High Energy Asymptotics and Trace Formulas for the Perturbed Harmonic Oscillator

by

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Contents

1 Abstract

A one-dimensional quantum harmonic oscillator perturbed by a smooth compactly supported potential is considered. For the corresponding eigenvalues λ_n, a complete asymptotic expansion for large n is obtained, and the coefficients of this expansion are expressed in terms of heat invariants. A sequence of trace formulas is obtained, expressing regularised sums of integer powers of eigenvalues λ_n also in terms of heat invariants.

The approach to this problem follows that employed by Gel'fand, Levitan and Dikii, see [9], [10], [7], on the corresponding problem for Sturm-Liouville differential operators. In [7] can be found the analogous results for the high energy asymptotic expansion and sequence of trace formulas for the Sturm-Liouville problem.

The approach employed here differs in that the coefficients of the asymptotic expansion are given in terms of heat invariants. The motivation for this is the recent result of I. Polterovich. In [18] is obtained an explicit formula for each heat invariant. This allows one to calculate the coefficients of the asymptotic expansion of λ_n for large n in terms of integrals of the perturbing potential and its derivatives.

A first result of this work is to establish the existence of an asymptotic expansion for λ_n in terms of the eigenvalues λ_n^0 of the Hamiltonian of the one dimensional harmonic oscillator.

One has the asymptotic expansion

$$\lambda_n \sim \lambda_n^0 + \frac{c_1}{\sqrt{\lambda_n^0}} + \frac{c_2}{\lambda_n^0} + \frac{c_3}{\lambda_n^0\sqrt{\lambda_n^0}} + \cdots, \ n \to \infty,$$

for some undetermined coefficients $\{c_j\} \in \mathbb{R}$. These coefficients $\{c_j\}$ are then determined in terms of heat invariants of the operator.

The trace formulas are obtained by considering the function $Z(s) = \sum \lambda_n^{-s}$, for $\operatorname{Re} s > \frac{1}{2}$ and its meromorphic continuation. This is known as the Zeta function of the operator. The method used here is identical to that employed in [7].

2 Acknowledgements

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3 Introduction and Background

In this work trace formulas and a high energy asymptotic expansion are established for the one dimensional perturbed harmonic oscillator, where the perturbation has compact support. The approach to this problem follows that employed by Gel'fand, Levitan and Dikii, see [9], [10], [7], on the corresponding problem for Sturm-Liouville differential operators. However, the method here differs in that the results are expressed in terms of the heat invariants of the operator. Indeed, part of the motivation for this work came from recent advances in the computation of heat invariants due to I. Polterovich, see [18].

This section gives a brief description of heat invariants - where they arise and how they are computed - followed by an outline of Gel'fand, Levitan and Dikii's work and results for Sturm-Liouville differential operators.

3.1 Local heat invariants

Define the Schrödinger operator $h = -\Delta + q(x)$ in $L^2(\mathbb{R}^n)$, for $n \geq 1$ arbitrary, where $q \in C^\infty(\mathbb{R}^n)$ and its derivatives are uniformly bounded in $\mathbb{R}^n$. Consider the operator e^{-th}, $t \geq 0$, and denote its integral kernel by $e^{-th}(x,y)$. It is well known that the following asymptotic expansion holds true as $t \to +0$:

$$e^{-th}(x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j a_j(x), \tag{3.1}$$

locally uniformly in $\mathbb{R}^n$, see Appendix D for more details on the proof of (3.1) in the one dimensional case. The coefficients $\{a_j(x)\}$ of (3.1) are called the local heat invariants of the operator h and are polynomials in q and its derivatives.

Heat invariants were first given by a recursive system of differential equations, see [16]. Later, Gilkey presented a way to obtain recursive formulas for heat invariants, see [11], based on a method for derivation of heat kernel asymptotics due to Seeley, [25].

Much work has been done on computing heat invariants. It is a well known problem in spectral geometry and further references can be found in [18]. However, only recently has an explicit formula for each heat invariant been available.

This explicit formula for the computation of the local heat invariants was obtained in [18]. Below is Theorem 1.2.1 of [18] adapted to the case of one dimensional Schrödinger operators.

Theorem 3.1. *One has*

$$a_j[q(x)] = \sum_{k=0}^{j-1} \frac{(-1)^j \Gamma(j+\frac{1}{2})(-\frac{d^2}{dy^2}+q(y))^{k+j}(|x-y|^{2k})\,|_{y=x}}{4^k k!(k+j)!(j-k)!\,\Gamma(k+\frac{3}{2})}. \tag{3.2}$$

Theorem 3.1 was developed from results in [17], [19], where a method of computing heat invariants was obtained, based on the Agmon-Kannai asymptotic expansion of resolvent kernels of elliptic operators, see [2].

From (3.2) or otherwise, one obtains

$$a_0[q(x)] = 1, \quad a_1[q(x)] = -q(x), \quad a_2[q(x)] = \frac{1}{2}q^2(x) - \frac{1}{6}q''(x),$$
$$a_3[q(x)] = -\tfrac{1}{6}q^3 + \tfrac{1}{6}qq'' + \tfrac{1}{12}q'^2 - \tfrac{1}{60}q^{(4)},$$
$$a_4[q(x)] = \tfrac{1}{24}q^4 + \tfrac{1}{30}q'q''' + \tfrac{1}{60}qq^{(4)} + \tfrac{1}{40}(q'')^2 - \tfrac{1}{840}q^{(6)} - \tfrac{1}{12}q''q^2 - \tfrac{1}{12}q(q')^2.$$

3.2 Sturm-Liouville Differential Operators

The following is an outline of the work of Gel'fand, Levitan and Dikii, as seen in [7], in which a high energy asymoptotic expansion and trace formulae were obtained for Sturm-Liouville differential operators. This work provided much of the motivation and method for the approach to the analogous problem for the harmonic oscillator that is the focus of this text.

Consider $H_0 = -\frac{d^2}{dx^2}$ in $L^2(0,\pi)$ with Dirichlet boundary conditions. Then it is well known that H_0 has eigenvalues $\lambda_n^0 = n^2,\ n \in \mathbb{N}$. Now let $H = H_0 + q,\ q \in C_0^\infty(0,\pi)$ (the discussion here shall be limited to the results of [7] for the case where q and its derivatives vanish at the ends of the interval $(0,\pi)$). Let λ_n be the eigenvalues of H such that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Then it was already known that the following asymptotic expansion held true as $n \to \infty$,

$$\lambda_n \sim \lambda_n^0 + c_0 + \frac{c_1}{\lambda_n^0} + \frac{c_2}{(\lambda_n^0)^2} + \frac{c_3}{(\lambda_n^0)^3} + \cdots, \tag{3.3}$$

for some coefficients $\{c_j\}$.

In [7] are obtained equations for the computation of $c_0, c_1, \ldots$. In the proof and formulation of the results is employed the inverse of the series (3.3), which is also an asymptotic series

$$\psi(\lambda_n) = \lambda_n^0 = \lambda_n + b_0 + \frac{b_1}{\lambda_n} + \frac{b_2}{\lambda_n^2} + \frac{b_3}{\lambda_n^3} + \cdots$$

for some coefficients $\{b_j\}$.

Below is Theorem 3.1 of [7] reformulated in terms of heat invariants.

Theorem 3.2. *(Gel'fand) The formal asymptotic expansions*

$$\pi \frac{d}{d\zeta}\sqrt{-\psi(-\zeta)} \ \textit{and} \ \sum_{j=0}^{\infty} \frac{-1}{\zeta^j\sqrt{\zeta}} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-j)} \int_0^\pi a_j[q(x)]dx \tag{3.4}$$

coincide, where a_j are the heat invariants.

One has

$$\begin{aligned}
\sqrt{-\psi(-\zeta)} &= \sqrt{\zeta - b_0 + \frac{b_1}{\zeta} - \frac{b_2}{\zeta^2} + \frac{b_3}{\zeta^3} + \cdots} \\
&= \sqrt{\zeta}\sqrt{1 - \frac{b_0}{\zeta} + \frac{b_1}{\zeta^2} - \frac{b_2}{\zeta^3} + \frac{b_3}{\zeta^4} + \cdots} \\
&= \sqrt{\zeta} - \frac{\frac{1}{2}b_0}{\sqrt{\zeta}} + \frac{\frac{1}{2}b_1 - \frac{1}{8}b_0}{\zeta\sqrt{\zeta}} + \frac{\frac{1}{4}b_0b_1 - \frac{1}{2}b_2}{\zeta^2\sqrt{\zeta}} \cdots .
\end{aligned}$$

This is now differentiated with respect to ζ.

Although, heat invariants were known at the time there was no explicit formula available for their computation. In [7], the coefficients of the asymptotic eigenvalue expansion are calculated by means of recurrence formulae developed from expansions of the trace of the resolvent. In [12], it is shown how similar expansions for the resolvent together with further combinatorial techniques produces the explicit formula (3.1) for the computation of the heat invariants.

The first three coefficients of (3.3) are given by

$$\begin{aligned} c_0 &= \frac{1}{\pi}\int_0^{\pi} q(x)dx, \\ c_1 &= \frac{1}{4\pi}\int_0^{\pi} [q(x)-c_0]^2 dx, \\ c_2 &= \frac{1}{8\pi}\int_0^{\pi} [q(x)-c_0]^3 dx + \frac{1}{16\pi}\int_0^{\pi} [q'(x)]^2 dx. \end{aligned}$$

In [7] trace formulas are also obtained. Assume $\lambda_n > 0$ for all n. Let $Z(s) = \sum \lambda_n^{-s}$, for Re $s > \frac{1}{2}$. The function $Z(s)$ is called the "zeta function" of the operator H. In the case that $q \equiv 0$ then $Z(s) = \zeta(2s) = \sum_{n=1}^{\infty}(n^2)^{-s}$, where ζ is the Riemann zeta function. It is possible to consider the meromorphic continuation of $Z(s)$ towards the left. This was used in deriving the following theorem, which is Theorem 4.2 of [7],

Theorem 3.3. ***(Gel'fand)*** *If $q(x)$ and its derivatives vanish at the ends of the interval $(0,\pi)$, then*

$$Z(-k) = 0, \quad \forall\, k \in \mathbb{N}. \tag{3.5}$$

This result can be reformulated to give more explicit formulas. For negative integers s the values of $Z(s)$ can be regarded as regularizations of the traces $\sum_{n=1}^{\infty} \lambda_n^k$.

Consider the asymptotic expansion (3.3) of λ_n raised to the power $-s$.

One has

$$\begin{aligned}
\lambda_n^{-s} &\sim (\lambda_n^0)^{-s}(1+\frac{c_0}{\lambda_n^0}+\frac{c_1}{(\lambda_n^0)^2}+\frac{c_2}{(\lambda_n^0)^3}+\cdots)^{-s}\\
&= (\lambda_n^0)^{-s}(1-s\frac{c_0}{\lambda_n^0}-s\frac{c_1}{(\lambda_n^0)^2}+\frac{s(s+1)}{2}\frac{c_0^2}{(\lambda_n^0)^2}\\
&\quad -s\frac{c_2}{(\lambda_n^0)^3}+s(s+1)\frac{c_0c_1}{(\lambda_n^0)^3}-\frac{s(s+1)(s+2)}{6}\frac{c_0^3}{(\lambda_n^0)^3}+\cdots).
\end{aligned}$$

Thus,

$$\lambda_n^{-s} \sim \sum_{j=0}^{\infty} d_j(s)(\lambda_n^0)^{-s-j},\ n\to\infty,$$

where $d_j(s)$ are explicit polynomials in s and c_j. For example

$$d_0(s)=1,\qquad d_1(s)=-sc_0,\qquad d_2(s)=-sc_1+\frac{s(s+1)}{2}c_0^2,$$
$$d_3(s)=-sc_2+s(s+1)c_0c_1-\frac{s(s+1)(s+2)}{6}c_0^3.$$

It was shown in [7] for $k\in\mathbb{N}$ that

$$Z(-k)=\sum_{n=0}^{\infty}\{\lambda_n^k-\sum_{j=0}^{k}d_j(-k)(\lambda_n^0)^{k-j}\}-\frac{1}{2}d_k(-k)=0$$

For example, in the case $k=1$,

$$Z(-1)=\sum_{n=1}^{\infty}(\lambda_n-\lambda_n^0-c_0)-\frac{1}{2}c_0=0.$$

For the case $k=2$,

$$Z(-2)=\sum_{n=1}^{\infty}(\lambda_n^2-(\lambda_n^0)^2-2\lambda_n^0c_0-2c_1-c_0^2)-\frac{2c_1+c_0^2}{2}=0.$$

For $k=3$, one has

$$\begin{aligned}
Z(-3) &= \sum_{n=1}^{\infty}(\lambda_n^3-(\lambda_n^0)^3-(3c_1+3c_0^2)\lambda_n^0-(3c_2+6c_0c_1+c_0^3))\\
&\quad -\frac{(3c_2+6c_0c_1+c_0^3))}{2}=0.
\end{aligned}$$

In this way, expressions can be formed for all $Z(-k)$, $k\in\mathbb{N}$.

The analogous trace formulas for the one dimensional harmonic oscillator are given in section 4. The proof is given in section 7 and uses the same method as set out in [7] for the trace formulas of the Sturm-Liouville problem as described above.

4 Main Results

Let H_0 in $L^2(\mathbb{R})$ be the Hamiltonian of the one dimensional harmonic oscillator, that is $H_0 = -\frac{d^2}{dx^2} + x^2$. The eigenvalues of H_0 are given by $\lambda_n^0 = 2n-1$, where $n \in \mathbb{N}$.

Let $V \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp} V \subset (-R, R)$ for some $R > 0$, be a real valued function (potential in physical terminology), let $H = H_0 + V$. It is well known that V is a relatively compact perturbation of H_0. Therefore, by Theorem XIII.14 of [22], the spectrum of H is also discrete. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of H. Our main aim is to describe a high energy asymptotic expansion for the eigenvalues λ_n of $H_0 + V$ in terms of the unperturbed eigenvalues λ_n^0.

In section 6 the following preliminary result shall be established.

Lemma 4.1. *Let $V \in C_0^\infty(\mathbb{R})$. Then the following asymptotic expansion holds true as $t \to +0$:*

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim \sum_{j=0}^{\infty} \frac{1}{\sqrt{4\pi t}} t^j \int_{-\infty}^{\infty} (a_j[x^2 + V(x)] - a_j[x^2]) dx. \tag{4.1}$$

Formally, the formula (4.1) follows from subtracting (3.1) with $q(x) = x^2$ from (3.1) with $q(x) = x^2 + V(x)$ and integrating over x.

Here is the main result:

Theorem 4.2. *i) One has the asymptotic expansion*

$$\lambda_n \sim \lambda_n^0 + \frac{c_1}{\sqrt{\lambda_n^0}} + \frac{c_2}{\lambda_n^0} + \frac{c_3}{\lambda_n^0\sqrt{\lambda_n^0}} + \cdots, \quad n \to \infty, \tag{4.2}$$

for some coefficients $c_j \in \mathbb{R}$.

ii) The coefficients of the expansion (4.2) can be obtained by inverting the formal asymptotic series

$$\lambda_n^0 \sim \lambda_n + \frac{b_1}{\sqrt{\lambda_n}} + \frac{b_2}{\lambda_n\sqrt{\lambda_n}} + \frac{b_3}{(\lambda_n)^2\sqrt{\lambda_n}} + \cdots, \quad n \to \infty, \tag{4.3}$$

where

$$b_j = (\sqrt{\pi}\Gamma(\tfrac{3}{2} - j))^{-1} \int_{\mathbb{R}} (a_j[x^2 + V(x)] - a_j[x^2])dx. \tag{4.4}$$

In other words, Theorem 4.2 (ii) implies that each c_j of (4.2) is uniquely determined by a well defined algebraic procedure, namely series inversion.

In order to compute a coefficient c_j, one needs to know finitely many coefficients b_j. For example,

$$c_1 = -b_1, \quad c_2 = 0, \quad c_3 = -b_2, \quad c_4 = -\frac{1}{2}b_1^2,$$
$$c_5 = -b_3, \quad c_6 = -2b_1b_2, \quad c_7 = \frac{5}{8}b_1^3 - b_4.$$

One thing to note is the absence of negative integer powers of λ_n in the expansion (4.3). Inverting the expansion (4.2) gives

$$\lambda_n^0 \sim \lambda_n - \frac{c_1}{\sqrt{\lambda_n}} - \frac{c_2}{\lambda_n} - \frac{c_3}{\lambda_n\sqrt{\lambda_n}} - \frac{(c_4 + \frac{1}{2}c_1^2)}{\lambda_n^2} - \frac{c_5}{\lambda_n^2\sqrt{\lambda_n}} - \cdots \tag{4.5}$$

Hence, the absence of whole integer negative powers is equivalent to a series of identities for the coefficients c_j. For example, the first three identities of this type are

$$c_2 = 0, \quad c_1^2 + 2c_4 = 0, \quad c_6 + c_2^2 + 2c_1c_3 = 0.$$

The first few c_j are given below.

$$\begin{aligned}
c_1 &= \frac{1}{\pi}\int_{-\infty}^{\infty} V(x)dx, \\
c_2 &= 0, \\
c_3 &= \frac{1}{2\pi}\Big(\int_{-\infty}^{\infty} V(x)x^2dx + \frac{1}{2}\int_{-\infty}^{\infty} V^2(x)dx\Big), \\
c_4 &= -\frac{1}{2}c_1^2, \\
c_5 &= \frac{1}{8\pi}\int_{-\infty}^{\infty} [V^3 + 3V^2x^2 + 3Vx^4 + \frac{1}{2}(V')^2 - 2V]dx.
\end{aligned}$$

We also obtain trace formulas, which as mentioned before, are a direct analogue of the trace formulas for the Sturm-Liouville problem, see [10], [9], [7], and our proofs follow the method as described in [7]. Let us introduce the Zeta functions

$$Z(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \quad Z_0(s) = \sum_{n=1}^{\infty} (\lambda_n^0)^{-s}, \quad \operatorname{Re} s > 1. \tag{4.6}$$

If $\lambda_n = 0$ for some n, then the corresponding term in the sum $\sum_{n=1}^{\infty} \lambda_n^{-s}$ is omitted. If $\lambda_n < 0$ for some n, then λ_n^{-s} should be understood as $|\lambda_n|^{-s} e^{-i\pi s}$. Due to the explicit formula $\lambda_n^0 = 2n - 1$, we have $Z_0(s) = (1 - 2^{-s})\zeta(s)$, where $\zeta(s)$ is the Riemann Zeta function. By the properties of ζ, we conclude that $Z_0(s)$ has a meromorphic continuation into the entire complex plane with the only pole at $s = 1$, and this pole is simple with residue $\frac{1}{2}$ (see 23.2 of [1]). The real zeros of Z_0 are at $s = -2n$, $n = 0, 1, 2, \ldots$.

Theorem 4.3. *One has that $Z(s)$ is meromorphic for $s \in \mathbb{C}$ with its only poles located at $s = 1$ and $s = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ and*

$$Z(-k) - Z_0(-k) = 0, \;\; k \in \mathbb{N}. \tag{4.7}$$

The proof is given in section 7. Given the asymptotics (4.2), we can give explicit formulas for $Z(-k)$, $k \in \mathbb{N}$. Consider the asymptotic expansion (4.2) of λ_n raised to the power $-s$. One has

$$\begin{aligned}
\lambda_n^{-s} &\sim (\lambda_n^0)^{-s}\Big(1 + \frac{c_1}{(\lambda_n^0)^{\frac{3}{2}}} + \frac{c_2}{(\lambda_n^0)^2} + \frac{c_3}{(\lambda_n^0)^{\frac{5}{2}}} + \ldots\Big)^{-s} \\
&= (\lambda_n^0)^{-s}\Big(1 - \frac{sc_1}{(\lambda_n^0)^{\frac{3}{2}}} - \frac{sc_2}{(\lambda_n^0)^2} - \frac{sc_3}{(\lambda_n^0)^{\frac{5}{2}}} + \frac{s(s+1)c_1^2}{2(\lambda_n^0)^3} + \ldots\Big)
\end{aligned}$$

Thus,

$$\lambda_n^{-s} \sim \sum_{j=0}^{\infty} d_j(s)(\lambda_n^0)^{-s-(j/2)}, \quad n \to \infty, \tag{4.8}$$

where $d_j(s)$ are explicit polynomials in s and c_j. For example,

$$d_0(s) = 1, \quad d_1(s) = d_2(s) = 0, \quad d_3(s) = -sc_1,$$
$$d_4(s) = -sc_2, \quad d_5(s) = -sc_3,$$
$$d_6(s) = -sc_4 + \frac{s(s+1)}{2}c_1^2, \quad d_7(s) = -sc_5 + s(s+1)c_1c_2,$$
$$d_8(s) = -sc_6 + \frac{1}{2}s(s+1)c_2^2 + s(s+1)c_1c_3.$$

Using this notation, we have for any $k \in \mathbb{N}$, $\operatorname{Re} s > 1$:

$$Z(s) = \sum_{n=0}^{\infty}\{\lambda_n^{-s} - \sum_{j=0}^{2k+2} d_j(s)(\lambda_n^0)^{-s-(j/2)}\} + \sum_{j=0}^{2k+2} d_j(s)Z_0(s+(j/2)). \quad (4.9)$$

Now both sides of (4.9) can be meromorphically continued into the half-plane $\operatorname{Re} s > -k - \frac{1}{2}$. By Theorem 4.3, the l.h.s. of (4.9) is analytic at $s = -k$. By (4.8), the same applies to the first term in the r.h.s. of (4.9). Thus, the second term in the r.h.s. of (4.9) is also analytic at $s = -k$. As $Z_0(s)$ has a simple pole at $s = 1$ with residue $\frac{1}{2}$ (and no other poles), it follows that $d_{2k+2}(-k) = 0$ and

$$\lim_{s\to -k} d_{2k+2}(s)Z_0(s+k+1) = \frac{1}{2}d'_{2k+2}(-k).$$

Thus, we obtain

$$Z(-k) = \sum_{n=0}^{\infty}\{\lambda_n^k - \sum_{j=0}^{2k+1} d_j(-k)(\lambda_n^0)^{k-(j/2)}\} + \sum_{j=0}^{2k+1} d_j(-k)Z_0(-k+(j/2)) + \frac{1}{2}d'_{2k+2}(k).$$

Combined with (4.7), this yields a series of formulas for $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty}\{\lambda_n^k - \sum_{j=0}^{2k+1} d_j(-k)(\lambda_n^0)^{k-(j/2)}\} + \sum_{j=1}^{2k+1} d_j(-k)Z_0(-k+(j/2)) + \frac{1}{2}d'_{2k+2}(k) = 0. \quad (4.10)$$

In particular, for $k = 1, 2, 3$ we obtain

$$\sum_{n=0}^{\infty}(\lambda_n - \lambda_n^0 - \frac{c_1}{\sqrt{\lambda_n^0}}) + c_1 Z_0(\frac{1}{2}) = 0; \tag{4.11}$$

$$\sum_{n=0}^{\infty}(\lambda_n^2 - (\lambda_n^0)^2 - 2c_1\sqrt{\lambda_n^0} - \frac{2c_3}{\sqrt{\lambda_n^0}}) + 2c_1 Z_0(-\frac{1}{2}) + 2c_3 Z_0(\frac{1}{2}) - \frac{1}{2}c_1^2 = 0; \tag{4.12}$$

$$\sum_{n=0}^{\infty}(\lambda_n^3 - (\lambda_n^0)^3 - 3c_1(\lambda_n^0)^{3/2} - 3c_3(\lambda_n^0)^{1/2} - 3(c_4 + c_1^2) - 3c_5(\lambda_n^0)^{-1/2})$$
$$+ 3c_1 Z_0(-\frac{3}{2}) + 3c_3 Z_0(-\frac{1}{2}) + 3c_5 Z_0(\frac{1}{2}) - \frac{1}{2}c_1 c_3 = 0. \tag{4.13}$$

The above formulas can be rearranged, replacing asymptotic expansions in powers of λ_n^0 by expansions in powers of n. For example, (4.11) is equivalent to

$$\sum_{n=1}^{\infty}(\lambda_n - \lambda_n^0 - \frac{c_1}{\sqrt{2n}}) + \frac{c_1}{\sqrt{2}}\zeta(\frac{1}{2}) = 0. \tag{4.14}$$

Formula (4.14) in various equivalent forms appeared before in the literature; see for example [3], [4]. Formulas (4.10) with $k \geq 2$ are believed to be new.

Remark. The following formula is presented in [14] and appears to be similar to (4.12),

$$\sum_{n=1}^{N}[\lambda_n^2 - (\lambda_n^0)^2 - A\sqrt{n} - B\frac{1}{\sqrt{n}}] = AC_1 + BC_0, \tag{4.15}$$

where C_0 is defined as before and

$$C_1 = \lim_{N\to\infty}\Big[\frac{2}{3}N^{\frac{2}{3}} + \frac{1}{2}N^{\frac{1}{2}} - \sum_{n=1}^{N}\sqrt{n}\Big],$$
$$A = \frac{2\sqrt{2}}{\pi}\int_{-\infty}^{\infty}V(x)dx,$$
$$B = \frac{\sqrt{2}}{2\pi}\int_{-\infty}^{\infty}(V^2(x) + V(x)x^2 - V(x))dx.$$

However, the series in formula (4.15) does not converge. This can be seen by squaring (4.2) and converting it into the same form as (4.15). One has

$$\begin{aligned}\lambda_n^2 &= (\lambda_n^0)^2 + 2c_1\sqrt{\lambda_n^0} + \frac{2c_3}{\sqrt{\lambda_n^0}} + O(n^{-\frac{3}{2}})\\ &= (\lambda_n^0)^2 + 2\sqrt{2}c_1\Big(\sqrt{n} - \frac{1}{4\sqrt{n}}\Big) + \frac{\sqrt{2}c_3}{\sqrt{n}} + O(n^{-\frac{3}{2}}), \quad n \to \infty,\end{aligned}$$

where

$$\begin{aligned}c_1 &= \frac{1}{\pi}\int_{-\infty}^{\infty} V(x)dx,\\ c_3 &= \frac{1}{2\pi}\Big(\int_{-\infty}^{\infty} V(x)x^2dx + \frac{1}{2}\int_{-\infty}^{\infty} V^2(x)dx\Big).\end{aligned}$$

Comparing this with the terms of the series in (4.15), one has

$$\lambda_n^2 - (\lambda_n^0)^2 - A\sqrt{n} - B\frac{1}{\sqrt{n}} = -\frac{\sqrt{2}}{4\pi\sqrt{n}}\int_{-\infty}^{\infty} V^2(x)dx + O(n^{-\frac{3}{2}}),\ n \to \infty.$$

This implies that the series in (4.15) does not converge.

5 Proof of Theorem 4.2 (ii)

The proof is based on the following

Lemma 5.1. *Let $\lambda_n^0 = 2n-1$, $n \in \mathbb{N}$, and let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of real numbers such that $\lambda_n = \lambda_n^0 + O(1)$ as $n \to \infty$. Suppose that an asymptotic expansion*

$$\lambda_n^0 \sim \lambda_n + \sum_{j=1}^{\infty} p_j\lambda_n^{-\alpha_j} + \sum_{j=1}^{\infty} q_j\lambda_n^{-j}, \quad n \to \infty, \tag{5.1}$$

holds true, where $0 \le \alpha_1 < \alpha_2 < \cdots$ are some non-integer exponents and $\{p_j\} \subset \mathbb{R}$, $\{q_j\} \subset \mathbb{R}$. Then one has the asymptotic expansion

$$\sum_{n=1}^{\infty} e^{-t\lambda_n} \sim \frac{1}{2t} + \sum_j \frac{p_j}{2}\Gamma(1-\alpha_j)t^{\alpha_j} + \frac{1}{2}\log t\sum_{j=1}^{\infty} q_j\frac{(-1)^j}{(j-1)!}t^j + \sum_{k=1}^{\infty} r_kt^k \tag{5.2}$$

as $t \to +0$, where $\{r_k\} \subset \mathbb{R}$ are some constants.

Given Lemma 5.1 and part (i) of Theorem 4.2, the proof of Theorem 4.2 (ii) is immediate. Indeed, inverting the asymptotic expansion (4.2) yields the expansion of the form

$$\lambda_n^0 \sim \lambda_n + \sum_{j=1}^{\infty} b_j \lambda_n^{\frac{1}{2}-j} + \sum_{j=1}^{\infty} \widetilde{b}_j \lambda_n^{-j}, \quad n \to \infty$$

with some real coefficients $\{b_j\}$, $\{\widetilde{b}_j\}$. Now using Lemma 5.1 and the explicit formula $\sum_{n=1}^{\infty} e^{-t\lambda_n^0} = (2\sinh t)^{-1}$, we obtain the asymptotic expansion

$$\sum_{n=1}^{\infty}(e^{-t\lambda_n} - e^{-t\lambda_n^0}) \sim \frac{1}{\sqrt{t}} \sum_{j=1}^{\infty} \frac{b_j}{2} \Gamma(\frac{3}{2} - j) t^j$$
$$+ \frac{1}{2} \log t \sum_{j=1}^{\infty} \widetilde{b}_j \frac{(-1)^j}{(j-1)!} t^j + \sum_{k=1}^{\infty} \widetilde{r}_k t^k$$

with some real coefficients $\{\widetilde{r}_k\}$. Comparing this to (4.1), we see that all coefficients $\widetilde{b}_i$ vanish and the coefficients b_j are related to the heat invariants by formulas (4.4). This completes the proof of Theorem 4.2 (ii).

The important part of this section is the proof of Lemma 5.1. Broadly speaking, this Lemma can be regarded as a discrete analogue of the following version of Watson's Lemma:

Lemma 5.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a locally bounded measurable function, such that $\psi(\lambda) = 0$ for all λ near $-\infty$. Suppose that ψ has the following asymptotic expansion*

$$\psi(\lambda) = \sum_j p_j \lambda^{-\alpha_j} + \sum_j q_j \lambda^{-\beta_j} + O(\lambda^{-M}) \quad \lambda \to \infty, \tag{5.3}$$

where $\{\alpha_j\} \subset \mathbb{R} \setminus \mathbb{N}$, $\{\beta_j\} \subset \mathbb{N}$, $\{p_j\} \subset \mathbb{R}$, $\{q_j\} \subset \mathbb{R}$ are finite sets and $M > \max(\{\alpha_j\} \cup \{\beta_j\})$, $M \in (0, \infty) \setminus \mathbb{N}$. Then the following asymptotic formula for the Laplace transform of ψ holds true for $t \to +0$:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-t\lambda}\psi(\lambda)d\lambda &= \sum_i p_i \Gamma(1-\alpha_i) t^{\alpha_i - 1} + (\log t) \sum_j q_j \frac{(-1)^{\beta_j}}{(\beta_j - 1)!} t^{\beta_j - 1} \\ &+ \sum_{0 \le k < M-1} r_k t^k + O(t^{M-1}),\end{aligned} \tag{5.4}$$

with some coefficients $\{r_k\}$.

The proof of this lemma is carried out by straightforward computation. It is presented at the end of this section.

Proof of Lemma 5.1. 1. Let

$$N(\lambda) = \sharp\{n \mid \lambda_n < \lambda\}, \quad N_0(\lambda) = \sharp\{n \mid \lambda_n^0 < \lambda\}.$$

The main idea of the proof is to approximate $N(\lambda)$ by $N_0(\psi(\lambda))$, where ψ is a function constructed to have the asymptotic expansion

$$\psi(\lambda) = \lambda + \sum_{\alpha_j < M} p_j \lambda^{-\alpha_j} + \sum_{j<M} q_j \lambda^{-j} + O(\lambda^{-M}), \quad \lambda \to \infty,$$

where the exponents and coefficients are the same as those in (5.1). The function ψ is constructed in terms of its inverse as follows.

The formal inversion of the expansion (5.1) has the form

$$\lambda_n \sim \lambda_n^0 + \sum_{j=1}^{\infty} s_j (\lambda_n^0)^{-\eta_j}, \quad n \to \infty, \tag{5.5}$$

where $0 \le \eta_1 < \eta_2 < \dots$ and $\{s_j\} \subset \mathbb{R}$. Fix some sufficiently large $M \in (0,\infty) \setminus \mathbb{N}$; we have

$$\lambda_n = \lambda_n^0 + \sum_{\eta_j < M} s_j (\lambda_n^0)^{-\eta_j} + O((\lambda_n^0)^{-M}), \quad n \to \infty. \tag{5.6}$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that

(i) $\phi(\lambda) \ge 0$ for all $\lambda \in \mathbb{R}$ and $\phi(\lambda) = 0$ for all $\lambda \le 1$;

(ii) $\phi(\lambda)$ is strictly increasing for $\lambda > 1$;

(iii) $\phi(\lambda) = \lambda + \sum_{\eta_j < M} s_j \lambda^{-\eta_j}$ for all sufficiently large $\lambda > 0$.

Let $\psi \in C^\infty(0,\infty)$ be such that $\phi(\psi(\lambda)) = \lambda \;\forall \lambda > 0$. Finally, for $\lambda > 0$ let us write $N_0(\lambda) = \frac{1}{2}\lambda + \omega(\lambda)$, where $\omega(\lambda)$ is a 2-periodic function.

With this notation we have:

$$\begin{aligned}\sum_{n=1}^{\infty} e^{-t\lambda_n} &= \int_{-\infty}^{\infty} e^{-t\lambda} dN(\lambda) = t\int_{-\infty}^{\infty} e^{-t\lambda} N(\lambda)d\lambda \\ &= t\int_{-\infty}^{\infty} e^{-t\lambda}\left(N(\lambda) - N_0(\psi(\lambda))\right)d\lambda + \frac{1}{2}t\int_0^{\infty} e^{-t\lambda}\psi(\lambda)d\lambda \\ &\qquad + t\int_0^{\infty} e^{-t\lambda}\omega(\psi(\lambda))d\lambda \\ &\qquad\qquad =: F_1(t) + F_2(t) + F_3(t).\end{aligned}$$

Below we consider separately the integrals $F_1(t)$, $F_2(t)$ and $F_3(t)$.

2. Consider $F_2(t)$. By the construction of ψ, we have the asymptotics

$$\psi(\lambda) = \lambda + \sum_{\alpha_j < M} p_j \lambda^{-\alpha_j} + \sum_{j<M} q_j \lambda^{-j} + O(\lambda^{-M}) \quad \lambda \to \infty$$

with the same exponents and coefficients as in (5.1). By Lemma 5.2, one has:

$$\begin{aligned}F_2(t) \sim \frac{1}{2t} + \sum_{\alpha_j<M} \frac{p_j}{2}\Gamma(1-\alpha_j)t^{\alpha_j} + \frac{1}{2}(\log t)\sum_{j<M} q_j \frac{(-1)^j}{(j-1)!}t^j \\ + \sum_{0<k<M} r_k t^k + O(t^M), \qquad t \to +0. \quad (5.7)\end{aligned}$$

3. Consider $F_1(t)$. Let $f_n(t) = e^{-t\lambda_n} - e^{-t\phi(\lambda_n^0)}$. Then $F_1(t) = \sum_{n=1}^{\infty} f_n(t)$. By the construction of ϕ, we have $\lambda_n = \phi(\lambda_n^0) + O(n^{-M})$, as $n \to \infty$, and so $f_n(t) = e^{-t\lambda_n}(1 - e^{tO(n^{-M})})$.

Now f_n has infinitely many continuous derivatives. One has that

$$\begin{aligned} f_n(t) &= e^{-t\lambda_n} O(tn^{-M}) \\ \text{and} \qquad f_n^{(j)}(t) &= O(n^{-M+j}),\end{aligned}$$

for $t \in [0, \infty)$. Therefore, $\sum_{n=1}^{N} f_n^{(j)}(t)$ is uniformly convergent as $N \to \infty$ for $j < [M] - 1$ and $t \in [0, \infty)$.

By applying Theorem 7.17 of [23] to the sequence $\{\sum_{n=1}^{N} f_n^{(j)}(t)\}_{N=1}^{\infty}$, one has that $F_1(t)$ has at least $[M] - 1$ continuous derivatives in t on $[0, \infty)$ and therefore, by the Taylor formula,

$$F_1(t) = \sum_{0 \le k < [M]-1} F_1^{(k)}(0)t^k + o(t^{[M]-1}), \quad t \to +0. \tag{5.8}$$

4. Let us prove that F_3 has continuous derivatives in $t \in [0, \infty)$ of any order, and so

$$F_3(t) \sim \sum_{k=0}^{\infty} F_3^{(k)}(0)t^k, \quad t \to +0. \tag{5.9}$$

Fix $N \in \mathbb{N}$. Changing the variable, we obtain

$$\begin{aligned} F_3(t) = t\int_0^\infty e^{-t\lambda}\omega(\psi(\lambda))d\lambda = t\int_0^\infty e^{-t\phi(\mu)}\phi'(\mu)\omega(\mu)d\mu \\ = -\int_0^\infty (e^{-t\phi(\mu)})'\omega(\mu)d\mu \end{aligned}$$

Note that if $\tilde{\omega}$ is a 2 periodic function then $\int_0^\mu \tilde{\omega}(\lambda)d\lambda = \tilde{\omega}_1(\mu) + \mu\int_0^2 \tilde{\omega}(\lambda)d\lambda$ where $\tilde{\omega_1}$ is a 2 periodic function. From the definition of ω, one has $\int_0^2 \omega(\lambda)d\lambda = 0$. Therefore, integrating by parts N times gives

$$F_3(t) = (-1)^{N+1}\int_0^\infty (e^{-t\phi(\mu)})^{(N+1)}(\omega_N(\mu) + p_N(\mu))d\mu, \tag{5.10}$$

where ω_N is a 2 periodic function and p_N is a polynomial of degree $N-1$. However, integrating by parts again one obtains

$$(-1)^{N+1}\int_0^\infty (e^{-t\phi(\mu)})^{(N+1)}p_N(\mu)d\mu = \int_0^\infty (e^{-t\phi(\mu)})'(p_N^{(N)}(\mu))d\mu = 0;$$

the boundary terms at $\mu = 0$ vanish by the assumption $\phi(\mu) = 0$ for $\mu \le 1$. Thus,

$$F_3(t) = (-1)^{N+1}\int_0^\infty (e^{-t\phi(\mu)})^{(N+1)}\omega_N(\mu)d\mu. \tag{5.11}$$

Using the property (iii) of ϕ, we obtain

$$(e^{-t\phi(\mu)})^{(N+1)} = e^{-t\phi(\mu)}\{(-t)^{N+1}(\phi'(\mu))^{N+1} + \sum_{l=1}^{N} t^l O(\mu^{l-N-2-\gamma_1})\}.$$

Substituting this into (5.11), the first term will have the form

$$\int_0^\infty e^{-t(\lambda+O(1))}t^{N+1}O(1)d\lambda, \tag{5.12}$$

while the other terms will be of the form

$$\int_0^\infty e^{-t(\lambda+O(1))} t^l O(\lambda^{l-N-2-\eta_1}) d\lambda, \tag{5.13}$$

for $l \in 1, 2, \ldots, N$.

Both (5.12) and (5.13) can be differentiated N times in t before there is a problem with the convergence of the integral at $t = 0$. The derivatives are also continuous.

It follows that F_3 has at least N continuous derivatives on $[0, \infty)$. As $N \in \mathbb{N}$ can be taken arbitrary large, this proves the statement.

5. Combining (5.7) – (5.9), and using the fact that M can be taken arbitrary large, we get the desired statement. □

Proof of Lemma 5.2. 1. The proof can be performed by explicit computation, checking that each term in the asymptotics (5.3) gives the desired contribution to (5.4). It is enough to consider the Laplace transforms of

$$\begin{aligned}
\psi_1(\lambda) &= \lambda^{-\alpha}\theta(\lambda-1),\ \alpha \in \mathbb{R}\setminus\mathbb{N}, && (5.14)\\
\psi_2(\lambda) &= \lambda^{-\beta}\theta(\lambda-1),\ \beta \in \mathbb{N}, && (5.15)\\
\psi_3(\lambda) &= O(\lambda^{-M}),\ \lambda \to \infty,\ M \in (0,\infty)\setminus\mathbb{N},\ M > \max\{\alpha,\beta\}, && (5.16)
\end{aligned}$$

where $\theta(\lambda) = (1 + \mathrm{sign}(\lambda))/2$ is the Heaviside function.

2. Consider (5.14). One has

$$\begin{aligned}
\int_{-\infty}^\infty e^{-t\lambda}\psi_1(\lambda)d\lambda &= \int_{-\infty}^\infty e^{-t\lambda}\lambda^{-\alpha}\theta(\lambda-1)d\lambda\\
&= \int_1^\infty e^{-t\lambda}\lambda^{-\alpha}d\lambda && (5.17)
\end{aligned}$$

Changing the variable gives,

$$\int_1^\infty e^{-t\lambda}\lambda^{-\alpha}d\lambda = t^{\alpha-1}\int_t^\infty e^{-r}r^{-\alpha}dr. \tag{5.18}$$

Note the Gamma function present in (5.4). Now for Re $z < 1$, the Euler definition of the Gamma function gives

$$
\begin{aligned}
\Gamma(1-z) &= \int_0^\infty e^{-\lambda}\lambda^{-z}d\lambda \\
&= \int_t^\infty e^{-\lambda}\lambda^{-z}d\lambda + \int_0^t e^{-\lambda}\lambda^{-z}d\lambda \\
&= \int_t^\infty e^{-\lambda}\lambda^{-z}d\lambda + t^{1-z}\int_0^1 e^{-t\lambda}\lambda^{-z}d\lambda \\
&= \int_t^\infty e^{-\lambda}\lambda^{-z}d\lambda + t^{1-z}\sum_{j=0}^{[M]-1}\frac{(-1)^j t^j}{j!}\frac{1}{j+1-z} \\
&\quad +O(t^{[M]-z+1}),\ t\to +0, \qquad (5.19)
\end{aligned}
$$

where $[M]$ denotes the integer part of M.

The above expression can be meromorphically continued to the right and will hold true for $z \in \mathbb{C}\setminus\mathbb{N}$ such that $\operatorname{Re} z < [M]+1$.

Using (5.19) and (5.18), one has

$$
\begin{aligned}
&\int_{-\infty}^\infty e^{-t\lambda}\psi_1(\lambda)d\lambda \\
&= t^{\alpha-1}\Gamma(1-\alpha) - \sum_{j=0}^{[M]-1}\frac{(-1)^j t^j}{j!}\frac{1}{j+1-\alpha} + O(t^{[M]}),\ t\to +0. \qquad (5.20)
\end{aligned}
$$

3. Consider (5.15). Integrating by parts, one has

$$
\begin{aligned}
&\int_{-\infty}^\infty e^{-t\lambda}\psi_2(\lambda)d\lambda \\
&= \int_1^\infty e^{-t\lambda}\lambda^{-\beta}d\lambda \\
&= \frac{(-1)e^{-t}}{\beta-1} + \frac{(-1)t}{\beta-1}\int_1^\infty \lambda^{-\beta+1}e^{-t\lambda}d\lambda \\
&= e^{-t}\Big(\frac{(-1)}{\beta-1} + \frac{(-1)^2 t}{(\beta-1)(\beta-2)} + \cdots + \frac{(-1)^{\beta-1}t^{\beta-2}}{(\beta-1)!}\Big) \\
&\quad +\frac{(-1)^{\beta-1}t^{\beta-1}}{(\beta-1)!}\int_1^\infty \lambda^{-1}e^{-t\lambda}d\lambda. \qquad (5.21)
\end{aligned}
$$

Taking the last intgral seperately, one has

$$
\begin{aligned}
&\int_1^\infty \lambda^{-1}e^{-t\lambda}d\lambda \\
= \;& t\Big(\int_0^\infty \log\lambda e^{-t\lambda}d\lambda - \int_0^1 \log\lambda e^{-t\lambda}d\lambda\Big) \\
= \;& \int_0^\infty \log\frac{\lambda}{t}e^{-\lambda}d\lambda - t\int_0^1\Big(\sum_{j=0}^{[M]-1}\frac{(-1)^j t^j\lambda^j}{j!} + O((t\lambda)^{[M]})\Big)\log\lambda d\lambda \\
= \;& -\log t\int_0^\infty e^{-\lambda}d\lambda + \int_0^\infty \log\lambda e^{-\lambda}d\lambda \\
& + \sum_{j=0}^{[M]-2}\frac{(-1)^j t^{j+1}}{j!}\int_0^1 \lambda^j\log\lambda d\lambda + O(t^{[M]}).
\end{aligned} \tag{5.22}
$$

Substituting (5.22) into (5.21) and expanding the exponential term, one has

$$
\int_{-\infty}^\infty \psi_2(\lambda)d\lambda = \sum_{j=0}^{[M]-1} a_j t^j + \frac{(-1)^\beta t^{\beta-1}}{(\beta-1)!}\log t + O(t^{[M]}),\ t\to+0, \tag{5.23}
$$

for some constants $\{a_j\}$.

4. Finally, consider (5.16). By the conditions of the lemma, assume $\psi_3(\lambda)$ is measurable, bounded and $\psi_3(\lambda)=0$ for all $\lambda<a$, for some $a\in\mathbb{R}$. Let $\tilde\psi_3(\lambda)=\psi_3(\lambda)\lambda^M$. Then $\tilde\psi_3(\lambda)=O(1)$, as $\lambda\to\infty$. It is also bounded.

One has

$$
\begin{aligned}
\int_{-\infty}^\infty e^{-t\lambda}\psi_3(\lambda)d\lambda \;=\; & \int_a^\infty\Big(e^{-t\lambda}-\sum_{j=0}^{[M]-1}\frac{(-1)^j t^j\lambda^j}{j!}\Big)\frac{\tilde\psi_3(\lambda)}{\lambda^M}d\lambda \\
& + \sum_{j=0}^{[M]-1}\frac{(-1)^j t^j}{j!}\int_a^\infty \psi_3(\lambda)\lambda^j d\lambda,
\end{aligned} \tag{5.24}
$$

It suffices to show that the first integral on the RHS above is $O(t^{M-1})$.

Assume $t \in [0,1]$. By a change of variable one has

$$\begin{aligned}
&\int_a^\infty \left(e^{-t\lambda} - \sum_{j=0}^{[M]-1} \frac{(-1)^j t^j \lambda^j}{j!}\right) \frac{\tilde{\psi}_3(\lambda)}{\lambda^M} d\lambda \\
= \;& t^{M-1} \int_{at}^\infty \left(e^{-\lambda} - \sum_{j=0}^{[M]-1} \frac{(-1)^j \lambda^j}{j!}\right) \frac{\tilde{\psi}_3(\frac{\lambda}{t})}{\lambda^M} d\lambda \\
\leq \;& C t^{M-1} \int_{-|a|}^\infty \left|\left(e^{-\lambda} - \sum_{j=0}^{[M]-1} \frac{(-1)^j \lambda^j}{j!}\right) \frac{1}{\lambda^M}\right| d\lambda, \qquad (5.25)
\end{aligned}$$

where C is some constant.

Substituting (5.25) into (5.24) gives

$$\int_{-\infty}^\infty e^{-t\lambda} \psi_3(\lambda) d\lambda = \sum_{j=0}^{[M]-1} b_j t^j + O(t^{M-1}), \; t \to +0, \qquad (5.26)$$

for some constants $\{b_j\}$.

Now (5.20), (5.23) and (5.26) establish that the Laplace transforms of (5.14), (5.15) and (5.16), respectively, give the correct contribution to (5.4). This completes the proof of Lemma 5.2.

□

6 Proof of Lemma 4.1

We shall require the following:

Lemma 6.1. *Let $V \in C_0^\infty(\mathbb{R})$. Then the following expansion holds true uniformly in any bounded interval $\Omega \subset \mathbb{R}$:*

$$e^{-tH}(x,x) \sim \sum_{j=0}^\infty \frac{1}{\sqrt{4\pi t}} t^j a_j[x^2 + V(x)], \quad t \to +0, \qquad (6.1)$$

where the heat invariants $a_j[V(x)]$ are polynomials in $V(x)$ and its derivatives.

Proof The asymptotic expansion (6.1) needs to be proved to hold uniformly in any interval. For simplicity of notation, we will prove this for $(-1,1)$. Let $q(x) \in C_0^\infty(\mathbb{R})$ be a scalar potential with the following properties:

$$q(x) = x^2 \text{ for } |x| \leq 2 \text{ and } q(x) = 0 \text{ for } |x| \geq 3. \tag{6.2}$$

Let $\tilde{H} = -\frac{d^2}{dx^2} + q + V$ be the corresponding operator. All the coefficieints of the operator $\tilde{H}$ are in C_0^∞, and so the exisiting proofs (see Appendix D) of the heat kernel expansion are valid. One has

$$e^{-t\tilde{H}}(x,x) \sim \frac{1}{(4\pi t)^{\frac{1}{2}}} \sum_{j=0}^{\infty} t^j a_j[x^2 + V(x)], \qquad t \to +0, \tag{6.3}$$

uniformly in $x \in (-1,1)$. Thus, it suffices to prove the estimate

$$\sup_{|x|<1} |e^{-t\tilde{H}}(x,x) - e^{-tH}(x,x)| = O(t^\infty), \qquad t \to +0. \tag{6.4}$$

Using a standard representation for operators of this type, one has

$$e^{-tH} - e^{-t\tilde{H}} = \int_0^t e^{-(t-s)\tilde{H}}(q(x) - x^2)e^{-sH} ds.$$

From this, one obtains

$$\begin{aligned} & |e^{-t\tilde{H}}(x,x) - e^{-tH}(x,x)| \\ = \; & \left| \int_0^t ds \int_{-\infty}^{\infty} dy \; e^{-(t-s)\tilde{H}}(x,y)(q(y) - y^2)e^{-sH}(y,x) \right| \\ = \; & \left| \int_0^t ds \int_{|y|>2} dy \; e^{-(t-s)\tilde{H}}(x,y)(q(y) - y^2)e^{-sH}(y,x) \right| \end{aligned} \tag{6.5}$$

From here using the diamagnetic inequality, see [26]:

$$|e^{-tH}(x,y)| \leq (4\pi t)^{-1} \exp\Big(- \frac{(x-y)^2}{4t} - tm \Big), \; t > 0, \; x, y \in \mathbb{R}, \tag{6.6}$$

where $m = \inf_{\mathbb{R}}(V(x) + x^2)$, one obtains

$$\begin{aligned} \sup_{|x|<1} |e^{-t\tilde{H}}(x,x) - e^{-tH}(x,x)| & \leq \sup_{|x|<1} \frac{C}{t^2} \int_0^t ds \int_{|y|>2} dy (y^2+1) e^{-\frac{(x-y)^2}{4(t-s)} - \frac{(x-y)^2}{4s}} \\ & = O(t^\infty), \; t \to +0. \qquad \square \end{aligned}$$

Proof of Lemma 4.1 For simplicity of notation, let us assume that the support of V belongs to the interval $(-1,1)$. For any $r>0$, denote by χ_r the characteristic function of the interval $(-r,r)$ and let $\tilde{\chi}_r = 1-\chi_r$. By Lemma 6.1, it suffices to prove that

$$\operatorname{Tr}(\tilde{\chi}_2(e^{-tH}-e^{-tH_0})) = O(t^\infty),\ t\to +0. \tag{6.7}$$

We have that

$$e^{-tH}-e^{-tH_0} = \int_0^t e^{-(t-s)H_0}Ve^{-sH}ds.$$

Therefore,

$$\begin{aligned}
& \operatorname{Tr}(\tilde{\chi}_2(e^{-tH}-e^{-tH_0}))| \\
\le\ & \|\tilde{\chi}_2(e^{-tH}-e^{-tH_0})\tilde{\chi}_2\|_{S_1} \\
\le\ & \int_0^t ds\|\tilde{\chi}_2 e^{-(t-s)H}\chi_1 V\chi_1 e^{-sH}\tilde{\chi}_2\|_{S_1} \\
\le\ & \int_0^t ds\|\tilde{\chi}_2 e^{-(t-s)H}\chi_1 V\|_{S_2}\|\chi_1 e^{-sH}\tilde{\chi}_2\|_{S_2},
\end{aligned} \tag{6.8}$$

where $|\operatorname{Tr}(AB)| \le \|AB\|_{S_1} \le \|A\|_{S_2}\|B\|_{S_2}$ has been used,with S_1, S_2 the Trace and Hilbert-Schmidt classes respectively, see Appendix A.

Using the diagmagnetic inequality (6.6) we can make the following calculation

$$\begin{aligned}
\|\tilde{\chi}_2 e^{-tH}\chi_1 V\|_{S_2}^2 &= \int\int_{\substack{|x|>2\\|y|<1}} dxdy|e^{-tH}(x,y)V(y)|^2 \\
&\le \frac{e^{-2tm}}{(4\pi t)^2}\int\int_{\substack{|x|>2\\|y|<1}} dxdye^{-\frac{(x-y)^2}{2t}}V^2(y) \\
&\le \frac{2e^{-2tm}}{(4\pi t)^2}\sup_{|y|<1}\int_{|x|>2} e^{-\frac{(x-y)^2}{2t}}V^2(y)dx \\
&\le \frac{4e^{-2tm}}{(4\pi t)^2}\sup_{|y|<1}V^2(y)\int_2^\infty e^{-\frac{(x-1)^2}{2t}}dx \\
&= O(t^\infty),\ t\to +0,.
\end{aligned} \tag{6.9}$$

A similar calculation shows that $\|\chi_1 e^{-tH}\tilde{\chi}_2\|_{S_2} = O(t^\infty)$, as $t\to +0$.

Using the above two estimates in (6.8) we obtain the required result. □

7 Proof of trace formulas

This follows the arguments of [7]. First let us assume $\lambda_n \neq 0$ for all n. Fix $k \in \mathbb{N}$ and consider formula (4.9) reproduced below:

$$Z(s) = \sum_{n=0}^{\infty}\{\lambda_n^{-s} - \sum_{j=0}^{2k+2} d_j(s)(\lambda_n^0)^{-s-(j/2)}\} + \sum_{j=0}^{2k+2} d_j(s)Z_0(s+(j/2)), \quad (7.1)$$

where $\operatorname{Re} s > 1$. The second term in the r.h.s. is meromorphic in $\mathbb{C}$ with possible poles at $s = 1-\frac{j}{2}$, $j = 0, 1, 2, \ldots, 2k+2$. The first term in the r.h.s. of (7.1) admits analytic continuation into the half-plane $\operatorname{Re} s > -k-\frac{1}{2}$. As k can be taken arbitrary large, it follows that Z admits a meromorphic continuation into the whole complex plane, all poles of Z are simple and located at the points $s = 1-\frac{j}{2}$, $j = 0, 1, 2, \ldots$.The function $Z_0(s)$ is meromorphic and its only pole is at $s = 1$.

For $c > 1$, one has the following integral representation (see 13.2.9 of [1]),

$$\sum_{n=1}^{\infty}(e^{-t\lambda_n} - e^{-t\lambda_n^0}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\Gamma(z)(Z(z) - Z_0(z))dz. \quad (7.2)$$

By the meromorphic continuation of $Z(s)$ the contour of integration can be shifted to the left of zero, see Appendix B for an explanation of why this is possible.

One has

$$\begin{aligned}\sum_{n=1}^{\infty}(e^{-t\lambda_n} - e^{t\lambda_n^0}) &= \sum_{j=0}^{2N+2} \operatorname*{Res}_{s=1-\frac{j}{2}}[\Gamma(s)(Z(s) - Z_0(s))t^s] \\ &\quad + \frac{1}{2\pi i}\int_{-N-\varepsilon-i\infty}^{-N-\varepsilon+i\infty} \Gamma(s)(Z(s) - Z_0(s))t^{-s}ds, \quad (7.3)\end{aligned}$$

where the last integral is of the order $O(t^{N+\varepsilon})$, see Appendix B.

It is known that $\Gamma(s)$ has a pole of order 1 with residue $(-1)^k/k!$ at $s = -k$, $k = 0, 1, 2, 3, \ldots$. By (7.1) it is also possible for $Z(s)$ to have a pole of order 1 at $s = -k$, $k = 1, 2, 3, \ldots$. Let $k \in \mathbb{N}$, if $Z(s)$ has a pole at $s = -k$

then $\Gamma(s)Z(s)$ has a pole of order 2. Thus

$$\begin{aligned}
&\Gamma(s)Z(s)t^{-s} \\
= \ & \Big(\frac{a_2}{(s+k)^2}+\frac{a_1}{(s+k)}+f(s)\Big)(e^{-(s+k)\ln t}t^k) \\
= \ & \Big(\frac{a_2}{(s+k)^2}+\frac{a_1}{(s+k)}+f(s)\Big)(t^k(1+(s+k)\ln t+O((s+k)^2)),
\end{aligned}$$

where $a_2,\ a_1$ are constants and $f(s)$ is regular in a neighbourhood of $s=-k$. This implies that $\mathrm{Res}_{s=-k}(\Gamma(s)Z(s)t^{-s}) = a_2t^k\ln t + t^ka_1$. By comparison with (4.1) , where there are no terms involving $\ln t$, one can draw the conclusion that $Z(s)$ is regular for $s=-k,\ k\in\mathbb{N}$. Now, for $k\in\mathbb{N}$, one has that

$$\operatorname*{Res}_{s=-k}(\Gamma(s)(Z(s)-Z_0(s))t^{-s}) = \frac{(-1)^k}{k!}(Z(-k)-Z_0(-k))t^k.$$

Again by comparison with (4.1), one can draw the conclusion that $Z(-k)-Z_0(-k)=0$, for all $k\in\mathbb{N}$.

To consider the case where one of the eigenvalues of H vanishes: $\lambda_m=0$, repeat the arguments of the proof for the sequence $\{\lambda_n\}$, $n\in\mathbb{N}\setminus\{m\}$. For example (7.1) becomes

$$Z(s)=\sum_{n\in\mathbb{N}\setminus\{m\}}^{\infty}\{\lambda_n^{-s}-\sum_{j=0}^{2k+2}d_j(s)(\lambda_n^0)^{-s-(j/2)}\}+\sum_{j=0}^{2k+2}d_j(s)Z_0(s+(j/2)). \quad (7.4)$$

Then one simply uses that $e^{-t\lambda_m}=1$ in (7.2). One obtains the same set of results, apart from the formula $Z(0)=0$; this should be replaced by $Z(0)=-1$. This completes the proof of Theorem 4.3. □

8 Proof of Theorem 4.2 (i)

Let us define two solutions $\psi_\pm^0=\psi_\pm^0(x,\lambda)$ of the equation $-\psi''+x^2\psi=\lambda\psi$ by

$$\psi_+^0(x,\lambda)=U(-\frac{\lambda}{2},x\sqrt{2}),\quad \psi_-^0(x,\lambda)=U(-\frac{\lambda}{2},-x\sqrt{2}),$$

where U is the parabolic cylinder function (see [1, §19]). For any $x \in \mathbb{R}$, the solutions $\psi^0_\pm(x,\lambda)$ are entire functions of λ. For any $\lambda \in \mathbb{C}$, the solutions $\psi^0_\pm(x,\lambda)$ have the asymptotics

$$\psi^0_+(x,\lambda) = \psi^0_-(-x,\lambda) = (x\sqrt{2})^{(\lambda-1)/2}e^{-x^2/2}(1+o(1)), \quad x \to +\infty,$$

and the Wronskian $w_0(\lambda) = W(\psi^0_-, \psi^0_+) = (\psi^0_-)'_x\psi^0_+ - \psi^0_-(\psi^0_+)'_x$ is given by

$$w_0(\lambda) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1-\lambda}{2})} = \frac{2}{\sqrt{\pi}}\Gamma(\tfrac{1+\lambda}{2})\cos(\tfrac{\pi\lambda}{2}). \tag{8.1}$$

At the eigenvalues $\lambda^0_n = 2n-1$, the Wronskian $w_0(\lambda)$ vanishes and we have

$$\psi^0_+(x,\lambda^0_n) = (-1)^{n+1}\psi^0_-(x,\lambda^0_n) = 2^{-(n-1)/2}e^{-x^2/2}H_{n-1}(x), \tag{8.2}$$

where H_n is the n'th Hermite polynomial.

Next, let $\psi_\pm = \psi_\pm(x,\lambda)$ be the solutions of the equation $-\psi'' + (x^2 + V(x))\psi = \lambda\psi$, normalised by

$$\begin{aligned}
\psi_+(x,\lambda) &= \psi^0_+(x,\lambda), \quad x > \sup\operatorname{supp} V,\\
\psi_-(x,\lambda) &= \psi^0_-(x,\lambda), \quad x < \inf\operatorname{supp} V.
\end{aligned}$$

The eigenvalues λ_n coincide with the zeros of the Wronskian $w(\lambda) = W(\psi_-, \psi_+)$. Theorem 4.2 (i) will follow from the three lemmas below.

Let Ω be the half-strip

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0, |\operatorname{Im}\lambda| \leq 1\},$$

for $\lambda \in \Omega$ let us denote by $\sqrt{\lambda}$ a branch of the square root, so that $\operatorname{Re}\sqrt{\lambda} \geq 0$.

Lemma 8.1. *The Wronskian $w(\lambda)$ is analytic in $\lambda \in \Omega$. The following asymptotic expansion holds true:*

$$w(\lambda) \sim \frac{2}{\sqrt{\pi}}\Gamma(\tfrac{1+\lambda}{2})\left(\cos(\tfrac{\pi\lambda}{2})\sum_{j=0}^{\infty}\frac{Q_j}{(\sqrt{\lambda})^j} + \sin(\tfrac{\pi\lambda}{2})\sum_{j=0}^{\infty}\frac{P_j}{(\sqrt{\lambda})^j}\right), \tag{8.3}$$

as $|\lambda| \to \infty$, $\lambda \in \Omega$. Here $Q_j, P_j \in \mathbb{C}$ are some coefficients, $Q_0 = 1$, $P_0 = 0$.

Lemma 8.2. *Let w be a function analytic in the half-strip Ω and having the asymptotic expansion (8.3) with $Q_0 = 1$, $P_0 = 0$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the zeros of w in Ω, enumerated such that $\operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_2 \leq \cdots$. Then for some $k \in \mathbb{Z}$ one has the asymptotic expansion*

$$\lambda_{n-k} \sim \lambda_n^0 + \frac{c_1}{\sqrt{\lambda_n^0}} + \frac{c_2}{\lambda_n^0} + \frac{c_3}{\lambda_n^0\sqrt{\lambda_n^0}} + \dots \tag{8.4}$$

with some coefficients $\{c_j\} \in \mathbb{C}$.

Note that the coefficients $\{c_j\}$ above will actually be real because the zeros of w are the eigenvalues of H.

It is necessary to show that if Lemma 8.2 is applied to the Wronskian $w(\lambda) = W(\psi_-, \psi_+)$, then one has $k = 0$ in (8.4). This will follow from

Lemma 8.3. *For the eigenvalues $\lambda_1 < \lambda_2 < \cdots$ of H, we have $\lambda_n = \lambda_n^0 + o(1)$, $n \to \infty$.*

8.1 Proof of Lemma 8.1

Let $x > \sup\operatorname{supp} V$; then

$$w(\lambda) = (\psi_-(x,\lambda))'_x\psi_+^0(x,\lambda) - \psi_-(x,\lambda)(\psi_+^0(x,\lambda))'_x. \tag{8.5}$$

We will use this formula and construct ψ_- in a standard way as a solution to the integral equation

$$\psi_-(x,\lambda) = \psi_-^0(x,\lambda) + \int_{-\infty}^{x} G_\lambda(x,y)V(y)\psi_-(y,\lambda)dy, \tag{8.6}$$

where the integral kernel $G_\lambda(x,y)$ is given by

$$G_\lambda(x,y) = -\frac{1}{w_0(\lambda)}(\psi_+^0(x,\lambda)\psi_-^0(y,\lambda) - \psi_-^0(x,\lambda)\psi_+^0(y,\lambda)). \tag{8.7}$$

The kernel $G_\lambda(x,y)$ is an entire function of λ. One can see this from examining seperately the component functions of $G_\lambda(x,y)$.

The function

$$\psi_+^0(x,\lambda)\psi_-^0(y,\lambda) - \psi_-^0(x,\lambda)\psi_+^0(y,\lambda) \tag{8.8}$$

is an entire function of λ because the functions $\psi_{\pm}^{0}(x,\lambda)$ are entire.

By (8.1), the function $\frac{1}{w_0(\lambda)}$ is analytic everywhere except at the points λ_n^0 where it has simple poles. However, at these points the function (8.8) vanishes due to the relation (8.2). Therefore, $G_\lambda(x,y)$ is indeed analytic.

Let $R>0$ be sufficiently large so that $\operatorname{supp} V \subset (-R,R)$. Denote $\Delta = [-2R,2R]$; let $L_\lambda : C(\Delta) \to C(\Delta)$ be the Volterra type integral operator from (8.6),

$$L_\lambda : f(x) \mapsto \int_{-2R}^{x} G_\lambda(x,y)V(y)f(y)dy. \tag{8.9}$$

Then the solution of the integral equation (8.6) can be written as

$$\psi_- = \sum_{n=0}^{\infty} L_\lambda^n \psi_-^0,$$

where the convergence of the series shall be justified later, and so for the Wronskian (8.5) we have the series representation

$$w(\lambda) = \sum_{n=0}^{\infty} W(L_\lambda^n \psi_-^0, \psi_+^0)(x), \quad x \in (R,2R).$$

Lemma 8.4. *For any $n \in \mathbb{N}$ and any $x \in (R,2R)$, the Wronskian $W(L_\lambda^n\psi_-^0,\psi_+^0)(x)$ is analytic in $\lambda \in \Omega$ and one has the estimate*

$$|W(L_\lambda^n\psi_-^0,\psi_+^0)(x)| \le \frac{C(\lambda)^n}{n!}|\Gamma(\tfrac{1+\lambda}{2})|, \quad C(\lambda) = O(|\lambda|^{-n/2}), \quad |\lambda| \to \infty, \lambda \in \Omega. \tag{8.10}$$

The asymptotic expansion

$$W(L_\lambda^n\psi_-^0,\psi_+^0)(x) \sim \Gamma(\tfrac{1+\lambda}{2})\left(\cos(\tfrac{\pi\lambda}{2})\sum_{j=n}^{\infty}\frac{Q_j^{(n)}}{(\sqrt{\lambda})^j} + \sin(\tfrac{\pi\lambda}{2})\sum_{j=n}^{\infty}\frac{P_j^{(n)}}{(\sqrt{\lambda})^j}\right), \tag{8.11}$$

with some coefficients $Q_j^{(n)}$, $P_j^{(n)}$ holds true as $|\lambda| \to \infty$, $\lambda \in \Omega$.

Clearly, Lemma 8.1 follows from Lemma 8.4.

8.2 Proof of Lemma 8.4:

1. It is convenient to introduce two linear combinations e_+ and e_- of of the solutions $\psi^0_\pm$:

$$e_+(x,\lambda) = \frac{\sqrt{\pi}2^{(1-\lambda)/4}}{\cos(\frac{\pi\lambda}{2})\Gamma(\frac{1+\lambda}{4})}\left(e^{-i\pi(\lambda+1)/4}\psi^0_+(x,\lambda) + e^{i\pi(\lambda+1)/4}\psi^0_-(x,\lambda)\right), \quad (8.12)$$

$e_-(x,\lambda) = e_+(-x,\lambda)$. The solutions $e_\pm(x,\lambda)$ are analytic in $\lambda \in \Omega$ (with removeable singularities at λ^0_n — see (8.2)). These solutions are chosen so that they have convenient asymptotic expansions as $\lambda \to \infty$, $\lambda \in \mathbb{R}$. Let

$$e_\pm(x,\lambda) = e^{u_r(x,\lambda)\pm iu_i(x,\lambda)}e^{\pm i\sqrt{\lambda}x}, \quad (8.13)$$

here u_r, u_i are the functions v_r, v_i, given in 19.9.5 of [1], with $x\sqrt{2}$ and $\sqrt{\frac{\lambda}{2}}$ substituted in for x and p.

Let $\Delta = (-2R, 2R)$. One has the following asymptotic expansions:

$$\begin{aligned} u_r(x,\lambda) &\sim +\frac{(\frac{\sqrt{2}}{2}x)^2}{(\sqrt{2\lambda})^2} + \frac{(\frac{\sqrt{2}}{2}x)^4}{(\sqrt{2\lambda})^4} - \frac{9(\frac{\sqrt{2}}{2}x)^2 - \frac{16}{3}(\frac{\sqrt{2}}{2}x)^6}{(\sqrt{2\lambda})^6} - \ldots \\ u_i(x,\lambda) &\sim -\frac{\frac{2}{3}(\frac{\sqrt{2}}{2}x)^3}{\sqrt{2\lambda}} + \frac{\frac{\sqrt{2}}{2}x + \frac{2}{5}(\frac{\sqrt{2}}{2}x)^5}{(\sqrt{2\lambda})^3} + \frac{\frac{16}{3}(\frac{\sqrt{2}}{2}x)^3 - \frac{4}{7}(\frac{\sqrt{2}}{2}x)^7}{(\sqrt{2\lambda})^5} - \ldots \end{aligned}$$

as $\lambda \to \infty$, $\lambda \in \Omega$ for $x \in \Delta$.

For our purposes it suffices to rewrite (8.13) as

$$e_\pm(x,\lambda) \sim e^{\pm i\sqrt{\lambda}x}\left(1 + \sum_{j=1}^\infty \frac{R^\pm_j(x)}{(\sqrt{\lambda})^j}\right), \quad \lambda \to \infty, \quad \lambda \in \Omega, \quad (8.14)$$

for some $R^\pm_j$, which are polynomials in x.

The functions $e_\pm$ have oscillatory terms of the type $e^{\pm i\sqrt{\lambda}x}$ present in their asymptotic expansions. These terms do not make a contribution to the resulting expansion of λ_n. However, oscillatory terms of the type $e^{i\pi\lambda}$ appear in the asymptotic expansion for $w(\lambda)$ and these do give a contribution to the asymptotic expansion of λ_n.

From the recursion formula (see (19.6.2) of [1]), one has that

$$(\psi^0_\pm(x,\lambda))'_x = x\psi^0_\pm(x,\lambda) \mp \sqrt{2}\psi^0_\pm(x,\lambda+2).$$

Using (8.12), this allows one to compute

$$(e_\pm(x,\lambda))'_x = xe_\pm(x,\lambda) \pm 2i\frac{\Gamma(\frac{3\pm\lambda}{4})}{\Gamma(\frac{1\pm\lambda}{4})}e_\pm(x,\lambda+2). \tag{8.15}$$

Using Stirling's formula, which is given by 6.1.37 of [1] as

$$\Gamma(\lambda) \sim e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}[1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}+\cdots], \tag{8.16}$$
$$z \to \infty,\ |\arg z| < \pi,$$

one has that

$$\frac{\Gamma(\frac{3+\lambda}{4})}{\Gamma(\frac{1+\lambda}{4})} \sim \frac{\sqrt{\lambda}}{2}(1+\sum_{j=1}^{\infty}\frac{\gamma_j}{\lambda^j}),\ \ |\lambda| \to \infty,\ \lambda \in \Omega, \tag{8.17}$$

for some constants γ_j.

By (8.15) with (8.17) and (8.14), one has that

$$(e_\pm(x,\lambda))'_x \ \sim e^{\pm i\sqrt{\lambda}x}\left(\pm i\sqrt{\lambda}+\textstyle\sum_{j=0}^{\infty}\frac{\widetilde{R}_j^\pm(x)}{(\sqrt{\lambda})^j}\right), \quad \lambda \to \infty,\ \lambda \in \Omega, \tag{8.18}$$

where $\widetilde{R}_j^\pm$ are polynomials in x.

2. Let us first prove the bound (8.10). We have

$$W(L_\lambda^n\psi_-^0,\psi_+^0)(x) = (L_\lambda^n\psi_-^0(x,\lambda))'_x\psi_+^0(x,\lambda) - L_\lambda^n\psi_-^0(x,\lambda)(\psi_+^0(x,\lambda))'_x; \tag{8.19}$$

let us obtain appropriate bounds for each term in the r.h.s. of (8.19). Expressing $\psi_\pm^0$ in terms of $e_\pm$,

$$\psi_\pm^0(x,\lambda) = \frac{1}{2\sqrt{\pi}i}2^{(\lambda-1)/4}\Gamma(\tfrac{1\pm\lambda}{4})\left(e^{i\pi(\lambda+1)/4}e_\mp(x,\lambda) - e^{-i\pi(\lambda+1)/4}e_\pm(x,\lambda)\right), \tag{8.20}$$

and using (8.14), (8.18), we obtain

$$\|\psi_\pm^0(\cdot,\lambda)\|_{C(\Delta)} = O(|2^{\lambda/4}\Gamma(\tfrac{1\pm\lambda}{4})|),\quad |\lambda| \to \infty,\quad \lambda \in \Omega, \tag{8.21}$$

$$\|(\psi_\pm^0(\cdot,\lambda))'_x\|_{C(\Delta)} = O(|\lambda^{1/2}2^{\lambda/4}\Gamma(\tfrac{1\pm\lambda}{4})|),\quad |\lambda| \to \infty,\quad \lambda \in \Omega. \tag{8.22}$$

Note the factors present in the above formulas. The following estimate, from Stirling's formula (8.16), explains the presence of $\Gamma(\frac{1\pm\lambda}{2})$ in (8.10). One has

$$\frac{(\Gamma(\frac{1\pm\lambda}{4})2^{\frac{\lambda}{4}})^2}{\Gamma(\frac{1\pm\lambda}{2})} = O(|\lambda|^{-\frac{1}{2}}),\ \ |\lambda| \to \infty,\ \lambda \in \Omega. \tag{8.23}$$

Next, expressing the kernel $G_\lambda(x,y)$ in terms of $e_\pm$,

$$G_\lambda(x,y) = \frac{1}{4i}\frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})}(e_+(x,\lambda)e_-(y,\lambda) - e_-(x,\lambda)e_+(y,\lambda)), \tag{8.24}$$

and using the asymptotics (8.14), we obtain

$$\sup_{|x|\le R}\sup_{|y|\le R}|G_\lambda(x,y)| = O(|\lambda|^{-1/2}), \quad |\lambda|\to\infty, \quad \lambda\in\Omega.$$

Using this estimate and the fact that L_λ is a Volterra type operator (see Appendix E for description of norm estimates for the powers of Volterra operaotrs) we obtain

$$\|L_\lambda^n\|_{C(\Delta)\to C(\Delta)} \le \frac{C(\lambda)^n}{n!}, \quad C(\lambda) = O(|\lambda|^{-1/2}), \quad |\lambda|\to\infty, \quad \lambda\in\Omega. \tag{8.25}$$

Finally, in order to estimate the term $(L_\lambda^n\psi_-^0)'_x$, let us introduce the operator $L'_\lambda : C(\Delta)\to C(\Delta)$ by

$$L'_\lambda : f(x)\mapsto \int_{-R}^{x}\frac{\partial G_\lambda(x,y)}{\partial x}V(y)f(y)dy.$$

Then $(L_\lambda^n\psi_-^0(x,\lambda))'_x = L'_\lambda L_\lambda^{n-1}\psi_-^0(x,\lambda)$. Using (8.24) with the asymptotics (8.14) and (8.18), one obtains

$$\|L'_\lambda\|_{C(\Delta)\to C(\Delta)} = O(1), \quad |\lambda|\to\infty, \quad \lambda\in\Omega. \tag{8.26}$$

Combining (8.19), (8.21)–(8.23), (8.24)–(8.26) one obtains (8.10).

3. Let us prove the asymptotic expansion (8.11). Using (8.20), we obtain

$$\begin{aligned} W(L_\lambda^n\psi_-^0,\psi_+^0) = \frac{1}{4\pi}2^{(\lambda-1)/2}\Gamma(\tfrac{1+\lambda}{4})^2\big(&W(L_\lambda^n e_+,e_+) + W(L_\lambda^n e_-,e_-) \\ &- e^{-i\pi(1+\lambda)/2}W(L_\lambda^n e_-,e_+) - e^{i\pi(1+\lambda)/2}W(L_\lambda^n e_+,e_-)\big)\end{aligned}$$

Denote

$$g_n^\pm(x,\lambda) = \frac{L_\lambda^n e_\pm(x,\lambda)}{e_\pm(x,\lambda)}; \tag{8.27}$$

by (8.14), the denominator does not vanish for all sufficiently large λ. Using this notation, one obtains

$$
\begin{aligned}
& W(L_\lambda^n \psi_-^0, \psi_+^0) \\
= \;& \frac{1}{4\pi} 2^{(\lambda-1)/2} \Gamma(\tfrac{1+\lambda}{4})^2 \Big(W(g_n^+ e_+, e_+) + W(g_n^- e_-, e_-) \\
& - e^{-i\pi(1+\lambda)/2} W(g_n^- e_-, e_+) - e^{i\pi(1+\lambda)/2} W(g_n^+ e_+, e_-) \Big) \\
= \;& \frac{1}{4\pi} 2^{(\lambda-1)/2} \Gamma(\tfrac{1+\lambda}{4})^2 \Big((g_n^+(x,\lambda))_x' e_+(x,\lambda) e_+(x,\lambda) \\
& \qquad + (g_n^-(x,\lambda))_x' e_-(x,\lambda) e_-(x,\lambda) \\
& \qquad - e^{-i\pi(1+\lambda)/2} (g_n^-(x,\lambda))_x' e_-(x,\lambda) e_+(x,\lambda) \\
& \qquad - e^{-i\pi(1+\lambda)/2} g_n^-(x,\lambda) W(e_-, e_+) \\
& \qquad - e^{i\pi(1+\lambda)/2} (g_n^+(x,\lambda))_x' e_+(x,\lambda) e_-(x,\lambda) \\
& \qquad - e^{i\pi(1+\lambda)/2} g_n^+(x,\lambda) W(e_+, e_-) \Big).
\end{aligned}
$$

From (8.14) the asymptotics of $e_\pm(x,\lambda)$ and the estimate (8.23), one has

$$
\begin{aligned}
W(L_\lambda^n \psi_-^0, \psi_+^0) = \;& \frac{1}{4\pi} 2^{(\lambda-1)/2} \Gamma(\tfrac{1+\lambda}{4})^2 \big(-e^{-i\pi(1+\lambda)/2} g_n^-(x,\lambda) W(e_-, e_+) \\
& - e^{i\pi(1+\lambda)/2} g_n^+(x,\lambda) W(e_+, e_-) \big) \\
& + (g_n^+(x,\lambda))_x' O(|\Gamma(\tfrac{1+\lambda}{2})|) + (g_n^-(x,\lambda))_x' O(|\Gamma(\tfrac{1+\lambda}{2})|). \qquad (8.28)
\end{aligned}
$$

Using (8.12) and (8.1) one can calculate that

$$
W(e_+, e_-) = 4i\sqrt{\pi}\, \Gamma(\tfrac{1+\lambda}{2}) 2^{\frac{1-\lambda}{2}} (\Gamma(\tfrac{1+\lambda}{4}))^{-2}.
$$

Substituting this into (8.28), one has

$$
\begin{aligned}
W(L_\lambda^n \psi_-^0, \psi_+^0) = \;& \frac{-i}{\sqrt{\pi}} \Gamma(\tfrac{1+\lambda}{2}) (e^{i\pi(1+\lambda)/2} g_n^+ - e^{-i\pi(1+\lambda)/2} g_n^-) \\
& + (g_n^+(x,\lambda))_x' O(|\Gamma(\tfrac{1+\lambda}{2})|) + (g_n^-(x,\lambda))_x' O(|\Gamma(\tfrac{1+\lambda}{2})|). \qquad (8.29)
\end{aligned}
$$

It suffices to show that if $x \in (R, 2R)$, then $g_n^\pm$ have the asymptotic expansions

$$
g_n^\pm(x,\lambda) \sim \sum_{j=n}^{\infty} \frac{S_j^\pm(x)}{(\sqrt{\lambda})^j}, \quad |\lambda| \to \infty, \quad \lambda \in \Omega \qquad (8.30)
$$

for some coefficients $S_j^{\pm} \in C^\infty(\mathbb{R})$, and that for any $x \in (R, 2R)$,

$$(g_n^{\pm}(x,\lambda))'_x = O(|\lambda|^{-\infty}), \quad |\lambda| \to \infty, \quad \lambda \in \Omega. \tag{8.31}$$

By the definition of $g_n^{\pm}$, we have

$$g_{n+1}^{\pm}(x,\lambda) = \int_{-R}^{x} \frac{G_\lambda(x,y)}{e_{\pm}(x,\lambda)} g_n^{\pm}(y,\lambda) e_{\pm}(y,\lambda) V(y) dy. \tag{8.32}$$

The expansion (8.30) shall be proved by induction. We shall only consider the case of g_n^+ but the same argument can be applied to g_n^-. For $n = 0$, one has $g_0^+ = 1$, which clearly satisfies (8.30). Assume (8.30) holds for some fixed n. Now substitute into (8.32) the expression (8.24) for $G_\lambda(x,y)$ and the asymptotics (8.14). Using the inductive assumption (8.30), one has an expression of the form

$$\begin{aligned} g_{n+1}^{+}(x,\lambda) &= \int_{-R}^{x} \frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})} \Big(\sum_{j=n}^{N} \frac{T_j(y)}{(\sqrt{\lambda})^j} + e^{-2i\sqrt{\lambda}x} e^{2i\sqrt{\lambda}y} \sum_{j=n}^{N} \frac{\tilde{T}_j(y,x)}{(\sqrt{\lambda})^j} \Big) V(y) dy \\ &\quad + \int_{-R}^{x} \frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})} O(|\sqrt{\lambda}|^{-N-1}) V(y) dy, \end{aligned}$$

for any integer $N > n$, where, by the inductive assumption and the asymptotics (8.14), one has $T_j \in C^\infty(\mathbb{R})$ and $\tilde{T}_j \in C^\infty(\mathbb{R}, \mathbb{R})$. By repeatedly integrating by parts and using $x \notin \operatorname{supp} V = (-R, R)$, one can show

$$\int_{-R}^{x} \frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})} e^{-2i\sqrt{\lambda}x} e^{2i\sqrt{\lambda}y} \sum_{j=n}^{N} \frac{\tilde{T}_j(y,x)}{(\sqrt{\lambda})^j} V(y) dy = O(|\lambda|^{-\infty}). \tag{8.33}$$

This implies that

$$g_{n+1}^{+}(x,\lambda) = \int_{-R}^{x} \frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})} \sum_{j=n}^{N} \frac{T_j(y)}{(\sqrt{\lambda})^j} V(y) dy + O(|\sqrt{\lambda}|^{-N-2}). \tag{8.34}$$

It follows from this and the asymptotics of the Gamma function that g_{n+1}^+ can be expressed in the form (8.30). This establishes (8.30) for all n.

The asymptotics (8.31) follows by differentiation of (8.32). Having set $x \notin \operatorname{supp} V = (-R, R)$, one has

$$\frac{d}{dx} g_{n+1}^{\pm}(x,\lambda) = \int_{-R}^{x} \Big(\frac{d}{dx} \Big(\frac{G_\lambda(x,y)}{e_{\pm}(x,\lambda)} \Big) \Big) g_n^{\pm}(y,\lambda) e_{\pm}(y,\lambda) V(y) dy.$$

Using (8.24), the expression for $G_\lambda(x,y)$, one has

$$\begin{aligned}
&\frac{d}{dx}g^{\pm}_{n+1}(x,\lambda) \\
&= \frac{1}{4i}\frac{\Gamma(\frac{1+\lambda}{4})}{\Gamma(\frac{3+\lambda}{4})}\int_{-R}^{x}\Big(\mp\frac{e'_{\mp}(x,\lambda)e_{\pm}(y,\lambda)}{e_{\pm}(x,\lambda)} \pm \frac{e_{\mp}(x,\lambda)e_{\pm}(y,\lambda)e'_{\pm}(x,\lambda)}{e^2_{\pm}(x,\lambda)}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times g^{\pm}_{n}(y,\lambda)e_{\pm}(y,\lambda)V(y)dy.
\end{aligned}$$

Note the above expression only contains the terms $e_{\pm}(y,\lambda)$ and not $e_{\mp}(y,\lambda)$. If one substitutes in the asymptotics (8.14) and (8.18), one produces an expression where all the terms are of a similar form to (8.33), that is they contain an exponential term of the form $e^{i\sqrt{\lambda}y}$, which does not cancel. Therefore, because $x \notin \operatorname{supp} V = (-R,R)$, integrating by parts will give (8.31).

That $Q_0 = 1$ and $P_0 = 0$ in (8.11) follows from (8.1). □

8.3 Proof of Lemma 8.2

1. Let $\Omega_n = \{z|\ |\mathrm{Im}\, z| \le 1,\ 2(n-1) \le \mathrm{Re}\, z \le 2n\}$. Fix ε such that $0<\varepsilon<1$ and let $\Gamma_{n,\varepsilon} = \{z|\ |z-\lambda^0_n| = \varepsilon\}$ with $B_{n,\varepsilon}$ denoting the interior of $\Gamma_{n,\varepsilon}$. Denote

$$\widetilde{w}(\lambda) = \frac{\sqrt{\pi}}{2}\frac{w(\lambda)}{\Gamma(\frac{1+\lambda}{2})};$$

the zeros of w in Ω coincide with the zeros of $\widetilde{w}$. From the conditions of Lemma 8.2, one has

$$\widetilde{w}(\lambda) = \cos\frac{\pi\lambda}{2} + O(\frac{1}{\sqrt{\lambda}}), \qquad |\lambda| \to \infty,\ \lambda \in \Omega. \tag{8.35}$$

We can now apply Rouché's Theorem to Ω_n and $B_{n,\varepsilon}$, see Appendix F. It follows that there exists $N \in \mathbb{N}$ such that for all $n \ge N$, the domain Ω_n contains exactly one zero of $\widetilde{w}$ and this zero is located in $B_{n,\varepsilon}$.

2. Let the half strip $\{z|\ |\mathrm{Im}\, z| \le 1,\ \mathrm{Re}\, z \le 2(N-1)\}$ contain m zeros of $\widetilde{w}$. Suppose the zeros λ_n of w are enumerated as $\mathrm{Re}\,\lambda_1 \le \mathrm{Re}\,\lambda_2 \le \cdots$ then it follows that for $k = N-m-1$, we have $\lambda_{n-k} \in B_{n,\varepsilon},\ n \ge N$.

3. Fix $\varepsilon > 0$. By the previous arguments and Cauchy's theorem, it follows that for all sufficiently large n we have

$$\lambda_{n-k} = \frac{1}{2\pi i}\int_{\Gamma_{n,\varepsilon}} \lambda\frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)}d\lambda. \tag{8.36}$$

4. Let us discuss the asymptotic expansion for $\frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)}$, $\lambda \in \Gamma_{n,\varepsilon}$, as $n \to \infty$. The notation $\mathcal{F}_i(\lambda)$, where $i \in \mathbb{N}$ is an index, shall be used in place of any function which has an asymptotic expansion of the following form

$$\mathcal{F}_i(\lambda) \sim \frac{c_{1i}}{\lambda^{\frac{1}{2}}} + \frac{c_{2i}}{\lambda} + \frac{c_{3i}}{\lambda^{\frac{3}{2}}} + \cdots, \ \lambda \to \infty,$$

for some coefficients $\{c_{ji}\} \in \mathbb{C}$.

Using this notation, one has $\widetilde{w}(\lambda) = \cos\frac{\pi\lambda}{2} + \cos\frac{\pi\lambda}{2}\mathcal{F}_1(\lambda) + \sin\frac{\pi\lambda}{2}\mathcal{F}_2(\lambda)$. Then (using Appendix C.1)

$$\begin{aligned}\widetilde{w}'(\lambda) = -\frac{\pi}{2}\sin\frac{\pi\lambda}{2} - \frac{\pi}{2}\sin\frac{\pi\lambda}{2}\mathcal{F}_1(\lambda) + \cos\frac{\pi\lambda}{2}\mathcal{F}_1'(\lambda) \\ + \frac{\pi}{2}\cos\frac{\pi\lambda}{2}\mathcal{F}_2(\lambda) + \sin\frac{\pi\lambda}{2}\mathcal{F}_2'(\lambda), \ \lambda \to \infty, \lambda \in \Omega. \quad (8.37)\end{aligned}$$

From here, one has

$$\begin{aligned}\frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)} &= \frac{\cos\frac{\pi\lambda}{2}\big(\mathcal{F}_1'(\lambda) + \frac{\pi}{2}\mathcal{F}_2(\lambda) + \tan\frac{\pi\lambda}{2}(\mathcal{F}_2'(\lambda) - \frac{\pi}{2} - \frac{\pi}{2}\mathcal{F}_1(\lambda))\big)}{\cos\frac{\pi\lambda}{2}(1 + \mathcal{F}_1(\lambda) + \tan\frac{\pi\lambda}{2}\mathcal{F}_2(\lambda))} \\ &= \big(\mathcal{F}_1'(\lambda) + \frac{\pi}{2}\mathcal{F}_2(\lambda) + (\tan\frac{\pi\lambda}{2})(\mathcal{F}_2'(\lambda) - \frac{\pi}{2} - \frac{\pi}{2}\mathcal{F}_1(\lambda))\big) \\ &\quad \times(1 + \sum_{n=1}^{\infty}(-1)^n(\mathcal{F}_1(\lambda) + \tan\frac{\pi\lambda}{2}\mathcal{F}_2(\lambda))^n),\end{aligned}$$

where $\lambda \in \Gamma_{n,\varepsilon}$ for sufficiently large n. Note $\tan\frac{\pi\lambda}{2}$ is bounded on $\Gamma_{n,\varepsilon}$.

For arbitrary N and $\lambda \in \Gamma_{n,\varepsilon}$, this expansion can be written as

$$\begin{aligned}\frac{\widetilde{w}'(\lambda)}{\widetilde{w}(\lambda)} = -\frac{\pi}{2}\tan\frac{\pi\lambda}{2} + \mathcal{F}_3(\lambda) + \frac{1}{\lambda}\mathcal{F}_4(\lambda)\tan\frac{\pi\lambda}{2} \\ + \sum_{j=2}^{N}(\tan\frac{\pi\lambda}{2})^j(\sqrt{\lambda})^{-j+2}\mathcal{F}_{j+3}(\lambda), \ |\lambda| \to \infty, \ \lambda \in \Omega, \quad (8.38)\end{aligned}$$

5. Now (8.38) can be substituted into (8.36) to give

$$\begin{aligned}\lambda_{n-k} = \frac{1}{2\pi i}\int_{\Gamma_{n,\varepsilon}} \lambda\Big(-\frac{\pi}{2}\tan\frac{\pi\lambda}{2} + f_3(\lambda) + \frac{1}{\lambda}f_4(\lambda)\tan\frac{\pi\lambda}{2} \\ + \sum_{j=2}^{N}(\tan\frac{\pi\lambda}{2})^j(\sqrt{\lambda})^{-j+2}f_{j+3}(\lambda) + O(|\lambda|^{-\frac{N}{2}})\Big)d\lambda. \quad (8.39)\end{aligned}$$

where the functions $f_j(\lambda)$ are analytic in $\lambda \in \Omega$ and satisfy $f_j(\lambda)+O(|\lambda|^{-\frac{N}{2}}) = \mathcal{F}_j(\lambda)$.

This can be integrated term by term to produce an asymptotic expansion for λ_{n-k}. The remainder term gives,

$$\int_{\Gamma_{n,\varepsilon}} \lambda O(|\lambda|^{-\frac{N}{2}})d\lambda = O(|\lambda_n^0|^{-\frac{N}{2}+1}). \tag{8.40}$$

For the other terms of (8.39), by Cauchy's theorem only those with $\tan \frac{\pi\lambda}{2}$ in the numerator will contribute to the integral. By the calculus of residues,

$$\frac{1}{2\pi i}\int_{\Gamma_{n,\varepsilon}} -\frac{\pi\lambda}{2}\tan\frac{\pi\lambda}{2}\, d\lambda = \lambda_n^0 \tag{8.41}$$

and for the other terms

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{n,\varepsilon}} \lambda\frac{\tan^j \frac{\pi\lambda}{2}}{\lambda^{\frac{m}{2}}}d\lambda &= \frac{1}{2\pi i}\int_{\Gamma_{0,\varepsilon}} \frac{\tan^j(\frac{\pi}{2}(\lambda+\lambda_n^0))}{(\lambda+\lambda_n^0)^{\frac{m}{2}-1}}d\lambda & (8.42)\\ &= (\lambda_n^0)^{-\frac{m}{2}+1}\sum_{k=0}^{j-1} a_k(\lambda_n^0)^{-k} & (8.43)\end{aligned}$$

where $\{a_k\} \in \mathbb{C}$, depends on j and m.

Thus, from (8.39), (8.40), (8.41) and (8.42) one has

$$\lambda_{n-k} \sim \lambda_n^0 + c_0 + \frac{c_1}{\sqrt{\lambda_n^0}} + \frac{c_2}{\lambda_n^0} + \frac{c_3}{\lambda_n^0\sqrt{\lambda_n^0}} + \cdots, \; n \to \infty.$$

The radius ε of $B_{n,\varepsilon}$ can be taken arbitrarily small and still λ_{n-k} will be contained within $B_{n,\varepsilon}$ for sufficiently large n. This implies $c_0 = 0$. □

8.4 Proof of Lemma 8.3

1. Consider the family of operators $H_\alpha = H_0 + \alpha V$, $\alpha \in [0,1]$, on $L^2(\mathbb{R})$, with $w(\lambda,\alpha)$ the corresponding Wronskian. Let $\lambda_1(\alpha) < \lambda_2(\alpha) < \ldots$, be the eigenvalues of H_α and hence the zeros of $w(\lambda,\alpha)$. Thus, $\lambda_n(0) = \lambda_n^0 = 2n-1$ and $\lambda_n(1)$, is an eigenvalue of H.

2. Consider the eigenvalues $\lambda_n(\alpha)$, which as mentioned are the zeros of the Wronskian $w(\lambda,\alpha)$. Since all zeros of the Wronskian are simple and the

Wronskian is analytic in α and λ, we have $\frac{\partial w}{\partial \lambda}(\lambda, \alpha) \neq 0$. Thus, by the implicit function theorem, the functions $\lambda_n = \lambda_n(\alpha)$ are continous in α.

3. Let

$$\widetilde{w}(\lambda, \alpha) = \frac{\sqrt{\pi}}{2} \frac{w(\lambda, \alpha)}{\Gamma(\frac{1+\lambda}{2})}.$$

Then $\widetilde{w}(\lambda, \alpha)$ and $w(\lambda, \alpha)$ have the same zeros in Ω.

Below we prove that the following asymptotic expansion holds uniformly in α,

$$\widetilde{w}(\lambda, \alpha) = \cos\frac{\pi\lambda}{2} + O(\frac{1}{\sqrt{\lambda}}), \qquad |\lambda| \to \infty,\ \lambda \in \Omega. \tag{8.44}$$

Let $\psi_\pm(\alpha, x, \lambda)$ be the solutions to

$$-\psi_\pm''(\alpha, x, \lambda) + x^2\psi_\pm(\alpha, x, \lambda) + \alpha V(x)\psi_\pm(\alpha, x, \lambda) = \lambda\psi_\pm(\alpha, x, \lambda),\ x \in \mathbb{R},\ \lambda \in \mathbb{C},$$

normalised by

$$\begin{aligned} \psi_+(\alpha, x, \lambda) &= \psi_+^0(x, \lambda), \quad x > \operatorname{sup\,supp} V, \\ \psi_-(\alpha, x, \lambda) &= \psi_-^0(x, \lambda), \quad x < \operatorname{inf\,supp} V. \end{aligned}$$

Then $\psi_\pm(1, x, \lambda) = \psi_\pm(x, \lambda)$ and $\psi_\pm(0, x, \lambda) = \psi_\pm^0(x, \lambda)$. By substituting αV for V in (8.9), the definition of L_λ, we have that

$$\psi_-(\alpha, x, \lambda) = \sum_{n=0}^{\infty} \alpha^n L_\lambda^n \psi_-^0. \tag{8.45}$$

By (8.45) and again fixing $x \in (R, 2R)$, we have that

$$w(\lambda, \alpha) = w_0(\lambda) + \sum_{n=1}^{\infty} \alpha^n W(L_\lambda^n \psi_-^0, \psi_+^0)(x).$$

By (8.1) and Lemma 8.4, one has (8.44).

4. Let Ω_n and $B_{n,\varepsilon}$ be defined as in the proof of Lemma 8.2. Fix ε such that $0 < \varepsilon < 1$. Using (8.44) one can apply Rouché's Theorem, see Appendix F. It follows that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all $\alpha \in [0, 1]$, the domain Ω_n contains exactly one zero of $\widetilde{w}(\lambda, \alpha)$ and this zero is located in $B_{n,\varepsilon}$.

5. Let $m(\alpha)$ denote the number of zeros of $\widetilde{w}(\lambda, \alpha)$ contained in the half strip $\{z| \, |\operatorname{Im} z| \leq 1, \ \operatorname{Re} z \leq 2(N-1)\}$. For $k(\alpha) = N - 1 - m(\alpha)$ the zero located in $B_{n,\varepsilon}$ is $\lambda_{n-k(\alpha)}(\alpha)$ for all $n \geq N$.

It shall be shown below that $m(\alpha)$ is constant for $\alpha \in [0, 1]$. By definition $m(0) = N - 1$, as it corresponds to the number of $\lambda_n^0 < 2(N-1)$. Therefore, $k(\alpha) = 0$ and $\lambda_n(\alpha) \in B_{n,\varepsilon}$ for all $n \geq N$ and $\alpha \in [0, 1]$. Specifically, $\lambda_n \in B_{n,\varepsilon}$ for all $n \geq N$. Once this is established, it proves Lemma 8.3.

That $m(\alpha)$ is constant shall be proven by contradiction. The proof relies on two facts.

i) $\lambda_n(\alpha) \in \mathbb{R}$ for $n \in \mathbb{N}$, as $\lambda_n(\alpha)$ is an eigenvalue of the self adjoint operator H_α.

ii) $\lambda_n(\alpha)$ is continuous in $\alpha \in [0, 1]$.

By i) and ii) for $m(\alpha)$ to vary, it must be that there exists $\alpha_0 \in [0, 1]$ and $j \in \mathbb{N}$ such that $\lambda_j(\alpha_0) \in \Omega_N$ but $\lambda_j(\alpha_0) \notin B_{N,\varepsilon}$. By part 4. this is impossible. Therefore, $m(\alpha)$ is constant for $\alpha \in [0, 1]$. $\square$

A The Trace and Hilbert-Schmidt classes

The Trace and Hilbert-Schmidt classes, S_1 and S_2 respectively, are defined by

$$\begin{aligned} S_1 &= \{A \in S_\infty | \sum_{n=1}^{\infty} s_n(A) < \infty\} \\ S_2 &= \{A \in S_\infty | \sum_{n=1}^{\infty} (s_n(A))^2 < \infty\} \end{aligned}$$

where $s_n(A)$ is defined to be the square root of the n'th eigenvalue of A^*A and S_∞ is the space of compact operators.

If $a(x,y) \in L^2(\mathbb{R} \times \mathbb{R})$ is the integral kernel of $A \in S_2$ then one has

$$\|A\|_{S_2} = \Big(\int\int |a(x,y)|^2 dxdy\Big)^{\frac{1}{2}}.$$

B Shifting the Contour of Integration

In this section we present a more detailed explanation of the shifting of contour argument used in section 7, notably to produce the equation (7.3). Here we are concerned with the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\Gamma(z)(Z(z) - Z_0(z))dz. \tag{B.1}$$

By the meromorphic continuation of $Z(s)$ the contour of integration can be shifted to the left of zero. This uses the fact that $\Gamma(z)$ decreases exponentially as $z \to \pm i\infty$ while $Z(z)$ increases at most as a power of z.

That $\Gamma(z)$ decreases exponentially can be seen from [1], 6.1.45, which states

$$(2\pi)^{-\frac{1}{2}}|\Gamma(x+iy)| \sim e^{-\frac{1}{2}\pi|y|}|y|^{-\frac{1}{2}+x}, \text{ as } |y| \to \infty. \tag{B.2}$$

To show that $Z(z)$ increases at most as a power of z, as $z \to \pm i\infty$, one can use the representation (7.1), which states

$$Z(z) = \sum_{n=0}^{\infty}\{\lambda_n^{-z} - \sum_{j=0}^{2k+2} d_j(z)(\lambda_n^0)^{-z-(j/2)}\} + \sum_{j=0}^{2k+2} d_j(z)Z_0(z+(j/2)), \tag{B.3}$$

where the coefficients $\{d_j(z)\}$ are polynomials in z. If $Z_0(z)$ is polynomially bounded as $z \to \pm i\infty$ then all the terms on the right hand side of (B.3) are polynomially bounded.

To prove this properety of Z_0, one can use the following representation, given by 23.2.3 of [1],

$$Z_0(z) = (1-2^{-z})\zeta(z) = (1-2^{-z})\Big(\frac{1}{z-1} + \frac{1}{2} + \sum_{k=1}^{n} \frac{B_{2k}}{2k} \begin{pmatrix} z+2k-2 \\ 2k-1 \end{pmatrix} - \begin{pmatrix} z+2n \\ 2n+1 \end{pmatrix} \int_1^\infty \frac{B_{2n+1}(x-[x])}{x^{z+2n+1}} dx\Big), \quad \text{(B.4)}$$

where $s \neq 1$, $n \in \mathbb{N}$, Re $s > -2n$, B_{2k} are the Bernoulli numbers and $B_{2n+1}(x)$ are the Bernoulli polynomials.

The Binomial coefficients in (B.4) can be shown to be polynomially bounded, as $z \to \pm i\infty$, from their definition in terms of Gamma functions, that is

$$\begin{pmatrix} z \\ w \end{pmatrix} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)},$$

and the formula (B.2). This proves that $Z(z)$ is polynomially bounded as $z \to \pm i\infty$.

Returning to (B.1), as the contour is shifted to the left, the poles of the integrand have to be taken into account. For the functions Z and Z_0, as defined in section 7, one has

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z}\Gamma(z)(Z(z) - Z_0(z))dz \\ = & \sum_{j=0}^{2N+2} \operatorname*{Res}_{s=1-\frac{j}{2}} [\Gamma(s)(Z(s) - Z_0(s))t^s] \\ & + \frac{1}{2\pi i} \int_{-N-\varepsilon-i\infty}^{-N-\varepsilon+i\infty} \Gamma(s)(Z(s) - Z_0(s))t^{-s}ds, \end{aligned}$$

where the last integral is of the order $O(t^{N+\varepsilon})$.

This can be shown by changing the variable,

$$\begin{aligned}
&\int_{-N-\varepsilon-i\infty}^{-N-\varepsilon+i\infty} \Gamma(s)(Z(s)-Z_0(s))t^{-s}ds \\
= \;& t^{N+\varepsilon}\int_{-i\infty}^{i\infty} \Gamma(s-N-\varepsilon)(Z(s-N-\varepsilon)-Z_0(s-N-\varepsilon))t^{-s}ds \\
\leq \;& t^{N+\varepsilon}\int_{-i\infty}^{i\infty} |\Gamma(s-N-\varepsilon)(Z(s-N-\varepsilon)-Z_0(s-N-\varepsilon))|ds \\
= \;& O(t^{N+\varepsilon}).
\end{aligned}$$

Here the final integral has been estimated from the facts, shown above, that $\Gamma(s)$ decreases exponentially as $s \to \pm i\infty$, while $Z(s)$ and $Z_0(s)$ increase only polynomially as $s \to \pm i\infty$.

Therefore, one can produce an asymptotic expansion for (B.1).

C Asymptotic Expansions

In the following definition of an asymptotic expansion, taken from [8], a_n denotes a constant, $\phi_n(x)$ a numerical function of x defined in $\mathbb{C}$, and ϕ_n an element of an asymptotic sequence. An asymptotic sequence is such that $\phi_{n+1} = o(\phi_n)$ as $x \to x_0$, for each n and some limit point $x_0 \in \mathbb{C}$.

The formal series $\sum a_n\phi_n(x)$, is said to be an asymptotic expansion to N terms of $f(x)$ as $x \to x_0$ if

$$f(x) = \sum_{n=1}^{N} a_n\phi_n(x) + o(\phi_N), \qquad x \to x_0.$$

An asymptotic expansion to any number of terms is written as

$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Throughout this work, only the cases of ϕ_n being powers or logarithms are dealt with.

C.1 Analytic Functions in a Strip: Possibility of Differentiation

Let us consider a function $f(z)$ which is analytic in the half strip $\Omega = \{z \in \mathbb{C} | \mathrm{Re} z \geq 0, |\mathrm{Im} z| < 1\}$. Suppose f admits asymptotic expansion of the form

$$f(z) \sim g(z) \sum_j P_j z^{-\alpha_j} + O(|z|^{-M}) \tag{C.1}$$

as $|z| \to \infty$, $z \in \Omega$, where $\{\alpha_j\} \subset \mathbb{R}$, $M > \max\{\alpha_j\}$, $g(z)$ is a function analytic in Ω and $P_j \in \mathbb{C}$ are some coefficients and the branch of $z^{-\alpha_j}$ is fixed by $z^{-\alpha_j} = |z|^{-\alpha_j} e^{i\alpha_j \arg z}$, $\arg z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

It shall be shown that differentiating the asymptotic expansion of f is legitimate. That is, the derivative of f admits the asymptotic expansion

$$f'(z) \sim \sum_j P_j \frac{d}{dz}(g(z) z^{-\alpha_j}) + O(|z|^{-M}), \tag{C.2}$$

as $|z| \to \infty$, $z \in \Omega$ and $|\mathrm{Im} z| < 1$.

It is possible to show (C.2) from Cauchy's integral formula. Consider a belonging to the interior of Ω and define $\Gamma_a = \{z | \, |z - a| = \varepsilon\}$ to be a positively oriented contour, where $\varepsilon > 0$ is chosen such that $\Gamma_a \in \Omega$. One has

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\Gamma_a} \frac{f(z)}{z - a} dz \\ f'(a) &= \frac{1}{2\pi i} \int_{\Gamma_a} \frac{f(z)}{(z - a)^2} dz \end{aligned} \tag{C.3}$$

Substituting (C.1) into (C.3) gives

$$f'(a) = \frac{1}{2\pi i} \sum_j \int_{\Gamma_a} \frac{P_j g(z) z^{-\alpha_j}}{(z-a)^2} dz + \frac{1}{2\pi i} \int_{\Gamma_a} \frac{O(|z|^{-M})}{(z-a)^2} dz$$

Now (C.2) follows from Cauchy's integral formula.

Thus, we have shown that differentiating the asymptotic expansion of f gives the asymptotic expansion for its derivative.

D Heat Invariants

Define the Schrödinger operator $h = -\frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbb{R})$, where $q \in C^\infty(\mathbb{R})$ and its derivatives are uniformly bounded. Consider the operator e^{-th}, $t \geq 0$, and denote its integral kernel by $e^{-th}(x,y)$. Then one has

$$e^{-th}(x,x) \sim (4\pi t)^{-\frac{1}{2}} \sum_{j=0}^{\infty} t^j a_j(x), \qquad t \to +0, \tag{D.1}$$

locally uniformly. As desribed in the introduction, the coefficients $\{a_j(x)\}$ of (D.1) are called the local heat invarinats of the operator h and are polynomials in q and its derivatives.

In this section shall be presented an outline of the proof for the existence of the asymptotic expansion (D.1) with some unknown coefficients a_j. The formula (3.1) for these coefficients was obtained by a difficult combinatorial analysis due to Polterovich, see [18].

The discussion here of the existence of (D.1), describes the method as shown in [12], but different approaches to this can be found, for example see [2].

This approach uses the iterated resolvent identity which as its name suggests is an iteration of the usual resolvent identity

$$R(z) = R_0(z) - R(z)VR_0(z),$$

where $R_0(z) = (H_0 - z)^{-1}$, $R(z) = (H - z)^{-1}$ and applies to all operators $H_0 = H_0^*$, $H = H_0 + V$ and $V = V^*$ is bounded. In order to prove (D.1), the iterated resolent identity shall be defined for $H_0 = h_0 = -\frac{d^2}{dx^2}$ and h as defined above. One needs the operators X_m, $m \geq 1$, defined recursively by

$$X_0 = I, \ X_{m+1} = X_m h_0 - hX_m, \tag{D.2}$$

on domain $\cap_{n\geq 0}\mathrm{Dom}(h_0^n)$. These operators are also used in [2] but are defined equivalently by multiple commutators which is a more complicated form. One has,

$$\begin{aligned}
X_1 &= -q, \\
X_2 &= q\frac{d^2}{dx^2} + \Big(-\frac{d^2}{dx^2} + q\Big)q \\
&= q\frac{d^2}{dx^2} - \frac{d^2}{dx^2}q + q^2 \\
&= -2\Big(\frac{dq}{dx}\Big)\frac{d}{dx} - \Big(\frac{d^2q}{dx^2}\Big) + q^2.
\end{aligned}$$

The iterated resolvent identity is

$$R(z) = \sum_{m=0}^{M} X_m R_0^{m+1}(z) + R(z)X_{M+1}R_0^{M+1}(z) \tag{D.3}$$

and holds true for any $M \geq 1$.

This can be proven by induction. The $M = 0$ case is the usual resolvent identity. Assume (D.3) holds for some fixed N. The following computation shows it holds for all $M \geq 1$,

$$\begin{aligned}
& R(z)X_{N+1}R_0^{N+1}(z) \\
= \;& R(z)X_{N+1}(h_0 - z)R_0^{N+2}(z) \\
= \;& R(z)\big[(h-z)X_{N+1} - (h-z)X_{N+1} + X_{N+1}(h_0 - z)\big]R_0^{N+2}(z) \\
= \;& R(z)\big[(h-z)X_{N+1} + X_{N+1}h_0 - hX_{N+1}\big]R_0^{N+2}(z) \\
= \;& X_{N+1}R_0^{N+2}(z) + R(z)X_{N+2}R_0^{N+2}(z).
\end{aligned}$$

Now to prove (D.1), one can use the itereated resolvent identity in the following way. Fix $c < 0$, $c < \inf \sigma(h)$, and $t > 0$. Multiply (D.3) by e^{-tz} and integrate over z from $c - i\infty$ to $c + i\infty$ to obtain

$$e^{-th} = \sum_{m=0}^{M} \frac{t^m}{m!} X_m e^{-th_0} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R(z)X_{M+1}R_0^{M+1}(z)e^{-tz}dz. \tag{D.4}$$

By Lemma 3.2 of [12] one has that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R(z)X_{M+1}R_0^{M+1}(z)e^{-tz}dz = O(t^{(M+2)/2-1/2}).$$

Now we can look at the first term on the RHS of (D.4). The operators X_m have the form

$$X_m = \sum_{j=0}^{m-1} b_{mj}(x) \frac{d^j}{dx^j} \tag{D.5}$$

where b_{mj} are polynomials in q and the derivatives of q. The explicit expression for the integral kernel of e^{-th_0} is

$$e^{-th_0}(x,y) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}} \tag{D.6}$$

Using (D.5) and (D.6) it is easy to compute the m'th term in the sum in (D.4). One has

$$X_m e^{-th_0}(x,x) = (4\pi t)^{-\frac{1}{2}} \sum_{j=0}^{m-1} b_{jm}(x) \frac{d^j}{dx^j} e^{-\frac{|x-y|^2}{4t}} |_{x=y}$$

For $j = 1$, one has

$$\begin{aligned} & b_{1m}(x) \frac{d}{dx} e^{-\frac{|x-y|^2}{4t}} |_{x=y} \\ = \; & b_{1m}(x) \frac{d}{dx} \left(1 - \frac{(x-y)^2}{4t} + \frac{1}{2} \frac{(x-y)^4}{(4t)^2} + \cdots \right) |_{x=y} \\ = \; & 0. \end{aligned}$$

For $j = 2$, one has

$$b_{2m}(x) \frac{d^2}{dx^2} \left(1 - \frac{(x-y)^2}{4t} + \frac{1}{2} \frac{(x-y)^4}{(4t)^2} + \cdots \right) |_{x=y} = b_{2m}(x) \frac{1}{2t}$$

From this it is clear that only even derivatives will give a non zero contribution. Therefore for the m'th term in (D.4) one has

$$t^m X_m e^{-th_0}(x,x) = (4\pi t)^{-\frac{1}{2}} \sum_{j=[m/2]+1}^{m} t^j f_{mj}(x), \; t > 0,$$

where f_{mj} are some polynomials in q and the derivatives of q. Converting (D.4), with the understanding that for large enough N the second term on the RHS becomes the remainder term, to integral kernels and substituting in the above one arrives at (D.1).

E Estimates for Volterra Type Operators

In this section we describe some basic properties of Volterra operators (see [20], for example). Volterra operators are integral operators of the form $(K\phi)(x) = \int_a^x k(x,t)\phi(t)dt$, where $k : [a,b] \times [a,b] \to \mathbb{C}$ is a measurable function and $x \in (a,b)$.

We are going to consider the case where k is continuous and bounded on $[a,b] \times [a,b]$ and K maps $C(a,b)$ to itself. We are interested in finding solutions to equations of the form

$$\phi(x) - \int_a^x k(x,t)\phi(t)dt = f(x), \qquad a \leq x \leq b. \tag{E.1}$$

That is $(I-K)\phi = f$. If $\sum_{n=0}^{\infty} \|K^n\|_{C(a,b)\to C(a,b)}$ converges, then the series $\sum_{n=0}^{\infty} K^n$ converges in $B(C(a,b))$, the space of bounded linear maps from $C(a,b)$ to itself, and $(I-K)^{-1} = \sum_{n=0}^{\infty} K^n$. Therefore, the equation (E.1) has the unique solution $\phi = \sum_{n=0}^{\infty} K^n f$.

To ascertain the conditions for the series $\sum_{n=0}^{\infty} \|K^n\|_{C(a,b)\to C(a,b)}$ to converge, we shall produce an estimate for $\|K^n\|_{C(a,b)\to C(a,b)}$. Firstly, for K^2 one has

$$\begin{aligned}
(K^2\phi)(x) = K(K\phi)(x) &= \int_a^x k(x,s)(K\phi)(s)ds \\
&= \int_a^x \int_a^s k(x,s)k(s,t)\phi(t)dtds \\
&= \int_a^x \int_t^x k(x,s)k(s,t)\phi(t)dsdt \\
&= \int_a^x k_2(x,t)\phi(t)dt
\end{aligned} \tag{E.2}$$

where $k_2(x,t) = \int_t^x k(x,s)k(s,t)ds$ is the kernel of K^2. Denote $M := sup\{|k(x,t)| : -a \leq t \leq x \leq b\}$. From (E.2) we have that

$$|k_2(x,t)| \leq \int_t^x M^2 = M^2(x-t).$$

For all $n \in \mathbb{N}$, denote the kernel of K^n by $k_n(x,t)$ then

$$k_{n+1}(x,t) = \int_t^x k_n(x,s)k(s,t)ds, \; a \leq t \leq x \leq b$$

We shall show for any $n \in N$, that one has

$$|k_n(x,t)| \leq \frac{M^n(x-t)^{n-1}}{(n-1)!}, \; a \leq t \leq x \leq b. \tag{E.3}$$

Proceed by induction in n. For $n = 1$, (E.3) holds. Assume that (E.3) holds for some fixed n. Then

$$\begin{aligned} |k_{n+1}(x,t)| &= |\int_t^x k_n(x,s)k(s,t)dt| \\ &\leq |\int_t^x \frac{M^n(x-t)^{n-1}}{(n-1)!}.M \; dt| \\ &= \frac{M^{n+1}(x-t)^n}{n!} \end{aligned}$$

This proves (E.3) for all $n \in N$.

Using (E.3) one has

$$\begin{aligned} \|K^n\|_{C(a,b)\to C(a,b)} &\leq \sup_{a\leq x\leq b} \int_a^x |k_n(x,t)|dt \\ &\leq \sup_{a\leq x\leq b} \int_a^x \frac{M^n(x-t)^{n-1}}{(n-1)!}dt \\ &= \frac{M^n(b-a)^n}{n!}. \end{aligned}$$

This implies that $\sum_{n=0}^{\infty} K^n$ converges for any value of M. Thus, the equation (E.1) has a unique solution for all Volterra operators of the type K.

F Rouché's Theorem

1. Fix ε such that $0 < \varepsilon < 2$ and let $\Gamma_{n,\varepsilon} = \{\lambda | \, |\lambda - (2n-1)| = \varepsilon\}$. Consider the analytic function

$$w(\lambda) = \cos\frac{\pi\lambda}{2} + O(\frac{1}{\sqrt{\lambda}}), \quad |\lambda| \to \infty. \tag{F.1}$$

It shall be shown that there exists $N \in \mathbb{N}$ such that if $n > N$ then $w(\lambda)$ has the same number of zeros as $\cos\frac{\pi\lambda}{2}$ within $\Gamma_{n,\varepsilon}$.

This is essential done by an application of Rouché's theorem, a simplified version of which appears below.

Theorem F.1 (Rouché). *Let Γ be a circle in a region Ω. Suppose ω and ω_0 are two holomorphic functions on Ω. If $|\omega_0(z) - \omega(z)| < |\omega_0(z)|\ \forall\ z \in \Gamma$ then ω and ω_0 have an identical number of zero's within Γ, counted according to their multiplicities.*

For the full theorem and proof see 10.43 of [24].

If $|\cos\frac{\pi\lambda}{2}|$ is bounded from below on $\Gamma_{n,\varepsilon}$ independently of n, then one can apply Rouché's theorem to $w(\lambda)$ and $\cos\frac{\pi\lambda}{2}$. This will give the desired result.

2. Let us show $|\cos\frac{\pi\lambda}{2}|$ is bounded from below on $\Gamma_{n,\varepsilon}$. Let $a + ib = \frac{\pi\lambda}{2}$ where $a,\ b \in \mathbb{R}$. Then $|\cos\frac{\pi\lambda}{2}| \geq |e^b - e^{-b}|$. This implies that away from the real line $|\cos\frac{\pi\lambda}{2}|$ is bounded from below. At the points of $\Gamma_{n,\varepsilon}$ on the real line $|\cos\frac{\pi\lambda}{2}| = |\cos\frac{\pi\varepsilon}{2}| > 0$. Therefore, by continuity $|\cos\frac{\pi\lambda}{2}|$ is bounded from below on $\Gamma_{n,\varepsilon}$.

References

[1] M. Abromowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, (1972).

[2] S. Agmon, Y. Kannai, *On the asymptotic behaviour of spectral functions and resolvent kernels of elliptic operators*, Israel J. Math. **5** (1967),1-30.

[3] E. V. Aleksandrova, O. V. Bochkareva, V. E. Podol'skiĭ. *Summation of regularized traces of the singular Sturm-Liouville operator.* (Russian) Differ. Uravn. **33** no. 3,291–295 (1997). English translation in : Differential Equations **33** no. 3 287–291 (1997).

[4] E. V. Aleksandrova, *Trace formulas in problems of oscillations in rods and tubes and also for some classes of singular operators* (Russian) Ph.D. thesis, Moscow Lomonosov State University, Moscow, (1997).

[5] L.A. Dikii, *On a Formula of Gel'fand-Levitan*, (Russian) Uspehi Matem. Nauk (N.S.) **8** no. 2(54) (1953), 119-123.

[6] L.A. Dikii, *The Zeta Function of an Ordinary Differential Equation on a Finite Interval,* (Russian) Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 187-200.

[7] L.A. Dikii, *Trace Formulas for Sturm-Liouville Differential Operators,* (Russian) Uspehi Mat. Nauk (N.S.) **13** no. 3(81) (1958), 111-143. English translation in: American Mathematical Society Translation, (1961), **18** ser 2 pp 81-115.

[8] A. Erdélyi, *Asymptotic Expansions,* Dover Publications (1956).

[9] I.M. Gel'fand, B.M. Levitan, *On a Simple Identity for the Characteristic Values of a Differential Operator of the Second Order,* (Russian) Doklady Akad. Nauk SSSR (N.S.) **88** (1953), 593-596.

[10] I.M. Gel'fand, *On Identities for Eigenvalues of a Differential Operator of Second Order,* (Russian) Uspehi Mat. Nauk (N.S.) **11** no. 1 (67) (1956), 191-198.

[11] P. Gilkey, *The Spectral Geometry of a Riemannian Manifold,* Journal of Differential Geometry **10** (1975), 601-618.

[12] M. Hitrik, I. Polterovich, *Resolvent Expansions and Trace Regularizations for Schrödinger Operators,* Advances in Differential Equations and Mathematical Physics (Birmingham, AL, 2002), 161-173, Contemp. Math., 327, Amer. Math. Soc., Providence, RI, (2003).

[13] M. Hitrik, I. Polterovich, *Regularized traces and Taylor Expansions for the Heat Semigroup,* J. London Math. Soc. (2) **68** (2003), no. 2, 402-418.

[14] G.V. Kozlov, V.A. Lyubishkin, *Regularized Higher Order Traces for a Harmonic Oscillator,* (Russian) Differentsial'nye Uravneniya **29** no. 1 (1993), 63, 180; translation in Differential Equations **29** (1993), no. 1, 51-53.

[15] V.A. Lyubishkin, *Regularized Traces of Singular Differential Operators*, Abstracts and Proceedings of the All Union School of Young Scientists on Functional Methods in Applied Mathematics and Mathematical Physics (1988), no. 42, 124-126.

[16] S. Minakshisundaram and A. Pleijel, *Some Properties of the Eigenfunction of the Laplace-Operator on Riemannian Manifolds*, Canadian Journal of Mathematics **1** (1949), 242-256.

[17] I. Polterovich, *A Commutator Method for Computation of Heat Invariants*, Indag. Math. (N.S.) **11** (2000), no. 1, 139-149.

[18] I. Polterovich, *Heat invariants of Riemannian Manifolds*, Israel J. Math. **119** (2000), 239-252.

[19] I. Polterovich, *From Agmon-Kannai expansion to Korteweg-de Vries Hierarchy*, Letters in Mathematical Physics 49 (1999), 71-77.

[20] D. Porter, D.S.G. Stirling, *Integral Equations*, Cambridge University Press, (1990).

[21] A. Pushnitski, I. Sorrell, *High Energy Asymptotics and Trace Formulae for the Perturbed Harmonic Oscillator*, (2005) accepted for publication Annales H. Poincare.

[22] M. Reed, B. Simon, Methods of modern mathematical physics, Vol. IV Analysis of operators, Academic Press, (1978).

[23] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, (1976).

[24] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, (1986).

[25] R.Seeley, *Complex Powers of an Elliptic Operator*, Proceedings of Symposia in Pure Mathematics **10** (1967), 288-307.

[26] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, (1979).