THE OPTIMAL CONTROL OF HEREDITARY SYSTEMS

by

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SUMMARY

This thesis considers the optimal control of systems governed by hereditary systems. In particular, the thesis examines the numerical solutions of these optimal control problems, but some theoretical results are obtained.

Gradient, conjugate gradient and second order methods for integro-differential systems are presented here together with a proof of the convergence of the ε-method and the minimum principle for these systems. In addition, gradient, conjugate gradient and second order methods for time lag systems are discussed and some results on other hereditary processes are presented.

The implementation of the numerical methods for time lag and integro-differential systems is examined at length, and several numerical examples are discussed. Some consideration is given to systems having state variable inequality constraints.

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CONTENTS

CHAPTER I

Introduction and Terminology

Mathematical models have long been used to describe processes which may occur in such fields as economics, medical science, engineering and biology. A large proportion of these processes can be modelled by means of a set of ordinary differential equations. A typical example is a process whose state may be described by a set of parameters $x_1, x_2 \ldots x_n$, which are termed the state variables. It may be possible to determine an empirical or theoretical relationship between the rate of change of each of these variables and the values of these parameters. This relationship might be of the form:

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), x_2(t), \ldots . x_n(t)) \qquad i = 1 \ldots, n \, . \tag{1.1}$$

It is likely that the values of some of the parameters determining the evolution of the system are at the operator's disposal. These parameters are termed control variables. The mathematical model may then be of the form:

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \ldots, x_n(t), u_1(t), \ldots u_r(t)) \quad i=1, \ldots, n \, . \tag{1.2}$$

In addition, the system equations may be explicitly time dependent and so the model becomes

$$\frac{dx_i(t)}{dt} = f_i(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_r(t),t) \tag{1.3}$$

$$i = 1,\ldots.n \;.$$

This may be more concisely expressed by using vector notation as:

$$\dot{x}(t) = f(x(t),u(t),t) \tag{1.4}$$

where $x(t)$ is an n-vector, $u(t)$ an r-vector and $f(x,u,t)$ is an n-vector function of the state, control and time.

The operator may wish to choose a control which in some sense is the best control. To do this, he would have to consider what his objectives are, and bearing these in mind, choose a performance index which accurately measures the sense in which he wishes to optimise the process.

A typical statement of an optimal control problem for these systems is for given t_0, x_0, f, ψ, ϕ such that

$t_0 \in R^1$,

$x_0 \in R^n$,

f is a function from $R^n \times R^m \times R^1$ into R^n,

ψ is a function from R^n into R^q,

and ϕ is a function from R^n into R,

choose the control $u(t)$ $t_0 \leqslant t \leqslant t_f$ which minimises $\phi(x(t_f),t_f)$ subject to

$$\dot{x}(t) = f(x(t),u(t),t) \qquad t_0 \leqslant t \leqslant t_f$$

$$x(t_0) = x_0$$

and

$$\psi(x(t_f),t_f) = 0 \;.$$

This problem, or others similar, has been investigated by many authors and several methods of calculating optimal controls numerically have been described [1] - [8].

Ordinary differential equations (1.1) have been extensively analysed by many workers: see for example ref. [9]. For some sytems, however, the ordinary differential equation is an inadequate model. One alternative is to model the system in terms of a distributed parameter system. The corresponding optimal control problem has been investigated by Holliday [10] among others.

Another class of problem, where the model (1.5), (1.6) is insufficient, is that of hereditary systems, in which the dynamics are dependent on the past history of the state and control, as well as their present values.

Some examples of models which could be used to describe such processes are:

i) differential-delay or time lag systems

$$\dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \qquad t_0 \leq t \leq t_f \tag{1.7}$$

where τ is a known constant greater than zero, and

$$x(t) = \sigma(t) \qquad t_0-\tau \leq t \leq t_0$$

with $\sigma(t)$ a known function of time;

ii) neutral systems

$$\dot{x}(t) = f(x(t),\dot{x}(t-\tau),x(t-\tau),u(t),t) \tag{1.8}$$

$$\text{for } t_0 \leq t \leq t_f$$

and

$$x(t) = \sigma(t) \qquad t_0-\tau \leq t \leq t_0$$
$$\dot{x}(t) = \dot{\sigma}(t) \qquad t_0-\tau \leq t \leq t_0$$

where τ is a known positive constant, and $\sigma(t)$ a known function of time;

iii) integro-differential systems

$$\dot{x}(t) = f\left(x(t),u(t), \int_{t_0}^{t} g(x(s),u(s),s,t)ds\right) \tag{1.9}$$

for $t_0 \leq t \leq t_f$

with $x(t_0)$ known.

Processes governed by ordinary differential equations have been the subject of investigations for many years. In contrast, hereditary systems are of more recent origin. They arise naturally in population dynamics. Cooke [11] suggested the following model for the size, $x(t)$, of a population with constant gestation period τ, constant birth rate α, and fixed life span σ;

$$\dot{x}(t) = \alpha\{x(t-\tau)-x(t-\tau-\sigma)\} .$$

Volterra [12] investigated the dynamics of a predator-prey population and derived a pair of coupled integro-differential equations:

$$\dot{x}(t) = \{\alpha-\beta y(t)-\int_{-h}^{0} F(-\theta)y(t+\theta)d\theta\}x(t)$$

$$\dot{y}(t) = \{\delta-\varepsilon x(t)-\int_{-h}^{0} G(-\theta)y(t+\theta)d\theta\}y(t)$$

where x represents the prey population and y the predator population. More recently, Bellman and Cooke [13] have given a comprehensive treatment of differential-difference equations and Halanay [14] has written on differential delay equations.

The optimal control problem for hereditary systems has a fairly short history. The first major contribution was probably the extension of Pontryagin's Maximum Principle [1] to time lag systems by Kharatishvilli [15], [16]. Computational methods for generating optimal controls for time lag systems have been given by T. E. Mueller [17], Sebesta [18] and Eller [19] among others. T. E. Mueller gives an algorithm for linear differential delay systems with a quadratic performance index, and Sebesta gives a similar algorithm for more general systems. Eller derives a set of partial differential equations whose solution yields a feedback control for linear time lag systems.

Little work seems to have been done on any of the other forms of hereditary systems. C. E. Mueller [20] derives feedback controls for a wide class of linear hereditary systems and discusses extensions to non-linear equations. Banks and Jacobs [21] and Kushner and Barnea [22] discuss the optimal control of systems governed by linear functional-differential equations. Oguztoreli [23] has given results for a large class of optimal control problems of hereditary systems and has an extensive bibliography.

This thesis describes methods of calculating optimal controls for hereditary processes. In chapter 2 we give the derivation of the gradient, conjugate gradient, and second order methods for the optimal control of time lag systems. The chapter continues by giving a discussion of processes with inequality constraints, and concludes with a description of numerical techniques suitable for calculating the optimal control of time lag systems in the presence of inequality constraints.

Chapter 3 begins by describing the ε-technique as applied to integro-differential systems and we present gradient, conjugate gradient and second order methods for these processes.

Chapter 4 gives a brief discussion of some results on other forms of hereditary processes, such as neutral systems and systems governed by integral equations.

In chapter 5 we apply the techniques described in chapters 2 and 3 to examples of time lag and integro-differential systems with and without inequality constraints.

We will now discuss some of the terminology which will be used. We have already classified several types of hereditary systems in (1.7) to (1.9) as time lag, neutral and integro-differential systems. This classification of the state equations can be further divided into linear and non-linear systems, in the usual way. For example, a linear time-lag system could be written as:

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)u(t) \qquad t_0 \leqslant t \leqslant t_f$$

$$x(t) = \phi(t) \qquad t_0-\tau \leqslant t \leqslant t_0$$

where $A(t)$ and $B(t)$ are $n \times n$ matrices and

$C(t)$ is an $n \times r$ matrix.

Similarly, a linear integro-differential system could be written as

$$\dot{x}(t) = A(t)x(t) + C(t)u(t) + \int_{t_0}^{t} \{B(s,t)x(s) + D(s,t)u(s)\}ds \qquad t_0 \leqslant t \leqslant t_f$$

$$x(t_0) = x_0 \quad .$$

Here $A(t)$ and $B(s,t)$ are $n \times n$ matrices and

$C(t)$ and $D(s,t)$ are $n \times r$ matrices.

Optimal control systems can also be classified by their performance index. A performance index which may be written as

$$J = \tfrac{1}{2}\, x^T(t_f)Px(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \quad ,$$

where P and $Q(t)$ are $n \times n$ matrices and $R(t)$ is an $r \times r$ matrix, will be termed quadratic. $R(t)$ will be termed the control cost matrix.

Much of the research on optimal control, whether on hereditary or lumped parameter systems, has been centred on linear systems with a quadratic performance index. These will be referred to as linear-quadratic systems.

The majority of the new results presented in this thesis are contained in chapter 3, where a gradient method, conjugate gradient method, second order iterative method and a minimum principle for integro-differential systems are derived.

In addition, the gradient method for systems governed by integral equations described in chapter 4 is new, as is the second order Runge Kutta method for integrating integro-differential equations described in appendix D.

A comparison of these numerical methods, both for integro-differential and time lag systems is also presented, together with a comparison of transformation techniques for dealing with constrained optimisation problems.

CHAPTER II

Optimal Control of Time Lag Systems

2.1 Introduction

In this chapter, we will outline some iterative procedures for the optimal control of systems described by

$$\dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \qquad t_0 \leq t \leq t_f \tag{2.1}$$

$$x(t) = \phi(t) \qquad t_0-\tau \leq t \leq t_0 \;, \tag{2.2}$$

with a scalar performance index

$$J = G(x(t_f),t_f)\,, \tag{2.3}$$

which is to be minimised.

Here $x(t)$ is an n-vector, $u(t)$ an r-vector, τ a constant delay and $\phi(t)$ a known function of time.

Some of the earliest work on systems governed by time lag equations of this type was done by Kharatishvilli [15]. He extended Pontryagin's maximum principle to cover systems with a single delay in the state, as in equation (2.3). Chyung [24] derived necessary conditions for linear systems with single delays and, under additional conditions, proved existence and sufficiency conditions for optimal controls.

Chyung and Lee [25] later derived necessary and sufficient conditions for the optimal control of linear systems with multiple delays in the state and having a quadratic performance index. Kharatishvilli [16] extended his maximum principle to differential delay equations with multiple delays. Much of this work has been discussed and extended by Oguztoreli [23].

Computational algorithms for finding the optimal control of differential-delay systems have been presented by Mueller, Sebesta, McKinnon, Ray and Soliman, and Sebesta and Asher. Mueller's [17] algorithm is applicable to linear-quadratic systems with a fixed lag. Sebesta [18] gives an extension of the gradient method of Bryson and Denham [4] to systems with time varying lags. This work has been further extended by Sebesta and Asher [26] to systems with time and state dependent lags. McKinnon's [27] algorithm is a second order algorithm and Ray and Soliman [28] outline a conjugate gradient method.

Most of the above methods are based upon the maximum principle. An alternative approach is given by Huang [29] who extends the ε-method of Balakrishnan [6] to systems with multiple time lags. The advantage of this method is that the state equations do not have to be solved. For systems described by ordinary differential equations or time lag equations, which can usually be integrated fairly easily, this method is probably inferior [30] to the gradient, conjugate gradient and second order methods. It has, however, been used to solve some problems [31], [32] and leads to an interesting derivation of the maximum principle. A further extension to systems represented by integro-differential equations is given in a following chapter.

2.2 Gradient Methods

Consider the system described by

$$\dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \qquad t_0 \leqslant t \leqslant t_f \tag{2.4}$$

$$x(t) = \phi(t) \qquad t_0-\tau \leqslant t \leqslant t_0 \quad ,$$

where $x(t)$ is an n-vector, $u(t)$ an r-vector and $\phi(t)$ a known continuous n-vector function of time.

We seek to minimise the performance index

$$J = G(x(t_f),t_f) \tag{2.5}$$

where t_f is unspecified but subject to the following stopping condition

$$K(x(t_f),t_f) = 0 \cdot \tag{2.6}$$

It is assumed that $f(x(t),x(t-\tau),u(t),t)$ is defined and continuous for all $x \epsilon R^n, u \epsilon R^r$ and $t \epsilon R$ and possesses continuous derivatives. The scalar functions $G(x(t_f),t_f)$ and $K(x(t_f),t_f)$ have similar properties.

We choose an initial control $u^*(t)$ and then the corresponding response $x^*(t)$ and terminal time t_f^* are found by integrating (2.4) until (2.6) is satisfied.

We now seek a modification $\delta u(t)$ to the control such that the new control $u^*(t) + \delta u(t)$ gives an improved value for J.

We start by linearizing (2.4) about the nominal pair (x^*,u^*) to give

$$\delta\dot{x}(t) = A_1(t)\delta x(t) + A_2(t)\delta x(t-\tau) + B(t)\delta u(t) \tag{2.7}$$

$$\text{for } t_0 \leqslant t \leqslant t_f$$

$$\delta x(t) = 0 \qquad t_0-\tau \leqslant t \leqslant t_0$$

where the matrices $A_1(t)$, $A_2(t)$ and $B(t)$ are defined as

$$A_1(t) = \left[\frac{\partial f_i}{\partial x_j(t)} (x^*(t), x^*(t-\tau), u^*(t), t) \right]$$

$$A_2(t) = \left[\frac{\partial f_i}{\partial y_j} (x^*(t), y, u^*(t), t) \right] \qquad y = x^*(t-\tau)$$

$$B(t) = \left[\frac{\partial f_i}{\partial u_j(t)} (x^*(t), x^*(t-\tau), u^*(t), t) \right]$$

where f_i is the i^{th} component of $f(x(t),x(t-\tau),u(t),t)$ and x_j the j^{th} component of x, etc.

The superscript * denotes evaluation along the nominal trajectory. In future, we will denote partial derivatives with respect to the lagged state by the subscript τ.

Thus our definition of $A_2(t)$ above may be written as

$$A_2(t) = f_\tau(x^*(t), x^*(t-\tau), u^*(t), t).$$

We define the Hamiltonian by

$$H(x(t),x(t-\tau),u(t),\lambda(t),t) = \lambda^T(t) f(x(t),x(t-\tau),u(t),t), \qquad (2.8)$$

where

$$\dot{\lambda}(t) = -A_1^T(t)\lambda(t) - A_2^T(t+\tau)\lambda(t+\tau) \qquad t_0 \leq t \leq t_f^* \qquad (2.9)$$

$$\lambda(t) \equiv 0 \qquad t > t_f^*$$

$$\lambda(t_f) = \left[\frac{\partial G}{\partial x} - (\dot{G}/\dot{K}) \frac{\partial K}{\partial x} \right]^*_{t=t_f^*} .$$

Premultiply (2.7) by $\lambda^T(t)$ and postmultiply the transpose of (2.9) by $\delta x(t)$ to give,

$$\lambda^T(t)\delta\dot{x}(t) = \lambda^T(t)A_1(t)\delta x(t) + \lambda^T(t)A_2(t)\delta x(t-\tau)$$
$$+ \lambda^T(t)B(t)\delta u(t) \qquad (2.10)$$

$$\dot{\lambda}^T(t)\delta x(t) = -\lambda^T(t)A_1(t)\delta x(t) - \lambda^T(t+\tau)A_2(t+\tau)\delta x(t). \qquad (2.11)$$

Adding (2.10) and (2.11) gives

$$\frac{d}{dt}\{\lambda^T(t)\delta x(t)\} = \lambda^T(t)A_2(t)\delta x(t-\tau) + \lambda^T(t)B(t)\delta u(t)$$
$$- \lambda^T(t+\tau)A_2(t+\tau)\delta x(t) \qquad (2.12)$$

Integrating (2.12) over $[t_0,t_f]$, noting that $\delta x(t_0) = 0$, gives:

$$\lambda^T(t_f^*)\delta x(t_f^*) = \int_{t_0}^{t_f^*}\{\lambda^T(t)A_2(t)\delta x(t-\tau)-\lambda^T(t+\tau)A_2(t+\tau)\delta x(t)\}dt$$
$$+ \int_{t_0}^{t_f^*} \lambda^T(t)B(t)\delta u(t)dt\,. \qquad (2.13)$$

But

$$\int_{t_0}^{t_f^*} \lambda^T(t+\tau)A_2(t+\tau)\delta x(t)dt = \int_{t_0}^{t_f^*} \lambda^T(t)A_2(t)\delta x(t-\tau)dt$$

as $\lambda(t) \equiv 0$ for $t > t_f^*$ and $\delta x(t) \equiv 0$ for $t \leq t_o$.

Hence (2.13) becomes:

$$\lambda^T(t_f^*)\delta x(t_f^*) = \int_{t_0}^{t_f^*} \lambda^T(t)B(t)\delta u(t)dt\,,$$

or in terms of the Hamiltonian,

$$\lambda^T(t_f^*)\delta x(t_f^*) = \int_{t_0}^{t_f^*} H_u^T(t)\delta u(t)dt\,. \qquad (2.14)$$

The first order change in the performance index J, due to the modification, $\delta u(t)$, to the control is given by

$$\Delta J = (G_x^*)^T \delta x(t_f) + \dot{G}^* \Delta t_f . \tag{2.15}$$

The first order change in value of the stopping condition is given by

$$\Delta K = (K_x^*)^T \delta x(t_f) + \dot{K}^* \Delta t_f . \tag{2.16}$$

We set ΔK to zero to ensure the stopping condition remains zero.

So from (2.16)

$$\Delta t_f = -\frac{1}{\dot{K}^*} (K_x^*)^T \delta x(t_f) .$$

Substituting in (2.15) gives:

$$\Delta J = \{G_x^* - (\dot{G}^*/\dot{K}^*) K_x^*\}^T \delta x(t_f) .$$

But from our definition of the terminal condition of (2.9)

$$\Delta J = \lambda^T(t_f^*) \delta x(t_f^*),$$

and so from (2.14)

$$\Delta J = \int_{t_0}^{t_f^*} H_u^T(t) \delta u(t) dt . \tag{2.17}$$

We wish to minimise this expression for ΔJ, but first we have to constrain $\delta u(t)$ so that the linearisation (2.7) is accurate. We choose a $\beta > 0$ and a positive definite symmetric $r \times r$ matrix $W(t)$ and constrain $\delta u(t)$ by

$$\beta = \int_{t_0}^{t_f} \delta u^T(t) W(t) \delta u(t) dt . \tag{2.18}$$

Adjoining the equality constraint (2.18) to the expression for ΔJ gives

$$\Delta J_A = \int_{t_0}^{t_f} H_u^T(t)\delta u(t)dt + \mu\{\beta - \int_{t_0}^{t_f} \delta u^T(t)W(t)\delta u(t)dt\} . \qquad (2.19)$$

From the calculus of variations, we see that (2.19) is minimised by

$$\delta u(t) = \tfrac{1}{2}\mu W^{-1}(t)H_u(t) . \qquad (2.20)$$

Substituting this into (2.17) gives

$$\beta = \frac{1}{4\mu^2} \int_{t_0}^{t_f} H_u^T(t)W^{-1}(t)H_u(t)dt$$

$$\mu^2 = \frac{1}{4\beta} \int_{t_0}^{t_f} H_u^T(t)W^{-1}(t)H_u(t)dt$$

$$\mu = \pm \sqrt{I}/2\sqrt{\beta}$$

where

$$I = \int_{t_0}^{t_f} H_u^T(t)W^{-1}(t)H_u(t) ,$$

therefore

$$\delta u(t) = \pm \{\beta/I\}^{\frac{1}{2}} W^{-1}(t)H_u(t) . \qquad (2.21)$$

It can easily be seen that, substituting (2.21) into (2.17), the minus sign gives ΔJ as negative, as required.

We can now choose the change in control to be

$$\delta u(t) = - \{\beta/I\}^{\frac{1}{2}} W^{-1}(t)H_u(t) \tag{2.22}$$

and repeat until satisfactory convergence is obtained.

Alternatively, in the case of the final time being specified, we may proceed as follows:

Noting that $\Delta t_f = 0$, we change the final time condition on $\lambda(t)$ to

$$\lambda(t_f) = \left.\frac{\partial G^*}{\partial x}\right|_{t_f} \tag{2.23}$$

$$\lambda(t) = 0 \qquad\qquad t > t_f \quad .$$

In this case, the change in performance index, to first order, is given by

$$\Delta J = \frac{\partial G^T}{\partial x} (t_f)^* \, \delta x(t_f) \tag{2.24}$$

So we see that, as in (2.17),

$$\Delta J = \int_{t_0}^{t_f} H_u^T(t)\delta u(t)dt \quad . \tag{2.25}$$

Instead of constraining the control by means of equation (2.18) we can set

$$\delta u(t) = - \varepsilon \, H_u(t) \qquad\qquad \varepsilon > 0 \quad . \tag{2.26}$$

If ε is chosen small enough and the linearization (2.7) is valid, then this value of $\delta u(t)$ will ensure a decrease in the performance index. We have two alternatives for setting ε.

The first is to choose a fixed ε, suitable for the problem, and make a fixed step, $- \varepsilon H_u(t)$ at each iteration. This will be termed the fixed step method.

The second alternative is to perform a linear search on ε, so as to find the minimum of the performance index along each search direction. This is done by choosing an initial $\varepsilon > 0$ and correcting the control by $- \varepsilon H_u(t)$. If this results in a decrease in J, ε is increased by some factor and the controls recorrected. J is again evaluated and the process repeated until an increase in performance index is found. The ε giving the minimum value of J is then found by quadratic interpolation and the new control calculated.

Should the first step, $- \varepsilon H_u(t)$, fail to give an improved cost, the ε is reduced by some factor until some improvement is found, and the ε giving the minimum value can again be found by interpolation.

For the latter method, at each evaluation of the performance index, the state equations have to be integrated. As there will be at least three performance index evaluations per iteration, it may appear that the time involved in integrating the state equations would make this method slow. In practice, however, for the fixed step gradient method, and the gradient method for varying final time, the change in performance index index has to be monitored, as it is often necessary to modify ε and β(or $W(t)$) respectively.

2.3 Conjugate Gradient Methods

The conjugate gradient method is an algorithm which is similar to the steepest descent method described in the previous section, but requiring some additional computation and storage. Instead of simply searching along the direction of steepest slope, progressive improvements are made to the search directions at each iteration, in the hope that better convergence will result.

It may be summarised as follows:

a) the first search direction, S_1, is the same as the steepest ascent method, i.e.

$$S_1 = - H_u(x_1, x_{\tau 1}, u_1, \lambda_1, t) \; . \tag{2.27}$$

The algorithm then proceeds by the following steps:

b) the (i-1)th step taken is

$$u_i(t) = u_{i-1}(t) + \varepsilon_{i-1} S_{i-1}(t) , \tag{2.28}$$

where ε_{i-1} is chosen by a one dimensional search along S_{i-1} to minimise $J(u_i)$;

c) the state and adjoint equations are integrated and the gradient at (x_i, u_i) is calculated by

$$g_i(t) = -H_u \; (x_i, x_{\tau i}, u_i, \lambda_i, t) \; ; \tag{2.29}$$

d) the i^{th} conjugate gradient search direction is calculated as follows

$$S_i = g_i + \beta_{i-1} \, S_{i-1} \tag{2.30}$$

where

$$\beta_{i-1} = \frac{|g_i, g_i|}{|g_{i-1}, g_{i-1}|} \qquad i > 1$$

$$\beta_0 = 0 \; ,$$

where

$$|a,b| = \int_{t_0}^{t_f} a^T(t)b(t)dt$$

This generates a new search direction and we return to step (b) and repeat until satisfactory convergence is obtained.

The conjugate gradient algorithm was originally applied to the minimisation of functions in n-space. It can be shown that a quadratic function of n-variables can be minimised by such a procedure in n-iterations from any starting point. The proof of this is given in appendix A. The proof assumes that the one dimensional searches along each direction are perfect, i.e. the exact minimum is found along each search direction. Lasden, Warren and Mitter [5] applied the method to lumped parameter systems. They also prove, under certain assumptions, that the conjugate gradient method always generates directions of descent.

In n-space, if the function to be minimised is non-quadratic, then the conjugate gradient method, in general, will not converge in n-iterations, and so will have exhausted its potential. It is therefore advantageous to make a steepest descent step after n-iterations, i.e. to restart the algorithm. For optimal control problems, the dimension n must be arbitrarily imposed. Pierson [33] compares the conjugate gradient method and the conjugate gradient with restart every four or five iterations and obtains improved convergence with the latter approach.

2.4 Second Order Methods

Several authors [5], [34] have reported poor convergence near to the optimum for steepest descent and conjugate gradient methods. These remarks are made on lumped parameter systems but it is expected that they apply equally well to hereditary systems. Accordingly, second order methods have been developed in an effort to improve the convergence near to the optimum. Merriam [35] derived a second order method for lumped parameter systems, and later Mitter [36] presented a more general discussion of second order algorithms. McKinnon [27] extended the approach of Merriam to non-linear systems with time lag.

Freeman [8] derived an algorithm, based on a contraction mapping principle, for linear-quadratic systems without any delay. This scheme does not always converge, but Freeman established conditions for convergence. More recently, Allwright [7] has published a method similar to Freeman's, but with guaranteed convergence for all positive definite control cost matrices. Numerical results presented by Allwright suggest that even when Freeman's method converges, Allwright's scheme gives better convergence.

The algorithms of Freeman and Allwright are for linear systems , but their approach is particularly attractive in the derivation of second order methods for non-linear hereditary systems, and has been used by Connor [37] and Connor and Hood [38], and will be described in this section.

We consider the system represented by the following differential-difference equation

$$\dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \qquad t_0 \leq t \leq t_f \tag{2.31}$$
$$x(t) = \phi(t) \qquad t_0-\tau \leq t \leq t_0$$

where, as before, $x(t)$ is an n-vector, $u(t)$ an r-vector and $\phi(t)$ a known function of time.

We wish to minimise the functional

$$J = G(x(t_f),t_f) + \int_{t_0}^{t_f} F(x(t),u(t),t)dt \, . \tag{2.32}$$

It is assumed that each element of $u(t)$ is measurable and square integrable on $[t_0,t_f]$. This assumption is needed for the application of the contraction mapping principle. In addition, it is assumed that $f(x(t),x(t-\tau),u(t),t)$ and $F(x(t),u(t),t)$ are defined and continuous for all $x(t)$ in R^n, $u(t)$ in R^r and t in R, and have continuous derivatives up to third order.

The function $G(x(t_f),t_f)$ is continuous in x and t_f and has continuous derivatives up to third order.

We define the Hamiltonian by

$$H(x(t),x(t-\tau),u(t),\lambda(t),t)$$

$$= F(x(t),u(t),t) + \lambda^T(t)f(x(t),x(t-\tau),u(t),t) \, , \tag{2.33}$$

and consider the augmented functional

$$J_A = G(x(t_f),t_f) + \int_{t_0}^{t_f} \{H(x(t),x(t-\tau),u(t),\lambda(t),t) - \lambda^T(t)\dot{x}(t)\}dt \, . \tag{2.34}$$

Taking variations $\xi(t)$, $\eta(t)$ in $x(t)$ and $u(t)$ respectively and expanding J_A to second order terms gives,

$$\Delta J_A = \langle G_x(t_f),\xi(t_f)\rangle + \langle \tfrac{1}{2} G_{xx}(t_f)\xi(t_f),\xi(t_f)\rangle$$

$$+ \langle \int_{t_0}^{t_f} \langle H_u(t),\eta(t)\rangle dt - \langle \lambda(t_f),\xi(t_f)\rangle$$

$$+ \int_{t_0}^{t_f} \langle H_x(t) + \dot{\lambda}(t), \xi(t)\rangle dt + \int_{t_0}^{t_f} \langle H_\tau(t), \xi(t-\tau)\rangle dt$$

$$+ \tfrac{1}{2} \int_{t_0}^{t_f} \{\langle H_{xx}(t)\xi(t), \xi(t)\rangle + \langle H_{\tau\tau}(t)\xi(t-\tau), \xi(t-\tau)\rangle$$

$$+ \langle H_{uu}(t)\eta(t), \eta(t)\rangle + 2 \langle H_{ux}(t)\xi(t), \xi(t)\rangle$$

$$+ 2 \langle H_{u\tau}(t)\xi(t-\tau), \eta(t)\rangle + 2 \langle H_{x\tau}(t)\xi(t-\tau), \xi(t)\rangle \}dt \, . \tag{2.35}$$

We also have the following identity

$$\int_{t_0}^{t_f} \langle H_\tau(t), \xi(t-\tau)\rangle dt = \int_{t_0-\tau}^{t_0} \langle H_\tau(t+\tau), \xi(t)\rangle dt$$

$$+ \int_{t_0}^{t_f-\tau} \langle H_\tau(t+\tau), \xi(t)\rangle dt \, , \tag{2.36}$$

But we have

$$\xi(t) = 0 \qquad t_0 - \tau \leq t \leq t_0 \, ,$$

so we may eliminate the first term of the right hand side of (2.36). Using (2.36), we may write

$$\int_{t_0}^{t_f} \langle H_x(t) + \dot{\lambda}(t), \xi(t)\rangle dt + \int_{t_0}^{t_f} \langle H_\tau(t), \xi(t-\tau)\rangle dt$$

$$= \int_{t_0}^{t_f-\tau} \{H_x(t) + H_\tau(t+\tau) + \dot{\lambda}(t)\}, \xi(t)\rangle$$

$$+ \int_{t_f-\tau}^{t_f} \langle H_x(t) + \dot{\lambda}(t), \xi(t)\rangle dt \, . \tag{2.37}$$

We now define $\lambda(t)$ to satisfy the following:

$$\dot{\lambda}(t) = - H_x(t) - H_\tau(t+\tau) \qquad t_0 \leqslant t \leqslant t_f-\tau \tag{2.38}$$

$$\dot{\lambda}(t) = - H_x(t) \qquad t_f-\tau < t \leqslant t_f \tag{2.39}$$

$$\lambda(t_f) = G_x(t_f) . \tag{2.40}$$

Using (2.38) - (2.40) in (2.35) we may write

$$\Delta J_A = \tfrac{1}{2}\langle G_{xx}(t_f)\xi(t_f),\xi(t_f)\rangle + \int_{t_0}^{t_f} \langle H_u(t),\eta(t)\rangle dt$$

$$+ \tfrac{1}{2} \int_{t_0}^{t_f} \{\langle H_{xx}(t)\xi(t),\xi(t)\rangle + \langle H_{\tau\tau}\xi(t-\tau),\xi(t-\tau)\rangle$$

$$+ \langle H_{uu}(t)\eta(t),\eta(t)\rangle + 2\langle H_{ux}\ \xi(t),\eta(t)\rangle$$

$$+ 2\langle H_{u\tau}(t)\xi(t-\tau),\eta(t)\rangle + 2\langle H_{x\tau}(t)\xi(t-\tau),\xi(t)\rangle\}dt . \tag{2.41}$$

If we assume a nominal control $u_0(t)$, we may solve equations (2.31) to give the nominal state $x_0(t)$ and solve (2.38) - (2.40) in backward time for $\lambda_0(t)$.

In general these solutions will not satisfy the normal optimality condition, $H_u(t) = 0$.

We seek a control correction $\eta(t)$ minimising the expression for ΔJ_A given in (2.41). We have to minimise this expression subject to the following constraints:

$$\dot{\xi}(t) = f_x(t)\xi(t) + f_\tau(t)\xi(t-\tau) + f_u(t)\eta(t) \tag{2.42}$$

$$\text{for} \qquad t_0 \leq t \leq t_f$$

and

$$\xi(t) = 0 \qquad\qquad t_0-\tau \leq t \leq t_0$$

where $f_x(t)$, $f_\tau(t)$, $f_u(t)$ are all evaluated along the nominal trajectory. It can be shown [see appendix B] that the solution of (2.42) may be written

$$\xi(t) = \int_{t_0}^{t} N(\sigma,t)\ f_u(\sigma)\eta(\sigma)d\sigma\ , \tag{2.43}$$

where $N(\sigma,t)$ is an n x n matrix satisfying a certain differential equation, and from (2.43) write

$$\xi(t-\tau) = \int_{t_0}^{t-\tau} N(\sigma,t-\tau)f_u(\sigma)\eta(\sigma)d\sigma\ . \tag{2.44}$$

Using Freeman's approach, we rewrite (2.43) and (2.44) in the form

$$\xi(t) = L\eta(t) \tag{2.45}$$

$$\xi(t-\tau) = L\eta(t-\tau) = \hat{L}\eta(t)\ . \tag{2.46}$$

Let

$$Q(t) = H_{xx}(t) + G_{xx}(t_f)\delta(t-t_f) \tag{2.47}$$

where $\delta(t)$ is the Dirac function.

We have to minimise

$$\Delta J_A = \int_{t_0}^{t_f} \{<H_u(t),\eta(t)> + \tfrac{1}{2}<Q(t)\xi(t),\xi(t)>$$

$$+ \tfrac{1}{2}\langle H_{\tau\tau}(t)\xi(t-\tau),\xi(t-\tau)\rangle + \tfrac{1}{2}\langle H_{uu}(t)\eta(t),\eta(t)\rangle$$

$$+ \langle H_{ux}(t)\xi(t),\eta(t)\rangle + \langle H_{u\tau}(t)\xi(t-\tau),\eta(t)\rangle$$

$$+ \langle H_{x\tau}(t)\xi(t-\tau),\xi(t)\rangle\}dt \quad . \tag{2.48}$$

Using (2.45) and (2.46) we may rewrite ΔJ_A in the form

$$2\Delta J_A = |(R + H_{uu})\eta,\eta| + |2H_u,\eta| \tag{2.49}$$

where

$$R\eta = (L^*QL)\eta + (\hat{L}^*H_{\tau\tau}\hat{L})\eta + 2(H_{ux}L)\eta + 2(H_{u\tau}\hat{L})\eta + 2(L^*H_{x\tau}\hat{L})\eta \, .$$

Here $|\cdot\, ,\cdot|$ denotes the inner product on the control Hilbert space and is given by

$$|a,b| = \int_{t_0}^{t_f} \langle a(t),b(t)\rangle dt \quad .$$

L^* and $\hat{L}^*$ denote the adjoint operators of L and $\hat{L}$ respectively. The derivation of the above results is given in appendix C.

We will write J' for ΔJ_A for ease of notation. Let $\eta_0(t)$ be the optimum value of $\eta(t)$. We give a small variation $\bar{\eta}(t)$ to $\eta_0(t)$ and determine a necessary condition for the optimality of $\eta_0(t)$.

$$2(J' + \Delta J') = |(R + H_{uu})(\eta_0 + \bar{\eta}),(\eta_0 + \bar{\eta})| + 2|H_u,\eta_0 + \bar{\eta}| \, . \tag{2.50}$$

Expanding to the first order in $\bar{\eta}$ gives

$$2\Delta J' = |(R + H_{uu})\eta_0,\bar{\eta}| + |(R + H_{uu})\bar{\eta},\eta_0| + 2|H_u,\bar{\eta}|$$

or

$$2\Delta J' = |\{(R + H_{uu}) + (R + H_{uu})^*\}\eta_0,\bar{\eta}| + 2|H_u,\bar{\eta}| \, . \tag{2.51}$$

Hence, a first order condition for η_0 to be the optimum value for

η is:

$$\{(R + H_{uu}) + (R + H_{uu})^*\}\eta_0 = -2H_u \ . \tag{2.52}$$

As H_{uu} is self adjoint we may write (2.52) as:

$$2H_{uu}\,\eta_0 = -2H_u - (R + R^*)\eta_0 \ . \tag{2.53}$$

Finally, (2.53) may be written as an integral equation:

$$\eta_0 = -H_{uu}^{-1}\{H_u + \tfrac{1}{2}(R + R^*)\eta_0\} \ . \tag{2.54}$$

Let us write (2.54) in the more compact form:

$$\eta_0 = C\eta_0 \ .$$

The above equation can be used to provide an iterative procedure for generating a control increment η_0 and is based on Freeman's |8| approach.

If the operator C is a contraction operator then the procedure defined by

$$\eta_{i+i} = C\eta_i \tag{2.55}$$

will converge to η_0 for any starting point.

If H_{uu} is small compared to $(R + R^*)$, then the convergence of (2.55) will be poor and it may fail to converge entirely. In an attempt to improve this we now follow Allwright's [7] argument.

We may rewrite (2.49) as

$$J(\eta) = \tfrac{1}{2}|P\eta, \eta| + |H_u, \eta| \ . \tag{2.56}$$

We see from (2.52) that the first order condition for η to be optimal is

$$(P + P^*)\eta + 2H_u = 0 \ . \tag{2.57}$$

This suggests using the generalized Newton Raphson technique. Noting that

$$P + P^* = R + R^* + 2H_{uu} \qquad (2.58)$$

the Newton Raphson algorithm may be written as

$$\eta_{n+1} - \eta_n = -\left[H_{uu} + \tfrac{1}{2}(R + R^*)\right]^{-1} \left[H_u + \tfrac{1}{2}(P + P^*)\,\eta_n\right]. \qquad (2.59)$$

Unfortunately, this cannot be implemented as $(R + R^*)$ is an infinite dimensional operator and so, in general, its inverse cannot be found. We follow Allwright's suggestion and approximate $\left[H_{uu} + \tfrac{1}{2}(R + R^*)\right]$ by $\left[H_{uu} + \Theta I\right]$ where Θ is the upper bound for $\tfrac{1}{4}||R + R^*||$. This approximation leads to the algorithm

$$\eta_{n+1} = \eta_n - \left[H_{uu} + \Theta I\right]^{-1} \left[H_u + \tfrac{1}{2}(P + P^*)\eta_n\right] \qquad (2.60)$$

which defines our alternative algorithm.

Note that setting $\Theta = 0$ in (2.60) leads to

$$\begin{aligned}
\eta_{n+1} &= \eta_n - H_{uu}^{-1}\left[H_u + H_{uu}\,\eta_n + \tfrac{1}{2}(R + R^*)\eta_n\right] \\
&= \eta_n - H_{uu}^{-1} H_u - H_{uu}^{-1} H_{uu}\,\eta_n - \tfrac{1}{2}H_{uu}^{-1}(R + R^*)\eta_n \\
&= -H_{uu}^{-1}\left[H_u + (R + R^*)\eta_n\right].
\end{aligned}$$

Allwright also makes the following suggestion for determining Θ. Set Θ to zero initially, giving Freeman's algorithm, and adjust Θ adaptively to optimise the convergence rate, which might be measured by the rate of decrease of the norm of the gradient.

The two contraction mapping algorithms defined by (2.55) and (2.60) do not require optimisation along search directions, as in the gradient methods described in an earlier section.

2.5 Processes with Inequality Constraints

In addition to satisfying dynamic constraints, some processes have to satisfy inequality constraints of the form:

$$C(x(t),u(t),t) \geq 0 \qquad t_0 \leq t \leq t_f$$

or

$$S(x(t),t) \geq 0 \qquad t_0 \leq t \leq t_f \quad .$$

These are termed control inequality constraints and state inequality constraints respectively. Such problems may arise, for example, in a re-entry vehicle entering the earth's atmosphere. The speed of re-entry must not exceed a certain value, otherwise the vehicle would break up. Alternatively, a component may not be able to exceed a certain level of performance, and so it is subject to some form of inequality constraint. Trajectories satisfying the constraints will be termed feasible, and the set of all feasible trajectories will be called the feasible region.

Bryson, Denham and Dreyfus |39| derive necessary conditions for lumped parameter systems with control and state inequality constraints.

Consider the problem of minimising

$$J = G(x(t_f),t_f)$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t)$$

$$x(t_0) = x_0$$

and

$$C(x(t),u(t),t) \geq 0 \ .$$

Bryson et al show that there are no discontinuities in the adjoint multiplier $\lambda(t)$ although there may be discontinuities in $u(t)$ and $\dot{\lambda}(t)$.

Now suppose that the control inequality constraint is replaced by a state variable inequality constraint of the form

$$S(x(t),t) \geqslant 0 \ .$$

Let t_1 be the point at which the trajectory enters the constraint boundary, and t_2 be the time at which it leaves the boundary. Between t_1 and t_2, the state variables are related by

$$S(x(t),t) = 0 \ .$$

As S vanishes identically along the constraint boundary, then its time derivatives must also vanish;

thus

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial x}\right) \dot{x}(t) = 0 \ .$$

But from the state equation we may write

$$\frac{dS}{dt} = \left(\frac{\partial S}{\partial t}\right) + \left(\frac{\partial S}{\partial x}\right) f(x(t),u(t),t) \ .$$

Thus dS/dt may be an explicit function of the control $u(t)$. If it is not an explicit function of $u(t)$, dS/dt may be differentiated repeatedly until it is. The derivative at which the control first appears explicitly defines the order of the constraint:

i.e. if

$$\frac{\partial}{\partial u} S^{(q)}(x(t),t) \neq 0 \quad \text{identically}$$

but

$$\frac{\partial}{\partial u} S^{(i)}(x(t),t) \equiv 0 \qquad i < q$$

then the constraint is said to be of order q. Here the superscript i denotes the i^{th} time derivative.

Thus we have $S^{(q)}(x(t),u(t),t)$ playing the same role as $C(x(t),u(t),t)$ did earlier but, in addition, at the entry points the following tangency conditions have to be satisfied

$$S(x(t_1),t_1) = 0$$

$$S'(x(t_1),t_1) = 0$$

$$S^{(q-1)}(x(t_1),t_1) = 0 .$$

These conditions lead to discontinuities in the influence functions $\lambda(t)$ at t_1. The influence functions are still continuous, however, at the exit point. Bryson et al apply their necessary conditions to two analytic examples, and in an appendix, show that the influence functions are non-unique along the constraint boundary. They can, in fact, have their points of discontinuity at the exit point instead of the entry point, or they can have discontinuities at exit and entry points.

Speyer and Bryson [40] show that this non-uniqueness of the influence functions results from neglecting to make use of a state space of reduced dimension along the constraint boundary, and present a new set of necessary conditions.

Jacobson, Lele and Speyer [41] suggest that the necessary conditions of [39] and [40] under-specify the behaviour of the influence functions at entry and exit points and derive another set of necessary conditions.

The results mentioned above are all for lumped parameter systems. Similar results for time lag systems have been given by Budelis and Bryson [49], and Sebesta and Asher [26].

Budelis and Bryson derive necessary conditions for an extremal path for processes governed by time lag systems and subject to control inequality constraints.

Asher and Sebesta present necessary conditions for a time lag system with control inequality constraints. Their derivation involves transforming the control variable inequality constraint to an equality constraint by adding a slack variable, a device which will be described below in the discussion of Jacobson's transformation method.

2.6 Numerical Techniques for Inequality Constrained Optimal Control

a) Direct Methods

Denham and Bryson [42] describe a steepest descent method for lumped systems with modification on the constraint boundary. These modifications are necessary because the control increments on the boundary are not independent of the state, but are related by

$$C(x + \delta x, u + \delta u, t) = 0$$

or

$$S^{(q)}(x + \delta x, u + \delta u, t) = 0 .$$

Their modifications also take into account discontinuities at the junction points.

b) Penalty Function Techniques

Probably the most widely used of the indirect methods are the penalty function techniques. The constrained problem is replaced by an unconstrained problem with the same system dynamics but with a different performance index. The new index is formed by adding a penalty term to the original performance index. This term has the property that it is small when the constraint is satisfied and takes on large values in the non-feasible region. Penalty function methods are applicable to lumped and time-lag systems.

There are two different types of penalty functions in common use; the interior penalty functions and the enterior penalty functions.

Consider the following problem:

Minimise

$$G(x(t_f),t_f)$$

subject to

$$\dot{x}(t) = f(x(t),x(t-\tau),u(t),t) \qquad t_0 \leqslant t \leqslant t_f$$

$$x(t) = \phi(t) \qquad t_0-\tau \leqslant t \leqslant t_0$$

and

$$C(x(t),u(t),t) \geqslant 0 \quad .$$

This inequality constrained problem is converted to a problem without constraints by adding a penalty function to the objective function to form the new objective function,

$$P_I(\Gamma_k) = G(x(t_f),t_f) + \Gamma_k \int_{t_0}^{t_f} \frac{dt}{C(x,u,t)} \tag{2.61}$$

where Γ_k is a positive scalar. This penalty term is an interior penalty function. The computation of an optimal control proceeds as follows:

An initial control is chosen such that the resulting trajectory does not violate the constraint. A sequence of Γ_k's is set up, such that $\Gamma_k > \Gamma_{k+1} > 0$ and $\lim_{k\to\infty} \Gamma_k = 0$. The optimal control minimising $P_I(\Gamma_k)$ is found for each Γ_k. As Γ_k is reduced, more effort is being made to minimise the original performance index, and the trajectory is allowed to get closer to the constraint boundary, assuming it is profitable to do so. Lasden, Warren and Rice [43] prove that, for lumped parameter systems, the sequence of unconstrained solutions converges to the solution of the constrained problem as $k \to \infty$ and this can be extended to hereditary systems. Unfortunately, the numerical procedures for finding the optimal control all use discrete approximations to the continuous problems. It is thus possible for the trajectory to cross a constraint boundary in between two points of discretization and not get heavily penalized. Note that outside the feasible region, the penalty function (2.61) is negative, therefore once the trajectory has crossed the constraint boundary, it will tend to stay there. This would obviously cause a breakdown of the method, and so any changes in control have to be monitored to ensure they do not violate the constraint boundary.

Alternatively we can formulate another unconstrained problem whose performance index is given as

$$P_E[\Gamma_k] = G[x(t_f),t_f] + \Gamma_k^{-1} \int_{t_0}^{t_f} h(C)[C(x,u,t)]^2 \, dt \tag{2.62}$$

where $h(a) = \begin{cases} 1 & a < 0 \\ 0 & a \geq 0 \end{cases}$

and $\Gamma_k > \Gamma_{k+1} > 0$ and $\lim_{k \to \infty} \Gamma_k = 0$.

This is exterior penalty function method. It has the advantage that the initial control, and any subsequent changes, do not have to be monitored. Lele and Jacobson [44] show that, for lumped parameter systems, the sequence of unconstrained minima approaches a solution to the constrained problem as $k \to \infty$. The proof of convergence of the two penalty function methods may be trivially extended to time lag systems.

c) Jacobson's Transformation Technique

An alternative approach is to transform the constrained problem into an unconstrained problem of increased dimension by the introduction of slack variables, an approach described by Jacobson and Lele [45] for ordinary systems with state space constraints. An advantage of this technique is that any nominal control gives a feasible trajectory. Another feature is that the transformed problem exhibits singular arcs corresponding to arcs lying on the constraint boundary in the original problem. This prohibits the use of second order methods but the gradient and conjugate gradient methods are still applicable.

Consider the problem of minimising

$$J = G(x(t_f),t_f)$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t)$$

$$x(t_0) = x_0$$

and

$$S(x(t),t) \geqslant 0 \quad .$$

We assume here that $u(t)$ is a scalar control function and $S(x(t),t)$ a scalar q^{th} order constraint.

The state variable inequality constraint is converted to an equality constraint by the introduction of a slack variable $\alpha(t)$

$$S(x(t),t) - \tfrac{1}{2}\,\alpha^2(t) = 0 \quad . \tag{2.63}$$

If this equality can be enforced for all t in the interval

$[t_0, t_f]$, then the state variable inequality constraint will obviously be satisfied. Differentiating (2.63) with respect to t:

$$\left.\begin{array}{l} S^{(1)}(x(t),t) - \alpha(t)\alpha_1(t) = 0 \\ S^{(2)}(x(t),t) - \alpha_1^2(t) - \alpha(t)\alpha_2(t) = 0 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \\ S^{(q)}(x(t),u(t),t) - (\text{terms involving } \alpha,\alpha_1,---\alpha_q) = 0 \end{array}\right\} \quad (2.64)$$

Here $\alpha_1(t) = \dot{\alpha}(t)$

and $\alpha_j(t) = \dot{\alpha}_{j-1}(t)$.

Solving the final equation in (2.64) for u(t)

$$u(t) = F(x(t),\alpha_q,\alpha_{q-1},----\alpha_1,\alpha,t) \quad . \qquad (2.65)$$

Substituting (2.65) for u(t) in the original problem gives the following unconstrained problem:

Minimise

$$J = G(x(t_f),t_f)$$

subject to

$$\dot{x}(t) = f(x(t),G(x,\alpha_q,----,\alpha_1,\alpha,t),t) \qquad (2.66)$$

$$\dot{\alpha}(t) = \alpha_1(t)$$

$$\dot{\alpha}_1(t) = \alpha_2(t)$$

$$- \quad - \quad - \quad -$$

$$\dot{\alpha}_{q-1}(t) = \alpha_q(t)$$

$$x(t_0) = x_0 \quad .$$

The initial conditions on $\alpha(t),----,\alpha_{q-1}(t)$ are chosen to satisfy

(2.63) and (2.64), i.e.

$$\alpha(t_0) = \pm\sqrt{S(x_0,t_0)\times 2}$$

$$\alpha_1(t_0) = S^1(x_0,t_0)/\alpha(t_0)$$

etc. and $\alpha_q(t)$ is treated as the control variable.

Jacobson's transformation technique can be applied to time lag systems and we illustrate this with an example.

Consider minimising

$$J = 9x_1^2(t_f) + 6x_1(t_f)x_2(t_f) + 3x_2^2(t_f) + x_3(t_f) \tag{2.67}$$

where $\quad t_f = 1 \quad$ and

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{2.68}$$

$$\dot{x}_2(t) = -x_2(t) - x_2(t-\tfrac{1}{2}) + u(t) \qquad 0 \leq t \leq 1$$

$$\dot{x}_3(t) = 10x_1^2(t) + 10x_2^2(t) + u^2(t),$$

with

$$x_1(t) = 1$$

$$x_2(t) = 0 \qquad -\tfrac{1}{2} \leq t \leq 0$$

$$x_3(t) = 0,$$

and

$$x_2(t) + 0.3 \geq 0 . \tag{2.69}$$

Introduce the slack variable $\alpha(t)$ and rewrite (2.69) as

$$x_2(t) + 0.3 - \tfrac{1}{2}\alpha^2(t) = 0 \tag{2.70}$$

Differentiating (2.70) with respect to time:

$$\dot{x}_2(t) - \alpha(t)\alpha_1(t) = 0 .$$

Substituting for $\dot{x}_2(t)$

$$-x_2(t) - x_2(t-\tfrac{1}{2}) + u(t) - \alpha(t)\alpha_1(t) = 0$$

or

$$u(t) = x_2(t) + x_2(t-\tfrac{1}{2}) + \alpha(t)\alpha_1(t) .$$

Let $x_4(t) = \alpha(t)$ and let $\alpha_1(t)$ be the new control, then our unconstrained problem becomes:

Minimise

$$J = 9x_1^2(t_f) + 6x_1(t_f)x_2(t_f) + 3x_2^2(t_f) + x_3(t_f)$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2})$$

$$\dot{x}_2(t) = x_4(t)\alpha_1(t)$$

$$\dot{x}_3(t) = 10x_1^2(t) + 10x_2^2(t) + \{x_2(t) + x_2(t-\tfrac{1}{2}) + x_4(t)\alpha_1(t)\}^2$$

$$\dot{x}_4(t) = \alpha_1(t) ,$$

with

$$x_1(t) = 1 \qquad -\tfrac{1}{2} \leq t \leq 0$$

$$x_2(t) = 0 \qquad -\tfrac{1}{2} \leq t \leq 0$$

$$x_3(t) = 0 \qquad -\tfrac{1}{2} \leq t \leq 0$$

and $$x_4(t) = \pm\sqrt{0.6} \qquad -\tfrac{1}{2} \leq t \leq 0 .$$

Unfortunately, the application of Jacobson's transformation technique to time lag systems can yield an unconstrained problem whose dynamics are governed not by differential difference equations, but by neutral systems. Such problems are more difficult to solve. Connor [48] describes a gradient method for these systems which is discussed further in a following chapter.

For an example of a neutral system arising from the application of the transformation technique we return to the example, (2.67) and (2.68), already examined. We replace the first order constraint (2.69) by the second order constraint

$$x_1(t) - 0.7 \geq 0 \quad , \tag{2.71}$$

Converting (2.71) to an equality constraint by the addition of a slack variable

$$x_1(t) - 0.7 - \tfrac{1}{2}\alpha^2(t) = 0 \quad .$$

Differentiating

$$\dot{x}_1(t) - \alpha(t)\dot{\alpha}(t) = 0 \quad .$$

Substituting for $\dot{x}_1(t)$ from (2.68)

$$x_2(t) + x_2(t-\tfrac{1}{2}) - \alpha(t)\dot{\alpha}(t) = 0 \quad .$$

Differentiating again

$$\dot{x}_2(t) + \dot{x}_2(t-\tfrac{1}{2}) - \dot{\alpha}^2(t) - \alpha(t)\ddot{\alpha}(t) = 0 \quad .$$

Substituting for $\dot{x}_2(t)$ from (2.68) gives

$$-x_2(t) - x_2(t-\tfrac{1}{2}) + u(t) + \dot{x}_2(t-\tfrac{1}{2}) - \dot{\alpha}^2(t) - \ddot{\alpha}(t)\ (t) = 0$$

or

$$u(t) = x_2(t) + x_2(t-\tfrac{1}{2}) - \dot{x}_2(t-\tfrac{1}{2}) + \dot{\alpha}^2(t) + \ddot{\alpha}(t)\alpha(t) \quad .$$

Let $\ddot{\alpha}(t)$ be the new controller $m(t)$ and let

$$x_4(t) = \alpha(t)$$

$$x_5(t) = \dot{\alpha}(t)$$

then the new system dynamics become

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2})$$

$$\dot{x}_2(t) = x_5^2(t) + x_4(t)m(t) - \dot{x}_2(t-\tfrac{1}{2})$$

$$\dot{x}_3(t) = 10x_1^2(t) + 10x_2^2(t) +$$

$$\{x_2(t) + x_2(t-\tfrac{1}{2}) - \dot{x}_2(t-\tfrac{1}{2}) + x_5^2(t) + x_4(t)m(t)\}^2$$

$$\dot{x}_4(t) = x_5(t)$$

$$\dot{x}_5(t) = m(t) .$$

The initial conditions on x_4 and x_5 are found in the usual manner. Note that the equations describing the dynamics of $x_2(t)$ and $x_3(t)$ both contain derivaties of x_2 with lagged arguement on the right hand side, and thus the new unconstrained problem is of the neutral type rather than the simpler time lag systems discussed in this chapter.

CHAPTER III

Integro-Differential Systems

3.1 Introduction

In the previous chapter, some iterative techniques for finding an optimal control, for systems governed by differential delay equations, have been described. These delay equations may be used to model processes whose rate of change depends on the present values of the state and control, and on the values of the state and control at some previous time(s).

It is a natural extension to consider processes governed by a system of integro-differential equations of the form:

$$\dot{x}(t) = f(x(t),u(t),v(t),t) \qquad t_0 \leqslant t \leqslant t_f$$

$$x(t_0) = x_0 \tag{3.1}$$

where

$$v(t) = \int_{t_0}^{t} g(x(s),u(s),s,t)ds \ .$$

The optimal control of such systems has not been widely studied. C. E. Mueller [20] discusses some numerical methods of generating a feedback control for function-differential equations. In one chapter, he describes the application of his technique to systems governed by equations of the form :

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(H(t)) + \int_{H(t)}^{t} L_0(t,s)x(s)ds + B(t)u(t) + v(t) \tag{3.2}$$

where $\quad t_0 \leqslant H(t) \leqslant t \qquad$ for all t on $(t_0,t_f]$

and $\quad \dot{H}(t) \geqslant 0 \quad .$

Although generally the system described by (3.2) cannot be written in the form (3.1), all of the examples Mueller gives are of the form (3.1).

Connor [47] derives a set of necessary conditions for systems similar to (3.1) and he also describes [46] a gradient method for linear systems with a quadratic performance index. Connor and Hood [38] present a second order method for differential-integral systems.

We begin by extending Balakrishnan's ε-method [6] to integro-differential systems, and use this approach to prove a maximum principle.

3.2 The ε-Problem

Consider the following special case of system (3.1):

$$\dot{x}(t) = f(x(t),u(t),t) + \int_{t_0}^{t} g(x(s),u(s),s,t)ds \tag{3.3}$$

$$x(t_0) = x_0 , \tag{3.4}$$

where $f(x,u,t)$ and $g(x,u,s,t)$ are continuous in all their arguements and continuously differentiable with respect to x. Here x is an n-vector and u an r-vector.

We wish to minimise the performance index

$$J(x,u) = \int_{t_0}^{t_f} F(x(t),u(t),t)dt \tag{3.5}$$

over the class of all functions $x(t),u(t)$ satisfying (3.3) and (3.4) such that $x(t)$ is absolutely continuous, the derivative $\dot{x}(t)$ is square integrable over $[t_0,t_f]$ and $u(t)$ is an admissible control.

It is assumed that $F(x(t),u(t),t)$ is a scalar function continuous in all variables and continuously differentiable with respect to x.

Let U be some convex subset of R^r, then the control $u(t)$ will be termed admissible if it is measurable and square integrable on $[t_0,t_f]$ and $u(t)$ is contained in U for all t on $[t_0,t_f]$. The set of all admissible controls will be denoted by Ω. It is assumed throughout that (3.3) has a unique solution for each admissible control.

We will further assume that for all u in Ω, there exists $M > 0$ such that:

$$||f(x,u(t),t)|| \leq M\{||x|| + 1\}$$

$$||f(x,u(t),t) - f(y,u(t),t)|| \leq M||x - y|| \tag{3.6}$$

$$\int_t^{t_f} ||g(x,u(t),t,s)||ds \leq M\{||x|| + 1\}$$

$$\int_t^{t_f} ||g(x,u(t),t,s) - g(y,u(t),t,s)||ds \leq M||x - y|| \tag{3.7}$$

$$|F(x,u(t),t) - F(y,u(t),t)| \leq M||x - y|| \tag{3.8}$$

for all t in $[t_0,t_f]$ and all $x,y \in R^n$.

(Note: see Appendix F for remarks on these assumptions).

We will define $\mathcal{A}$ to be the set of all admissible states and $\mathcal{B}$ to be the subset of $\Omega \times \mathcal{A}$ such that $[u,x]$ in $\Omega \times \mathcal{A}$ satisfy equation (3.3).

We now formulate the ε problem. For each $\varepsilon > 0$, we seek the minimum of the following functional:

$$\overline{J}(\varepsilon,x,u) = \int_{t_0}^{t_f} F(x(t),u(t),t)dt + \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||\dot{x}(t) - f(x(t),u(t),t)$$

$$- \int_{t_0}^{t} g(x(s),u(s),s,t)ds||^2 \, dt \tag{3.9}$$

over the class of absolutely continuous functions $x(t)$ satisfying (3.4) and the class of admissible controls. We call this the ε problem.

Suppose that for each $\varepsilon > 0$, there exists an absolutely continuous function $x_0(t,\varepsilon)$ satisfying (3.4) and an admissible control $u_0(t,\varepsilon)$ such that $\bar{J}(\varepsilon,x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon))$ attains the minimum, over $\mathfrak{R}\times\Omega$ of $\bar{J}(\varepsilon,x,u)$, then we may prove the following theorem:

<u>Theorem 1</u>

For each $\varepsilon > 0$, let $x_0(t,\varepsilon)$ be as defined above, and let $\bar{J}(\varepsilon,x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon))$ be the minimum. If $\hat{x}(t,\varepsilon)$ is the solution of (3.3) and (3.4) for the control of $u_0(t,\varepsilon)$ then

$$\lim_{\varepsilon\to 0} \bar{J}(\varepsilon,x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)) = \lim_{\varepsilon\to 0} \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t)dt$$

$$= \inf \int_{t_0}^{t_f} F(x(t),u(t),t)dt \tag{3.10}$$

where the infimum is over admissible controls and absolutely continuous $x(t)$ satisfying (3.3) and (3.4).

<u>Proof</u>

Let $0 < \varepsilon_2 < \varepsilon_1$

and let

$$z_0(t,\varepsilon) = \dot{x}_0(t,\varepsilon) - f(x_0(t,\varepsilon),u_0(t,\varepsilon),t)$$

$$- \int_{t_0}^{t} g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)ds \tag{3.11}$$

Now, by definition

$$\frac{1}{2\varepsilon_2} \int_{t_0}^{t_f} ||z_0(t,\varepsilon_2)||^2\, dt + \int_{t_0}^{t_f} F(x_0(t,\varepsilon_2),u_0(t,\varepsilon_2),t)dt$$

is the infimum of $\bar{J}(\varepsilon_2, x(\cdot), u(\cdot))$ over $\mathcal{A} \times \mathcal{B}$.

Thus we have that

$$\frac{1}{2\varepsilon_2}\int_{t_0}^{t_f} ||z_0(t,\varepsilon_2)||^2\, dt + \int_{t_0}^{t_f} F(x_0(t,\varepsilon_2),u_0(t,\varepsilon_2),t)dt$$

$$\leq \frac{1}{2\varepsilon_2}\int_{t_0}^{t_f} ||z_0(t,\varepsilon_1)||^2\, dt + \int_{t_0}^{t_f} F(x_0(t,\varepsilon_1),u_0(t,\varepsilon_1),t)dt$$

and similarly we have

$$\frac{1}{2\varepsilon_1}\int_{t_0}^{t_f} ||z_0(t,\varepsilon_1)||^2\, dt + \int_{t}^{t_f} F(x_0(t,\varepsilon_1),u_0(t,\varepsilon_1),t)dt$$

$$\leq \frac{1}{2\varepsilon_1}\int_{t_0}^{t_f} ||z_0(t,\varepsilon_2)||^2\, dt + \int_{t_0}^{t_f} F(x_0(t,\varepsilon_2),u_0(t,\varepsilon_2),t)dt$$

So we have inequalities of the form

$$\frac{1}{2\varepsilon_1} A_1 + B_1 \leq \frac{1}{2\varepsilon_1} A_2 + B_2 \tag{3.12}$$

$$\frac{1}{2\varepsilon_2} A_2 + B_2 \leq \frac{1}{2\varepsilon_2} A_1 + B_1 \tag{3.13}$$

and $\quad \dfrac{1}{2\varepsilon_2} > \dfrac{1}{2\varepsilon_1}$.

So from (3.12) and (3.13)

$$A_1\left(\frac{1}{2\varepsilon_1} - \frac{1}{2\varepsilon_2}\right) \leq A_2\left(\frac{1}{2\varepsilon_1} - \frac{1}{2\varepsilon_2}\right) ,$$

and so, since $\quad \dfrac{1}{2\varepsilon_1} < \dfrac{1}{2\varepsilon_2}$, we have

$$A_1 \geq A_2 .$$

Similarly

$$B_1 \leq B_2 .$$

Thus

$$\int_{t_0}^{t_f} ||z_0(t,\varepsilon_2)||^2 \, dt \leq \int_{t_0}^{t_f} ||z_0(t,\varepsilon_1)||^2 \, dt$$

and

$$\int_{t_0}^{t_f} F(x_0(t,\varepsilon_1),u_0(t,\varepsilon_1),t)dt \leq \int_{t_0}^{t_f} F(x_0(t,\varepsilon_2),u(t,\varepsilon_2),t)dt .$$

So we have that $\int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt$ is monotonically increasing as $\varepsilon \to 0$ and that $\int_{t_0}^{t_f} ||z_0(t,\varepsilon)||^2 \, dt$ is monotonically decreasing as $\varepsilon \to 0$. We now show that in fact $\int_{t_0}^{t_f} ||z_0(t,\varepsilon)||^2 \, dt$ decreases monotonically to zero.

Suppose $\int_{t_0}^{t_f} ||z_0(t,\varepsilon)||^2 \, dt$ has an infimum α, as $\varepsilon \to 0$. Obviously $\alpha \geq 0$, so assume that $\alpha > 0$.

Let $h(\varepsilon) = \inf_{\mathfrak{A} \times \Omega} \{\bar{J}(\varepsilon,x(\cdot,\varepsilon),u(\cdot,\varepsilon))\}$ for each ε.

We know that, by definition,

$$h(\varepsilon) \leq \inf \int_{t_0}^{t_f} F(x,u,t)dt \triangleq F_0$$

say, where the infimum is taken over admissible u(t) and absolutely continuous x(t) satisfying (3.3) and (3.4), i.e. over $\mathcal{B}$.

$$h(\varepsilon) = \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||z_0(t,\varepsilon)||^2 dt$$

$$+ \int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt$$

$$\geq \alpha/2\varepsilon + \int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt .$$

So
$$F_0 \geq \alpha/2\varepsilon + \int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt$$

But by choosing ε small enough, we can make $\alpha/2\varepsilon$ as large as we like, giving a contradiction, thus $\alpha = 0$, and we have shown that

$\int_{t_0}^{t_f} ||z_0(t,\varepsilon)||^2 dt$ decreases monotonically to zero.

Let $\hat{x}(t,\varepsilon)$ be the solution of (3.3) and (3.4) using the control $u_0(t,\varepsilon)$. We now show that $\hat{x}(t,\varepsilon)$ converges to $x_0(t,\varepsilon)$ uniformly on $[t_0,t_f]$ as $\varepsilon \to 0$.

From (3.11), we have that

$$\dot{x}_0(t,\varepsilon) = z_0(t,\varepsilon) + f(x_0(t,\varepsilon),u_0(t,\varepsilon),t) + \int_{t_0}^{t} g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)ds . \tag{3.14}$$

By the definition of $\hat{x}(t,\varepsilon)$

$$\dot{\hat{x}}(t,\varepsilon) = f(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t)$$

$$+ \int_{t_0}^{t} g(\hat{x}(s,\varepsilon),u_0(t,\varepsilon),s,t)ds \ , \qquad (3.15)$$

and so by (3.14) and (3.15)

$$||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)|| = ||\int_{t_0}^{t} \{f(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma)$$

$$- f(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma) + \int_{t_0}^{\sigma} [g(\hat{x}(s,\varepsilon),u_0(s,\varepsilon),s,t)$$

$$- g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)]ds - g_0(\sigma,\varepsilon)\}d\sigma||$$

$$\leq \int_{t_0}^{t} ||g_0(\sigma,\varepsilon)||d\sigma + \int_{t_0}^{t} ||f(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma)$$

$$- f(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma)||d\sigma$$

$$+ \int_{t_0}^{t} \int_{t_0}^{\sigma} ||g(\hat{x}(s,\varepsilon),u_0(s,\varepsilon),s,\sigma) - g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,\sigma)||ds\ d\sigma \ .$$

But

$$\int_{t_0}^{t} \int_{t_0}^{\sigma} ||g(\hat{x}(s,\varepsilon),u_0(s,\varepsilon),s,\sigma) - g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,\sigma)||ds\ d\sigma$$

$$= \int_{t_0}^{t} \int_{\sigma}^{t} ||g(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s) - g(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s)||ds\ d\sigma \ .$$

Now for t in $[t_0,t_f]$

$$\int_\sigma^t ||g(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma, s) - g(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s)||ds$$

$$\leq \int_\sigma^{t_f} ||g(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s) - g(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s)||ds .$$

But by (3.7)

$$\int_\sigma^{t_f} ||g(\hat{x}(\sigma.\varepsilon),u_0(\sigma,\varepsilon),\sigma,s) - g(x_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma,s)||ds$$

$$\leq M||\hat{x}(\sigma,\varepsilon) - x_0(\sigma,\varepsilon)|| .$$

So

$$||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)|| \leq \int_{t_0}^t ||z_0(\sigma,\varepsilon)||d\sigma$$

$$+ \int_{t_0}^t ||f(\hat{x}(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma) - f(\hat{x}_0(\sigma,\varepsilon),u_0(\sigma,\varepsilon),\sigma)||d\sigma$$

$$+ \int_{t_0}^t M ||\hat{x}(\sigma,\varepsilon) - x_0(\sigma,\varepsilon)||d\sigma ,$$

which by (3.6) becomes

$$||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)|| \leq \int_{t_0}^t ||z_0(\sigma,\varepsilon)||d\sigma$$

$$+ 2\int_{t_0}^t M||\hat{x}(\sigma,\varepsilon) - x_0(\sigma,\varepsilon)||d\sigma . \tag{3.16}$$

We have already seen that $\displaystyle\int_{t_0}^{t_f} ||z_0(\sigma,\varepsilon)||^2 \, d\sigma$

tends to zero as ε tends to zero, and thus

$$\int_{t_0}^{t_f} ||z_0(\sigma,\varepsilon)|| d\sigma \quad \to \quad 0 \quad \text{as } \varepsilon \to 0$$

and

$$M_\varepsilon(t) = \int_{t_0}^{t} ||z_0(\sigma,\varepsilon)|| d\sigma \quad \to \quad 0 \text{ as } \varepsilon \to 0.$$

Let $\quad V_\varepsilon(t) = ||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)||$

then we may write (3.16) as

$$V_\varepsilon(t) \leqslant M_\varepsilon(t) + 2\int_{t_0}^{t} M \, V_\varepsilon(\sigma) d\sigma$$

$$\leqslant M_\varepsilon(t_f) + 2\int_{t_0}^{t} M \, V_\varepsilon(\sigma) d\sigma \quad .$$

Thus by Gronwall's inequality

$$V_\varepsilon(t) \leqslant M_\varepsilon(t_f) \exp\{2\int_{t_0}^{t} M \, d\sigma\}$$

for all $\quad t$ in $[t_0,t_f]$.

Hence, $\quad V_\varepsilon(t) \to 0 \quad$ as $\quad \varepsilon \to 0$

for all $\quad t$ in $[t_0,t_f]$

or, in other words

$$\lim_{\varepsilon\to 0}||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)|| = 0 \tag{3.17}$$

uniformly on $[t_0,t_f]$, and so $\hat{x}(t,\varepsilon)- x_0(t,\varepsilon)$ converges uniformly to 0 on $[t_0,t_f]$.

By assumption (3.8)

$$\lim_{\varepsilon\to 0}|F(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t) - F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)|$$

$$\leq M\,||\hat{x}(t,\varepsilon) - x_0(t,\varepsilon)||, \tag{3.18}$$

so by (3.17) and (3.18)

$$\lim_{\varepsilon\to 0}|F(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t) - F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)| = 0$$

uniformly on $[t_0,t_f]$.

This last result implies

$$\lim_{\varepsilon\to 0} \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t)dt = \lim_{\varepsilon\to 0}\int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt .$$

By definition, for each ε, we have that

$$\bar{J}(\varepsilon,x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)) \leq J(\varepsilon,\hat{x}(\cdot,\varepsilon),u_0(\cdot,\varepsilon)).$$

Taking the limit as ε tends to 0:

$$\lim_{\varepsilon\to 0} \{\frac{1}{2\varepsilon} \int_{t_0}^{t_f}||z_0(t,\varepsilon)||^2\, dt + \int_{t_0}^{t_f} F(x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt\}$$

$$\leq \lim_{\varepsilon\to 0} \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon),u_0(t,\varepsilon),t)dt .$$

and so we have that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||\hat{z}_0(t,\varepsilon)||^2 \, dt = 0 \; . \tag{3.19}$$

Thus we have proved that

$$\lim_{\varepsilon\to 0} \bar{J}(\varepsilon, x_0(\cdot,\varepsilon), u_0(\cdot,\varepsilon)) =$$

$$\lim_{\varepsilon\to 0} \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon), u_0(t,\varepsilon), t) dt \; . \tag{3.20}$$

But, by definition

$$\bar{J}(\varepsilon, x_0(\cdot,\varepsilon), u_0(\cdot,\varepsilon)) \leqslant \inf_{\mathcal{B}} \int_{t_0}^{t_f} F(x(t), u(t), t) dt$$

$$\leqslant \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon), u_0(t,\varepsilon), t) dt \tag{3.21}$$

for any $\varepsilon < 0$.

Therefore, by (3.20) and (3.21)

$$\lim_{\varepsilon\to 0} \bar{J}(\varepsilon, x_0(\cdot,\varepsilon), u_0(\cdot,\varepsilon)) = \inf_{\mathcal{B}} \int_{t_0}^{t_f} F(x(t), u(t), t) dt$$

$$= \lim_{\varepsilon\to 0} \int_{t_0}^{t_f} F(\hat{x}(t,\varepsilon), u_0(t,\varepsilon), t) dt$$

and the theorem is proved.

3.3 Necessary Conditions for the ε-Problem

As before we assume that for each ε-problem there exists an admissible control $u_0(t,\varepsilon)$ and an absolutely continuous function $x_0(t,\varepsilon)$ such that $x_0(t_0,\varepsilon) = x_0$ and $\bar{J}(\varepsilon,x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon))$ is the minimum of $\bar{J}(\varepsilon,x(\cdot),u_0(\cdot))$.

Let $w(t)$ be an n-vector valued function which has derivatives of all orders on $[t_0,t_f]$ and such that $w(t_0) = 0$.

Let θ be a real variable and let

$$x(t) = x_0(t,\varepsilon) + \theta w(t) .$$

If the pair $(x_0(t,\varepsilon),u_0(t,\varepsilon))$ gives the minimum of $\bar{J}(\varepsilon,x(\cdot),u(\cdot))$ over $\mathcal{A} \times \Omega$, then

$$\frac{d}{d\theta}\left[\bar{J}(\varepsilon,x(\cdot),u_0(t,\varepsilon))\right]\Bigg|_{\theta=0} = 0 . \tag{3.22}$$

Let $z_0(t,\varepsilon)$ be as defined in (3.11) then, by (3.22)

$$\frac{1}{\varepsilon}\int_{t_0}^{t_f} \Big< z_0(t,\varepsilon),\dot{w}(t) - f_x(x_0(t,\varepsilon),u_0(t,\varepsilon),t)w(t)$$

$$- \int_{t_0}^{t} g_x(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)w(s)ds \Big> dt$$

$$+ \int_{t_0}^{t_f} \Big< F_x(x_0(t,\varepsilon),u_0(t,\varepsilon),t),w(t) \Big> dt = 0 \tag{3.23}$$

where

$$< a,b > = a^T b ,$$

$$f_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$$F_x = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)^T$$

$$g_x = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} .$$

We may rewrite (3.23) as

$$\frac{1}{\varepsilon} \int_{t_0}^{t_f} < z_0(t,\varepsilon), \dot{w}(t) > dt$$

$$= - \int_{t_0}^{t_f} < F_x(x_0(t,\varepsilon), u_0(t,\varepsilon), t), w(t) > dt$$

$$+ \frac{1}{\varepsilon} \int_{t_0}^{t_f} < z_0(t,\varepsilon), f_x(x_0(t,\varepsilon), u_0(t,\varepsilon), t) w(t) > dt$$

$$+ \frac{1}{\varepsilon} \int_{t_0}^{t_f} \int_{t_0}^{t} < z_0(t,\varepsilon), g_x(x_0(s,\varepsilon), u_0(s,\varepsilon), s, t) w(s) > ds\, dt \, . \quad (3.24)$$

We may substitute the following identities into equation (3.24)

$$\int_{t_0}^{t_f} < \Xi_0(t,\varepsilon),\dot{w}(t) > dt = < \Xi_0(t_f,\varepsilon),w(t_f) > - < \Xi_0(t_0,\varepsilon),w(t_0) >$$

$$- \int_{t_0}^{t_f} < \dot{\Xi}_0(t,\varepsilon),w(t) > dt$$

and

$$\int_{t_0}^{t_f} \int_{t_0}^{t} < \Xi_0(t,\varepsilon),g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)w(s) > ds\ dt$$

$$= \int_{t_0}^{t_f} \int_{t}^{t_f} < \Xi_0(s,\varepsilon),g(x_0(t,\varepsilon),u_0(t,\varepsilon),t,s)w(t) > ds\ dt \quad ,$$

and noting that $w(t_0) = 0$, we obtain

$$\frac{1}{\varepsilon} \int_{t_0}^{t_f} < \dot{\Xi}_0(t,\varepsilon),w(t) > dt = \frac{1}{\varepsilon} < \Xi_0(t_f,\varepsilon),w(t_f) >$$

$$- \int_{t_0}^{t_f} \Bigg\{ < \frac{1}{\varepsilon}\ f_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t)\Xi_0(t,\varepsilon),w(t) >$$

$$+ \int_{t}^{t_f} < \frac{1}{\varepsilon} g_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t,s)\Xi_0(s,\varepsilon),w(t) > ds$$

$$- < F_x(x_0(t,\varepsilon),u_0(t,\varepsilon),t),w(t) > \Bigg\}\ dt \ . \tag{3.25}$$

Now, since (3.25) holds for all w(t) with the previously indicated property, we have that

$$\dot{z}_0(t,\varepsilon) = - f_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t)z_0(t,\varepsilon)$$

$$- \int_t^{t_f} g_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t,s)z_0(s,\varepsilon))ds$$

$$+ \varepsilon F_x(x_0(t,\varepsilon),u_0(t,\varepsilon),t) \tag{3.26}$$

and

$$z_0(t_f,\varepsilon) = 0 . \tag{3.27}$$

The existence of the derivative of z_0 may be shown as in ref. 6.

We set $\psi(t,\varepsilon) = z_0(t,\varepsilon)/\varepsilon$, then (3.26) and (3.27) may be be written as:

$$\dot{\psi}(t,\varepsilon) = - f_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t)\psi(t,\varepsilon)$$

$$- \int_t^{t_f} g_x^T(x_0(t,\varepsilon),u_0(t,\varepsilon),t,s)\psi(s,\varepsilon)ds$$

$$+ F_x(x_0(t,\varepsilon),u_0(t,\varepsilon),t) ; \tag{3.28}$$

and

$$\psi(t_f,\varepsilon) = 0 . \tag{3.29}$$

Now by (3.26), $\dot{z}_0(t,\varepsilon)$ exists and is finite, and hence we have that $z_0(t,\varepsilon)$ is absolutely continuous, and therefore $\psi(t,\varepsilon)$ is absolutely continuous.

Now let

$$H(\varepsilon,\Xi(t),x(t),u(t),t) = -F(x(t),u(t),t)$$

$$+ < \Xi(t)/\varepsilon, f(x(t),u(t),t) >$$

$$+ \int_t^{t_f} < \Xi(s)/\varepsilon, g(x(t),u(t),t,s) > ds, \quad (3.30)$$

where $\Xi(t)$ is defined by

$$\Xi(t) = \dot{x}(t) - f(x(t),u(t),t) - \int_{t_0}^{t} g(x(t),u(t),s,t)ds.$$

We can easily show that

$$\overline{J}(\varepsilon,x_0(t,\varepsilon),u(t),t) = \int_{t_0}^{t_f} F(x_0(t,\varepsilon),u(t),t)dt$$

$$+ \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon) - f(x_0(t,\varepsilon),u(t),t) - \int_{t_0}^{t} g(x_0(s,\varepsilon),u(s),s,t)ds||^2 dt$$

$$= \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon)||^2 dt - \frac{1}{2\varepsilon} \int_{t_0}^{t_f} || f(x_0(t,\varepsilon),u_0(t,\varepsilon),t)$$

$$- \int_{t_0}^{t} g(x_0(s,\varepsilon),u_0(s,\varepsilon),s,t)ds||^2 dt$$

$$+ \frac{1}{2\varepsilon} \int_{t_0}^{t_f} ||f(x_0(t,\varepsilon),u_0(t,\varepsilon),t) - f(x_0(t,\varepsilon),u(t),t)$$

$$+ \int_{t_0}^{t} \{g(x_0(t,\varepsilon),u_0(t,\varepsilon),s,t) - g(x_0(t,\varepsilon),u(t),s,t)\}ds||^2 dt$$

$$- \int_{t_0}^{t_f} H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), u(t), t)\,dt \,. \tag{3.31}$$

Thus, given any $\varepsilon > 0$, (3.31) attains its infimum for over $\mathfrak{A}\times\Omega$ for $u(\cdot) = u_0(\cdot,\varepsilon)$ if

$$\int_{t_0}^{t_f} H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), u(t), t)\,dt$$

attains its supremum at $u(\cdot) = u_0(\cdot,\varepsilon)$.

Thus we have proved the following:

<u>Theorem 2</u>

If $u_0(t,\varepsilon)$ and $x_0(t,\varepsilon)$ are solutions of the ε-problem, there exists an n-vector valued function $\psi(t,\varepsilon)$, defined and absolutely continuous on $[t_0,t_f]$, satisfying (3.28) and (3.29) and not identically zero on $[t_0,t_f]$ such that

$$\int_{t_0}^{t_f} H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), u(t), t)\,dt$$

$$\leq \int_{t_0}^{t_f} H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), u_0(t,\varepsilon), t)\,dt \tag{3.32}$$

for all admissible controls $u(t)$.

We can demonstrate a pointwise form of the above theorem as follows:

We wish to show that

$$H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), v, t) \leq H(\varepsilon, Z_0(t,\varepsilon), x_0(t,\varepsilon), u_0(t,\varepsilon), t)$$

almost everywhere on $[t_0,t_f]$, for all $v \in U$. We define E to be the subset of $[t_0,t_f]$ on which

$$H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),v,t) > H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),u_0(t,\varepsilon),t) .$$

Define a new control $w(t)$ as follows:

$$w(t) = v \qquad \text{for } t \text{ in } E$$

$$w(t) = u_0(t,\varepsilon) \qquad \text{for } t \text{ in the complement of } E$$

then $w(t)$ is an admissible control.

Now we have that

$$H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),w(t),t) > H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),u_0(t,\varepsilon),t)$$

on E and

$$H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),w(t),t) = H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),u_0(t,\varepsilon),t)$$

on the complement of E, and so we have

$$\int_{t_0}^{t_f} H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),w(t),t)dt$$

$$> \int_{t_0}^{t_f} H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),u_0(t,\varepsilon),t)dt$$

which contradicts (3.32) unless E has measure zero. Thus, for any v in U

$$H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),u_0(t,\varepsilon),t)$$

$$\geq H(\varepsilon,\mathcal{Z}_0(t,\varepsilon),x_0(t,\varepsilon),v,t) \qquad (3.33)$$

almost everywhere on $[t_0,t_f]$.

3.4 The Limiting Case

We now examine the behaviour of the ε-maximum principle as ε goes to zero. As before, we assume the existence of the solution to each ε-problem.

Let $\{\varepsilon_k\}$ be a sequence decreasing monotonically to zero. From Theorem 1 we know that

$$\lim_{k\to\infty} \bar{J}(\varepsilon_k, x_0(\cdot,\varepsilon_k), u_0(\cdot,\varepsilon_k)) = \inf \int_{t_0}^{t_f} F(x,u,t)dt \ ,$$

where the infimum is taken over all absolutely continuous $x(t)$ and admissible controls satisfying (3.3) and (3.4).

We assume that there is an admissible control $u_0(t)$ such that for all t in $[t_0,t_f]$,

$$\lim_{k\to\infty} u_0(t,\varepsilon_k) = u_0(t) \ .$$

Let $\hat{x}(t,\varepsilon_k)$ be the solution of (3.3) and (3.4) with control $u_0(t,\varepsilon_k)$. Now from (3.17) we see that the sequence $\{\eta_k(t)\}$, where

$$\eta_k(t) = ||x_0(t,\varepsilon_k) - \hat{x}(t,\varepsilon_k)||,$$

converges to zero uniformly on $[t_0,t_f]$ as $k \to \infty$.

All convergent sequences are bounded. Let A be the bound of $\{\eta_k(t)\}$ for all t on $[t_0,t_f]$.

We may write $\hat{x}(t,\varepsilon_k)$ as

$$\hat{x}(t,\varepsilon_k) = \int_{t_0}^{t} \dot{\hat{x}}(s,\varepsilon_k)ds + x_0$$

$$= \int_{t_0}^{t} \{f(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma) + \int_{t_0}^{\sigma} g(x(s),u(s),s,\sigma)ds\}d\sigma + x_0$$

Therefore

$$||\hat{x}(t,\varepsilon_k)|| \leqslant ||x_0||$$

$$+ \int_{t_0}^{t} ||f(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma) + \int_{t_0}^{\sigma} g(\hat{x}(s,\varepsilon_k),u_0(s,\varepsilon_k),s,\sigma)ds||d\sigma$$

$$\leqslant ||x_0|| + \int_{t_0}^{t} ||f(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma||d\sigma$$

$$+ \int_{t_0}^{t} \int_{t_0}^{\sigma} ||g(\hat{x}(s,\varepsilon_k),u_0(s,\varepsilon_k),s,\sigma)||ds\ d\sigma .$$

But we have the following:

$$\int_{t_0}^{t} \int_{\sigma}^{t} ||g(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma,s)||ds\ d\sigma$$

$$\leqslant \int_{t_0}^{t} \int_{\sigma}^{t_f} ||g(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma,s)||ds\ d\sigma$$

for $\qquad t_0 \leqslant t \leqslant t_f$;

and

$$\int_{t_0}^{t} \int_{t_0}^{\sigma} ||g(\hat{x}(s,\varepsilon_k),u_0(s,\varepsilon_k),s,\sigma)||ds\ d\sigma$$

$$= \int_{t_0}^{t} \int_{\sigma}^{t} ||g(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma,s||ds\ d\sigma,$$

so we may write

$$||\hat{x}(t,\varepsilon_k)|| \leqslant \int_{t_0}^{t} ||f(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma)||d\sigma +$$

$$\int_{t_0}^{t} \int_{\sigma}^{t_f} ||g(\hat{x}(\sigma,\varepsilon_k),u_0(\sigma,\varepsilon_k),\sigma,s)|| ds\, d\sigma + ||x_0|| .$$

Using assumptions (3.6) and (3.7) in the above expression

$$||\hat{x}(t,\varepsilon_k)|| \leq M \int_{t_0}^{t} ||\hat{x}(\sigma,\varepsilon_k) + 1|| d\sigma + M \int_{t_0}^{t} ||\hat{x}(\sigma,\varepsilon_k) + 1|| d\sigma$$

or $$||\hat{x}(t,\varepsilon_k)|| \leq 2M(t - t_0) + 2M \int_{t_0}^{t} ||\hat{x}(\sigma,\varepsilon_k)|| d\sigma.$$

By Gronwall's inequality:

$$||\hat{x}(t,\varepsilon_k)|| \leq 2M(t - t_0)\exp\int_{t_0}^{t} 2M d\sigma$$

and so $||\hat{x}(t,\varepsilon_k)||$ is certainly bounded for all t in $[t_0,t_f]$, by B say.

Thus from the triangular inequality

$$||x_0(t,\varepsilon_k)|| \leq ||\hat{x}(t,\varepsilon_k)|| + ||x_0(t,\varepsilon_k) - \hat{x}(t,\varepsilon_k)|| ,$$

we have that

$$||x_0(t,\varepsilon_k)|| \leq A + B$$

for $t_0 \leq t \leq t_f$ and for all k.

Thus the sequence of functions $x_0(t,\varepsilon_k)$ is uniformly bounded on $[t_0,t_f]$. In other words, each element of

$$x_0(t,\varepsilon_k) = [x_0^1(t,\varepsilon_k), \ldots\ldots, x_0^n(t,\varepsilon_k)]^T$$

is uniformly bounded.

We proceed now by showing that the family $\{x_0^i(t,\varepsilon_k)\}$ indexed by k is equicontinuous. In order to show equicontinuity, we first show that

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2\, dt$$

is bounded.

We may write

$$\dot{x}_0(t,\varepsilon_k) = z_0(t,\varepsilon_k) + f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$+ \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds ,$$

then

$$||\dot{x}_0(t,\varepsilon_k)|| \leqslant ||z_0(t,\varepsilon_k)|| + ||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$+ \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||,$$

therefore

$$||\dot{x}_0(t,\varepsilon_k)||^2 \leqslant ||z_0(t,\varepsilon_k)||^2 + 2||z_0(t,\varepsilon)|| \times$$

$$||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t) + \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||$$

$$+ \quad ||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t) + \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||^2.$$

But

$$2||z_0(t,\varepsilon_k)||.||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$+ \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds|| \leqslant ||z_0(t,\varepsilon_k)||^2$$

$$+ ||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t) + \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||^2.$$

Hence $$||\dot{x}_0(t,\varepsilon_k)||^2 \leqslant 2\Big[||z_0(t,\varepsilon_k)||^2 + ||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$+ \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||^2\Big].$$

Integrating gives

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2 \, dt \le 2 \int_{t_0}^{t_f} ||g_0(t,\varepsilon_k)||^2 \, dt$$

$$+ 2 \int_{t_0}^{t_f} ||f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$+ \int_{t_0}^{t} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds||^2 \, dt \quad ,$$

i.e.

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2 \, dt \le 2 \int_{t_0}^{t_f} ||g_0(t,\varepsilon_k)||^2 \, dt + 4 \int_{t_0}^{t_f} M^2\{|| x_0(t,\varepsilon_k)|| + 1\}^2 dt$$

by assumptions (3.6) and (3.7).

But we have already proved that $x_0(t,\varepsilon_k)$ is bounded, therefore we can find a number α such that

$$\int_{t_0}^{t_f} M^2\{||x_0(t,\varepsilon_k)|| + 1\}^2 dt \le \alpha \ .$$

We know that $\int_{t_0}^{t_f} ||g_0(t,\varepsilon_k)||^2 dt$ is a convergent sequence and hence has an upper bound, say β, thus

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2 dt \le 4\alpha + 2\beta$$

and hence is bounded.

We now proceed to show equicontinuity.

Recall the Schwarz inequality

$$\left| \int_a^b f_1(t)f_2(t)dt \right|^2 \leq \int_a^b f_1^2(t)dt \int_a^b f_2^2(t)dt \text{ .}$$

Now we can write

$$||x_0(t_1,\varepsilon_k) - x_0(t_2\varepsilon_k)||^2 = || \int_{t_1}^{t_2} \dot{x}_0(t,\varepsilon_k)dt||^2$$

and hence by the Schwarz inequality with

$$f_1 = ||\dot{x}_0(t,\varepsilon_k)|| \quad \text{and} \quad f_2 = 1 \text{ ,}$$

$$||x_0(t_1,\varepsilon_k) - x_0(t_2,\varepsilon_k)||^2$$

$$\leq \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} ||\dot{x}_0(t,\varepsilon_k)||^2 \, dt$$

$$\leq (t_2 - t_1) \int_{t_1}^{t_2} ||\dot{x}_0(t,\varepsilon_k)||^2 \, dt \text{ .}$$

Let the bound for

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2 \, dt \quad \text{be} \quad \Lambda,$$

then for each i, $1 < i < n$

$$|x^i(t_1,\varepsilon_k) - x^i(t_2,\varepsilon_k)| \leq \Lambda(t_2 - t_1) \tag{3.34}$$

Thus, given any $\eta > 0$, we may choose $\delta = \eta/\Lambda$ so that for all

t_1, t_2 satisfying $|t_1 - t_2| < \delta$, we may write

$$|x^i(t_1,\varepsilon_k) - x^i(t_2,\varepsilon_k)| \leqslant \Lambda|T_2 - t_1| \leqslant \Lambda\eta/\Lambda ,$$

i.e.

$$|x^i(t_1,\varepsilon_k) - x^i(t_2\varepsilon_k)| \leqslant \eta .$$

Hence, $\{x^i(t,\varepsilon_k)\}$ is an equicontinuous family for all i. Hence, by Arzela's theorem, $\{x^i(t,\varepsilon_k)\}$ is relatively compact and therefore contains a uniformly convergent subsequence, converging to a function $x_0^i(t)$ which is continuous.

Let

$$x_0(t) = |x_0^1(t), \; . \; . \; . \; . \; , \; x_0^n(t)|^T .$$

We assumed at the beginning of section 3.4 that there is an admissible control $u_0(t)$ such that for all t in $[t_0,t_f]$

$$\lim_{k\to\infty} u_0(t,\varepsilon_k) = u_0(t).$$

From (3.19) we see that

$$\lim_{k\to\infty} \mathcal{Z}_0(t,\varepsilon_k) = 0 \qquad \text{almost everywhere on } [t_0,t_f] .$$

This is equivalent to

$$\lim_{k\to\infty} |\dot{x}_0(t,\varepsilon_k) - f(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t)$$

$$- \int_{t_0}^{t_f} g(x_0(s,\varepsilon_k),u_0(s,\varepsilon_k),s,t)ds| = 0 \qquad (3.35)$$

almost everywhere on $[t_0,t_f]$.

But

$$\left.\begin{array}{l} \lim\limits_{k\to\infty} x_0(t,\varepsilon_k) = x_0(t) \\ \\ \lim\limits_{k\to\infty} u_0(t,\varepsilon_k) = u_0(t) \end{array}\right\} \tag{3.36}$$

By the continuity of f(x,u,t) and g(x,u,s,t) and by (3.36) we may rewrite (3.35) as

$$\lim_{k\to\infty} \dot{x}_0(t,\varepsilon_k) = f(x_0(t),u_0(t),t) + \int_{t_0}^{t} g(x_0(s),u_0(s),s,t)ds \ . \tag{3.37}$$

Now, since $x_0(t,\varepsilon_k)$ is absolutely continuous, we have that

$$x_0(t,\varepsilon_k) = x_0 + \int_{t_0}^{t} \dot{x}_0(s,\varepsilon_k)ds \ . \tag{3.38}$$

Taking the limit of (3.38)

$$x_0(t) = x_0 + \lim_{k\to\infty} \int_{t_0}^{t} \dot{x}_0(s,\varepsilon_k)ds \ . \tag{3.39}$$

We have already shown that

$$\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon_k)||^2 \ dt = \beta_k$$

where β_k is finite.

Hence a constant function Y_k can be found such that

$$|x_0^i(t,\varepsilon_k)| < y_k^i \;, \quad t \in [t_0, t_f]$$

where y_k^i is the i^{th} component of the n-vector Y_k.

Hence by Lebesgue's dominated convergence theorem, we may write (3.39) as

$$x_0(t) = x_0 + \int_{t_0}^{t} \lim_{k\to\infty} \dot{x}_0(s,\varepsilon_k)\, ds \;. \tag{3.40}$$

Hence, from (3.37), we may write (3.40) as

$$x_0(t) = x_0 + \int_{t_0}^{t} f(x_0(\sigma),u_0(\sigma),\sigma)d\sigma$$

$$+ \int_{t_0}^{t} \int_{t_0}^{\sigma} g(x_0(s),u_0(s),s,\sigma)ds\, d\sigma \;. \tag{3.41}$$

But (3.41) implies that

$$\dot{x}_0(t) = f(x_0(t),u_0(t),t) + \int_{t_0}^{t} g(x_0(s),u_0(s),s,t)ds \tag{3.42}$$

almost everywhere on $[t_0,t_f]$.

Also, since

$$\lim_{k\to\infty} \frac{1}{2\varepsilon_k} \int_{t_0}^{t_f} ||\bar{z}_0(t,\varepsilon_k)||^2\, dt = 0 \quad ,$$

we may write

$$\lim_{k\to\infty} \bar{J}(\varepsilon_k,x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k))$$

$$= J(x_0(\cdot),u_0(\cdot)) = \inf_{\mathcal{B}} J(x(\cdot),u(\cdot)) \;.$$

Now, by continuity properties

$$\lim_{k\to\infty} f_x^T(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t) = f_x^T(x_0(t),u_0(t),t),$$

$$\lim_{k\to\infty} g_x^T(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t,s) = g_x^T(x_0(t),u_0(t),t,s),$$

and

$$\lim_{k\to\infty} F_x(x_0(t,\varepsilon_k),u_0(t,\varepsilon_k),t) = F_x(x_0(t),u_0(t),t).$$

Hence, from (3.28), there exists a $\psi(t)$ such that

$$\lim_{k\to\infty} \psi(t,\varepsilon_k) = \psi(t),$$

where

$$\dot{\psi}(t) = - f_x^T(x_0(t),u_0(t),t)\psi(t)$$

$$- \int_{t_0}^{t} g_x^T(x_0(t),u_0(t),t,s)\psi(s)ds + F_x(x_0(t),u_0(t),t) \qquad (3.43)$$

for almost all t on $[t_0,t_f]$.

In addition, $\psi(t)$ satisfies

$$\psi(t_f) = 0 \qquad (3.44)$$

Now from (3.30)

$$\lim_{k\to\infty} H(\varepsilon_k, Z_0(t,\varepsilon_k), x_0(t,\varepsilon_k), u(t))$$

$$= \langle \psi(t), f(x_0(t),u(t),t) + \int_{t}^{t_f} g(x_0(t),u(t),t,s)ds$$

$$+ F(x_0(t),u(t),t) \rangle,$$

and

$$\lim_{k\to\infty} H(\varepsilon_k, \bar{z}_0(t,\varepsilon_k), x_0(t,\varepsilon_k), u_0(t,\varepsilon_k))$$

$$= < \psi(t), f(x_0(t),u_0(t),t) + \int_t^{t_f} g(x_0(t),u_0(t),t,s)ds$$

$$+F(x_0(t),u_0(t),t) > .$$

Thus the ε-maximum problem, in the limit, becomes

$$\max_{u\varepsilon U} H(\psi(t),x_0(t),u,t) = H(\psi(t),x_0(t),u_0(t),t) \tag{3.45}$$

where $\psi(t)$ is as defined by (3.43) and (3.44).

3.5 Gradient Methods

The development of numerical methods for finding the optimal control of integro-differential systems is similar to the discussion in section 2.2 and following sections. The calculation of the gradient to the Hamiltonian is somewhat different and so a full discussion of the steepest descent method is given here.

We consider the following system:

$$\dot{x}(t) = f(x(t),w(t),u(t),t) \qquad t_0 \leq t \leq t_f , \tag{3.46}$$

where $x(t)$ is an n-vector, $u(t)$ an r-vector control function and $w(t)$ is a p-vector defined by

$$w_i(t) = \int_{t_0}^{t} g_i(x(s),u(s),s,t)ds \qquad 1 \leq i \leq p . \tag{3.47}$$

We seek to minimise the function

$$J = G(x(t_f),t_f) \tag{3.48}$$

where t_f is the known terminal time. We also have the initial condition

$$x(t_0) = x_0 \tag{3.49}$$

where x_0 is known.

In the usual way, a nominal control $u^*(t)$ is chosen and the corresponding response $x^*(t)$ is derived from integrating (3.46).

We now seek an incremental control $\delta u(t)$ such that the control $u(t) + \delta u(t)$ gives an improved value for J.

Equation (3.46) is linearised about the nominal pair (x^*,u^*) to give

$$\delta\dot{x} = A(t)\delta x(t) + B(t)\delta u(t) + C(t)\delta w(t) \tag{3.50}$$

with $\delta x(t_0) = 0$,

where

$$A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}^* (x^*(t), w^*(t), u^*(t), t)$$

$$B(t) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_r} \end{pmatrix}^* (x^*(t), w^*(t), u^*(t), t)$$

$$C(t) = \begin{pmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_p} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial w_1} & \cdots & \frac{\partial f_n}{\partial w_p} \end{pmatrix}^* (x^*(t), w^*(t), u^*(t), t)$$

Also we have from (3.47) that, to first order in $\delta x, \delta u$:

$$\delta w(t) = \int_{t_0}^{t} \{F_1(t,s)\delta x(s) + F_2(t,s)\delta u(s)\}ds \tag{3.51}$$

where $$F_1(t,s) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{pmatrix}^* (x^*(s), u^*(s), s, t)$$

$$F_2(t,s) = \begin{pmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_r} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial u_1} & \cdots & \frac{\partial g_p}{\partial u_r} \end{pmatrix}^* (x^*(s), u^*(s), s, t) \; .$$

Using (3.51) in (3.50) gives

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t) + \int_{t_0}^{t} \{S_1(t,s)\delta x(s) + S_2(t,s)\delta u(s)\}ds \;, \qquad (3.52)$$

where

$$S_i(t,s) = C(t)F_i(t,s) \qquad i = 1, 2.$$

We now introduce the adjoint system of equations:

$$\dot{\lambda}(t) = - A^T(t)\lambda(t) - \int_{t}^{t_f} S_1^T(s,t)\lambda(s)ds \qquad t_0 \leq t \leq t_f \quad (3.53)$$

with

$$\lambda(t_f) = \frac{\partial G}{\partial x}(t_f) \; .$$

Premultiplying (3.52) by $\lambda^T(t)$ and postmultiplying the transpose of (3.53) by $\delta x(t)$:

$$\lambda^T(t)\delta\dot{x}(t) = \lambda^T(t)A(t)\delta x(t) + \lambda^T(t)B(t)\delta u(t)$$

$$+ \lambda^T(t) \int_{t_0}^{t} \{S_1(t,s)\delta x(s) + S_2(t,s)\delta u(s)\}ds , \quad (3.54)$$

and

$$\dot{\lambda}^T(t)\delta x(t) = -\lambda^T(t)A(t)\delta x(t) - \int_{t}^{t_f} \lambda^T(s)S_1(s,t)\delta x(t)ds . \quad (3.55)$$

Adding (3.54) and (3.55):

$$\frac{d}{dt}\left[\lambda^T(t)\delta x(t)\right] = \lambda^T(t)B(t)\delta u(t) - \int_{t}^{t_f} \lambda^T(s)S_1(s,t)\delta x(t)ds$$

$$+ \int_{t_0}^{t} \lambda^T(t)\{S_1(t,s)\delta x(s) + S_2(t,s)\delta u(s)\}ds . \quad (3.56)$$

Integrating (3.56) between t_0 and t_f and recalling the identity

$$\int_{t_0}^{t_f} \int_{t_0}^{t} P(s,t)ds\ dt = \int_{t_0}^{t_f} \int_{t}^{t_f} P(t,s)ds\ dt , \quad (3.57)$$

we may write

$$\lambda^T(t_f)\delta x(t_f) = \int_{t_0}^{t_f} M^T(t)\delta u(t)dt$$

where

$$M(t) = B^T(t)\lambda(t) + \int_{t}^{t_f} S_2^T(s,t)\lambda(s)ds . \quad (3.58)$$

But we see that the first order change in J is given by

$$\Delta J = \left(\frac{\partial G^*}{\partial x}\right)^T(t_f)\delta x(t_f) = \lambda^T(t_f)\delta x(t_f) ,$$

so we have

$$\Delta J = \int_{t_0}^{t_f} M^T(t)\delta u(t)dt \quad . \tag{3.59}$$

Thus we see that if we choose an incremental control

$$\delta u(t) = - \varepsilon M(t),$$

and ε is chosen small enough to ensure the validity of the linearisation (3.52), then there will be a decrease in the performance index J.

The discussion on methods of choosing ε at the end of section 2.2 again applies here. Similarly, the conjugate gradient method can be developed in a similar fashion to section 2.3 and so no further discussion will be given here. We will continue by deriving a second order method for integro-differential systems.

3.6 Second Order Methods

We will extend the second order methods, described in section 2.5, to integro-differential systems. Such an extension has been given by Connor and Hood [38], but we consider here a more general system.

We consider the system represented by the integro-differential equation:

$$\dot{x}(t) = f(x(t),w(t),u(t),t) \qquad t_0 \leqslant t \leqslant t_f \tag{3.60}$$

with

$x(t_0) = x_0$ specified, and where $x(t)$ is an n-vector, $u(t)$ an r-vector control function and $w(t)$ is as defined in (3.47).

It is desired to minimise the functional

$$J = G\{x(t_f),t_f\} + \int_{t_0}^{t_f} F(x(t),u(t),t)dt \quad . \tag{3.61}$$

It is assumed that each element of $u(t)$ is measurable and square integrable on $[t_0,t_f]$. Moreover it is assumed that $f(x,u,t)$, $g(x,u,t,s)$ and $F(x,u,t)$ are defined and continuous for all x in R^n, u in R^r, w in R^p and s,t in R and have continuous first and second derivatives. The function $G(x,t_f)$ is assumed continuous for all x in R^n and has continuous first and second derivatives.

We adjoin the dynamic constraint (3.60) to (3.61) in the usual way and we seek to minimise the functional

$$J_A = G(x(t_f),t_f) + \int_{t_0}^{t_f} \Big[F(x(t),u(t),t) - \lambda^T(t)\dot{x}(t)$$

$$+ \lambda^T(t)f(x(t),w(t),u(t),t)\Big]dt \quad . \tag{3.62}$$

Taking variations $\xi(t)$, $\eta(t)$ in $x(t)$, $u(t)$ respectively gives to second order in the variations :

$$\Delta J_A = \langle G_x(x(t_f),t_f),\xi(t_f)\rangle + \tfrac{1}{2}\langle G_{xx}(t_f)\xi(t_f),\xi(t_f)\rangle$$

$$+ \int_{t_0}^{t_f} \{\langle F_x(x,u,t),\xi(t)\rangle + \langle F_u(x,u,t),\eta(t)\rangle\}dt$$

$$+ \int_{t_0}^{t_f} \{\langle f_x^T(x,w,u,t)\lambda(t),\xi(t)\rangle + \int_{t_0}^{t} \langle S_1^T(t,s)\lambda(t),\xi(s)\rangle ds\}dt$$

$$+ \int_{t_0}^{t_f} \{<f_u^T(x,w,u,t)\lambda(t),\eta(t)> + \int_{t_0}^{t} <S_2^T(t,s)\lambda(t),\eta(s)>ds\}dt$$

$$+ \tfrac{1}{2} \int_{t_0}^{t_f} \{<H_{xx}(x,w,u,t)\xi(t),\xi(t) \quad +$$

$$<H_{ww}(x,w,u,t)\delta w(t),\delta w(t)> + <H_{uu}(x,w,u,t)\eta(t),\eta(t)>$$

$$+ 2<H_{xu}(x,w,u,t)\eta(t),\xi(t)> + 2<H_{uw}(x,w,u,t)\delta w(t),\eta(t)>$$

$$+ 2<H_{xw}(x,w,u,t)\delta w(t),\xi(t)>\}dt - <\lambda(t_f),\xi x(t_f)>$$

$$+ \int_{t_0}^{t_f} <\dot{\lambda}(t),\xi x(t)>dt \quad , \tag{3.63}$$

where

$$H(x(t),w(t),u(t),t) = F(x(t),u(t),t) + \lambda^T(t)f(x(t),w(t),u(t),t)$$

and $w(t)$, $S_1(t,s)$ and $S_2(t,s)$ are as defined in (3.51) and (3.52).

We may simplify (3.63) further by requiring that:

$$\dot{\lambda}(t) = - F_x(x,u,t) - f_x^T(x,w,u,t)\lambda(t) - \int_{t}^{t_f} S_1^T(s,t)\lambda(s)ds \tag{3.64}$$

and

$$\lambda(t_f) = G_x(x(t_f),t_f) \; . \tag{3.65}$$

Using (3.64) and (3.65) in (3.63), we see that we have to minimise

$$\Delta J_A = \tfrac{1}{2}<G_{xx}(t_f)\xi(t_f),\xi(t_f)> + \int_{t_0}^{t_f} <F_u(x,u,t),\eta(t)>dt$$

$$+ \int_{t_0}^{t_f} \{<f_u^T(x,w,u,t)\lambda(t),\eta(t)> + \int_{t_0}^{t} <S_2(t,s)\lambda(t),\eta(s)>ds\}dt$$

$$+ \tfrac{1}{2} \int_{t_0}^{t_f} \{<H_{xx}(x,w,u,t)\xi(t),\xi(t)>$$

$$+ <H_{ww}(x,w,u,t)\delta w(t),\delta w(t)> + <H_{uu}(x,w,u,t)\eta(t),\eta(t)$$

$$+ 2<H_{xu}(x,w,u,t)\eta(t),\xi(t)> + 2<H_{uw}(x,w,u,t)\delta w(t),\eta(t)>$$

$$+ 2<H_{xw}(x,w,u,t)\delta w(t),\xi(t)>\}dt \qquad (3.66)$$

subject to

$$\dot{\xi}(t) = f_x(x,w,u,t)\xi(t) + f_u(x,w,u,t)\eta(t)$$

$$+ \int_{t_0}^{t} \{S_1(t,s)\xi(s) + S_2(t,s)\eta(s)\}ds \qquad (3.67)$$

(see derivation of equation (3.52))

and

$$\xi(t_0) = 0 \ . \qquad (3.68)$$

From (3.51)

$$\delta w(t) = \int_{t_0}^{t} \{F_1(t,s)\xi(s) + F_2(t,s)\eta(s)\}ds \ .$$

But we can show that we may write the solution, $\xi(t)$, of (3.67) and (3.68) as

$$\xi(t) = \int_{t_0}^{t} M(\sigma,t)\eta(\sigma)d\sigma$$

where $M(\sigma,t)$ is an n x m matrix (see appendix B).

Therefore

$$\delta w(t) = \int_{t_0}^{t} \{F_1(t,s) \int_{t_0}^{s} M(\sigma,s)\eta(\sigma)d\sigma + F_2(t,s)\eta(s)\}ds$$

which we may rewrite as

$$\delta w(t) = \int_{t_0}^{t} \{\int_{s}^{t} F_1(t,\sigma)\ M(s,\sigma)d\sigma + F_2(t,s)\}\eta(s)\ ds\,. \tag{3.69}$$

We write this in operator form as

$$\delta w(t) = L_2\eta[t]\,. \tag{3.70}$$

In a similar fashion we may write

$$\xi(t) = L_1\eta[t]\,. \tag{3.71}$$

If we let

$$\hat{G}(t) = H_{xx}(t) + G_{xx}(t_f)\delta(t-t_f)$$

where $\delta(t)$ is the Dirac function, then we may rewrite the problem defined by (3.66)-(3.68) as:

Find the $\eta(t)$, $t_0 \leqslant t \leqslant t_f$, which minimises

$$2\Delta J_A = |(R + H_{uu})\eta,\eta| + 2|H_{uu},\eta| \tag{3.72}$$

where $|\cdot\,,\,\cdot|$ denotes the inner product in the control space defined by

$$|\alpha,\beta| = \int_{t_0}^{t_f} \alpha^T(t)\beta(t)dt \,, \tag{3.73}$$

and where

$$R = L_1^*\hat{G}L_1 + L_2^*H_{ww}L_2 + 2L_1^*H_{xw}L_2 + 2H_{ux}L_1 + 2H_{uw}L_2 \,, \tag{3.74}$$

and L* denotes the adjoint operator of L with respect to the inner product (3.73), and is derived in appendix C.

We can now follow the arguement given in Chapter 2 from equation (2.49) onwards and derive the two second order techniques defined by:

$$\eta_{n+1} = -H_{uu}^{-1}H_u - \tfrac{1}{2}H_{uu}^{-1}(R + R^*)\eta_n \tag{3.75}$$

and

$$\eta_{n+1} = \eta_n - \left[H_{uu} + \theta I\right]^{-1}\left[H_u + \tfrac{1}{2}(P + P^*)\eta_n\right] \tag{3.76}$$

where $P = R + H_{uu}$.

CHAPTER IV

Some Other Hereditary Systems

4.1 Introduction

We give here details of some results on hereditary processes which are not covered by the earlier chapters. Probably the most important of these systems are neutral systems which, as we saw in chapter 2, arise naturally from applying Jacobson's transformation technique to time lag systems. Several results on neutral systems have been presented by Connor. In [48] he derives a gradient method for neutral systems, and in [50] gives the results of applying this gradient method to some examples of neutral systems. In [51], an extension of the ε-method to neutral systems is given and in [52] the time optimal control of neutral systems with amplitude and rate limited controls is considered.

Aggarwal [53] discusses the feedback control of linear systems with distributed delay. He shows that this type of system can be used to represent linear time lag systems. He goes on to compare his feedback control for linear time lag systems with optimal controls obtained numerically from gradient type methods.

An extension of the gradient method to systems governed by integral equations has been given by Connor and Hood [54]. The problem considered is as follows:

find the control $u(t)$ minimising

$$J = \int_{t_0}^{t_f} F(x(t),u(t),t)dt \tag{4.1}$$

subject to

$$x(t) = f(t) + \int_{t_0}^{t} K(t,x(s),u(s),s)ds \ , \tag{4.2}$$

where the final time t_f, is specified by the scalar stopping condition

$$\int_{t_0}^{t_f} G(x(s),u(s)s)ds = \beta \ , \tag{4.3}$$

where $\beta > 0$ is a specified constant and $G(x(s),u(s),s)$ is positive for $t_0 < s < \infty$. The state and control are assumed to be scalar, although the extension to vectors is straightforward, and $f(t)$,

$K(x,u,t)$, $\frac{\partial K}{\partial x}(x,u,t)$, $F(x,u,t)$, $\frac{\partial F}{\partial x}(x,u,t)$, $G(x,u,t)$ and

$\frac{\partial G}{\partial x}(x,u,t)$ are all considered to be continuous in all their arguments.

Integral systems arise naturally from integro-differential systems in the following way:

consider the scalar equation

$$\dot{x}(t) = f(x(t),u(t),w(t),t) \tag{4.4}$$

$$x(t_0) = x_0$$

and

$$w(t) = \int_{t_0}^{t} g(x(s),u(s),s,t)ds \ . \tag{4.5}$$

We may write (4.4) as

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),u(s),w(s),s)ds \tag{4.6}$$

Equations (4.5) and (4.6) are a pair of coupled integral equations and the numerical techniques for the solution of integral equations may be applied. Mocarsky [55] examines the convergence of step by step methods of solution of systems of the form:

$$w(t) = \int_{t_0}^{t} g(x(s),s,t)ds \tag{4.7}$$

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),w(s),s)ds . \tag{4.8}$$

Obviously systems of ordinary differential equations may be written in the form of an integral equation.

4.2 The gradient method for systems governed by integral equations

Consider the problem of finding the control which minimises

$$J = \int_{t_0}^{t_f} F(x(t),u(t),t)dt \tag{4.9}$$

subject to

$$x(t) = f(t) + \int_{t_0}^{t} K(t,x(s),u(s),s)ds \tag{4.10}$$

where f(t) is continuous for $t_0 \leq t \leq t_f$ and $F(x,u,t)$, $K(t,x,u,s)$, $\frac{\partial K}{\partial x}$ and $\frac{\partial F}{\partial x}$ are continuous in all their arguments. We consider the case of the terminal time, t_f, being fixed. The state, x(t), is an n-vector and the control, u(t), is an r-vector.

We take a nominal pair (x^*,u^*) and consider perturbations $(\xi(t),\eta(t))$ about this nominal trajectory.

Linearising equation (4.10):

$$\xi(t) = \int_{t_0}^{t} \{A(t,s)\xi(s) + B(t,s)\eta(s)\}ds \tag{4.11}$$

where $A(t,s) = \left[\partial K_i/\partial x_j\right](t, x^*(s), u^*(s), s)$

$$B(t,s) = \left[\partial K_i/\partial u_j\right](t, x^*(s), u^*(s), s)$$

and A and B are evaluated along the nominal trajectory. Defining the adjoint variable, $\lambda(t)$, as the solution of

$$\lambda(t) = F_x(t) + \int_{t}^{t_f} A^T(s,t)\lambda(s)ds \tag{4.12}$$

we see that the first order variation in the performance index due to the perturbation (ξ,η) is given by

$$\Delta J = \int_{t_0}^{t_f} \{F_x^T(t)\xi(t) + F_u^T(t)\eta(t)\}dt\ . \tag{4.13}$$

Multiplying (4.11) by $\lambda^T(t)$ and integrating over $[t_0,t_f]$

$$\int_{t_0}^{t_f} \lambda^T(t)\xi(t)dt = \int_{t_0}^{t_f} \int_{t_0}^{t} \{\lambda^T(t)A(t,s)\xi(s)$$

$$+ \lambda^T(t)B(t,s)\eta(s)\}ds\ dt\ . \tag{4.14}$$

Multiplying (4.12) by $\xi(t)$ and integrating over $[t_0,t_f]$

$$\int_{t_0}^{t_f} \lambda^T(t)\xi(t)dt = \int_{t_0}^{t_f} \{F_x^T(t)\xi(t)$$

$$+ \int_{t_0}^{t_f} \lambda^T(t)A(s,t)\xi(s)ds\}dt \, . \qquad (4.15)$$

Subtracting (4.14) from (4.15) and using the following identity

$$\int_{t_0}^{t_f} \int_{t_0}^{t} \lambda^T(t)A(t,s)\xi(s)ds \ dt$$

$$= \int_{t_0}^{t_f} \int_{t}^{t_f} \lambda^T(s)A(s,t)\xi(t)ds \ dt \, , \qquad (4.16)$$

we have

$$\int_{t_0}^{t_f} F_x^T(t)\xi(t)dt = \int_{t_0}^{t_f} \int_{t_0}^{t} \lambda^T(t)B(t,s)\eta(s)ds \ dt \, ; \qquad (4.17)$$

so by substituting (4.17) into (4.13) we have

$$\Delta J = \int_{t_0}^{t_f} \{F_u^T(t)\eta(t) + \int_{t_0}^{t} \lambda^T(t)B(t,s)\eta(s)ds\}dt \, , \qquad (4.18)$$

or using the identity (4.16)

$$\Delta J = \int_{t_0}^{t_f} \{F_u(t) + \int_{t}^{t_f} \lambda^T(s)B(s,t)ds\}\eta(t)dt \qquad (4.18)$$

and so, in the usual manner, we have the direction:

$$-\{F_u(t) + \int_t^{t_f} B^T(s,t)\lambda(s)ds\}$$

as the direction of steepest descent / which can be used to generate either a conjugate gradient method or a steepest descent method as described in the previous chapters.

4.3 The Gradient Method for Neutral Systems

This will not be described in detail as it is basically similar to the gradient methods for time lag and integro-differential systems, although the adjoint system is unusual.

The problem considered in [48] is that of minimising the scalar performance index

$$J = \phi[x(t_f),t_f] \tag{4.19}$$

where

$$\dot{x}(t) = f(x(t),x(t-\tau),\dot{x}(t-\tau),u(t),t) \qquad t_0 \leq t \leq t_f \tag{4.20}$$

$$x(t) = X(t) \qquad t_0-\tau \leq t \leq t_0$$

where $\tau > 0$ is a known constant, $X(t)$ is continuously differentiable, and the final time t_f is defined by the scalar stopping condition

$$\Omega[x(t_f),t_f] = 0 . \tag{4.21}$$

The state equation is linearized about the nominal control and state in the usual manner to

$$\dot{\xi}(t) = A_1(t)\xi(t) + A_2(t)\xi(t-\tau) + A_3(t)\dot{\xi}(t-\tau) + B(t)\eta(t) \tag{4.22}$$

where

$$A_1(t) = \left[\frac{\partial f_i}{\partial x_j}\right](x^*(t), x^*(t-\tau), \dot{x}^*(t-\tau), u(t), t)$$

$$A_2(t) = \left[\frac{\partial f_i}{\partial x_j(t-\tau)}\right](x^*(t), x^*(t-\tau), \dot{x}^*(t-\tau), u(t), t)$$

$$A_3(t) = \left[\frac{\partial f_i}{\partial \dot{x}_j(t-\tau)}\right](x^*(t), x^*(t-\tau), \dot{x}^*(t-\tau), u(t), t)$$

and

$$B(t) = \left[\frac{\partial f_i}{\partial u_j(t)}\right](x^*(t), x^*(t-\tau), \dot{x}^*(t-\tau), u(t), t).$$

A_1, A_2, A_3 and B are evaluated along the nominal trajectory.

We define the adjoint system of equations as:

$$\dot{\lambda}(t) = -A_1^T(t)\lambda(t) - A_2^T(t+\tau)\lambda(t+\tau)$$

$$+ \frac{d}{dt}\left[A_3^T(t+\tau)\lambda(t+\tau)\right] \quad . \tag{4.23}$$

We define $\lambda_\phi(t)$ to be the solution of (4.23)

with $\lambda_\phi(t) = 0 \qquad t > t_f$ (4.24)

$$\lambda_\phi(t_f) = \left[\frac{\partial \phi}{\partial x}\right]_{t=t_f}$$

and $\lambda_\Omega(t)$ to be the solution of (4.23)

with $\lambda_\Omega(t) = 0 \qquad t > t_f$

$$\lambda_\Omega(t_f) = \left[\frac{\partial \Omega}{\partial x}\right]_{t=t_f} \quad . \tag{4.25}$$

From equation (4.22) and (4.23)

$$\frac{d}{dt}\left[\lambda^T(t)\xi(t)\right] = \lambda^T(t)B(t)\eta(t) + p(t)\dot{\xi}(t-\tau) + \dot{p}(t+\tau)\xi(t) \tag{4.26}$$

where $$p_\phi(t) = \lambda_\phi^T(t)A_3(t) .$$

Integrating (4.26) we obtain

$$\lambda^T(t_f)\xi(t_f) = \int_{t_0}^{t_f}\lambda^T(t)B(t)\eta(t)dt + \int_{t_0}^{t_f}\frac{d}{dt}\left[p(t)\xi(t-\tau)\right]dt . \tag{4.27}$$

But we see that the first order change in the performance index is

$$\Delta\phi = \frac{\partial\phi}{\partial x}^T \delta x + \dot{\phi}\Delta t_f$$

and the first order change in the stopping criterion is

$$\Delta\Omega = \frac{\partial\Omega}{\partial x}^T \delta x + \dot{\Omega}\Delta t_f .$$

But we require $\Delta\Omega$ to be zero so we may write

$$\Delta\phi = \left(\frac{\partial\phi}{\partial x} - \frac{\dot{\phi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right)^T \delta x(t_f) . \tag{4.28}$$

From our definitions of $\lambda_\phi(t)$ and $\lambda_\Omega(t)$ we may write (4.28) as

$$\Delta\phi = \left(\lambda_\phi(t_f) - \frac{\dot{\phi}}{\dot{\Omega}}\lambda_\Omega(t_f)\right)^T \delta x(t_f)$$

or by (4.17)

$$\Delta\phi = \int_{t_0}^{t_f} \lambda_{\phi\Omega}^T(t)B(t)\eta(t)dt + \int_{t_0}^{t_f} \frac{d}{dt}\left[p_{\phi\Omega}(t)\xi(t-\tau)\right]dt \tag{4.29}$$

where

$$\lambda_{\phi\Omega}(t) = \left[\lambda_\phi(t) - \frac{\dot{\phi}}{\dot{\Omega}}\bigg|_{t_f} \lambda_\Omega(t)\right] .$$

Connor [48] derives his control perturbation from equation (4.29), and considers the possibility of discontinuities in $\dot{x}(t)$, and hence in the second integral on the right hand side of equation (4.29). The second integral may however be determined as a function of $\eta(t)$ by writing the solution of (4.22) in terms of the transition matrix, as is done in the second order methods described in chapters two and three.

CHAPTER V

Results and Conclusions

5.1 Introduction

We give here details of the application of some of the algorithms described in earlier chapters to particular optimal control problems. The numerical work has been done using the Loughborough University I.C.L. 1904A computer.

We will consider first the results obtained for time lag systems. For these systems, a fourth order Runge Kutta integration technique was used to solve the differential-difference equations which occur in the algorithms. A convergence criterion was not used in these examples. Instead, the algorithm was run for a number of iterations large enough to guarantee convergence. This makes comparison of methods simpler.

Unless stated otherwise, steepest descent refers to the technique incorporating the one dimensional search for the optimum along successive directions of steepest descent. The conjugate gradient method with restart has restart after five iterations. In some problems convergence has been obtained in under five iterations or the restart has given similar results to the conjugate gradient method and so the restart results are not given.

5.2 Differential-Difference Systems

Problem 1

Minimise

$$J = 9x_1^2(t_f) + 6x_1(t_f)x_2(t_f) + 3x_2^2(t_f)$$

$$+ \int_0^{t_f} \left[10x_1^2(t) + 10x_2^2(t) + u^2(t)\right] dt \tag{5.1}$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{5.2}$$

$$\dot{x}_2(t) = (1-x_1^2(t))x_2(t) - x_1(t) - x_1(t-\tfrac{1}{2}) + u(t) \tag{5.3}$$

with initial conditions

$$x_1(t) = 1 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0 \tag{5.4}$$

$$x_2(t) = 0 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0 \tag{5.5}$$

and the final time is given as:

$$t_f = 1 \ .$$

This example is from McKinnon who used his own second order method to find an optimal control for this system. For comparison the steepest descent and conjugate gradient methods of chapter 2 were applied to this problem. All used an initial control

$$u(t) \equiv 0 \qquad 0 \leqslant t \leqslant 1$$

Iteration Number	Value of Performance Index J		
	McKinnon's Method	Steepest Descent	Conjugate Gradient
0	42.76	42.76	42.76
1	11.8996	13.667	13.667
2	11.70	12.957	12.82
3	11.683	12.536	11.725
4	11.683	12.225	11.696
7	-	11.821	11.671
10	-	11.693	11.671

The optimal trajectory given by the conjugate gradient method is shown in fig. 1.

Problem 2

Minimise

$$J = 9x_1^2(t_f) + 6x_1(t_f)x_2(t_f) + 3x_2^2(t_f)$$

$$\int_0^{t_f} \left[10x_1^2(t) + 10x_2^2(t_f) + u^2(t)\right] dt \tag{5.6}$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{5.7}$$

$$\dot{x}_2(t) = -x_2(t) - x_2(t-\tfrac{1}{2}) + u(t) \tag{5.8}$$

and the initial conditions

$$x_1(t) = 1 \qquad -\tfrac{1}{2} \leq t \leq 0 \tag{5.9}$$

$$x_2(t) = 0 \qquad -\tfrac{1}{2} \leq t \leq 0 \tag{5.10}$$

with final time specified:

$$t_f = 1 \quad .$$

The results of applying the steepest descent and conjugate gradient methods to this problem may be summarized as follows for an initial control $u(t) \equiv 1$.

Iteration Number	Performance Index J	
	Steepest Descent	Conjugate Gradient
0	40.6289	40.6289
1	10.6656	10.6656
2	10.4031	10.3821
4	10.3875	10.3658
6	10.3661	10.3658

The optimal trajectory calculated by the conjugate gradient method for this problem is plotted in figure 2.

Problem 3

This is the same as problem 2 but with the additional constraint:

$$x_2(t) \geqslant -0.3 \tag{5.11}$$

This problem is also discussed by McKinnon [27] and he uses his second order method to synthesize an optimal control for this problem. The inequality constraint (5.11) is allowed for by adding a penalty term

$$\int_0^1 \left(\frac{x_2(t)}{0.3} \right)^{2\omega} dt \tag{5.12}$$

to the performance index (5.6).

The computation is started with $\omega = 2$ increasing the value to $\omega = 3$ after three iterations. The constraint imposed by the penalty term (5.12) is in fact

$$-0.3 \leqslant x_2(t) \leqslant 0.3$$

but the upper limit is not approached.

The conjugate gradient and steepest ascent methods were used on this problem with the same penalty term.

From an initial control

$$u(t) \equiv 0$$

the following results were obtained.

Iteration Number	Value of Performance Index J		
	McKinnon's Method	Steepest Descent	Conjugate Gradient
0	19.000	19.000	19.000
1	16.0796	12.7620	12.7620
2	12.2832	12.513	12.4991
3	12.0610	12.2863	12.2051
4	12.0202	12.1682	12.1187
5	12.0187	12.0931	12.0172

The optimal trajectory given by the conjugate gradient method for this problem is shown in fig. 3.

An alternative approach is to apply Jacobson's transformation technique to this example. The transformed problem becomes: minimise

$$J = 9x_1^2(t_f) + 6x_1(t_f) + 3x_2^2(t_f)$$

$$+ \int_0^{t_f} \left[10x_1^2(t) + 10x_2^2(t) + (x_3(t)m(t) + x_2(t) + x_2(t-\tfrac{1}{2}))^2\right] dt \quad (5.13)$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{5.14}$$

$$\dot{x}_2(t) = x_3(t)m(t) \tag{5.15}$$

$$\dot{x}_3(t) = m(t) \tag{5.16}$$

with initial conditions:

$$x_1(t) = 1.0 \qquad -\tfrac{1}{2} \leq t \leq 0$$

$$x_2(t) = 0.0 \qquad -\tfrac{1}{2} \leq t \leq 0$$

$$x_3(t) = \sqrt{0.6} \qquad -\tfrac{1}{2} \leq t \leq 0$$

and $\quad t_f = 1$

The control u(t) in the untransformed problem is related to m(t) by:

$$u(t) = x_3(t)m(t) + x_2(t) + x_2(t-\tfrac{1}{2})$$

The results for the transformed problem, starting with initial control $m(t) \equiv 1$, were:

Iteration Number	Performance Index J		
	Steepest Descent	Conjugate Gradient	With Restart
0	66.1964	66.1964	66.1964
2	11.4946	11.5006	11.5006
4	11.2887	11.2851	11.2851
6	11.2714	11.2713	11.2710
8	11.2690	11.2661	11.2657
10	11.2673	11.2654	11.2654

The optimal trajectory calculated by the conjugate gradient method is shown in figure 4.

The interior penalty function technique was also applied to this problem by adding a term

$$\int_0^{t_f} \frac{\varepsilon}{x_2(t) + 0.3}\, dt \tag{5.17}$$

to the performance index (5.6). Care had to be taken, when this technique was used, that the constraint boundary was not violated between time step points. It was thus necessary to monitor the steps taken along each search direction. Because of this, it was pointless using a one dimensional search for the optimum along each of the search directions. Consequently only one step was taken along each search direction unless the constraint was violated or the performance index increased in the value. The step length was repeatedly halved until the constraint remained unviolated and the value of the performance was decreased.

The results obtained with $u(t) \equiv 0$

Iteration	Value of Performance Index			
Number	ε = .1	ε = .01	ε = .001	ε = .0001
0	19.3333	19.0333	19.0033	19.0003
1	14.9376	12.7793	12.4644	14.8642
2	14.6543	12.7742	12.4121	12.8159
3	14.6179	12.7713	12.4028	12.3433
4	14.5441	12.6893	12.3648	12.2854
5	14.0770	12.4689	12.3535	12.2756
6	13.9376	12.4385	12.3138	12.2546
7	13.6977	12.3570	12.3099	12.0888

The optimal state and control trajectories for ε = .0001 are shown in figure 5.

Finally, the exterior penalty function technique was applied to this problem. A term

$$\int_0^{t_f} S(x_2)(x_2(t) + 0.3)^2 \, dt \tag{5.18}$$

where

$$S(x_2) = 0 \qquad x_2(t) + 0.3 \geq 0$$

$$S(x_2) = 10^4 \qquad x_2(t) + 0.3 < 0$$

was added to the performance index (5.6).

With the exterior penalty function, we do not need to monitor the constraint boundary; consequently the steepest descent and conjugate were used with the following results:

Initial control $\equiv 0$

Iteration Number	Value of Performance Index		
	Steepest Descent	Conjugate Gradient	With Restart
0	19.000	19.000	19.000
1	15.515	15.515	15.515
2	13.337	13.755	13.755
3	12.803	12.728	12.728
4	12.429	12.338	12.338
5	12.246	12.198	12.198
6	12.202	12.179	12.175
7	12.185	12.176	11.868
8	12.018	12.173	11.742
9	11.946	12.168	11.669
10	11.940	12.158	11.640

The optimal trajectory generated by the restart method is shown in figure 6.

Problem 4

Minimise

$$J = 6x_1^2(t_f) + 2x_1(t_f)x_2(t_f) + x_2^2(t_f) + \int_0^{t_f} [x_1^2(t) + x_2^2(t) + u^2(t)]\,dt \tag{5.19}$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{5.20}$$

$$\dot{x}_2(t) = (1-x_1^2(t))x_2(t)-x_1(t)-x_1(t-\tfrac{1}{2})+u(t) \tag{5.21}$$

with initial conditions

$$x_1(t) = 1.0 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0$$

$$x_2(t) = 0.0 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0$$

and the state variable inequality constraint

$$x_2(t) \geqslant 0.5 - 2.0(t-1\tfrac{1}{2})^2 \tag{5.22}$$

The final time is specified

$$t_f = 3$$

Using Jacobson's transformation technique, this problem was transformed to:

Choose the control m(t) minimising

$$J = 6x_1^2(t_f) + 2x_1(t_f)x_2(t_f) + x_2^2(t_f) + \int_0^{t_f} [x_1^2(t) + x_1^2(t) + u^2(t)]\,dt \tag{5.23}$$

subject to

$$\dot{x}_1(t) = x_2(t) + x_2(t-\tfrac{1}{2}) \tag{5.24}$$

$$\dot{x}_2(t) = -4(t-1\tfrac{1}{2}) + x_3(t)m(t) \tag{5.25}$$

$$\dot{x}_3(t) = m(t) \tag{5.26}$$

with initial conditions

$$x_1(t) = 1 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0$$

$$x_2(t) = 0 \qquad -\tfrac{1}{2} \leqslant t \leqslant 0$$

$$x_3(t) = 2\sqrt{2} \qquad -\tfrac{1}{2} \leqslant t \leqslant 0$$

where the control u(t) of the untransformed problem is given by

$$u(t) = x_1(t) + x_1(t-\tfrac{1}{2}) -4(t-1\tfrac{1}{2}) + x_3(t)m(t) - (1-x_1^2(t))x_2(t) \tag{5.27}$$

The results obtained may be summarised as:

Initial control $m(t) = 2t-3$

Iteration Number	Value of Performance Index		
	Steepest Descent	Conjugate Gradient	With Restart
0	33.9375	33.9375	33.9375
2	12.7006	12.8637	12.8637
4	11.5044	11.6679	11.6679
6	11.4959	11.2199	11.1945
8	11.4956	11.2135	11.1908
10	11.4955	11.2111	11.1264
12	11.4954	11.2038	9.8576
14	11.0505	11.1894	9.8548

The optimal trajectories generated by the steepest descent and conjugate gradient with restart are shown in figures 7 and 8.

The interior penalty function technique was also applied to the problem. The dynamics remained as defined in equations (5.20) and (5.21) but a penalty term,

$$\frac{\varepsilon}{x_2(t) - 0.5 + 2(t - 1.5)^2} ,$$

was added to the performance index (5.19).

A single step steepest descent method was used on this problem, with each step monitored to ensure the constraint boundary was not violated. The results for $\varepsilon = 10^{-6}$ are shown in the following table.

Iteration Number	Performance Index $\varepsilon = 10^{-6}$
0	35.7787
2	35.3728
4	22.3471
6	20.9810
8	20.5310
10	20.1468
12	20.0028

The value of the performance index was not affected by removal of the penalty term. These results were obtained using the following initial control,

$$\begin{aligned} u(t) &= 1.6 + 1.33t & 0 \leq t \leq 1.5 \\ u(t) &= 8.1 - 3t & 1.5 \leq t \leq 2.7 \\ u(t) &= 0 & 2.7 < t \quad . \end{aligned}$$

The trajectory corresponding to iteration number 12 is shown in fig. 9.

To test the sensitivity of the results shown in fig. 9 to the initial control, the following alternative initial control was also used.

$$u(t) = 1.6 + 2t/1.5 \qquad 0 \leq t \leq 1.3$$

$$u(t) = 7.3 - 3t \qquad 1.3 < t \leq 3$$

The results corresponding to this control are given below.

Iteration Number	Performance Index $\varepsilon = 10^{-6}$
0	23.9750
2	20.4872
4	17.1166
6	17.0761
8	16.3802
10	16.1610

and the corresponding trajectory for iteration 10 is shown in fig. 10. A modification of the monitoring system at iteration 10 allowed further progress to be made towards the minimum, and this is indicated in the following table.

Iteration Number	Performance Index $\varepsilon = 10^{-6}$
11	16.0422
12	15.7409
13	15.3185

The removal of the penalty function term does not affect the value of the performance index. The trajectory corresponding to iteration 13 is shown in fig. 10a.

A description of the monitoring systems used in the above investigations will now be given.

First Method

(1) Set new control $u_{new}(t) = u_{old}(t) - \eta H_u(t)$.

(2) Integrate state equations using $u_{new}(t)$ and check if constraint is violated.

(3) If constraint is violated set $\eta = 0.75\eta$ and go to step (1), otherwise go to step 4.

(4) Check to see if new performance index is less or equal to the old performance index. If yes, the search is ended and we calculate a new search direction, $H_u(t)$. If no, set $\eta = 0.75\eta$ and

$$u_{new}(t) = u_{old}(t) - \eta H_u(t).$$

(5) Integrate state equations and go to (4).

On the completion of this monitoring η is reset to some user input value.

Second Method

Due to the poor performance of the interior penalty function method, a modification of the above monitoring technique was attempted. This will now be described.

Suppose we have discretized the state and control functions to $x(i)$ and $u(i)$, $i = 1, \ldots,$ NSTEP, then the method is:

(1) Set new control $u_{new}(i) = u_{old}(i) - \eta H_u(i)$.

(2) Integrate state equations using $u_{new}(t)$.

(3) For $j = 1, \ldots,$ NSTEP, check if the constraint is violated at the jth point, then for $i = 1$ to j set $H_u(i) = .95 H_u(i)$.

If no constraint violation has occurred for $j = 1, \ldots,$ NSTEP, go to step (1).

Steps (4) and (5) are as in the first monitoring technique.

It is difficult to justify theoretically the above method of monitoring the control increments as (a) the "shape" of $H_u(t)$, which the gradient method calculates, is deformed, (b) the method will be strongly dependent on the discretization used. For these reasons this method was not used by itself but was used in conjunction with the original monitoring system. When the original method had converged to a control, one iteration was performed using the modified monitoring system. With this new control the method proceeds using the original monitoring. It was hoped that this would overcome the method's tendency to converge quickly to a poor control. The only problem tested where this modification gave any improvement was problem 4 using the second initial control. The trajectory generated is shown in fig. 10a. The trajectory before application of the modified monitor is shown in fig. 10.

The exterior penalty function technique was applied to this problem by adding the term,

$$\int_0^{t_f} S(x_2)(x_2(t) - 0.5 + 2(t - 1.5)^2)^2 dt \quad ,$$

to the performance index (5.19) where,

$$S(x_2) = 0 \qquad x_2(t) \geq 0.5 - 2(t - 1.5)^2$$

$$S(x_2) = 10^4 \qquad x_2 < 0.5 - 2(t - 1.5)^2$$

For an initial control:

$$u(t) = 1 + 4t/3 \qquad 0 \leq t \leq 1.5$$

$$u(t) = 7.5 - 3t \qquad 1.5 \leq t \leq 2.5$$

$$u(t) = 0.0 \qquad 2.5 \leq t$$

the results obtained are given below.

Iteration Number	Performance Index		
	Steepest Descent	Conjugate Gradient	With Restart
0	618.000	618.000	618.000
1	25.152	25.1522	25.1522
2	23.927	23.9582	23.9582
3	23.883	23.9223	23.9223
4	23.866	23.9174	23.9174
5	22.197	23.6097	23.6097
6	22.099	23.5993	16.7080
7	22.081	23.5989	15.1765
8	20.441	-	14.291
9	19.671	-	14.2629
10	19.629	-	14.2523
11	19.628	-	14.2213
12	19.626	-	14.1725
13	19.628	-	13.6596

The optimal trajectory calculated by the exterior penalty function technique with the conjugate gradient method with restart is shown in fig. 11. Removal of the penalty function for this trajectory gives a performance index of 13.6556. In general the removal of the penalty terms from the final performance index calculated made little difference to its value, although the removal of McKinnon's penalty function from the conjugate gradient solution to problem 3 reduced the performance index from 12.01722 to 11.4528.

5.3 Integro-Differential Systems

As for the differential-difference systems, the steepest descent method referred to in the following incorporates the one-dimensional search for an optimum along successive directions of steepest descent, and similarly no convergence criteria was used. The integro-differential equations were integrated by a second order Runge Kutta method (Appendix D). The ε technique was also used on the following problem. The resulting ε problem was optimised by using Powell's function minimisation technique.

Problem 1

This problem was investigated by C. E. Mueller [20].

Minimise

$$J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt \tag{5.28}$$

subject to

$$\dot{x}(t) = 1 + x(t) + u(t) + 4\int_0^t x(s)ds \tag{5.29}$$

$$x(0) = 1$$

The results of using steepest descent and conjugate gradient method on this problem, starting with a control $u(t) \equiv 1$, may be summarized as:

Iteration Number	Performance Index	
	Steepest Descent	Conjugate Gradient
0	40.4183	40.4183
1	7.2375	7.2375
2	7.1931	7.1984
3	7.1451	7.1839
4	6.8019	6.9786
5	6.8019	6.8067

The ε method was also used on this problem.

The state and control were approximated by

$$x(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 \tag{5.30}$$

$$u(t) = b_0 + b_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5$$

Taking ε as 10^{-4} gave the following values of a_i and b_i.

i	a_i	b_i
0	1	-5.016016
1	-3.040163	12.885894
2	7.060962	-16.219442
3	-5.235314	10.876497
4	3.373973	-2.529044

Using this control function and integrating equation (5.29) gives a performance index

$$J = 6.7883$$

The trajectories generated by the steepest descent and ε-method are plotted in figures 12 and 13 respectively.

Problem 2

Minimise:

$$J = \int_0^1 [x^2(t) + u^2(t)]\,dt \tag{5.31}$$

subject to

$$\dot{x}(t) = x(t) + u(t) + 4\cos\pi t + 4\int_0^t (1-t)(\tfrac{1}{2}-s)x(s)\,ds \tag{5.32}$$

with

$$x(0) = 1$$

The conjugate gradient and steepest descent techniques were applied to this problem and the following results obtained from an initial control $u(t) \equiv 1$.

Iteration Number	Performance Index	
	Steepest Descent	Conjugate Gradient
0	19.9734	19.9734
1	6.2456	6.2456
2	6.1960	6.1970
3	6.1951	6.1949
4	6.1951	6.1920

Balakrishnan's ε-method was also used on this problem with the representation (5.30) used in problem 1.

The following notation will be used:

J_ε : the optimum value of the ε-problem

i.e. $$J = J(x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)) + \frac{1}{\varepsilon}\int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon) - f(x_0,u_0,t) - \int_{t_0}^{t} g(x_0,u_0,s,t)ds||^2 dt .$$

J_A : the value of the performance index of the optimal control problem using the ε-problem state $x_0(t,\varepsilon)$,

i.e. $$J_A = J(x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)) .$$

J_B : this is a measure of the error in the approximation of the solution of the system equations,

i.e. $$J_B = \int_{t_0}^{t_f} ||\dot{x}_0(t,\varepsilon) - f(x_0,u_0,t) - \int_{t_0}^{t} g(x_0,u_0,s,t)ds||^2 dt .$$

J : the value of the performance index using the state obtained from integrating the system equations using $u_0(t,\varepsilon)$, the ε-problem control as input.

i.e. $$J = J(\hat{x}_\varepsilon, u_0(\cdot,\varepsilon)) .$$

The following results were obtained:

$\underline{\varepsilon = 0.1}$

$J_\varepsilon = 9.469632$ $J_A = 5.683853$ $J_B = .378578$

i	a_i	b_i
0	1.00	-2.985935
1	3.008772	3.015719
2	-2.022025	.247812
3	-.59691	.133773
4	-.032428	-.036405

$\underline{\varepsilon = .005}$

$J_\varepsilon = 34.02192$ $J_A = 6.761882$ $J_B = .01381$

i	a_i	b_i
0	1.00	-3.669798
1	1.28163	-.326371
2	2.162075	11.44347
3	-5.923631	-5.207725
4	2.21542	-2.312005

$\underline{\varepsilon = .0001}$

$J_\varepsilon = 127.2522 \qquad J_A = 6.85565 \qquad J_B = .01204$

i	a_i	b_i
0	1.00	-3.711
1	1.224185	-.366723
2	2.190146	11.5
3	-5.958365	-5.14017
4	2.1694	-2.5404

$\underline{\varepsilon = .000002}$

$J_\varepsilon = 5854.4987 \qquad J_A = 6.855324 \qquad J_B = .011695$

i	a_i	b_i
0	1.00	-3.696992
1	1.220504	-.435944
2	2.221066	11.570538
3	-6.008471	-5.146686
4	2.177623	-2.610154

The solutions for the controls from the ε-problems with ε = .005 and ε = 0.1 were used to obtain the state by solving the system equation. This procedure gave values of J = 6.7604 and J = 6.2210 respectively.

These solutions are shown in figures 15 and 15a. In figure 15a the state, $x_0(t,\varepsilon)$, given by the ε-problem representation is shown for comparison. The trafectory generated by the conjugate gradient method is shown in figure 14.

In addition, a least squares best fit solution, using cubic polynomials, was obtained from the conjugate gradient method solution to this problem. This was used as a starting point for the ε-method with ε = .0001 with the following result:

$\underline{\varepsilon = .0001}$

Best fit:

i	a_i	b_i
0	1.00	-3.54508
1	2.31264	6.48314
2	.77904	-4.33907
3	-2.5535	1.40715

Resulting ε-problem:

$J_\varepsilon = 3179.1195 \qquad J_A = 6.224872 \qquad J_B = .317289$

After optimisation:

$J_\varepsilon = 2801.5663 \qquad J_A = 6.726589 \qquad J_B = .279484$

i	a_i	b_i
0	1.00	-3.674138
1	2.552373	6.602254
2	.334119	-4.303243
3	-2.664663	1.410132
4	-.000297	-1.466552

Problem 3

Minimise

$$J = 6x^2(t_f) + \int_1^{t_f} (x^2 + u^2)dt \tag{5.33}$$

subject to

$$\dot{x}(t) = x(t) + u(t) - \int_1^t x \sin\left\{4\pi \frac{(s-1)(t-s)}{t^2}\right\} ds \tag{5.34}$$

with $x(1) = 1$

and $t_f = 2$

With an initial control $u(t) \equiv 1$, the following results were obtained:

Iteration Number	Performance Index	
	Steepest Descent	Conjugate Gradient
0	57.2592	57.2592
1	18.95	18.95
2	1.8294	9.0905
3	1.8293	1.7988
4	1.8293	1.7988

The trajectory generated by the conjugate gradient method is shown in figure 16.

Balakrishnan's ε-method was also used on this problem. All the runs used $\varepsilon = .0001$ with the representation:

$$x(t) = 1 + a_1(t-1) + a_2(t-1)^2 + a_3(t-1)^3 + a_4(t-1)^4$$

$$u(t) = b_0 + b_1(t-1) + b_2(t-1)^2 + b_3(t-1)^3 + b_4(t-1)^4 .$$

Starting from $a_i = b_i = 0$, except $a_1 = -0.8$, $b_0 = -2$, $b_1 = 1.75$, the following result was obtained:

$J_\varepsilon = 6.300556 \qquad J_A = 2.5906 \qquad J_B = .000371$

i	a_i	b_i
0	1.00	-2.10221
1	-1.104105	1.911175
2	.436346	-.031694
3	-.051326	-.019148
4	.020068	-.088719

When this control was used to integrate the system equation, the value of the performance index obtained was $J = 2.588$. This is shown in figure 17.

A least squares best fit approximation to the solution given by the conjugate gradient method was obtained and used as a starting point.

<u>Best fit coefficients</u>

i	a_i	b_i
0	1.00	-1.92347
1	-.8665	1.94706
2	.3975	-.36973
3	-.4248	.12339

Resulting ε-problem:

$J_\varepsilon = 184.0148 \qquad J_A = 1.915 \qquad J_B = .01821$.

After optimisation, the following results were obtained:

$J_\varepsilon = 180.95087 \qquad J_A = 1.801 \qquad J_B = .017915$

i	a_i	b_i
0	1.00	-1.924
1	-.871	1.947
2	.398	-.369
3	-.425	.123
4	0.	-.0186

A less detailed representation was used on this problem with $b_2, b_3, b_4 \equiv 0$.

Starting from $a_i, b_i = 0$ gave the following:

$J_\varepsilon = 14.9375$ $J_A = 4.5075$ $J_B = .001053$.

i	a_i	b_i
0	1.00	-1.250317
1	-.25632	.367569
2	.11588	0.0
3	-.211147	0.0
4	-.000829	0.0

Starting from $a_1 = -.8$, $b_0 = -2$, $b_1 = 1.75$:

$J_\varepsilon = 33.322$ $J_A = 2.8018$ $J_B = .003052$.

i	a_i	b_i
0	1.00	2.0804
1	-1.07538	1.903
2	.40697	0.0
3	.000156	0.0
4	.038517	0.0

CONCLUSIONS

We have examined the application of the conjugate gradient and steepest descent methods to four differential-difference systems and three integro-differential systems. Of the four differential-difference systems, two had inequality constraints. These two constrained problems were converted to unconstrained problems by means of Jacobson's transformation technique and by the interior and exterior penalty function techniques.

In addition, a penalty function due to McKinnon was used to solve problem 3. Thus there are effectively ten unconstrained problems which have been solved by the three gradient methods.

Ranking the methods by the final value of the performance index produces the following:

differential-difference systems

Problem	Position		
	Steepest Descent	Conjugate Gradient	with Restart
Problem 1	3	1 =	1 =
Problem 2	3	1 =	1 =
Problem 3 with McKinnon P.F.	3	1 =	1 =
Problem 3 with Jacobson's Trans.	3	1 =	1 =
Problem 3 with Exterior P.F.	2	3	1
Problem 4 with Jacobson's Trans.	2	3	1
Problem 4 with Exterior P.F.	2	3	1

integro-differential systems

Problem	Position		
	Steepest Descent	Conjugate Gradient	with Restart
Problem 1	1	2 =	2 =
Problem 2	3	1 =	1 =
Problem 3	Inconclusive	-	-

The most significant fact to emerge from this ranking is that the conjugate gradient with restart is the best method for all but one of the problems studied and in some cases is far superior to the other two methods, in particular problem 4 using Jacobson's transformation technique. The conjugate gradient with restart was always at least as good as the conjugate gradient method in all the examples tested, the latter seems to choose poor search directions after six or seven iterations.

Comparing these methods with McKinnon's second order method, on problem 1 the second order method is superior, but on problem 3 the second order method and the conjugate gradient method seem equally efficient.

As would be expected from three basically similar methods, the time per iteration for the steepest descent, conjugate gradient and restart are very much the same. Thus it takes the same amount of computing time to perform six iterations with a conjugate gradient as it would take with the steepest descent method. The only exception to this is when one method has converged earlier than the other, and the converged algorithm spends, at each iteration, a long time performing a fruitless one-dimensional search. This search is halted by a limit set by the programmer, and the next iteration is started.

To compare the timing of the three algorithms it is sufficient to compare the number of iterations required for convergence. For all the examples tested, after six iterations have elapsed, the performance index given by the conjugate gradient method with restart after an iteration has been completed, is lower than the performance index given by the other methods after the corresponding iteration. For the simpler problems where convergence has been reached in less than six iterations, the conjugate gradient has not restarted. The conjugate gradient does still seem superior to the steepest descent for these problems.

In addition, we have considered the ε-method for integro-differential systems. This method gave the best solution of problem 1 in section 5.3. A comparison of figures 12 and 13 shows that the ε-method gives a control whose value at t_f is closer to zero than that given by the gradient methods. The true optimal control for this problem should be zero at t_f. Had an initial control, $u(t) \equiv 0$, been chosen then the gradient methods would not have moved away from $u(t_f) = 0$ and they may have been superior to the ε-method. For problems 2 and 3 of section 5.3 the ε-method is inferior to any of the gradient methods - markedly so in the case of problem 3. The ε method did, however, have a shorter running time than the gradient methods for all the problems tested.

The performance of the ε-method when applied to problem 2 was not as good as in problem 1. Except for the $\varepsilon = .1$ solution, the performance index given by the controls generated by the ε-method were inferior to those of the gradient method solutions. For $\varepsilon = .005$ and smaller, it is obvious that a lot of effort is going into keeping J_B small, and it is tempting to think that relaxing ε might in fact improve the performance. This, superficially at least, is borne out by the solution obtained to the ε-problem with $\varepsilon = 0.1$. Closer inspection of figure 15a shows that the state trajectory given by the polynomial representation and that given by integration of the state equations, using the control given by the ε-method, are markedly different. This is emphasised by the difference between J_A and J.

In the examples tested in [30], where the true solution for the control and state to the optimal control problem was in fact a set of polynomials in t, the ε-method coupled with Powell's method performed well. The coefficients of the polynomials given by the ε-method were correct to four significant places with $\varepsilon = 10^{-5}$. In this reference, all of the integration of the performance index and the adjoined system equations was done analytically. Because of the greater complexity of the examples examined here, all of the integration in the ε-method solutions was done numerically. This obviously could lead to inaccuracies. Consequently, in addition to the normal runs, using a time step of $h = .02$, several runs were made with a timestep $h = .002$. This change of timestep size did not make any significant change to the results. However, to ensure that the numerical integration was accurate, each solution for $h = .02$ was used as a starting point for the same ε-problem, but with $h = .002$. Two iterations of Powell's method were then performed. In no instance did this procedure lead to a significant modification of the original solution.

The ε-problems, although quadratic in the fitting parameters, were obviously difficult to optimise. For instance, in the solution to problem 2 with $\varepsilon = .0001$, the solution to the ε-problem starting from $a_i = b_i = 0$, and the solution starting from the best fit to the conjugate gradient solution are obviously different. Further, examination of the $\varepsilon = .000002$ solution to problem 2 shows that this is in fact a better solution to the $\varepsilon = .0001$ solution than the one given. When applied to problem 3, Powell's method did not move far from the starting point. Consequently, in order to obtain a reasonable solution, the starting point used was a good linear approximation to the conjugate gradient solution. This strong dependence of the method on the starting point used suggests that the contours of the ε-problem are sets of ellipsoids with a large eccentricity. This, together with the usual problem in computing of rounding error, and the fact that there is probably some "noise" from the integration, could lead to poor convergence. This is also probably the reason for the poor convergence seen in problem 2.

In an attempt to cure this, some attempts were made to rescale the variables in both problems, but with no improvement in convergence.

The shorter running time of the ε-method as compared with the gradient methods is due to the comparatively lengthy numerical integration of the integro-differential equations, a problem which is increased by the fact that all of the gradient methods tested search for an optimum along each gradient direction, each search involving an integration of the state equations. Similarly, whatever optimisation procedure is used to solve the ε-problem, the performance index of the epsilon problem will have to be repeatedly evaluated, and each evaluation will involve an integration of the state equation for each set of values of a_i, b_i. However, in this case we need only integrate an explicit function of time instead of solving the state equations using, for example, a Runge Kutta method. It is possible that a one step gradient method would be faster than the gradient methods investigated. It would still be necessary to monitor the step along each gradient direction to ensure that an iteration does not give an increase of the performance index. It is also likely that the reduction in the number of searches along the gradient direction would be at least partially compensated by the increased number of iterations required.

The problems on which the epsilon method and the gradient methods have been compared are simple. It seems likely that an increase in the complexity of the problems would affect the epsilon method more adversely than it would affect the gradient methods.

An examination of the problems 3 and 4 of section 5.2 gives a comparison of the transformation and penalty function techniques that have been used. It can be seen from these problems that the Jacobson transformation technique is by far the most effective method tested for solving constrained optimisation problems. It has the additional advantages that: a) there is no need to search for a feasible initial control, b) all the trajectories generated are feasible. This contrasts with the interior penalty function technique which requires a feasible initial control; and the exterior penalty function technique which can generate optimal controls that are infeasible.

In problem 3, Jacobson's transformation gives a trajectory that quickly approaches the constraint boundary, and follows it closely up to termination. McKinnon's penalty function gives a similar trajectory and performance index, but does not follow the constraint boundary as closely as the Jacobson transformation solution. The most noticeable feature of the trajectory generated by this method is the "kink" in the control at $t = 0.5$. This kink is noticeable in McKinnon's own solution to this problem, reproduced here in fig. 3a. Both the interior and exterior penalty function methods approach the constraint boundary more slowly, but the exterior penalty function solution does follow the constraint boundary fairly closely.

For problem 4, McKinnon's penalty function technique was not applicable, and again the Jacobson transformation technique was the most successful technique used on the problem. The performance of the interior penalty function method was very disappointing, and failed, in general, to change the nominal control in any significant way (see fig. 10a for an exception to this generalization). This poor performance, together with the difficulty of selecting a suitable nominal control, suggests that the Jacobson transformation technique is of more general application than the interior penalty function method for optimal control problems with state space inequality constraints.

However, Jacobson's method cannot always be used and, in this event, the exterior penalty is the better of the two penalty function techniques. It has the advantage that it may be used in conjunction with the conjugate gradient method with restart which, on the basis of this investigation, is an effective algorithm for solving an optimal control problem. It would be useful to have a direct method of solving constrained problems, similar to that of Bryson and Denham [42] for ordinary differential equation systems, but no such extension is currently available.

Finally, the main new theoretical results obtained in this thesis are those of Chapter 3, namely the gradient method, the ε-method and a minimum principle for integro-differential systems. A conjugate gradient method for these systems is also indicated. The numerical properties of these techniques have been investigated in Chapter 5. A second order method for integro-differential equations is also presented but no numerical results have been obtained using this method. Integral systems are studied in Chapter 4, where a gradient method for these systems is proposed, but no numerical experience has yet been obtained with this approach. The problem of state space inequality constraints is investigated using Jacobson's transformation technique and the exterior and interior penalty function techniques, and a critical numerical comparison is reported in Chapter 5.

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APPENDIX A

The Conjugate Gradient Method in R_n

Consider the minimisation of the general quadratic function

$$V(x) = c + a^T x + \tfrac{1}{2} x^T H x \tag{A.1}$$

where H is a constant n x n positive definite matrix, c a scalar constant and a a constant n-vector. We wish to find the n-vector x^* which gives $V(x)$ its minimum value.

Definition

Two non-zero vectors d_i, d_j are said to be conjugate with respect to the positive definite matrix H if

$$d_i^T H d_j = 0 \quad .$$

A set of non-zero vectors $\{d_i;\ i = 1,\ldots,r;\ r < n\}$ is said to be a set of mutually conjugate vectors if

$$d_i^T H d_j = 0 \qquad \text{for all } i,j = 1,\ldots,r \qquad i \neq j\ .$$

Assume that there exists a set $\{s_i\}$ of n mutually conjugate directions, then we may prove the following:

<u>Lemma</u> The set $\{s_i\}$ spans R_n.

<u>Proof</u>

Let the set of scalars $\{\alpha_i;\ i = 1,\ldots,n\}$ be such that

$$\alpha_1 s_1 + \alpha_2 s_2 + \ldots\ldots + \alpha_n s_n = 0 \tag{A.2}$$

Multiplying (A.2) by $s_i^T H$ for $1 \leqslant i \leqslant n$

$$\alpha_i s_i^T H s_i = 0 \qquad 1 \leqslant i \leqslant n$$

But H is positive definite and s_i is non-zero, thus

$$\alpha_1 = 0 \qquad 1 \leqslant i \leqslant n$$

therefore, $\{s_i\}$ is a set of n linearly independent vectors and thus spans R_n.

We may now prove the following theorem:

<u>Theorem</u>

If each of the n mutually conjugate directions $\{s_i\}$ is used once and only once as a search direction, then the successive linear searches for a minimum along each direction will lead to the minimum of V(x) from any starting point.

<u>Proof</u>

Choose the arbitrary starting point x_0.

Form

$$x_{i+1} = x_i + \lambda_{i+1} s_{i+1} \qquad i \geq 1 \tag{A.3}$$

where λ_{i+1} is chosen so that $V(x_i + \lambda_{i+1} s_{i+1})$ is a minimum along s_{i+1}.

Thus

$$g_{i+1}^T s_{i+1} = 0 \quad , \tag{A.4}$$

where g_{i+1} is the gradient of $V(x)$ at x_{i+1}.

By (A.1) we see that

$$\begin{aligned} g_i &= a + Hx_i \\ &= a + H(x_0 + \sum_{j=1}^{i} \lambda_j s_j) \quad , \end{aligned} \tag{A.5}$$

therefore $$s_i^T g_i = s_i^T a + s_i^T H(x_0 + \sum_{j=1}^{i} \lambda_j s_j)$$

$$= s_i^T H(H^{-1}a + x_0) + \lambda_i s_i^T H s_i \quad .$$

But by (A.4) $\quad s_i^T g_i = 0 \quad ,$

therefore $$\lambda_i = - \frac{s_i^T H(H^{-1}a + x_0)}{s_i^T H s_i} \quad . \tag{A.6}$$

And so by (A.6) we may write

$$x_i = x_0 - \sum_{j=1}^{i} \frac{s_j H(H^{-1}a + x_0)}{s_j^T H s_j} \quad . \tag{A.7}$$

The set $\{s_j\}$ spans R_n (see lemma) and therefore we may write

$$v = \sum_{j=1}^{n} \alpha_j s_j \, , \tag{A.8}$$

where v is an arbitrary vector.

Multiply (A.8) by $s_i^T H$

$$s_i^T H v = \alpha_i \, s_i^T \, H \, s_i \quad , \tag{A.9}$$

so from (A.8) and (A.9)

$$v = \sum \frac{s_j^T H v}{s_j^T H s_j} \tag{A.10}$$

for any vector v.

Comparing (A.7) with (A.10) and setting i equal to n

$$x_n = x_0 - (H^{-1} a + x_0) = H^{-1} a \quad .$$

It can readily be seen that x^*, the value of x giving the minimum of $V(x)$, is $x^* = H^{-1} a$.

Thus $\qquad x_n = x^*$.

We now seek a method of generating the n conjugate directions $\{s_i\}$. We proceed as follows:

Starting at x_0, calculate the gradient, g_0, of $V(x)$.

Set $\qquad s_1 = - g_0$.

For $\qquad i = 1,2,\ldots.n, \qquad$ form

$$x_i = x_{i-1} + \lambda_i s_i$$

with λ_i such that $V(x_{i-1} + \lambda_i s_i)$ is a minimum along s_i. Calculate g_i, the gradient of $V(x)$ at x_i and set

$$s_{i+1} = - g_i + \gamma_i s_i \ .$$

We wish to choose γ_i so that

$$s_{i+1}^T H s_j = 0 \qquad j = 1,2,\ldots i \ . \tag{A.11}$$

From our definition of x_i,

$$s_i = 1/\lambda_i(x_i - x_{i-1}) \ ,$$

therefore $\quad Hs_i = 1/\lambda_i(Hx_i - Hx_{i-1})$.

Substituting from equation (A.5)

$$Hs_i = 1/\lambda_i(g_i - g_{i-1}) \ . \tag{A.12}$$

Now assume we are at stage r and we have $\{s_i;\ i = 1,\ldots r\}$, a set of known conjugate gradients, and we know that

$$s_i^T g_i = 0 \ . \tag{A.13}$$

Now, for $k < r$

$$s_k^T g_r = s_k^T(a + Hxr)$$

$$= s_k^T[a + H(x_0 + \sum_{i=1}^{k} \lambda_i s_i + \sum_{i=k+1}^{r} \lambda_i s_i)]$$

$$= s_k^T[a + H(x_k + \sum_{k+1}^{r} \lambda_i s_i)]$$

$$= s_k^T g_k + s_k^T H \sum_{k+1}^{r} \lambda_i s_i \ ,$$

By (A.13) and by the conjugacy of $\{s_i\}$, this last expression is zero, thus

$$s_k^T\, g_r = 0 \qquad k \leqslant r \quad . \tag{A.14}$$

Now $$s_i = -g_{i-1} + \gamma_{i-1}\, s_{i-1} \quad .$$

For $i < r$ we have, by (A.14)

$$s_i\, g_r = 0 = g_r^T(-g_{i-1} + \gamma_{i-1}\, s_{i-1})$$

$$= g_r^T\, g_{i-1} + \gamma_{i-1}\, g_r^T\, s_{i-1}$$

$$= -g_r^T\, g_{i-1} \quad ,$$

i.e.

$$g_r^T\, g_{i-1} = 0 \qquad i \leqslant r \, . \tag{A.15}$$

We require that

$$s_{r+1}^T\, Hs_r = 0 \, .$$

Now

$$s_{r+1} = -g_r + \gamma_r\, s_r \quad .$$

Therefore by (A.12)

$$s_{r+1}^T\, Hs_r = (-g_r + \gamma_r\, s_r)^T \;\; 1/\lambda_r(g_r - g_{r-1}) \, .$$

Expanding, we have

$$s_{r+1}^T\, Hs_r = (-g_r^T\, g_r + \gamma_r\, s_r\, g_r + g_r^T\, g_{r-1} - \gamma_r\, s_r\, g_{r-1})/\lambda_r \quad .$$

By (A.14) and (A.15)

$$s_{r+1} Hs_r = 1/\lambda_r(-g_r^T g_r - \gamma_r s_r^T g_{r-1})$$

$$= 1/\lambda_r(-g_r^T g_r - \gamma_r g_{r-1}^T(-g_{r-1} + \gamma_{r-1} s_{r-1})).$$

Thus s_{r+1} is conjugate to s_r if

$$\gamma_r = \frac{g_r^T \quad g_r}{g_{r-1}^T g_{r-1}} .$$

Finally we have to prove that, in addition to s_{r+1} being conjugate to s_r, it is conjugate to s_j, $j \leqslant r$.

$$s_{r+1}^T Hs_j = (-g_r^T + \gamma_r s_r^T)Hs_j$$

$$= -g_r^T Hs_j + \gamma_r s_r^T Hs_j .$$

By conjugacy of s_r and s_j

$$s_{r+1}^T Hs_j = -g_r^T Hs_j .$$

From (A.12)

$$s_{r+1}^T Hs_j = -g_r^T(g_j - g_{j-1})/\lambda_{j-1} .$$

Thus by (A.15)

$$s_{r+1}^T Hs_j = 0 .$$

Thus $\{s_i \; ; \; i = 1,\ldots.,r+1\}$ is a set of conjugate directions.

This method will continue to generate conjugate directions until it reaches the minimum point of $V(x)$. From the theorem proved earlier, this minimum is reached in at most n iterations.

APPENDIX B

Variation of Parameters Solution

1. Time Lag Equations

Consider the set of differential time lag equations

$$\dot{\xi}(t) = A(t)\xi(t) + B(t)\xi(t-\tau) + C(t)\eta(t) \qquad t_0 \leqslant t \leqslant t_f \;, \tag{B.1}$$

where $\xi(t)$ is a state n-vector and $\eta(t)$ a control r-vector. $A(t)$ and $B(t)$ are n x n matrices and $C(t)$ is an n x r matrix. We also have the initial condition:

$$\xi(t) = 0 \qquad t_0 - \tau \leqslant t \leqslant t_0 \;.$$

From (B.1) we may write

$$N(s,t)\dot{\xi}(s) = N(s,t)A(s)\xi(s) + N(s,t)B(s)\xi(s-\tau) + N(s,t)C(s)\eta(s) \tag{B.2}$$

where $N(s,t)$ is an n x n matrix.

Integrating (B.2) between t_0 and t

$$\int_{t_0}^{t} N(s,t)\dot{\xi}(s) = \int_{t_0}^{t} N(s,t)\left[A(s)\xi(s) + B(s)\xi(s-\tau) + C(s)\eta(s)\right]ds \;. \tag{B.3}$$

We have the following identity

$$\int_{t_0}^{t} N(s,t)\dot{\xi}(s)ds = N(t,t)\xi(t) - N(t_0,t)\xi(t_0)$$
$$\int_{t_0}^{t} \frac{\partial N}{\partial s}(s,t)\xi(s)ds \;,$$

Recalling that

$$\xi(t_0) = 0 ,$$

we may write (B.3) as

$$N(t,t)\xi(t) - \int_{t_0}^{t} \frac{\partial N}{\partial s}(s,t)\xi(s)ds$$

$$= \int_{t_0}^{t} \big[N(s,t)A(s)\xi(s) + N(s,t)B(s)\xi(s-\tau)$$

$$+ N(s,t)\eta(s)\big]ds . \tag{B.4}$$

If we set

$$N(s,t) = 0 \quad \text{for} \quad s > t , \tag{B.5}$$

we may write

$$\int_{t_0}^{t} N(s,t)B(s)\xi(s-\tau)ds = \int_{t_0}^{t} N(s+\tau,t)B(s+\tau)\xi(s)ds .$$

Equation (B.4) becomes

$$N(t,t)\xi(t) = \int_{t_0}^{t} \frac{\partial N}{\partial s}(s,t)\xi(s)ds \tag{B.6}$$

$$+ \int_{t_0}^{t} \big[N(s,t)A(s) + N(s+\tau,t)B(s+\tau)\big]\xi(s)ds$$

$$+ \int_{t_0}^{t} N(s,t)\ (s)ds .$$

If we set

$$N(t,t) = I \tag{B.7}$$

and

$$\frac{\partial N}{\partial s}(s,t) = -N(s,t)A(s) - N(s+\tau,t)B(s+\tau), \tag{B.8}$$

we may write

$$\xi(t) = \int_{t_0}^{t} N(s,t)\eta(s)ds \tag{B.9}$$

where N(s,t) satisfies (B.5),(B.7) and (B.8).

2. Integro-Differential Equations

Consider now the set of integro-differential equations

$$\dot{\xi}(t) = A(t)\xi(t) + C(t)\eta(t) + \int_{t_0}^{t} \left[B(\sigma,t)\xi(\sigma) + D(\sigma,t)\eta(\sigma)\right] d\sigma \tag{B.10}$$

for t in $[t_0,t_f]$, and

$$\xi(t_0) = 0 .$$

A(t) and B(σ,s) are n x n matrices and C(t) and D(σ,t) are n x r matrices.

We may write

$$N(s,t)\dot{\xi}(s) = N(s,t)A(s)\xi(s) + N(s,t)C(s)\eta(s) + \int_{t_0}^{s} \left[N(s,t)B(\sigma,s)\xi(\sigma) + N(s,t)D(\sigma,s)\eta(\sigma)\right] d\sigma . \tag{B.11}$$

Integrating (B.11) between t_0 and t

$$\int_{t_0}^{t} N(s,t)\dot{\xi}(s)ds = \int_{t_0}^{t} N(s,t)\left[A(s)\xi(s) + C(s)\eta(s)\right]ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} N(s,t)\left[B(\sigma,s)\xi(\sigma) + S(\sigma,s)\eta(\sigma)\right]d\sigma \; . \qquad (B.12)$$

But we may write the following identity

$$\int_{t_0}^{t}\int_{t_0}^{s} F(\sigma,s,t)d\sigma\; ds = \int_{t_0}^{t}\int_{\sigma}^{t} F(\sigma,s,t)ds\; d\sigma$$

$$= \int_{t_0}^{t}\int_{s}^{t} F(s,\sigma,t)d\sigma\; ds \; . \qquad (B.13)$$

Using (B.13) we may rewrite (B.12) as

$$\int_{t_0}^{t} N(s,t)\dot{\xi}(s)ds = \int_{t_0}^{t} \left[N(s,t)A(s) + \int_{s}^{t} N(\sigma,t)B(s,\sigma)d\sigma\right]\xi(s)ds$$

$$+ \int_{t_0}^{t} \left[N(s,t)C(s) + \int_{s}^{t} N(\sigma,t)D(s,\sigma)d\sigma\right]\eta(s)ds \; . \qquad (B.14)$$

Integrating the left hand side of (B.14), recalling that $\xi(t_0) = 0$, gives

$$N(t,t)\xi(t) = \int_{t_0}^{t} \Big[\frac{\partial N}{\partial s}(s,t) + N(s,t)A(s)$$

$$+ \int_{s}^{t} N(\sigma,t)B(s,\sigma)d\sigma\Big]\xi(s)ds$$

$$+ \int_{t_0}^{t} \left[N(s,t)C(s) + \int_{s}^{t} N(\sigma,t)D(s,\sigma)d\sigma \right] \eta(s)ds \; .$$

By setting

$$N(t,t) = I \tag{B.15}$$

$$\frac{\partial N}{\partial s}(s,t) = -N(s,t)A(s) - \int_{s}^{t} N(\sigma,t)B(s,\sigma)d\sigma \; , \tag{B.16}$$

we may write

$$\xi(t) = \int_{t_0}^{t} M(s,t)\eta(s)ds \, , \tag{B.17}$$

where

$$M(s,t) = N(s,t)C(s) + \int_{s}^{t} N(\sigma,t)D(s,\sigma)d\sigma \; .$$

APPENDIX C

Derivation of Adjoint Operators

We define the adjoint L^*, of the operator $L: H_m \to H_n$, with respect to the inner products $|\cdot,\cdot|_m$ and $|\cdot,\cdot|_n$ as the operator satisfying

$$|L^*a, b|_m = |a, Lb|_n \tag{C.1}$$

for arbitrary a, b in H_n, H_m, respectively.

The inner product used is

$$|a, b|_n = \int_{t_0}^{t_f} \langle a(t), b(t)\rangle_n dt$$

where $\langle a, b\rangle_n$ is the usual scalar product in n space.

Adjoints for time lag systems

The operators L and $\hat{L}$ are given in equation (2.43) and (2.44) as

$$(L\eta)(t) = \int_{t_0}^{t} N(\sigma,t)f_u(\sigma)\eta(\sigma)d\sigma \tag{C.2}$$

and

$$(\hat{L}\eta)(t) = \int_{t_0}^{t} N(\sigma,t-\tau)f_u(\sigma)\eta(\sigma)d\sigma \;. \tag{C.3}$$

By our definition of the adjoint operator in (C.1) we wish to find an operator L^* satisfying

$$\int_{t_0}^{t_f} <(L^*\xi)(t),\eta(t)>dt = \int_{t_0}^{t_f} \xi^T(t) \int_{t_0}^{t} N(\sigma,t)f_u(\sigma)\eta(\sigma)d\sigma \tag{C.4}$$

identically.

But we may write the right hand side of (C.4) as

$$\int_{t_0}^{t_f}\int_{t_0}^{t} \xi^T(t)N(\sigma,t)f_u(\sigma)\eta(\sigma)d\sigma\ dt$$

$$= \int_{t_0}^{t_f} \int_{\sigma}^{t_f} \xi^T(t)N(\sigma,t)f_u(\sigma)\eta(\sigma)dt\ d\sigma$$

$$= \int_{t_0}^{t_f} \int_{t}^{t_f} \xi^T(\sigma)N(t,\sigma)f_u(t)\eta(t)d\sigma\ dt$$

$$= \int_{t_0}^{t_f} \eta^T(t) \int_{t}^{t_f} f_u(t)N^T(t,\sigma)\xi(\sigma)d\sigma\ dt$$

$$= |\eta(t),(L^*\xi)(t)|$$

$$= |(L^*\xi)(t),\eta(t)| \ .$$

Thus we see that L^* is in fact the operator we are looking for, and may be written

$$(L^*\xi)(t) = \int_{t}^{t_f} f_u^T(t)N^T(t,\sigma)\xi(\sigma)d\sigma \ , \tag{C.5}$$

and similarly

$$(\hat{L}^*\xi)(t) = \int_{t}^{t_f} f_u^T(t)N^T(t,\sigma-\tau)\xi(\sigma)d\sigma \ . \tag{C.6}$$

Adjoints for Integro-Differential Systems

By an argument identical to that used above we may write

$$(L_1^*\xi)(t) = \int_t^{t_f} M^T(t,\sigma)\xi(\sigma)d\sigma \, . \tag{C.6}$$

For the adjoint of L_2 we proceed as follows

$$(L_2\eta)(t) = \int_{t_0}^{t} \{F_1(t,s)\int_s^t M(s,\sigma)d\sigma + F_2(t,s)\}\eta(s)ds \, .$$

Therefore $\qquad |\xi,\, L_2\eta|$

$$= \int_{t_0}^{t_f} \xi^T(t)\int_{t_0}^{t} \{F_1(t,s)\int_s^t M(s,\sigma)d\sigma + F_2(t,s)\}\eta(s)ds\; dt$$

$$= \int_{t_0}^{t_f} \int_s^{t_f} \xi^T(t)\{F_1(t,s)\int_s^t M(s,\sigma)d\sigma + F_2(t,s)\}\eta(s)dt\; ds$$

$$= \int_{t_0}^{t_f} \int_t^{t_f} \xi^T(s)\{F_1(s,t)\int_t^s M(t,\sigma)d\sigma + F_2(s,t)\}\eta(t)ds\; dt$$

$$= \int_{t_0}^{t_f} \eta^T(t)\int_t^{t_f} \{\int_t^s M^T(t,\sigma)F_1^T(s,t)d\sigma + F_2^T(s,t)\}\xi(s)ds\; dt \, ,$$

so we may now write

$$(L_2^*\xi)(t) = \int_t^{t_f} \{\int_t^s M^T(t,\sigma)F_1^T(s,t)d\sigma + F_2^T(s,t)\}\xi(s)ds \, . \tag{C.7}$$

APPENDIX D

Derivation of a second order Runge-Kutta method for Integro-Differential Equations

We wish to integrate an integro-differential equation of the following form:

$$\dot{x}(t) = f(x(t),w(t),t)$$

where

$$w(t) = \int_0^t g(x(s),s,t)ds$$

using a second order Runge-Kutta method.

By a Taylor series expansion we have that

$$x(t+h) = x(t) + h\dot{x}(t) + \tfrac{1}{2}h^2\ \ddot{x}(t) + h^3/3!\ x(t) + \ldots\ldots \quad . \quad (1)$$

Now

$$\dot{x}(t) = f(x,w,t) \ ;$$

$$\ddot{x}(t) = f_x(x,w,t)f(x,w,t) + f_w(x,w,t)\Big[g(x(t),t,t) + \int_0^t g_t(x(s),s,t)ds\Big] + f_t(x,w,t) \ ,$$

$$\dddot{x}(t) = f_{xx}\ f^2 + 2f_{xt}\ f + f_x\Big[f_xf + f_t + f_w(g + \int_0^t g_t\ ds)\Big]$$

$$+ 2f_{xw}\ f\Big[g + \int_0^t g_t\ ds\Big] + 2f_{wt}\Big[g + \int_0^t g_t\ ds\Big]$$

$$+ f_w\Big[g_xf + g_s + 2g_t + \int_0^t g_{tt}\ ds\Big] + f_{ww}\Big[g + \int_0^t g_t\ ds\Big]^2 + f_{tt} \ .$$

We will use the following integration formula

$$x(t+h) - x(t) = w_1 k_1 + w_2 k_2 ,$$

where

$$k_1 = hf(x(t),w(t),t)$$

$$k_2 = hf(x(t) + a_1 k_1, w^*, t+a_1 h)$$

where

$$w^* = \int_0^t g(x(s),s,t+a_1 h)ds + mhg(x+a_2 k_1, t+a_2 h, t+a_2 h) .$$

We wish to choose a_1, a_2, w_1, w_2 and m so as to best fit the true solution.

Expanding k_2 by Taylor series up to order h^3

$$k_2 = hf(x(t),w(t),t) + h\{f_x a_1 k_1 + f_w(a_1 h \int_0^t g_t \; ds + mhg(x,t,t))$$

$$+ \tfrac{1}{2}h(f_{xx} a_1^2 k_1^2 + 2f_{xt} a_1^2 hk_1 + 2f_{xw}(a_1 h \int_0^t g_t \; ds + mhg)a_1 k_1$$

$$+ f_{ww}^2 (a_1 h \int_0^t g_t \; ds + mhg)^2 + 2f_{wt}(a_1 h \int_0^t g_t \; ds + mhg)a_1 h + f_{tt} a_1^2 h^2$$

$$+ f_w(a_1^2 h^2 \int_0^t g_{tt} \; ds + 2mhg_x a_2 k_1 + 2mhg_s a_2 h + 2mhg_t a_2 h)\} .$$

We may thus write

$$w_1 k_1 + w_2 k_2 = (w_1 + w_2)hf(x,w,t)$$

$$+ w_2 h^2 [a_1 f_x f + f_w(a_1 \int_0^t g_t + mg) + a_1 f_t] + w_2 \tfrac{1}{2}h^3[f_{xx} a_1^2 f^2 + 2f_{xt} a_1^2 f$$

$$+ 2f_{xw}(a_1 \int_0^t g_t \, ds + mhg)a_1 f + f_{ww}^2(a_1 \int_0^t g_t \, ds + mg)^2$$

$$+ 2f_{wt}(a_1 \int_0^t g_t \, ds + mg)a_1 + f_{tt}a_1^2 + f_w(a_1^2 \int_0^t g_{tt} \, ds + 2mg_x a_2 f \quad (2)$$

$$+ 2mg_s a_2 + 2mg_t a_2)\Big] \; .$$

Thus by choosing w_1, w_2, a_1, a_2 and m such that:

$$w_1 + w_2 = 1 \quad (3)$$

$$w_2 a_1 = \tfrac{1}{2} \quad (4)$$

$$w_2 m = \tfrac{1}{2} \quad (5)$$

then the expansions (1) and (2) match up to order h^2. Further, if we set

$$w_2 a_1^2 = \tfrac{1}{2} \quad (6$$

$$w_2 m a_2 = 1/6 \quad (7$$

we are left with a truncation error:

$$h^3/6 \; f_x\left[f_x f + f_t + f_w(g + \int_0^t g_t \, ds)\right]$$

+ terms of higher order.

Solving (3) - (7)

$$a_1 = 2/3$$

$$a_2 = 1/3$$

$$w_1 = 1/4$$

$$w_2 = 3/4$$

$$m = 2/3$$

APPENDIX E

FLOWCHARTS FOR GRADIENT AND CONJUGATE GRADIENT METHODS

We give here a more detailed description of the programming aspects of the methods used. Sections A and C of the flowchart describe the generation of the search directions for the conjugate gradient method, section B describes the 1-dimensional search used. This search routine consisted of stepping along the search direction until the minimum was bracketed, and then using cubic interpolation to locate the minimum.

No details are given in the flowcharts of the integration routines. For the time-delay systems, a fourth order Runge-Kutta method was used, whereas for the integro-differential systems the second order Runge-Kutta method described in Appendix D was used.

The following flowchart describes the conjugate gradient method. The gradient method used can be generated by setting $\beta_i = 0$ throughout.

Conjugate gradient method

Set iteration counter $i = 1$,
Set $\beta_0 = 0$, $s_0(t) = 0$

(A)

(A)

Use nominal control $u_i(t)$

integrate the state equations
and evaluate performance index.
Store value of performance index
as J_1.

Integrate adjoint equations
and hence evaluate the
gradient: $g_i(t)$

Set: $\beta_{i-1} = 0$, $i = 1$

$\beta_{i-1} = \langle g_i, g_i \rangle / \langle g_{i-1}, g_{i-1} \rangle$, $i > 1$.

Calculate the conjugate
gradient direction:

$s_i = -g_i + \beta_{i-1} s_{i-1}$

(B)

(C)

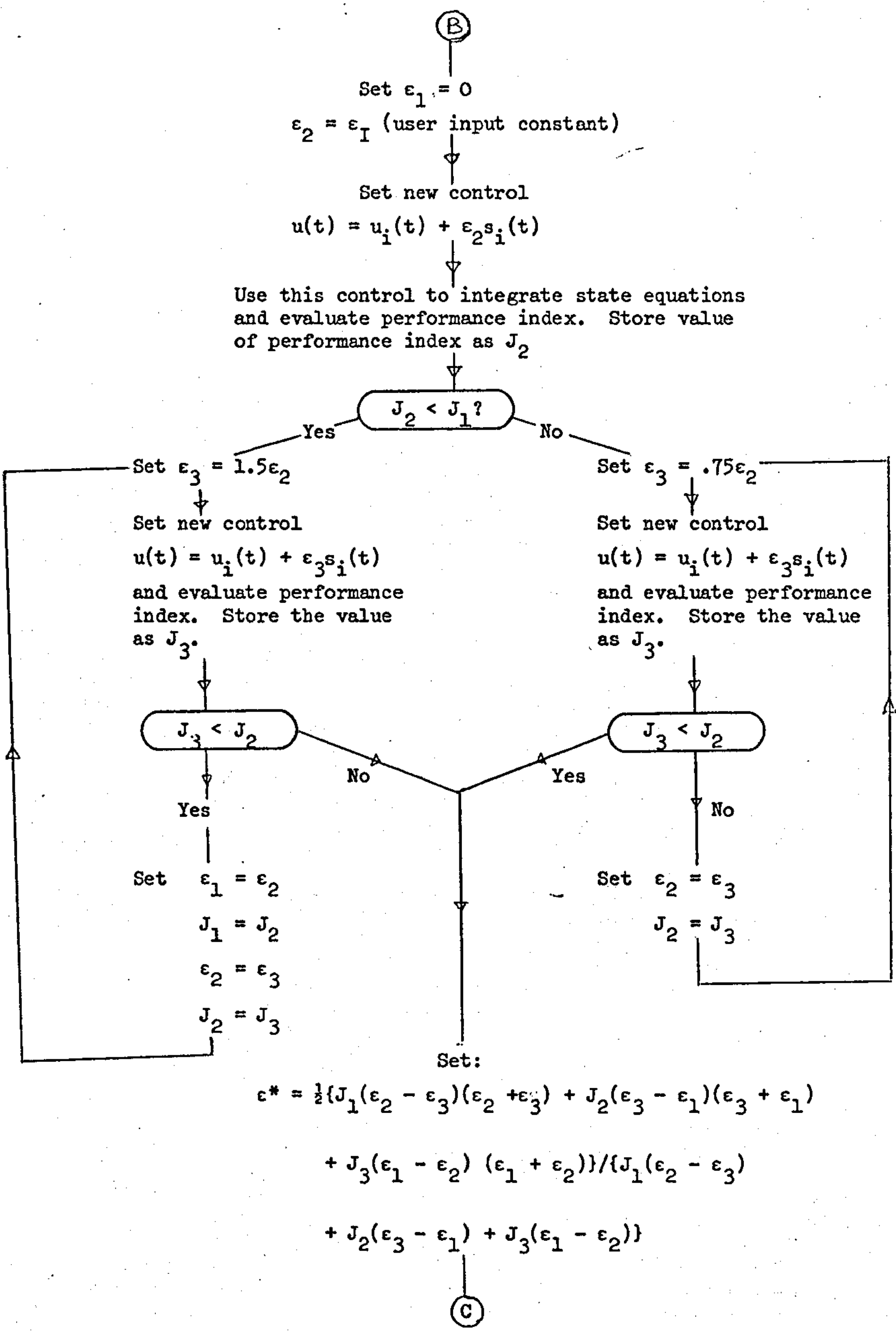
Set new control

$u_{i+1}(t) = u_i(t) + \varepsilon^* s_i(t)$

Set $i = i + 1$

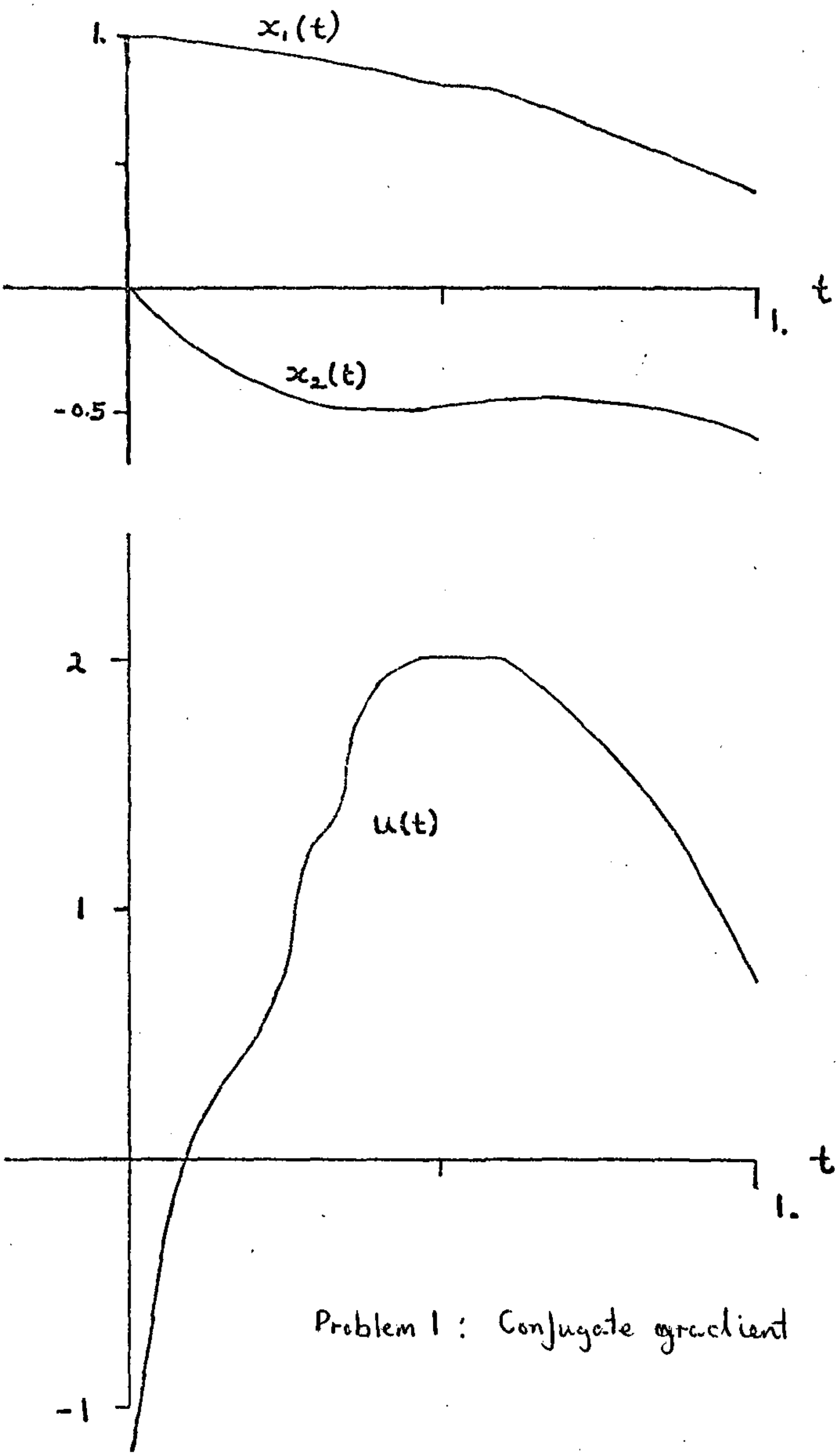
(A)

One dimensional search procedure



APPENDIX F

Conditions (3.6) - (3.8) may be derived directly, by continuity and differentiability arguments, from the assumption that the response x of the system always satisfies the condition that $x(t)$ is contained in a closed and bounded subset of R^n, i.e. $x(t)$ is in X for all t in $[t_0, t_f]$, where X is a closed and bounded subset of R^n. (see "Ordinary differential equations and stability theory: an introduction", by D. A. Sanchez, page 124).



Problem 1 : Conjugate gradient

Fig 1

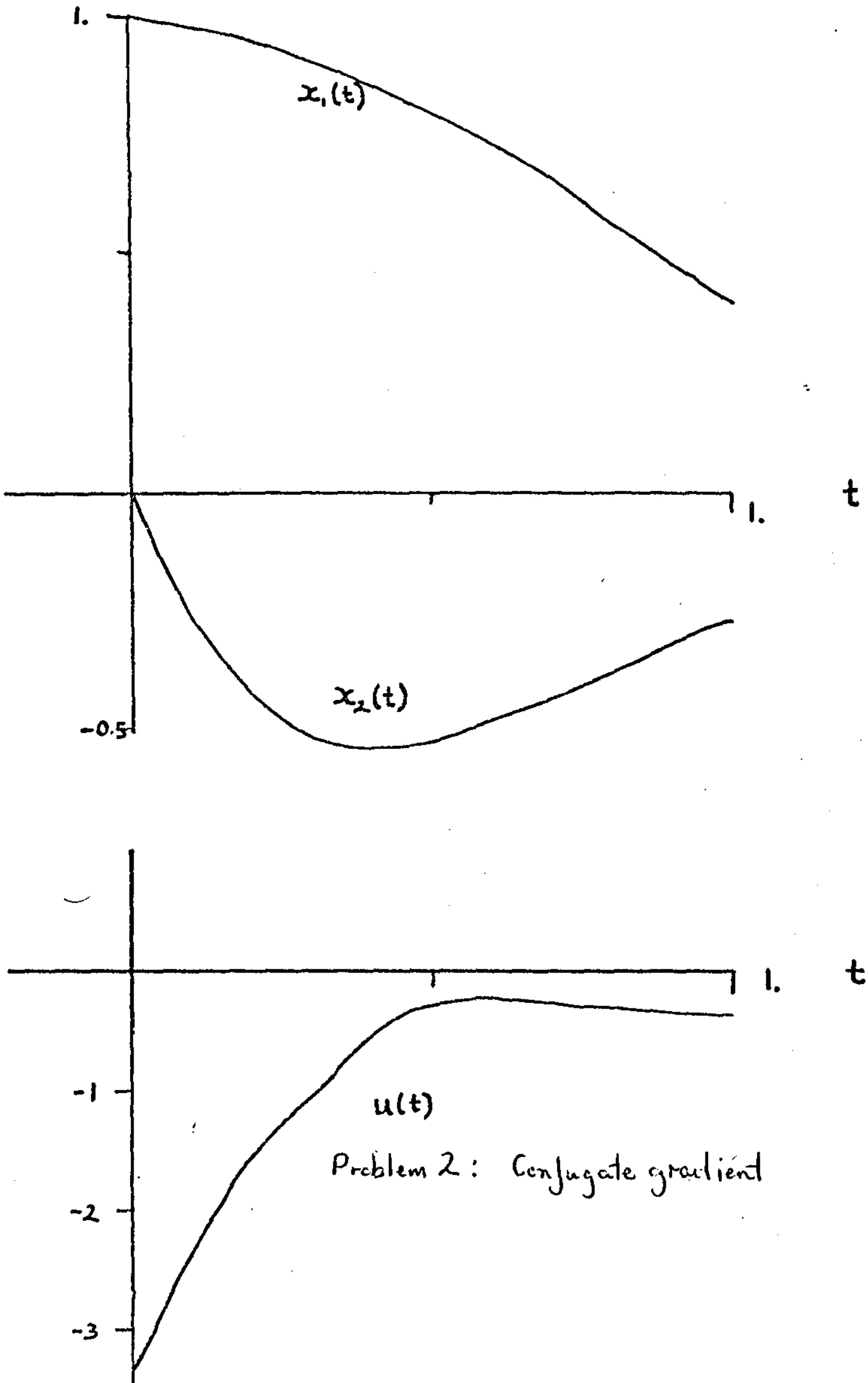


Fig 2

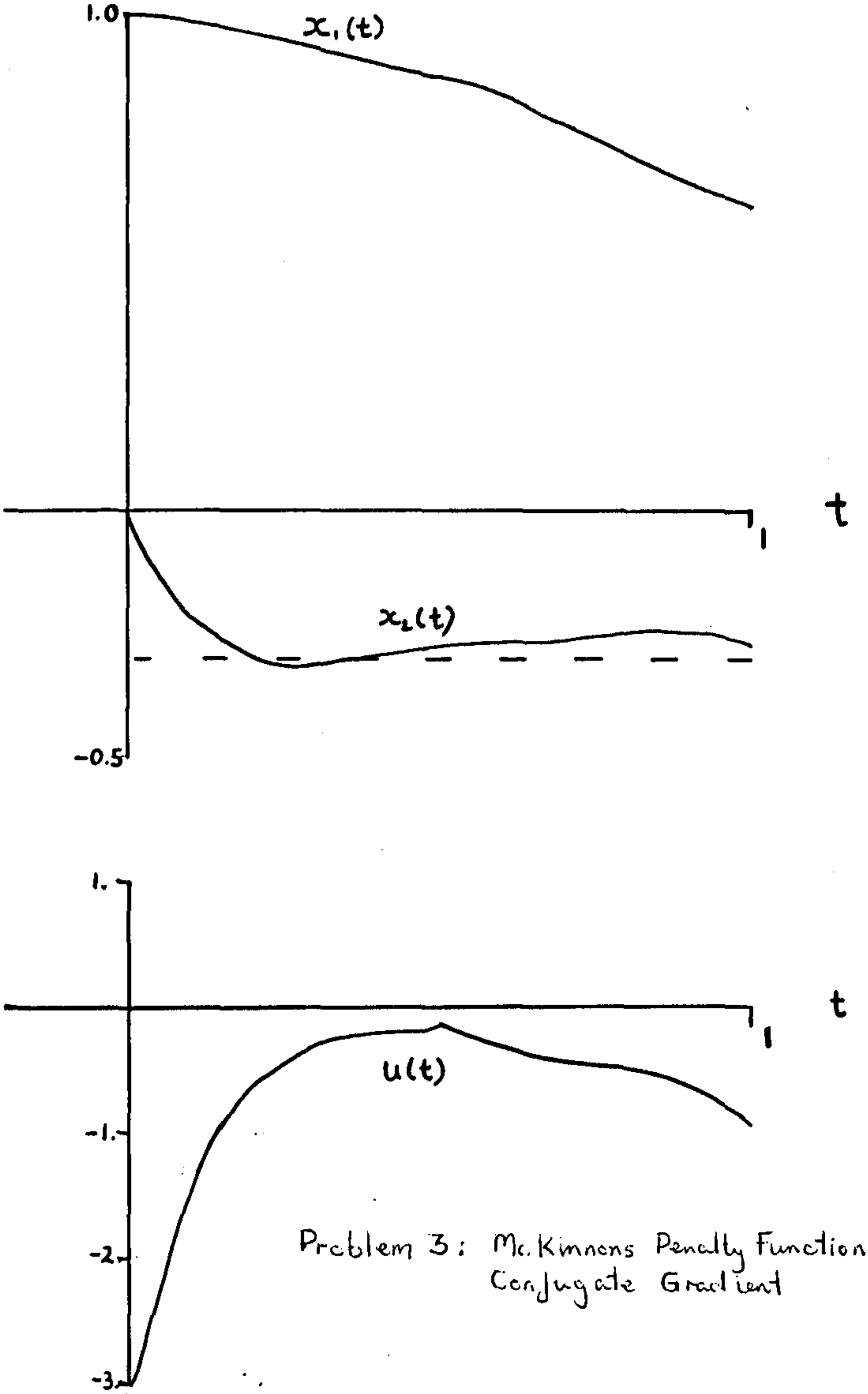

1.0
$x_1(t)$
t
1
$x_2(t)$
-0.5
1.
t
1
u(t)
-1.
-2.
-3.
Problem 3: Mc.Kinnons Penalty Function
Conjugate Gradient

Fig 3

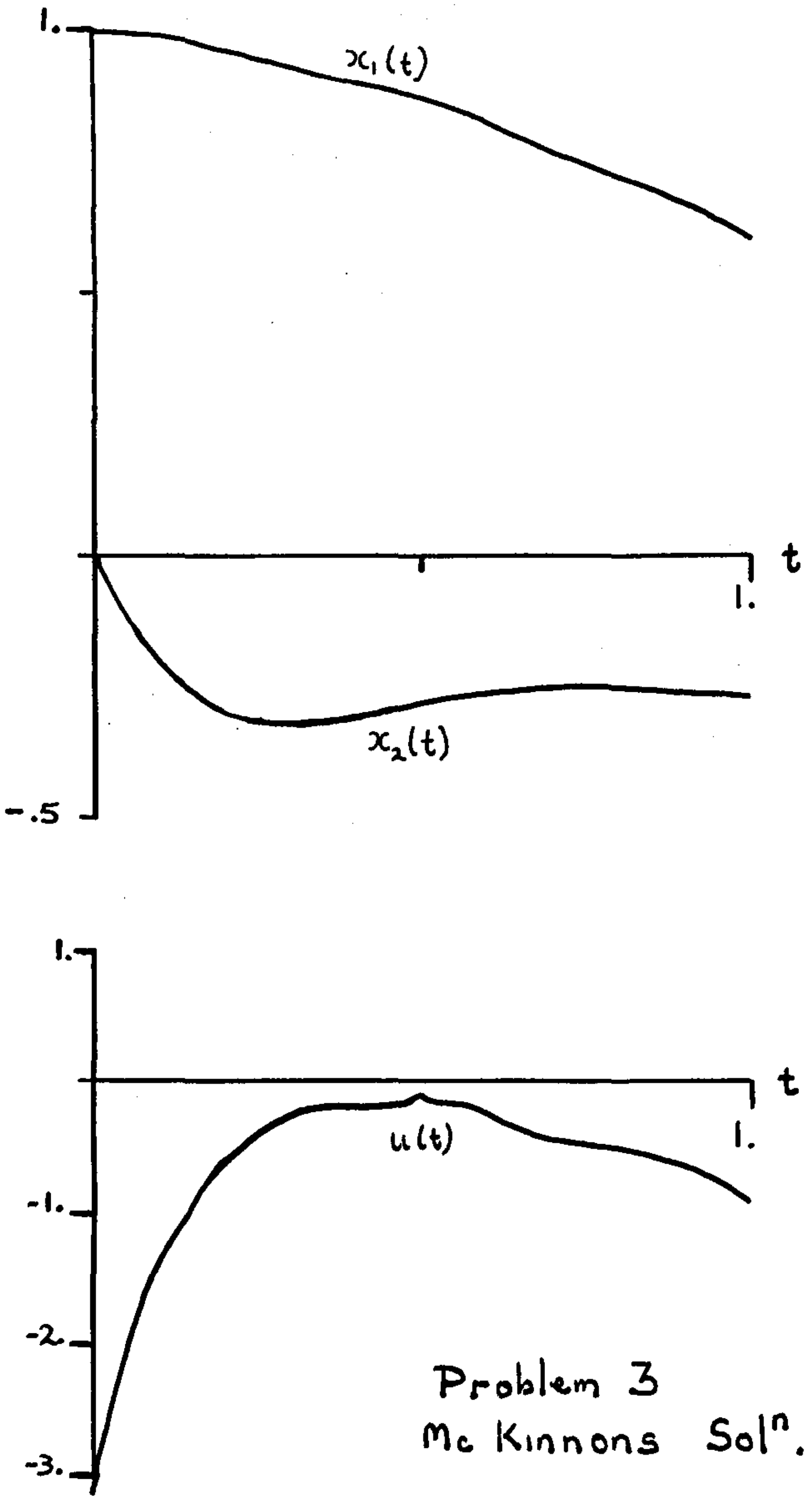
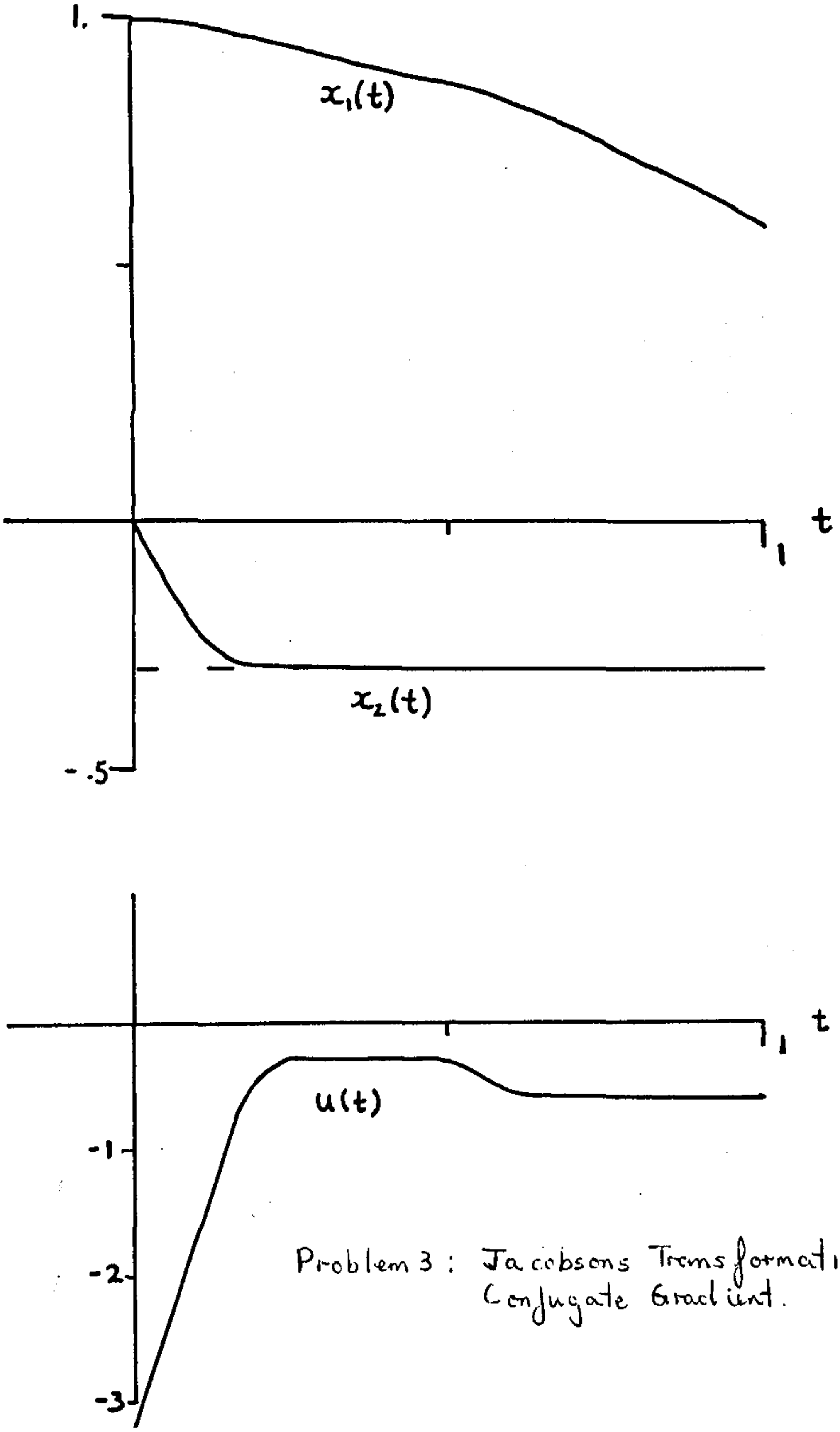
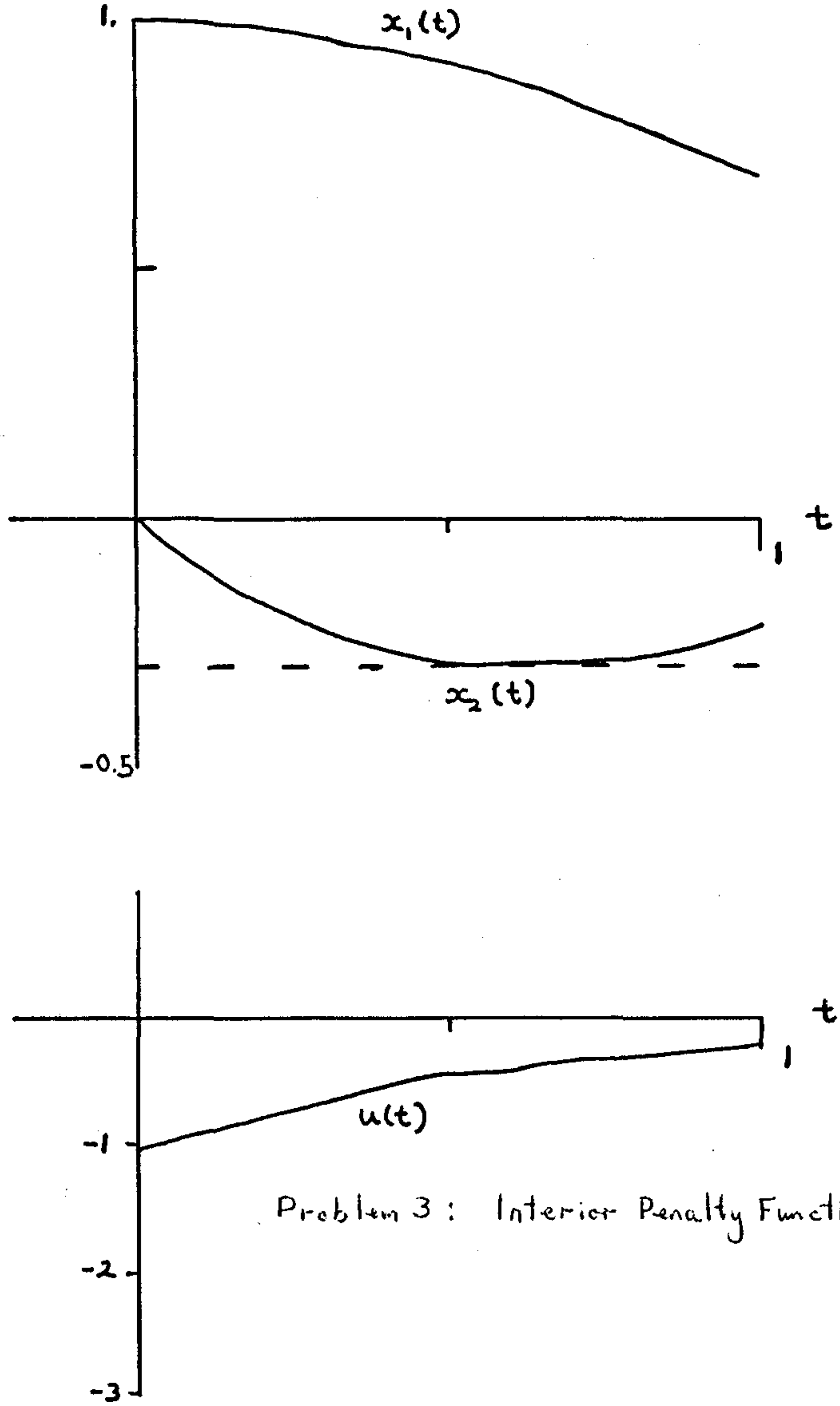


Fig 3a



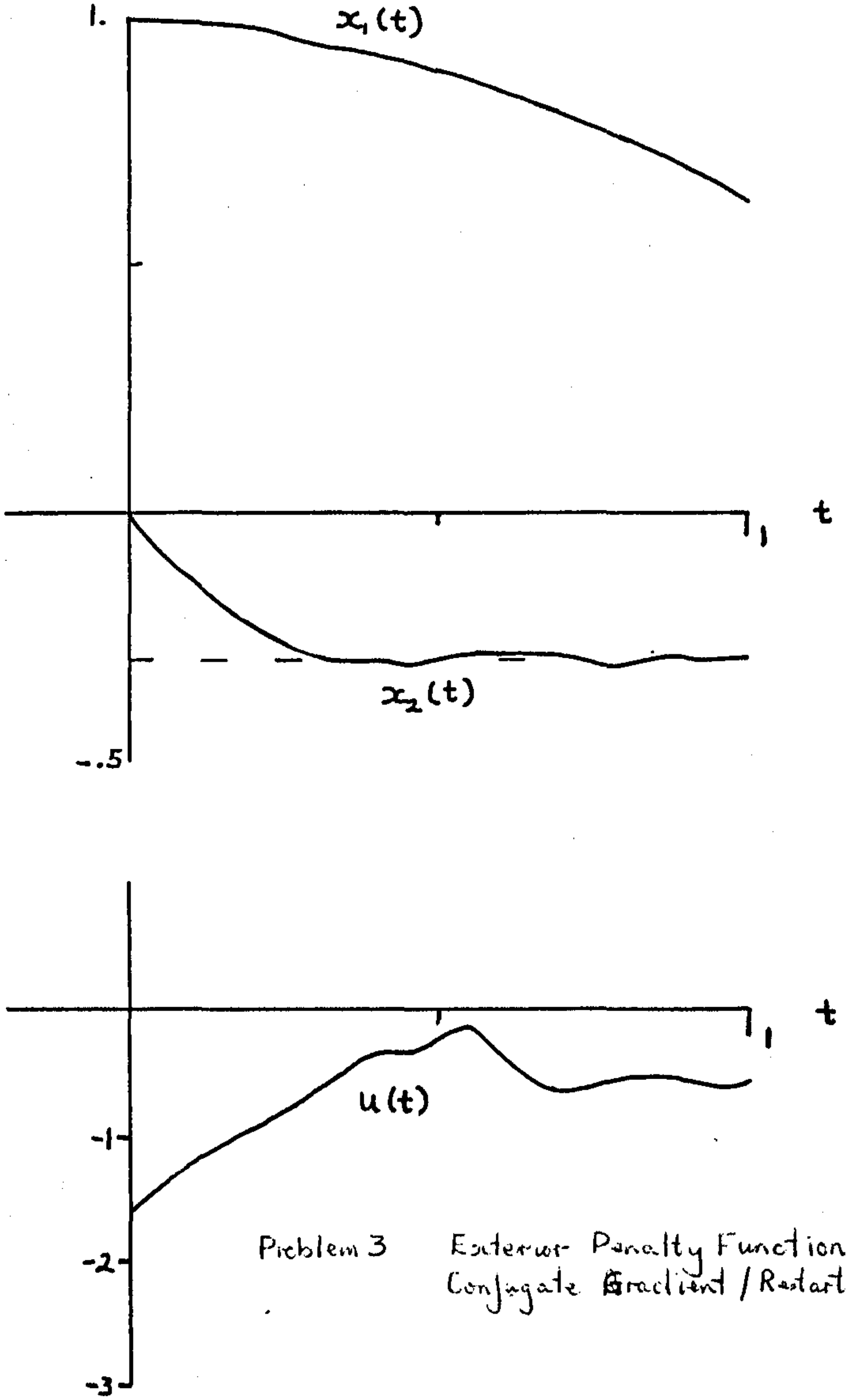
Problem 3: Jacobsons Transformation
Conjugate Gradient.

Fig. 4



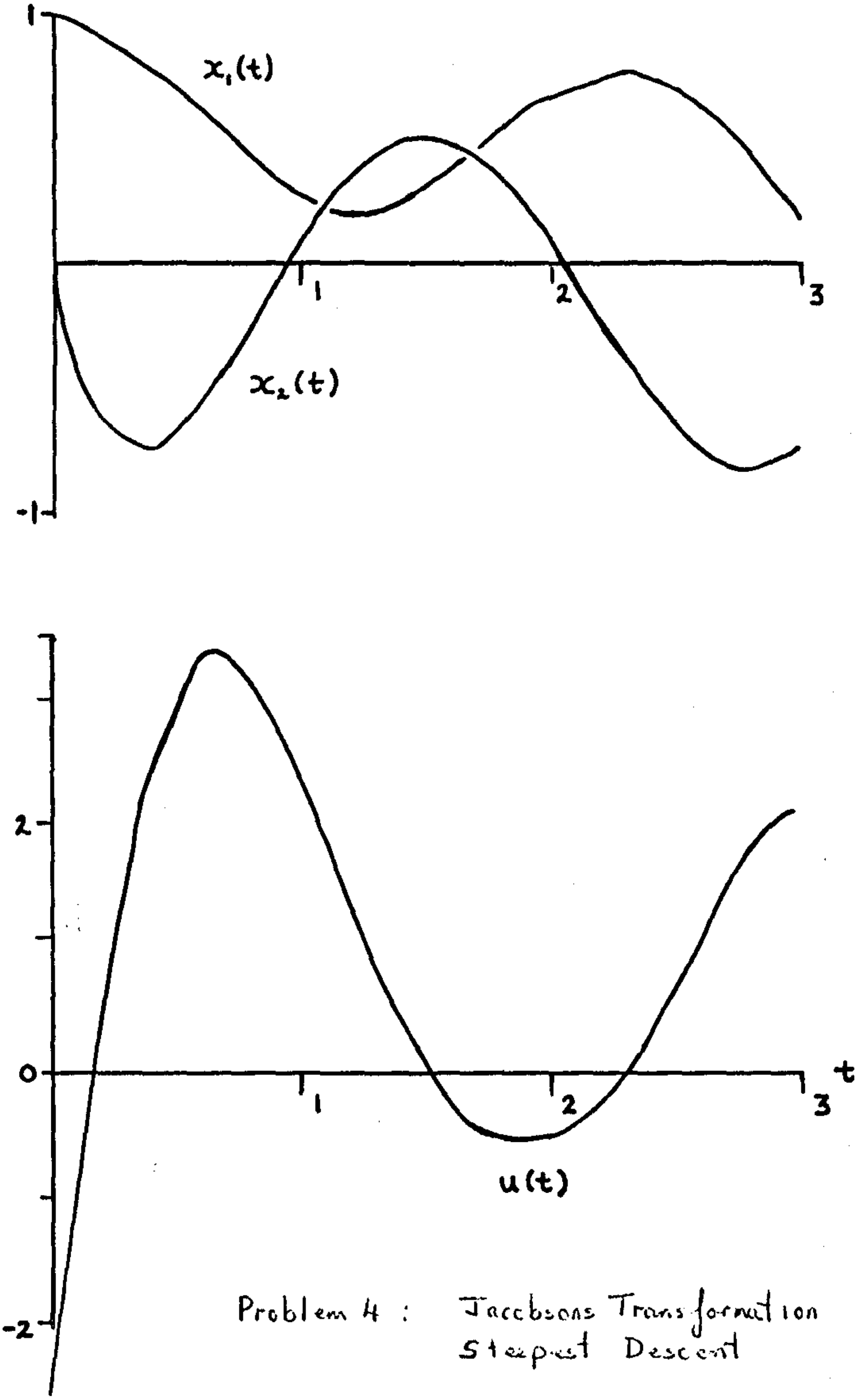
Problem 3 : Interior Penalty Function

Fig 5



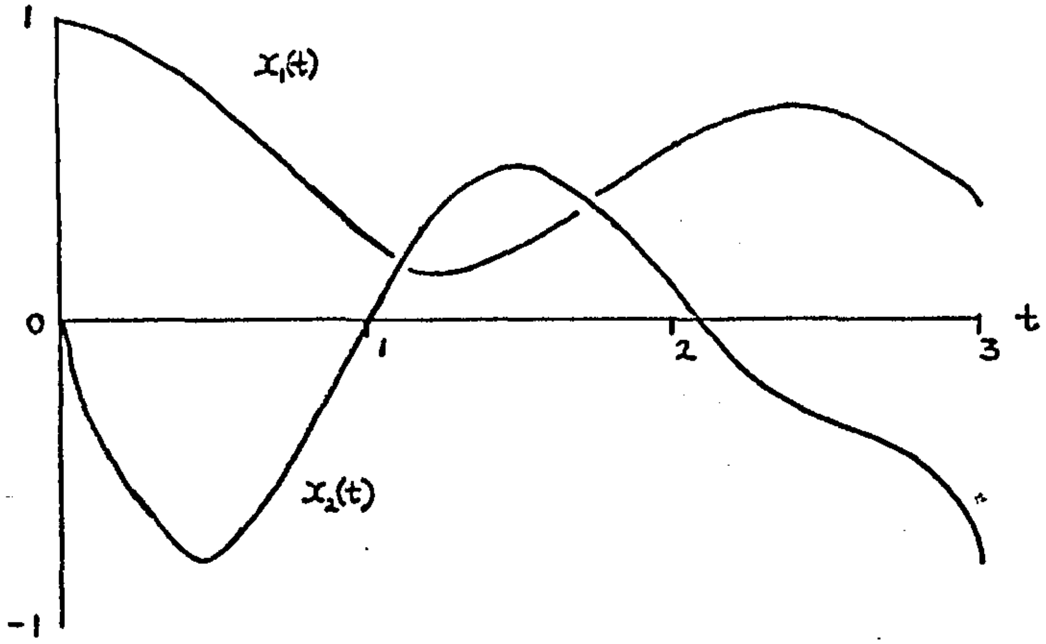
Problem 3 Exterior Penalty Function
Conjugate Gradient / Restart

Fig 6



1
-1
$x_1(t)$
$x_2(t)$
1
2
3
2
0
-2
t
u(t)
Problem 4 : Jacobsons Transformation
Steepest Descent

Fig 7


1
0
-1
$x_1(t)$
$x_2(t)$
1
2
3
t

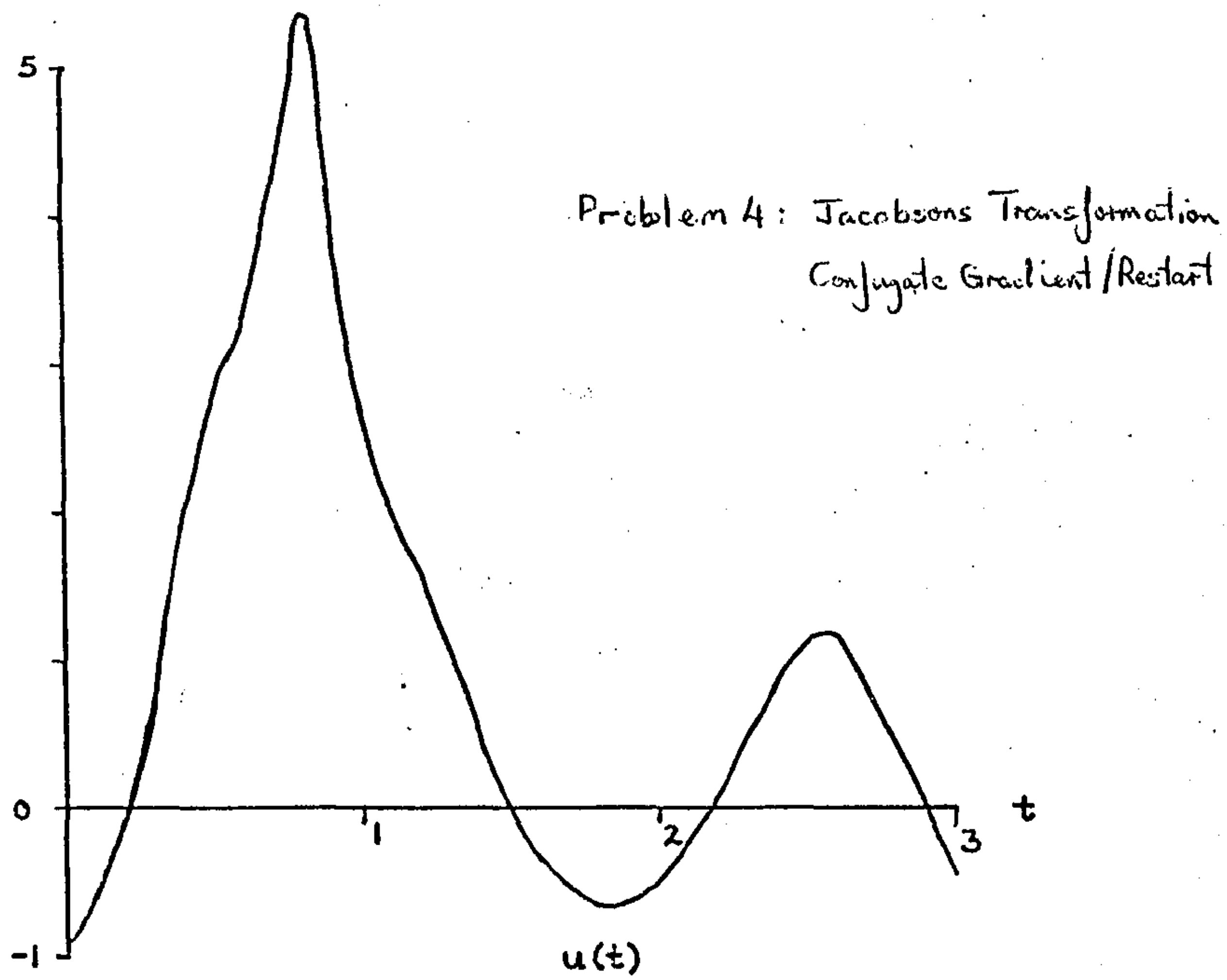

5
0
-1
1
2
3
t
Problem 4: Jacobsons Transformation
Conjugate Gradient / Restart
u(t)

Fig 8

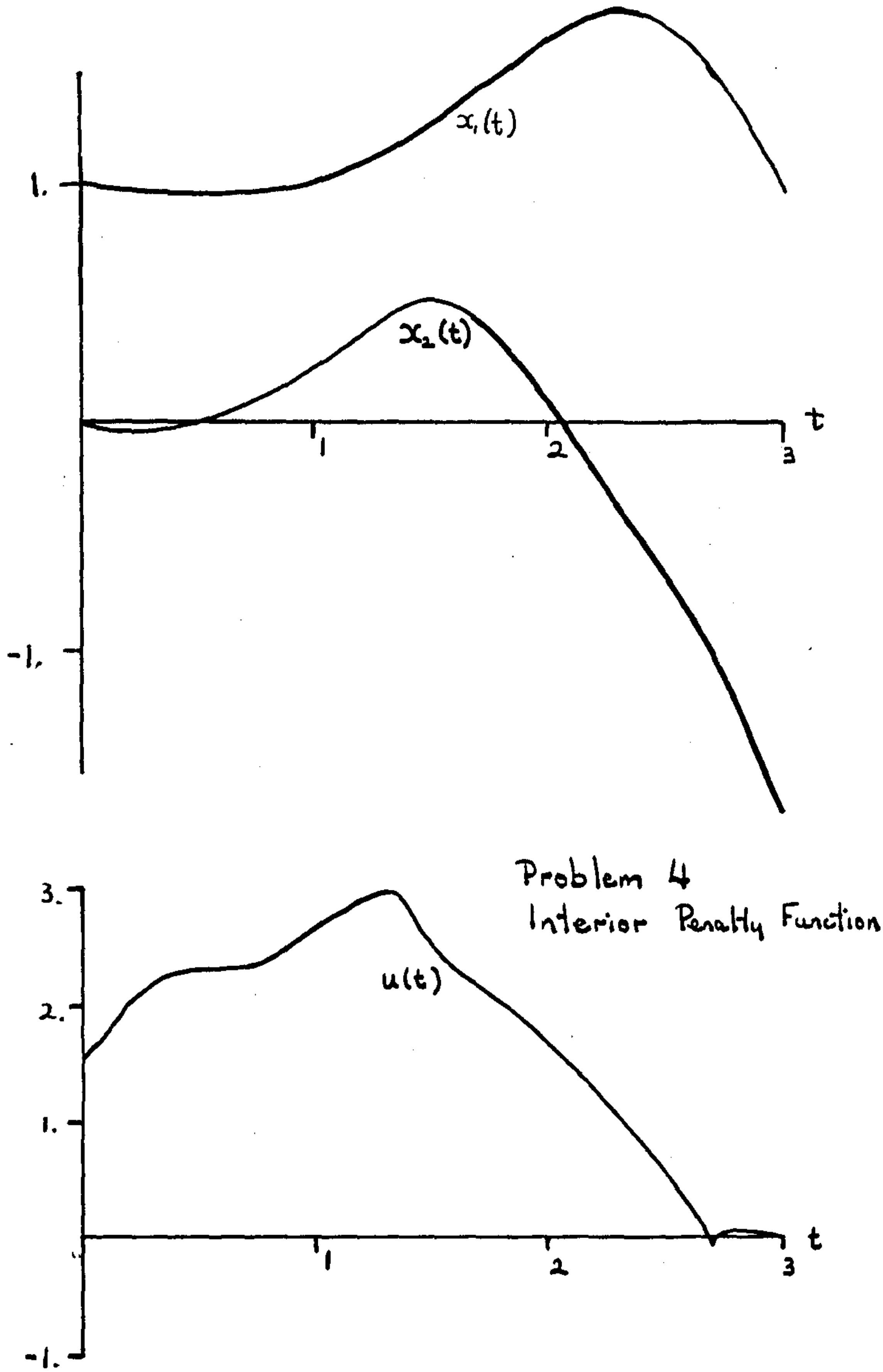
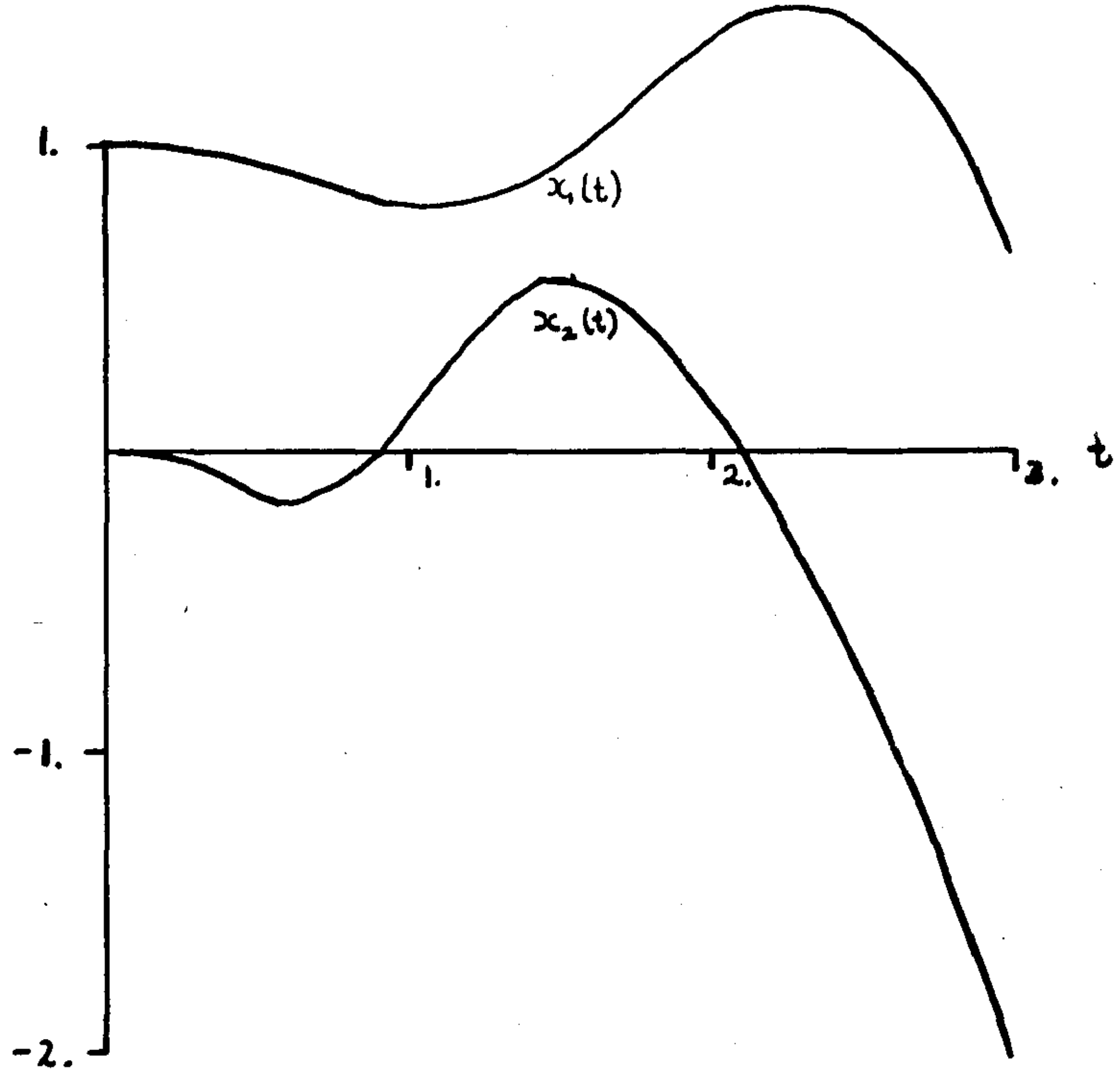
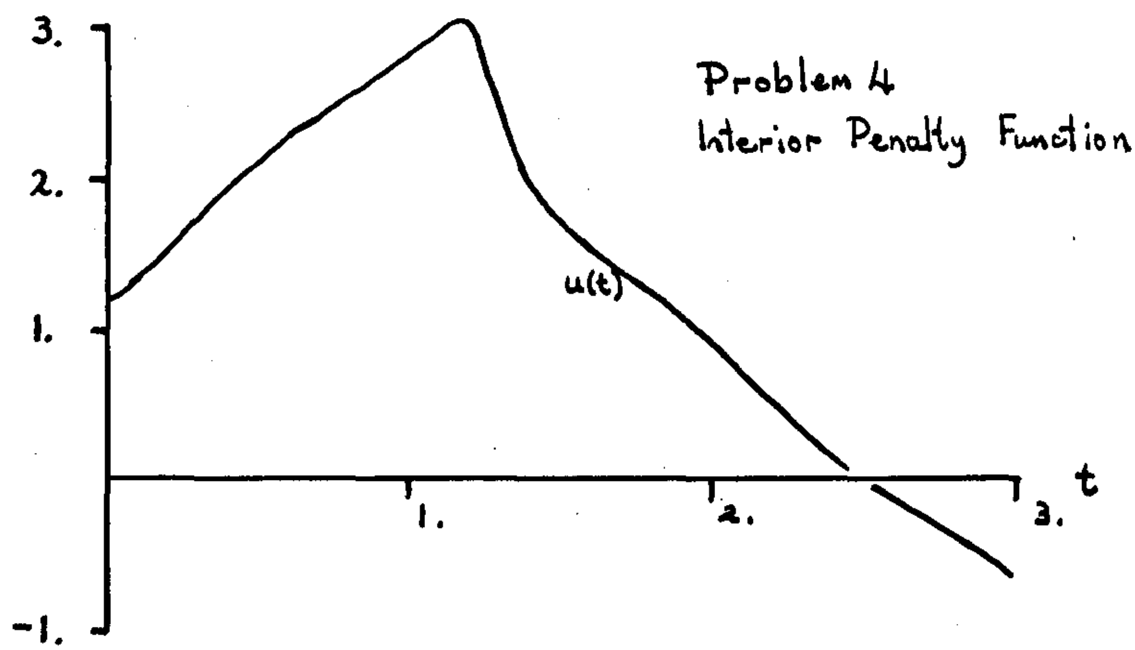

1.
-1.
$x_1(t)$
$x_2(t)$
1
2
3
t
Problem 4
Interior Penalty Function
3.
2.
1.
-1.
$u(t)$
1
2
3
t

Fig 9





Problem 4
Interior Penalty Function

Fig 10

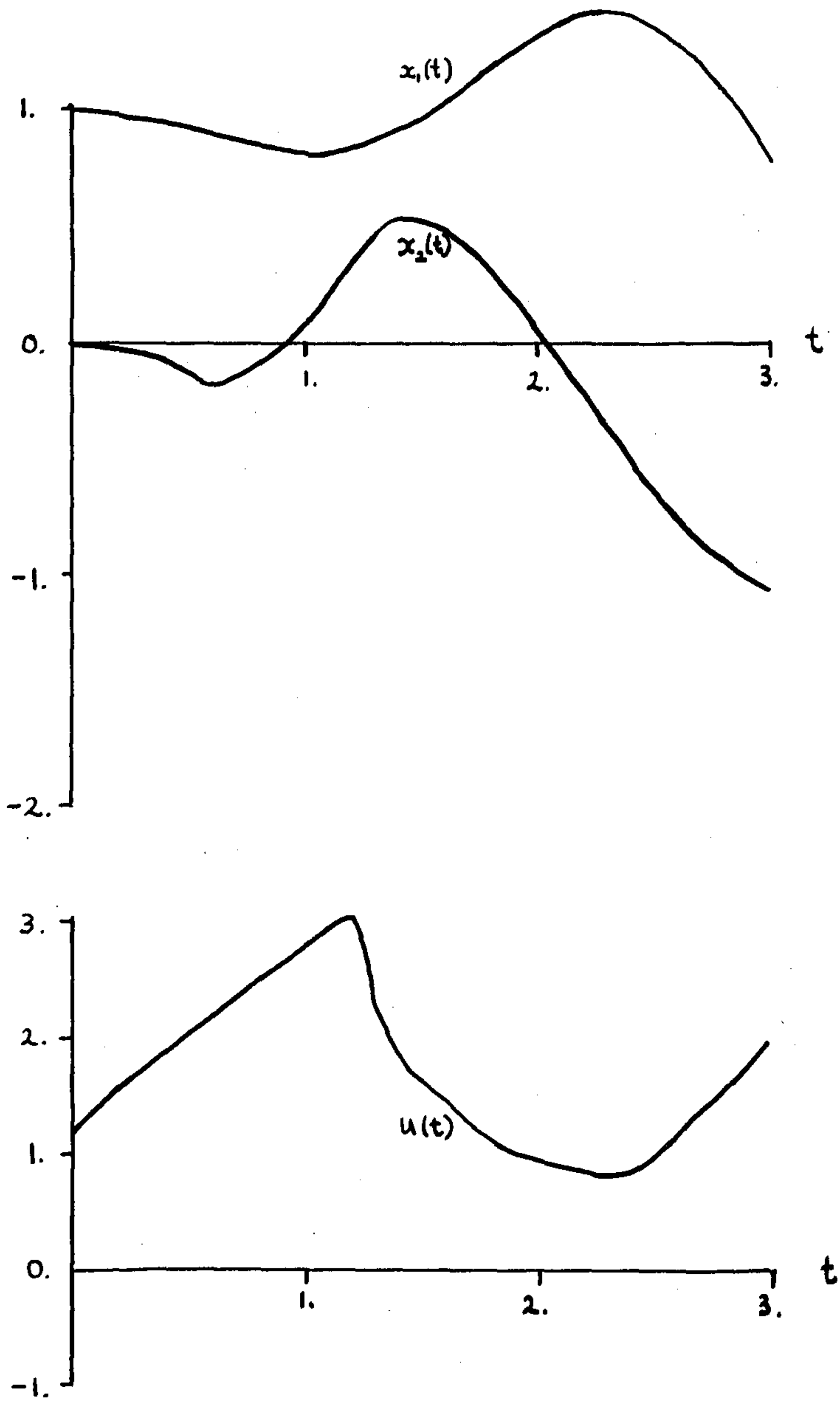
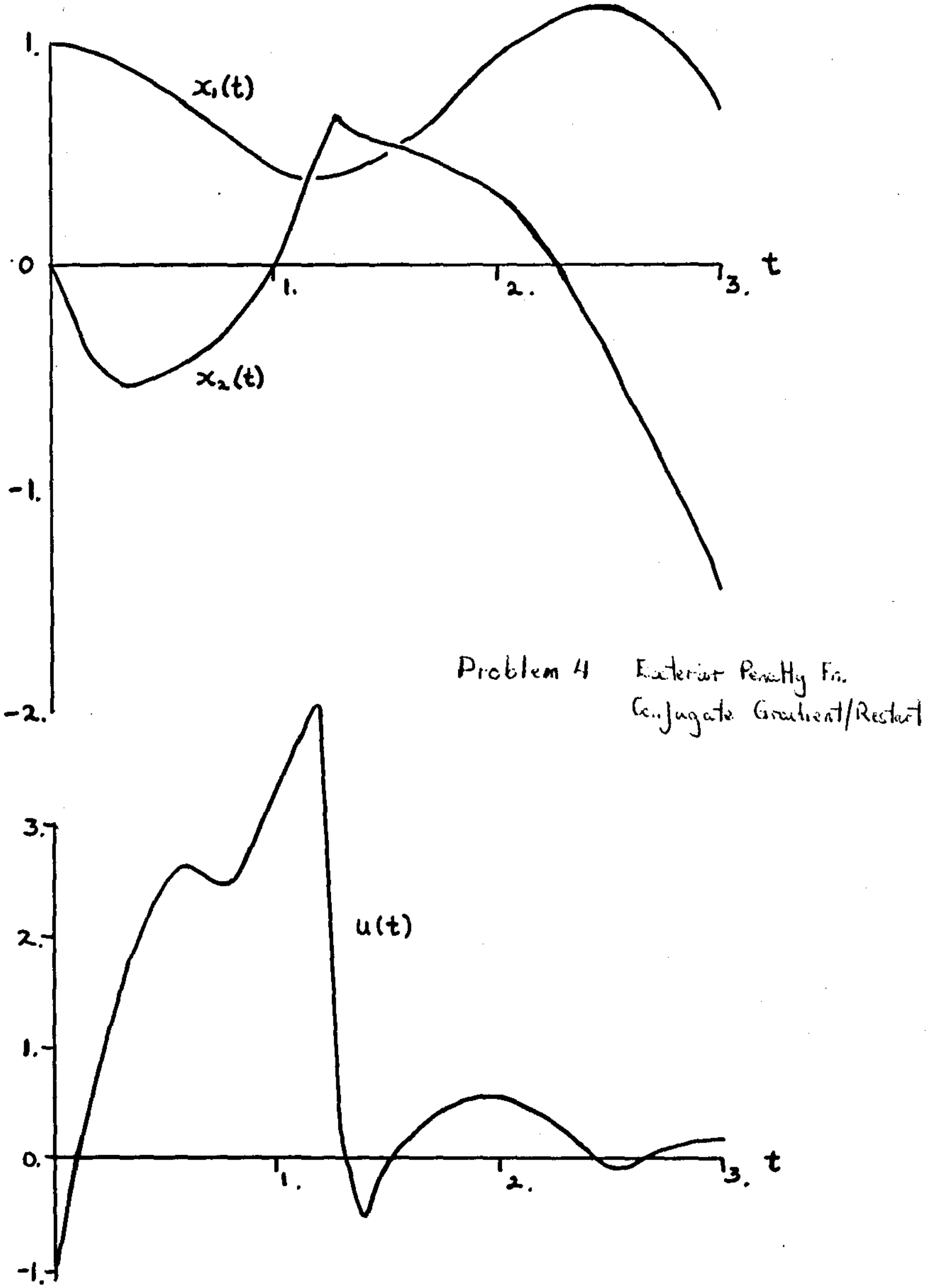


Fig 10a



1.
0
-1.
-2.
1.
2.
3.
t
$x_1(t)$
$x_2(t)$
Problem 4 Exterior Penalty Fn.
Conjugate Gradient/Restart
3.
2.
1.
0.
-1.
u(t)

Fig 11

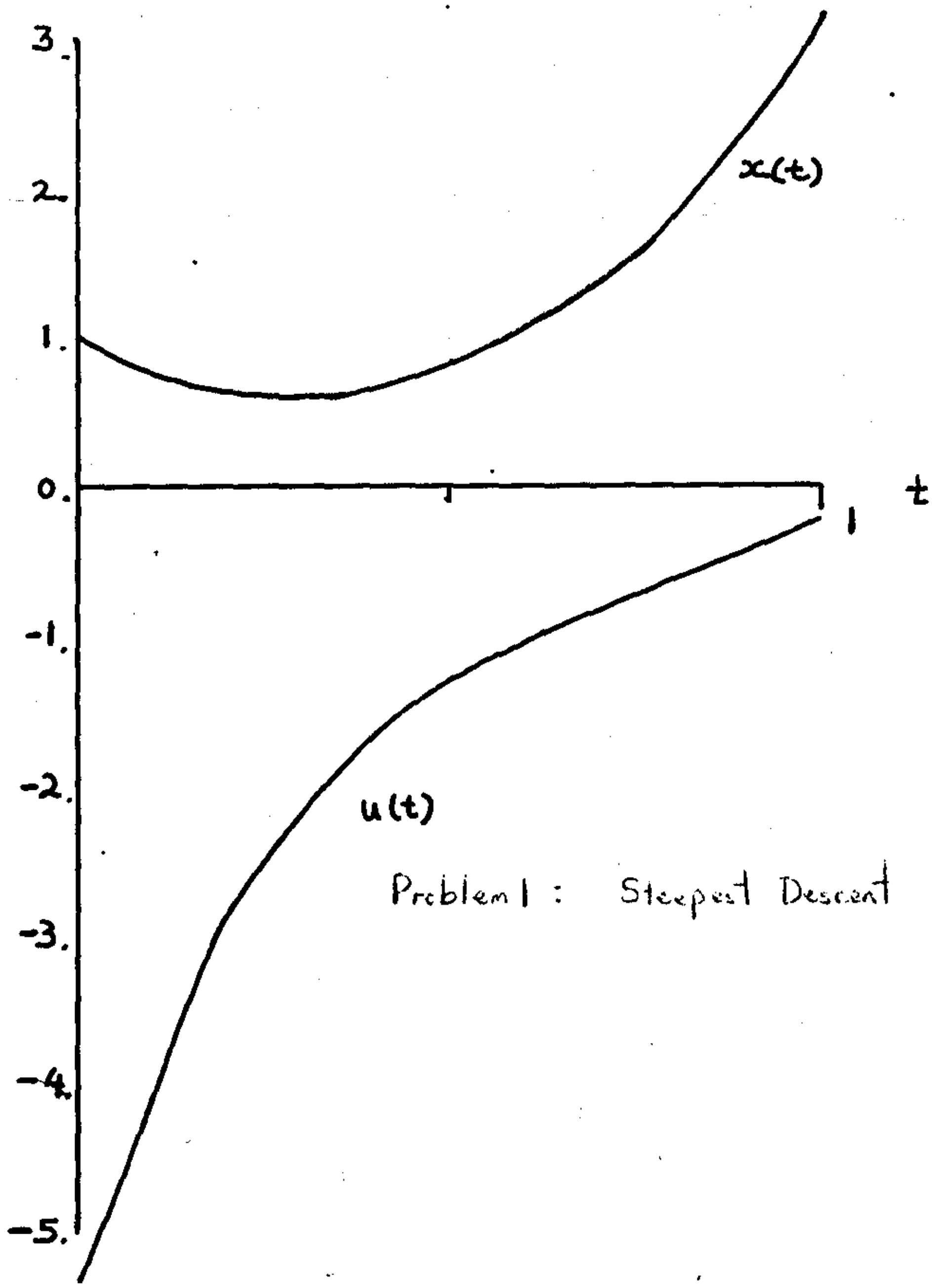


Fig 12

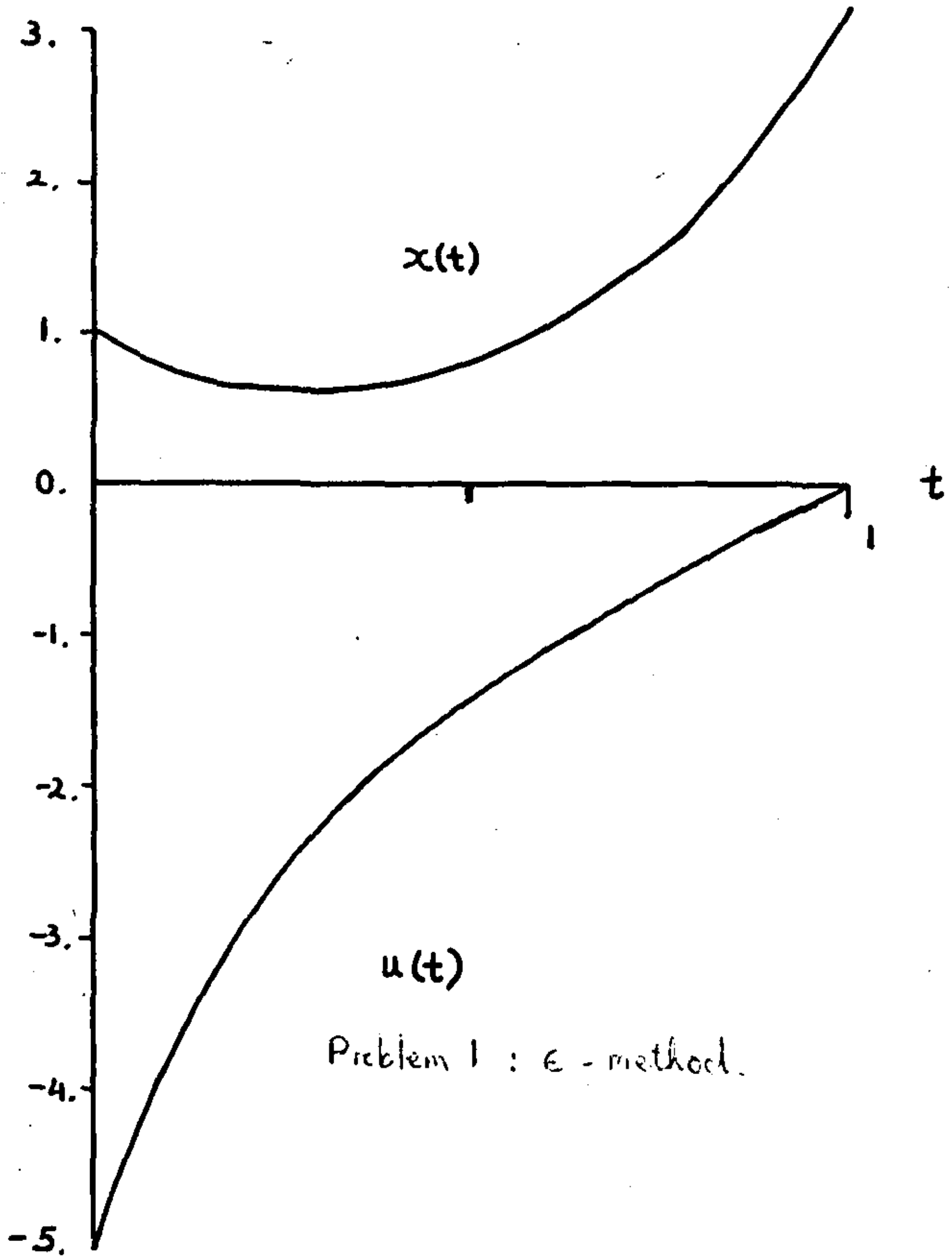


Fig 13

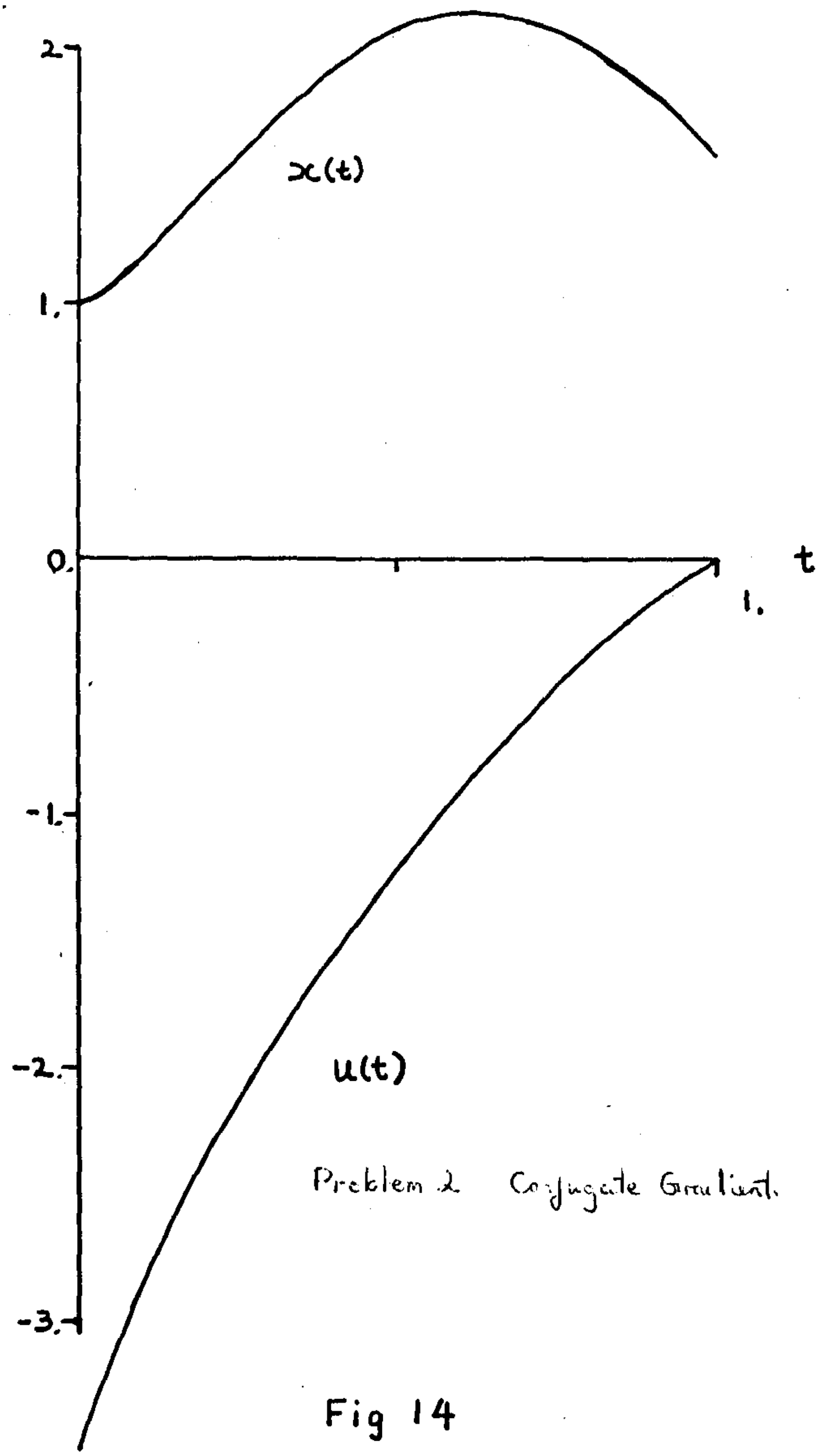


Fig 14

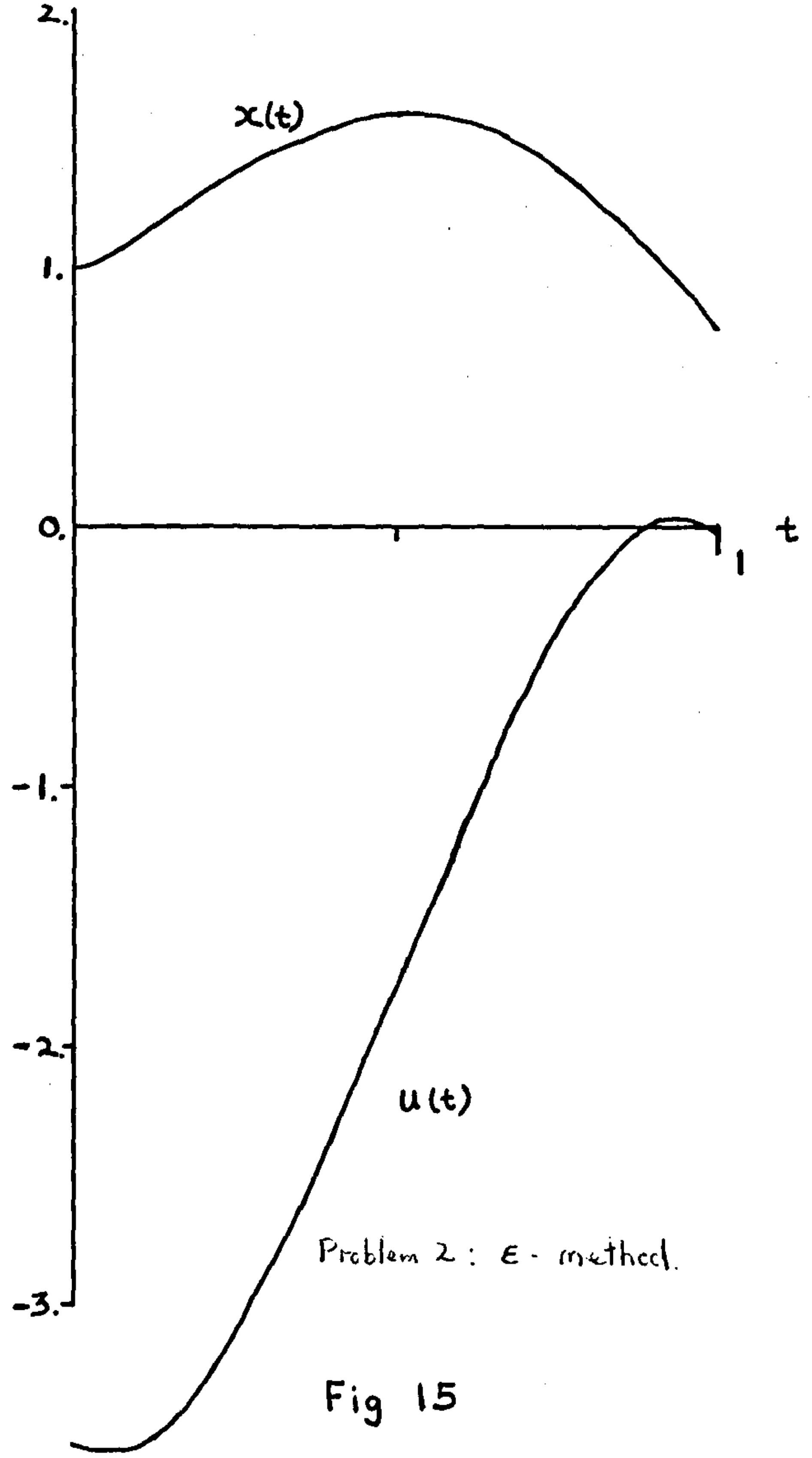


Fig 15

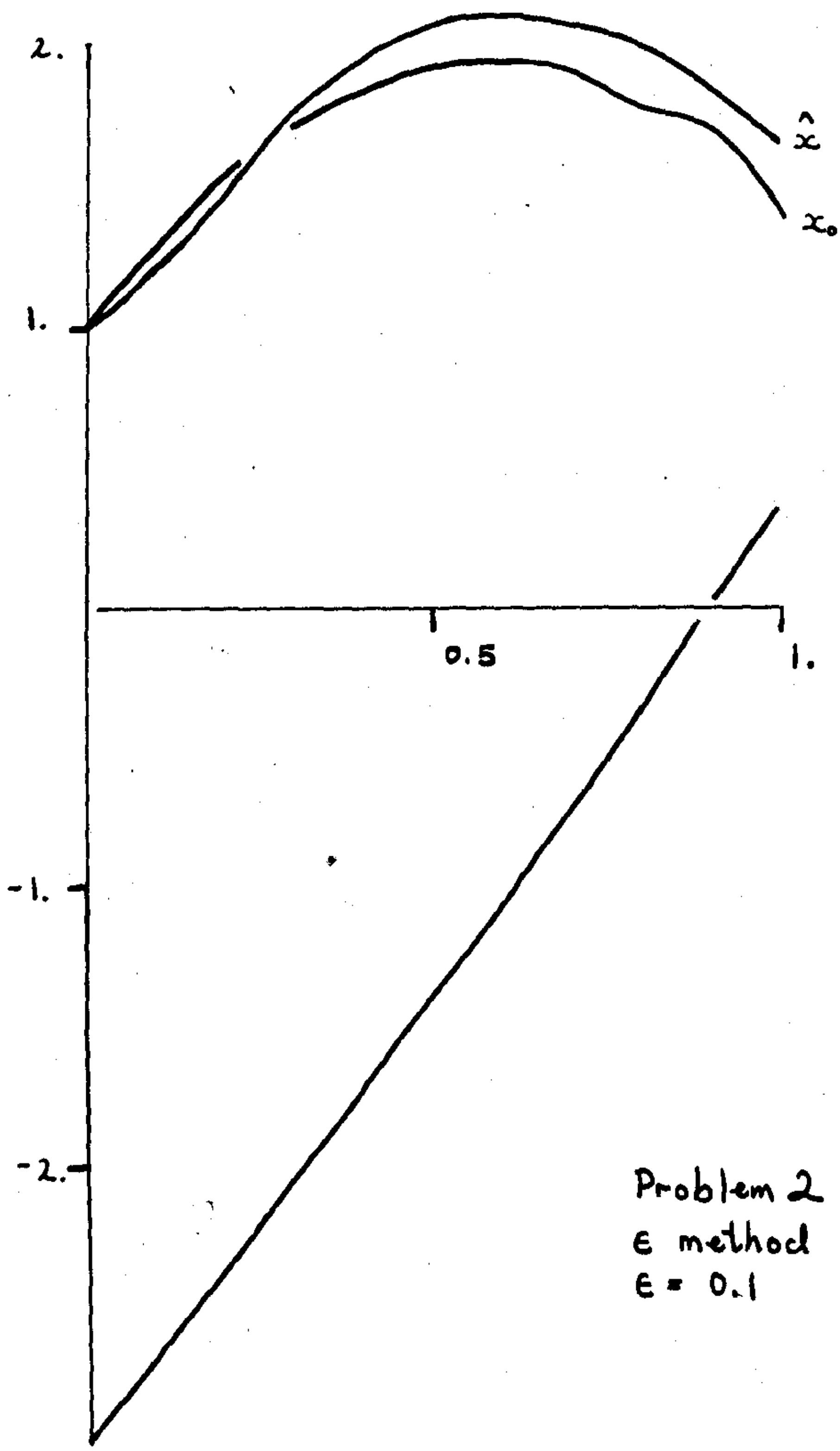


Fig 15a

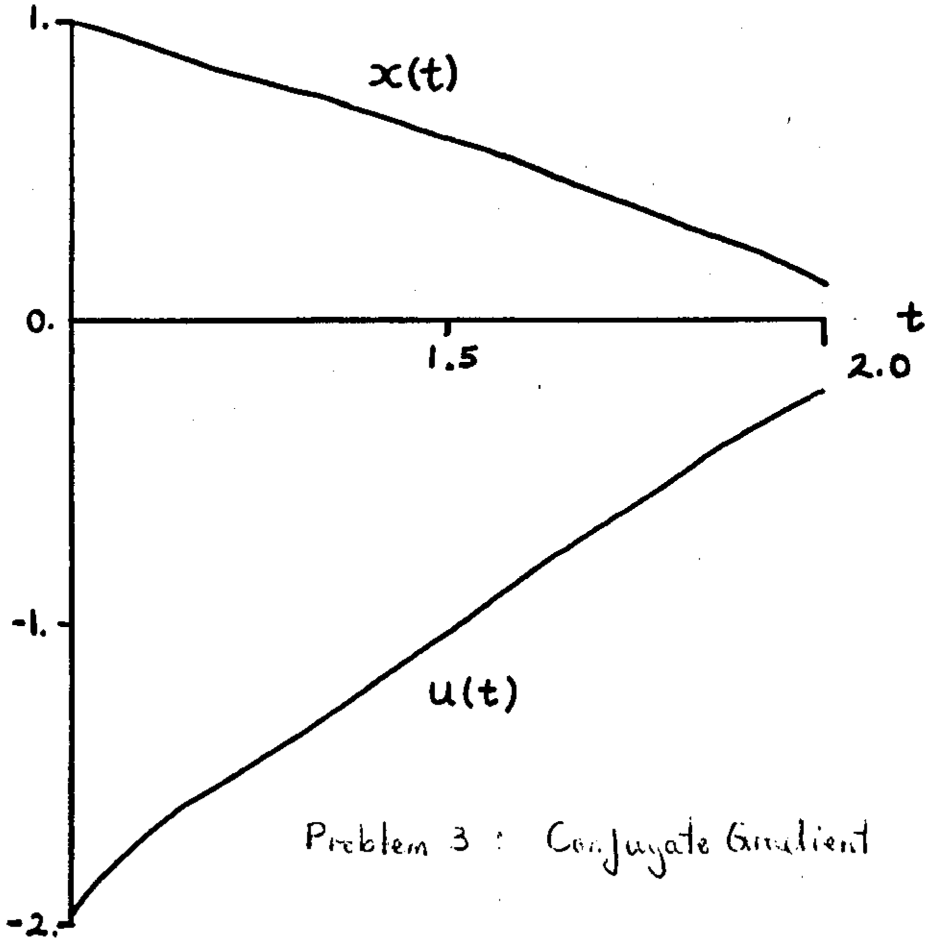


Fig 16

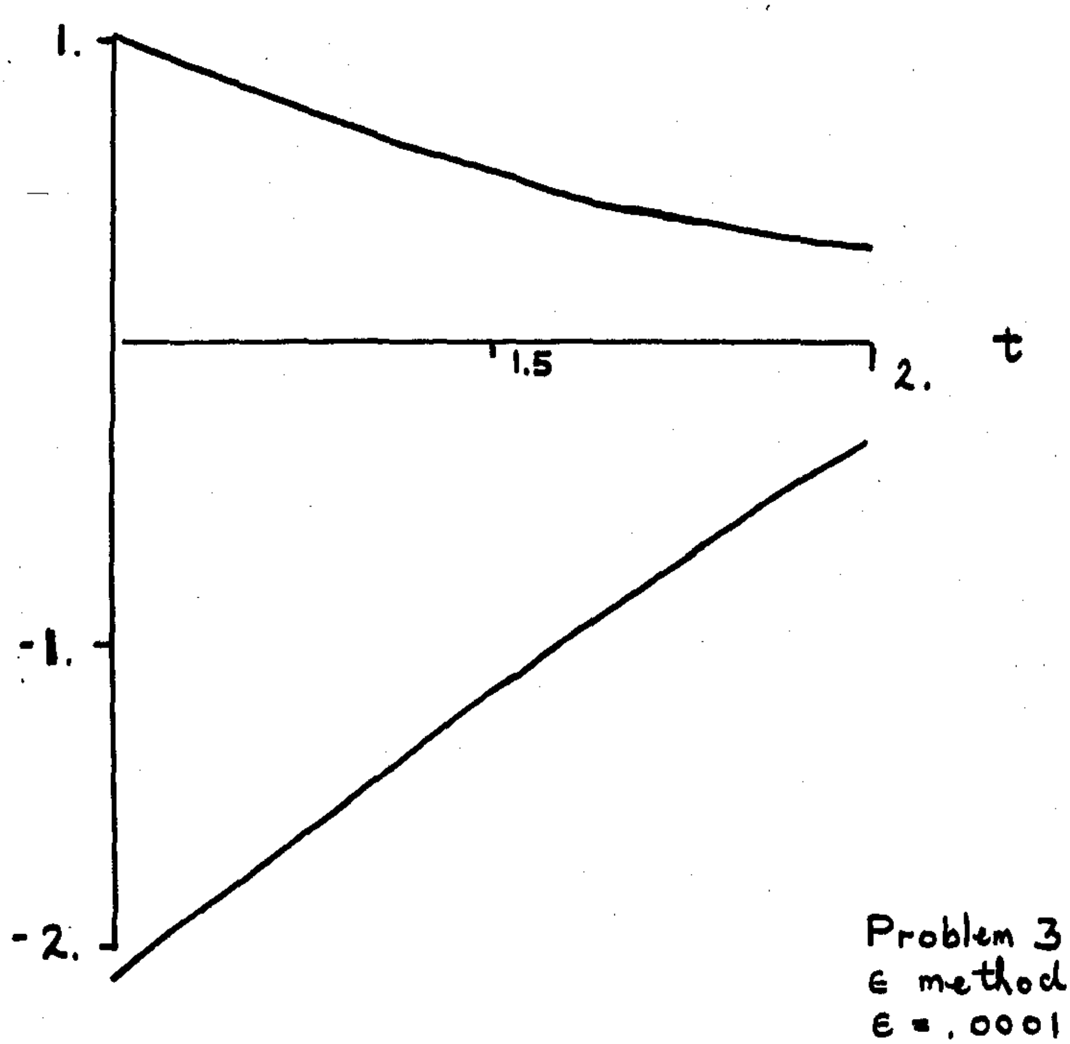


Fig 17