation (4.36) can be further decomposed. It may be symmetrised on ABC, and contracted with ε^{BC}. These operations respectively yield

$$\nabla_{(A}{}^{\dot{P}}\nabla_{B|\dot{P}|}\xi_{C)} = -\ \Psi_{ABCD}\ \xi^D - \frac{1}{2}\ \Sigma_{(AB}\ \xi_{C)} \qquad (4.38)$$

$$+ K_{(A}{}^{\dot{Q}}{}_{B}{}^{E}\nabla_{|E\dot{Q}|}\xi_{C)} - \overline{K}_{\dot{P}(A}{}^{\dot{P}\dot{Q}}\nabla_{B|\dot{Q}|}\xi_{C)}$$

$$\nabla_{(A}{}^{\dot{P}}\nabla_{B)\dot{P}}\xi^B = -\ \frac{1}{2}\Omega\,\xi_A + \frac{3}{4}\ \Sigma_{AB}\ \xi^B \qquad (4.39)$$

$$+ K_{(A}{}^{\dot{Q}}{}_{B)}{}^{E}\ \nabla_{E\dot{Q}}\ \xi^B - \overline{K}_{\dot{P}(A}{}^{\dot{P}\dot{Q}}\nabla_{B)\dot{Q}}\xi^B$$

The Bianchi identities for the curvature (2.17) can be written in the form

$$\nabla^{\delta}\ \varepsilon_{\delta\lambda}{}^{\mu\nu}\ R_{\mu\nu\alpha\beta} = -2\ \varepsilon_{\delta\lambda}{}^{\mu\nu}\ R^{\delta}{}_{\kappa\alpha\beta} \qquad (4.40)$$

where the spinor equivalent of $\varepsilon_{\delta\lambda}{}^{\mu\nu}$ is given by (Penrose, 1960)

$$\varepsilon_{C\dot{Y}D\dot{Z}}^{A\dot{W}B\dot{X}} = i\ \delta^A_C\ \delta^B_D\ \delta^{\dot{W}}_{\dot{Z}}\ \delta^{\dot{X}}_{\dot{Y}} - i\ \delta^A_D\ \delta^B_C\ \delta^{\dot{W}}_{\dot{Y}}\ \delta^{\dot{X}}_{\dot{Z}} \qquad (4.41)$$

Using this and (4.13), the spinor equivalent of (4.40) is given by

$$\nabla^{A\dot X} R_{A\dot W B\dot X C\dot Y D\dot Z} = [K_{B\dot P E}{}^{F}\,\delta_{\dot W}{}^{\dot Q} - K_{E\dot W B}{}^{F}\,\delta_{\dot P}{}^{\dot Q} + \bar K_{\dot P B\dot W}{}^{\dot Q}\,\delta_{E}{}^{F} - \bar K_{\dot W E\dot P}{}^{\dot Q}\,\delta_{B}{}^{F}]\, R^{E\dot P}{}_{F\dot Q C\dot Y D\dot Z} \tag{4.42}$$

Upon symmetrizing (4.42) with respect to CD and using (4.31), one obtains an expression which can be decomposed first by symmetrizing with respect to BCD and second by contracting with ε^{BC}. The symmetrization and contraction respectively yield the following identities, which are the spinor equivalents of the Bianchi identities for the curvature (spinor indices have been relabelled):

$$\begin{aligned}\nabla^{E}{}_{\dot X}\Psi_{EABC} &- \nabla_{(A}{}^{\dot W}\Phi_{BC)\dot X\dot W} - \nabla_{(A}{}^{\dot W}\, i\,\Theta_{BC)\dot X\dot W} - \tfrac{1}{2}\nabla_{(A|\dot X|}\Sigma_{BC)} \\ &= [\Psi^{EF}{}_{(AB} + \Sigma^{(E}{}_{(A}\delta^{F}{}_{B} + \tfrac{\Lambda}{3}\delta^{E}{}_{(A}\delta^{F}{}_{B}]\,[2K_{|E\dot X|C)F} - K_{C)\dot X EF}] \\ &- [\Phi_{(AB}{}^{\dot P\dot W} + i\,\Theta_{(AB}{}^{\dot P\dot W}]\,[2\bar K_{|\dot P|C)\dot X\dot W} - \bar K_{|\dot X|C)\dot P\dot W}]\end{aligned} \tag{4.43}$$

$$\begin{aligned}\nabla^{A\dot Z}(\Phi_{AB\dot Y\dot Z} &+ i\,\Theta_{AB\dot Y\dot Z}) - \tfrac{1}{2}\nabla_{B\dot Y}\Omega + \nabla^{A}{}_{\dot Y}\Sigma_{BA} \\ &= -\Psi_{EFGB}K^{E}{}_{\dot Y}{}^{FG} - (\Phi_{AB\dot W\dot X} + i\,\Theta_{AB\dot W\dot X})(\bar K_{\dot Y}{}^{A\dot W\dot X} - 2\bar K^{\dot X A\dot W}{}_{\dot Y}) \\ &+ \tfrac{1}{2}\Sigma_{EF}K_{B\dot Y}{}^{EF} + \Sigma_{EB}K_{F\dot X}{}^{EF} + \tfrac{2}{3}\Omega K_{A\dot X B}{}^{A}\end{aligned} \tag{4.44}$$

The Bianchi identities for the torsion (2.16) can be written in the form

$$\nabla^{\lambda}\,\varepsilon_{\delta\lambda}{}^{\mu\nu}\,K_{\mu\nu\kappa} = \frac{1}{2}\,\varepsilon_{\delta\lambda}{}^{\mu\nu}\,R_{\mu\nu}{}^{\lambda}{}_{\kappa} + \varepsilon_{\delta\lambda}{}^{\mu\nu}\,K_{\mu\nu}{}^{\alpha}\,(K_{\alpha}{}^{\lambda}{}_{\kappa} - K^{\lambda}{}_{\alpha\kappa})$$

The spinor equivalent of this equation is given, with the aid of (4.31) and (4.41), by

$$\nabla^{E}{}_{\dot{P}}\,(K_{A}{}^{\dot{P}}{}_{B}{}^{\dot{X}}{}_{E}{}^{\dot{W}} - K_{E}{}^{\dot{W}}{}_{B}{}^{\dot{X}}{}_{A}{}^{\dot{P}}) \tag{4.45}$$

$$= \Sigma_{AB}\,\varepsilon^{\dot{W}\dot{X}} - \varepsilon_{AB}\,\bar{\Sigma}^{\dot{W}\dot{X}} - 2i\;\Theta_{AB}{}^{\dot{W}\dot{X}} + \frac{1}{2}\,\varepsilon_{AB}\,\varepsilon^{\dot{W}\dot{X}}\,(\Omega - \bar{\Omega})$$

$$- (K^{E\dot{P}F\dot{Q}}{}_{B}{}^{\dot{X}} - K^{F\dot{Q}E\dot{P}}{}_{B}{}^{\dot{X}})(K_{A\dot{P}E}{}^{\dot{W}}{}_{F\dot{Q}} - K_{E}{}^{\dot{W}}{}_{A\dot{P}F\dot{Q}})$$

The curvature components that enter this equation are those belonging to the spinor equivalents of the tensor $D_{\mu\nu}$ defined in Chapter 3. Explicit expressions may be given for these spinors by taking various combinations of contractions and symmetrizations of (4.45). For example the spinor $\Theta_{AB}{}^{\dot{W}\dot{X}}$ is given by symmetrizing with respect to AB and $\dot{W}\dot{X}$. After a series of laborious calculations, which involve the decomposition of the contortion spinor (4.13), one obtains

$$2i\;\Theta_{AB}{}^{\dot{W}\dot{X}} = \nabla^{E(\dot{W}}K_{E}{}^{\dot{X})}{}_{AB} + \nabla_{(A|\dot{P}|}\,\bar{K}^{\dot{P}}{}_{B)}{}^{\dot{W}\dot{X}} \tag{4.46a}$$

$$- 2K_{(E}{}^{\dot{W}}{}_{F)(A}K^{E\dot{X}F}{}_{B)} + 2\bar{K}^{(\dot{P}}{}_{A}{}^{\dot{Q})(\dot{W}}\bar{K}_{\dot{P}B}{}^{\dot{X})}{}_{\dot{Q}}$$

$$-K_{E\dot{P}}{}^{E}{}_{(A}\left[\bar{K}^{\dot{P}}{}_{B)}{}^{\dot{W}\dot{X}} + \bar{K}^{(\dot{W}}{}_{B)}{}^{\dot{X})\dot{P}}\right] + \bar{K}^{\dot{P}E}{}_{\dot{P}}{}^{(\dot{W}}\left[K_{E}{}^{\dot{X})}{}_{AB} + K_{(A}{}^{\dot{X}}{}_{B)E}\right]$$

$$-K_{(A}{}^{\dot{P}E}{}_{B)}\left[\bar{K}_{\dot{P}E}{}^{\dot{W}\dot{X}} + \bar{K}^{(\dot{W}}{}_{\dot{P}E}{}^{\dot{X})}\right] + \bar{K}^{(\dot{W}}{}_{E\dot{P}}{}^{\dot{X})}\left[K^{E\dot{P}}{}_{AB} + K_{(A}{}^{\dot{P}E}{}_{B)}\right]$$

$$-K^{E(\dot{W}}{}_{F(A}K_{B)}{}^{\dot{X})F}{}_{E} + \bar{K}_{\dot{P}(A}{}^{\dot{Q}(\dot{W}}\bar{K}^{\dot{X})}{}_{B)\dot{Q}}{}^{\dot{P}}$$

$$2\,\Sigma_{AB} = \nabla^{E}{}_{\dot{P}}\,[K_{E}{}^{\dot{P}}{}_{AB} - 2\,K_{(A}{}^{\dot{P}}{}_{B)E}] \qquad \text{(b)}$$

$$-2\,K^{E\dot{P}F}{}_{(A}K_{B)\dot{P}EF} + K_{E\dot{P}}{}^{E}{}_{(A}\bar{K}^{\dot{Q}}{}_{B)\dot{Q}}{}^{\dot{P}} + K_{(A}{}^{\dot{P}E}{}_{B)}\bar{K}_{\dot{Q}E\dot{P}}{}^{\dot{Q}}$$

$$+\,K^{E}{}_{\dot{P}F(A}K_{B)}{}^{\dot{P}F}{}_{E} + \bar{K}^{\dot{P}}{}_{(A}{}^{\dot{Q}\dot{W}}\,\bar{K}_{|\dot{W}|B)\dot{P}\dot{Q}}$$

$$-\,\bar{K}_{\dot{P}E\dot{Q}}{}^{\dot{P}}\,[K^{E\dot{Q}}{}_{AB} + K_{(A}{}^{\dot{Q}E}{}_{B)}] + \bar{K}^{\dot{P}E}{}_{\dot{P}\dot{Q}}\,[K_{E}{}^{\dot{Q}}{}_{AB} + K_{(A}{}^{\dot{Q}}{}_{B)E}]$$

$$+\,2\,K_{(E}{}^{\dot{P}}{}_{F)(A}K^{E}{}_{|\dot{P}|}{}^{F}{}_{B)} - 2\,\bar{K}^{(\dot{P}}{}_{(A}{}^{\dot{Q})\dot{W}}\,\bar{K}_{|\dot{P}|B)\dot{W}\dot{Q}}$$

$$2\,(\Omega - \bar{\Omega}) = \nabla_{A\dot{X}}\left[K_{E}{}^{\dot{X}AE} - \bar{K}_{\dot{P}}{}^{A\dot{X}\dot{P}}\right] + K_{A\dot{X}BC}\,K^{A\dot{X}BC} - \bar{K}_{\dot{W}E\dot{X}\dot{Y}}\,\bar{K}^{\dot{W}E\dot{X}\dot{Y}} \qquad \text{(c)}$$

Variations of the spin coefficients and the dyad components of the curvature spinor under a number of dyad transformations are given in Appendix A.

CHAPTER 5

THE GENERALISED NEWMAN-PENROSE IDENTITIES

The generalised Newman-Penrose identities given in this chapter are the intrinsic components of the commutation relations, the Ricci identities, the Bianchi identities for the torsion and the Bianchi identities for the curvature. They may be obtained either by taking the tetrad components of these tensor identities or by taking the dyad components of their spinor equivalents. In either case it is convenient to introduce the intrinsic derivatives

$$\nabla_m \equiv e_m{}^{\mu} \, \nabla_\mu \tag{5.1}$$

$$\nabla_{a\dot{x}} = \zeta_a{}^A \, \bar{\zeta}_{\dot{x}}{}^{\dot{X}} \, \nabla_{A\dot{X}} \tag{5.2}$$

denoting their components separately by

$$D = \ell^\mu \, \nabla_\mu = o^A \, \bar{o}^{\dot{X}} \, \nabla_{A\dot{X}} \tag{5.3a}$$

$$\Delta = n^\mu \, \nabla_\mu = \iota^A \, \bar{\iota}^{\dot{X}} \, \nabla_{A\dot{X}} \tag{b}$$

$$\delta = m^\mu \, \nabla_\mu = o^A \, \bar{\iota}^{\dot{X}} \, \nabla_{A\dot{X}} \tag{c}$$

$$\bar{\delta} = \bar{m}^\mu \, \nabla_\mu = \iota^A \, \bar{o}^{\dot{X}} \, \nabla_{A\dot{X}} \tag{d}$$

5.1 The Commutation Relations

When operating on a scalar field ϕ, the tetrad components of the commutators (3.11) are given by

$$\nabla_{[m}\nabla_{n]}\phi = (\gamma^{p}{}_{[mn]} - K_{[nm]}{}^{p})\,\nabla_{p}\,\phi \tag{5.4}$$

The dyad equivalent of this equation is given by taking the dyad components of equation (4.18), which gives the commutators in spinor form. We obtain

$$\begin{aligned}
&(\nabla_{a\dot{w}}\nabla_{b\dot{x}} - \nabla_{b\dot{x}}\nabla_{a\dot{w}})\phi \\
&\quad = \varepsilon^{ef}\,[(\Gamma_{eab\dot{x}} + K_{b\dot{x}ae})\,\nabla_{f\dot{w}} - (\Gamma_{eba\dot{w}} + K_{a\dot{w}be})\,\nabla_{f\dot{x}}]\phi \\
&\quad + \varepsilon^{\dot{r}\dot{s}}\,[(\overline{\Gamma}_{\dot{r}\dot{w}\dot{x}b} + \overline{K}_{\dot{x}b\dot{w}\dot{r}})\,\nabla_{a\dot{s}} - (\overline{\Gamma}_{\dot{r}\dot{x}\dot{w}a} + \overline{K}_{\dot{w}a\dot{x}\dot{r}})\,\nabla_{b\dot{s}}]\phi
\end{aligned} \tag{5.5}$$

Using the definitions (5.3), the equations (5.4) or (5.5) can be expanded in the form

$$(\Delta D - D\Delta)\phi = (\gamma^0 + \overline{\gamma^0})D\phi + (\varepsilon^0 + \overline{\varepsilon^0})\Delta\phi - (\tau^0 + \overline{\pi^0})\overline{\delta}\phi - (\overline{\tau^0} + \pi^0)\delta\phi \tag{5.6a}$$

$$(\delta D - D\delta)\phi = (\overline{\alpha^0} + \beta^0 - \overline{\pi^0})D\phi + \kappa^0\,\Delta\phi - \sigma^0\overline{\delta}\phi - (\overline{\rho^0} + \varepsilon^0 - \overline{\varepsilon^0})\delta\phi \tag{b}$$

$$(\delta\Delta - \Delta\delta)\phi = -\overline{\nu^0}\,D\phi - (\overline{\alpha^0} + \beta^0 - \tau^0)\Delta\phi + \overline{\lambda^0}\overline{\delta}\phi + (\mu^0 - \gamma^0 + \overline{\gamma^0})\delta\phi \tag{c}$$

$$(\overline{\delta}\delta - \delta\overline{\delta})\phi = (\overline{\mu^0} - \mu^0)D\phi + (\overline{\rho^0} - \rho^0)\Delta\phi + (\beta^0 - \overline{\alpha^0})\overline{\delta}\phi + (\alpha^0 - \overline{\beta^0})\delta\phi \tag{d}$$

5.2 The Ricci Identities

If the Ricci identity (2.13) is applied to the four tetrad vectors $e_m{}^\mu$ and then tetrad components are taken, we obtain (with the aid of (3.18))

$$\nabla_\ell \gamma_{mnk} - \nabla_k \gamma_{mn\ell} =$$

$$\gamma_{pmk} \gamma^p{}_{n\ell} - \gamma_{pm\ell} \gamma^p{}_{nk} - 2\gamma_{mn}{}^p (\gamma_{p[k\ell]} - K_{[\ell k]p})$$

$$+ A_{k\ell mn} + B_{k\ell mn} + \frac{D}{12} \epsilon_{k\ell mn}$$

$$+ R_{\ell[m}\eta_{n]k} - R_{k[m}\eta_{n]\ell} - \frac{1}{3} R\eta_{\ell[m}\eta_{n]k} \tag{5.7}$$

The dyad equivalent of this expression is derived by applying the spinor Ricci identities (4.36) and (4.37) to the basis spinors $\zeta_a{}^A$ and taking dyad components. We obtain

$$\nabla_{a\dot{w}} \Gamma_{cdb\dot{x}} - \Gamma_{b\dot{x}} \Gamma_{cda\dot{w}} = (\Gamma^e{}_{da\dot{w}} \Gamma_{ecb\dot{x}} - \Gamma^e{}_{db\dot{x}} \Gamma_{eca\dot{w}}$$

$$+ \Gamma^e{}_{ba\dot{w}} \Gamma_{cde\dot{x}} - \Gamma^e{}_{ab\dot{x}} \Gamma_{cde\dot{w}}) + (\Gamma_{cda}{}^{\dot{r}} \overline{\Gamma}_{\dot{r}\dot{w}\dot{x}b} - \Gamma_{cdb}{}^{\dot{r}} \overline{\Gamma}_{\dot{r}\dot{x}\dot{w}a})$$

$$+ \epsilon_{\dot{w}\dot{x}} [\Gamma_{cd(a}{}^{\dot{s}} \overline{K}^{\dot{r}}{}_{b)\dot{r}\dot{s}} - \Gamma_{cde\dot{r}} K_{(a}{}^{\dot{r}}{}_{b)}{}^e] + \epsilon_{ab} [\Gamma_{cd}{}^f{}_{(\dot{w}} K^e{}_{\dot{x})ef} - \Gamma_{cde\dot{r}} \overline{K}_{(\dot{w}}{}^e{}_{\dot{x})}{}^{\dot{r}}]$$

$$+ \epsilon_{\dot{w}\dot{x}} [\Psi_{cdab} + \frac{\Omega}{3} \epsilon_{c(a}\epsilon_{b)d} - \frac{1}{2} (\Sigma_{c(a}\epsilon_{b)d} + \Sigma_{d(a}\epsilon_{b)c})] + \epsilon_{ab}(\Phi_{cd\dot{w}\dot{x}} + i\,\theta_{cd\dot{w}\dot{x}}) \tag{5.8}$$

We may now expand equations (5.7) and (5.8) explicitly in terms of the notations given in Chapters 3, 4 and 5. The following 18 equations are derived either by expanding (5.7) and taking certain linear combinations or, more directly, by simply expanding equation (5.8).

$$\begin{aligned} D\rho - \bar{\delta}\kappa = {} & \rho(\rho+\varepsilon+\bar{\varepsilon}) + \sigma\bar{\sigma} - \tau\bar{\kappa} - \kappa(3\alpha+\bar{\beta}-\pi) + \Phi_{00} \\ & -\rho(\rho_1-\varepsilon_1+\bar{\varepsilon}_1) - \sigma\bar{\sigma}_1 + \tau\bar{\kappa}_1 + \kappa(\alpha_1+\bar{\beta}_1-\pi_1) + i\Theta_{00} \end{aligned} \tag{5.9a}$$

$$\begin{aligned} D\sigma - \delta\kappa = {} & \rho\sigma + \sigma(\bar{\rho}+3\varepsilon-\bar{\varepsilon}) - \tau\kappa - \kappa(\bar{\alpha}+3\beta-\bar{\pi}) + \Psi_0 \\ & -\rho\sigma_1 - \sigma(\bar{\rho}_1+\varepsilon_1-\bar{\varepsilon}_1) + \tau\kappa_1 + \kappa(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) \end{aligned} \tag{b}$$

$$\begin{aligned} D\tau - \Delta\kappa = {} & \rho(\tau+\bar{\pi}) + \sigma(\bar{\tau}+\pi) - \tau(\bar{\varepsilon}-\varepsilon) - \kappa(3\gamma+\bar{\gamma}) + \Psi_1 + \Phi_{01} \\ & -\rho(\tau_1+\bar{\pi}_1) - \sigma(\bar{\tau}_1+\pi_1) + \tau(\bar{\varepsilon}_1+\varepsilon_1) + \kappa(\gamma_1+\bar{\gamma}_1) + i\Theta_{01} + \Phi_0 \end{aligned} \tag{c}$$

$$\begin{aligned} D\alpha - \bar{\delta}\varepsilon = {} & \alpha(\rho+\bar{\varepsilon}-\varepsilon) + \beta\bar{\sigma} - \gamma\bar{\kappa} - \varepsilon(\alpha+\bar{\beta}-\pi) + \rho\pi - \kappa\lambda + \Phi_{10} \\ & -\alpha(\rho_1+\bar{\varepsilon}_1-\varepsilon_1) - \beta\bar{\sigma}_1 + \gamma\bar{\kappa}_1 + \varepsilon(\alpha_1+\bar{\beta}_1-\pi_1) + i\Theta_{10} \end{aligned} \tag{d}$$

$$\begin{aligned} D\beta - \delta\varepsilon = {} & \alpha\sigma + \beta(\bar{\rho}-\bar{\varepsilon}) - \gamma\kappa - \varepsilon(\bar{\alpha}-\bar{\pi}) + \sigma\pi - \kappa\mu + \Psi_1 \\ & -\alpha\sigma_1 - \beta(\bar{\rho}_1-\bar{\varepsilon}_1+\varepsilon_1) + \gamma\kappa_1 + \varepsilon(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) - \Phi_0 \end{aligned} \tag{e}$$

$$\begin{aligned} D\gamma - \Delta\varepsilon = {} & \alpha(\tau+\bar{\pi}) + \beta(\bar{\tau}+\pi) - \gamma(\varepsilon+\bar{\varepsilon}) - \varepsilon(\gamma+\bar{\gamma}) + \tau\pi - \kappa\nu + \Psi_2 - \Lambda + \Phi_{11} \\ & -\alpha(\tau_1+\bar{\pi}_1) - \beta(\bar{\tau}_1+\pi_1) + \gamma(\varepsilon_1+\bar{\varepsilon}_1) + \varepsilon(\gamma_1+\bar{\gamma}_1) + i\Theta_{11} - i\Sigma \end{aligned} \tag{f}$$

$$D\lambda - \bar{\delta}\pi = \mu\bar{\sigma} + \lambda(\rho-3\varepsilon+\bar{\varepsilon}) - \pi(-\alpha+\bar{\beta}-\pi) - \nu\bar{\kappa} + \Phi_{20}$$

$$-\mu\bar{\sigma}_1 - \lambda(\rho_1-\varepsilon_1+\bar{\varepsilon}_1) + \pi(\alpha_1+\bar{\beta}_1-\pi_1) + \nu\bar{\kappa}_1 + i\Theta_{20} \tag{g}$$

$$D\mu - \delta\pi = \mu(\bar{\rho}-\varepsilon-\bar{\varepsilon}) + \lambda\sigma - \pi(\bar{\alpha}-\beta-\bar{\pi}) - \nu\kappa + \Psi_2 + 2\Lambda$$

$$-\mu(\bar{\rho}_1+\varepsilon_1-\bar{\varepsilon}_1) - \lambda\sigma_1 + \pi(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) + \nu\kappa_1 - \Phi_1 + 2i\Sigma \tag{h}$$

$$D\nu - \Delta\pi = \mu(\pi+\bar{\tau}) + \lambda(\bar{\pi}+\tau) - \pi(\bar{\gamma}-\gamma) - \nu(3\varepsilon+\bar{\varepsilon}) + \Psi_3 + \Phi_{21}$$

$$-\mu(\pi_1+\bar{\tau}_1) - \lambda(\bar{\pi}_1+\tau_1) + \pi(\bar{\gamma}_1+\gamma_1) + \nu(\varepsilon_1+\bar{\varepsilon}_1) + i\Theta_{21} - \Phi_2 \tag{i}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \lambda(\bar{\mu}+3\gamma-\bar{\gamma}) + \pi\nu + \nu(3\alpha+\bar{\beta}-\bar{\tau}) - \Psi_4$$

$$+\mu\lambda_1 + \lambda(\bar{\mu}_1+\gamma_1-\bar{\gamma}_1) - \pi\nu_1 - \nu(\alpha_1+\bar{\beta}_1-\bar{\tau}_1) \tag{j}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha}+\beta) + \sigma(\bar{\beta}-3\alpha) + \tau(\rho-\bar{\rho}) + \kappa(\mu-\bar{\mu}) - \Psi_1 + \Phi_{01}$$

$$-\rho(\bar{\alpha}_1-\beta_1) - \sigma(\bar{\beta}_1-\alpha_1) - \tau(\rho_1-\bar{\rho}_1) - \kappa(\mu_1-\bar{\mu}_1) + i\Theta_{01} - \Phi_0 \tag{k}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha(\bar{\alpha}-\beta) + \beta(\bar{\beta}-\alpha) + \gamma(\rho-\bar{\rho}) + \varepsilon(\mu-\bar{\mu}) + \rho\mu - \sigma\lambda - \Psi_2 + \Lambda + \Phi_{11}$$

$$-\alpha(\bar{\alpha}_1-\beta_1) - \beta(\bar{\beta}_1-\alpha_1) - \gamma(\rho_1-\bar{\rho}_1) - \varepsilon(\mu_1-\bar{\mu}_1) + i\Theta_{11} + i\Sigma \tag{l}$$

$$\delta\lambda - \bar{\delta}\mu = \mu(\bar{\beta}+\alpha) + \lambda(\bar{\alpha}-3\beta) + \pi(\mu-\bar{\mu}) + \nu(\rho-\bar{\rho}) - \Psi_3 + \Phi_{21}$$

$$-\mu(\bar{\beta}_1-\alpha_1) - \lambda(\bar{\alpha}_1-\beta_1) - \pi(\mu_1-\bar{\mu}_1) - \nu(\rho_1-\bar{\rho}_1) + i\Theta_{21} + \Phi_2 \qquad (5.9)(m)$$

$$\delta\nu - \Delta\mu = \mu(\mu+\gamma+\bar{\gamma}) + \lambda\bar{\lambda} - \pi\bar{\nu} - \nu(3\beta+\bar{\alpha}-\tau) + \Phi_{22}$$

$$-\mu(\mu_1-\gamma_1+\bar{\gamma}_1) - \lambda\bar{\lambda}_1 + \pi\bar{\nu}_1 + \nu(\beta_1+\bar{\alpha}_1-\tau_1) + i\Theta_{22} \qquad (n)$$

$$\delta\gamma - \Delta\beta = \alpha\bar{\lambda} + \beta(\mu-\gamma+\bar{\gamma}) - \gamma(\beta+\bar{\alpha}-\tau) - \epsilon\bar{\nu} + \mu\tau - \nu\sigma + \Phi_{12}$$

$$-\alpha\bar{\lambda}_1 - \beta(\mu_1-\gamma_1+\bar{\gamma}_1) + \gamma(\beta_1+\bar{\alpha}_1-\tau_1) + \epsilon\bar{\nu}_1 + i\Theta_{12} \qquad (o)$$

$$\delta\tau - \Delta\sigma = \rho\bar{\lambda} + \sigma(\mu-3\gamma+\bar{\gamma}) - \tau(\bar{\alpha}-\beta-\tau) - \kappa\bar{\nu} + \Phi_{02}$$

$$-\rho\bar{\lambda}_1 - \sigma(\mu_1-\gamma_1+\bar{\gamma}_1) + \tau(\bar{\alpha}_1+\beta_1-\tau_1) + \kappa\bar{\nu}_1 + i\Theta_{02} \qquad (p)$$

$$\Delta\rho - \bar{\delta}\tau = -\rho(\bar{\mu}-\gamma-\bar{\gamma}) - \sigma\lambda + \tau(\bar{\beta}-\alpha-\bar{\tau}) + \kappa\nu - \Psi_2 - 2\Lambda$$

$$+\rho(\bar{\mu}_1+\gamma_1-\bar{\gamma}_1) + \sigma\lambda_1 - \tau(\bar{\beta}_1+\alpha_1-\bar{\tau}_1) - \kappa\nu_1 - \Phi_1 - 2i\Sigma \qquad (q)$$

$$\Delta\alpha - \bar{\delta}\gamma = -\alpha(\bar{\mu}-\bar{\gamma}) - \beta\lambda + \gamma(\bar{\beta}-\bar{\tau}) + \epsilon\nu + \rho\nu - \tau\lambda - \Psi_3$$

$$+\alpha(\bar{\mu}_1+\gamma_1-\bar{\gamma}_1) + \beta\lambda_1 - \gamma(\bar{\beta}_1+\alpha_1-\bar{\tau}_1) - \epsilon\nu_1 - \Phi_2 \;. \qquad (r)$$

5.3 The Bianchi Identities for the Torsion

The Bianchi identity for the torsion (2.16), which generalises a symmetry property of the curvature tensor in V_4, written out in terms of its tetrad components takes the form

$$\nabla_{[\ell}K_{mn]k} = \frac{1}{2}B_{[\ell mn]k} + \frac{D}{24}\varepsilon_{\ell mnk} + \frac{1}{2}R_{[\ell m}{}^{n}{}_{n]k}$$

$$- \gamma_{pk[\ell}K_{mn]}{}^{p} + \gamma_{p[\ell m}K_{n]k}{}^{p} + \gamma_{p[\ell m}K^{p}{}_{n]k}$$

$$+ K_{[\ell m}{}^{p}K_{n]kp} - K_{pk[\ell}K_{mn]}{}^{p} \qquad (5.10)$$

The dyad equivalent of equation (5.10) is derived by taking the dyad components of equation (4.45) which gives the Bianchi identities for the torsion in spinor form. We obtain the following lengthy expression

$$\varepsilon_{\dot{w}\dot{x}}\nabla^{e\dot{r}}K_{a\dot{r}eb} - \varepsilon_{ab}\nabla^{e\dot{r}}\overline{K}_{\dot{w}e\dot{r}\dot{x}} + \nabla_{e\dot{x}}K^{e}{}_{\dot{w}ab} - \nabla_{b\dot{r}}\overline{K}^{\dot{r}}{}_{a\dot{w}\dot{x}}$$

$$= \Sigma_{ab}\varepsilon_{\dot{w}\dot{x}} - \varepsilon_{ab}\overline{\Sigma}_{\dot{w}\dot{x}} - 2i\,\Theta_{ab\dot{w}\dot{x}} + \frac{1}{2}\varepsilon_{ab}\varepsilon_{\dot{w}\dot{x}}(\Omega - \overline{\Omega})$$

$$+ \varepsilon_{\dot{w}\dot{x}}[(\Gamma_{f}{}^{ef\dot{r}} + \overline{\Gamma}_{\dot{s}}{}^{\dot{r}\dot{s}e})K_{a\dot{r}eb} + \Gamma_{a}{}^{ef\dot{r}}K_{e\dot{r}fb} + \Gamma_{b}{}^{ef\dot{r}}K_{a\dot{r}fe}]$$

$$- \varepsilon_{ab}[(\overline{\Gamma}_{\dot{s}}{}^{\dot{r}\dot{s}e} + \Gamma_{f}{}^{ef\dot{r}})\overline{K}_{\dot{w}e\dot{r}\dot{x}} + \overline{\Gamma}_{\dot{w}}{}^{\dot{r}\dot{s}e}\overline{K}_{\dot{r}e\dot{s}\dot{x}} + \overline{\Gamma}_{\dot{x}}{}^{\dot{r}\dot{s}e}\overline{K}_{\dot{w}e\dot{s}\dot{r}}]$$

$$-2K_{e\dot{w}f(a}\Gamma_{b)}{}^{fe}{}_{\dot{x}} - \overline{\Gamma}_{\dot{w}}{}^{\dot{r}}{}_{\dot{x}}{}^{e}K_{e\dot{r}ab} - \Gamma_{f}{}^{ef}{}_{\dot{x}}K_{e\dot{w}ab}$$

$$+2\overline{K}_{\dot{r}a\dot{s}(\dot{w}}\overline{\Gamma}_{\dot{x})}{}^{\dot{s}\dot{r}}{}_{b} + \Gamma_{a}{}^{e}{}_{b}{}^{\dot{r}}\overline{K}_{\dot{r}e\dot{w}\dot{x}} + \overline{\Gamma}_{\dot{s}}{}^{\dot{r}\dot{s}}{}_{b}\overline{K}_{\dot{r}a\dot{w}\dot{x}}$$

$$+ \epsilon_{\dot{w}\dot{x}} K^{e\dot{r}f}{}_{b} K_{a\dot{r}ef} - 2 K^{(e}{}_{\dot{x}}{}^{f)}{}_{b} K_{e\dot{w}af} + K_{a\dot{x}}{}^{ef} K_{e\dot{w}bf}$$

$$- \epsilon_{ab} \bar{K}^{\dot{r}e\dot{s}}{}_{\dot{x}} \bar{K}_{\dot{w}e\dot{r}\dot{s}} + 2 \bar{K}^{(\dot{r}}{}_{b}{}^{\dot{s})}{}_{\dot{x}} \bar{K}_{\dot{r}a\dot{w}\dot{s}} - \bar{K}_{\dot{w}b}{}^{\dot{r}\dot{s}} \bar{K}_{\dot{r}a\dot{x}\dot{s}}$$

$$+2 \bar{K}_{(\dot{r}|a|\dot{x})\dot{w}} K_{e}{}^{\dot{r}e}{}_{b} - 2 K_{(e|\dot{w}|b)a} \bar{K}_{\dot{r}}{}^{e\dot{r}}{}_{\dot{x}} + 2K_{[a}{}^{\dot{r}}{}_{e]b} \bar{K}_{\dot{w}}{}^{e}{}_{\dot{r}\dot{x}} - 2\bar{K}_{[\dot{w}}{}^{e}{}_{\dot{r}]\dot{x}} K_{a}{}^{\dot{r}}{}_{eb} \tag{5.11}$$

Written out explicitly, equation (5.10) yields the following set of equations which may also be obtained by expanding equation (5.11) and taking certain linear combinations.

$$\begin{aligned} D(\rho_1-\bar{\rho}_1) + \delta\bar{\kappa}_1 - \bar{\delta}\kappa_1 = {} & 2i\Theta_{00} + \rho(\rho_1+2\epsilon_1) - \bar{\rho}(\bar{\rho}_1+2\bar{\epsilon}_1) + (\epsilon+\bar{\epsilon})(\rho_1-\bar{\rho}_1) \\ & - \kappa(2\alpha_1-\bar{\tau}_1) + \bar{\kappa}(2\bar{\alpha}_1-\tau_1) + (3\bar{\alpha}+\beta-\bar{\pi})\bar{\kappa}_1 - (3\alpha+\bar{\beta}-\pi)\kappa_1 \\ & + \bar{\sigma}\sigma_1 - \sigma\bar{\sigma}_1 - (\rho_1-\bar{\rho}_1)(\rho_1+\bar{\rho}_1+\epsilon_1+\bar{\epsilon}_1) \\ & +\kappa_1(3\alpha_1+\bar{\beta}_1-\bar{\tau}_1-\pi_1) - \bar{\kappa}_1(3\bar{\alpha}_1+\beta_1-\tau_1-\bar{\pi}_1) \end{aligned} \tag{5.12a}$$

$$\begin{aligned} D(\bar{\alpha}_1-\beta_1) - \delta(\rho_1+\bar{\epsilon}_1-\epsilon_1) + \bar{\delta}\sigma_1 = {} & -2i\Theta_{01} + 2\Phi_0 + (\rho+\bar{\rho}+\epsilon-\bar{\epsilon})(\bar{\alpha}_1-\beta_1) \\ & -(\rho-\bar{\rho})\tau_1+\overline{\rho\pi_1}+\overline{\pi\rho_1}-(\bar{\alpha}+\beta)\rho_1 - \rho(\bar{\alpha}_1+\beta_1) + (\bar{\alpha}+\beta-\bar{\pi})(\epsilon_1-\bar{\epsilon}_1) \\ & + \kappa(\mu_1+\gamma_1-\bar{\gamma}_1)-\overline{\kappa\lambda_1}-\overline{\lambda\kappa_1} + \bar{\mu}\kappa_1 + (3\alpha-\bar{\beta}-\pi)\sigma_1 + \sigma(\alpha_1+\bar{\beta}_1-\pi_1) \\ & - (\rho_1+\bar{\rho}_1)(\bar{\alpha}_1-\beta_1) + (\rho_1-\bar{\rho}_1)(\tau_1+\bar{\pi}_1) - \kappa_1(\bar{\mu}_1+\gamma_1-\bar{\gamma}_1)+\bar{\kappa}_1\bar{\lambda}_1 \\ & + (2\bar{\alpha}_1-\bar{\pi}_1)(\rho_1+\bar{\epsilon}_1-\epsilon_1) - (2\alpha_1-\pi_1)\sigma_1 \end{aligned} \tag{b}$$

$$
\begin{aligned}
D(\mu_1-\bar{\mu}_1) + \delta(\alpha_1+\bar{\beta}_1-\pi_1) - \bar{\delta}(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) = 2i\Theta_{11} &+ 6i\Sigma - \Phi_1 + \bar{\Phi}_1 \\
&+ (\rho+\bar{\rho}-\varepsilon-\bar{\varepsilon})(\mu_1-\bar{\mu}_1)+(\rho-\bar{\rho})(\gamma_1+\bar{\gamma}_1)+\mu(\rho_1+\bar{\varepsilon}_1-\varepsilon_1) \\
&\qquad - \bar{\mu}(\bar{\rho}_1+\varepsilon_1-\bar{\varepsilon}_1) \\
&- \kappa\nu_1 + \overline{\kappa\nu}_1 - \lambda\sigma_1 + \overline{\lambda\sigma}_1 + (\bar{\alpha}-\beta)(\alpha_1+\bar{\beta}_1-\pi_1) \\
&\qquad - (\alpha-\bar{\beta})(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) \\
&- \bar{\pi}(2\bar{\beta}_1-\pi_1)+\pi(2\beta_1-\bar{\pi}_1)-(\rho_1+\bar{\rho}_1)(\mu_1-\bar{\mu}_1)-(\rho_1-\bar{\rho}_1)(\gamma_1+\bar{\gamma}_1) \\
&+\kappa_1\nu_1 - \bar{\kappa}_1\bar{\nu}_1 - (\bar{\alpha}_1-\beta_1)(\alpha_1+\bar{\beta}_1-\pi_1) + (\alpha_1-\bar{\beta}_1)(\bar{\alpha}_1+\beta_1-\bar{\pi}_1)
\end{aligned}
\tag{c}
$$

$$
\begin{aligned}
D(\tau_1-\bar{\alpha}_1-\beta_1)-\Delta\kappa_1+\delta(\varepsilon_1+\bar{\varepsilon}_1) = 2i\Theta_{01} &+ 2\Phi_0-\bar{\rho}(\bar{\alpha}_1+\beta_1+\bar{\pi}_1)-\sigma(\alpha_1+\bar{\beta}_1+\pi_1) \\
&+\tau(\rho_1+2\varepsilon_1)+\bar{\pi}(\rho_1-\bar{\rho}_1)+\bar{\tau}\sigma_1+(\varepsilon-\bar{\varepsilon})(\tau_1-\bar{\alpha}_1-\beta_1) \\
&+(\bar{\alpha}+\beta-\bar{\pi})(\varepsilon_1+\bar{\varepsilon}_1)+\kappa(\mu_1+\bar{\gamma}_1-\gamma_1)+(\mu-\bar{\gamma}-3\gamma)\kappa_1 + \overline{\kappa\lambda}_1+\overline{\lambda\kappa}_1 \\
&+\bar{\rho}_1(\bar{\alpha}_1+\beta_1+\bar{\pi}_1)-\rho_1(\tau_1+\bar{\pi}_1)-\sigma_1(\bar{\tau}_1-\alpha_1-\bar{\beta}_1)-(\varepsilon_1+\bar{\varepsilon}_1)(\bar{\alpha}_1+\beta_1-\bar{\pi}_1) \\
&-(\varepsilon_1-\bar{\varepsilon}_1)(\tau_1-\bar{\alpha}_1-\beta_1)\mp(\mu_1-2\gamma_1)\kappa_1-\bar{\lambda}_1\bar{\kappa}_1
\end{aligned}
\tag{d}
$$

$$D(\mu_1+\bar{\gamma}_1-\gamma_1)+\Delta(\bar{\rho}_1+\epsilon_1-\bar{\epsilon}_1)-\delta(\bar{\tau}_1+\pi_1) = -2i\Theta_{11}+6i\Sigma-\Phi_1-\bar{\Phi}_1$$

$$+\bar{\rho}(\mu_1+2\bar{\gamma}_1)-\mu(\bar{\rho}_1+2\epsilon_1)-(\epsilon+\bar{\epsilon})(\mu_1+\bar{\gamma}_1-\gamma_1)+(\gamma+\bar{\gamma})(\bar{\rho}_1+\epsilon_1-\bar{\epsilon}_1)$$

$$-(\tau+\bar{\alpha}-\beta-\bar{\pi})(\bar{\tau}_1+\pi_1)-(\bar{\tau}-\pi)(\bar{\alpha}_1+\beta_1)-(\tau+\bar{\pi})(\alpha_1-\bar{\beta}_1)+\overline{\tau\pi}_1$$

$$-\pi\tau_1-\overline{\kappa\nu}_1+\nu\kappa_1$$

$$-\overline{\lambda\sigma}_1+\sigma\lambda_1-\bar{\rho}_1(\gamma_1+\bar{\gamma}_1)+\mu_1(\epsilon_1+\bar{\epsilon}_1)-2\epsilon_1\gamma_1+2\bar{\epsilon}_1\bar{\gamma}_1$$

$$+(\tau_1-\bar{\pi}_1)(\bar{\tau}_1+\pi_1)$$

$$+(\tau_1+\bar{\pi}_1)(\alpha_1-\bar{\beta}_1)+\bar{\lambda}_1\bar{\sigma}_1-\sigma_1\lambda_1 \qquad \text{(e)}$$

$$D\lambda_1+\Delta\bar{\sigma}_1-\bar{\delta}(\bar{\tau}_1+\pi_1) = 2i\Theta_{20}-\lambda(\bar{\rho}_1+2\epsilon_1)+\bar{\sigma}(\mu_1+2\bar{\gamma}_1)-\bar{\tau}(\bar{\tau}_1+2\bar{\beta}_1)+\pi(\pi_1+2\alpha_1)$$

$$+(\bar{\alpha}-\beta)(\bar{\tau}_1+\pi_1)+(\rho+\bar{\epsilon}-3\epsilon)\lambda_1-(\bar{\mu}+\gamma-3\bar{\gamma})\bar{\sigma}_1-\bar{\kappa}\nu_1+\nu\bar{\kappa}_1$$

$$-\lambda_1(\rho_1-\bar{\rho}_1-3\epsilon_1+\bar{\epsilon}_1)-\bar{\sigma}_1(\mu_1-\bar{\mu}_1+3\bar{\gamma}_1-\gamma_1)-(\bar{\tau}_1+\pi_1)(\alpha_1-\bar{\beta}_1)$$

$$+\bar{\tau}_1^2-\pi_1^2 \qquad \text{(f)}$$

$$D\nu_1+\Delta(\alpha_1+\bar{\beta}_1-\pi_1)-\bar{\delta}(\gamma_1+\bar{\gamma}_1) = 2i\Theta_{21}-2\Phi_2-\bar{\mu}(\bar{\tau}_1+\alpha_1+\bar{\beta}_1)-\lambda(\tau_1+\bar{\alpha}_1+\beta_1)$$

$$+\pi(\mu_1+2\gamma_1)+\bar{\tau}(\mu_1-\bar{\mu}_1)+\bar{\pi}\lambda_1-(\gamma-\bar{\gamma})(\alpha_1+\bar{\beta}_1-\pi_1)$$

$$-(\bar{\tau}-\alpha-\bar{\beta})(\gamma_1+\bar{\gamma}_1)+\nu(\rho_1+\bar{\epsilon}_1-\epsilon_1)+(\rho-\bar{\epsilon}-3\epsilon)\nu_1+\bar{\nu}\bar{\sigma}_1+\overline{\sigma\nu}_1$$

$$+\bar{\mu}_1(\bar{\tau}_1+\alpha_1+\bar{\beta}_1)-\mu_1(\bar{\tau}_1+\pi_1)+\lambda_1(\bar{\alpha}_1+\beta_1-\bar{\pi}_1)+(\gamma_1+\bar{\gamma}_1)(\bar{\tau}_1-\alpha_1-\bar{\beta}_1)$$

$$+(\gamma_1-\bar{\gamma}_1)(\alpha_1+\bar{\beta}_1-\pi_1)-(\rho_1-2\epsilon_1)\nu_1-\bar{\sigma}_1\bar{\nu}_1 \qquad \text{(g)}$$

$$\Delta(\rho_1-\bar{\rho}_1)+\delta(\bar{\tau}_1-\alpha_1-\bar{\beta}_1)-\bar{\delta}(\tau_1-\bar{\alpha}_1-\beta_1) = -2i\Theta_{11}-6i\Sigma-\Phi_1+\bar{\Phi}_1$$

$$-(\mu+\bar{\mu}-\gamma-\bar{\gamma})(\rho_1-\bar{\rho}_1)-(\mu-\bar{\mu})(\varepsilon_1+\bar{\varepsilon}_1)-\rho(\mu_1+\bar{\gamma}_1-\gamma_1)+\bar{\rho}(\bar{\mu}_1+\gamma_1-\bar{\gamma}_1)$$

$$+\nu\kappa_1-\overline{\nu\kappa}_1+\sigma\lambda_1-\overline{\sigma\lambda}_1-(\alpha-\bar{\beta})(\tau_1-\bar{\alpha}_1-\beta_1)+(\bar{\alpha}-\beta)(\bar{\tau}_1-\alpha_1-\bar{\beta}_1)$$

$$+\bar{\tau}(2\bar{\alpha}_1-\tau_1)-\tau(2\alpha_1-\bar{\tau}_1)+(\mu_1+\bar{\mu}_1)(\rho_1-\bar{\rho}_1)+(\mu_1-\bar{\mu}_1)(\varepsilon_1+\bar{\varepsilon}_1)$$

$$-\kappa_1\nu_1+\bar{\kappa}_1\bar{\nu}_1+(\alpha_1-\bar{\beta}_1)(\tau_1-\bar{\alpha}_1-\beta_1)-(\bar{\alpha}_1-\beta_1)(\bar{\tau}_1-\alpha_1-\bar{\beta}_1) \tag{h}$$

$$\Delta(\alpha_1-\bar{\beta}_1)-\delta\lambda_1+\bar{\delta}(\mu_1+\bar{\gamma}_1-\gamma_1)=-2i\Theta_{21}-2\Phi_2-(\mu+\bar{\mu}+\gamma-\bar{\gamma})(\alpha_1-\bar{\beta}_1)$$

$$-(\mu-\bar{\mu})\pi_1+\overline{\mu\tau}_1+\overline{\tau\mu}_1-(\alpha+\bar{\beta})\mu_1-\mu(\alpha_1+\bar{\beta}_1)-(\bar{\tau}-\alpha-\bar{\beta})(\gamma_1-\bar{\gamma}_1)$$

$$+\nu(\rho_1+\varepsilon_1-\bar{\varepsilon}_1)-\overline{\nu\sigma}_1-\overline{\sigma\nu}_1+\bar{\rho}\nu_1-(\tau+\bar{\alpha}-3\beta)\lambda_1-\lambda(\tau_1-\bar{\alpha}_1-\beta_1)$$

$$+(\mu_1+\bar{\mu}_1)(\alpha_1-\bar{\beta}_1)+(\mu_1-\bar{\mu}_1)(\bar{\tau}_1+\pi_1)-\nu_1(\bar{\rho}_1+\varepsilon_1-\bar{\varepsilon}_1)+\bar{\nu}_1\bar{\sigma}_1$$

$$-(\bar{\tau}_1-2\bar{\beta}_1)(\mu_1+\bar{\gamma}_1-\gamma_1)+(\tau_1-2\beta_1)\lambda_1 \tag{i}$$

$$\Delta(\mu_1-\bar{\mu}_1)-\delta\nu_1+\overline{\delta\nu}_1=-2i\Theta_{22}-\mu(\mu_1+2\gamma_1)+\bar{\mu}(\bar{\mu}_1+2\bar{\gamma}_1)-(\gamma+\bar{\gamma})(\mu_1-\bar{\mu}_1)$$

$$+\nu(2\beta_1-\bar{\pi}_1)-\bar{\nu}(2\bar{\beta}_1-\pi_1)+(\bar{\tau}-\alpha-3\bar{\beta})\bar{\nu}_1-(\tau-\bar{\alpha}-3\beta)\nu_1$$

$$-\bar{\lambda}\lambda_1+\lambda\bar{\lambda}_1+(\mu_1-\bar{\mu}_1)(\mu_1+\bar{\mu}_1+\gamma_1+\bar{\gamma}_1)$$

$$+\nu_1(\tau_1-\bar{\alpha}_1-3\beta_1+\bar{\pi}_1)-\bar{\nu}_1(\bar{\tau}_1-\alpha_1-3\bar{\beta}_1+\pi_1) \tag{j}$$

Since the above identities are a feature of U_4 only, some comments on their interpretation are appropriate. They can, for example, simply be thought of as defining the new components of the curvature tensor which vanish in a V_4. In most applications the torsion tensor is determined algebraically by the spin density of the matter field. Thus once a tetrad or dyad basis is adopted, the components of the contortion tensor are determined. These can then be inserted into the above identities to determine the components of the curvature tensor which are peculiar to a U_4. On the other hand the above identities may also be regarded as integrability conditions on the components of the contortion tensor. It is this interpretation which has led to their description as "Bianchi identities for the torsion".

5.4 The Bianchi Identities for the Curvature

The Bianchi identity for the curvature (2.17) can be written in terms of its tetrad components as

$$\nabla_{[\ell}R_{mn]pq} = \gamma^{k}{}_{p[\ell}R_{mn]qk} - \gamma^{k}{}_{q[\ell}R_{mn]pk}$$

$$-2\,\gamma^{k}{}_{[mn}R_{\ell]kpq} + 2\,K_{[nm}{}^{k}R_{\ell]kpq} \tag{5.13}$$

The dyad equivalents of equation (5.13) consists of two equations obtained by taking the dyad components of equations (4.43) and (4.44) which give the Bianchi identities for the curvature in spinor form. We obtain

$$\nabla^{e}{}_{\dot{x}}\,\Psi_{abce} - \nabla_{(a}{}^{\dot{r}}\,[\Phi_{bc)\dot{x}\dot{r}} + i\,\Theta_{bc)\dot{x}\dot{r}}] - \frac{1}{2}\nabla_{(a|\dot{x}|}\Sigma_{bc)}$$

$$= [3\Psi_{ef(ab}\Gamma_{c)}{}^{ef}{}_{\dot{x}} + \Psi_{abce}\,\Gamma^{e}{}_{f}{}^{f}{}_{\dot{x}}] - 2\,\Gamma^{e}{}_{(ab}{}^{\dot{r}}[\Phi_{c)e\dot{x}\dot{r}} + i\,\Theta_{c)e\dot{x}\dot{r}}]$$

$$- \overline{\Gamma}_{\dot{x}}{}^{\dot{r}\dot{s}}{}_{(a}[\Phi_{bc)\dot{r}\dot{s}} + i\,\Theta_{bc)\dot{r}\dot{s}}] - \overline{\Gamma}^{\dot{r}}{}_{\dot{s}}{}^{\dot{s}}{}_{(a}[\Phi_{bc)\dot{x}\dot{r}} + i\,\Theta_{bc)\dot{x}\dot{r}}]$$

$$+ [\Psi^{ef}{}_{(ab} + \Sigma^{(e}{}_{(a}\delta^{f)}{}_{b} + \frac{\Omega}{3}\,\delta^{e}{}_{(a}\delta^{f}{}_{b}][2K_{|e\dot{x}|c)f} - K_{c)\dot{x}ef}]$$

$$- [\Phi_{(ab}{}^{\dot{r}\dot{s}} + i\,\Theta_{(ab}{}^{\dot{r}\dot{s}}][2\overline{K}_{|\dot{r}|c)\dot{x}\dot{s}} - \overline{K}_{|\dot{x}|c)\dot{r}\dot{s}}] - \Sigma_{e(a}\Gamma_{b}{}^{e}{}_{c)\dot{x}}$$

$$\tag{5.14}$$

$$\nabla^{e\dot{r}}(\Phi_{ea\dot{x}\dot{r}} + i\ \Theta_{ea\dot{x}\dot{r}}) - \frac{1}{2}\nabla_{a\dot{x}}\,\Omega - \nabla_{e\dot{x}}\,\Sigma_{a}{}^{e}$$

$$= [(\Phi_{ea\dot{x}\dot{r}} + i\ \Theta_{ea\dot{x}\dot{r}})\overline{\Gamma}_{s}{}^{\dot{r}\dot{s}e} + (\Phi_{ea\dot{r}\dot{s}} + i\ \Theta_{ea\dot{r}\dot{s}})\overline{\Gamma}_{x}{}^{\dot{r}\dot{s}e}]$$

$$+ [(\Phi_{ef\dot{x}\dot{r}} + i\ \Theta_{ef\dot{x}\dot{r}})\Gamma_{a}{}^{ef\dot{r}} + (\Phi_{ea\dot{x}\dot{r}} + i\ \Theta_{ea\dot{x}\dot{r}})\Gamma_{f}{}^{ef\dot{r}}]$$

$$+ \Psi_{efga}\,K^{e}{}_{\dot{x}}{}^{fg} - (\Phi_{ea\dot{r}\dot{s}} + i\ \Theta_{ea\dot{r}\dot{s}})(\overline{K}_{\dot{x}}{}^{e\dot{r}\dot{s}} - 2\overline{K}^{\dot{s}e\dot{r}}{}_{\dot{x}})$$

$$+ 2\,\Sigma_{e(f}\Gamma_{a)}{}^{ef}{}_{\dot{x}} + \Sigma_{ea}\,K_{f\dot{x}}{}^{ef} + \frac{1}{2}\Sigma_{ef}\,K_{a\dot{x}}{}^{ef} + \frac{2}{3}\,\Omega\,K_{e\dot{x}a}{}^{e}$$

(5.15)

The explicit expansions of the Bianchi identities for the curvature can now be written down in terms of the notations in chapters 3, 4 and 5. In V_4, linear combinations are usually taken such that the component Λ is eliminated from the first eight equations. In U_4 however, the equations are all considerably more lengthy and preference is therefore given to combinations that yield the shortest equations. The difference that this introduces is only in the components of the Ricci tensor, and therefore in the vacuum case the equations are in the standard form.

The Bianchi identities for the curvature are given below as twelve complex equations. These were obtained either by expanding equation (5.13) and taking certain linear combinations or by doing likewise with equations (5.14) and (5.15).

They could also have been obtained by taking intrinsic derivatives of the Ricci identities (5.9) and then taking linear combinations using the commutation relations (5.6) together with the Bianchi identities for the torsion (5.12). This approach has in fact been used to confirm the results obtained by the previously described method, thus providing an independent check on the system of equations (5.6), (5.9), (5.12) and (5.16).

$$
\begin{aligned}
&\bar{\delta}\Psi_0-D(\Psi_1-\Phi_{01}-i\Theta_{01}+\Phi_0)-\delta(\Phi_{00}+i\Theta_{00})\\
&\quad=(4\alpha-\pi-2\alpha_1+\pi_1)\Psi_0-2(2\rho+\varepsilon-\rho_1)\Psi_1+(3\kappa-\kappa_1)\Psi_2\\
&\quad-(2\bar{\alpha}+2\beta-\bar{\pi}-2\bar{\alpha}_1+\bar{\pi}_1)(\Phi_{00}+i\Theta_{00})+2(\bar{\rho}+\varepsilon-\bar{\rho}_1)(\Phi_{01}+i\Theta_{01})\\
&\quad+2\sigma(\Phi_{10}+i\Theta_{10})-2\kappa(\Phi_{11}+i\Theta_{11})-(\bar{\kappa}-\bar{\kappa}_1)(\Phi_{02}+i\Theta_{02})\\
&\quad-2\kappa_1(\Lambda+i\Sigma)-2(\varepsilon-\rho_1)\Phi_0+(\kappa-\kappa_1)\Phi_1
\end{aligned}
\tag{5.16(a)}
$$

$$
\begin{aligned}
&\Delta\Psi_0-\delta(\Psi_1+\Phi_{01}+i\Theta_{01}+\Phi_0)+D(\Phi_{02}+i\Theta_{02})\\
&\quad=(4\gamma-\mu-2\gamma_1+\mu_1)\Psi_0-2(2\tau+\beta-\tau_1)\Psi_1+(3\sigma-\sigma_1)\Psi_2\\
&\quad-(\bar{\lambda}-\bar{\lambda}_1)(\Phi_{00}+i\Theta_{00})+2(\bar{\pi}-\beta-\bar{\pi}_1)(\Phi_{01}+i\Theta_{01})+2\sigma(\Phi_{11}+i\Theta_{11})\\
&\quad+(2\varepsilon-2\bar{\varepsilon}+\bar{\rho}+2\bar{\varepsilon}_1-\bar{\rho}_1)(\Phi_{02}+i\theta_{02})-2\kappa(\Phi_{12}+i\Theta_{12})\\
&\quad-2\sigma_1(\Lambda+i\Sigma)-2(\beta-\tau_1)\Phi_0+(\sigma-\sigma_1)\Phi_1
\end{aligned}
\tag{b}
$$

$$\bar{\delta}(\Psi_1-\Phi_0)-D(\Psi_2-\Phi_{11}-i\Theta_{11}-\Lambda-i\Sigma)-\delta(\Phi_{10}+i\Theta_{10})$$

$$=\lambda\Psi_0+(2\alpha-2\pi-2\alpha_1+\pi_1)\Psi_1-(3\rho-2\rho_1)\Psi_2+(2\kappa-\kappa_1)\Psi_3$$

$$-\mu(\Phi_{00}+i\Theta_{00})+\pi(\Phi_{01}+i\theta_{01})-(2\bar{\alpha}-\bar{\pi}-2\bar{\alpha}_1+\bar{\pi}_1)(\Phi_{10}+i\theta_{10})$$

$$+2(\bar{\rho}-\bar{\rho}_1)(\Phi_{11}+i\theta_{11})+\sigma(\Phi_{20}+i\theta_{20})-\kappa(\Phi_{21}+i\Theta_{21})-(\bar{\kappa}-\bar{\kappa}_1)(\Phi_{12}+i\Theta_{12})$$

$$-2\rho_1(\Lambda+i\Sigma)-(2\alpha-2\alpha_1+\pi_1)\Phi_0+\rho\Phi_1-\kappa_1\Phi_2 \qquad \text{(c)}$$

$$\Delta(\Psi_1-\Phi_0)-\delta(\Psi_2+\Phi_{11}+i\theta_{11}-\Lambda-i\Sigma)+D(\Phi_{12}+i\theta_{12})$$

$$=\nu\Psi_0-(2\mu-2\gamma-\mu_1+2\gamma_1)\Psi_1-(3\tau-2\tau_1)\Psi_2+(2\sigma-\sigma_1)\Psi_3$$

$$-\mu(\Phi_{01}+i\Theta_{01})-(\bar{\lambda}-\bar{\lambda}_1)(\Phi_{10}+i\theta_{10})+2(\bar{\pi}-\bar{\pi}_1)(\Phi_{11}+i\Theta_{11})$$

$$+\pi(\Phi_{02}+i\Theta_{02})+(\bar{\rho}-2\bar{\epsilon}-\bar{\rho}_1+2\bar{\epsilon}_1)(\Phi_{12}+i\Theta_{12})+\sigma(\Phi_{21}+i\Theta_{21})-\kappa(\Phi_{22}+i\Theta_{22})$$

$$-2\tau_1(\Lambda+i\Sigma)-(2\gamma+\mu_1-2\gamma_1)\Phi_0+\tau\Phi_1-\sigma_1\Phi_2 \qquad \text{(d)}$$

$$\bar{\delta}(\Psi_2+\Phi_{11}+i\Theta_{11}-\Lambda-i\Sigma)-D(\Psi_3+\Phi_2)-\Delta(\Phi_{10}+i\Theta_{10})$$

$$=(2\lambda-\lambda_1)\Psi_1-(3\pi-2\pi_1)\Psi_2-(2\rho-2\epsilon-\rho_1+2\epsilon_1)\Psi_3+\kappa\Psi_4$$

$$-\nu(\Phi_{00}+i\Theta_{00})+\lambda(\Phi_{01}+i\Theta_{01})+(\bar{\mu}-2\bar{\gamma}-\bar{\mu}_1+2\bar{\gamma}_1)(\Phi_{10}+i\Theta_{10})$$

$$+2(\bar{\tau}-\bar{\tau}_1)(\Phi_{11}+i\Theta_{11})+\tau(\Phi_{20}+i\Theta_{20})-(\bar{\sigma}-\bar{\sigma}_1)(\Phi_{12}+i\Theta_{12})-\rho(\Phi_{21}+i\theta_{21})$$

$$-2\pi_1\Lambda+2(\pi-\pi_1)i\chi+\lambda_1\Phi_0-\pi\Phi_1+(2\epsilon+\rho_1-2\epsilon_1)\Phi_2 \qquad \text{(e)}$$

$$\Delta(\Psi_2-\Phi_{11}-i\Theta_{11}-\Lambda-i\Sigma)-\delta(\Psi_3+\Phi_2)+\bar{\delta}(\Phi_{12}+i\Theta_{12})$$

$$= (2\nu-\nu_1)\Psi_1-(3\mu-2\mu_1)\Psi_2+(2\beta-2\tau-2\beta_1+\tau_1)\Psi_3+\sigma\Psi_4$$

$$-\nu(\Phi_{01}+i\Theta_{01})-(\bar{\nu}-\bar{\nu}_1)(\Phi_{10}+i\Theta_{10})+2(\bar{\mu}-\bar{\mu}_1)(\Phi_{11}+i\Theta_{11})+\lambda(\Phi_{02}+i\Theta_{02})$$

$$-(2\bar{\beta}-\bar{\tau}-2\bar{\beta}_1+\bar{\tau}_1)(\Phi_{12}+i\Theta_{12})+\tau(\Phi_{21}+i\Theta_{21})-\rho(\Phi_{22}+i\Theta_{22})$$

$$-2\mu_1(\Lambda+i\Sigma)\quad+\nu_1\Phi_0-\mu\Phi_1+(2\beta-2\beta_1+\tau_1)\Phi_2 \qquad \text{(f)}$$

$$\bar{\delta}(\Psi_3+\Phi_{21}+i\Theta_{21}-\Phi_2)-D\Psi_4-\Delta(\Phi_{20}+i\Theta_{20})$$

$$= (3\lambda-\lambda_1)\Psi_2-2(2\pi+\alpha-\pi_1)\Psi_3+(4\varepsilon-\rho-2\varepsilon_1+\rho_1)\Psi_4$$

$$-2\nu(\Phi_{10}+i\Theta_{10})+2\lambda(\Phi_{11}+i\Theta_{11})+(2\gamma-2\bar{\gamma}+\bar{\mu}+2\bar{\gamma}_1-\bar{\mu}_1)(\Phi_{20}+i\Theta_{20})$$

$$+2(\bar{\tau}-\alpha-\bar{\tau}_1)(\Phi_{21}+i\Theta_{21})-(\bar{\sigma}-\bar{\sigma}_1)(\Phi_{22}+i\Theta_{22})$$

$$-2\lambda_1(\Lambda+i\Sigma)-(\lambda-\lambda_1)\Phi_1+2(\alpha-\pi_1)\Phi_2 \qquad \text{(g)}$$

$$
\begin{aligned}
&\Delta(\Psi_3-\Phi_{21}-i\Theta_{21}-\Phi_2)-\delta\Psi_4+\bar{\delta}(\Phi_{22}+i\Theta_{22})\\
&\quad=(3\nu-\nu_1)\Psi_2-2(2\mu+\gamma-\mu_1)\Psi_3+(4\beta-\tau-2\beta_1+\tau_1)\Psi_4\\
&\qquad-2\nu(\Phi_{11}+i\Theta_{11})-(\bar{\nu}-\bar{\nu}_1)(\Phi_{20}+i\Theta_{20})+2\lambda(\Phi_{12}+i\Theta_{12})\\
&\qquad+2(\bar{\mu}+\gamma-\bar{\mu}_1)(\Phi_{21}+i\Theta_{21})+(\bar{\tau}-2\bar{\beta}-2\alpha-\bar{\tau}_1+2\bar{\beta}_1)(\Phi_{22}+i\Theta_{22})\\
&\qquad-2\nu_1(\Lambda+i\Sigma)-(\nu-\nu_1)\Phi_1+2(\gamma-\mu_1)\Phi_2
\end{aligned}
\tag{h}
$$

$$
\begin{aligned}
&D(\Phi_{11}+i\Theta_{11}+\Phi_1+3\Lambda+3i\Sigma)-\delta(\Phi_{10}+i\Theta_{10})-\bar{\delta}(\Phi_{01}+i\Theta_{01}+2\Phi_0)+\Delta(\Phi_{00}+i\Theta_{00})\\
&\quad=\lambda_1\Psi_0-(2\alpha_1+\pi_1)\Psi_1+(\rho_1+2\varepsilon_1)\Psi_2-\kappa_1\Psi_3\\
&\qquad-(\mu+\bar{\mu}-2\gamma-2\bar{\gamma}-\bar{\mu}_1+2\bar{\gamma}_1)(\Phi_{00}+i\Theta_{00})+(\pi-2\alpha-2\bar{\tau}+2\bar{\tau}_1)(\Phi_{01}+i\Theta_{01})\\
&\qquad-(2\bar{\alpha}+2\tau-\bar{\pi}-2\bar{\alpha}_1+\bar{\pi}_1)(\Phi_{10}+i\Theta_{10})+2(\rho+\bar{\rho}-\bar{\rho}_1)(\Phi_{11}+i\Theta_{11})\\
&\qquad+(\bar{\sigma}-\bar{\sigma}_1)(\Phi_{02}+i\Theta_{02})+\sigma(\Phi_{20}+i\Theta_{20})-(\bar{\kappa}-\bar{\kappa}_1)(\Phi_{12}+i\Theta_{12})-\kappa(\Phi_{21}+i\Theta_{21})\\
&\qquad-4(\rho_1-\varepsilon_1)(\Lambda+i\Sigma)-(4\alpha-2\pi-2\alpha_1+3\pi_1)\Phi_0+(2\rho-\rho_1+2\varepsilon_1)\Phi_1-(2\kappa+\kappa_1)\Phi_2
\end{aligned}
\tag{i}
$$

$$
\begin{aligned}
&\Delta(\Phi_{01}+i\Theta_{01}-2\Phi_0)-\delta(\Phi_{11}+i\Theta_{11}-3\Lambda-3i\Sigma-\Phi_1)-\bar{\delta}(\Phi_{02}+i\Theta_{02})+D(\Phi_{12}+i\Theta_{12}) \\
&\quad = \nu_1\Psi_0-(\mu_1+2\gamma_1)\Psi_1+(\tau_1+2\beta_1)\Psi_2-\sigma_1\Psi_3 \\
&\qquad +(\bar{\nu}-\bar{\nu}_1)(\Phi_{00}+i\Theta_{00})-(\bar{\lambda}-\bar{\lambda}_1)(\Phi_{10}+i\Theta_{10})-(\mu+2\bar{\mu}-2\gamma-2\bar{\mu}_1)(\Phi_{01}+i\Theta_{01}) \\
&\qquad -2(\tau-\bar{\pi}+\bar{\pi}_1)(\Phi_{11}+i\Theta_{11})+(\pi+2\bar{\beta}-2\alpha-\bar{\tau}-2\bar{\beta}_1+\bar{\tau}_1)(\Phi_{02}+i\Theta_{02}) \\
&\qquad +(2\rho+\bar{\rho}-2\bar{\epsilon}-\bar{\rho}_1+2\bar{\epsilon}_1)(\Phi_{12}+i\Theta_{12})+\sigma(\Phi_{21}+i\Theta_{21})-\kappa(\Phi_{22}+i\Theta_{22}) \\
&\qquad +4(\beta_1-\tau_1)(\Lambda+i\Sigma)+(2\mu-4\gamma-3\mu_1+2\gamma_1)\Phi_0+(2\tau+2\beta_1-\tau_1)\Phi_1-(2\sigma+\sigma_1)\Phi_2
\end{aligned} \tag{j}
$$

$$
\begin{aligned}
&\Delta(\Phi_{10}+i\Theta_{10})-\delta(\Phi_{20}+i\Theta_{20})-\bar{\delta}(\Phi_{11}+i\Theta_{11}-3\Lambda-3i\Sigma+\Phi_1)+D(\Phi_{21}+i\Theta_{21}+2\Phi_2) \\
&\quad = \lambda_1\Psi_1-(2\alpha_1+\pi_1)\Psi_2+(\rho_1+2\epsilon_1)\Psi_3-\kappa_1\Psi_4 \\
&\qquad +\nu(\Phi_{00}+i\Theta_{00})-\lambda(\Phi_{01}+i\Theta_{01})-(2\mu+\bar{\mu}-2\bar{\gamma}-\bar{\mu}_1+2\bar{\gamma}_1)(\Phi_{10}+i\Theta_{10}) \\
&\qquad +2(\pi-\bar{\tau}+\bar{\tau}_1)(\Phi_{11}+i\Theta_{11})-(\tau+2\bar{\alpha}-2\beta-\bar{\pi}-2\bar{\alpha}_1+\bar{\pi}_1)(\Phi_{20}+i\Theta_{20}) \\
&\qquad +(\bar{\sigma}-\bar{\sigma}_1)(\Phi_{12}+i\Theta_{12})+(\rho+2\bar{\rho}-2\epsilon-2\bar{\rho}_1)(\Phi_{21}+i\Theta_{21})-(\bar{\kappa}-\bar{\kappa}_1)(\Phi_{22}+i\Theta_{22}) \\
&\qquad -4(\alpha_1-\pi_1)(\Lambda+i\Sigma)-(2\lambda+\lambda_1)\Phi_0+(2\pi+2\alpha_1-\pi_1)\Phi_1+(2\rho-4\epsilon-3\rho_1+2\epsilon_1)\Phi_2
\end{aligned} \tag{k}
$$

$$
\begin{aligned}
&\Delta(\Phi_{11}+i\Theta_{11}-\Phi_1+3\Lambda+3i\Sigma)-\bar{\delta}(\Phi_{12}+i\Theta_{12})-\delta(\Phi_{21}+i\Theta_{21}-2\Phi_2)+D(\Phi_{22}+i\Theta_{22}) \\
&\quad = \nu_1\Psi_1-(\mu_1+2\gamma_1)\Psi_2+(2\beta_1+\tau_1)\Psi_3-\sigma_1\Psi_4 \\
&\qquad +\nu(\Phi_{01}+i\Theta_{01})+(\bar{\nu}-\bar{\nu}_1)(\Phi_{10}+i\Theta_{10})-2(\mu+\bar{\mu}-\bar{\mu}_1)(\Phi_{11}+i\Theta_{11})-\lambda(\Phi_{02}+i\Theta_{02}) \\
&\qquad -(\bar{\lambda}-\bar{\lambda}_1)(\Phi_{20}+i\Theta_{20})+(2\bar{\beta}+2\pi-\bar{\tau}-2\bar{\beta}_1+\bar{\tau}_1)(\Phi_{12}+i\Theta_{12}) \\
&\qquad -(\tau-2\beta-2\bar{\pi}+2\bar{\pi}_1)(\Phi_{21}+i\Theta_{21})+(\rho+\bar{\rho}-2\epsilon-2\bar{\epsilon}-\bar{\rho}_1+2\bar{\epsilon}_1)(\Phi_{22}+i\Theta_{22}) \\
&\qquad +4(\mu_1-\gamma_1)(\Lambda+i\Sigma)-(2\nu+\nu_1)\Phi_0+(2\mu-\mu_1+2\gamma_1)\Phi_1-(4\beta-2\tau-2\beta_1+3\tau_1)\Phi_2
\end{aligned} \tag{l}
$$

Equations (5.6), (5.9), (5.12) and (5.16) give a complete set of identities generalising those of Newman and Penrose for a V_4. An alternative approach to the extension of the N-P identities has recently been given by Gambini and Herrera (1980). However they have chosen to work with the torsion tensor and have written its components explicitly. It has been found here to be more convenient to work with the contortion tensor since its components can be regarded as corrections to the V_4 spin coefficients and this interpretation can be used to simplify the notation. Furthermore Gambini and Herrera have not given the Bianchi identities for the torsion explicitly in terms of tetrad or dyad components.

CHAPTER 6

ON THE INTEGRATION OF THE GENERALISED N-P IDENTITIES

Now that a N-P type formalism for space-times with torsion has been developed, we must face the problem of getting it to work in the integration of the identities in order to obtain metrics. This being one of the main uses of the N-P formalism.

Although we are guided by the standard V_4 approach, we can expect some modifications due to the introduction of torsion which is usually specified algebraically, the increase in the number of field variables and the introduction of the Bianchi identities for the torsion.

The purpose of this chapter is to show how the formalism may be applied, by obtaining a number of simple exact solutions. The field equations of gravitational theories that include torsion, like for example the Einstein-Cartan theory, determine the non-zero components of the Ricci tensor and the torsion tensor. In this chapter however, no field equations are considered. Instead some simple geometries are derived by imposing certain severe conditions on the torsion and curvature. The geometries obtained need not be solutions of any specific theory. Consequently no physical importance is attached to the results obtained.

A distinction is drawn here between geometries and exact solutions of physical field equations. A specific choice of metric and torsion fields is referred to as a geometry. Assumptions on the torsion are trivial to impose because the generalised identities hold for any metric and torsion. The conditions on the curvature are nontrivial since they require the metric and torsion to satisfy certain differential equations.

6.1 Trivector Torsion

We first assume that the contortion tensor is a trivector, i.e. totally antisymmetric. In this case we may describe it in terms of the vector S_μ defined by

$$S_\mu = \frac{1}{3!}\,\varepsilon_{\mu\nu\kappa\lambda}\,K^{\nu\kappa\lambda}, \qquad K_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\,S^\lambda.$$

With this form for the contortion tensor, the contortion components (3.7) are given in terms of the four independent components $\rho_1 - \bar{\rho}_1$, $\mu_1 - \bar{\mu}_1$, and α_1 by

$$\kappa_1 = 0, \quad \rho_1 = -\bar{\rho}_1, \quad \sigma_1 = 0, \quad \tau_1 = -2\bar{\alpha}_1$$

$$\varepsilon_1 = -\tfrac{1}{2}\rho_1, \quad \beta_1 = \bar{\alpha}_1, \quad \gamma_1 = -\tfrac{1}{2}\mu_1$$

$$\pi_1 = -2\alpha_1, \quad \lambda_1 = 0, \quad \mu_1 = -\bar{\mu}_1, \quad \nu_1 = 0$$

Using the identity

$$\varepsilon_{\kappa\lambda\mu\nu} = -\,4!\;i\;\ell_{[\kappa}n_\lambda m_\mu \bar{m}_{\nu]}$$

together with equation (3.9d) it can be shown that S_μ is given by

$$S_\mu = i\,\mu_1\,\ell_\mu + 2\,i\,\alpha_1\,m_\mu - 2\,i\,\bar{\alpha}_1\,\bar{m}_\mu - i\,\rho_1\,n_\mu$$

With these independent components, the Bianchi identities for the torsion reduce to the following equations

$$D\rho_1 = (\varepsilon+\bar{\varepsilon})\rho_1 - 2\kappa\alpha_1 + 2\overline{\kappa\alpha_1} + i\theta_{00}$$

$$\delta\rho_1 = (\bar{\alpha}+\beta)\rho_1 - 2\sigma\alpha_1 + 2\overline{\rho\alpha_1} + i\theta_{01} - \Phi_0$$

$$D\mu_1 + 2\delta\alpha_1 - 2\overline{\delta\alpha_1} = (\rho+\bar{\rho}-\varepsilon-\bar{\varepsilon})\mu_1 + (\mu+\bar{\mu})\rho_1 + 2(\bar{\alpha}-\beta-\bar{\pi})\alpha_1$$
$$-2\,(\alpha-\bar{\beta}-\pi)\bar{\alpha}_1 + i\theta_{11} + 3\Sigma - \tfrac{1}{2}\,(\Phi_1-\bar{\Phi}_1)$$

$$2D\alpha_1 = \pi\rho_1 - 2\,(\varepsilon-\bar{\varepsilon})\alpha_1 + \bar{\kappa}\mu_1 + i\theta_{10} - \bar{\Phi}_0$$

$$D\mu_1 + 2\delta\alpha_1 - \Delta\rho_1 = (\bar{\rho}-\varepsilon-\bar{\varepsilon})\mu_1 + (\mu-\gamma-\bar{\gamma})\rho_1 + 2(\tau+\bar{\alpha}-\beta-\bar{\pi})\alpha_1$$
$$-2(\bar{\tau}-\pi)\bar{\alpha}_1 - i\theta_{11} + 3\Sigma - \tfrac{1}{2}(\Phi_1+\bar{\Phi}_1)$$

$$2\bar{\delta}\alpha_1 = \lambda\rho_1 - 2(\bar{\alpha}-\beta)\alpha_1 + \bar{\sigma}\mu_1 + i\theta_{20}$$

$$2\Delta\alpha_1 = \nu\rho_1 - 2(\gamma-\bar{\gamma})\alpha_1 + \bar{\tau}\mu_1 + i\theta_{21} - \Phi_2$$

$$\Delta\rho_1 - 2\delta\alpha_1 + 2\overline{\delta\alpha_1} = -(\mu+\bar{\mu}-\gamma-\bar{\gamma})\rho_1 - (\rho+\bar{\rho})\mu_1 + 2(\alpha-\bar{\beta}+\bar{\tau})\bar{\alpha}_1$$
$$-2(\bar{\alpha}-\beta+\tau)\alpha_1 - i\theta_{11} - 3\Sigma - \tfrac{1}{2}(\Phi_1-\bar{\Phi}_1)$$

$$\bar{\delta}\mu_1 = -(\alpha+\bar{\beta})\mu_1 - 2\bar{\mu}\alpha_1 + 2\lambda\bar{\alpha}_1 - i\theta_{21} - \Phi_2$$

$$\Delta\mu_1 = -(\gamma+\bar{\gamma})\mu_1 + 2\nu\bar{\alpha}_1 - 2\bar{\nu}\alpha_1 - i\theta_{22}$$

These equations are now further reduced by assuming that S_μ is a null vector and that it is continuously defined. The latter condition is equivalent to assuming that the spin of the source field which gives rise to the contortion in space-time is locally aligned. It is then possible to align the tetrad vector ℓ_μ with S_μ, and this implies the conditions

$$\rho_1 = 0, \qquad \alpha_1 = 0$$

Therefore $\mu_1 - \bar{\mu}_1 = -2\,(\gamma_1 - \bar{\gamma}_1)$ is now the only remaining non-zero contortion component.

At this stage, conditions are imposed on the curvature tensor in order to obtain a fairly simple geometry. It is assumed that the only non-zero component of the Ricci tensor is Φ_{22}, i.e.

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = \Phi_{12} = \Lambda = \Phi_0 = \Phi_1 = \Phi_2 = 0,$$

and that the only non-zero components of the tensor $C_{\kappa\lambda\mu\nu}$ are Ψ_4 θ_{22}. The latter condition can be expressed in the form

$$A_{\kappa\lambda\mu\nu}\,\ell^\nu = 0, \quad B_{\kappa\lambda\mu\nu}\,\ell^\nu = 0, \qquad D = 0$$

With these constraints, the Bianchi identities for the torsion imply that

$$\kappa = 0, \quad \rho = 0, \quad \sigma = 0, \quad \tau = 0$$

$$D\mu_1 = -(\varepsilon+\bar{\varepsilon})\mu_1$$

$$\delta\mu_1 = -(\bar{\alpha}+\beta)\mu_1$$

$$\Delta\mu_1 = -(\gamma+\bar{\gamma})\mu_1 - i\theta_{22}$$

Thus ℓ_μ is tangent to an autoparallel null congruence for which the expansion, twist and shear are all zero.

Note that for a trivector torsion, autoparallels and extremals coincide (see theorem in Section 2.2).

The Bianchi identities for the curvature are now reduced to

$$D\Psi_4 = 4\varepsilon\Psi_4$$

$$\delta\Psi_4 - \bar{\delta}(\Phi_{22}+i\theta_{22}) = -4\beta\Psi_4 + 2(\alpha+\bar{\beta})(\Phi_{22}+i\theta_{22})$$

$$D(\Phi_{22}+i\theta_{22}) = -2(\varepsilon+\bar{\varepsilon})(\Phi_{22}+i\theta_{22})$$

So far only the direction of the vector ℓ_μ has been specified, and we can therefore make use of the tetrad transformations (given in Appendix A)

$$\tilde{\ell}_\mu = R\,\ell_\mu$$

$$\tilde{m}_\mu = e^{is}\,(m_\mu + RT\ell_\mu)$$

$$\tilde{n}_\mu = R^{-1}\,n_\mu + \bar{T}\,m_\mu + T\bar{m}_\mu + RT\bar{T}\,\ell_\mu$$

Since κ^0 and ρ^0 are zero, we have the result from section 3.1 that ℓ_μ is proportional to the gradient of a scalar field. The freedom in R above can then be used to make ℓ_μ equal to a gradient, which imposes the conditions

$$\varepsilon + \bar{\varepsilon} = 0, \quad \bar{\alpha} + \beta = 0$$

It can be seen from equations (A.4) that if these are to be preserved then R must be restricted by

$$DR = 0, \quad \delta R = 0$$

The condition $\varepsilon+\bar{\varepsilon}=0$ implies that the autoparallel congruence to which ℓ_μ is tangent is affinely parameterised. It is also possible to use the freedom in S to put both $\varepsilon-\bar{\varepsilon}= 0$ and $\bar{\alpha}-\beta= 0$. To show this, we have from (A.4) that

$$\tilde{\varepsilon}-\bar{\tilde{\varepsilon}} = \varepsilon-\bar{\varepsilon} + iDS$$

$$\tilde{\beta}-\bar{\tilde{\alpha}} = e^{is} (\beta-\bar{\alpha}+i\delta S)$$

Thus we want

$$DS = i(\varepsilon-\bar{\varepsilon})$$

$$\delta S = i(\beta-\bar{\alpha})$$

These can only hold simultaneously provided the commutation relation [(5.6(b))]

$$(\delta D - D\delta)S = - \bar{\pi}DS - (\varepsilon-\bar{\varepsilon})\delta S$$

is satisfied identically. This condition implies that

$$\delta\varepsilon - \delta\bar{\varepsilon} - D\beta + D\bar{\alpha} = -\bar{\pi}\,(\varepsilon-\bar{\varepsilon})-(\varepsilon-\bar{\varepsilon})(\beta-\bar{\alpha})$$

which can be shown to be identically satisfied using the Ricci identities (5.9)(d) and (e). In order to preserve the conditions $\varepsilon-\bar{\varepsilon}= 0$ and $\bar{\alpha}-\beta= 0$, the function S must in future be restricted by

$$DS = 0, \quad \delta S = 0$$

We now have $\varepsilon=\alpha=\beta= 0$. It is now also possible to use the freedom in T to put $\pi=\lambda= 0$. This can be done following analogously the steps above except that in this case the transformations (A.6) are used. T must in future be restricted by

$$DT = 0, \quad \delta T = 0$$

The remaining, non-zero, spin coefficients are now given by

$$\gamma = \gamma^{0} - \tfrac{1}{2}\, i\, M$$

$$\nu = \nu^{0}$$

$$\mu = \mu^{0} + i\, M$$

where

$$\mu_1 = -2\gamma_1 = i\, M .$$

If we denote the affine parameter along ℓ_μ by r and put $\ell_\mu = \partial_\mu u$, we may adopt the Robinson and Trautman [1960] coordinate system $(x^1, x^2, x^3, x^4) = (u, r, x, y)$. The tetrad can then be written in the form

$$\ell^\mu = \delta_2{}^\mu, \qquad \ell_\mu = \delta_\mu{}^1$$

$$n^\mu = \delta_1{}^\mu + U\,\delta_2{}^\mu + X^3\,\delta_3{}^\mu + X^4\,\delta_4{}^\mu$$

$$m^\mu = \omega\,\delta_2{}^\mu + \xi^3\,\delta_3{}^\mu + \xi^4\,\delta_4{}^\mu$$

and the metric equations (obtained by substituting the coordinates into the commutations relations (5.6)) become

$$D\xi^i = 0$$

$$D\omega = 0$$

$$DX^i = 0$$

$$DU = -(\gamma^0 + \bar{\gamma}^0)$$

$$\delta U - \Delta\omega = -\bar{\nu}^0 + (\mu^0 - \gamma^0 + \bar{\gamma}^0)\omega$$

$$\delta X^i - \Delta\xi^i = (\mu^0 - \gamma^0 + \bar{\gamma}^0)\xi^i$$

$$\delta\bar{\xi}^i - \bar{\delta}\xi^i = 0$$

$$\delta\bar{\omega} - \bar{\delta}\omega = \mu^0 - \bar{\mu}^0$$

where i = 3, 4, X^i and U are real scalar functions, ω and ξ^i are complex scalar functions. The metric equations imply that ξ^i, ω

and X^i are all independent of r. The above form for the tetrad vectors is preserved under the following coordinate transformations [Newman and Tamburino, 1962]

$$\tilde{u} = \tilde{u}(r, x^3, x^4)$$

$$\tilde{r} = r + h(u, x^3, x^4)$$

$$\tilde{X}^i = \tilde{X}^i(u, x^3, x^4)$$

and

$$\tilde{u} = f(u), \ \tilde{r} = r/\partial_u f(u), \ \ \tilde{\ell}_\mu = \partial_u f(u)\ \ell_\mu, \ \tilde{n}_\mu = \frac{n_\mu}{\partial_u f(u)}$$

It is convenient at this stage to introduce the complex coordinate $Z = x + iy$. The coordinate transformation $\tilde{Z} = \tilde{Z}(u, z, \bar{z})$ can then be used to put

$$\xi^3 = P, \qquad \xi^4 = i\,P$$

This coordinate is now defined up to the transformation $\tilde{Z} = \tilde{Z}(u,z)$ and the u dependence of this can be used to put

$$X^i = 0$$

(The rigorous mathematical steps of the Z-coordinate transformations are given in Trim, 1972 and Collinsen and Morris, 1972). The remaining freedom in Z is given by $\tilde{Z} = \tilde{Z}(z)$.

The Ricci identities now reduce to:

$$D\gamma = 0, \qquad D\mu = 0, \qquad D\nu = 0$$

$$\delta\gamma = 0, \qquad \bar{\delta}\gamma = 0, \qquad \bar{\delta}\mu = 0$$

$$\bar{\delta}\nu = \Psi_4$$

$$\delta\nu - \Delta\rho = \mu(\mu+\gamma+\bar{\gamma}) - 2\mu\mu_1 + \Phi_{22} + i\theta_{22}$$

where

$$D = \frac{\partial}{\partial r}, \quad \Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r}, \quad \delta = \omega\frac{\partial}{\partial r} + 2P\frac{\partial}{\partial \bar{z}}$$

Thus the Bianchi identities for the torsion and the Ricci identities imply that $\mu^0 = \mu^0(u, \bar{z})$. We may therefore use the remaining freedom in T to put

$$\mu^0 = 0$$

Similarly we may use the remaining freedom in S to put

$$\gamma^0 - \bar{\gamma}^0 = 0$$

and the transformation

$$\tilde{u} = f(u), \quad \text{etc.}$$

to make

$$\gamma^0 + \bar{\gamma}^0 = 0$$

The coordinate transformation $\tilde{r} = r + h(u, z, \bar{z})$ can be used to put

$$\omega = 0$$

(See Collinson and Morris, 1972). The metric equations imply that $P = P(\bar{z})$ and therefore the remaining z-coordinate freedom $\tilde{z} = \tilde{z}(z)$ can be used to set

$$P = 2^{-\frac{1}{2}}$$

The tetrad is now given by

$$\ell^\mu = \delta_2{}^\mu \qquad \ell_\mu = \delta_\mu{}^1$$

$$n^\mu = \delta_1{}^\mu + U\,\delta_2{}^\mu \qquad n_\mu = -U\,\delta_\mu{}^1 + \delta_\mu{}^2$$

$$m^\mu = 2^{-\frac{1}{2}}\,(\delta_3{}^\mu + i\delta_4{}^\mu) \qquad m_\mu = -2^{-\frac{1}{2}}\,(\delta_\mu{}^3 + i\delta_\mu{}^4)$$

The non-zero spin coefficients and contortion components are

$$\mu = \mu_1 = -2\gamma = -2\gamma_1 = i\,M$$

$$\nu = \nu^0$$

The remaining metric equations, Ricci identities, Bianchi identities for the torsion and curvature are now given by

$$DU = 0, \qquad 2^{\frac{1}{2}}\,\frac{\partial}{\partial z}\,U = -\nu$$

$$D\nu = 0, \qquad 2^{\frac{1}{2}}\,\frac{\partial}{\partial z}\,\nu = \Psi_4, \qquad 2^{\frac{1}{2}}\,\frac{\partial}{\partial \bar{z}}\,\nu = M^2 + \Phi_{22}$$

$$DM = 0, \qquad \frac{\partial}{\partial z} M = 0, \qquad \frac{\partial}{\partial u} M = - \theta_{22}$$

$$D\Psi_4 = 0, \qquad D(\Phi_{22} + i\theta_{22}) = 0, \quad \frac{\partial}{\partial \bar{z}} \Psi_4 - \frac{\partial}{\partial z} (\Phi_{22} + i\theta_{22}) = 0$$

Substituting for ν above we obtain the non-zero components of the curvature tensor

$$\Psi_4 = -2 \frac{\partial^2}{\partial z^2} U$$

$$\Phi_{22} = -M^2 - 2 \frac{\partial^2}{\partial z \partial \bar{z}} U$$

$$\theta_{22} = - \frac{\partial}{\partial u} M.$$

where $M = M(u)$ and $U = U(u, z, \bar{z})$. The last equation above is now identically satisfied. The metric, obtained through the completeness relation (3.3), is the plane wave metric given by

$$ds^2 = 2\, du\, dr - 2U\,(u, z, \bar{z})\, du^2 - dz\, d\bar{z}$$

and contains a single arbitrary function $U(u, z, \bar{z})$. Finally, the contortion tensor is given by

$$K_{\lambda\mu\nu} = 6\, i\, M(u)\ \ell_{[\lambda} m_{\mu} \bar{m}_{\nu]}$$

It is interesting to note that for this solution if $U = 0$ the metric is flat, but there remains a finite curvature generated through the presence of torsion.

6.2 Semisymmetric Torsion

It is possible to repeat the calculation of section 6.1, replacing the assumption that the contortion is a trivector by the assumption that it has a semisymmetric form given by

$$K_{\lambda\mu\nu} = -2\ (g_{\lambda\mu}\ a_\nu - g_{\lambda\nu}\ a_\mu)$$

where $a_\mu = \frac{1}{6} K_{\alpha\mu}{}^{\alpha}$.

In addition, a_μ is assumed to be a continuous null vector. The tetrad vector ℓ_μ is aligned with a_μ, and the contortion tensor then has only one independent real component given by

$$\mu_1 + \bar{\mu}_1 = -2(\gamma_1 + \bar{\gamma}_1) = -4\ a_\alpha\ n^\alpha.$$

We may now proceed to make the remaining assumptions of Section 6.1, and we find that the Bianchi identities for the torsion imply that

$$\kappa = 0, \quad \rho = 0, \quad \sigma = 0, \quad \tau = 0$$

$$D\mu_1 = -\ (\varepsilon+\bar{\varepsilon})\mu_1$$

$$\delta\mu_1 = -\ (\bar{\alpha}+\beta)\mu_1$$

$$\theta_{22} = 0$$

Thus again ℓ_μ is tangent to an autoparallel null congruence for which the expansion, twist and shear are all zero. We therefore proceed exactly as before obtaining the same metric, but this time the contortion tensor is given in terms of

$$a_\mu = -\tfrac{1}{2} N(u)\, \ell_\mu ,$$

where $N = \mu_1$ is a real function, and the non-zero components of the curvature tensor are

$$\Psi_4 = -2 \frac{\partial^2}{\partial z^2} U$$

$$\Phi_{22} = N^2 - \frac{\partial N}{\partial u} - 2 \frac{\partial^2}{\partial z \partial \bar{z}} U$$

6.3 Discussion

It is possible to combine the two geometries obtained in Sections 6.1 and 6.2. We then have that the contortion tensor

$$K_{\lambda\mu\nu} = N(u)\ (g_{\lambda\mu}\ \ell_\nu - g_{\lambda\nu}\ \ell_\mu) + M(u)\ \ell^\kappa\ \varepsilon_{\kappa\lambda\mu\nu}$$

is consistent with the metric

$$ds^2 = 2\ du\ dr - 2\ U(u,\ z,\ \bar{z})du^2 - dz\ d\bar{z}$$

and the non-zero components of the curvature tensor

$$\Psi_4 = -2\ \frac{\partial^2}{\partial z^2}\ U$$

$$\Phi_{22} = N^2 - M^2 - \frac{\partial}{\partial u}\ N - 2\ \frac{\partial^2}{\partial z \partial \bar{z}}\ U$$

$$\Theta_{22} = -\ \frac{\partial}{\partial u}\ M.$$

It is also instructive to compare the latter with their values in a V_4. These are given (using the identities in Appendix B) by

$$\Psi_4 = \Psi_4{}^o$$

$$\Phi_{22} = \Phi_{22}{}^o + N^2 - M^2 - \frac{\partial N}{\partial u}$$

It can clearly be seen now that in the associated V_4, the geometries obtained correspond to the familiar plane-parallel wave solutions of the Einstein-Maxwell field equations describing null electromagnetic and gravitational plane waves.

The authors Baker, Atkins and Davis (1978) have attempted to reinterpret plane null electromagnetic solutions, of the standard Einstein-Maxwell equations, in U_4 in terms of the two types of torsion tensors considered above. They have considered a flat U_4 consistent with a plane wave metric. In fact the arguments presented here show that the plane-wave metric which they introduce is the only possible metric that is compatible with their other assumptions.

In the combined solution above, if U_4 is flat then the associated V_4 is conformally flat with

$$\Phi_{22}{}^{o} = M^2 - N(u)^2 - \frac{\partial N(u)}{\partial u}$$

$$U = \tfrac{1}{2}\left(M^2 - N(u)^2 - \frac{\partial N(u)}{\partial u}\right) z\bar{z} + f_1(u)\, z + f_2(u)\, \bar{z} + f_3(u)$$

and we have the same result as Baker, Atkins and Davis except that this solution is obtained with greater ease and generality (since the initial assumptions of this chapter lead uniquely to a plane-wave metric).

Although the formal results of this chapter are not physically important, they should be considered as a convenient step in developing the formalism. The techniques developed here should also be useful for the generation of realistic exact solutions of gravitational field theories involving torsion. In fact the results of the trivector torsion case will be useful later because this is exactly the type of torsion induced by a Neutrino field in the Einstein-Cartan theory.

CHAPTER 7

THE EINSTEIN-CARTAN THEORY

The Einstein-Cartan theory, as proposed by E Cartan (1922, 1923), is a slight modification of Einstein's theory. It incorporates spin from the beginning as a dynamical quantity in the theory of gravitation. In it, spin and the geometrical concept of torsion are related algebraically. As a consequence of this relation, the torsion must be zero outside matter, and the vacuum field equations are identical to those of standard general relativity.

The purpose of this brief chapter is simply to present the field equations of the Einstein-Cartan theory. It contains no new material, but must necessarily be included since in the following two chapters the formalism developed in the first half of this thesis is applied to particular source fields in the framework of the Einstein-Cartan theory.

Physical and mathematical arguments in favour of the Einstein-Cartan theory have been given in Hehl (1973, 1974), Trautman (1972, a,b,c; 1973, a,b), Kuchowicz (1975) and Hehl et al (1976). The latter contains further relevant references in connection with this theory and with other theories involving torsion. This chapter follows mainly the notation of Hehl (1973, 1974).

Let us assume that space-time is described by the Riemann-Cartan manifold U_4 which has a locally Lorentz metric and a linear connection which is not necessarily symmetric. We start with a

Lagrangian density for the matter field in the framework of special relativity theory. By applying the principal of minimal gravitational coupling, we are led to the desired matter Lagrangian for the Einstein-Cartan theory. This procedure merely involves introducing the U_4 metric $g_{\mu\nu}$ and replacing partial derivatives by covariant ones with respect to the connection $\Gamma_{\mu\nu}{}^{\lambda}$.

If ψ denotes the matter field, then its Lagrangian density in special relativity is given by $\mathcal{L} = \mathcal{L}(\psi, \partial\psi, \eta)$ where η is the constant Minkowski metric. In general relativity, minimal coupling leads to the Lagrangian density $\mathcal{L}(\psi, \nabla^0\psi, g) = \mathcal{L}(\psi, \partial\psi, g, \partial g)$. The corresponding total action function over the sum of two scalar densities is given by

$$W = \int d_4x \; [\mathcal{L}(\psi, \partial\psi, g, \partial g) - \frac{1}{2K}\mathcal{R}(g, \partial g, \partial\partial g)]$$

Here, $\mathcal{R} = \sqrt{-g}\;R$ is the familiar density of the curvature scalar of the Riemannian space-time V_4 and $k = 8\pi\;G\;c^{-4}$.

In the Einstein-Cartan theory, field equations are derivable through the total action function

$$W = \int d_4x \; [\mathcal{L}(\psi, \partial\psi, g, \partial g, K) - \frac{1}{2K}\mathcal{R}(g, \partial g, \partial\partial g, K, \partial K)] \qquad (7.1)$$

where $\mathcal{L}(\psi, \nabla\psi, g) = \mathcal{L}(\psi, \partial\psi, g, \partial g, K)$ is the Lagrangian density of the matter field with non-zero spin density, $\mathcal{R} = \sqrt{-g}\;R$ is the density of the curvature scalar of the Riemann-Cartan space-time U_4 and K is a symbolic representation of the contortion tensor $K_{\mu\nu}{}^{\lambda}$.

It should be noticed that the contortion tensor appearing in (7.1) is treated as a new independent variable. Therefore the action function depends on the independent variables

$$\psi,\ g_{\mu\nu}\ \text{(10 components)},\qquad K_{\mu\nu}{}^{\lambda}\ \text{(24 components)} \tag{7.2}$$

If we vary the action function (7.1) with respect to the independent variables (7.2), we obtain the matter field equations

$$\frac{\delta\mathcal{L}}{\delta\psi} = 0 \tag{7.3}$$

and the 10 plus 24 independent field equations

$$\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} = \frac{1}{2k}\frac{\delta\mathcal{R}}{\delta g_{\mu\nu}}$$

$$\frac{\delta\mathcal{L}}{\delta K_{\mu\nu}{}^{\kappa}} = \frac{1}{2k}\frac{\delta\mathcal{R}}{\delta K_{\mu\nu}{}^{\kappa}}\,.$$

Now the dynamical <u>metric</u> energy-momentum tensor of the matter field is defined by

$$\sigma^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \tag{7.4}$$

and the dynamical spin angular momentum tensor is defined by

$$\tau_{\kappa}{}^{\nu\mu} = \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta K_{\mu\nu}{}^{\kappa}} \tag{7.5}$$

These can be determined for any given matter field if a suitable Lagrangian is known. It is convenient at this point to introduce the differential operator

$$\overset{*}{\nabla}_{\mu} = \nabla_{\mu} + K_{\alpha\mu}{}^{\alpha} \tag{7.6}$$

It can be shown by direct calculation (Hehl, 1974) that

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta g_{\mu\nu}} = -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R + \overset{*}{\nabla}_{\kappa} (K^{\mu\nu\kappa} + g^{\mu\nu} K^{\lambda\kappa}{}_{\lambda} - g^{\mu\kappa} K^{\lambda\nu}{}_{\lambda})$$

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta K_{\mu\nu}{}^{\kappa}} = -K_{\kappa}{}^{\nu\mu} + K^{\nu}{}_{\kappa}{}^{\mu} + \delta^{\mu}_{\kappa} K_{\lambda}{}^{\mu\lambda} - g^{\mu\nu} K_{\lambda\kappa}{}^{\lambda}$$

It can also be shown that the first of these expressions is symmetric as required. Substituting these expressions together with the definitions (7.4) and (7.5) into the field equations above we obtain

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \overset{*}{\nabla}_{\kappa} (K^{\mu\nu\kappa} + g^{\mu\nu} K^{\lambda\kappa}{}_{\lambda} - g^{\mu\kappa} K^{\lambda\nu}{}_{\lambda}) = -k\sigma^{\mu\nu} \tag{7.7}$$

$$\frac{1}{2} (K_{\kappa\nu\mu} - K_{\nu\kappa\mu} - g_{\mu\kappa} K_{\lambda\nu}{}^{\lambda} + g_{\mu\nu} K_{\lambda\kappa}{}^{\lambda}) = -k\, \tau_{\kappa\nu\mu} \tag{7.8}$$

These field equations can be rewritten in the convenient form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k \sum_{\mu\nu} \tag{7.9}$$

$$K_{\mu\nu\kappa} = -k\,(\tau_{\mu\nu\kappa} - \tau_{\nu\kappa\mu} + \tau_{\kappa\mu\nu} + g_{\mu\nu}\,\tau_{\alpha\kappa}{}^{\alpha} - g_{\mu\kappa}\,\tau_{\alpha\nu}{}^{\alpha}) \quad (7.10)$$

where $\sum_{\mu\nu}$ is the canonical energy momentum tensor defined by

$$\textstyle\sum^{\mu\nu} = \sigma^{\mu\nu} + \overset{*}{\nabla}_{\kappa}\,(\tau^{\mu\nu\kappa} - \tau^{\nu\kappa\mu} + \tau^{\kappa\mu\nu}) \quad (7.11)$$

Equations (7.9) and (7.10) are the Einstein-Cartan field equations.

The equation (7.10) is an algebraic relation between the torsion of the manifold and the dynamical spin density of the source field indicating that torsion is present at all points of space-time where spin is present. Thus for a particular matter field the contortion tensor is uniquely determined, and outside of matter the torsion vanishes.

Equation (7.9) can be seen to resemble Einstein's familiar equation. However contortion and spin angular momentum components are also included. It is important to notice that the components of this equation are not necessarily symmetric. However the anti-symmetric part of this equation is identically satisfied as a consequence of the field equation (7.10) or (7.8) and the definition (7.11). In fact the vanishing of the anti-symmetric part of (7.9) can be interpreted as a conservation equation for the local spin-angular momentum of the field (Kuchowicz, 1975).

Hehl (1974) has shown that in Einstein-Cartan theory, the process of minimal coupling cannot be applied to the lagrangian of an electromagnetic field as this would lead to the violation of gauge invariance. Therefore, in order to preserve gauge invariance, we assume that the e-m field does not produce nor feel torsion. Hence, when considering the applications of the spin coefficient formalism in Einstein-Cartan theory, it would not be fruitful to investigate electromagnetic fields in vacuum. Torsion would not be present in this case and solutions would be equal to those of general relativity.

In order to demonstrate the applicability of the formalism developed in this thesis we consider classical neutrino fields and semi-classical spin fluids. Since these sources give rise to torsion in Einstein-Cartan theory, they can be used to demonstrate some new features and additional difficulties that arise in the generalised formalism. These are illustrated in the following two chapters.

CHAPTER 8

CLASSICAL NEUTRINO FIELDS IN THE EINSTEIN-CARTAN THEORY

A classical Neutrino field is a particularly simple example of a source field with non-zero spin density. In the Einstein-Cartan theory, this spin density is related algebraically to the torsion of the manifold U_4. We assume that a Neutrino is a Dirac particle with zero mass, zero charge and spin $\frac{1}{2}$. Such a particle is described by a single two-component spinor which satisfies the Weyl equation. The latter is simply the Dirac equation with the mass and charge parameters put to zero. A Neutrino field can then be treated as a classical field. Such an approach has already been considered in general relativity. This work has been reviewed by Kuchowicz (1974). In this chapter, the same approach is considered in the Einstein-Cartan theory.

In general relativity, Neutrino fields are known to have a number of anomalous properties (Griffiths, 1980a). For example, the sign of the energy density of a Neutrino field is usually observer dependent and it may also change as the field propagates. The complete energy-momentum tensor may even vanish while the Neutrino current vector remains non-zero. Exact solutions for which this occurs are known as ghost Neutrino's. By using the spin coefficient approach developed here, Griffiths (1981) has shown that these anomalies do not occur as readily in the Einstein-Cartan theory (see also, Griffiths 1980b). This chapter is a review of his work.

8.1 Field Equations

The field equations for Neutrino fields in the Einstein-Cartan theory are derivable through the Lagrangian density

$$\mathcal{L} = 2\sqrt{-g}\, i\, \sigma^{\mu}{}_{A\dot{X}} \left(\phi^{A} \nabla_{\mu} \bar{\phi}^{\dot{X}} - \bar{\phi}^{\dot{X}} \nabla_{\mu} \phi^{A}\right) \tag{8.1}$$

where $\phi_A(x^\mu)$ is a two component spinor defining the Neutrino field. Upon varying this lagrangian, we obtain via (7. 3) the Neutrino-Weyl equation

$$\sigma^{\mu}{}_{A\dot{X}} \nabla_{\mu} \phi^{A} = 0 \qquad \text{or} \qquad \nabla_{A\dot{X}} \phi^{A} = 0 \tag{8.2}$$

the dynamical energy momentum tensor

$$\sigma^{\mu\nu} = i\, \sigma_{\nu A\dot{X}} \left(\phi^{A} \nabla_{\mu} \bar{\phi}^{\dot{X}} - \bar{\phi}^{\dot{X}} \nabla_{\mu} \phi^{A}\right) + i\, \sigma_{\mu A\dot{X}} \left(\phi^{A} \nabla_{\nu} \bar{\phi}^{\dot{X}} - \bar{\phi}^{\dot{X}} \nabla_{\nu} \phi^{A}\right) \tag{8.3}$$

and the dynamical spin angular momentum tensor

$$\tau_{\lambda\mu\nu} = -\varepsilon_{\lambda\mu\nu\kappa}\, \sigma^{\kappa}{}_{A\dot{X}}\, \phi^{A} \bar{\phi}^{\dot{X}}.$$

This can be written in the form

$$\tau_{\lambda\mu\nu} = -\varepsilon_{\lambda\mu\nu\kappa}\, J^{\kappa} \tag{8.4}$$

where J^{μ} is the Neutrino current vector given by

$$J^{\mu} = \sigma^{\mu}{}_{A\dot{X}}\, \phi^{A}\, \bar{\phi}^{\dot{X}} \tag{8.5}$$

The canonical energy-momentum tensor (7.11) takes the form

$$\Sigma_{\mu\nu} = 2i\ \sigma_{\mu A\dot{X}}\ (\phi^{A}\ \nabla_{\nu}\ \bar{\phi}^{\dot{X}} - \bar{\phi}^{\dot{X}}\ \nabla_{\nu}\ \phi^{A}) \tag{8.6}$$

(8.4) and (8.6) may now be substituted into the field equations (7. 9) and (7.10). We obtain

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -2ik\ \sigma_{\mu A\dot{X}}\ (\phi^{A}\ \nabla_{\nu}\ \bar{\phi}^{\dot{X}} - \bar{\phi}^{\dot{X}}\ \nabla_{\nu}\ \phi^{A}) \tag{8.7}$$

$$K_{\lambda\mu\nu} = k\ \varepsilon_{\lambda\mu\nu\kappa}\ J^{\kappa} \tag{8.8}$$

Note that the contortion tensor in (8.8) is linearly related to the current vector. Therefore any solutions of these field equations describing Neutrino fields in the Einstein-Cartan theory do not reduce to those of general relativity in the limit when contortion vanishes, as this would also imply the vanishing of the

Neutrino field. In addition, since the contortion tensor is totally antisymmetric, we have that extremals of the metric are also autoparallels of the connection.

8.2 General Properties

We now introduce a spinor dyad (o^A, ι^A) or an equivalent tetrad of complex null vectors $(\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu)$. The basis spinor o^A may be aligned with the Neutrino spinor ϕ^A:

$$\phi^A = \phi\, o^A \tag{8.9}$$

and the current vector J^μ is then given by

$$J^\mu = \phi\bar{\phi}\, \ell^\mu \tag{8.10}$$

With this notation, the Neutrino-Weyl equation (8.2) can be written as two complex scalar equations

$$D\phi = (\rho - \varepsilon)\phi \tag{8.11}$$

$$\delta\phi = (\tau - \beta)\phi \tag{8.12}$$

and the contortion tensor, obtained from the field equation (8.8), becomes

$$K_{\lambda\mu\nu} = k\, \phi\, \overline{\phi}\, \varepsilon_{\lambda\mu\nu\kappa}\, \ell^{\kappa} \tag{8.13}$$

It can easily be shown, using (3.7) or (4.17)(c), that the only non-zero contortion components are

$$\mu_1 = -2\, \gamma_1 = -\, i\, k\, \phi\, \overline{\phi} \tag{8.14}$$

The components of the Ricci tensor, defined by (3.23) or (4.32), may be evaluated using the field equation (8.7) together with the Neutrino-Weyl equations (8.11) and (8.12). We obtain

$$\begin{aligned}
\Phi_{00} &= 0 \\
\Phi_{01} &= -\tfrac{1}{2}\, i\, k\, \phi\, \overline{\phi}\, \kappa \\
\Phi_{02} &= -\, i\, k\, \phi\, \overline{\phi}\, \sigma \\
\Phi_{11} &= \tfrac{1}{2}\, i\, k\, \phi\, \overline{\phi}\, (\overline{\rho} - \rho) \\
\Phi_{12} &= \tfrac{1}{2}\, i\, k\, [\phi\, \delta\, \overline{\phi} + \phi\, \overline{\phi}\, (\overline{\alpha} - 2\tau)] \\
\Phi_{22} &= i\, k\, [\phi\, \Delta\, \overline{\phi} - \overline{\phi}\, \Delta\, \phi + \phi\, \overline{\phi}\, (\overline{\gamma} - \gamma)] \\
\\
\Lambda &= 0 \\
\\
\Phi_0 &= \tfrac{1}{2}\, i\, k\, \phi\, \overline{\phi}\, \kappa \\
\Phi_1 &= i\, k\, \phi\, \overline{\phi}\, \rho \\
\Phi_2 &= \tfrac{1}{2}\, i\, k\, [\phi\, \overline{\delta}\, \phi + \phi\, \overline{\phi}\, \alpha]
\end{aligned} \tag{8.15}$$

The contortion components (8.14) are now inserted into the Bianchi identities for the torsion (5.12 a-j) and the components of the curvature tensor Θ_{AB} and Σ are then given by

$$i\,\Theta_{00} = 0$$

$$i\,\Theta_{01} = \tfrac{1}{2}\, i\, k\, \phi\, \bar{\phi}\, \kappa$$

$$i\,\Theta_{02} = i\, k\, \phi\, \bar{\phi}\, \sigma$$

$$i\,\Theta_{11} = \tfrac{1}{2}\, i\, k\, \phi\, \bar{\phi}\, (\rho + \bar{\rho})$$

$$i\,\Theta_{12} = \tfrac{1}{2}\, i\, k\, [\phi\, \delta\, \bar{\phi} + \phi\, \bar{\phi}\, (\bar{\alpha} + 2\tau)]$$

$$i\,\Theta_{22} = i\, k\, [\phi\, \Delta\, \bar{\phi} + \bar{\phi}\, \Delta\, \phi + \phi\, \bar{\phi}\, (\gamma + \bar{\gamma})]$$

$$i\,\Sigma = 0 \tag{8.16}$$

Note that the identities involving the antisymmetric part of the Ricci tensor Φ_A are automatically satisfied as required (see Chapter 7).

In order to relate the basis tetrad more directly with the Neutrino field, we choose the basis spinor 0^A so that

$$\phi^A = 0^A, \quad \text{i.e. } \phi = 1 \text{ in (8.9).}$$

The tetrad is now defined up to transformations of the form (see Appendix A).

$$\tilde{\ell}^\mu = \ell^\mu,\ \tilde{m}^\mu = m^\mu + T\ell^\mu,\ \tilde{n}^\mu = n^\mu + \bar{T}\, m^\mu + T\, \bar{m}^\mu + T\,\bar{T}\,\ell^\mu \tag{8.17}$$

where T is arbitrary. A tetrad defined in this way is referred to as a Neutrino tetrad. The non-zero contortion components μ_1 and γ_1 are now given by

$$\mu_1 = - 2\gamma_1 = - i k$$

and the Neutrino-Weyl equations (8.11) and (8.12) are expressed as the following conditions on the spin coefficients

$$\varepsilon = \rho, \quad \beta = \tau \tag{8.18}$$

We may now substitute the components of the curvature tensor (8.15) and (8.16), together with the above conditions on the spin coefficients and contortion components, into the last four Bianchi identities for the curvature. The latter do not contain derivatives of the components of the generalised Weyl tensor. (5.16) (i) is identically satisfied with the aid of the Ricci identity (5.9)(a). The Ricci identities (5.9) (c, d and k) may be used to simplify (5.16) (j and k). Similarly, the remaining identity (5.16) (l) may also be simplified using the Ricci identities (5.9) (f, l and q). We obtain

$$(5.16)(j) \rightarrow \overline{\delta}\sigma - \Delta\kappa = 2\sigma\alpha + \kappa\,(\overline{\mu} - 2\gamma - \overline{\gamma} + \tfrac{3}{2}\,i\,k) + \Psi_1$$

$$(5.16)(k) \rightarrow \overline{\delta}\sigma - \Delta\kappa = 2\sigma\alpha + \kappa\,(\overline{\mu} - 2\gamma - \overline{\gamma} + \tfrac{1}{6}\,i\,k) + \Psi_1$$

$$(5.16)(l) \rightarrow 2\,ik\,(\rho + \overline{\rho}) = 0$$

These equations are only consistent if

$$\kappa = 0 \tag{8.19}$$

$$\rho + \bar{\rho} = 0 \tag{8.20}$$

$$\bar{\delta}\sigma = 2\sigma\alpha + \Psi_1 \tag{8.21}$$

The first two of these conditions imply that the Neutrino current is necessarily tangent to a congruence of expansion-free null Autoparallels.

With these further conditions, the Ricci identities (5.9) (a and b) become

$$D\rho = 0 \tag{8.22}$$

$$\rho^2 + \sigma\bar{\sigma} = 0 \tag{8.23}$$

$$D\sigma = 4\rho\sigma + \Psi_0 \tag{8.24}$$

and we may now deduce the following

<u>Theorem 8.1</u> Neutrino fields in the Einstein Cartan theory necessarily propagate along expansion-free null autoparallels, whose twist and shear are equal and are constant along the autoparallels.

<u>Theorem 8.2:</u> The congruence tangent to the Neutrino current vector is twist-free if and only if it is shear-free, and is then necessarily aligned with the repeated principal null congruence of an algebraically special gravitational field.

The proof of theorem 8.2 follows from equations (8.23), (8.21) and (8.24). It should be noticed that the conditions (8.19) and (8.23) automatically satisfy two of the three conditions that are required for the sign of the energy density of the Neutrino field to be constant with respect to all observers (Griffiths 1980a). Thus the energy non-definiteness that occurs as an anomaly in general relativity does not occur as readily in Einstein-Cartan theory.

The converse of theorem 8.2 can also be obtained as follows:

<u>Theorem 8.3:</u> If the Neutrino current vector is aligned with a repeated principal null congruence of an algebraically special gravitational field, then the shear and twist of the congruence must vanish.

<u>Proof:</u> Under the conditions of the theorem:

$$\Psi_0 = \Psi_1 = 0$$

We assume that ρ and σ are non-zero and look for a contradiction. With this assumption it is possible to use the tetrad freedom (8.17), which is written explicitly in (A.6), to put

$$\tau = 0$$

in addition to the conditions (8.18 - 24). The Bianchi identity for the curvature (5.16b) immediately implies that

$$\Psi_2 = \frac{1}{3} i\, k\, \rho,$$

and the Ricci identities (5.9k, d, l, q, p) now reduce to

$$\delta\rho = \rho\bar{\alpha} - \sigma\alpha$$

$$D\alpha = -\rho(\alpha - 2\pi) + \bar{\sigma}\,\bar{\alpha}$$

$$\delta\alpha = \alpha\bar{\alpha} + \rho\,(2\mu - \bar{\mu} + 2\gamma + \frac{2}{3} i\, k) - \sigma\lambda$$

$$\Delta\rho = -\rho(\bar{\mu} - \gamma - \bar{\gamma} - \frac{2}{3} i\, k) - \sigma\lambda$$

$$\Delta\sigma = \rho\bar{\lambda} - \sigma\,(\mu - 3\gamma + \bar{\gamma} + 2\, i\, k)$$

It can be shown now, by applying the commutator (5.6b) to $\bar{\sigma}$ and using the above identities that

$$\rho\bar{\alpha} - \sigma\alpha = 0$$

Thus $\delta\rho = 0$ (from above), and the commutator (5.6d) applied to ρ implies that $\Delta\rho = 0$, and hence

$$\sigma\lambda = -\rho\,(\bar{\mu} - \gamma - \bar{\gamma} - \frac{2}{3} i\, k) \quad \text{or} \quad \rho\bar{\lambda} = -\sigma\,(\mu - \gamma - \bar{\gamma} + \frac{2}{3} i\, k)$$

From (8.23) and (8.21) we have that

$$\delta\sigma = -2\,\sigma\,\bar{\alpha}$$

and therefore the commutator relation (5.6d) applied to σ implies that

$$ik\rho\sigma = 0$$

contradicting the assumption and thus completing the proof.

Note that theorems 8.2 and 8.3 bear some resemblance to the Goldberg-Sachs theorem for vacuum space-times in the general theory of relativity.

8.3 Exact Solutions

We now follow the methods of Collinson and Morris (1972) in order to obtain all possible exact solutions corresponding to Neutrino pure radiation fields. Such fields are characterised by the fact that the symmetric energy-momentum tensor only possesses a Φ_{22} component (Griffiths and Newing, 1971). Hence for a Neutrino tetrad, (8.15), (8.18) and (8.20) imply that

$$\kappa = \sigma = \rho = \varepsilon = 0, \quad \alpha = 2\bar{\tau}, \quad \beta = \tau.$$

Theorem 8.2 and the Ricci identities (5.9 l, p and q) immediately give

$$\tau = \alpha = \beta = 0, \quad \Psi_0 = \Psi_1 = \Psi_2 = 0.$$

It is now possible to use the tetrad transformation (8.17) to put

$$\pi = 0$$

and such transformations in future must be restricted by DT = 0 (see (A.6)).

Under these conditions, we have from Section 3.1 that the Neutrino current vector is equal to the gradient of a scalar field i.e. $\ell_\mu = \partial_\mu u$. The coordinates $(x^1, x^2, x^3, x^4) = (u,r,x,y)$ are now introduced where r is an affine parameter along the auto-parallel tangent to ℓ_μ. Just as in Section 6.1, the tetrad can be written in the form

$$\ell^\mu = \delta^\mu{}_2 , \qquad \ell_\mu = \delta_\mu{}'$$

$$n^\mu = \delta^\mu{}_1 + U\,\delta^\mu{}_2 + X^i\,\delta^\mu{}_i \qquad (i = 3,4)$$

$$m^\mu = \qquad \omega\,\delta^\mu{}_2 + \xi^i\,\delta^\mu{}_i \qquad (i = 3,4)$$

where X^i and U are real scalar functions, ω and ξ^i are complex scalar functions. It is convenient to put

$$z = x + i\,y.$$

The coordinates are then defined up to the transformations

$$\tilde{r} = r + h\,(u, z, \bar{z}) \tag{8.25)(a}$$

$$\tilde{z} = \tilde{z}\,(u, z, \bar{z}) \tag{b}$$

$$\tilde{u} = f\,(u), \quad \tilde{r} = r/\partial_u f(u), \quad \tilde{\ell}_\mu = \partial_u\, f(u)\ell_\mu, \quad \tilde{n}_\mu = n_\mu/\partial_u\, f(u) \tag{c}$$

The metric equations are given by

$$DU = -\gamma - \bar{\gamma}$$

$$DX^i = 0$$

$$D\omega = 0$$

$$D\xi^i = 0$$

$$\delta U - \Delta\omega = -\bar{\nu} + \overline{\lambda\omega} + (\mu - \gamma + \bar{\gamma} + 2ik)\omega$$

$$\delta X^i - \Delta\xi^i = \overline{\lambda\xi}^i + (\mu - \gamma + \bar{\gamma} + 2ik)\xi^i$$

$$\bar{\delta}\omega - \delta\bar{\omega} = \bar{\mu} - \mu - 2ik$$

$$\bar{\delta}\xi^i - \delta\bar{\xi}^i = 0$$

Therefore ξ^i, ω and X^i are independent of r. The coordinate transformation (8.25)(b) can now be used to put (Newman and Tamburino, 1962)

$$\xi^3 = P, \quad \xi^4 = iP \quad \text{and} \quad X^i = 0.$$

Note that P is complex and the last metric equation implies that $\bar{\delta}P = 0$. (The remaining freedom in $\tilde{z}$ is given by $\tilde{z} = \tilde{z}(z)$).

The Ricci identities reduce to

$$D\gamma = 0$$

$$D\lambda = 0$$

$$D\mu = 0$$

$$D\nu = \Psi_3$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma} - 2ik) - \Psi_4$$

$$\delta\lambda - \bar{\delta}\mu = -\Psi_3$$

$$\delta\nu - \Delta\mu = \mu(\mu + \gamma + \bar{\gamma} + 2ik) + \lambda\bar{\lambda} + 2ik\bar{\gamma}$$

$$\delta\gamma = 0$$

$$\bar{\delta}\gamma = \Psi_3$$

where

$$D = \frac{\partial}{\partial r}, \quad \Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r}, \quad \delta = \omega\frac{\partial}{\partial r} + 2P\frac{\partial}{\partial \bar{z}} \cdot$$

Thus the Ricci identities imply that λ is independent of r and therefore the $\bar{z}$ dependence of T in (A.6) can be used to put

$$\lambda = 0.$$

T must in future be restricted by

$$DT = 0, \quad \delta T = 0 \qquad \text{i.e. } T = T(u, z)$$

The Bianchi identities for the curvature now reduce to

$$D\Psi_3 = 0$$

$$\delta\Psi_3 = 0$$

$$\bar{\delta}\Psi_3 - D\Psi_4 = 0$$

$$\Delta\Psi_3 - \delta\Psi_4 = -2\,(2\mu + \gamma + ik)\,\Psi_3,$$

which together with the Ricci identities and the metric equations imply (noting that $\lambda = 0$, $X^i = 0$).

$$U = -\,(\gamma + \bar{\gamma})r + V$$

$$\nu = \Psi_3 r + N \tag{8.26}$$

$$\bar{N} = \frac{\partial\omega}{\partial u} - 2P\,\frac{\partial V}{\partial \bar{z}} + (\mu + 2\bar{\gamma} + 2ik)\omega$$

$$\mu - \gamma + \bar{\gamma} + 2ik = -\,\frac{\partial}{\partial u}\,(\mathrm{Log}\ P) \tag{8.27}$$

$$\mu - \bar{\mu} + 2ik = 2P\,\frac{\partial\bar{\omega}}{\partial\bar{z}} - 2\bar{P}\,\frac{\partial\omega}{\partial z}$$

$$\Psi_3 = 2\bar{P}\,\frac{\partial\mu}{\partial z}\,, \qquad \frac{\partial\mu}{\partial z} = \frac{\partial\gamma}{\partial z} \tag{8.28}$$

$$\Psi_4 = \bar{\omega}\,\Psi_3 + 2\bar{P}\,\frac{\partial}{\partial z}\,\Psi_3 r + 2\bar{P}\,\frac{\partial N}{\partial z} \tag{8.29}$$

$$2ik\bar{\gamma} = \omega\,\Psi_3 + 2P\,\frac{\partial N}{\partial\bar{z}} - \frac{\partial\mu}{\partial u} - \mu(\mu + \gamma + \bar{\gamma} + 2ik) \tag{8.30}$$

where V (real), N, μ and ω are functions of u, z and $\bar{z}$, $P = P(u, \bar{z})$ and $\gamma = \gamma(u,z)$. There are now two cases to consider according to whether or not Ψ_3 is zero.

<u>Case (1).</u> $\Psi_3 = 0$

(8.28) implies that $\mu = \mu(u, \bar{z})$ and therefore the remaining tetrad freedom in T can be used to put

$$\mu^0 = 0, \text{ i.e. } \mu = -ik$$

T is now a function of u only. We may also use the coordinate transformation (8.25)(a) to put

$$\omega = 0$$

The real part of equation (8.27) now implies that

$$P = \frac{e^{ia(u)}}{\sqrt{2}}$$

where the remaining coordinate transformation $\tilde{z} = \tilde{z}(z)$ has been used to make P a function of u only. Also, since γ is now a function of u only (see (8.28)), we may use the freedom in (8.25)(c) to put

$$\gamma + \overline{\gamma} = 0.$$

Equations (8.26), (8.27), (8.29) and (8.30) together with (8.15) imply that

$$\gamma = \frac{1}{2} i \left(\frac{\partial a(u)}{\partial u} + k\right)$$

$$\nu = -\sqrt{2}\, e^{-ia(u)} \frac{\partial U}{\partial z}$$

$$\Psi_4 = -2e^{-2ia(u)} \frac{\partial^2 U}{\partial z^2}$$

$$\Phi_{22} = k \left(\frac{\partial a(u)}{\partial u} + k\right) = -2 \frac{\partial^2 U}{\partial z \partial \overline{z}} - k^2,$$

and all the other curvature components are zero.

The tetrad yields the familiar plane wave metric

$$ds^2 = 2\,du\,dr - 2U\,du^2 - dx^2 - dy^2$$

with

$$U = m(u)z\bar{z} + F$$

where

$$m(u) = -\frac{1}{2}k\left(\frac{\partial a(u)}{\partial u} + 2k\right)$$

and $F = F(u, z, \bar{z})$ is a real arbitrary solution of Laplace's equation $\frac{\partial^2 F}{\partial z \partial \bar{z}} = 0$. The Weyl tensor in this case is of type N or O.

This solution has also been obtained independently by Dereli and Tucker (1981) using a different approach.

Case (2). $\Psi_3 \neq 0$

Because γ is a function of u and z only, we note from (8.28) that $\frac{\partial^2 \mu}{\partial \bar{z} \partial z} = 0$. The remaining freedom in T can now be used to put

$$\mu = \mu\,(u,z),$$

and (8.28) then implies that

$$\gamma = \mu + i\,g(u) + \frac{1}{2}ik$$

where g(u) is real. We may also use the transformation (8.25)(a) to put

$$\omega = \frac{1}{2}(\bar{\mu} - ik)Q$$

where Q = Q(u, z) is defined such that

$$\frac{\partial Q}{\partial z} = \frac{1}{\bar{P}}$$

Thus, from (8.27) we have

$$\mu = -2ig + ik + \frac{\partial}{\partial u}\left(\log \frac{\partial Q}{\partial z}\right)$$

The only remaining equation is (8.30) which, after a lengthy calculation, can be written as

$$\Phi_{22} + i\Theta_{22} = 2ik\left(\bar{\mu} - ig - \frac{1}{2}ik\right) = (\mu + ik)\,\overline{PQ}\,\frac{\partial \bar{\mu}}{\partial \bar{z}}$$

$$+ (\bar{\mu} - ik)\,\bar{P}Q\,\frac{\partial \mu}{\partial z} - 4P\bar{P}\,\frac{\partial^2 V}{\partial z \partial \bar{z}} + \bar{\mu}(\mu + 2ik)$$

This defines V in terms of Q and g up to an arbitrary solution of Laplace's equation $\frac{\partial^2 V}{\partial z \partial \bar{z}} = 0$. The Weyl tensor is of type III and is given by (8.28) and (8.29).

8.4 Ghost Neutrinos

In general relativity, the term Ghost Neutrinos has been given to those solutions of Einstein's equation for which the energy-momentum tensor of the Neutrino field vanishes, while its current vector remains non-zero. Letelier (1975) has shown that certain restrictions on the curvature tensor imply the non-existence of ghost neutrinos in Einstein-Cartan space-time. He considers the following conditions

$$R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} = 0 \tag{A}$$

$$R_{\kappa\lambda\mu\nu} = \overset{\circ}{R}_{\kappa\lambda\mu\nu} \tag{B}$$

$$R_{\kappa\lambda\mu\nu} = 0 \tag{C}$$

These are shown to imply the vanishing of the Neutrino current vector and therefore ghost neutrinos do not exist in Einstein-Cartan spaces subject to these restrictions.

Griffiths (1980,b) has pointed out that the restrictions (A), (B) and (C) are unnecessarily strong. He argues that the only necessary condition for "Ghostness" is $R_{\mu\nu} = 0$ and goes on to present an exact solution satisfying this condition.

For a Neutrino tetrad, the condition $R_{\mu\nu} = 0$ implies, from Section (8.2) that

$$\kappa = \rho = \sigma = \tau = \alpha = \gamma - \overline{\gamma} = 0.$$

These are just the conditions considered in the previous section but with the additional constraint $\gamma = \overline{\gamma}$. The latter in fact immediately implies that $\Psi_3 = 0$ above. Thus the only possible ghost solutions in the Einstein-Cartan theory are the type N solutions given above with the additional constraint

$$2 \frac{\partial^2 U}{\partial z \partial \overline{z}} + k^2 = 0.$$

These solutions have a zero canonical energy-momentum tensor and therefore do not contribute to the curvature of space-time through the gravitational field equations. Some curvature however is still generated by the torsion. Thus the characterisation of "ghostness" is not as clear in the Einstein-Cartan theory as in general relativity. A number of different characterisations have been suggested and analysed by Letelier (1980).

CHAPTER 9

SEMI-CLASSICAL SPIN FLUIDS IN THE EINSTEIN-CARTAN THEORY

The analysis of "perfect fluid source fields with spin" is relevant to the derivation of cosmological models based on the Einstein-Cartan theory. It is sufficient at this stage to restrict ourselves to a semi-classical spin fluid. The fluid generalises the "perfect fluid" of general relativity by permitting a non-vanishing spin density, which we assume to be aligned so that the spin varies continuously from one fluid element to another.

9.1 The Weyssenhoff Model

In the Weyssenhoff model, it is postulated (Hehl et al 1976) that the canonical energy-momentum tensor and the spin angular momentum tensor take the forms

$$\Sigma_{\mu\nu} = P_\mu U_\nu + p(U_\mu U_\nu - g_{\mu\nu}) \qquad (9.1)$$

$$\tau_{\mu\nu}{}^{\kappa} = \tau_{\mu\nu}\, U^{\kappa}, \quad \tau_{\mu\nu} = \tau_{[\mu\nu]} \qquad (9.2)$$

where U^μ is the normalised fluid four-velocity ($U^\mu U_\mu = +1$ for our signature --- +), P^μ is the linear momentum density, p is the pressure and $\tau_{\mu\nu}$ the spin density. One effect of spin is that P^μ and U^μ are not necessarily parallel and we may write

$$P^\mu = e\, U^\mu + f^\mu \qquad (9.3)$$

where f^{μ} is a space-like vector satisfying $f^{\mu}U_{\mu} = 0$ and

$$e = P^{\mu}U_{\mu} \tag{9.4}$$

is the fluid energy density (which is taken to be non-zero). It should be noticed that the components of f^{μ} define the anti-symmetric part of $\sum_{\mu\nu}$:-

$$\textstyle\sum_{[\mu\nu]} = f_{[\mu}U_{\nu]}, \quad f_{\mu} = 2U^{\nu} \textstyle\sum_{[\mu\nu]}$$

The spin angular momentum tensor is chosen to satisfy the Weyssenhoff restriction

$$\tau_{\mu\alpha} \; U^{\alpha} = 0 \tag{9.5}$$

The latter is a generalisation from special relativity, where $\tau_{\mu\nu}$ is orthogonal to U^{μ} (see box 5.6 in Misner et al, 1973). An alternative restriction is given by

$$\tau_{\mu\alpha} \; P^{\alpha} = 0 \tag{9.6}$$

although this is not considered here.

The above characterisation of a "perfect fluid with spin" is an extension, from special relativity, of the semi-classical model of a spin fluid given by Weyssenhoff and Raabe (1947). Kuchowicz (1976) has called this model the Weyssenhoff fluid.

The Einstein-Cartan field equations (7. 9) and (7.10) can now be written for Weyssenhoff fluids in the form

$$R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R = -k \left[(e + p) U_\mu U_\nu - p g_{\mu\nu} + f_{(\mu} U_{\nu)}\right] \quad (9.7)$$

$$R_{[\mu\nu]} = -k f_{[\mu} U_{\nu]} \quad (9.8)$$

$$K_{\mu\nu\kappa} = -k (\tau_{\mu\nu} U_\kappa - \tau_{\nu\kappa} U_\mu + \tau_{\kappa\mu} U_\nu) \quad (9.9)$$

Note that (7. 9) has been separated into its symmetric and antisymmetric parts. The antisymmetric part (9.8) defines the conservation of angular momentum within the field (see Chapter 7).

9.2 General Properties

We begin by defining a continuous intrinsic angular momentum four-vector as follows

$$\tau_\mu := \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} U^\alpha \tau^{\beta\gamma} \quad (9.10)$$

This is clearly a space-like vector, orthogonal to U^μ. As a consequence of the Weyssenhoff restriction (9.5) the tensor $\tau^{\mu\nu}$ can then be regained from the expression

$$\tau_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} U^\alpha \tau^\beta \quad (9.11)$$

Also as a result of this restriction, $\tau_{\mu\nu}$ and τ_{μ} have only three independent components.

The tetrad of null vectors (ℓ^{μ}, n^{μ}, m^{μ}, $\overline{m}^{\mu}$) are now introduced at all points in space-time. It is possible in general to rotate and align these with the vectors U^{μ} and τ^{μ} as follows

$$U^{\mu} = \frac{1}{\sqrt{2}} (\ell^{\mu} + n^{\mu}) \tag{9.12}$$

$$\tau^{\mu} = \frac{S}{K} (\ell^{\mu} - n^{\mu}) \tag{9.13}$$

The torsion is then determined by a single function S, while the tetrad must in future be restricted by transformations of the form

$$\tilde{\ell}^{\mu} = \ell^{\mu}, \ \tilde{m}^{\mu} = e^{i\phi} m^{\mu}, \quad \tilde{n}^{\mu} = n^{\mu} \tag{9.14}$$

As a consequence of (9.13), $\tau_{\mu\nu}$ is given in terms of the tetrad by

$$\tau_{\mu\nu} = -2\sqrt{2}\, i \, \frac{S}{K} m_{[\mu} \overline{m}_{\nu]} \tag{9.15}$$

It is easily shown, using (3.7) or (4.17)(c), that the only non-zero contortion components are given by

$$\rho_1 = \mu_1 = 2\epsilon_1 = 2\gamma_1 = iS \tag{9.16}$$

These can now be substituted directly into the Bianchi identities for the torsion (5.12 a-j). The components of the curvature tensor

Θ_{AB}, Φ_A and Σ are then given by

$$\Theta_{00} = DS - (\rho + \bar{\rho} + \varepsilon + \bar{\varepsilon})S$$

$$\Theta_{01} = \frac{1}{2}(\kappa - \tau - 2\bar{\pi})S$$

$$\Theta_{02} = 0$$

$$\Theta_{11} = \frac{1}{2}(\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})S + \frac{1}{4}\Theta_{00} + \frac{1}{4}\Theta_{22}$$

$$\Theta_{12} = \frac{1}{2}(\bar{\nu} - \bar{\pi} - 2\tau)\ S$$

$$\Theta_{22} = -\Delta S - (\mu + \bar{\mu} + \gamma + \bar{\gamma})S$$

$$\Sigma = \frac{1}{3}\Theta_{11}$$

$$\Phi_0 = -\frac{1}{2}(\kappa + \tau)\ iS$$

$$\Phi_1 = -(\varepsilon + \bar{\varepsilon} - \gamma - \bar{\gamma})iS - \frac{1}{2}i\,\Theta_{00} + \frac{1}{2}i\Theta_{22}$$

$$\Phi_2 = \frac{1}{2}(\nu + \pi)\ iS \qquad (9.17)$$

The last three of these components describe the antisymmetric part of the Ricci tensor, which according to (9.8) is related to the vector f^μ defining the antisymmetric part of the canonical energy-momentum tensor. Hence expressions for these components can also be obtained via this relation. By writing f^μ in terms of the tetrad components

$$f^\mu = b(\ell^\mu - n^\mu) + cm^\mu + \bar{c}\,\bar{m}^\mu$$

where b is real, (9.8) implies that

$$\Phi_0 = \frac{1}{4\sqrt{2}}\ k\ \bar{c}$$

$$\Phi_1 = \frac{1}{2\sqrt{2}}\ k\ b$$

$$\Phi_2 = -\frac{1}{4\sqrt{2}}\ k\ c$$

Comparing this with the components above, we obtain the conditions

$$b = 0, \quad c = 2\sqrt{2}\ (\bar{\kappa} + \bar{\tau})iS/k = -2\sqrt{2}\ (\nu + \pi)iS/k$$

Because we have interpreted (9.8) to define angular momentum conservation, these conditions are regarded as the constraint equations describing the conservation of angular momentum within the fluid. Note that for the classical Neutrino field considered in Chapter 8 these equations were identically satisfied as a consequence of the Neutrino field equations. However, no analogous field equations are considered here, so these additional conditions are necessary. They immediately imply that

$$\kappa + \tau + \bar{\nu} + \bar{\pi} = 0 \tag{9.18}$$

$$\Theta_{00} - \Theta_{22} = -2\ (\varepsilon + \bar{\varepsilon} - \gamma - \bar{\gamma})S \tag{9.19}$$

which are equivalent to the constraints

$$\Phi_1 = 0, \quad \Phi_0 + \Phi_2 = 0 \tag{9.20}$$

They also imply that the vector f^μ takes the form

$$f^\mu = 2\sqrt{2}\,\frac{iS}{k}\,[(\bar{\kappa} + \bar{\tau})m^\mu - (\kappa + \tau)\bar{m}^{\,\mu}] \tag{9.21}$$

and we obtain:-

<u>Theorem 9.1</u> If the canonical energy-momentum tensor has a non-zero antisymmetric part, then the vector f^μ is orthogonal to τ^μ as well as U^μ.

It should also be noticed that, using (9.17), the condition (9.19) can be written as

$$DS + \Delta S = (\rho + \bar{\rho} - \mu - \bar{\mu} - \varepsilon - \bar{\varepsilon} + \gamma + \bar{\gamma})S$$

which can be conveniently rewritten in the form

$$\nabla_\mu\,(SU^\mu) = 0 \tag{9.22}$$

This can be seen to imply that the magnitude of the intrinsic angular momentum of each particle of the fluid is conserved along its world-line. The other constraint (9.19) is related to the condition that the direction of the intrinsic angular momentum is conserved, as can be seen from (9.12) and the identity

$$U^\alpha\,\nabla_\alpha\,(\ell_\mu - n_\mu) = (\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})U_\mu - \frac{1}{\sqrt{2}}\,(\bar{\kappa} + \bar{\tau} + \nu + \pi)m_\mu$$
$$- \frac{1}{\sqrt{2}}\,(\kappa + \tau + \bar{\nu} + \bar{\pi})\bar{m}_\mu$$

The remaining components of the Ricci tensor, defined by (3.23) or (4.32), can be obtained from (9.7). These are given, using the above constraints, by

$$\Phi_{00} = 2\Phi_{11} = \Phi_{22} = \frac{1}{4} k\ (e + p)$$

$$\Phi_{01} = \Phi_{12} = \frac{1}{2} (\kappa + \tau) iS$$

$$\Phi_{02} = 0$$

$$\Lambda = \frac{1}{24} k\ (e - 3p) \tag{9.23}$$

At this point it is convenient to introduce the acceleration vector (Ellis 1971) defined by

$$\dot{U}_\mu = U^\nu \nabla_\nu U_\mu$$

This measures the acceleration, of the fluid elements, due to non-gravitational forces. Note that as a consequence of the Weyssenhoff restriction (9.5) and the field equation (9.9) it can be seen that components of the contortion do not enter this equation. In terms of spin coefficients we obtain using (9.16)

$$\dot{U}_\mu = \frac{1}{2} (\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})(\ell_\mu - n_\mu) - (\bar{\kappa} + \bar{\tau})m_\mu - (\kappa + \tau)\bar{m}_\mu \tag{9.24}$$

By comparing this with (9.21) we obtain the following

<u>Theorem 9.2</u> The vector f_μ, if non-zero, is orthogonal to the acceleration vector of the fluid.

<u>Theorem 9.3</u> If the acceleration vector of the fluid is zero, then the canonical energy-momentum tensor is symmetric and the momentum vector is parallel to the velocity vector.

The converse of theorem 9.3 is not generally true because of the presence of the first term in (9.24). However it is instructive to note that if the canonical energy-momentum tensor is not symmetric, then the momentum vector is not parallel to the velocity vector and the fluid must be accelerating.

It should be noticed that the three theorems given above are consequences of the field equation (9.8), which is interpreted to represent the conservation of angular momentum within the fluid. It is possible to take certain linear combinations of the components (5.16)(i)-(ℓ) of the Bianchi identities for the curvature and obtain equations in which only the new components Θ_{AB}, Φ_{AB} and Σ appear in the derivatives on the left hand side. These equations are peculiar to U_4 only and are now found to be automatically satisfied for Weyssenhoff fluids in the Einstein-Cartan theory.

The acceleration vector $\dot{U}_\mu$ defined earlier is one of four kinematic quantities which will be of assistance in the interpretation of the exact solutions presented in Section 9.3. The other three are defined by (Ellis 1971).

<u>The volume expansion</u> $\theta = \nabla_\mu U^\mu$

<u>The shear tensor</u> $\sigma_{\mu\nu} = \nabla_{(\nu}U_{\mu)} - \dot{U}_{(\mu}U_{\nu)} - \frac{1}{3}\theta(g_{\mu\nu} - U_\mu U_\nu)$

<u>The vorticity tensor</u> $\omega_{\mu\nu} = \nabla_{[\nu}U_{\mu]} - \dot{U}_{[\mu}U_{\nu]}$

These can be given in terms of the spin coefficients associated with the tetrad defined by (9.12) and (9.13) as follows

$$\theta = -\frac{1}{\sqrt{2}}(\rho + \bar{\rho} - \mu - \bar{\mu} - \varepsilon - \bar{\varepsilon} + \gamma + \bar{\gamma}) \qquad (9.25)$$

$$\begin{aligned}\sigma_{\mu\nu} = {} & \frac{1}{3\sqrt{2}}(\rho + \bar{\rho} - \mu - \bar{\mu} + 2\varepsilon + 2\bar{\varepsilon} - 2\gamma - 2\bar{\gamma})\{g_{\mu\nu} - U_\mu U_\nu + 3m_{(\mu}\bar{m}_{\nu)}\} \\ & + \frac{1}{\sqrt{2}}(\bar{\kappa} + \nu - \alpha - \bar{\beta})\{\ell_{(\mu}m_{\nu)} - n_{(\mu}m_{\nu)}\} \\ & + \frac{1}{\sqrt{2}}(\kappa + \bar{\nu} - \bar{\alpha} - \beta)\{\ell_{(\mu}\bar{m}_{\nu)} - n_{(\mu}\bar{m}_{\nu)}\} \\ & + \frac{1}{\sqrt{2}}(\sigma - \bar{\lambda})\, m_\mu m_\nu + \frac{1}{\sqrt{2}}(\bar{\sigma} - \lambda)\, \bar{m}_\mu \bar{m}_\nu \end{aligned} \qquad (9.26)$$

$$\begin{aligned}\omega_{\mu\nu} = {} & -\frac{1}{\sqrt{2}}(\bar{\kappa} + \nu + \alpha + \bar{\beta})\{\ell_{[\mu}m_{\nu]} - n_{[\mu}m_{\nu]}\} \\ & - \frac{1}{2}(\kappa + \bar{\nu} + \bar{\alpha} + \beta)\{\ell_{[\mu}\bar{m}_{\nu]} - n_{[\mu}\bar{m}_{\nu]}\} \\ & - \frac{1}{\sqrt{2}}(\rho - \bar{\rho} + \mu - \bar{\mu})\, m_{[\mu}\bar{m}_{\nu]} \end{aligned} \qquad (9.27)$$

It can be seen that the contortion components (9.13) only appear in the vorticity tensors which can be decomposed with the aid of (9.15) as follows

$$\omega_{\mu\nu} = \overset{0}{\omega}_{\mu\nu} + k\ \tau_{\mu\nu} \tag{9.28}$$

It is well known (Kuchowicz 1976) that if the vector U^{μ} is hypersurface orthogonal then $\overset{0}{\omega}_{\mu\nu} = 0$ and $\dot{U}_{\mu} = 0$.

9.3 Exact Solutions

In this section the following assumptions are made in order to simplify the field equations to a point at which they can be easily integrated. The exact solutions obtained are not claimed to be general in any sense but they are believed to be new.

Assumptions:-

1. The equation of state of the fluid corresponds to stiff matter ($p = e$). Using the above notation this implies that

$$\Phi_{11} + 3\Lambda = 0$$

2. The momentum vector is parallel to the velocity vector so that the energy-momentum tensor is symmetric ($f^{\mu} = 0$). Equations (9.18) and (9.21) now imply that

$$\kappa + \tau = \nu + \pi = 0$$

3. ℓ^{μ} is a tangent vector to an expansion-free and shear-free null autoparallel congruence. In terms of the spin coefficients this implies that (see Section 3.1)

$$\rho + \bar{\rho} = \sigma = \kappa = 0$$

4. Both ℓ^{μ} and n^{μ} are tangent to repeated principal null directions of the generalised Weyl tensor. This implies that

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$$

and therefore the generalised Weyl tensor is either of type D or 0.

We now use the tetrad freedom (9.14) to put

$$\bar{\alpha} - \beta = 0$$

so that in future such transformations must be restricted by

$$\delta\phi = 0$$

With the above conditions we now turn to the field equations. The Ricci identities (5.9)(a) and (p) immediately imply that

$$\Phi_{11} = -\frac{1}{2}\rho^2, \quad \lambda = 0 \tag{9.29}$$

It was mentioned in the last section that those linear combinations of the components (5.16)(i) - (l) of the Bianchi identities for the curvature which vanish in V_4 are automatically satisfied for Weyssenhoff fluids in the Einstein-Cartan theory. The remaining combinations of these equations now reduce to

$$(D + \Delta)\rho = -(\varepsilon + \bar{\varepsilon} - \gamma - \bar{\gamma} + \mu + \bar{\mu})\rho$$

$$(D - \Delta)\rho = -(\varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma})\rho + \frac{iS}{\rho}(\Psi_2 - \bar{\Psi}_2) - \frac{4}{3} S\, \Theta_{11}$$

$$\delta\rho = 0$$

These together with the Ricci identities (5.9)(a), (g), (j), (k), (l) and (q) imply that

$$\alpha = \beta = \pi = \nu = 0$$

$$\Psi_2 = \rho\mu + 2\rho^0(\varepsilon + \bar{\varepsilon}) - \frac{1}{3}\rho^2 + \frac{4}{3} i\, \Theta_{11}$$

$$\rho^0 \left(\Psi_2 - \bar{\Psi}_2 + \frac{4}{3} i\, \Theta_{11}\right) = 0$$

$$2\rho^0(\varepsilon + \bar{\varepsilon} - \gamma) - (\mu^0 - \bar{\mu}^0)\varepsilon = 0 \qquad (9.30)$$

The remaining field equations are now given by

<u>The commutation relations:</u>

$$(\Delta D - D\Delta) = (\gamma^0 + \bar{\gamma}^0)D + (\varepsilon^0 + \bar{\varepsilon}^0)\Delta \qquad (9.31)(a)$$

$$(\delta D - D\delta) = (\rho^0 - \varepsilon^0 + \bar{\varepsilon}^0)\,\delta \qquad (b)$$

$$(\delta\Delta - \Delta\delta) = (\mu^0 - \gamma^0 + \overline{\gamma}^0)\delta \tag{c}$$

$$(\overline{\delta}\delta - \delta\overline{\delta}) = -(\mu^0 - \overline{\mu}^0)D - 2\rho^0\Delta \tag{d}$$

The Ricci identities:

$$D\rho = (\rho^0 - iS)(\varepsilon + \overline{\varepsilon}) + 2i\ \Theta_{11} \tag{9.32)(a}$$

$$\Delta\rho = +\ \rho(\gamma + \overline{\gamma} - \mu - \overline{\mu}) - 2\rho^0\ (\varepsilon + \overline{\varepsilon}) - 2i\Theta_{11} \tag{b}$$

$$D\mu = (2\rho^0 - \mu)(\varepsilon + \overline{\varepsilon}) + 2i\ \Theta_{11} \tag{c}$$

$$\Delta\mu = \rho^2 - \mu^2 - (\mu - 2iS)(\gamma + \overline{\gamma}) - 2i\ \Theta_{11} \tag{d}$$

$$D\gamma - \Delta\varepsilon = (2\rho^0 - \gamma)(\varepsilon + \overline{\varepsilon}) - \varepsilon(\gamma + \overline{\gamma}) + \rho(\mu - \rho) + 2i\ \Theta_{11} \tag{e}$$

$$\delta\varepsilon = \overline{\delta}\varepsilon = 0 \tag{f}$$

$$\delta\gamma = \overline{\delta}\gamma = 0 \tag{g}$$

$$\delta\rho = \overline{\delta}\rho = 0 \tag{h}$$

$$\overline{\delta}\mu = 0 \tag{i}$$

The Bianchi identities for the torsion:

$$DS = -(\varepsilon + \overline{\varepsilon})S + 2\ \Theta_{11} \tag{9.33)(a}$$

$$\Delta S = (\gamma + \overline{\gamma} - \mu - \overline{\mu})S - 2\ \Theta_{11} \tag{b}$$

$$2\Theta_{11} = \Theta_{00} + 2(\varepsilon + \overline{\varepsilon})S = \Theta_{22} + 2(\gamma + \overline{\gamma})S = 6\Sigma \tag{c}$$

The Bianchi identities for the curvature:

$$\delta\Theta_{00} = 0 \tag{9.34)(a}$$

$$\delta\Theta_{22} = 0 \qquad \text{(b)}$$

$$\rho^0\,[D(\varepsilon + \bar{\varepsilon}) + (\varepsilon + \bar{\varepsilon})^2 - \frac{1}{2}\rho(\mu^0 - \bar{\mu}^0 - 2\rho^0)] = 0 \qquad \text{(c)}$$

$$\rho^0\Delta(\varepsilon + \bar{\varepsilon}) - \rho^0(\varepsilon + \bar{\varepsilon})(2\varepsilon + 2\bar{\varepsilon} - \gamma - \bar{\gamma}) - \frac{1}{4}\rho(\mu^0 - \bar{\mu}^0)(\mu^0 - \bar{\mu}^0 - 2\rho^0) = 0 \qquad \text{(d)}$$

$$\rho^0\,[i\Theta_{11} + \rho^0(\varepsilon + \bar{\varepsilon}) + \frac{1}{4}\rho\,(\mu + \bar{\mu})] = 0 \qquad \text{(e)}$$

$$(\mu^0 - \bar{\mu}^0)\,[i\Theta_{11} + \rho^0(\varepsilon + \bar{\varepsilon}) + \frac{1}{4}\rho\,(\mu + \bar{\mu})] = 0 \qquad \text{(f)}$$

$$\delta\mu = 0 \qquad \text{(g)}$$

From (9.30) it can be seen that there are two cases to consider according to whether or not ρ^0 is zero.

Case (i) $\rho^0 = 0$

In this case equation (9.29) reduces to $\Phi_{11} = \frac{1}{2}S^2$, so the following solutions do not reduce to perfect fluid solutions in general relativity in the limit when torsion vanishes. The energy density e and the spin density S are now related by

$$S^2 = \frac{1}{2}\,k\,e\,.$$

Therefore the "effective" energy density and the "effective" pressure defined by (Arkuszewski, Kopcynski and Ponomariev 1974)

$$e_{eff} = e - \frac{2}{k}S^2$$

$$P_{eff} = P - \frac{2}{k} S^2$$

both vanish in this case.

Equation (9.34)(d) immediately implies that

$$\mu^0 = \overline{\mu^0}$$

and all the remaining Bianchi identities for the curvature are automatically satisfied. It can now be shown by using the equations (B.1) given in Appendix B that

$$\overset{0}{R}_{\kappa\lambda\mu\nu} = 0$$

so the metric is flat and the curvature is generated purely by the torsion. It is also possible to use the remaining tetrad freedom (9.14) to put

$$\varepsilon^0 = \overline{\varepsilon}^0, \gamma^0 = \overline{\gamma^0}$$

and the scalar ϕ must in future be treated as a constant.

It is now possible to choose two null cordinates $x^1 = u$ and $x^2 = v$ and two space-like coordinates $x^3 = x$, $x^4 = y$ such that the tetrad takes the form

$$\ell_\mu = B^{-1}\, \delta^1_\mu \qquad \ell^\mu = A\, \delta^\mu_2$$

$$n_\mu = A^{-1}\, \delta^2_\mu \qquad n^\mu = B\, \delta^\mu_1$$

$$m^\mu = \xi^3\, \delta^\mu_3 + \xi^4\, \delta^\mu_4$$

where A, B are real and ξ^i (i = 1, 2) are complex functions. This tetrad is preserved under the following coordinate transformations

$$u' = f_1(u) + g_1\,(x,y)$$

$$v' = f_2(v) + g_2\,(x,y)$$

$$x' = x'\,(x,y)$$

$$y' = y'\,(x,y)$$

The intrinsic derivatives when acting on scalars are now given by

$$D = A\,\frac{\partial}{\partial v}\,, \quad \Delta = B\,\frac{\partial}{\partial u}\,, \quad \delta = \xi^3\,\frac{\partial}{\partial x} + \xi^4\,\frac{\partial}{\partial y}$$

The metric equations are given by

$$DB = -2\varepsilon^0\,B$$

$$\Delta A = 2\gamma^0\,A$$

$$\delta A = \delta B = 0$$

$$\left.\begin{aligned} D\xi^i &= 0 \\ \Delta\xi^i &= -\,\mu^0\,\xi^i \\ \bar{\delta}\,\xi^i - \delta\bar{\xi}^i &= 0 \end{aligned}\right\} \quad i = (3,4)$$

We can now make a coordinate transformation and put

$$\xi^3 = \frac{1}{\sqrt{2}}\,P, \qquad \xi^4 = \frac{i}{\sqrt{2}}\,P$$

where P is a real function and the metric equations immediately imply that

$$A(u,v), \quad B = B(u,v), \quad P = P(u)$$

$$\varepsilon^0 = -\frac{1}{2} A \frac{\partial}{\partial v} \log B$$

$$\gamma^0 = \frac{1}{2} B \frac{\partial}{\partial u} \log A$$

$$\mu^0 = -B \frac{d}{du} \log P$$

The remaining Ricci identities give

$$\frac{\partial^2}{\partial u \partial v} (\log A + \log B) = 0$$

$$\frac{d^2}{du^2} (P^{-1}) + \frac{d}{du} (p^{-1}) \frac{\partial}{\partial u} (\log A + \log B) = 0$$

The first of these implies that a coordinate transformation can be used to put

$$AB = 1$$

and then the second implies that

$$P = (au + b)^{-1}$$

where a and b are real constants. Thus the metric is given by

$$ds^2 = 2\, dudv - (au + b)^2 (dx^2 + dy^2)$$

and clearly for the case $a = 0$ it is possible to put $b = 1$. In the case $a \neq 0$ one can easily put $a = 1$, $b = 0$. However even though the metric is flat, it is not in general possible to make a coordinate transformation to put $a = 0$, $b = 1$. Because of our choice of tetrad vectors we do not have the required coordinate freedom to allow us to do this. The four-velocity of the fluid is given by

$$U^{\mu} = \frac{1}{\sqrt{2}} (A^{-1} \delta^{\mu}_{1} + A \delta^{\mu}_{2})$$

where $A = A(u, v)$ is totally arbitrary. The intrinsic angular momentum four-vector is given by

$$\tau^{\mu} = - \frac{S}{k} (A^{-1} \delta^{\mu}_{1} - A \delta^{\mu}_{2})$$

where $S = S(u,v)$ satisfies the equation

$$\frac{\partial}{\partial v} (SA) + \frac{\partial}{\partial u} (SA^{-1}) = - \frac{2a}{(au + b)} SA^{-1}$$

which is the condition (9.19). The non-zero components of the curvature tensor are

$$\Phi_{00} = 2\Phi_{11} = \Phi_{22} = -6\Lambda = S^2$$

$$2\Theta_{11} = \Theta_{00} + 2S \frac{\partial A}{\partial v} = \Theta_{22} - 2S \frac{\partial}{\partial u} (A^{-1}) = 6\,\Sigma = \frac{\partial}{\partial v} (SA)$$

$$\Psi_2 = \frac{i\,a}{(au + b)} SA^{-1} + \frac{2i}{3} [\frac{\partial}{\partial v} (SA) - S^2]$$

Because of the arbitrariness of the function $A(u,v)$, a wide class of solutions having in general non-zero acceleration, volume expansion, and shear have been obtained. The kinematic quantities are given by

$$\dot{U}_\mu = \frac{1}{2}\left[\frac{\partial A}{\partial v} - \frac{\partial}{\partial u}(A^{-1})\right] \quad (A\,\delta_\mu^1 - A^{-1}\,\delta_\mu^2)$$

$$\theta = \frac{1}{\sqrt{2}}\left[\frac{2a}{(au+b)} A^{-1} + \frac{\partial A}{\partial v} + \frac{\partial}{\partial u}(A^{-1})\right]$$

$$\sigma_{\mu\nu} = \frac{1}{3\sqrt{2}}\left[\frac{a}{(au+b)} A^{-1} - \frac{\partial A}{\partial v} - \frac{\partial}{\partial u}(A^{-1})\right]$$

$$X[(A\delta_\mu^1 - A^{-1}\,\delta_\mu^2)(A\delta_\nu^1 - A^{-1}\delta_\nu^2) - (au+b)^2\,\delta_\mu^3\,\delta_\nu^4 - (au+b)^2\,\delta_\mu^4\delta_\nu^4]$$

$$\omega_{\mu\nu}^{0} = 0$$

Case (ii) $\rho^0 \neq 0$

If the commutator (9.31)(d) is applied to ρ then it can be shown using the equations (9.30) and (9.32)(a) and (b) that

$$(\mu - \bar{\mu} - 2\rho)(\mu + \bar{\mu} - 4\varepsilon - 4\bar{\varepsilon}) = 0$$

It is easily shown with the aid of equations (9.32)(c) and (9.34) (c) that the case $(\mu - \bar{\mu} - 2\rho) \neq 0$ leads to a contradiction. We therefore conclude that

$$\mu - \bar{\mu} = 2\rho$$

Equation (9.30) now immediately implies that

$$\gamma = \bar{\varepsilon}$$

If the commutator (9.31)(d) is now applied to μ then it can be shown by using equations (9.32)(a) and (b), together with the above condition, that

$$\mu = \rho$$

Finally, it is possible to use the remaining tetrad freedom to put

$$\varepsilon = \bar{\varepsilon}, \quad \gamma = \bar{\gamma}.$$

All the Bianchi identities for the curvature are now automatically satisfied.

In this case it is convenient to introduce three space-like coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ and one time-like coordinate $x^4 = t$ defined such that

$$\ell^\mu = A\ \delta_4{}^\mu + B\ \delta_3{}^\mu$$

$$n^\mu = A\ \delta_4{}^\mu - B\ \delta_3{}^\mu$$

$$m^\mu = \xi^i\ \delta_i{}^\mu \qquad\qquad i = 1, 2, 3, 4$$

where $A = A\ (z,t)$ and $B = B\ (z, t)$. This tetrad is preserved under the following coordinate transformations

$$x' = x'\ (x,y)$$

$$y' = y'\ (x,y)$$

$$z' = f_1(z) + g_1(x,y)$$

$$t' = f_2(t) + g_2(x,y)$$

The intrinsic derivatives are given by

$$D + \Delta = 2A \frac{\partial}{\partial t}$$

$$D - \Delta = 2B \frac{\partial}{\partial z}$$

$$\delta = \xi^i \frac{\partial}{\partial x^i} \qquad i = 1, 2, 3, 4$$

and the metric equations and Ricci identities now take the form

$$B \frac{\partial A}{\partial z} = -2 \varepsilon A$$

$$\frac{\partial B}{\partial t} = 0$$

$$A \frac{\partial}{\partial t} \xi^i = - \rho \xi^i + \delta A \delta_4^i \qquad i = 1, 2, 3, 4$$

$$B \frac{\partial}{\partial z} \xi^i = \delta B \delta_3^i \qquad i = 1, 2, 3, 4$$

$$\bar{\delta} \xi^i - \delta \bar{\xi}^i = -4 \rho^0 A \delta_4^i \qquad i = 1, 2, 3, 4$$

$$\frac{\partial \rho^0}{\partial t} = 0 \qquad B \frac{\partial \rho^0}{\partial z} = 2 \rho^0 \varepsilon$$

$$\frac{\partial \rho}{\partial t} = 0 \qquad B \frac{\partial \rho}{\partial z} = -2 \rho \varepsilon \qquad \delta\rho = 0$$

$$\frac{\partial \varepsilon}{\partial t} = 0 \qquad B \frac{\partial \varepsilon}{\partial z} = -2 \varepsilon^2 \qquad \delta\varepsilon = 0$$

The first of the metric equations above together with the Ricci identities $\frac{\partial \varepsilon}{\partial t} = \delta\varepsilon = 0$ immediately imply that

$$\xi^3 = 0$$

It is possible to make a coordinate transformation and put

$$B = 1 \qquad A = A(z)$$

Integrating the above equations we obtain

$$A = \frac{1}{az + b} \qquad 2\varepsilon = \frac{a}{az + b}$$

$$\rho = \frac{ic}{az + b} \qquad \rho^0 = i\ (az + b)f$$

$$S = \frac{c}{az + b} - (az + b)f \tag{9.35}$$

where a, b and c are real constants, one of which may be taken to be unity if it is non-zero. (The constant c is related to the energy density of the fluid by

$$e = \frac{2c^2}{k(az + b)^2}$$

and is therefore required to be non-zero).

$f = f\ (x,y)$ is a real function yet to be determined.

A coordinate transformation can be made to put

$$\xi^2 = i\ \xi^1$$

The functions $\xi^i = \xi^i\ (x,y,t)$ $(i = 1, 2, 4)$ are now given by

$$\xi^1 = -i\ \xi^2 = Pe^{-ict}, \qquad \xi^4 = Qe^{-ict}$$

where P = P(x,y) and Q = Q(x,y) are complex. However a further transformation can also be used to make P into a real function. The remaining metric equations now give

$$Q = \frac{1}{c}\left(i\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$2cf\,(x,y) = -P^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log P \tag{9.36}$$

The latter equation may be taken to define the function f when P is considered to be an arbitrary function. Alternatively it may be considered as an equation for P when c and f are defined by the torsion according to (9.35).

The metric is thus given by

$$ds^2 = \frac{(az+b)^2}{2}\left[dt + \frac{1}{Pc}\left(\frac{\partial P}{\partial y}dx - \frac{\partial P}{\partial x}dy\right)\right]^2 - \frac{1}{2P^2}(dx^2 + dy^2) - \frac{1}{2}dz^2$$

The four-velocity of the fluid and its intrinsic angular momentum four-vector are given by

$$U^\mu = \frac{\sqrt{2}}{(az+b)}\,\delta_4^{\ \mu}\,, \quad \tau^\mu = \frac{2S}{k}\,\delta_3^{\ \mu}$$

The non-zero components of the curvature tensor are

$$\Phi_{00} = 2\,\Phi_{11} = \Phi_{22} = -6\Lambda = \frac{c^2}{(az+b)^2}$$

$$\Theta_{00} = \Theta_{22} = -\frac{2ac}{(az+b)^2}\,, \quad \Theta_{11} = 3\Sigma = -af\,(x,y)$$

$$\Psi_2 = -\frac{2}{3}\frac{c^2}{(ax+b)^2} + \frac{2}{3}a\;i\,f\,(x,y)$$

Finally, the kinematic quantities are

$$\dot{U}_\mu = + \frac{2a}{(az + b)} \delta_\mu^{\;3}$$

$$\Theta = 0, \qquad \sigma_{\mu\nu} = 0$$

$$\omega^{\;0}_{\mu\nu} = \frac{(az + b)\ f(x,y)}{\sqrt{2}\ \ P^2} (\delta_\mu^{\;2}\ \delta_\nu^{\;1} - \delta_\mu^{\;1}\ \delta_\nu^{\;2})$$

Thus the solution in general has non-zero acceleration and vorticity. It has three arbitrary parameters a, b and c and an arbitrary function P(x,y).

The above solution is seen to reduce to a stiff matter algebraic type D solution of a perfect fluid in general relativity in the limit when torsion vanishes. This occurs if $a = 0$ and $f = c/b^2$ and then, the acceleration vector is zero. Therefore the resulting solution belongs to the same class of solutions considered by Wainwright (1970). By putting $b = 1$ in this case, a simple solution of (9.36) may be obtained in the form

$$P(x) = d \cosh x + \sqrt{d^2 + 2c^2} \sinh x$$

where d is an arbitrary constant.

It is of interest to note that if $e = p = 0$ then the above equations do not imply the vanishing of the torsion. For a physical spin fluid however, the energy must be non-zero. (We are not considering "ghost" Weyssenhoff fluids!).

9.4 Discussion

The exact solutions given above are believed to be new. Most previously known solutions for Weyssenhoff fluids in Einstein-Cartan theory have been reviewed by Kuchowicz (1976). These have all been described in terms of possible cosmological models and interest is focused on the question as to whether or not such models have singularities. They all have zero acceleration and vorticity, and have been obtained using more familiar and standard techniques. The solutions presented above generally have non-zero acceleration and are therefore possibly the first "tilted" cosmological models in Einstein-Cartan theory, although they are restricted to the case of stiff matter.

A number of authors however have considered semi-classical spin fluid theory in a slightly different form to that which has been used in this chapter. For example Kuchowicz (1976) appears to specify only the symmetric energy momentum tensor and therefore does not obtain the antisymmetric equation (9.8) as an additional field equation. However not all the results described in his review have been obtained using his approach. Also a number of authors have considered the restriction

$$\tau_{\mu\alpha} P^{\alpha} = 0 \qquad (9.37)$$

instead of the Weyssenhoff restriction (9.5). Rosenbaum, Ryan and Shepley (1979) have obtained a number of simple exact solutions using this condition.

The general results described in Section 9.2 do not apply to these two alternative approaches. However for the exact solutions given in Section 9.3, $f_{\mu} = 0$, so the energy-momentum tensor is symmetric and the restriction (9.37) is identical to (9.5). Thus the exact solutions obtained here are also solutions of these alternative theories.

CHAPTER 10

CONCLUSION

The main purpose of this work has been to develop an alternative set of techniques for the analysis of source fields in U_4 theories of gravity. The generalised N-P identities are derivable from two equivalent mathematical approaches. These involve the use of tetrads and spinor dyads and are given exhaustively in this thesis. Note however that the tetrad approach can also be developed in terms of exterior differential forms. This would involve the extension to U_4 of the complex vectorial formalism of Debever (1964) and Cahen, Debever and Defrise (1967).

As the formalism developed is an extension to U_4 of an established technique in general relativity, it is tempting to simply add torsion to all the main results of the N-P formalism for Einstein's theory. However it is not obvious that this would be useful. Each application of the formalism should achieve some goal of the torsion theories. For example, it was hoped that the anomalous properties of the classical Neutrino field in Einstein's theory would be removed in the Einstein-Cartan theory. The techniques developed here have proven to be ideal for the investigation of such a proposal.

The generalised formalism has in fact been extremely successful when applied to the Neutrino field in Einstein-Cartan theory. In this case it probably offers the best set of techniques for obtaining exact solutions. The contortion tensor for the field has a particularly simple form and all exact solutions for the case of pure radiation fields are given in Chapter 8. Although the anomalies mentioned above still persist, they do not occur as readily in Einstein-Cartan theory. For example the only possible ghost Neutrino solutions have a plane wave metric.

It has been difficult to generate exact solutions for the case of Weyssenhoff fluids in Einstein-Cartan theory using the formalism. This is not surprising since the N-P formalism has not been particularly convenient for the analysis of perfect fluids in G.R.. Severe assumptions have had to be made in order to obtain the very special solutions presented in Chapter 9. However, further progress can still be made along the lines developed. In particular one might consider spinning dust and then, matter corresponding to a more general equation of state for the fluid. It should also be possible to obtain exact solutions in the case when the generalised Weyl tensor $A_{\kappa\lambda\mu\nu}$ is zero. Such solutions are seen to generalise the conformally flat perfect fluid solutions in Einstein's theory. On an optimistic note, the solutions obtained for Weyssenhoff fluids are the first to describe "tilted cosmological models" in the Einstein-Cartan theory.

The applications of the formalism in Chapters 8 and 9 have clarified the role of the Bianchi identities for the torsion. After the tetrad or dyad basis is adapted in each case, the non-zero components of the contortion tensor can be inserted into these identities. They then yield the components of the curvature that are peculiar to U_4. These components together with the components of the Ricci tensor can then be inserted into the appropriate Bianchi identities for the curvature and yield conditions on certain spin coefficients. This general procedure will probably be appropriate to most future applications of the formalism. Further conditions on the spin coefficients appear through the matter field equations or the conservation theorems. The generalised identities are then considerably reduced. In order to obtain a particular solution it is necessary to make assumptions on certain spin coefficients and also on the "free" gravitational field described by the components of $A_{\kappa\lambda\mu\nu}$ which are not determined by the field equations.

For future applications of the formalism, there are a number of different directions open to us. In connection with these, a few comments on alternative torsion theories of gravity are appropriate. As already mentioned in Chapter 7, one troublesome point in the usual form of the Einstein-Cartan theory is that the electro-magnetic field is decoupled from torsion. If, in general, particles with spin are to generate and react to torsion then it is reasonable to expect that photons should be coupled to it. The authors Hojman et al (1978) have suggested a way of achieving this and have also presented a dynamical theory (the HRRS theory) which allows the propagation of torsion in a vacuum. It should be pointed out however that the HRRS theory is refuted by the results of solar system experiments (Ni 1979). Nevertheless the work of Hojman et al has generated considerable interest on gravity theories which permit torsion to propagate in a vacuum (see for example the review by Neville 1980). It is believed that the N-P type formalism presented in this thesis may provide a convenient method for the analysis and generation of exact solutions in these theories.

We now turn to gravitation theories formulated in the Weitzenbock space-time (A_4) (Weitzenbock 1923). This space-time is obtained from a Riemann-Cartan space-time U_4 by simply putting the curvature tensor to zero. This definition is equivalent to the notion of absolute parallelism in U_4. One of these so called teleparallel theories formulated in A_4 that might be considered in future applications of the formalism is that proposed by Hayashi and Shirafuji (1979). The generalised identities in such theories are drastically reduced and the Bianchi identities for the torsion take over the role of the V_4 Bianchi identities for the curvature in that they are the only set of integrability conditions. Teleparallel theories obviously do not reduce to Einstein's theory and they therefore offer one application of the formalism not motivated through the successes of the N-P formalism in general relativity.

Finally, there are also torsion theories in which a variety of alternative lagrangians are proposed (see the recent review by Wallner 1980). These are also worthy of consideration in applications of the generalised formalism.

APPENDIX A

Consider the most general transformation of the basis dyad spinors

$$\tilde{o}^A = a\, o^A + b\, \iota^A \tag{A.1(a)}$$

$$\tilde{\iota}^A = c\, o^A + d\, \iota^A \tag{b}$$

where a, b, c and d are complex scalar functions which, from the orthogonality relation (4.8) satisfy

$$ad - bc = 1$$

The corresponding general transformation of the tetrad vectors is easily shown to be

$$\tilde{\ell}^\mu = a\bar{a}\, \ell^\mu + a\bar{b}\, m^\mu + b\bar{a}\, \bar{m}^\mu + b\bar{b}\, n^\mu \tag{A.2(a)}$$

$$\tilde{m}^\mu = a\bar{c}\, \ell^\mu + a\bar{d}\, m^\mu + b\bar{c}\, \bar{m}^\mu + b\bar{d}\, n^\mu \tag{b}$$

$$\tilde{n}^\mu = c\bar{c}\, \ell^\mu + c\bar{d}\, m^\mu + d\bar{c}\, \bar{m}^\mu + d\bar{d}\, n^\mu \tag{c}$$

A special case of interest occurs when b = 0 in these transformations. In this case, the direction of ℓ^μ is preserved and if we put

$$a = R^{\frac{1}{2}}\, e^{\frac{i}{2}S}$$

$$c = R^{\frac{1}{2}}\, \overline{T}\, e^{-\frac{i}{2}S}$$

where $R > 0$ and S are real functions and T is complex then

$$\tilde{\ell}^{\mu} = R\, \ell^{\mu} \tag{A.3)(a}$$

$$\tilde{m}^{\mu} = e^{iS}\, (m^{\mu} + RT\, \ell^{\mu}) \tag{b}$$

$$\tilde{n}^{\mu} = R^{-1}\, n^{\mu} + \overline{T}\, m^{\mu} + T\, \overline{m}^{\mu} + RT\overline{T}\, \ell^{\mu} \tag{c}$$

This transformation is called the <u>null rotation</u> about ℓ^{μ} and corresponds to a Lorentz transformation in the tangent space, leaving fixed the direction of ℓ^{μ}.

The behaviour of the variables in the spin coefficient formalism under (A.3) is listed below:

<u>With $T = 0$:</u>

$$\begin{aligned}
\tilde{\kappa} &= R^2 e^{iS}\, \kappa & \tilde{\kappa}_1 &= R^2 e^{iS}\, \kappa_1 \\
\tilde{\tau} &= e^{iS}\, \tau & \tilde{\tau}_1 &= e^{iS}\, \tau_1 \\
\tilde{\sigma} &= R\, e^{2iS}\, \sigma & \tilde{\sigma}_1 &= R\, e^{2iS}\, \sigma_1 \\
\tilde{\rho} &= R\rho & \tilde{\rho}_1 &= R\, \rho_1 \\
\tilde{\pi} &= e^{-iS}\, \pi & \tilde{\pi}_1 &= e^{-iS}\, \pi_1 \\
\tilde{\nu} &= R^{-2} e^{-iS}\, \nu & \tilde{\nu}_1 &= R^{-2} e^{-iS}\, \nu_1 \\
\tilde{\mu} &= R^{-1}\, \mu & \tilde{\mu}_1 &= R^{-1}\, \mu_1 \\
\tilde{\lambda} &= R^{-1} e^{-2iS}\, \lambda & \tilde{\lambda}_1 &= R^{-1} e^{-2iS}\, \lambda_1 \\
\tilde{\varepsilon} &= R\varepsilon + \tfrac{1}{2}\, DR + \tfrac{i}{2}\, RDS & \tilde{\varepsilon}_1 &= R\, \varepsilon_1 \\
\tilde{\gamma} &= R^{-1}\gamma + \tfrac{1}{2}\, R^{-2}\, \Delta R + \tfrac{i}{2}\, R^{-1}\Delta S & \tilde{\gamma}_1 &= R^{-1}\, \gamma_1 \\
\tilde{\beta} &= e^{iS}\, (\beta + \tfrac{1}{2}\, R^{-1}\delta R + \tfrac{i}{2}\, \delta S) & \tilde{\beta}_1 &= e^{iS}\, \beta_1 \\
\tilde{\alpha} &= e^{-iS}\, (\alpha + \tfrac{1}{2}\, R^{-1}\overline{\delta}R + \tfrac{i}{2}\, \overline{\delta}S) & \tilde{\alpha}_1 &= e^{-iS}\, \alpha_1
\end{aligned} \tag{A.4}$$

$$\tilde{\Phi}_{00} = R^2 \, \Phi_{00} \qquad \tilde{\Theta}_{00} = R^2 \, \Theta_{00}$$

$$\tilde{\Phi}_{01} = R \, e^{iS} \, \Phi_{01} \qquad \tilde{\Theta}_{01} = R \, e^{iS} \, \Theta_{01}$$

$$\tilde{\Phi}_{02} = e^{2iS} \, \Phi_{02} \qquad \tilde{\Theta}_{02} = e^{2iS} \, \Theta_{02}$$

$$\tilde{\Phi}_{11} = \Phi_{11} \qquad \tilde{\Theta}_{11} = \Theta_{11}$$

$$\tilde{\Phi}_{12} = R^{-1} \, e^{iS} \, \Phi_{12} \qquad \tilde{\Theta}_{12} = R^{-1} \, e^{iS} \, \Theta_{12}$$

$$\tilde{\Phi}_{22} = R^{-2} \, \Phi_{22} \qquad \tilde{\Theta}_{22} = R^{-2} \, \Theta_{22}$$

$$\tilde{\Psi}_0 = R^2 \, e^{2iS} \, \Psi_0$$

$$\tilde{\Psi}_1 = R \, e^{iS} \, \Psi_1 \qquad \tilde{\Phi}_0 = R \, e^{iS} \, \Phi_0$$

$$\tilde{\Psi}_2 = \Psi_2 \qquad \tilde{\Phi}_1 = \Phi_1$$

$$\tilde{\Psi}_3 = R^{-1} \, e^{-iS} \, \Psi_3 \qquad \tilde{\Phi}_2 = R^{-1} \, e^{-iS} \, \Phi_2$$

$$\tilde{\Psi}_4 = R^{-2} \, e^{-2iS} \, \Psi_4 \tag{A.5}$$

<u>With R = 1, S = 0:</u>

$\tilde{\kappa} = \kappa$

$\tilde{\tau} = \tau + T\rho + \overline{T}\sigma + T\overline{T}\kappa$

$\tilde{\sigma} = \sigma + T\kappa$

$\tilde{\rho} = \rho + \overline{T}\kappa$

$\tilde{\varepsilon} = \varepsilon + \overline{T}\kappa$

$\tilde{\gamma} = \gamma + T\alpha + \overline{T}\,(\beta + \tau) + T\overline{T}\,(\rho + \varepsilon) + \overline{T}^2\sigma + T\overline{T}\kappa$

$\tilde{\beta} = \beta + \overline{T}\sigma + T\varepsilon + T\overline{T}\kappa$

$\tilde{\alpha} = \alpha + \overline{T}(\rho + \varepsilon) + \overline{T}^2\,\kappa$

$\tilde{\pi} = \pi + 2\overline{T}\varepsilon + \overline{T}^2\kappa + D\overline{T}$

$$\tilde{\nu} = \nu + T\lambda + \overline{T}\,(\mu + 2\gamma) + T\overline{T}\,(\pi + 2\alpha) + \overline{T}^2(\tau + 2\beta) + T\overline{T}^2(\rho + 2\varepsilon) + \overline{T}^3\sigma + T\overline{T}^3\kappa + \Delta\overline{T} + T\,\overline{\delta}\,\overline{T} + \overline{T}\,\delta\,\overline{T} + T\overline{T}\,D\overline{T}$$

$\tilde{\mu} = \mu + T\pi + 2\overline{T}\beta + 2\,T\overline{T}\varepsilon + \overline{T}^2\sigma + T\overline{T}^2\kappa + \delta\overline{T} + TD\overline{T}$

$\tilde{\lambda} = \lambda + \overline{T}\,(\pi + 2\alpha) + \overline{T}^2\,(\rho + 2\varepsilon) + \overline{T}^3\kappa + \overline{\delta}\overline{T} + \overline{T}D\overline{T}$

$\tilde{\kappa}_1 = \kappa_1$

$\tilde{\tau}_1 = \tau_1 + T\,\rho_1 + \overline{T}\,\sigma_1 + T\overline{T}\,\kappa_1$

$\tilde{\sigma}_1 = \sigma_1 + T\,\kappa_1$

$\tilde{\rho}_1 = \rho_1 + \overline{T}\,\kappa_1$

$\tilde{\varepsilon}_1 = \varepsilon_1 + \overline{T}\,\kappa$

$\tilde{\gamma}_1 = \gamma_1 + T\alpha_1 + \overline{T}\,(\beta_1 + \tau_1) + T\overline{T}\,(\rho_1 + \varepsilon_1) + \overline{T}^2\,\sigma_1 + T\overline{T}\,\kappa_1$

$\tilde{\beta}_1 = \beta_1 + \overline{T}\,\sigma_1 + T\varepsilon_1 + T\overline{T}\,\kappa_1$

$\tilde{\alpha}_1 = \alpha_1 + \overline{T}\,(\rho_1 + \varepsilon_1) + \overline{T}^2\,\kappa_1$

$$\tilde{\pi}_1 = \pi_1 + 2\bar{T}\,\varepsilon_1 + \bar{T}^2\,\kappa_1$$

$$\tilde{\nu}_1 = \nu_1 + T\lambda_1 + \bar{T}\,(\mu_1+2\gamma_1) + T\bar{T}\,(\pi_1+2\alpha_1) + \bar{T}^2(\tau_1+2\beta_1) + T\bar{T}^2(\rho_1+2\varepsilon_1)$$
$$+ \bar{T}^3\,\sigma_1 + T\bar{T}^3\,\kappa_1$$

$$\tilde{\mu}_1 = \mu_1 + T\,\pi_1 + 2\bar{T}\,\beta_1 + 2\,T\bar{T}\,\varepsilon_1 + \bar{T}^2\,\sigma_1 + T\bar{T}^2\kappa_1$$

$$\tilde{\lambda}_1 = \lambda_1 + \bar{T}\,(\pi_1+2\alpha_1) + \bar{T}^2\,(\rho_1+2\varepsilon_1) + \bar{T}^3\,\kappa_1$$

(A.6)

$$\tilde{\Phi}_{00} = \Phi_{00}$$

$$\tilde{\Phi}_{01} = \Phi_{01} + T\Phi_{00}$$

$$\tilde{\Phi}_{02} = \Phi_{02} + 2T\Phi_{01} + T^2\Phi_{00}$$

$$\tilde{\Phi}_{11} = \Phi_{11} + T\Phi_{10} + \bar{T}\Phi_{01} + T\bar{T}\Phi_{00}$$

$$\tilde{\Phi}_{12} = \Phi_{12} + 2T\Phi_{11} + \bar{T}\Phi_{02} + 2T\bar{T}\Phi_{01} + T^2\Phi_{10} + T^2\bar{T}\Phi_{00}$$

$$\tilde{\Phi}_{22} = \Phi_{22} + 2T\Phi_{21} + 2\bar{T}\Phi_{12} + 4T\bar{T}\Phi_{11} + T^2\Phi_{20} + \bar{T}^2\Phi_{02} + 2T^2\bar{T}\Phi_{10} + $$
$$+ 2T\bar{T}^2\Phi_{01} + T^2\bar{T}^2\Phi_{00}$$

$$\tilde{\Theta}_{00} = \Theta_{00}$$

$$\tilde{\Theta}_{01} = \Theta_{01} + T\,\Theta_{00}$$

$$\tilde{\Theta}_{02} = \Theta_{02} + 2T\,\Theta_{01} + T^2\,\Theta_{00}$$

$$\tilde{\Theta}_{11} = \Theta_{11} + T\,\Theta_{10} + \bar{T}\,\Theta_{01} + T\bar{T}\,\Theta_{00}$$

$$\tilde{\Theta}_{12} = \Theta_{12} + 2T\,\Theta_{11} + \bar{T}\,\Theta_{02} + 2T\bar{T}\,\Theta_{01} + T^2\,\Theta_{10} + T^2\,\bar{T}\,\Theta_{00}$$

$$\tilde{\Theta}_{22} = \Theta_{22} + 2T\,\Theta_{21} + 2\bar{T}\,\Theta_{12} + 4T\bar{T}\,\Theta_{11} + T^2\,\Theta_{20} + \bar{T}^2\,\Theta_{02} +$$
$$+ 2T\,\bar{T}^2\Theta_{10} + 2T\bar{T}^2\,\Theta_{01} + T^2\bar{T}^2\,\Theta_{00}$$

$$\tilde{\Psi}_0 = \Psi_0$$

$$\tilde{\Psi}_1 = \Psi_1 + \bar{T}\,\Psi_0$$

$$\tilde{\Psi}_2 = \Psi_2 + 2\bar{T}\,\Psi_1 + \bar{T}^2\,\Psi_0$$

$$\tilde{\Psi}_3 = \Psi_3 + 3\bar{T}\,\Psi_2 + 3\bar{T}^2\Psi_1 + \bar{T}^3\,\Psi_0$$

$$\tilde{\Psi}_4 = \Psi_4 + 4\bar{T}\Psi_3 + 6\bar{T}^2\Psi_2 + 4\,\bar{T}^3\Psi_1 + \bar{T}^4\Psi_0$$

$$\tilde{\Phi}_0 = \Phi_0$$
$$\tilde{\Phi}_1 = \Phi_1 + 2\overline{T}\,\Phi_0$$
$$\tilde{\Phi}_2 = \Phi_2 + \overline{T}\,\Phi_1 + \overline{T}^2\Phi_0 \tag{A.7}$$

APPENDIX B

It is useful to know the way in which a component of the curvature tensor differs from its value in the associated V_4. This then clearly describes the effect of torsion. Therefore, in the spirit of the spin coefficient formalism, the following expansions are given using the notations in Chapters 3, 4 and 5. They have been obtained by taking the tetrad components of equation (2.18) but may also be derived from the dyad components of its spinor equivalent. The expansions of the new components that vanish in a V_4, could alternatively have been obtained directly from the Bianchi identity for the torsion given in §5.3.

$$\Psi_0 = \Psi_0^o + D\sigma_1 - \delta\kappa_1 - (\bar{\rho}^o + 3\epsilon^o - \bar{\epsilon}^o)\sigma_1 - \sigma^o(\rho_1 + 2\epsilon_1) - (\bar{\pi}^o - 3\beta^o - \bar{\alpha}^o)\kappa_1$$
$$+ \kappa^o(\tau_1 + 2\beta_1) - 2\sigma_1\epsilon_1 + 2\beta_1\kappa_1 \qquad \text{(B.1)(a)}$$

$$4\Psi_1 = 4\Psi_1^o + D(\tau_1 + 2\beta_1) - \Delta\kappa_1 - \delta(\rho_1 + 2\epsilon_1) + \bar{\delta}\sigma_1$$
$$-(\tau^o + \bar{\pi}^o - \bar{\alpha}^o - \beta^o)(\rho_1 + 2\epsilon_1) + (\rho^o - \bar{\rho}^o - \epsilon^o + \bar{\epsilon}^o)(\tau_1 + 2\beta_1) - (\bar{\tau}^o + 3\pi^o + 3\alpha^o - \bar{\beta}^o)\sigma_1$$
$$- 2\sigma^o(\pi_1 + 2\alpha_1) + (3\mu^o - \bar{\mu}^o + 3\gamma^o + \bar{\gamma}^o)\kappa_1 + 2\kappa^o(\mu_1 + 2\gamma_1)$$
$$+ 2\beta_1\rho_1 - 2(\pi_1 + \alpha_1)\sigma_1 - 2\epsilon_1\tau_1 + 2(\mu_1 + \gamma_1)\kappa_1 \qquad \text{(b)}$$

$$6\Psi_2 = 6\Psi_2^{\ o} + D(\mu_1 + 2\gamma_1) - \Delta(\rho_1 + 2\epsilon_1) - \delta(\pi_1 + 2\alpha_1) + \bar{\delta}(\tau_1 + 2\beta_1)$$
$$+ (2\mu^o - \bar{\mu}^o + \gamma^o + \bar{\gamma}^o)(\rho_1 + 2\epsilon_1) + (2\rho^o - \bar{\rho}^o + \epsilon^o + \bar{\epsilon}^o)(\mu_1 + 2\gamma_1)$$
$$- (2\pi^o + \bar{\tau}^o + \alpha^o - \bar{\beta}^o)(\tau_1 + 2\beta_1) - (2\tau^o + \pi^o - \bar{\alpha}^o + \beta^o)(\pi_1 + 2\alpha_1) \qquad \text{(c)}$$
$$+ 3\nu^o\kappa_1 + 3\kappa^o\nu_1 - 3\sigma^o\lambda_1 - 3\lambda^o\sigma_1$$
$$+ 2(\mu_1 + \gamma_1)\rho_1 - 2(\pi_1 + \alpha_1)\tau_1 + 2\nu_1\kappa_1 - 2\lambda_1\sigma_1 + 2\epsilon_1\mu_1 - 2\beta_1\pi_1$$

$$4\Psi_3 = 4\Psi_3^o + D\nu_1 - \Delta(\pi_1 + 2\alpha_1) - \delta\lambda_1 + \bar{\delta}(\mu_1 + 2\gamma_1)$$
$$- (\bar{\tau}^o + \pi^o - \alpha^o - \bar{\beta}^o)(\mu_1 + 2\gamma_1) + (\mu^o - \bar{\mu}^o - \gamma^o + \bar{\gamma}^o)(\pi_1 + 2\alpha_1) - (3\tau^o + \bar{\pi}^o - \bar{\alpha}^o + 3\beta^o)\lambda_1$$
$$- 2\lambda^o(\tau_1 + 2\beta_1) + (3\rho^o - \bar{\rho}^o + 3\epsilon^o + \bar{\epsilon}^o)\nu_1 + 2\nu^o(\rho_1 + 2\epsilon_1)$$
$$+ 2\alpha_1\mu_1 - 2(\tau_1 + \beta_1)\lambda_1 - 2\gamma_1\pi_1 + 2(\rho_1 + \epsilon_1)\nu_1 \qquad \text{(d)}$$

$$\Psi_4 = \Psi_4^o - \Delta\lambda_1+\bar{\delta}\nu_1-(\bar{\mu}^o+3\gamma^o-\bar{\gamma}^o)\lambda_1 - \lambda^o(\mu_1+2\gamma_1) - (\bar{\tau}^o-3\alpha^o-\bar{\beta}^o)\nu_1$$

$$+ \nu^o(\pi_1+2\alpha_1) - 2\lambda_1\gamma_1+2\alpha_1\nu_1 \qquad \text{(B.1)(e)}$$

$$\begin{aligned}12i\Sigma = {} & D(\mu_1-\bar{\mu}_1-\gamma_1+\bar{\gamma}_1) - \Delta(\rho_1-\bar{\rho}_1-\epsilon_1+\bar{\epsilon}_1) - \delta(\bar{\tau}_1+\pi_1-\alpha_1-\bar{\beta}_1)+\bar{\delta}(\tau_1+\bar{\pi}_1-\bar{\alpha}_1-\beta_1)\\
& - (\mu^o+\bar{\mu}^o-\gamma^o-\bar{\gamma}^o)(\rho_1-\bar{\rho}_1-\epsilon_1+\bar{\epsilon}_1) - (\rho^o+\bar{\rho}^o-\epsilon^o-\bar{\epsilon}^o)(\mu_1-\bar{\mu}_1-\gamma_1+\bar{\gamma}_1)\\
& - (\bar{\tau}^o-\pi^o+\alpha^o-\bar{\beta}^o)(\tau_1+\bar{\pi}_1-\bar{\alpha}_1-\beta_1) + (\tau^o-\bar{\pi}^o+\bar{\alpha}^o-\beta^o)(\bar{\tau}_1+\pi_1-\alpha_1-\bar{\beta}_1)\\
& - (\mu_1-2\gamma_1)\rho_1 + (\bar{\mu}_1-2\bar{\gamma}_1)\bar{\rho}_1+2\epsilon_1\mu_1-2\bar{\epsilon}_1\bar{\mu}_1+\lambda_1\sigma_1-\bar{\lambda}_1\bar{\sigma}_1\\
& + (\pi_1-2\alpha_1)\tau_1 - (\bar{\pi}_1-2\bar{\alpha}_1)\bar{\tau}_1-2\beta_1\pi_1+2\bar{\beta}_1\bar{\pi}_1 - \nu_1\kappa_1+\bar{\nu}_1\bar{\kappa}_1 \qquad \text{(B.2)}\end{aligned}$$

$$\begin{aligned}2i\Theta_{00} = {} & D(\rho_1-\bar{\rho}_1)-\bar{\delta}\kappa_1+\delta\bar{\kappa}_1-(\epsilon^o+\bar{\epsilon}^o)(\rho_1-\bar{\rho}_1)-\rho^o(\rho_1+2\epsilon_1)+\bar{\rho}^o(\bar{\rho}_1+2\bar{\epsilon}_1)-\bar{\sigma}^o\sigma_1+\sigma^o\bar{\sigma}_1\\
& + \bar{\kappa}^o(\tau_1-2\bar{\alpha}_1)-\kappa^o(\bar{\tau}_1-2\alpha_1)-(\pi^o-3\alpha^o-\bar{\beta}^o)\kappa_1+(\bar{\pi}^o-3\bar{\alpha}^o-\beta^o)\bar{\kappa}_1\\
& - 2\epsilon_1\rho_1+2\bar{\epsilon}_1\bar{\rho}_1+2\alpha_1\kappa_1-2\bar{\alpha}_1\bar{\kappa}_1 \qquad \text{(B.3)(a)}\end{aligned}$$

$$\begin{aligned}4i\Theta_{01} = {} & D(\tau_1-2\bar{\alpha}_1)-\Delta\kappa_1+\delta(\rho_1+2\bar{\epsilon}_1)-\bar{\delta}\sigma_1-(\bar{\alpha}^o+\beta^o)(\rho_1+2\bar{\epsilon}_1)-\bar{\pi}^o(\rho_1-2\bar{\rho}_1-2\bar{\epsilon}_1)\\
& - \tau^o(\rho_1+2\epsilon_1)-\rho^o(\tau_1+2\beta_1)+\bar{\rho}^o(\tau_1+2\bar{\pi}_1+2\bar{\alpha}_1)-(\mu^o-\bar{\mu}^o-3\gamma^o-\bar{\gamma}^o)\kappa_1\\
& - (\pi^o+\bar{\tau}^o-3\alpha^o+\bar{\beta}^o)\sigma_1+2\sigma^o(\alpha_1+\bar{\beta}_1)-2\bar{\kappa}^o\bar{\lambda}_1-2\bar{\lambda}^o\bar{\kappa}_1\\
& - (\epsilon^o-\bar{\epsilon}^o)(\tau_1-2\bar{\alpha}_1)+2\kappa^o(\gamma_1-\bar{\gamma}_1) \qquad \text{(b)}\\
& - 2\beta_1\rho_1+2\bar{\pi}_1\bar{\rho}_1+2\alpha_1\sigma_1-2\epsilon_1\tau_1+2\gamma_1\kappa_1-2\bar{\lambda}_1\bar{\kappa}_1\end{aligned}$$

$$\begin{aligned}2i\Theta_{02} = {} & -D\bar{\lambda}_1-\Delta\sigma_1+\delta(\tau_1+\bar{\pi}_1)-\bar{\lambda}^o(\rho_1+2\bar{\epsilon}_1)+(\bar{\rho}^o+\epsilon^o-3\bar{\epsilon}^o)\bar{\lambda}_1+\sigma^o(\bar{\mu}_1+2\gamma_1)-(\mu^o-3\gamma^o+\bar{\gamma}^o)\sigma_1\\
& + (\bar{\alpha}^o-\beta^o)(\tau_1+\bar{\pi}_1)-\tau^o(\tau_1+2\beta_1)+\bar{\pi}^o(\bar{\pi}_1+2\bar{\alpha}_1)-\kappa^o\bar{\nu}_1+\bar{\nu}^o\kappa_1\\
& + 2\gamma_1\sigma_1-2\bar{\epsilon}_1\bar{\lambda}_1-2\beta_1\tau_1+2\bar{\alpha}_1\bar{\pi}_1 \qquad \text{(c)}\end{aligned}$$

$$
\begin{aligned}
4i\Theta_{11} = {} & D(\gamma_1-\bar{\gamma}_1)-\Delta(\varepsilon_1-\bar{\varepsilon}_1)+\delta(\alpha_1+\bar{\beta}_1)-\bar{\delta}(\bar{\alpha}_1+\beta_1)-\mu^o(\rho_1+\varepsilon_1+\bar{\varepsilon}_1)+\bar{\mu}^o(\bar{\rho}_1+\varepsilon_1+\bar{\varepsilon}_1) \\
& -\rho^o(\mu_1+\gamma_1+\bar{\gamma}_1)+\bar{\rho}^o(\bar{\mu}_1+\gamma_1+\bar{\gamma}_1)+\lambda^o\sigma_1-\bar{\lambda}^o\bar{\sigma}_1+\sigma^o\lambda_1-\bar{\sigma}^o\bar{\lambda}_1 \\
& -\pi^o(\tau_1-\bar{\alpha}_1+\beta_1)+\bar{\pi}^o(\bar{\tau}_1-\alpha_1+\bar{\beta}_1)-\tau^o(\pi_1+\alpha_1-\bar{\beta}_1)+\bar{\tau}^o(\bar{\pi}_1+\bar{\alpha}_1-\beta_1) \\
& +\nu^o\kappa_1-\bar{\nu}^o\bar{\kappa}_1+\kappa^o\nu_1-\bar{\kappa}^o\bar{\nu}_1-(\bar{\alpha}^o-\beta^o)(\alpha_1+\bar{\beta}_1)+(\alpha^o-\bar{\beta}^o)(\bar{\alpha}_1+\beta_1) \\
& +(\varepsilon^o+\bar{\varepsilon}^o)(\gamma_1-\bar{\gamma}_1)+(\gamma^o+\bar{\gamma}^o)(\varepsilon_1-\bar{\varepsilon}_1) \qquad \text{(B.3)(d)} \\
& -\mu_1\rho_1+\bar{\mu}_1\bar{\rho}_1+\lambda_1\sigma_1-\bar{\lambda}_1\bar{\sigma}_1+\nu_1\kappa_1-\bar{\nu}_1\bar{\kappa}_1-\pi_1\tau_1+\bar{\pi}_1\bar{\tau}_1
\end{aligned}
$$

$$
\begin{aligned}
4i\Theta_{12} = {} & -D\bar{\nu}_1+\Delta(\bar{\pi}_1-2\beta_1)-\bar{\delta}\bar{\lambda}_1+\delta(\bar{\mu}_1+2\gamma_1)-(\rho^o-\bar{\rho}^o+\varepsilon^o+3\bar{\varepsilon}^o)\bar{\nu}_1 \\
& -\tau^o(2\mu_1-\bar{\mu}_1+2\gamma_1)+\bar{\pi}^o(\bar{\mu}_1+2\bar{\gamma}_1)+(\bar{\alpha}^o+\beta^o)(\bar{\mu}_1+2\gamma_1)-\mu^o(\bar{\pi}_1+2\tau_1+2\beta_1) \\
& +\bar{\mu}^o(\bar{\pi}_1+2\bar{\alpha}_1)+2\nu^o\sigma_1+2\sigma^o\nu_1+(\bar{\tau}^o+\pi^o+\alpha^o-3\bar{\beta}^o)\bar{\lambda}_1 \\
& -2\bar{\lambda}^o(\alpha_1+\bar{\beta}_1)-(\gamma^o-\bar{\gamma}^o)(\bar{\pi}_1-2\beta_1)+2\bar{\nu}^o(\varepsilon_1-\bar{\varepsilon}_1) \\
& -2\tau_1\mu_1+2\bar{\alpha}_1\bar{\mu}_1+2\nu_1\sigma_1-2\bar{\beta}_1\bar{\lambda}_1+2\bar{\gamma}_1\bar{\pi}_1-2\bar{\varepsilon}_1\bar{\nu}_1 \qquad \text{(e)}
\end{aligned}
$$

$$
\begin{aligned}
2i\Theta_{22} = {} & -\Delta(\mu_1-\bar{\mu}_1)+\delta\nu_1-\bar{\delta}\bar{\nu}_1-(\gamma^o+\bar{\gamma}^o)(\mu_1-\bar{\mu}_1)-\mu^o(\mu_1+2\gamma_1)+\bar{\mu}^o(\bar{\mu}_1+2\bar{\gamma}_1)-\bar{\lambda}^o\lambda_1+\lambda^o\bar{\lambda}_1 \\
& -(\tau^o-\bar{\alpha}^o-3\beta^o)\nu_1+(\bar{\tau}^o-\alpha^o-3\bar{\beta}^o)\bar{\nu}_1+\bar{\nu}^o(\pi_1-2\bar{\beta}_1)-\nu^o(\bar{\pi}_1-2\beta_1) \\
& -2\gamma_1\mu_1+2\bar{\gamma}_1\bar{\mu}_1+2\beta_1\nu_1-2\bar{\beta}_1\bar{\nu}_1 \qquad \text{(f)}
\end{aligned}
$$

$$
\begin{aligned}
2\Phi_{00} = 2\Phi^o_{00} & + D(\rho_1+\bar{\rho}_1)-\delta\bar{\kappa}_1-\bar{\delta}\kappa_1-(\varepsilon^o+\bar{\varepsilon}^o)(\rho_1+\bar{\rho}_1)-\rho^o(\rho_1+2\varepsilon_1)-\bar{\rho}^o(\bar{\rho}_1+2\bar{\varepsilon}_1)-\bar{\sigma}^o\sigma_1-\sigma^o\bar{\sigma}_1 \\
& -(\pi^o-3\alpha^o-\bar{\beta}^o)\kappa_1-(\bar{\pi}^o-3\bar{\alpha}^o-\beta^o)\bar{\kappa}_1+\bar{\kappa}^o(\tau_1+2\bar{\alpha}_1)+\kappa^o(\bar{\tau}_1+2\alpha_1) \\
& -2\varepsilon_1\rho_1-2\bar{\varepsilon}_1\bar{\rho}_1+2\alpha_1\kappa_1+2\bar{\alpha}_1\bar{\kappa}_1 \qquad \text{(B.4)(a)}
\end{aligned}
$$

$$
\begin{aligned}
4\Phi_{01} = 4\Phi^{o}_{01} &+ D(\tau_1+2\bar{\alpha}_1)-\Delta\kappa_1+\delta(\rho_1-2\bar{\varepsilon}_1)-\bar{\delta}\sigma_1-(\bar{\alpha}^{o}+\beta^{o})(\rho_1-2\bar{\varepsilon}_1)-\bar{\pi}^{o}(\rho_1+2\bar{\rho}_1+2\bar{\varepsilon}_1) \\
&- \tau^{o}(\rho_1+2\varepsilon_1)-\rho^{o}(\tau_1+2\beta_1)+\bar{\rho}^{o}(\tau_1-2\bar{\pi}_1-2\bar{\alpha}_1)-(\pi^{o}+\bar{\tau}^{o}-3\alpha^{o}+\bar{\beta}^{o})\sigma_1 \\
&+ 2\sigma^{o}(\alpha_1-\bar{\beta}_1)+2\bar{\kappa}^{o}\bar{\lambda}_1+2\bar{\lambda}^{o}\bar{\kappa}_1-(\varepsilon^{o}-\bar{\varepsilon}^{o})(\tau_1+2\bar{\alpha}_1)-(\mu^{o}-\bar{\mu}^{o}-3\gamma^{o}-\bar{\gamma}^{o})\kappa_1 \\
&+ 2\kappa^{o}(\gamma_1+\bar{\gamma}_1) \\
&- 2\beta_1\rho_1-2\bar{\pi}_1\bar{\rho}_1+2\alpha_1\sigma_1-2\varepsilon_1\tau_1+2\gamma_1\kappa_1+2\bar{\lambda}_1\bar{\kappa}_1
\end{aligned}
\tag{B.4)(b}
$$

$$
\begin{aligned}
2\Phi_{02} = 2\Phi^{o}_{02} &+ D\bar{\lambda}_1-\Delta\sigma_1+\delta(\tau_1-\bar{\pi}_1)-\bar{\lambda}^{o}(\rho_1-2\bar{\varepsilon}_1)-(\bar{\rho}^{o}+\varepsilon^{o}-3\bar{\varepsilon}^{o})\bar{\lambda}_1-\sigma^{o}(\bar{\mu}_1-2\gamma_1)-(\mu^{o}-3\gamma^{o}+\bar{\gamma}^{o})\sigma_1 \\
&+ (\bar{\alpha}^{o}-\beta^{o})(\tau_1-\bar{\pi}_1)-\tau^{o}(\tau_1+2\beta_1)-\bar{\pi}^{o}(\bar{\pi}_1+2\bar{\alpha}_1)+\kappa^{o}\bar{\nu}_1+\bar{\nu}^{o}\kappa_1 \\
&+ 2\gamma_1\sigma_1+ 2\bar{\varepsilon}_1\bar{\lambda}_1-2\beta_1\tau_1-2\bar{\alpha}_1\bar{\pi}_1
\end{aligned}
\tag{c}
$$

$$
\begin{aligned}
4\Phi_{11} = 4\Phi^{o}_{11} &+ D(\gamma_1+\bar{\gamma}_1)-\Delta(\varepsilon_1+\bar{\varepsilon}_1)+\delta(\alpha_1-\bar{\beta}_1)+\bar{\delta}(\bar{\alpha}_1-\beta_1)-\mu^{o}(\rho_1+\varepsilon_1-\bar{\varepsilon}_1)-\bar{\mu}^{o}(\bar{\rho}_1-\varepsilon_1+\bar{\varepsilon}_1) \\
&- \rho^{o}(\mu_1+\gamma_1-\bar{\gamma}_1)-\bar{\rho}^{o}(\bar{\mu}_1-\gamma_1+\bar{\gamma}_1)+\lambda^{o}\sigma_1+\bar{\lambda}^{o}\bar{\sigma}_1+\sigma^{o}\lambda_1+\bar{\sigma}^{o}\bar{\lambda}_1 \\
&-\pi^{o}(\tau_1+\bar{\alpha}_1+\beta_1)-\bar{\pi}^{o}(\bar{\tau}_1+\alpha_1+\bar{\beta}_1)-\tau^{o}(\pi_1+\alpha_1+\bar{\beta}_1)-\bar{\tau}^{o}(\bar{\pi}_1+\bar{\alpha}_1+\beta_1) \\
&+\nu^{o}\kappa_1+\bar{\nu}^{o}\bar{\kappa}_1+\kappa^{o}\nu_1+\bar{\kappa}^{o}\bar{\nu}_1-(\bar{\alpha}^{o}-\beta^{o})(\alpha_1-\bar{\beta}_1)-(\alpha^{o}-\bar{\beta}^{o})(\bar{\alpha}_1-\beta_1) \\
&+ (\varepsilon^{o}+\bar{\varepsilon}^{o})(\gamma_1+\bar{\gamma}_1)+(\gamma^{o}+\bar{\gamma}^{o})(\varepsilon_1+\bar{\varepsilon}_1) \\
&- \mu_1\rho_1-\bar{\mu}_1\bar{\rho}_1+\lambda_1\sigma_1+\bar{\lambda}_1\bar{\sigma}_1+\nu_1\kappa_1+\bar{\nu}_1\bar{\kappa}_1-\pi_1\tau_1-\bar{\pi}_1\bar{\tau}_1
\end{aligned}
\tag{d}
$$

$$
\begin{aligned}
4\Phi_{12} = 4\Phi^{o}_{12} &+ D\bar{\nu}_1-\Delta(\bar{\bar{\pi}}_1+2\beta_1)+ \bar{\delta}\bar{\lambda}_1-\delta(\bar{\mu}_1-2\gamma_1)+(\rho^{o}-\bar{\rho}^{o}+\varepsilon^{o}+3\bar{\varepsilon}^{o})\bar{\nu}_1 \\
&- \tau^{o}(2\mu_1+\bar{\mu}_1+2\gamma_1)-\bar{\pi}^{o}(\bar{\mu}_1+2\bar{\gamma}_1) \\
&- (\bar{\alpha}^{o}+\beta^{o})(\bar{\mu}_1-2\gamma_1)+\mu^{o}(\bar{\pi}_1-2\tau_1-2\beta_1)-\bar{\mu}^{o}(\bar{\pi}_1+2\bar{\alpha}_1)+2\nu^{o}\sigma_1+2\sigma^{o}\nu_1 \\
&- (\bar{\tau}^{o}+\pi^{o}+\alpha^{o}-3\bar{\beta}^{o})\bar{\lambda}_1-2\bar{\lambda}^{o}(\alpha_1-\bar{\beta}_1)+(\gamma^{o}-\bar{\gamma}^{o})(\bar{\pi}_1+2\beta_1)+2\bar{\nu}^{o}(\varepsilon_1+\bar{\varepsilon}_1) \\
&- 2\tau_1\mu_1-2\bar{\alpha}_1\bar{\mu}_1+2\nu_1\sigma_1+2\bar{\beta}_1\bar{\lambda}_1-2\bar{\gamma}_1\bar{\pi}_1+2\bar{\varepsilon}_1\bar{\nu}_1
\end{aligned}
\tag{e}
$$

$$
\begin{aligned}
2\Phi_{22} = 2\Phi^{o}_{22} &- \Delta(\mu_1+\overline{\mu}_1)+\delta\nu_1+\overline{\delta}\overline{\nu}_1-\mu^{o}(\mu_1+2\gamma_1)-\overline{\mu}^{o}(\overline{\mu}_1+2\overline{\gamma}_1)-(\gamma^{o}+\overline{\gamma}^{o})(\mu_1+\overline{\mu}_1) \\
&- \overline{\lambda}^{o}\lambda_1-\lambda^{o}\overline{\lambda}_1-(\tau^{o}-\overline{\alpha}^{o}-3\beta^{o})\nu_1-(\overline{\tau}^{o}-\alpha^{o}-3\overline{\beta}^{o})\overline{\nu}_1+\overline{\nu}^{o}(\pi_1+2\overline{\beta}_1)+\nu^{o}(\overline{\pi}_1+2\beta_1) \\
&- 2\gamma_1\mu_1-2\overline{\gamma}_1\overline{\mu}_1+2\beta_1\nu_1+2\overline{\beta}_1\overline{\nu}_1 \qquad \text{(B.4)(f)}
\end{aligned}
$$

$$
\begin{aligned}
12\Lambda = 12\Lambda^{o} &+ D(\mu_1+\overline{\mu}_1-\gamma_1-\overline{\gamma}_1)-\Delta(\rho_1+\overline{\rho}_1-\varepsilon_1-\overline{\varepsilon}_1)+\delta(\overline{\tau}_1-\pi_1+\alpha_1-\overline{\beta}_1)+\overline{\delta}(\tau_1-\overline{\pi}_1+\overline{\alpha}_1-\beta_1) \\
&+ (\gamma^{o}+\overline{\gamma}^{o}-\mu^{o}-\overline{\mu}^{o})(\rho_1+\overline{\rho}_1-\varepsilon_1-\overline{\varepsilon}_1)+(\varepsilon^{o}+\overline{\varepsilon}^{o}-\rho^{o}-\overline{\rho}^{o})(\mu_1+\overline{\mu}_1-\gamma_1-\overline{\gamma}_1) \\
&- (\overline{\tau}^{o}-\pi^{o}+\alpha^{o}-\overline{\beta}^{o})(\tau_1-\overline{\pi}_1+\overline{\alpha}_1-\beta_1)-(\tau^{o}-\overline{\pi}^{o}+\overline{\alpha}^{o}-\beta^{o})(\overline{\tau}_1-\pi_1+\alpha_1-\overline{\beta}_1) \\
&- (\mu_1-2\gamma_1)\rho_1-(\overline{\mu}_1-2\overline{\gamma}_1)\overline{\rho}_1+2\varepsilon_1\mu_1+2\overline{\varepsilon}_1\overline{\mu}_1+\lambda_1\sigma_1+\overline{\lambda}_1\overline{\sigma}_1 \\
&+ (\pi_1-2\alpha_1)\tau_1+(\overline{\pi}_1-2\overline{\alpha}_1)\overline{\tau}_1-2\beta_1\pi_1-2\overline{\beta}_1\overline{\pi}_1-\nu_1\kappa_1-\overline{\nu}_1\overline{\kappa}_1 \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
4\Phi_{o} &= D(\tau_1-2\beta_1)-\Delta\kappa_1-\delta(\rho_1-2\varepsilon_1)+\overline{\delta}\sigma_1-(\tau^{o}+\overline{\pi}^{o}-\overline{\alpha}^{o}-\beta^{o})\rho_1+\rho^{o}(\tau_1+2\beta_1)-(\overline{\rho}^{o}+\varepsilon^{o}-\overline{\varepsilon}^{o})(\tau_1-2 \\
&- (\overline{\tau}^{o}-\pi^{o}+3\alpha^{o}-\overline{\beta}^{o})\sigma_1+2\sigma^{o}\pi_1-2(\tau^{o}-\overline{\pi}^{o}+\overline{\alpha}^{o}+\beta^{o})\varepsilon_1-(\mu^{o}+\overline{\mu}^{o}-3\gamma^{o}-\overline{\gamma}^{o})\kappa_1-2\kappa^{o}\mu_1 \\
&+ 2\beta_1\rho_1+2(\pi_1-\alpha_1)\sigma_1-2\varepsilon_1\tau_1-2(\mu_1-\gamma_1)\kappa_1 \qquad \text{(B.6)(a)}
\end{aligned}
$$

$$
\begin{aligned}
2\Phi_1 = &- D\mu_1-\Delta\rho_1+\delta\pi_1+\overline{\delta}\tau_1+(\gamma^{o}+\overline{\gamma}^{o}-\overline{\mu}^{o})\rho_1-(\varepsilon^{o}+\overline{\varepsilon}^{o}-\overline{\rho}^{o})\mu_1+2\rho^{o}\gamma_1-2\mu^{o}\varepsilon_1 \\
&- \lambda^{o}\sigma_1+\sigma^{o}\lambda_1-(\alpha^{o}-\overline{\beta}^{o}+\overline{\tau}^{o})\tau_1-(\overline{\alpha}^{o}-\beta^{o}-\overline{\pi}^{o})\pi_1-2\tau^{o}\alpha_1+2\pi^{o}\beta_1+\nu^{o}\kappa_1-\kappa^{o}\nu_1 \\
&+ 2\gamma_1\rho_1-2\varepsilon_1\mu_1-2\alpha_1\tau_1+2\beta_1\pi_1 \qquad \text{(b)}
\end{aligned}
$$

$$
\begin{aligned}
4\Phi_2 = &- D\nu_1+\Delta(\pi_1-2\alpha_1)+\delta\lambda_1-\overline{\delta}(\mu_1-2\gamma_1)+(\overline{\tau}^{o}+\pi^{o}-\alpha^{o}-\overline{\beta}^{o})\mu_1-\mu^{o}(\pi_1+2\alpha_1) \\
&+ (\overline{\mu}^{o}+\gamma^{o}-\overline{\gamma}^{o})(\pi_1-2\alpha_1)-(\tau^{o}-\overline{\pi}^{o}+\overline{\alpha}^{o}-3\beta^{o})\lambda_1-2\lambda^{o}\tau_1+2(\pi^{o}-\overline{\tau}^{o}+\alpha^{o}+\overline{\beta}^{o})\gamma_1 \\
&+ (\rho^{o}+\overline{\rho}^{o}-3\varepsilon^{o}-\overline{\varepsilon}^{o})\nu_1+2\nu^{o}\rho_1 \\
&- 2\alpha_1\mu_1-2(\tau_1-\beta_1)\lambda_1+2\gamma_1\pi_1+2(\rho_1-\varepsilon_1)\nu_1 \qquad \text{(c)}
\end{aligned}
$$

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