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Physical properties of gravitational solitons

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PHYSICAL PROPERTIES
of
GRAVITATIONAL SOLITONS


by *Salvatore Micciché*

A Doctoral Thesis

*Submitted in partial fulfilment of the requirements
for the award of*

*Doctor of Philosophy
of Loughborough University*

September 1999

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Date	July 00
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*This work is dedicated to the Land where I came from
Unfriendly, savage, bitter, hard.
Inaccessible and mysterious. Strong.
Whipped by the sun, abandoned in the sea.
I belong to her.
In her my roots are.
Once upon a time we recognized each other.*

Keywords

1. General Relativity
2. Einstein's equations
3. Exact solutions
4. Gravitational Solitons
5. Inverse scattering technique
6. Alekseev's method
7. Electrovacuum solitons
8. Time Shift
9. Real Poles

Abstract

Soliton solutions of Einstein's field equations for space-times with two non-null, commuting Killing Vectors are exact solutions obtained using the solution-generating techniques that resemble the well-known Inverse Scattering Methods that have been widely used in the solution of certain nonlinear p d e.'s such as Korteweg-de Vries, Sine-Gordon, non-linear Schrödinger.

There exist two main soliton techniques in General Relativity. The Belinski-Zakharov technique allows for purely gravitational solutions. The Alekseev technique allows for solutions of the Einstein-Maxwell equations. In both techniques, solitons arise in connection with the poles of a certain so-called "dressing matrix".

In this thesis we consider these techniques with respect to three aspects

I) Time-Shift problem in complex-pole solitons

Solitons of the ordinary p.d.e.'s retain their shape and velocity on interaction but experience a characteristic time-shift. This effect is a clear result of the nonlinearity in the equations. It is not clear whether or not gravitational solitons also experience a similar behaviour.

In the literature there have been claims for the detection of such a behaviour. By considering the Weber-Wheeler-Bonnor solution, we have clarified that this effect is also shared by nonsolitonic solutions of linear equations. Thus this apparent shift must have some other physical explanation.

We have therefore turned to the analysis of a time-dependent nondiagonal soliton solution, namely an explicit 4-soliton solution, and we have shown that the two inner solitons do not suffer any time-shift when colliding.

II) Nature of the singularities in real-pole solitons

We have considered soliton solutions with real poles in the cosmological context. It is well known that these solutions contain singularities on certain null hypersurfaces. Using a Kasner seed solution, we have discussed the nature of these singularities with the following results:

- the soliton solution can not be extended up to the whole spacetime without the appearance of thin sheets of null matter located along the null hypersurfaces, or the introduction of a singular axis

- Matter-free extensions are admissible (i.e. the singularity is removable with a coordinate transformation) but other singularities are introduced. We also discuss a number of such extensions

III) Electromagnetic Solitons with real poles

It may be expected that soliton solutions of the Einstein-Maxwell equations should be equivalent to the purely gravitational ones, once the electromagnetic field is let vanish. However, not all the vacuum soliton solutions can be obtained in this way. Simple arguments show that the class of Alekseev's vacuum solitons is smaller than the Belinski-Zakharov's.

In particular, we will consider the problem of introducing real-pole solitons in the ambit of the Alekseev formalism. This is achieved by letting the dressing matrix have real poles that are different from those of its inverse. This possibility is new in this context, as in the usual Alekseev technique the poles are taken to be complex-conjugate with each other. We also show that this procedure allows for the generation of vacuum metrics only. Finally, at least in a few particular cases, we also show that the solutions thus generated are identical to the BZ ones

Acknowledgements

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Emilio. You know: if the craftsman will become a worker ... , it will be again your fault!

Cornelius. I wish to recall here the many conversations we had about physics and else, and how, in his peculiar way, he always transmitted something to me.

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¹from a bottle of MacCallan Scotch Whisky

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Chapter 1

Two Killing–Vector metrics: an introduction

The Einstein's theory of Gravitation, the theory of General Relativity, is based upon the equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4 \quad (1.1)$$

Here, $R_{\mu\nu}$ is the Ricci tensor constructed in terms of the metric $g_{\mu\nu}$ and $R = R_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar. $T_{\mu\nu}$ is the energy-momentum tensor. G is the gravitational coupling constant. The energy-momentum tensor contains all the information about the matter fields under consideration. The Ricci tensor contains all the (geometrical) information about the spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.2)$$

in which these matter fields exist.

The revolutionary content of these equations consists in that the spacetime is no longer considered as a fixed frame upon which the matter fields act. Instead, there exists a mutual interaction between spacetime and matter [1]. Moreover, even if $T_{\mu\nu}$ were zero, there may exist non trivial solutions of the Einstein's equations, thus indicating that even a vacuum spacetime can have a rather complicated structure.

Due to the symmetry properties of the Ricci tensor, the Einstein's equations (1.1) turn out to be 10 coupled nonlinear p.d.e.'s. Along with these, equations describing the matter fields must also be considered. Evidently, the complexity of the equations is such as to make it extremely difficult to obtain solutions, even in the simpler case when the matter fields vanish. However, according to the physical situation one is to describe, symmetries can be imposed which may result in a simplification of the equations.

1.1 Symmetries and Isometry Groups

In Riemannian spaces, the existence of symmetries can be expressed as follow [2, 3]: let us suppose that – at each point x^ν – there exist a vector $\xi^\mu = \xi^\mu(x^\nu)$ such that the metric is invariant under translations along it. This invariance is expressed by the following equation:

$$g_{\mu\nu,\rho} \xi^\rho + g_{\rho\nu} \xi^\rho_{,\mu} + g_{\mu\rho} \xi^\rho_{,\nu} = 0. \quad (1.3)$$

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September 1999

Certificate of Originality

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgments or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a higher degree.

..... (Signed)

. (Date)

This work is dedicated to the Land where I came from.

Unfriendly, savage, bitter, hard.

Inaccessible and mysterious. Strong.

Whipped by the sun, abandoned in the sea.

I belong to her.

In her my roots are.

Once upon a time we recognized each other.

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In this thesis we consider these techniques with respect to three aspects.

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II) Nature of the singularities in real-pole solitons

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- the soliton solution can not be extended up to the whole spacetime without the appearance of thin sheets of null matter located along the null hypersurfaces, or the introduction of a singular axis
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Chapter 1

Two Killing–Vector metrics: an introduction

The Einstein's theory of Gravitation, the theory of General Relativity, is based upon the equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \quad (1.1)$$

Here, $R_{\mu\nu}$ is the Ricci tensor constructed in terms of the metric $g_{\mu\nu}$ and $R = R_{\mu\nu} g^{\mu\nu}$ is the Ricci scalar. $T_{\mu\nu}$ is the energy-momentum tensor. G is the gravitational coupling constant. The energy-momentum tensor contains all the information about the matter fields under consideration. The Ricci tensor contains all the (geometrical) information about the spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.2)$$

in which these matter fields exist.

The revolutionary content of these equations consists in that the spacetime is no longer considered as a fixed frame upon which the matter fields act. Instead, there exists a mutual interaction between spacetime and matter [1]. Moreover, even if $T_{\mu\nu}$ were zero, there may exist non trivial solutions of the Einstein's equations, thus indicating that even a vacuum spacetime can have a rather complicated structure.

Due to the symmetry properties of the Ricci tensor, the Einstein's equations (1.1) turn out to be 10 coupled nonlinear p d e.'s. Along with these, equations describing the matter fields must also be considered. Evidently, the complexity of the equations is such as to make it extremely difficult to obtain solutions, even in the simpler case when the matter fields vanish. However, according to the physical situation one is to describe, symmetries can be imposed which may result in a simplification of the equations

1.1 Symmetries and Isometry Groups

In Riemannian spaces, the existence of symmetries can be expressed as follow [2, 3]: let us suppose that – at each point x^ν – there exist a vector $\xi^\mu = \xi^\mu(x^\nu)$ such that the metric is invariant under translations along it. This invariance is expressed by the following equation:

$$g_{\mu\nu,\rho} \xi^\rho + g_{\rho\nu} \xi^\rho_{,\mu} + g_{\mu\rho} \xi^\rho_{,\nu} = 0. \quad (1.3)$$

This is called the *Killing's equation*. The vector ξ^μ is called the *Killing vector*.

As an example, if the Killing vector is in the form $\xi^\mu = (0, 0, 0, 1)$, the Killing equation becomes:

$$\frac{\partial g_{\mu\nu}}{\partial x^4} = 0 \quad (1.4)$$

which express the invariance along x^4 -direction in a form which is familiar to everybody.

By using the properties of the Ricci tensor, it can be shown that an N -dimensional Riemannian space possesses at most $(N+1)N/2$ linearly independent Killing vectors. Thus, physical spacetimes for which $N = 4$, can admit at most 10 Killing vectors. The full 10 Killing vector symmetry can be found in flat spacetimes – i.e. spacetimes for which all the Christoffel symbols are null.

The above transformations, associated with Killing vectors, and that preserve the metric, are called *Isometries* (or *Motions*). They possess a Lie group structure: the relevant group is called the *Group of Isometry* and indicated with G_r , where r is the number of Killing vectors. The *structure constants* f_{ij}^k of the group are given by:

$$\xi_i^\mu \frac{\partial}{\partial x^\mu} \xi_j^\nu - \xi_j^\mu \frac{\partial}{\partial x^\mu} \xi_i^\nu = f_{ij}^k \xi_k^\nu, \quad i, j, k = 1, \dots, r. \quad (1.5)$$

Naturally, the structure constants are antisymmetric – i.e. $f_{ij}^k = -f_{ji}^k$ – and satisfy the Jacobi identity

$$f_{il}^k f_{jn}^\ell + f_{jl}^k f_{ni}^\ell + f_{nl}^k f_{ij}^\ell = 0. \quad (1.6)$$

After introducing the quantities

$$\xi_i = \xi_i^\mu \frac{\partial}{\partial x^\mu}, \quad (1.7)$$

equation (1.5) assumes the familiar form

$$[\xi_i, \xi_j] = f_{ij}^k \xi_k. \quad (1.8)$$

The ξ_i 's may be regarded as the generators of the group algebra

The *Regions of Transitivity* (or *Transitivity Surfaces*) are those regions of the spacetime whose points can be carried into one another by the symmetry transformations of the group.

As an example, the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{2M}{r}\right) dt^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1.9)$$

is clearly invariant under rotations of a generic angle ϕ about the axis $\theta = 0, \pi$, which means it admits the Killing vector ∂_ϕ . The regions of transitivity are the spheres $r = \text{const.}$

The spacetimes can be classified according to the kind of isometry group they admit, i.e. the number of Killing vectors, the form of the structure constants and

the regions of transitivity: the problem of the classification of all the non-isomorphic G_r , is a known and solved issue in the context of Group Theory.

For instance, there exist 9 non-isomorphic G_3 groups, the so-called *Bianchi Models*. A simpler example is provided by the G_2 groups. In this case we have only two Killing vectors ξ_1^μ and ξ_2^μ which can only give either:

$$[\xi_1, \xi_2] = 0, \quad (1.10)$$

or

$$[\xi_1, \xi_2] = c_1 \xi_1 + c_2 \xi_2. \quad (1.11)$$

By possibly redefining the Killing vectors, one of the constants c_i can always be let vanish. In the first case we have an abelian G_2 group.

1.2 Symmetries of Physical interest

Many metrics of physical interest possess an high degree of symmetry.

The first metric ever discovered, the *Schwarzschild Metric* [4] – see (1.9) – is a diagonal time-independent spherically-symmetric solution. It describes the exterior field surrounding a star. As such, it possesses 4 Killing vectors:

$$\begin{aligned} \xi_1 &= \sin \phi \partial_\theta + \cos \phi \cot \theta \partial_\phi, & \xi_3 &= \partial_\phi, \\ \xi_2 &= \cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi, & \xi_4 &= \partial_t. \end{aligned} \quad (1.12a)$$

The *Kerr Solution* [5]

$$\begin{aligned} ds^2 &= \Sigma \left(\frac{1}{\Delta} dr^2 + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\phi^2 - dt^2 + \frac{2Mr}{\Sigma} (a \sin^2 \theta d\phi - dt)^2, \\ \Sigma &= r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \end{aligned} \quad (1.13)$$

is a non-diagonal axisymmetric solution. It contains the Schwarzschild metric in the limit when a vanishes. It describes the exterior field surrounding a rotating star. It possesses 2 Killing vectors:

$$\xi_1 = \partial_t, \quad \xi_2 = \partial_\phi. \quad (1.14)$$

Both the Schwarzschild and the Kerr metrics have been generalized to the case when an electromagnetic field, generated by a point-like charge e , is taken in account. The *Reissner-Nordström Metric* [6, 7]

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r} + G \frac{e^2}{2r^2} \right)^{-1} dr^2 - \left(1 - \frac{2M}{r} + G \frac{e^2}{2r^2} \right) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \\ A^\mu &= \left(-\frac{e}{r}, 0, 0, 0 \right), \end{aligned} \quad (1.15)$$

generalizes the Schwarzschild metric and the *Kerr-Newman Metric* [8, 1]

$$\begin{aligned} ds^2 &= \Sigma \left(\frac{1}{\Delta} dr^2 + d\theta^2 \right) - \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - 2a \frac{2Mr - e^2}{\Sigma} \sin^2 \theta d\phi dt + \\ &\quad + \frac{(a^2 + r^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2, \end{aligned}$$

$$A_\mu = -e \frac{r}{\Sigma} [(dt)_\mu - a \sin^2 \theta (d\phi)_\mu], \quad (1.16a)$$

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2 - e^2, \quad (1.16b)$$

generalizes the Kerr metric.

The *Robertson-Walker Metric* is an isotropic spherically-symmetric time-dependent solution of Einstein's equations [2]:

$$ds^2 = -dt^2 + K(t)^2 (dX^2 + f(X)^2 (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (1.17a)$$

$$f(X) = \begin{cases} \sin X & \text{for } \kappa = +1 \\ X & \text{for } \kappa = 0 \\ \sinh X & \text{for } \kappa = -1 \end{cases} . \quad (1.17b)$$

Here, X is a space-like coordinate, $r^2 = x^2 + y^2 + z^2$ and κ is a parameter both related to the curvature of the spatial (cartesian) sector of the spacetime through

$$ds^2 = -dt^2 + K(t)^2 \frac{1}{(1 + \kappa r^2/4)^2} (dx^2 + dy^2 + dz^2) . \quad (1.18)$$

This metric possesses 6 Killing vectors [2]:

$$\xi_1 = \sin \phi \partial_\theta + \cos \phi \cot \theta \partial_\phi , \quad (1.19a)$$

$$\xi_2 = \cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi , \quad (1.19b)$$

$$\xi_3 = \partial_\phi , \quad (1.19c)$$

$$\xi_4 = \left[1 - \frac{1}{4} \kappa r^2\right] \partial_x + \frac{1}{2} \kappa x (x \partial_x + y \partial_y + z \partial_z) , \quad (1.19d)$$

$$\xi_5 = \left[1 - \frac{1}{4} \kappa r^2\right] \partial_y + \frac{1}{2} \kappa y (x \partial_x + y \partial_y + z \partial_z) , \quad (1.19e)$$

$$\xi_6 = \left[1 - \frac{1}{4} \kappa r^2\right] \partial_z + \frac{1}{2} \kappa z (x \partial_x + y \partial_y + z \partial_z) , \quad (1.19f)$$

Of these, the first 3 Killing vectors take in account the symmetry under rotations – i.e. the spatial isotropy – and the others the symmetry under translations – i.e. the homogeneity.

The class of *Kantowski-Sachs Metrics*:

$$ds^2 = dt^2 - a^2(t) dx^2 - b^2(t) (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1.20)$$

possesses 4 Killing vectors:

$$\begin{aligned} \xi_1 &= \sin \phi \partial_\theta + \cos \phi \cot \theta \partial_\phi , & \xi_3 &= \partial_\phi , \\ \xi_2 &= \cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi , & \xi_4 &= \partial_x . \end{aligned} \quad (1.21a)$$

These two classes of metrics are time-dependent – i.e. none of the Killing vectors is time-like – and have thus been considered as possible cosmological models [9, 10]. The Bianchi models mentioned above may also be considered for describing cosmological models. In fact, when all the Killing vectors ξ_a^i are taken to be space-like, these metrics can be put in the form

$$ds^2 = dt^2 - g_{ij}(t) (\xi_a^i dx^a) (\xi_b^j dx^b) , \quad a, b = 1, 2, 3 , \quad i, j = 1, 2, 3 \quad (1.22)$$

The Robertson-Walker metrics provide *Isotropic Spatially-Homogeneous Cosmological Models*. In an attempt to provide more accurate cosmologies, one can release the assumption of isotropy, i.e. the symmetry under rotations. This is partly done

in the models based upon the Kantowski-Sachs metrics which, in fact, have only 4 Killing vectors. The Bianchi Models retain the sole symmetry under translations and therefore may provide *Spatially Homogeneous Anisotropic Cosmological Models*. Further refinements in the construction of effective cosmological models lead to the relaxation of the spatial homogeneity – *Spatial Inhomogeneous Cosmologies*. The simplest way to do that is to break the symmetry in one spatial direction only. Therefore a symmetry under two Killing vectors is maintained.

Other than that, spacetimes with two Killing vectors may be considered in order to describe the interaction between gravitational waves. A typical line element, describing a gravitational plane wave propagating in u -direction, is given by.

$$ds^2 = 2du dv - e^{-U} (e^V \cosh W dx^2 - 2 \sinh W dx dy + e^{-V} dy^2) , \quad (1.23a)$$

$$u = \frac{1}{\sqrt{2}}(t - z) , \quad v = \frac{1}{\sqrt{2}}(t + z) , \quad (1.23b)$$

where U, V, W are functions depending of u only. These spacetimes admit a 5 Killing vector symmetry [11]

$$\xi_1 = \partial_x , \quad \xi_2 = \partial_y , \quad \xi_3 = \partial_v , \quad (1.24a)$$

$$\xi_4 = x \partial_v + P_-(u) \partial_x + N(u) \partial_y , \quad \xi_5 = y \partial_v + P_+(u) \partial_y + N(u) \partial_x , \quad (1.24b)$$

$$N(u) = \int du e^U \sinh W , \quad P_{\pm}(u) = \int du e^{U \pm V} \cosh W . \quad (1.24c)$$

When two such waves, propagating in u - and v -directions, come to interact, it is reasonable to suppose that the resulting metric will retain only a two Killing vector symmetry generated by ξ_1 and ξ_2 . In fact, it can be expected that, after the interaction, the metric will be still independent of x and y .

In these two cases, the two Killing vectors are considered to be both spacelike. As mentioned above – see the Kerr metric – situations with one space-like and one time-like Killing vector are also possible.

The discussion put forward so far should make evident the wide variety of physical problems that may be described by considering spacetimes with a two Killing vector symmetry – and of commuting Killing vectors in particular [10]. The next section will be devoted to a presentation of the main features – such as line element and field equations – of a spacetime with an abelian G_2 group of isometry.

1.3 The Abelian G_2 Spacetimes

The Line Element

Let us suppose that the 4-dimensional spacetime (1.2) admits an Abelian two-dimensional group of isometries for which the two-dimensional transitivity surfaces are not isotropic, i.e. the spacetime admit the existence of two commuting Killing vectors ξ_1^μ and ξ_2^μ acting orthogonally transitively.

The fact that the two ξ_a^μ 's commute with each other, implies that we can choose coordinates so that they both are coordinate vector fields: $\xi_a^\mu = \delta_a^\mu$.

Under these assumptions, it can be shown that the spacetimes can be rewritten as [51]:

$$ds^2 = -f \eta_{AB} dx^A dx^B - g_{ab} dx^a dx^b, \quad \eta_{AB} = \begin{pmatrix} -\epsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad (1.25a)$$

$$f = f(\alpha, \beta), \quad g_{ab} = g_{ab}(\alpha, \beta), \quad x^A = (\alpha, \beta), \quad (1.25b)$$

$$\det g_{ab} = \epsilon \alpha^2, \quad \epsilon = \pm 1, \quad A = 1, 2, \quad a = 3, 4, \quad (1.25c)$$

where the x^a 's are ignorable coordinates

We will not give a proof of this result. It is a well known result whose derivation can be found, for example, in [1].

Stationary Axisymmetric case

The stationary axisymmetric metrics are recovered from the above line element with the choice:

$$\epsilon = -1, \quad \xi_1 = \partial_t, \quad \xi_2 = \partial_\phi, \quad (1.26)$$

It is then convenient to choose coordinates ρ and z such that $\alpha = \rho$ and $\beta = z$.

In general, a metric is said to be *stationary* if it possesses a time-like Killing vector. It is *static* if this also is hypersurface orthogonal. As examples, the Kerr metric is stationary axisymmetric and the Schwarzschild, which is diagonal, is static spherically symmetric.

The stationary axisymmetric spacetimes are usually rewritten in the form [13, 14]:

$$ds^2 = \frac{1}{f} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2] - f (dt - \omega d\phi)^2. \quad (1.27)$$

Under certain assumptions, it can be shown – again see [1], theorem 7.1.1 – that the 2-planes orthogonal to the Killing vectors are integrable. These assumptions are satisfied in a wide range of spacetimes of physical interest. In particular, this is so for asymptotically flat spacetimes, i.e. $\gamma \rightarrow 0$, $\omega \rightarrow 0$, $f \rightarrow 1$, as $\rho \rightarrow \infty$.

Cosmological case

The cosmological case is recovered from (1.25) with the choice:

$$\epsilon = +1, \quad \xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad (1.28)$$

Here both ξ_1 and ξ_2 are spacelike. A useful choice for α and β is given by $\alpha = t$ and $\beta = z$. However, alternative expressions are sometimes necessary as will be seen below.

When $\epsilon = 1$, the above line element (1.25) can also be rewritten in the equivalent form [11]:

$$ds^2 = 2 e^{-M} du dv - e^{-U} \left(\frac{1}{\chi} (dx - \omega dy)^2 + \chi dy^2 \right), \quad (1.29)$$

where x and y are ignorable coordinates and u and v are two arbitrary null coordinates.

Cylindrically symmetric case

The Cylindrically symmetric metrics are recovered from (1.25) with the choice:

$$\epsilon = +1, \quad \xi_1 = \partial_x, \quad \xi_2 = \partial_\phi, \quad (1.30)$$

Useful coordinates are $\alpha = \rho$ and $\beta = t$.

The Field Equations

Let us consider the line element (1.29). With this, for the vacuum case, the Einstein's Field Equations (EFE) become [11]:

$$U_{,uv} = U_{,u} U_{,v} \quad (1.31a)$$

$$2 U_{,vv} = U_{,v}^2 + \frac{1}{\chi^2} (\chi_{,v}^2 + \omega_{,v}^2) - 2 U_{,v} M_{,u} \quad (1.31b)$$

$$2 U_{,uu} = U_{,u}^2 + \frac{1}{\chi^2} (\chi_{,u}^2 + \omega_{,u}^2) - 2 U_{,u} M_{,u} \quad (1.31c)$$

$$2 M_{,uv} = - U_{,u} U_{,v} + \frac{1}{\chi^2} (\chi_{,u} \chi_{,v} + \omega_{,u} \omega_{,v}) \quad (1.31d)$$

$$2 \chi_{,uv} = U_{,u} \chi_{,v} + \chi_{,u} U_{,v} + \frac{2}{\chi} (\chi_{,u} \chi_{,v} - \omega_{,u} \omega_{,v}) \quad (1.31e)$$

$$2 \omega_{,uv} = U_{,u} \omega_{,v} + \omega_{,u} U_{,v} + \frac{2}{\chi} (\chi_{,u} \omega_{,v} + \omega_{,u} \chi_{,v}) \quad (1.31f)$$

where $,u$ and $,v$ indicate differentiation with respect of u, v respectively. Notice that, differently from [11], we put.

$$\chi = e^{-V} \operatorname{sech} W, \quad \omega = e^{-V} \operatorname{tgh} W. \quad (1.32)$$

Consider $\alpha = e^{-U}$. Equation (1.31a) becomes $\alpha_{,uv} = 0$. This immediately implies that

$$\alpha = \mathcal{F}(u) + \mathcal{G}(v). \quad (1.33)$$

The β coordinate – which is harmonically conjugate to α – will be defined as $\beta = \mathcal{G}(v) - \mathcal{F}(u)$.

A typical choice which we will consider throughout most of the thesis is:

$$\alpha = \frac{1}{\sqrt{2}} (u + v), \quad \beta = \frac{1}{\sqrt{2}} (v - u), \quad (d\alpha^2 - d\beta^2) = 2 du dv. \quad (1.34)$$

However, in chapter 5 different choices will be required.

The class of Weyl metrics

Let us introduce the Weyl line element [2]

$$ds^2 = -e^{-2\psi} dt^2 + e^{-2\psi} (e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2) . \quad (1.35)$$

The Weyl metrics are static axisymmetric spacetimes. They are a particular case of (1.25) in which the choice (1.26) has been made and the diagonal case only is considered. With respect to (1.27), we have $\omega = 0$ and $f = e^{2\psi}$.

In this case, the vacuum field equations assume the simplified form [2]:

$$\psi_{,\rho\rho} + \frac{1}{\rho} \psi_{,\rho} + \psi_{,zz} = 0 , \quad \gamma_{,\rho} = \rho (\psi_{,\rho}^2 + \psi_{,z}^2) , \quad \gamma_{,z} = 2 \rho \psi_{,\rho} \psi_{,z} . \quad (1.36)$$

The equation for ψ is the main equation we need to solve. Once we get a solution for this, then it is straightforward to obtain a solution for γ .

Actually, the equation for ψ is the usual potential equation in flat space. One might then think that, given a classical solution ψ – Newtonian theory of Gravitation – one might obtain a relativistic solution by solving the equation for γ . Comments on this somehow misleading conjecture, have been considered in [2]

The class of Einstein–Rosen metrics

Let us introduce the Einstein–Rosen line element:

$$ds^2 = e^{2(\gamma-\psi)} (dt^2 - d\rho^2) - (\rho^2 e^{-2\psi} d\phi^2 + e^{+2\psi} dz^2) . \quad (1.37)$$

The Einstein–Rosen metrics are cylindrically symmetric spacetimes. They are a particular case of (1.25) in which the choice (1.30) has been made and the diagonal case only is considered.

In this case, the vacuum field equations assume the simplified form:

$$\psi_{,\rho\rho} + \frac{1}{\rho} \psi_{,\rho} - \psi_{,tt} = 0 \quad \gamma_{,\rho} = \rho (\psi_{,\rho}^2 + \psi_{,t}^2) , \quad \gamma_{,t} = 2 \rho \psi_{,\rho} \psi_{,t} . \quad (1.38)$$

The equation for ψ is the main equation we need to solve. Once we get a solution for ψ then it is straightforward to obtain a solution for γ , much as in the previous case

Metrics (1.35) and (1.37) have been written in a form such that the equation for ψ is linear: actually it is the usual wave equation rewritten in the appropriate coordinates. Having that, the general solution for ψ is a linear superposition of Bessel functions J_0 and Y_0 .

The Einstein–Rosen metrics can easily be converted to diagonal cosmological metrics

$$ds^2 = f(dt^2 - dZ^2) - g_{11} dx^2 - g_{22} dy^2 , \quad (1.39)$$

by performing the simple substitution of coordinates:

$$\rho_{cyl} \rightarrow t_{cosm} , \quad t_{cyl} \rightarrow z_{cosm} . \quad (1.40)$$

In fact, by rewriting the metric components as:

$$g_{11} \mapsto t^2 e^{-2\psi}, \quad f \mapsto e^{2(\gamma-\psi)}, \quad (1.41)$$

the line element (1.39) can be put in the form:

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dZ^2) - (t^2 e^{-2\psi} dx^2 + e^{2\psi} dy^2), \quad (1.42)$$

whose relevant field equation is now:

$$\psi_{,tt} - \psi_{,zz} + \frac{1}{t} \psi_{,t} = 0. \quad (1.43)$$

Again, this equation is linear and therefore ψ enjoys the linear superposition property.

1.4 Solution generating techniques

As mentioned above, the abelian G_2 spacetimes admit a set of equations which are completely integrable. Over the years various attempts have been performed in order to obtain solutions and to give a physical meaning to them.

Over the last two decades, a number of solution generating techniques have been developed. A striking account of these is given in [12], from the voice (or, by the pen) of the very authors.

On the other hand, a few solutions – whose construction was based on the intuition of the researcher, rather than on a systematic approach to the equations – had already been produced. In [15] a good review of the main results is given. It is by using one of these “naive” approaches, the *complex trick*, that the most important Kerr-Newman metric [8] was discovered.

The crucial events that facilitated a systematic study of spacetimes with two commuting Killing vectors were the discovery of the Ernst Equation and of the Geroch Group.

The Ernst Equation

Stationary Axisymmetric case

Let us consider an axisymmetric line element in the form (1.27). In [16, 17] F. Ernst has shown that the field equations for this line element are equivalent to the existence of two potentials \mathcal{E} and Φ that satisfy the equations:

$$(\operatorname{Re} \mathcal{E} + |\Phi|^2) \nabla^2 \mathcal{E} = (\nabla \mathcal{E} + 2 \bar{\Phi} \nabla \Phi) \cdot \nabla \mathcal{E}, \quad (1.44a)$$

$$(\operatorname{Re} \mathcal{E} + |\Phi|^2) \nabla^2 \Phi = (\nabla \mathcal{E} + 2 \bar{\Phi} \nabla \Phi) \cdot \nabla \Phi, \quad (1.44b)$$

where $\nabla = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial z} \right)$.

The functions \mathcal{E} , Φ are called the *Ernst Potentials*. The metric components are related to these through the relations – see [3], formulae (17.31), (17.32), (17.33):

$$\mathcal{E} = (f - |\Phi|^2) + i\varphi, \quad (1.45a)$$

$$\omega_{,\zeta} = -\frac{\rho}{f^2} (i\varphi_{,\zeta} + \bar{\Phi} \Phi_{,\zeta} - \Phi \bar{\Phi}_{,\zeta}), \quad \sqrt{2} \partial_{\zeta} = \partial_{\rho} - i\partial_z, \quad (1.45b)$$

$$\gamma_{,\zeta} = \sqrt{2} \frac{\rho}{f^2} \left(\frac{1}{4} (\mathcal{E}_{,\zeta} + 2 \bar{\Phi} \Phi_{,\zeta}) (\bar{\mathcal{E}}_{,\zeta} + 2 \Phi \bar{\Phi}_{,\zeta}) - f \bar{\Phi}_{,\zeta} \Phi_{,\zeta} \right). \quad (1.45c)$$

The function Φ is a scalar potential related to one of the components of $\Phi = \{\Phi_a\}$, in terms of which the electromagnetic field $F_{\mu\nu}$ and the energy-momentum tensor are defined:

$$F^*_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad \omega_{\mu\nu} = F_{\mu\nu} + i F^*_{\mu\nu}, \quad (1.46a)$$

$$\omega_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \omega^{\rho\sigma}, \quad \nabla_\mu \omega^{\nu\mu} = 0, \quad (1.46b)$$

$$T_{\mu\nu} = \frac{1}{2} F^*_{\mu}{}^\lambda \bar{F}_{\nu\lambda}, \quad \omega_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu, \quad \Phi_a = \Phi_\mu \xi_a^\mu, \quad (1.46c)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita symbol. It is assumed that all the functions in the above equations depend upon ρ and z only.

As pointed out in [3], equations (1.44) are a set of *elliptic* differential equations.

Cosmological and Cylindrically Symmetric case

When considering the line element for cosmological or cylindrical symmetric spacetimes in the form (1.29), it is also convenient to introduce the complex function – see [11] formulae (16.19) and subsequent:

$$\mathcal{Z} = (\tilde{\chi} + \mathcal{H}\bar{\mathcal{H}}) + i\tilde{\omega}. \quad (1.47)$$

It can then be shown that the potentials \mathcal{Z} and \mathcal{H} satisfy:

$$(\text{Re } \mathcal{Z} - |\mathcal{H}|^2) \nabla^2 \mathcal{Z} = (\nabla \mathcal{Z} - 2\bar{\mathcal{H}}\nabla \mathcal{H}) \cdot \nabla \mathcal{Z}, \quad (1.48a)$$

$$(\text{Re } \mathcal{Z} - |\mathcal{H}|^2) \nabla^2 \mathcal{H} = (\nabla \mathcal{Z} - 2\bar{\mathcal{H}}\nabla \mathcal{H}) \cdot \nabla \mathcal{H}, \quad (1.48b)$$

The pair of potentials \mathcal{Z} and \mathcal{H} are the cosmological analogue of the stationary axisymmetric potentials \mathcal{E} and Φ . The metric components χ, ω in (1.29) are obtained from $\tilde{\chi}$ and $\tilde{\omega}$ by way of the relations:

$$\omega_{,\tau} = \frac{1-\zeta^2}{\tilde{\chi}^2} (\tilde{\omega}_{,\zeta} - i\mathcal{H} \bar{\mathcal{H}}_{,\zeta} + i\bar{\mathcal{H}} \mathcal{H}_{,\zeta}), \quad \alpha = \sqrt{1-\tau^2} \sqrt{1-\zeta^2}, \quad (1.49a)$$

$$\omega_{,\zeta} = \frac{1-\tau^2}{\tilde{\chi}^2} (\tilde{\omega}_{,\tau} - i\mathcal{H} \bar{\mathcal{H}}_{,\tau} + i\bar{\mathcal{H}} \mathcal{H}_{,\tau}), \quad \beta = \tau \zeta, \quad (1.49b)$$

$$\chi = \alpha \frac{1}{\tilde{\chi}}. \quad (1.49c)$$

In the vacuum case, the M function is then easily obtained by using the field equations (1.31) – see [11] for the electrovacuum case.

It may be noticed that the equations (1.48) are a set of *hyperbolic* differential equations.

Vacuum spacetimes of this class present an interesting property. Let us introduce the complex function:

$$Z = \chi + i\omega, \quad (1.50)$$

so that the line element (1.29) becomes [11]:

$$ds^2 = 2e^{-M} du dv - 2e^{-U} \frac{1}{Z + \bar{Z}} (dx + iZ dy) (dx - i\bar{Z} dy). \quad (1.51)$$

The function Z is the so-called *Ernst function* – to be distinct from the Ernst potential \mathcal{Z} . It can be shown that the two main equations (1.31e) and (1.31f) take the nicer form – see [11], formulae (11.4) to (11.8).

$$\operatorname{Re} Z \nabla^2 Z = 2(\nabla Z)^2. \quad (1.52)$$

This is nothing but the vacuum analogue of equations (1.48). Naturally, the Ernst function Z and the Ernst potential \mathcal{Z} are interrelated by means of equations (1.49) – in the vacuum limit $\mathcal{H} = 0$:

$$\omega_{,\tau} = \frac{1 - \zeta^2}{\tilde{\chi}^2} \tilde{\omega}_{,\zeta}, \quad \omega_{,\zeta} = \frac{1 - \tau^2}{\tilde{\chi}^2} \tilde{\omega}_{,\tau}, \quad \chi = \alpha \frac{1}{\tilde{\chi}}. \quad (1.53)$$

Moreover, both Z and \mathcal{Z} satisfy the same vacuum Ernst equation (1.52). Equations (1.53) therefore represent an invariance transformation for (1.52). This is referred to as the *Neugebauer-Kramer Involution*. It is an auto-Bäcklund transformation for the vacuum Ernst equation.

The Geroch Group K

In [18], Geroch was able to give an algorithm for generating new vacuum solutions from a known one. Originally, he dealt with the case when the metrics admit one Killing vector. The new metric is given by [1].

$$g_{\mu\nu} = \sigma g_{0\mu\nu} + 2 \sin \theta \xi_{(\mu} \gamma_{\nu)} + \frac{\lambda \sin^2 \theta}{\sigma} \gamma_\mu \gamma_\nu, \quad (1.54a)$$

$$\sigma = (\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta, \quad (1.54b)$$

$$\gamma_\mu = 2\alpha_{,\mu} \cos \theta - \beta_{,\mu} \sin \theta. \quad (1.54c)$$

θ is an arbitrary parameter such that $\theta \in [0, \pi]$. Unfortunately, the application of a second such algorithm gives back the initial metric g_0 , i.e. the iteration of the procedure does not generate further solutions.

In [19], this method was generalized to the case when two Killing vectors may exist, such as to generate a two-parameter family of solutions. Indeed, it has also been proved [19] that an infinite dimensional group of transformations is generated by repeatedly applying the transformation (1.54).

The extreme power of such a technique was confirmed when Hauser and Ernst [33] discovered that all the asymptotically flat, stationary axisymmetric spacetimes can be generated in this way by starting from the Minkowski metric, as conjectured by Geroch himself. Furthermore, the Geroch's hypothesis has been generalized by N. R. Sigbatullin. He proved that arbitrary free gravitational, electromagnetic and neutrino fields – admitting an abelian G_2 of isometries – can be obtained from a Minkowski spacetime by means of some transformations which generalize the original Geroch's ones [59].

Solution Generating Techniques

Giving the Geroch set of transformations a group structure, enabled a number of new techniques to be discovered

- **The Kinnersley–Chitre Transformations (KC)**

In [20, 21], Kinnersley and Chitre used the Ernst equation to construct an infinite hierarchy of potentials

$$F(t) = \sum_{n=0}^{\infty} t^n H_n, \quad G(s, t) = \sum_{n,m=0}^{\infty} t^n s^m N_{mn}. \quad (1.55)$$

Here, t, s are complex parameters. F, G are 2×2 matrix functions analytic in t, s . Starting from F , an axisymmetric solution can be uniquely produced according to a certain procedure. F satisfies certain partial differential equations the solution of which, requires that some initial conditions – at $t = 0$ – be given. To fix these initial conditions is tantamount to specifying some known solution of Einstein's equations, i.e. a “seed” solution. Incidentally, the matrix function $F(t)$ is somehow related to the Ernst potentials.

Subsequently, they also showed in [22] that the solutions thus obtained, provided a representation of the Geroch group.

The above procedure has been used in [23] to generate the so-called Kinnersley-Chitre solutions.

- **The Hoenselaers–Kinnersley–Xanthopoulos Transformations (HKX)**

Starting from the KC transformations, Hoenselaers, Kinnersley and Xanthopoulos generated another set of transformations [24, 25, 26] – see also Hoenselaers in [12].

After translating the HKX transformation in the language of the Ernst equation, these are shown [33] to generate all the asymptotically flat, stationary axisymmetric spacetimes, when applied to a general Weyl metric. This had already been conjectured in [24, 26].

The HKX transformation has been used in [27, 28] to generate a solution representing two Kerr masses kept apart by their spin-spin interaction.

- **The Hauser–Ernst (HE) Homogeneous Hilbert Problem (HHP)**

In [29], Hauser and Ernst reformulated the KC (vacuum-to-vacuum) transformations in terms of a linear integral equation:

$$f(t) + \frac{1}{2\pi i} \int ds \frac{1}{s-t} K(s) (f(s) + s^{-1}I) = 0, \quad (1.56a)$$

$$F(t) = [I + t f(t)] F_0(t), \quad (1.56b)$$

where $F(t)$ is the usual KC function given in (1.55) and the Kernel function $K(s)$ is determined by the seed through the relations:

$$K(s) = F_0(s) \tau(s) F_0^{-1}(s), \quad \tau(s) \varepsilon = e^{\gamma(s)} \varepsilon - I, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1.57a)$$

In [30], the construction was generalized to the electrovacuum case.

Subsequently, the same authors gave proof that the above linear integral equation can be solved by setting an HHP [31]. This result was generalized to the

electrovacuum case in [32] A brief introduction of what an HHP is was given in Hauser's paper in [12].

Finally, we would like to quote the following sentence from [31] *"... Moreover, our HHP appears to be a link between the group theoretical and the recent soliton approaches to exact solution research, though the authors do not yet understand the details of this link"* Details on the soliton techniques will be introduced below.

An emphasis on the Ernst equations itself, rather than on its group properties, can be recognized in the attempts at obtaining solutions by way of rewriting the relevant equations in a more suitable form, or by way of Bäcklund transformations.

- **Harrison (HAR)**

In 1978, Harrison [34] introduced a Bäcklund transformation for the Ernst Equation. This was done by considering its prolongation structure – see also Harrison in [12] He also applied this technique to generate new solutions [35].

Invariance transformations for the Einstein-Maxwell equations had already been considered by Harrison himself in [36] – see also theorem (30.5) in [3].

- **Neugebauer-Kramer (NK)**

In 1979 another Backlund transformation was introduced by Neugebauer [37, 38]. As an application, in [39] the important Kerr-NUT solution was given – see also [3] formula (30.25) A generalization of the $N = 2$ Neugebauer transformation has been given in [44]. Applications of [37, 38] can also be found in [45, 46].

In 1969 the same authors [40, 41] had already introduced some invariance transformations of Einstein-Maxwell equations – see [3], formulae (30.25) and section 30.5. These transformations also contained, as particular case, some previous result they had obtained in [42, 43] – see [3], theorem (30.9).

The attempt to solve the Ernst equation by way of Bäcklund transformations has also had the merit of opening the door to the use, in GR, of techniques already known in the field of Integrable Systems: in particular, the inverse scattering techniques or soliton techniques and the related formalisms. With these, a stronger emphasis is given to the Einstein's and Einstein-Maxwell's equations.

- **The Belinski-Zakharov soliton technique (BZ)**

This is a solution-generating-technique that produces vacuum metrics [47, 48, 49]. Starting from some known solution – “seed” – the technique is based on the construction of a so-called “dressing” matrix which is a meromorphic function of an unphysical parameter that can subsequently be removed. It is essentially modelled upon the usual inverse-scattering techniques for solving nonlinear p.d e's such as the Korteweg-de Vries or Sine-Gordon equation. By analogy, the solutions thus generated are called “solitons”.

In [50] a generalization to the electrovacuum case is attempted.

A complete account of this technique will be given in chapter 2.

- **The Alekseev soliton technique (ALEK)**

This technique generalizes the previous one to the case when an electromagnetic field is also considered [51, 52, 53, 54]. A different linear-pair is constructed. Non-solitonic solutions of these – i.e. solutions in which the dressing matrix has not a meromorphic structure – have also been given in [55].

The previous construction has been generalized to the case in which a neutrino field is also considered [56].

Finally, solutions of the linear-pair have been shown to be equivalent to the solutions of a certain linear integral equation in [57, 58].

A complete account of this technique will also be given in chapter 2

- **The Sigbatullin soliton technique (SIGB)**

Sigbatullin [60] has also introduced a generating technique for the Einstein-Maxwell equations. This has been reviewed in [61, 62] – see also [15].

This method has lately been applied [63, 64], [65] to get asymptotically flat solutions representing the exterior field of a magnetized spinning mass.

As pointed out in [63, 66], the advantage of adopting Sigbatullin's generating technique, would consist in a more clear relation between the free parameters entering the solution and their physical meaning.

As brief and incomplete as it may be, this small account of solution-generating-techniques cannot be finished without having mentioned the results obtained by Bonnor and Ehlers. These might be regarded as typical of that pioneering age preceding the discover of Geroch group. They mainly are invariance transformations of the field equations – see also [3], chap. 30.

- **Ehlers Transformations**

In [67], Ehlers showed that, given a solution Z_0 of the Ernst equation, a new one is given by [11]:

$$Z = \imath \frac{aZ_0 + \imath b}{cZ_0 + \imath d} \quad (1.58)$$

where the constants a, b, c, d are the components of an $SL(2, \mathbb{R})$ matrix M that transforms the Killing vectors ∂_x and ∂_y of the line element (1.51) according to:

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \mapsto M \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.59)$$

The electrovacuum generalization was also provided – see theorem 30.3 in [12]. This transformation is a particular case of those given in [42].

- **Bonnor Transformations**

In [68], Bonnor showed how to generate magnetostatic solutions from electrostatic ones (*duality transformations*). In [69] he also showed how to generate static solution of Einstein-Maxwell equations starting from vacuum stationary axisymmetric ones. The same solutions as in [69] were also considered in [70]. These are now referred to as the *Bonnor-Melvin solutions*.

To conclude this short review, we also mention the solutions found by A. Tomimatsu and H. Sato [71, 72] as representative of the pioneering era which ended with the discover of the Geroch group.

Far from being exhaustive, the above summary rather reflects our personal taste and knowledge – and ignorance. Once again, we refer to both [3] and [12] for a more sound review.

We have tried to sketch a pattern from which the richness of the study of two Killing vector spacetimes were evident. Richness that entails the several approaches one might adopt in considering this sector of classical general relativity: from the point of view of both physics and mathematics. For example, the development of soliton techniques in GR – as well as of the Backlund transformation approach – tells of the strong links with the theory of Integrable Systems, not to mention about the group theory approach.

The existence of all the above techniques has posed the theoretical problem of their interrelationship. Since the commuting two Killing vector spacetimes have this underlying group structure – the Geroch group K – the different techniques might correspond to different representations of the group. Naturally, one is interested in understanding the actual way in which that occurs. Moreover, each technique allows for the generation of solutions that might be equivalent, although written in different forms: again, it is important to understand the relationship between these metric in order to avoid the reproduction of known results. A complete account of these topics is far beyond the scopes of these thesis. Again, we refer to [3] and [12], as well as the papers by Cosgrove [73, 74, 75], Kramer [76] and Kitchingham [77, 78].

In this thesis, we will only be interested in the solitonic techniques introduced by V.A. Belinski and V.E. Zakharov, for the vacuum case, and by G.A. Alekseev for the electrovacuum case.

1.5 Overview of the thesis

With regard to the problems addressed above, the equivalence between the HKX, the HHP, the HAR and the BZ techniques has been demonstrated by Cosgrove in [73, 75]. To give a flavour, the 4-soliton solution – with Minkowski seed – corresponds to a double Harrison transformation, and the two soliton solution – again with Minkowski seed – corresponds to Kerr metrics with or without horizons depending on whether the poles in the dressing matrix are real or complex. The last result had already been obtained in [49], by simply rewriting the soliton metric in Boyer–Lindquist coordinates. The interrelation with the Tomimatsu–Sato solutions has been given in [79] and also in [80]. For instance, from this we quote. “... *the fact that the Tomimatsu–Sato solution with zero angular momentum and with arbitrary integer distortion parameter is a particular case of the $2n$ -soliton static solution.*”

The Alekseev technique, as anticipated above, generalizes the BZ to the electrovacuum case. Alekseev himself showed in [51] that, when a Minkowski seed is considered, the 1-soliton solution corresponds to a Kerr–Newman metric without horizons, and the 2-soliton solution “... *describes the external field of two interacting charged rotating massive sources of Kerr–Newman type*”. Again, no horizon is

visible.

The interrelation between the two techniques has been considered by quite a few authors. The above comments suggest that a vacuum solution with one pole in the Alekseev formalism is equivalent to a solution with two poles in Belinski-Zakharov's. For the simplest case of one vacuum ALEK- and two BZ-solitons, this conjecture has been explicitly proven by P. Kordas in [15].

The above comment gives the opportunity of introducing one of the problems we will deal with in this thesis

As a matter of fact, the BZ technique allows for the generation of metrics which may have or may have not horizons. This possibility is ruled out in the Alekseev formalism which, instead, allows only for solutions without horizons, even when the vacuum limit is considered. We anticipate here that this is due to the fact that the poles entering the meromorphic dressing matrix must be complex.

In chapter 6, we will show that it is possible to reformulate the Alekseev formalism in such a way as to permit real poles in the dressing matrix and therefore the metric so generated might have horizons. However, we also show that this generalization can work only in the vacuum case. In this case the solutions thus produced are equivalent to the BZ ones. Electrovacuum solutions without horizons are still the only electrovacuum solitons one is able to generate.

The problem of singularities will be also inspected with reference to another issue. It is known that cosmological solitons have singularities along certain null hypersurfaces. The nature of these has been studied in many papers: they are coordinate singularities only and can therefore be removed by an appropriate coordinate transformation. The resulting metric can subsequently be extended through the singularity. Many examples of possible extensions have been given in the literature.

In chapter 5 we will consider this problem with the aim of giving extensions that maintain a zero Ricci tensor. The extensions are given by matching the (transformed) soliton solution with appropriate metrics which, we show, must be different from the seed.

As mentioned above, soliton solutions without horizons are also admissible. When considered in the cosmological context, these solutions represent waves generated at $t = 0$ that propagate along a certain background given by the seed. This interpretation offers the opportunity of studying the intimate nature of gravitational solitons. Solitons of the usual nonlinear p.d.e.'s are characterized by a particular behaviour under interaction: they pass through each other without suffering any modification other than a small shift in the direction along which they propagate.

In chapter 4 we analyze whether or not gravitational solitons also display the same effect. The answer is negative, thus confirming the fact that gravitational solitons must be regarded as gravitational waves rather than solitons.

We will face these issues after having introduced the soliton techniques in chapter 2 and chapter 3. The first three chapters are therefore reviews of known techniques. However, chapters 4, 5 and 6 each contain original material which has been separately published in [113, 114] and [135, 136, 148, 149].

Chapter 2

Soliton Generating Techniques

The aim of this chapter is to give a brief, though self-contained and complete, introduction to solitons in GR. We will mainly quote the known results established in [48, 49], [51, 52, 53, 54] and reviewed in [81]. However, by doing so we will have the opportunity to fix some notation convention and emphasize results that will be used later on.

2.1 Solitons

The history of solitons dates back to one day on 1834 when a british gentleman interested in naval design, Sir J. Scott Russel, noticed something unusual on the canal between Edinburgh and Glasgow. He was walking along the river when he saw what he called a “*wave of translation*” [82]. In his words: “... *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. ...*”. The wave observed by Scott Russel is usually referred to as a “Solitary Wave”. This expression indicates a wave which propagates without change of form and has some localized shape. Moreover, the speed is proportional to the amplitude: the taller the faster.

The word “Soliton” was firstly used in the work of Zabusky and Kruskal in 1965 [83]. They used this word for a particular class of solitary waves, the peculiarity of which consists in their behaviour when interacting with each other: they pass through each other, the only modification being a small change in their phase. For instance, if $u(x, t) = u(x - vt)$ describes a soliton before its interaction, afterwards it can be described by $u(x - vt + \phi)$

The above shift is a clear sign that we are considering solutions for nonlinear equations. In fact, solutions of linear equations do linearly superpose and therefore phase shift effects can not be expected. Solitary waves are due to a very subtle mechanism of compensation between the derivative (dispersion) and the nonlinear terms in the equations of which they are solutions [82]. The fact that solitons retain their own features even after having collided with other solitons may indicate that the

energy they are carrying may be propagated in “*localized stable packets and without dispersion*” [83]. As mentioned in [84], this issue is obviously of great importance in physics [85, 86, 87], biology [88, 89] as well as for engineering applications [90, 91, 92, 93].

Finally, it must be emphasized that not all the nonlinear p.d.e.’s admitting solitary wave solutions have soliton solutions as well. In some cases, it happens that two colliding solitary waves after the collision slightly change their shape. This results in the appearance of small oscillations that arise to balance the loss in energy due to the modification of the solitary wave’s shape [83]

2.1.1 Inverse Scattering Approach

Inverse Scattering Problem

Given a certain differential operator $L(x, \partial_x)$, with eigenfunctions $\psi(x)$ and eigenvalues λ :

$$L \psi(x) = \lambda \psi(x), \quad (2.1)$$

the Inverse Scattering Problem can be posed as follows: *if we know the asymptotic form of ψ for all possible energies:*

$$\begin{cases} \psi(x) \stackrel{x \rightarrow +\infty}{=} a(k) e^{-ikx}, \\ \psi(x) \stackrel{x \rightarrow -\infty}{=} e^{-ikx} + b(k) e^{+ikx}, \end{cases} \quad |a|^2 + |b|^2 = 1, \quad (2.2)$$

how can we reconstruct $L(x, \partial_x)$?

For the Schrödinger equation

$$\psi_{xx}(x) + U(x) \psi(x) = \lambda \psi(x), \quad (2.3)$$

the problem is solved if one is able to find solutions for the Gelfand–Leviatan–Marchenko equation:

$$K(x, z) + G(x + z) = - \int dy K(x, y) G(y + z), \quad (2.4a)$$

$$G(x) = F(x) + \sum_k c_n^2 e^{-k_n x}, \quad (2.4b)$$

where: $F(x)$ is the Fourier Transform of the spectral data $b(k)$, c_n are the analogue of $b(k)$ for the discrete spectrum and $K(x, z)$ is a two variable function in terms of which $U(x)$ may be defined as:

$$U(x) = -2 \frac{\partial}{\partial x} K(x, z)|_{z=x}.$$

Inverse Scattering Techniques and Soliton Solutions

Let us now see how an IST might be used to get solutions for nonlinear p.d.e.’s. Hereafter we will specialize our discussion to the Korteweg–de Vries equation:

$$u_t - 6uu_x + u_{xxx} = 0. \quad (2.5)$$

When using an IST, the starting point is the existence of some scattering data, at least asymptotically. Thus, in order to apply an IST for solving KdV, we need a mechanism to construct such “scattering data” for KdV.

Consider the KdV in the usual form (2.5). By performing the transformation – *Miura Transformation*

$$u = v^2 + v_x, \quad (2.6)$$

it becomes:

$$(2v + \frac{\partial}{\partial x}) (v_t - 6v^2 v_x + v_{xxx}) = 0.$$

Therefore, if v is a solution of equation

$$v_t - 6v^2 v_x + v_{xxx} = 0, \quad (2.7)$$

then (2.6) generates solutions of KdV. Equation (2.7) is called “modified Korteweg-de Vries” (mKdV). Equation (2.6) can be regarded as a *Riccati equation* for v . It can therefore be linearized by means of the further transformation

$$v = \psi_x / \psi. \quad (2.8)$$

With this, equation (2.6) reads:

$$\psi_{xx}(x, t) + u(x, t) \psi(x, t) = 0. \quad (2.9)$$

Since KdV is invariant under transformations like – *Galilean Invariance*

$$u(x, t) \rightarrow \lambda + u(x + 6\lambda t, t), \quad (2.10)$$

we can use this freedom to put (2.9) in the form:

$$\psi_{xx}(x, t) + u(x, t) \psi(x, t) = \lambda \psi(x, t). \quad (2.11)$$

The Schrödinger-like equation (2.11), gives us a way to construct *scattering data*.

This is the first step of the procedure to obtain soliton solutions:

1. Let us suppose that we know $u(x, t)$ at a certain time $t = 0$

$$u(x, 0) = f(x). \quad (2.12)$$

2. Solve (2.11) at $t = 0$ to get an asymptotic ($x \approx \pm\infty$) solution $\psi_0 = \psi(x, 0)$ and $\lambda_0 = \lambda(0)$. This gives the initial *spectral data*

3. Construct asymptotic $\psi(x, t)$ and $\lambda(t)$ starting from ψ_0 and λ_0 .

4. Use IST to obtain $u(x, t)$.

Point 3) above is crucial: how can one have $\psi(x, t)$ and $\lambda(t)$ from ψ_0 and λ_0 ?

Indeed, it is possible to construct a linear differential equation whose solutions are $\psi(x, t)$ and $\lambda(t)$ with ψ_0 and λ_0 as initial conditions. This is called *Time-Evolution Equation* (TEE). For KdV, TEE can be obtained starting from:

$$\frac{\partial}{\partial x} (\psi_x R - \psi R_x) = \lambda_t \psi^2, \quad (2.13)$$

where $R = \psi_t - u_x \psi + 2(u - 2\lambda)\psi_x$. We will see later how this equation can be derived. It allows for the solutions – see [82] chapter 4.

$$\lambda(t) = \lambda_0, \quad b(k, t) = b(k, 0) e^{8ik^3 t}, \quad c_n(t) = c_n(0) e^{4kn^3 t}.$$

Soliton Solutions are that particular class of solutions of the Gelfand–Leviatan–Marchenko equation corresponding to discrete values of the spectral parameter λ .

In conclusion, the key-points to use an IST to get solutions for KdV are:

1. an initial condition (IC).
2. the existence of two linear equations (EQ1,2). The first one (EQ1), together with (IC), gives the scattering data at any fixed time, the latter (EQ2) gives their time-evolution.

Comments

To better understand the previous mechanism, let us see how (2.13) can be derived. Consider the time-independent Schrodinger equation (2.3) and differentiate with respect to both x and t :

$$\psi_{xxx} + u_x \psi + (\lambda + u)\psi_x = 0, \quad (2.14a)$$

$$\psi_{xxt} + (\lambda_t + u_t)\psi + (\lambda + u)\psi_t = 0. \quad (2.14b)$$

Let us define the quantity:

$$R = \psi_t - u_x \psi + 2(u - 2\lambda)\psi_x. \quad (2.15)$$

By using (2.14) it can be seen that R satisfies:

$$\frac{\partial}{\partial x} (\psi_x R - \psi R_x) = \psi^2 (\lambda_t + u_t + 6uu_x + u_{xxx}). \quad (2.16)$$

Let us now consider the pair of linear differential equations

$$\psi_{xx} + (\lambda + u)\psi = 0, \quad (2.17a)$$

$$\frac{\partial}{\partial x} (\psi_x R - \psi R_x) = \psi^2 (\lambda_t + u_t + 6uu_x + u_{xxx}). \quad (2.17b)$$

Equation (2.17b) reduces to (2.13) if the KdV is satisfied. As a result we have:

$$\psi_{xx} + (\lambda + u)\psi = 0, \quad \frac{\partial}{\partial x} (\psi_x R - \psi R_x) = \lambda_t \psi^2, \quad (2.18)$$

therefore KdV is the compatibility conditions for equations (2.18).

Summary

The mechanism to solve KdV (or any other nonlinear equation) by IST can be generalized as follow:

- Consider a solution of the nonlinear equation at any fixed time t_0 (IC).

- Construct a Pair of Linear equations (EQ1, EQ2) whose compatibility condition is given by the nonlinear one (NLE).
- Solve EQ1 + IC to get the scattering data at t_0 .
- Solve EQ2 to determine the evolution of the scattering data at any $t > t_0$.
- Apply the IST (i.e. solve the Gelfand–Leviatan–Marchenko linear integral equation) to determine the solution of the nonlinear equation at any subsequent time.

Soliton solutions correspond to distinct poles in the complex plane of the spectral parameter.

Solitons can also be obtained by considering different approaches to the solution of the relevant equations: Bäcklund and Darboux transformations as well as the Lax-Pair approach, the Zakharov–Shabat approach and the AKNS formalism. Details of these techniques can be found, for example, in [82, 83] and references therein. For a short review focused to the introduction of these techniques in General Relativity, we refer to the paper by Gurses in [12].

2.1.2 Solitons in General Relativity

Solitons in General Relativity are particular solutions of the Einstein Field Equations (EFE) for spacetimes with two Killing vectors. They are obtained by using a technique that resembles those described above in the following points:

1. Instead of solving the Einstein's nonlinear equations (EFE) we try to solve a pair of linear differential equations (whose compatibility conditions are given by EFE)

This is similar to the introduction of a Lax-Pair.

2. These equations involve a “spectral” parameter λ . Their unknown function is a $\Psi(\lambda)$ in terms of which the solution of NLE is given by simply algebraic operations.
3. The function $\Psi(\lambda)$ has meromorphic structure (i.e. discrete simple poles) over the complex plane λ .

This recalls the fact that, for classical soliton solutions, λ are eigenvalues belonging to the discrete energy spectrum.

One more comment: the classical technique is a typical initial value problem: we need (2.12) to obtain the relevant discrete spectrum. In GR, at least for the techniques we will consider, the spectral problem is completely determined by the structure of the linear pair. Nevertheless, to get a solution g , we will have to start from a previously known g_0 , the so-called Seed Metric. For this occurrence, Soliton Techniques in General Relativity are usually referred to as *Dressing Techniques*.

2.2 The Belinski–Zakharov Technique

In this section we will introduce the key features of the Belinski–Zakharov soliton technique, as introduced in two papers [48, 49] in 1978. This technique has been reviewed in [81].

2.2.1 Matrix Form for the Equations

Let us consider the 2 Killing Vector spacetime (1.29) in the form:

$$ds^2 = 2e^{-M} du dv - g_{ab} dx^a dx^b \quad a, b = 1, 2 \quad (2.19)$$

The functions g_{ab} can be interpreted as the components of the following 2×2 matrix.

$$\mathbf{g} = \frac{\alpha}{\chi} \begin{pmatrix} 1 & -\omega \\ -\omega & \chi^2 + \omega^2 \end{pmatrix} \quad \mathbf{g}^{-1} = \frac{1}{\alpha \chi} \begin{pmatrix} \chi^2 + \omega^2 & +\omega \\ +\omega & 1 \end{pmatrix} \quad (2.20)$$

with $\alpha = \sqrt{\det \mathbf{g}} = e^{-U}$. It is also convenient to introduce the following matrices:

$$\mathbf{A} = -\alpha \mathbf{g}_{,u} \cdot \mathbf{g}^{-1} \quad \mathbf{B} = +\alpha \mathbf{g}_{,v} \cdot \mathbf{g}^{-1} \quad (2.21)$$

where $,u$ and $,v$ indicate differentiation with respect of u, v respectively. Explicitly we have:

$$\mathbf{A} = \alpha \begin{pmatrix} -\frac{1}{\chi} \chi_{,u} - U_{,u} - \frac{\omega}{\chi^2} \omega_{,u} & -\omega_{,u} \frac{1}{\chi^2} \\ \frac{2\omega}{\chi} \chi_{,u} - \frac{\chi^2 - \omega^2}{\chi^2} \omega_{,u} & \frac{1}{\chi} \chi_{,u} - U_{,u} + \frac{\omega}{\chi^2} \omega_{,u} \end{pmatrix}, \quad (2.22a)$$

$$\mathbf{B} = \alpha \begin{pmatrix} +\frac{1}{\chi} \chi_{,v} + U_{,v} + \frac{\omega}{\chi^2} \omega_{,v} & +\omega_{,v} \frac{1}{\chi^2} \\ -\frac{2\omega}{\chi} \chi_{,v} + \frac{\chi^2 - \omega^2}{\chi^2} \omega_{,v} & -\frac{1}{\chi} \chi_{,v} + U_{,v} + \frac{\omega}{\chi^2} \omega_{,v} \end{pmatrix}. \quad (2.22b)$$

Proposition (2.2.1) Matrix Form for the Equations

With the above notation, Einstein's Field Equations became:

$$\left. \begin{matrix} (1.31a) \\ (1.31e) \\ (1.31f) \end{matrix} \right\} \rightarrow \mathbf{A}_{,v} - \mathbf{B}_{,u} = 0, \quad (2.23a)$$

$$(1.31b) \rightarrow M_{,u} = -\frac{\alpha_{,uu}}{\alpha_{,u}} + \frac{\alpha_{,u}}{\alpha} - \frac{1}{4\alpha\alpha_{,u}} \text{Tr} \mathbf{A}^2, \quad (2.23b)$$

$$(1.31c) \rightarrow M_{,v} = -\frac{\alpha_{,vv}}{\alpha_{,v}} + \frac{\alpha_{,v}}{\alpha} - \frac{1}{4\alpha\alpha_{,v}} \text{Tr} \mathbf{B}^2, \quad (2.23c)$$

$$(1.31d) \rightarrow M_{,uv} = -\frac{\alpha_{,u}}{\alpha} \frac{\alpha_{,v}}{\alpha} - \frac{1}{4} \text{Tr} \mathbf{A} \mathbf{B}. \quad (2.23d)$$

Proof

The proof is trivial and therefore will not be given.

□ q.e.d.

Equation (2.23d) becomes the integrability condition for (2.23b) and (2.23c). Equation (2.23a) is the most important and, as we will see, is actually *the* equation we *must* solve. Equation (2.23b) and (2.23c) can be easily (at least in principle!) solved, once A and B have been found. Therefore, M is uniquely determined by g_0 up to some integration constants

Proposition (2.2.2) Integrability Condition

Equation (2.21) admit the following integrability condition:

$$\alpha (A_{,v} + B_{,u}) + [A, B] - \alpha_{,v} A - \alpha_{,u} B = 0. \quad (2.24)$$

Proof

The proof is simply done by inspection and by using the following identities:

$$g_{,u}^{-1} = -g^{-1} g_{,u} g^{-1}, \quad g_{,v}^{-1} = -g^{-1} g_{,v} g^{-1}. \quad (2.25)$$

□ q.e.d

As a result, the two equations we are interested in are:

$$\begin{cases} (2.23a) \mapsto A_{,v} - B_{,u} = 0, \\ (2.24) \mapsto \alpha (A_{,v} + B_{,u}) + [A, B] - \alpha_{,v} A - \alpha_{,u} B = 0 \end{cases}$$

2.2.2 A Linear-Pair formulation

Let us introduce the two operators

$$\mathcal{D}_1 = \partial_u - \frac{2\lambda}{\lambda - \alpha} \alpha_{,u} \partial_\lambda, \quad \mathcal{D}_2 = \partial_v + \frac{2\lambda}{\lambda + \alpha} \alpha_{,v} \partial_\lambda, \quad (2.26)$$

where λ is an arbitrary (unphysical) parameter, hereafter referred to as the *spectral parameter*.

Proposition (2.2.3) Property of \mathcal{D}_1 and \mathcal{D}_2

The following implication is true:

$$[\mathcal{D}_1, \mathcal{D}_2] = 0 \iff \alpha_{,uv} = 0. \quad (2.27)$$

Proof

The proof follows immediately from the fact that:

$$(\mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1) = \frac{1}{\lambda^2 - \alpha^2} 4\lambda \alpha_{,uv} \partial_\lambda. \quad (2.28)$$

□ q.e.d.

Proposition (2.2.4) Invariance under Rescaling

\mathcal{D}_1 and \mathcal{D}_2 are invariant under the following substitution:

$$\lambda \mapsto \lambda' = \alpha^2 \frac{1}{\lambda} . \quad (2.29)$$

Proof

Let us suppose that \mathcal{D}_1 acts upon a generic function $f = f(\alpha, \beta, \lambda)$. When considering the transformation (2.29), the u -derivative and the λ -derivative transform as follows:

$$\partial_u \mapsto \partial_u + \frac{\partial \lambda'}{\partial \alpha} \alpha_{,u} \partial_{\lambda'} , \quad \partial_\lambda \mapsto \frac{\partial \lambda'}{\partial \lambda} \partial_{\lambda'} . \quad (2.30)$$

Therefore \mathcal{D}_1 becomes:

$$\mathcal{D}_1 \mapsto \partial_u - \frac{2 \lambda'}{\lambda' - \alpha} \alpha_{,u} \partial_{\lambda'} , \quad (2.31)$$

which proves the assertion. \square q.e.d.

Proposition (2.2.5) A Linear-Pair formulation

Let us introduce the complex matrix function $\Psi = \Psi(u, v, \lambda)$ and consider the following matrix equations.

$$\mathcal{D}_1 \Psi = \frac{1}{\lambda - \alpha} \mathbf{A} \Psi , \quad \mathcal{D}_2 \Psi = \frac{1}{\lambda + \alpha} \mathbf{B} \Psi , \quad (2.32)$$

where $\alpha, \mathbf{A}, \mathbf{B}$ are given in (2.21). Then (2.23a) and (2.24) are the integrability conditions for (2.32).

Proof

The integrability condition for the above linear pair is given by

$$\mathcal{D}_2(2.32.a) - \mathcal{D}_1(2.32.b) = 0 .$$

By using (2.27), and after some little algebra, we have that this is equivalent to prove:

$$0 = \left(-\alpha_{,v} \mathbf{A} - \alpha_{,u} \mathbf{B} + (\mathbf{A} \mathbf{B} - \mathbf{B} \mathbf{A}) + \alpha (\mathbf{A}_{,v} + \mathbf{B}_{,u}) \right) + \lambda (\mathbf{A}_{,v} - \mathbf{B}_{,u}) . \quad (2.33)$$

This is satisfied for arbitrary λ iff both (2.23a) and (2.24) are satisfied. Therefore the assertion is proved. \square q.e.d.

In this respect the system (2.32) is a Linear-Pair of equations for our non-linear-equation (2.23a)

Proposition (2.2.6) Form of the Solution

The matrix g , defined as:

$$g(u, v) = \Psi(u, v, 0), \quad (2.34)$$

is a solution for Einstein's equations (2.23).

Proof

Putting $\lambda = 0$ in (2.32) gives $g_{,u} = -\alpha^{-1} A g$ and $g_{,v} = +\alpha^{-1} B g$, which are identical to the definitions (2.21) of A and B . \square q.e.d.

2.2.3 The Dressing Ansatz

The previous arguments tell us that, whenever we have a solution for Ψ , then a solution for Einstein's equations is automatically ensured see (2.34). The problem is therefore that of finding such a solution Ψ . The following construction gives one possible way to construct it. We want to emphasize that, in principle, this is not the only possible solution to (2.32).

Step 1

Let us consider a known solution $g_0(u, v)$ of Einstein's equations. From this metric we can derive A_0 and B_0 and therefore we can get a solution Ψ_0 for the Linear Pair (2.32). Hereafter $g_0(u, v)$ will be called *Seed Metric*.

Step 2.a - Dressing Ansatz

We now look for a solution in the form:

$$\Psi(u, v, \lambda) = S(u, v, \lambda) \Psi_0(u, v, \lambda). \quad \begin{array}{l} \text{Dressing} \\ \text{Ansatz} \end{array} \quad (2.35)$$

By substituting (2.35) into (2.32) we can see that such a solution exists, provided that S satisfies:

$$\mathcal{D}_1 S(u, v, \lambda) = \frac{1}{\lambda - \alpha} (A S - S A_0), \quad (2.36a)$$

$$\mathcal{D}_2 S(u, v, \lambda) = \frac{1}{\lambda + \alpha} (B S - S B_0). \quad (2.36b)$$

Step 2.b - Reality of g

It may be noted that not all solutions of (2.36) are acceptable. In fact, we want g to be real. Thus we can only accept solutions such that:

$$\bar{S}(u, v, \bar{\lambda}) = S(u, v, \lambda), \quad \bar{\Psi}(u, v, \bar{\lambda}) = \Psi(u, v, \lambda). \quad (2.37)$$

Step 2.c - Symmetry of g

A necessary requirement for g is that $g_{ab} = g_{ba}$. Consider the new matrix S' defined by

$$S'(u, v, \lambda) = g S^\dagger(u, v, \lambda') g_0^{-1}, \quad \lambda' = \alpha^2 \frac{1}{\lambda}. \quad (2.38)$$

By using the invariance of \mathcal{D}_1 and \mathcal{D}_2 under the transformation (2.29), it can be shown that S' also satisfies (2.36), provided that g is symmetric. In fact, if this is the case, then $g (A^\dagger)^{-1} g^{-1} = A$. This ensures – together with the invariance of $D_{1,2}$ under (2.29) – that S' is a solution. As a result, we can be sure that g is symmetric if there exists a scalar function $h(u, v, \lambda)$ such that:

$$S'(u, v, \lambda) = h(u, v, \lambda) S(u, v, \lambda). \quad (2.39)$$

In general such a function will be different from 1. The correct expression for g will then be given by:

$$g_{ph} = h(u, v, \lambda) S(u, v, \lambda') g_0 S^t(u, v, \lambda), \quad (2.40a)$$

or

$$g_{ph} = h(u, v, \lambda) S(u, v, \lambda) g_0 S^t(u, v, \lambda'), \quad (2.40b)$$

the two expressions being equivalent due to the fact that g_{ph} is symmetric by construction

As suggested by (2.34) and (2.35), we might now want to put the new solution g_{ph} in the form:

$$g_{ph} = S(u, v, 0) g_0. \quad (2.41)$$

Therefore we must impose that the function $h(u, v, \lambda)$ satisfies the auxiliary condition:

$$\lim_{\lambda \rightarrow 0} h(u, v, \lambda) S(u, v, \lambda') = \mathbf{I} \quad (2.42)$$

By substituting (2.42) into (2.32) it is trivial to prove that (2.41) is a well defined solution.

Step 2.d - $\det g$

Finally, we require that $\det g = \alpha^2$. This implies a further condition upon S :

$$\det |S(u, v, 0)| = 1 \quad (2.43)$$

The Physical Metric**Proposition (2.2.7) Rescaling Property for g**

Any solution g of the linear pair (2.26) is defined up to a transformation:

$$g \mapsto g_{sc} = \frac{1}{h(u, v, 0)} g, \quad (2.44)$$

where $h(u, v, 0)$ is a function of u, v only.

Proof

Let us substitute (2.39) into (2.36). We have:

$$\mathcal{D}_1 h(u, v, \lambda) = 0, \quad \mathcal{D}_2 h(u, v, \lambda) = 0. \quad (2.45)$$

From the above equations it follows that:

$$\partial_u h(u, v, 0) = 0, \quad \partial_v h(u, v, 0) = 0. \quad (2.46)$$

We can now consider the scaled metric:

$$\mathbf{g}_{sc} = \frac{1}{h(u, v, 0)} \mathbf{g}, \quad \mathbf{g}_{sc} = \mathbf{S}(u, v, \frac{\alpha^2}{\lambda}) \mathbf{g}_0 \mathbf{S}^t(u, v, \lambda). \quad (2.47)$$

Since the scaling does not break the matrix structure, we have that \mathbf{g}_{sc} is symmetric. Moreover, the new matrices \mathbf{A}_{sc} and \mathbf{B}_{sc} associated to \mathbf{g}_{sc} are given by:

$$\mathbf{A}_{sc} = \mathbf{A} - \alpha (\log h)_{,u} \mathbf{I}, \quad \mathbf{B}_{sc} = \mathbf{B} - \alpha (\log h)_{,v} \mathbf{I}. \quad (2.48)$$

By using (2.39) we have:

$$\mathbf{A}_{sc} = \mathbf{A}, \quad \mathbf{B}_{sc} = \mathbf{B}, \quad (2.49)$$

and therefore the main equation (2.23a) is satisfied. \square q.e.d.

Proposition (2.2.8) The physical metric

Given any solution \mathbf{g} of the linear pair (2.26), it is possible to obtain a \mathbf{g}_{ph} with the correct determinant by simply considering:

$$\mathbf{g}_{ph} = \frac{\alpha}{\sqrt{\det \mathbf{g}}} \mathbf{g}. \quad (2.50)$$

Proof

The proof immediately follows from the previous assertion. In this case we have:

$$h = \frac{\alpha}{\sqrt{\det \mathbf{g}}}. \quad (2.51)$$

\square q.e.d.

2.2.4 Simple Poles Ansatz

In order to explicitly construct a solution, we may now introduce some assumptions about the pole structure of the \mathbf{S} matrix in the complex plane λ . By analogy with the situation encountered in section 2.1, we can impose that both \mathbf{S} and \mathbf{S}^{-1} have singularities in λ and that these are simple poles (i.e. assume that \mathbf{S} and \mathbf{S}^{-1} have meromorphic structure).

The Structure of the Singularities

Proposition (2.2.9) Structure of the Singularities

Let us suppose that S is not invertible at a number n of points $\mu[k]$. Let us suppose these to be simple poles for S^{-1} . Then S has simple poles at:

$$\mu[k] = \alpha^2 \frac{1}{\nu[k]}, \quad k = 1, \dots, n. \quad (2.52)$$

Proof

Let us consider the relation (2.38). Since $\det g = \det g_0$, we have:

$$\det S^t(\lambda) = \frac{1}{\det S(\alpha^2/\lambda)}. \quad (2.53)$$

The relation $S S^{-1} = I$ implies that $\det S = (\det S^{-1})^{-1}$. Therefore (2.53) becomes:

$$\det S^t(\lambda) = \det S^{-1}(\alpha^2/\lambda). \quad (2.54)$$

Let us now suppose S be not invertible in $\lambda = \mu[k]$. Consequently, $\det S(\mu[k]) = 0$ and $\det S^t(\mu[k]) = 0$. Thus, $\det S^{-1}(\alpha^2/\mu[k]) = 0$. This means that S^{-1} has poles at $\nu[k] = \alpha^2/\mu[k]$. \square q.e.d.

From (2.37) it can also be deduced that poles in S are either real or occur in complex-conjugate pairs. Consequently, for n complex pairs of poles, the general form of S and S^{-1} is given by.

$$S = I + \sum_{k=1}^n \left(\frac{1}{\lambda - \mu[k]} R_{[k]} + \frac{1}{\lambda - \bar{\mu}[k]} \bar{R}_{[k]} \right), \quad (2.55a)$$

$$S^{-1} = I + \sum_{k=1}^n \left(\frac{1}{\lambda - \nu[k]} Q_{[k]} + \frac{1}{\lambda - \bar{\nu}[k]} \bar{Q}_{[k]} \right), \quad (2.55b)$$

where the matrices $Q_{[k]}$ are related to the $R_{[k]}$'s through the condition $S S^{-1} = I$.

Proposition (2.2.10) Equations for the poles

The poles $\mu[k]$ are solutions of the following equations.

$$\mu[k]_{,u} + \frac{2\mu[k]}{\mu[k] - \alpha} \alpha_{,u} = 0, \quad \mu[k]_{,v} - \frac{2\mu[k]}{\mu[k] + \alpha} \alpha_{,v} = 0. \quad (2.56)$$

Proof

By substituting (2.55a) and (2.55b) into (2.36a) and (2.36b). We have:

$$D_1 S = \sum_{k=1}^n \frac{1}{(\lambda - \mu[k])^2} [\mu[k]_{,u} + \frac{2\lambda}{\lambda - \alpha} \alpha_{,u}] R[k] + \sum_{k=1}^n \frac{1}{\lambda - \mu[k]} D_1 R[k] + \quad (2.57a)$$

$$\sum_{k=1}^n \frac{1}{(\lambda - \bar{\mu}[k])^2} [\bar{\mu}[k]_{,u} + \frac{2\lambda}{\lambda - \alpha} \alpha_{,u}] \bar{R}[k] + \sum_{k=1}^n \frac{1}{\lambda - \bar{\mu}[k]} D_1 \bar{R}[k] ,$$

$$\frac{1}{\lambda - \alpha} (A S - S A_0) = \frac{1}{\lambda - \alpha} \sum_{k=1}^n \frac{1}{\lambda - \mu[k]} (A R[k] - R[k] A_0) +$$

$$\frac{1}{\lambda - \alpha} \sum_{k=1}^n \frac{1}{\lambda - \bar{\mu}[k]} (A \bar{R}[k] - \bar{R}[k] A_0) . \quad (2.57b)$$

After equating (2.57a) and (2.57b) we have to impose that the second order pole disappears, for the pole structure of the two members to be the same. This proves the assertion. \square q.e.d.

It is worth mentioning that, due to condition (2.27), equations (2.56) satisfies the integrability condition:

$$\mu[k]_{,uv} - \mu[k]_{,vu} = 0 . \quad (2.58)$$

Analogously for $\nu[k]$.

Proposition (2.2.11) Pole-Trajectories

The poles $\mu[k]$ and $\nu[k]$ have the following explicit expression.

$$\mu[k] = w[k] - \beta \pm \sqrt{(w[k] - \beta)^2 - \alpha^2} , \quad (2.59a)$$

$$\nu[k] = w[k] - \beta \mp \sqrt{(w[k] - \beta)^2 - \alpha^2} \quad (2.59b)$$

Proof

Equations (2.56) can be rewritten in the form

$$\frac{\alpha_{,u}}{\sqrt{\mu}} - \frac{1}{2} \frac{\mu_{,u}}{\sqrt{\mu} \mu} \alpha + \frac{1}{2} \frac{\mu_{,u}}{\sqrt{\mu}} = \left(\frac{\mu + \alpha}{\sqrt{\mu}} \right)_{,u} = 0 , \quad (2.60a)$$

$$\frac{\alpha_{,v}}{\sqrt{\mu}} - \frac{1}{2} \frac{\mu_{,v}}{\sqrt{\mu} \mu} \alpha - \frac{1}{2} \frac{\mu_{,v}}{\sqrt{\mu}} = \left(\frac{\mu - \alpha}{\sqrt{\mu}} \right)_{,v} = 0 , \quad (2.60b)$$

where the index $[k]$ has been dropped for simplicity. By integrating both members of both equations respectively in u and v one obtains:

$$\mu - \alpha = 2 \sqrt{\mu} C(u) , \quad \mu + \alpha = 2 \sqrt{\mu} D(v) ,$$

where C and D are arbitrary functions of u and v respectively. From this the following relations may be derived:

$$\mu^2 - 2\mu(2D^2 - \alpha) + \alpha^2 = 0 , \quad \mu^2 - 2\mu(2C^2 + \alpha) + \alpha^2 = 0 .$$

They are only consistent if $\alpha = D^2(v) - C^2(u)$. This must be compared with (1.33), thus yielding

$$C^2(u) = \frac{1}{2} w - \mathcal{F}, \quad D^2(u) = \frac{1}{2} w + \mathcal{G}, \quad (2.63a)$$

$$\mu^2 - 2\mu(w - \beta) + \alpha^2 = 0. \quad (2.63b)$$

where the second equation is obtained by using that β , harmonically conjugate to α , can also be written in the form: $\beta = \mathcal{F} - \mathcal{G}$.

The solution of the second equation above is clearly given by (2.59a). \square q.e.d.

Proposition (2.2.12) On the structure of $R_{[k]}$ and $Q_{[k]}$

$R_{[k]}$ and $Q_{[k]}$ must satisfy

$$\det R_{[k]} = 0, \quad \det Q_{[k]} = 0. \quad (2.64)$$

Proof

From the identity $S S^{-1} = I$ evaluated at $\lambda = \mu_{[k]}$, we have $R_{[k]} S^{-1}(\mu_{[k]}) = 0$. These can be considered as a system of linear equations. Therefore, in order to have a solution, we have to impose that the determinant of $R_{[k]}$ vanishes. By similarly evaluating the identity at $\lambda = \nu_{[k]}$, we can get a similar condition for $Q_{[k]}$. \square q.e.d.

According to the above result, we can look for solutions such that $R_{[k]}$ and $Q_{[k]}$ are in the form:

$$R_{[k]}_{ab} = n_{[k]_a} m_{[k]_b}, \quad Q_{[k]}_{ab} = p_{[k]_a} q_{[k]_b}, \quad (2.65a)$$

$$R_{[k]} = \mathbf{n}_{[k]} \otimes \mathbf{m}_{[k]}, \quad Q_{[k]} = \mathbf{p}_{[k]} \otimes \mathbf{q}_{[k]}. \quad (2.65b)$$

The Construction of $R_{[k]}$

Given a Seed metric g_0 we can (at least in principle) obtain Ψ_0 . Having that, we can introduce the matrix $M_{[k]} = \Psi_0^{-1}(\lambda = \mu_{[k]})$ that clearly satisfies the equations:

$$M_{[k],u} + \frac{1}{\mu_{[k]} - \alpha} M_{[k]} A_0 = 0, \quad M_{[k],v} + \frac{1}{\mu_{[k]} + \alpha} M_{[k]} B_0 = 0. \quad (2.66)$$

Proposition (2.2.13) The vectors $\mathbf{m}_{[k]}$

The 2-vectors $\mathbf{m}_{[k]}$ are given by:

$$\mathbf{m}_{[k]} = \kappa_{[k]} \cdot M_{[k]}. \quad (2.67)$$

where $\kappa_{[k]}$ is a completely arbitrary complex vector.

Proof

Equations (2.36) can be rewritten as:

$$\frac{1}{\lambda - \alpha} A = (D_1 S) S^{-1} + \frac{1}{\lambda - \alpha} S A_0 S^{-1} \quad (2.68a)$$

$$\frac{1}{\lambda + \alpha} B = (D_2 S) S^{-1} + \frac{1}{\lambda + \alpha} S B_0 S^{-1} \quad (2.68b)$$

We now require that the second member does not have second order poles. Then:

$$\mathbf{R}_{[k],u} \mathbf{S}^{-1}(\mu_{[k]}) + \frac{1}{\mu_{[k]} - \alpha} \mathbf{R}_{[k]} \mathbf{A}_0 \mathbf{S}^{-1}(\mu_{[k]}) = 0, \quad (2.69a)$$

$$\mathbf{R}_{[k],v} \mathbf{S}^{-1}(\mu_{[k]}) + \frac{1}{\mu_{[k]} + \alpha} \mathbf{R}_{[k]} \mathbf{B}_0 \mathbf{S}^{-1}(\mu_{[k]}) = 0. \quad (2.69b)$$

From the identity $\mathbf{S} \mathbf{S}^{-1} = \mathbf{I}$ we have $\mathbf{R}_{[k]} \mathbf{S}^{-1}(\mu_{[k]}) = 0$. We can now rewrite $\mathbf{S}^{-1}(\mu_{[k]})$ as $\mathbf{S}^{-1}(\mu_{[k]}) = \mathbf{T}_{[k]} \otimes \mathbf{P}_{[k]}$. Therefore the previous condition now reads:

$$\mathbf{m}_{[k]} \cdot \mathbf{T}_{[k]} = 0. \quad (2.70)$$

By substituting (2.65) and (2.70) into (2.69a) and (2.69b) we have:

$$\left[\mathbf{m}_{[k],u} + \frac{1}{\mu_{[k]} - \alpha} \mathbf{m}_{[k]} \cdot \mathbf{A}_0 \right] \cdot \mathbf{T}_{[k]} = 0, \quad (2.71a)$$

$$\left[\mathbf{m}_{[k],v} + \frac{1}{\mu_{[k]} + \alpha} \mathbf{m}_{[k]} \cdot \mathbf{B}_0 \right] \cdot \mathbf{T}_{[k]} = 0. \quad (2.71b)$$

Notice that equations (2.71) have the same structure of equations (2.66). Therefore the assertion is proved. \square q.e.d.

Proposition (2.2.14) The vectors $\mathbf{n}_{[k]}$

The 2-vectors $\mathbf{n}_{[k]}$ are given by:

$$\mathbf{n}_{[k]} = \sum_{j=1}^{2n} \frac{1}{\mu_{[k]}} \Gamma^{-1}_{[k,j]} N_{[k]_a}, \quad \mathbf{N}_{[k]} = \{N_{[k]_a}\}, \quad (2.72a)$$

$$\Gamma_{[k,j]} = \frac{1}{\mu_{[k]} \mu_{[j]} - \epsilon \alpha^2} \mathbf{N}_{[k]} \cdot \mathbf{m}_{[j]}, \quad \mathbf{N}_{[k]} = \mathbf{m}_{[k]} \cdot \mathbf{g}_0, \quad (2.72b)$$

where the sum is now extended to both the pole-trajectories and their complex-conjugate companions.

Proof

By substituting (2.55a) into (2.38) and putting $\lambda = \nu_{[k]} = \alpha^2/\lambda$, we have a set of n equations for $\mathbf{R}_{[k]}$:

$$\mathbf{R}_{[k]} \cdot \mathbf{g}_0 \cdot \left[\mathbf{I} + \sum_{j=1}^n \left(\frac{1}{\nu_{[k]} - \mu_{[j]}} \mathbf{R}_{[j]} + \frac{1}{\nu_{[k]} - \bar{\mu}_{[j]}} \mathbf{R}_{[j]}^t \right) \right] = 0. \quad (2.73)$$

By using $\mathbf{R}_{[k]} \mathbf{S}^{-1}(\mu_{[k]}) = 0$ into (2.73) we have:

$$\sum_{j=1}^n \frac{\mathbf{m}_{[k]} \cdot \mathbf{g}_0 \cdot \mathbf{m}_{[j]}}{\nu_{[k]} - \mu_{[j]}} n_{[j]_a} + \sum_{j=1}^n \frac{\mathbf{m}_{[k]} \cdot \mathbf{g}_0 \cdot \bar{\mathbf{m}}_{[j]}}{\nu_{[k]} - \bar{\mu}_{[j]}} \bar{n}_{[j]_a} = -m_{[k]_c} g_{0_{ca}}. \quad (2.74)$$

These are n equations for the n components $n_{[j]_a}$. The trick is now to renumber the poles without distinguishing between $\mu_{[k]}$ and $\bar{\mu}_{[k]}$ in (2.55a). By introducing $\Gamma_{[k,j]}$ and $\mathbf{N}_{[k]}$ defined above, equation (2.74) simplifies to $\sum_{j=1}^{2n} \mu_{[k]} \Gamma_{[k,j]} n_{[j]_a} = N_{[k]_a}$, which proves the assertion. \square q.e.d.

Proposition (2.2.15) Determinant of the n -soliton solution

The determinant of the unscaled metric is given by:

$$\det g = \prod_{k=1}^{2n} \frac{|\mu[k]|^2}{\alpha^2} \det g_0. \quad (2.75)$$

Proof

For the proof we refer to [49], where it is given by using the smart procedure of adding one pole at a time. \square q.e.d.

Proposition (2.2.16) The physical n -soliton solution

The physical metric describing an n -soliton solution is given by.

$$g_{ab} = \prod_{k=1}^{2n} \frac{|\mu[k]|}{\alpha} \left(g_{0ab} - \frac{1}{\mu[k]} \frac{1}{\mu[l]} \Gamma^{-1}_{[k,l]} N_{[l]a} N_{[k]b} \right). \quad (2.76)$$

where $N_{[k]b}$ and $\Gamma_{[k,l]}$ have been defined in (2.72b) and $\mu[k]$ are given by (2.59a) with $w[k]$ arbitrary complex constants.

Proof

The proof simply follows from the definitions (2.55a) and (2.34), along with the rescaling procedure earlier introduced. \square q.e.d.

Alternatively one can use the following form given in [95]:

$$g_{ab} = \prod_{k=1}^N \frac{|\mu[k]|}{\alpha} \left(g_{0ab} - \tilde{\Gamma}^{-1}_{[k,l]} N_{[k]a} N_{[l]b} \right), \quad (2.77a)$$

$$N_{[k]} = g_0 \cdot m_{[k]}, \quad \tilde{\Gamma}_{[k,l]} = \frac{\mu[k] \mu[l]}{\mu[k] \mu[l] - \epsilon \alpha^2} m_{[k]} \cdot g_0 \cdot m_{[l]}, \quad (2.77b)$$

where now $N = 2n$ and the definition for Γ as been slightly modified.

The form for the n -soliton solution given above deserves a few remarks. The seed metric g_0 enters this solution in a direct way and through Ψ_0 in a more involved one. In particular, Ψ_0 will be necessary for the construction of the $m_{[k]}$ vectors. However, it must be recalled that, given any g_0 , it is not simple to obtain the corresponding Ψ_0 . In fact, this involves the solution of some nonlinear equations. Nevertheless many explicit expression are now known. We will consider them in chapter 3.

A second problem is given by the explicit manipulation of the algebraic expressions in $S(\alpha, \beta, \lambda)$. According to the number of poles, we have to deal with $N \times N$ matrices (namely the $\Gamma_{[k,l]}$) that need to be inverted. Even for a 4×4 matrix, this may be not a simple task. In this respect, a significant result has been achieved by Alekseev [95], who managed to avoid finding inverses by rewriting the metric components as the ratio of two determinants of $N \times N$ matrices. This result will also be briefly sketched in chapter 3.

2.3 The Alekseev Technique

This section is devoted to a presentation of the Alekseev technique [51, 52, 53, 54]. It involves 3×3 matrices rather than 2×2 as in the previous case.

2.3.1 Matrix Form for the Einstein-Maxwell Equations

Proposition (2.3.1) Duality Form of the Einstein's Equations

Let us introduce:

$$K_{\mu\nu(a)} = \nabla_\mu \xi_{\nu a} - \nabla_\nu \xi_{\mu a}, \quad K^*_{\mu\nu(a)} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} K^{\rho\sigma}_{(a)}, \quad (2.78a)$$

$$\Omega_{\mu\nu(a)} = K_{\mu\nu(a)} + i K^*_{\mu\nu(a)}, \quad \mu, \nu, \rho, \sigma = 1, 2, 3, 4, \quad (2.78b)$$

where ξ_a^μ are the relevant Killing vectors. Let us suppose that there exist a self-dual bivector $S_{\mu\nu(a)}$ such that:

$$T_\mu{}^\nu \xi_{\nu a} = \epsilon_{\mu\nu\rho\sigma} \nabla^\nu S^{\rho\sigma}_{(a)}, \quad (2.79)$$

where $T_{\mu\nu}$ is the energy-matter tensor. Then the Einstein Field Equations are equivalent to the existence of a bivector

$$\mathcal{H}_{\mu\nu(a)} = \Omega_{\mu\nu(a)} + 4i(8\pi G) S_{\mu\nu(a)} \quad (2.80)$$

which has the following properties:

$$\mathcal{H}_{\mu\nu(a)} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{H}^{\rho\sigma}_{(a)}, \quad \epsilon^{\mu\nu\rho\sigma} \nabla_\nu \mathcal{H}_{\rho\sigma(a)} = 0, \quad (2.81a)$$

$$\exists H_{\mu(a)} \text{ s.t. } \mathcal{H}_{\mu\nu(a)} = \partial_\mu H_{\nu(a)} - \partial_\nu H_{\mu(a)}. \quad (2.81b)$$

Proposition (2.3.2) Duality Form of the Maxwell's Equations

The electromagnetic field can be described by a bivector $F_{\mu\rho}$ and a potential Φ_ν such that:

(i) the energy-momentum tensor is given by:

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\rho} F^\rho{}_\nu - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \right) \quad (2.82)$$

(ii) The Maxwell Equations are given by:

$$\nabla_{[\mu} F_{\nu\rho]} = 0, \quad \nabla_\mu F^{\nu\mu} = 0. \quad (2.83)$$

(iii) By introducing the dual variables $F^*_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ and $\omega_{\mu\nu} = F_{\mu\nu} + i F^*_{\mu\nu}$, the Maxwell Equations can be put in the form:

$$\omega_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \omega^{\rho\sigma}, \quad \nabla_\mu \omega^{\nu\mu} = 0, \quad \omega_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu. \quad (2.84)$$

Notice the close similarity between equations (2.81) and (2.84).

Proposition (2.3.3) Kinnersley Equations

Let us consider the line element (1.25), and introduce the matrices h_a^b , ϵ_{AB} , ϵ_{ab} defined by:

$$g_{ab} = h_a^c \epsilon_{cb}, \quad \epsilon_{ab} = \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a, b = 1, 2, \quad (2.85a)$$

$$\det h_a^b = \epsilon \alpha^2, \quad \epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A, B = 1, 2. \quad (2.85b)$$

Then the equations (2.81) and (2.84) became.

$$\partial_A H_a^b = -\frac{i}{\alpha} \epsilon_A^B h_a^c \partial_B H_c^b, \quad (2.86a)$$

$$\partial_A \Phi_a = -\frac{i}{\alpha} \epsilon_A^B h_a^c \partial_B \Phi_c, \quad \Phi_a = \Phi_\mu \xi_a^\mu, \quad (2.86b)$$

$$\partial_A H_a^b = \partial_A h_a^b - \frac{i}{\alpha} \epsilon_A^B h_a^c \partial_B h_c^b - 2 \bar{\Phi}^c \partial_A \Phi_a, \quad (2.86c)$$

where the indices A, B are raised and lowered by means of the matrix η_{AB} , and the indices a, b are lowered and raised by the matrix ϵ_{ab} .

The usual electromagnetic vector field $A_\mu = (0, 0, A_1, A_2)$ is related to $\Phi = \{\Phi_a\}$ through the relation: $A_a = \text{Re } \Phi_a$. It is also appropriate to emphasize that the function α must satisfy $\eta^{AB} \partial_A \alpha_{,B} = 0$, which is equivalent to the known $\alpha_{,uv} = 0$

Proposition (2.3.4) Matrix Form for the Einstein-Maxwell Equations

The equations (2.86a) and (2.86b) are equivalent to:

$$\eta^{AB} \partial_A U_B + \frac{i}{2\alpha} \epsilon^{AB} U_A U_B = 0, \quad (2.87a)$$

$$\epsilon^{AB} \partial_B U_A = 0. \quad (2.87b)$$

The equation (2.86c) is equivalent to

$$\partial_A (G - 4i\beta\Omega) = 2(U_A^\dagger \Omega - \Omega U_A), \quad (2.88a)$$

$$G U_A = -4i\epsilon \alpha \epsilon_A^B \Omega U_B, \quad (2.88b)$$

$$\text{Tr } U_A = 2i\beta_{,A}, \quad \text{Re}(\text{Tr } U_A) = 0, \quad (2.88c)$$

$$G = G^\dagger, \quad G^{55} = 1, \quad (2.88d)$$

where β is a function defined through $\beta_{,A} = \epsilon \epsilon_A^B \alpha_{,B}$. Moreover:

$$U_A = \begin{pmatrix} \partial_A H_a^b & \partial_A \Phi_a \\ 2\bar{\Phi}^c \partial_A H_c^b & 2\bar{\Phi}^c \partial_A \Phi_c \end{pmatrix}, \quad G = \begin{pmatrix} -4h^{ab} + 4\Phi^a \bar{\Phi}^b & -2\Phi^a \\ -2\bar{\Phi}^b & 1 \end{pmatrix}. \quad (2.89)$$

and Ω is a 3×3 matrix with all null entries, except for $\Omega_{12} = 1 = -\Omega_{21}$; 5 labels the electromagnetic components – the third row and column – of all matrices.

Proof

Let us consider U_A and G in the form:

$$U_A = \begin{pmatrix} H_{Aa}^b & \Phi_{Aa} \\ X_A^b & Y_A \end{pmatrix}, \quad G = \begin{pmatrix} -4 h^{ab} + 4 \Phi^a \bar{\Phi}^b & -2\Phi^a \\ -2 \bar{\Phi}^b & 1 \end{pmatrix}. \quad (2.90)$$

Let us substitute the above form of U_A into (2.88a). We can obtain the relations:

$$\Phi_{Aa} = \partial_A \Phi_a, \quad (2.91)$$

$$-4 \partial_A (h^{ab} + 4 \Phi^a \bar{\Phi}^b + i \beta \epsilon^{ab}) = -2 (H_A^{ab} + \bar{H}_A^{ba}). \quad (2.92)$$

The first relation defines one of the elements of the U_A given above. Let us now substitute U_A from (2.90) into (2.88b). After a little algebra we obtain:

$$X_A^b = 2 \bar{\Phi}^c \partial_A H_c^b, \quad (2.93a)$$

$$Y_A = 2 \bar{\Phi}^c \partial_A \Phi_c, \quad (2.93b)$$

$$-4 h^{ac} \Phi_{Ac} + 4 \Phi^a \bar{\Phi}^c \Phi_{Ac} - 2 \Phi^a Y_A = 4 i \epsilon \alpha \epsilon^{AC} \Phi_{Bc}, \quad (2.93c)$$

$$-4 h^{ac} H_{Ac}^b + 4 \Phi^a \bar{\Phi}^c H_{Ac}^b - 2 \Phi^a H_{Aa}^b = 4 i \epsilon \alpha \epsilon^{AC} H_{Bc}^b. \quad (2.93d)$$

Relations (2.93a) and (2.93b) define two more elements of U_A . By substituting (2.93b) into (2.93c) we get the other relation:

$$-4 h^{ac} \Phi_{Ac} = 4 i \epsilon \alpha \epsilon_A^C \epsilon^{ac} \Phi_{Bc}. \quad (2.94)$$

By substituting (2.93a) into (2.93d) we get the relation:

$$-4 h^{ac} H_{Ac}^b = 4 i \epsilon \alpha \epsilon_A^C \epsilon^{ac} H_{Bc}^b. \quad (2.95)$$

It is worth noticing that (2.94) and (2.95) correspond to the first two Kinnersley equations. This proves one first part of the assertion.

Let us now go back to (2.92). This equation gives information on the hermitian part of H_A^{ab} only. Therefore there is still an arbitrariness on the anti-hermitian part. We can use that to set:

$$H_A^{ab} = \partial_A (h^{ab} - \Phi^a \bar{\Phi}^b + i \beta \epsilon^{ab}) + \gamma_A \epsilon^{ab} + i S_A^{ab}, \quad (2.96)$$

where γ_A is a real 3-vector and S_A^{ab} is a real matrix as well. Let us substitute (2.96) into (2.94). After a little algebra we get the equation:

$$\begin{aligned} \partial_A h_a^b + i \beta_{,A} \epsilon_a^b - \partial_A (\Phi_a \bar{\Phi}^b) + \gamma_A \epsilon_a^b + i S_{Aa}^b &= \frac{i}{\alpha} \epsilon_A^B h_a^c (\partial_B h^{cb}) + \\ + \frac{1}{\alpha} \epsilon_A^B h_a^c \beta_{,B} \epsilon_c^b - \frac{i}{\alpha} \epsilon_A^B h_a^c \partial_B (\Phi_c \bar{\Phi}^b) + \frac{1}{\alpha} \epsilon_A^B h_a^c \gamma_B \epsilon_c^b + \frac{1}{\alpha} \epsilon_A^B h_a^c S_{Bc}^b. \end{aligned} \quad (2.97)$$

If we now use (2.95) into (2.97), we can obtain:

$$(\partial_A \Phi^a) \bar{\Phi}^b = \frac{i}{\alpha} \epsilon_A^B h_a^c (\partial_B \Phi_c) \bar{\Phi}^b. \quad (2.98)$$

By substituting (2.98) into (2.97) and by isolating its real and imaginary parts, we get the two equations:

$$\partial_A h_a^b - \Phi_a \partial_A \bar{\Phi}^b + \gamma_A \epsilon_a^b = -\frac{1}{\alpha} \epsilon_A^B h_a^c \beta_{,B} \epsilon_c^b - \frac{1}{\alpha} \epsilon_A^B h_a^c S_{Bc}^b, \quad (2.99)$$

$$\beta_{,A} \epsilon_a^b + S_{Aa}^b = \frac{1}{\alpha} \epsilon_A^B h_a^c (\partial_B h_c^b) + \frac{1}{\alpha} \epsilon_A^B h_a^c \Phi_c (\partial_B \bar{\Phi}^b) + \frac{1}{\alpha} \epsilon_A^B h_a^c \gamma_B \epsilon_c^b. \quad (2.100)$$

In order to solve the above equations let us make the ansatz.

$$\gamma_A \delta_a^b = -\frac{1}{\alpha} \epsilon_A^B h_a^c S_{Bc}^b \epsilon_{ba}. \quad (2.101)$$

The condition on the determinant of g can be rewritten in the form:

$$h_a^c h_c^b = -\epsilon \alpha^2 \delta_a^b, \quad (2.102)$$

$$(\partial_A h_a^c) h_c^b + h_a^c (\partial_A h_c^b) = -2 \epsilon \alpha \alpha_{,A} \delta_a^b, \quad (2.103)$$

where the second relation has been obtained from the first by differentiating it. By substituting (2.101) and (2.102) into (2.100) we get the relation:

$$\beta_{,A} \epsilon_a^b = \frac{1}{\alpha} \epsilon_A^B h_a^c (\partial_B h_c^b) + \frac{1}{\alpha} \epsilon_A^B h_a^c \Phi_c (\partial_B \bar{\Phi}^b). \quad (2.104)$$

By substituting (2.101) into (2.99) we get the relation:

$$\partial_B h_a^c = \Phi_a \partial_B \bar{\Phi}^c - \frac{1}{\alpha} \epsilon_B^D h_a^d \beta_{,D} \epsilon_d^c. \quad (2.105)$$

Let us substitute (2.103) into (2.104). With the help of (2.105) we finally get the important relation:

$$\boxed{\beta_{,A} = -\epsilon \epsilon_A^B \alpha_{,B}} \quad (2.106)$$

At this stage H_A^{ab} , defined in (2.96), reads:

$$H_{Aa}^b = \partial_A h_a^b + \frac{2}{\alpha} \epsilon_A^B h_a^c \partial_B h_c^b - 2 \bar{\Phi}^b \partial_A \Phi^a + \gamma_A \delta_a^b + \frac{2}{\alpha} \epsilon_A^B \gamma_B h_a^b. \quad (2.107)$$

From this and by using the above relations one can obtain:

$$\boxed{\text{Tr } \mathbf{U}_A = 2\gamma_A + \partial_A \text{Tr } \mathbf{h} = 2\gamma_A + 2i\beta_{,A}} \quad (2.108)$$

We can always use the arbitrariness in the definition of γ and set:

$$\boxed{\text{Re Tr } \mathbf{U}_A = 0} \quad \boxed{\text{Tr } \mathbf{U}_A = 2i\beta_{,A}} \quad (2.109)$$

The expression for H_A^{ab} will now be further simplified to:

$$H_{Aa}^b = \partial_A h_a^b + \frac{2}{\alpha} \epsilon_A^B h_a^c \partial_B h_c^b - 2 \bar{\Phi}^b \partial_A \Phi^a. \quad (2.110)$$

It is now trivial to show that $H_{Aa}^b = \partial_A H_a^b$. In fact, the proof simply comes from substituting (2.110) into (2.87b).

Finally, equation (2.87a) will follow as a byproduct of this construction. \square q.e.d.

Proposition (2.3.5) Rank of U_A

Condition (2.88b) is equivalent to:

$$\text{rank } U_A = 1. \quad (2.111)$$

Proof

The assertion follows from the previous proposition and by using the Kinnersley equations given in (2.86). \square q.e.d.

2.3.2 A Linear-Pair formulation

Let us consider the pair of linear equations given by:

$$\partial_A \Psi = \Lambda_A^B U_B \Psi, \quad \Lambda_A^B = \frac{1}{2i} \frac{(w - \beta) \delta_A^B + \epsilon \alpha \epsilon_A^B}{(w - \beta)^2 - \epsilon \alpha^2}, \quad (2.112)$$

where Ψ is a matrix function depending upon the relevant coordinates α and β and upon the spectral parameter w . More details on the construction of this linear pair are given in appendix A. Here we just emphasize that different spectral planes are adopted in the Alekseev and BZ formalisms. This is expressed by labelling the spectral parameters with different letters.

Proposition (2.3.6) A linear-pair formulation

The solutions of (2.87) are equivalent to those of the linear system (2.112).

Proof

Let us consider (A.8):

$$\left(\frac{\epsilon^{AB} D_A \Lambda_B^C}{\det \Lambda} \right) U_\gamma + \left(\frac{\epsilon^{AB} \Lambda_B^C}{\det \Lambda} \right) \partial_A U_C - \epsilon^{AB} U_A U_B = 0. \quad (2.113)$$

From (A.9b) we have.

$$\left(\frac{\epsilon^{AB} \Lambda_B^C}{\det \Lambda} \right) \partial_A U_C - \epsilon^{AB} U_A U_B = 0. \quad (2.114)$$

By inserting (A.9a) into (2.114) we have:

$$\eta^{AB} \partial_A U_B + \mathcal{F} \epsilon^{AB} \partial_A U_B - \frac{1}{2i\alpha} \epsilon^{AB} U_A U_B = 0, \quad (2.115)$$

where \mathcal{F} has been defined in appendix A. Clearly (2.115) is satisfied iff:

$$\eta^{AB} \partial_A U_B + \frac{i}{2\alpha} \epsilon^{AB} U_A U_B = 0, \quad \epsilon^{AB} \partial_A U_B = 0. \quad (2.116)$$

That concludes the proof. \square q.e.d.

Equations (2.88) can be rewritten – in terms of the matrices that enter the linear system (2.112) – by introducing the matrix function W :

$$W \equiv G + 4i(w - \beta) \Omega. \quad (2.117)$$

Proposition (2.3.7) Properties of W

Condition (2.88d) is equivalent to

$$W = W^\dagger, \quad W^{55} = 1, \quad \frac{\partial W}{\partial w} = 4i\Omega. \quad (2.118)$$

Proof

This trivially follows from (2.117) and (2.88d). \square q e d

Proposition (2.3.8) Equation for W

Conditions (2.88a) and (2.88b) are equivalent to:

$$\partial_A W + \Lambda_A^B (W U_B - U_B^\dagger W) = 0. \quad (2.119)$$

This equation admits the following first integral:

$$\Psi^\dagger W \Psi = K(w), \quad (2.120)$$

where K is an arbitrary hermitian matrix depending only upon w .

Proof

Let us consider (2.88a) and the antihermitian part of (2.88b):

$$(U_A^\dagger \Omega - \Omega U_A) = -\frac{1}{2} \partial_A (G - 4i\beta \Omega), \quad (2.121a)$$

$$G U_A - U_A^\dagger G = 2i\epsilon \alpha \epsilon_A^B \partial_B (G - 4i\beta \Omega). \quad (2.121b)$$

Let us consider the quantity

$$4i(w - \beta) (2.121a) + (2.121b). \quad (2.122)$$

By introducing the new function W defined (2.117), the above equation has the simple form:

$$W U_A - U_A^\dagger W = -2i(w - \beta) \partial_A W + 2i\epsilon \alpha \epsilon_A^B \partial_B W. \quad (2.123)$$

By considering (A.10), (A.12) and (2.112) we have:

$$\partial_A W + \Lambda_A^B (W U_B - U_B^\dagger W) = 0. \quad (2.124)$$

From (2.112) we also have:

$$U_B = (\Lambda^{-1})_A^B \partial_B \Psi \Psi^{-1}. \quad (2.125)$$

By inserting (2.125) into (2.119) we have:

$$\partial_A (\Psi^\dagger W \Psi) = 0 , \quad (2.126)$$

that is

$$\Psi^\dagger W \Psi = K(w) . \quad (2.127)$$

where K is a constant of integration

□ q.e.d

Equation (2.112) implies that any Ψ is defined up to a transformation:

$$\Psi \mapsto \tilde{\Psi} = \Psi C(w) , \quad (2.128)$$

where $C(w)$ is a matrix depending only upon the spectral parameter w . This transformation can not affect the final form of W , because this contains the (physical) metric components through the relation (2.117). In fact, by using the above transformation, (2.120) became:

$$\Psi^\dagger W \Psi \mapsto C(w)^\dagger \Psi^\dagger W \Psi C(w) = C(w)^\dagger K(w) C(w) = \tilde{K}(w) . \quad (2.129)$$

As we shall see later on, this freedom can be used to make a suitable choice of the matrix $K(w)$

It is now convenient to introduce the null coordinates associated with α and β :

$$\xi = \beta + j \alpha , \quad \eta = \beta - j \alpha , \quad (2.130)$$

where $j = i$ if $\epsilon = -1$ and $j = 1$ if $\epsilon = 1$. In this new coordinate system, equations (2.112) have the simpler form:

$$2i(w - \xi) \partial_\xi \Psi = U_\xi \Psi , \quad 2i(w - \eta) \partial_\eta \Psi = U_\eta \Psi . \quad (2.131)$$

In passing, notice that the equation (2.88c) yields

$$\text{Tr} U_\xi = i , \quad \text{Tr} U_\eta = i . \quad (2.132)$$

2.3.3 The Dressing Ansatz

Step 1

Let us consider a known solution g_0 (Seed Metric) of the Einstein–Maxwell field equations. From this metric it is possible to obtain the two matrix functions U_{0A} . Rather than applying the definition given in (2.89), it is convenient to construct G_0 as a first step and then use the equations (2.88a), (2.88b) and (2.88c) to get an explicit expression for U_{0A} . Given U_{0A} , it is then possible to solve the linear pair (2.112) to obtain Ψ_0 . This procedure may be rather difficult since it involves the solution of coupled nonlinear p.d.e.'s.

Step 2.a

We now look for solutions of the form:

$$\Psi(\alpha, \beta, w) = S(\alpha, \beta, w) \Psi_0(\alpha, \beta, w) , \quad \begin{array}{l} \text{Dressing} \\ \text{Ansatz} \end{array} \quad (2.133)$$

where \mathbf{S} is a matrix to be determined. By substituting (2.133) into (2.112) we have:

$$\partial_A \mathbf{S} = \Lambda_A^B (\mathbf{U}_B \mathbf{S} - \mathbf{S} \mathbf{U}_{0B}) . \quad (2.134)$$

Alternatively these equations might be rewritten in the ξ and η coordinates introduced above. We have:

$$2i(w - \xi) \partial_\xi \mathbf{S} = \mathbf{U}_\xi \mathbf{S} - \mathbf{S} \mathbf{U}_{0\xi} , \quad 2i(w - \eta) \partial_\eta \mathbf{S} = \mathbf{U}_\eta \mathbf{S} - \mathbf{S} \mathbf{U}_{0\eta} . \quad (2.135)$$

Step 2.b

By performing the two limits for $w \rightarrow \xi$ and $w \rightarrow \eta$ in (2.135) we have:

$$\mathbf{U}_\xi = \mathbf{S}(w = \xi) \mathbf{U}_{0\xi} \mathbf{S}^{-1}(w = \xi), \quad \mathbf{U}_\eta = \mathbf{S}(w = \eta) \mathbf{U}_{0\eta} \mathbf{S}^{-1}(w = \eta), \quad (2.136)$$

therefore:

$$\text{Tr } \mathbf{U}_\xi = \text{Tr } \mathbf{U}_{0\xi} , \quad \text{Tr } \mathbf{U}_\eta = \text{Tr } \mathbf{U}_{0\eta} , \quad (2.137a)$$

$$\text{rank } (\mathbf{U}_\xi) = \text{rank } (\mathbf{U}_{0\xi}) , \quad \text{rank } (\mathbf{U}_\eta) = \text{rank } (\mathbf{U}_{0\eta}) , \quad (2.137b)$$

where the last step comes from \mathbf{S} being invertible. Thus, if \mathbf{U}_0 satisfies the conditions (2.88c) and (2.111) the new matrix \mathbf{U} satisfies them, too.

Remark.

From (2.120) and (2.133) we have:

$$\Psi_0^\dagger \mathbf{W}_0 \Psi_0 = \mathbf{K}_0(w) , \quad \Psi_0^\dagger \mathbf{S}^\dagger \mathbf{W} \mathbf{S} \Psi_0 = \mathbf{K}(w) , \quad (2.138)$$

Incidentally, notice that if we now assumed that $\mathbf{K}(w) = \mathbf{K}_0(w)$, the above relations would yield:

$$\mathbf{S}^\dagger \mathbf{W} \mathbf{S} = \mathbf{W}_0 . \quad (2.139)$$

This is a simple relation whose importance will be illustrated in section 2.3.4 as well as in chapter 6.

The Physical Operators

Proposition (2.3.9) Limiting form of \mathbf{S}

In the limit in which $w \rightarrow \infty$ the matrix \mathbf{S} assumes the form:

$$\mathbf{S} = \mathbf{I} + \frac{1}{w} \mathbf{R} , \quad \lim_{w \rightarrow \infty} \mathbf{R} = \text{bounded} , \quad \mathbf{R} = \lim_{w \rightarrow \infty} w (\mathbf{S} - \mathbf{I}) . \quad (2.140)$$

Proof

Notice that $\Lambda_A^B \rightarrow 0$ as $w \rightarrow \infty$. Therefore, by using (2.134), we have:

$$\lim_{w \rightarrow \infty} \partial_A \mathbf{S} = 0 . \quad (2.141)$$

From this relation we argue that $\mathbf{S}(w \rightarrow \infty) = \text{const.}$ By using the fact that Ψ is defined up to a transformation like (2.151) we can always make this constant to be the identity. Therefore, the general form of \mathbf{S} is the one given in (2.140). \square q e d

Proposition (2.3.10) The U_A Matrix

The new matrix U_A is given by

$$U_A = U_{0_A} + 2i \partial_A R. \quad (2.142)$$

Proof

Let us consider (2.134). By putting (2.140) into (2.134) we have:

$$\partial_A R = w \Lambda_A^B (U_B - U_{0_B}) + \Lambda_A^B (U_B R - R U_{0_B}). \quad (2.143)$$

By using (2.112) we have:

$$\lim_{w \rightarrow \infty} w \Lambda_A^B = \frac{1}{2i} \delta_A^B, \quad \lim_{w \rightarrow \infty} \Lambda_A^B = 0. \quad (2.144)$$

Therefore the thesis is proven. \square q e d.

Proposition (2.3.11) The G Matrix

The new matrix G is given by:

$$G = G_0 - 4i (R \Omega + R^\dagger \Omega) + 4i \beta_0 \Omega \quad (2.145)$$

where $\beta_0 \in \mathbb{R}$ is an arbitrary constant of integration.

Proof

Let us consider (2.88a) both for the seed and the new metric:

$$\partial_A (G - 4i \beta \Omega) = 2 (U_A^\dagger \Omega - \Omega U_A), \quad (2.146a)$$

$$\partial_A (G_0 - 4i \beta \Omega) = 2 (U_{0_A}^\dagger \Omega - \Omega U_{0_A}). \quad (2.146b)$$

Let us substitute (2.142) into the above equations and consider their difference. We obtain:

$$\partial_A (G - 4i \beta \Omega) = \partial_A (G_0 - 4i \beta \Omega) - 4i \partial_A (\Omega R + R^\dagger \Omega). \quad (2.147)$$

By integrating the above equation we obtain:

$$G = G_0 - 4i (\Omega R + R^\dagger \Omega) + \text{const.} \quad (2.148)$$

In order to ensure that $G = G^\dagger$, this constant of integration can be chosen to be $4i \beta_0 \Omega$, where β_0 is just a real number. \square q e d.

It may be asked why a constant of integration is added to the 12 and 21 components only. Surely the electromagnetic components do not necessitate such a constant: in fact, the electromagnetic field $F_{\mu\nu}$ is defined in terms of the derivatives of Φ^a . Therefore, adding a constant to Φ^a would not affect $F_{\mu\nu}$ at all. On the other hand, the purely gravitational components 11 and 22 turn out to be real, regardless the

actual form of \mathbf{R} . Having that, a constant of integration is only needed in the remaining 12 and 21 components.

Proposition (2.3.12) The \mathbf{W} Matrix

The new matrix \mathbf{W} is given by:

$$\mathbf{W} = \mathbf{W}_0 - 4i (\mathbf{R} \Omega + \mathbf{R}^\dagger \Omega) + 4i \beta_0 \Omega . \quad (2.149)$$

Proof

This simply came from the definition of \mathbf{W} given in (2.117). \square q.e.d.

Equation (2.120) implies that the formalism admits two arbitrary first integrals: in fact it holds both for \mathbf{W} associated to the new metric and for the \mathbf{W}_0 associated to the seed Ψ_0 . We are then left with the two arbitrary matrices $\mathbf{K}(w)$ and $\mathbf{K}_0(w)$:

$$\Psi^\dagger \mathbf{W} \Psi = \mathbf{K}(w) , \quad \Psi_0^\dagger \mathbf{W}_0 \Psi_0 = \mathbf{K}_0(w) . \quad (2.150)$$

By using the fact that any Ψ is defined up to a transformation

$$\Psi \rightarrow \tilde{\Psi} = \Psi \mathbf{C}(w) , \quad (2.151)$$

one can use this freedom to choose a suitable $\mathbf{K}_0(w)$. The choice that has been considered by Alekseev is:

$$\mathbf{K}_0 = \begin{pmatrix} 4\epsilon & 0 & 0 \\ 0 & -4\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (2.152)$$

This is not the only possible choice, but it is rather convenient. We will clarify this point later on in this chapter and also in chapter 6.

2.3.4 The Simple Pole Ansatz

In this section we study solutions obtained by making the ansatz that \mathbf{S} and \mathbf{S}^{-1} only have simple poles (i.e. \mathbf{S} , \mathbf{S}^{-1} have meromorphic structure) in the complex plane w .

$$\mathbf{S} = \mathbf{I} + \sum_{k=1}^N \frac{1}{w - w[k]} \mathbf{R}_{[k]} , \quad \mathbf{S}^{-1} = \mathbf{I} + \sum_{k=1}^N \frac{1}{w - w[k]} \mathbf{Q}_{[k]} , \quad (2.153)$$

where N is the number of poles and $\mathbf{R}_{[k]}$ and $\mathbf{Q}_{[k]}$ are two 3×3 matrices to be determined through the condition $\mathbf{S} \mathbf{S}^{-1} = \mathbf{I}$.

Proposition (2.3.13) General form for $\mathbf{R}_{[k]}$ and $\mathbf{Q}_{[k]}$

The matrices $\mathbf{R}_{[k]}$ and $\mathbf{Q}_{[k]}$ must have the general form:

$$\mathbf{R}_{[k]} = \mathbf{n}_{[k]} \otimes \mathbf{m}_{[k]} , \quad \mathbf{Q}_{[k]} = \mathbf{p}_{[k]} \otimes \mathbf{q}_{[k]} . \quad (2.154)$$

Proof

The proof can easily be derived from that given for the BZ solitons – see Proposition (2.2.12). \square q.e.d.

Proposition (2.3.14) The vectors $\mathbf{n}_{[k]}$ and $\mathbf{q}_{[k]}$

The 3-vectors $\mathbf{n}_{[k]}$ and $\mathbf{q}_{[k]}$ are given by:

$$\mathbf{q}_{[k]} = - \sum_{j=1}^N \Gamma^{-1}_{[j,k]} \mathbf{m}_{[j]}, \quad \mathbf{n}_{[k]} = \sum_{j=1}^N \Gamma^{-1}_{[k,j]} \mathbf{p}_{[j]}, \quad \Gamma_{[k,j]} = \frac{\mathbf{p}_{[k]} \cdot \mathbf{m}_{[j]}}{w_{[j]} - \nu_{[k]}}. \quad (2.155)$$

Proof

Let us consider the identity $\mathbf{S} \mathbf{S}^{-1} = \mathbf{I}$. By substituting (2.153) into it we have:

$$\sum_{j=1}^N \frac{1}{w - \nu_{[j]}} \mathbf{Q}_{[j]} + \sum_{k=1}^N \frac{1}{w - w_{[k]}} \mathbf{R}_{[k]} + \sum_{k,j=1}^N \frac{1}{(w - w_{[k]})(w - \nu_{[j]})} \mathbf{R}_{[k]} \mathbf{Q}_{[j]} = 0$$

Let us now consider the limit when w approaches $w_{[k]}$. In order to avoid singularities we must set:

$$\mathbf{R}_{[k]} + \sum_{j=1}^N \frac{1}{w_{[k]} - \nu_{[j]}} \mathbf{R}_{[k]} \mathbf{Q}_{[j]} = 0. \quad (2.156)$$

By substituting (2.154) in the above expression we finally get the equation:

$$\mathbf{m}_{[k]} + \sum_{j=1}^N \frac{\mathbf{m}_{[k]} \cdot \mathbf{p}_{[j]}}{w_{[k]} - \nu_{[j]}} \mathbf{q}_{[j]} = 0. \quad (2.157)$$

By considering the identity $\mathbf{S}^{-1} \mathbf{S} = \mathbf{I}$ and the limit $w \rightarrow \nu_{[k]}$, one can analogously obtain the equation:

$$\mathbf{p}_{[k]} - \sum_{j=1}^N \frac{\mathbf{p}_{[k]} \cdot \mathbf{m}_{[j]}}{w_{[j]} - \nu_{[k]}} \mathbf{n}_{[j]} = 0. \quad (2.158)$$

From these, it is then straightforward to prove the assertion. \square q.e.d.

Proposition (2.3.15) One first property of the poles

The poles in \mathbf{S} and \mathbf{S}^{-1} do not depend upon α and β .

Proof

Let us consider (2.134) written in the form:

$$\Lambda_A^B \mathbf{U}_B = (\partial_A \mathbf{S} + \Lambda_A^B \mathbf{S} \mathbf{U}_{0B}) \mathbf{S}^{-1}. \quad (2.159)$$

By substituting (2.153) into (2.159) we have:

$$\Lambda_A^B U_B = \sum_{k=1}^N \frac{\partial_A w[k]}{(w - w[k])^2} R[k] S^{-1} + \sum_{k=1}^N \frac{1}{w - w[k]} (\partial_A R[k] + \Lambda_A^B R[k] U_{0B}) S^{-1},$$

where S^{-1} is evaluated at $w[k]$. The first member of the above equation does not contain poles in $w[k]$. Therefore the same must happen at the second member. As a result, we have to set:

$$\partial_A w[k] = 0, \quad (\partial_A R[k] + \Lambda_A^B R[k] U_{0B}) S^{-1}(w[k]) = 0. \quad (2.160)$$

Let us consider (2.134) written in the form:

$$\Lambda_A^B U_B = -S (\partial_A S^{-1} - \Lambda_A^B S U_{0B} S^{-1}). \quad (2.161)$$

Analogously, by considering the residue at the poles $\nu[k]$ we must set:

$$\partial_A \nu[k] = 0, \quad S(\nu[k]) (\partial_A Q[k] - \Lambda_A^B U_{0B} Q[k]) = 0. \quad (2.162)$$

The first equations in (2.160) and (2.162) prove the assertion. \square q.e.d.

Proposition (2.3.16) The vectors $m[k]$ and $p[k]$

The 3-vectors $m[k]$ and $p[k]$ are given by:

$$m[k] = \kappa[k] \cdot \Psi_0^{-1}(w[k]), \quad p[k] = \Psi_0(\nu[k]) \cdot \ell[k], \quad (2.163)$$

where $\kappa[k]$ and $\ell[k]$ are completely arbitrary constant vectors.

Proof

From the relations $S S^{-1} = I$ and $S^{-1} S = I$ we can obtain:

$$R[k] S^{-1}(w[k]) = 0, \quad S(\nu[k]) Q[k] = 0. \quad (2.164)$$

Let us set:

$$S^{-1}(w[k]) = N[k] \otimes M[k], \quad S(\nu[k]) = P[k] \otimes T[k]. \quad (2.165)$$

The condition $R[k] S^{-1}(w[k]) = 0$ now reads $m[k] \cdot N[k] = 0$. As a result, the second equation in (2.160) simplifies to:

$$\partial_A m[k] \cdot N[k] + \Lambda_A^B m[k] \cdot N[k] U_{0B} = 0. \quad (2.166)$$

Analogously, the condition $S(\nu[k]) Q[k] = 0$ now reads $T[k] \cdot p[k] = 0$. Therefore, the second equation in (2.162) simplifies to:

$$\partial_A T[k] \cdot p[k] - \Lambda_A^B U_{0B} T[k] \cdot p[k] = 0. \quad (2.167)$$

By confronting (2.166) and (2.167) with (2.112) – written for the seed metric – we have the assertion proved. \square q.e.d.

Proposition (2.3.17) A second property for the poles

Under the assumption that $\mathbf{K} = \mathbf{K}_0$, we have that the poles in \mathbf{S} and \mathbf{S}^{-1} must be complex-conjugate to each other: $\nu_{[k]} = \overline{w_{[k]}}$.

Proof

Let us consider the (2.139) in the form:

$$\mathbf{S}^\dagger \mathbf{W} = \mathbf{W}_0 \mathbf{S}^{-1}. \quad (2.168)$$

Let us substitute (2.153) into (2.168). We have:

$$\mathbf{W} + \sum \frac{1}{w - \overline{w_{[k]}}} \mathbf{R}_{[k]}^\dagger \mathbf{W} = \mathbf{W}_0 + \sum \frac{1}{w - \nu_{[k]}} \mathbf{W}_0 \mathbf{Q}_{[k]}.$$

Since $\mathbf{R}_{[k]}$ and $\mathbf{Q}_{[k]}$ have no poles, and since the structure of the singularities must be the same in the two members of the above equation, we have to conclude:

$$\nu_{[k]} = \overline{w_{[k]}} \quad (2.169)$$

This concludes the proof. \square q.e.d.

Proposition (2.3.18) Constraints on the free parameters

Under the assumption that $\mathbf{K} = \mathbf{K}_0$, we have that:

$$\ell_{[k]} = -4 \in \mathbf{K}_0 (\overline{w_{[k]}})^{-1} \overline{\kappa_{[k]}}. \quad (2.170)$$

Proof

Let us consider the (2.139) in the two equivalent forms:

$$\mathbf{W} = \mathbf{S}^{\dagger^{-1}} \mathbf{W}_0 \mathbf{S}^{-1}, \quad \mathbf{W}^{-1} = \mathbf{S} \mathbf{W}_0^{-1} \mathbf{S}^\dagger. \quad (2.171)$$

Let us substitute (2.153) into the second of equations (2.171). We have.

$$\begin{aligned} \mathbf{W}^{-1} = \mathbf{W}_0^{-1} + \sum_{j=1}^N \frac{1}{w - \overline{w_{[j]}}} \mathbf{W}_0^{-1} \mathbf{R}_{[j]}^\dagger + \\ + \sum_{j=1}^N \frac{1}{w - w_{[j]}} \left(\mathbf{R}_{[j]} \mathbf{W}_0^{-1} + \sum_{k=1}^N \frac{1}{w - \overline{w_{[k]}}} \mathbf{R}_{[j]} \mathbf{W}_0^{-1} \mathbf{R}_{[k]}^\dagger \right), \end{aligned} \quad (2.172)$$

where the α and β dependence has been dropped out. Let us now impose that the second member has not singularities at $w = w_{[k]}$. Therefore we have:

$$\mathbf{R}_{[j]} \mathbf{W}_0 (w_{[k]})^{-1} + \sum_{k=1}^N \frac{1}{w_{[k]} - \overline{w_{[k]}}} \mathbf{R}_{[j]} \mathbf{W}_0 (w_{[k]})^{-1} \mathbf{R}_{[k]}^\dagger = 0. \quad (2.173)$$

Let us insert in the above equation the explicit expression for $\mathbf{R}_{[j]}$. We are left with:

$$\mathbf{m}_{[j]} \cdot \mathbf{W}_0(w_{[k]})^{-1} + \sum_{k=1}^N \frac{\mathbf{m}_{[j]} \cdot \mathbf{W}_0(w_{[k]})^{-1} \cdot \overline{\mathbf{m}}_{[k]}}{w_{[k]} - \overline{w}_{[k]}} \overline{\mathbf{n}}_{[k]} = 0 . \quad (2.174)$$

By confronting this expression with (2.157) we have the assertion proved. \square q.e.d.

Up to this stage the vector $\kappa_{[k]}$ is completely arbitrary. It is trivial to notice that \mathbf{S} and \mathbf{S}^{-1} are invariant under the transformation:

$$\kappa_{[k]} \mapsto k_1 \kappa_{[k]} , \quad \ell_{[k]} \mapsto k_2 \ell_{[k]} , \quad (2.175)$$

where k_1 and k_2 are two arbitrary complex constants. In fact it can be seen that under such a transformation we also have:

$$\mathbf{m}_{[k]} \mapsto k_1 \mathbf{m}_{[k]}, \quad \mathbf{n}_{[k]} \mapsto k_1^{-1} \mathbf{n}_{[k]}, \quad \mathbf{p}_{[k]} \mapsto k_2 \mathbf{p}_{[k]}, \quad \mathbf{q}_{[k]} \mapsto k_2^{-1} \mathbf{q}_{[k]} . \quad (2.176)$$

Therefore we can use this freedom to put:

$$\kappa_{[k]} = (1 , \kappa_2[k] , \kappa_3[k]) , \quad \ell_{[k]} = (1 , \ell_2[k] , \ell_3[k]) , \quad (2.177)$$

where $\kappa_2[k]$, $\kappa_3[k]$, $\ell_2[k]$ and $\ell_3[k]$ are completely arbitrary complex constants. However for any arbitrary \mathbf{K}_0 it is not ensured that condition (2.170) will automatically give $\ell_{[k]}$ in the above form. However, this is so for the particular $\mathbf{K}_0(w)$ given in (2.152).

Chapter 3

Belinski–Zakharov Soliton Solutions

In this chapter we intend to give an account of the soliton metrics that may be explicitly constructed by applying the Belinski–Zakharov (BZ) generating technique described in chapter 2. In some case we will simply quote known solutions and rewrite them in our notation. In some other case we will derive again known solutions by using the Alekseev’s determinant method [95].

As mentioned at the end of section 2.2.4, the BZ soliton technique does not easily provide explicit solutions. This is mainly due to the Γ^{-1} in (2.76). To invert this matrix is not a simple task and it is hard to get soliton solutions with more than 4 poles. The problem has been partly solved by Alekseev [95], who managed to rewrite each metric component in terms of the ratio of two determinants. This considerably reduces the amount of computational work that has to be performed. Surprisingly enough, we have not found any paper referring to it. We have seen a similar approach only in [124].

Even more troublesome than this, there is also the problem of finding Ψ_0 , i.e. to solve explicitly the linear pair (2.32) for the seed. In this respect, we have found the works of Kitchingham [77] and Letelier [94] invaluable.

In sections 3.1 and 3.2 we will briefly deal with these two problems. Subsequently, sections 3.3, 3.4 and 3.5 will be devoted to the presentation of a few explicit soliton solutions and their properties.

3.1 On the generating function Ψ_0

As mentioned above, we will consider here the problem of finding the generating function Ψ_0 once a metric g_0 has been given. This is a crucial step in the construction of the soliton solutions, because all the relevant quantities are constructed in terms of the Ψ_0 associated with the seed.

In [77] a number of explicit cases are investigated, both diagonal and not diagonal. The analysis is carried out using the fact that the BZ inverse scattering technique is known to be explicitly related with the Kinnersley–Chitre transformations. Actually, in [77, 78] the KC transformations are also extended from the stationary axisymmetric case to the cosmological ones. Among the various results obtained, the one relevant to us is the explicit form of Ψ_0 for the Wainwright–Marshmann

metric [96, 97, 98], which we will consider in the next section

In [99] Belinski and Francaviglia show how to construct the Ψ_0 generating function associated to a Bianchi II metric. The case of Bianchi I metrics is recovered as a particular limit of this.

In [94] the diagonal case is considered in the cosmological and cylindrically symmetric context: $\epsilon = 1$. Given a general diagonal metric

$$g_0 = \begin{pmatrix} \alpha \phi & 0 \\ 0 & \alpha \frac{1}{\phi} \end{pmatrix}, \quad (3.1)$$

where ϕ is a function of α and β , it is shown that the associated generating function Ψ_0 is given by [94].

$$\Psi_0 = \begin{pmatrix} (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} \Upsilon(\alpha, \beta, \lambda) & 0 \\ 0 & (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} \frac{1}{\Upsilon(\alpha, \beta, \lambda)} \end{pmatrix}, \quad (3.2)$$

where $\Upsilon(\alpha, \beta, \lambda)$ fulfills the "initial condition" $\Upsilon(\alpha, \beta, 0) = \phi(\alpha, \beta)$ and satisfies the equations:

$$(\alpha \partial_\alpha - \lambda \partial_\alpha + 2\lambda \partial_\lambda) \log \Upsilon = \alpha \phi_{,\alpha}, \quad (3.3)$$

$$(\alpha \partial_\beta - \lambda \partial_\alpha) \log \Upsilon = \alpha \phi_{,\beta}. \quad (3.4)$$

Of course, the function ϕ satisfies the equation:

$$\phi_{,\alpha\alpha} + \frac{1}{\alpha} \phi_{,\alpha} - \phi_{,\beta\beta} = 0. \quad (3.5)$$

Moreover, it is shown that the linear superposition property enjoyed by those solutions – equation (3.5) is linear – for the generating functions Ψ_0 is expressed as follow: if $\phi^{(1)}$ and $\phi^{(2)}$ are two solutions to (3.5), then the other new solution $\phi = h_1 \phi^{(1)} + h_2 \phi^{(2)}$ is generated by the function $\Upsilon = \Upsilon_1^{h_1} \Upsilon_2^{h_2}$.

$$\phi = h_1 \phi^{(1)} + h_2 \phi^{(2)} \iff \Upsilon = \Upsilon_1^{h_1} \Upsilon_2^{h_2}. \quad (3.6)$$

This last property can be used to generate further functions for diagonal solutions.

To conclude this section, it is worth mentioning that the problem of finding Ψ_0 also exists for the Alekseev solution generating technique. As far as we know, explicit expressions of Alekseev's Ψ_0 are known for the Minkowski metric [51, 52, 53, 54], the Kasner metric [128] and the Bianchi VI₀ metric [104]. We have found an explicit expression for the Kasner metric – independently from [128] – which will be used in chapter 6. We also have been involved in the search of the Alekseev's Ψ_0 associated with the Bertotti–Robinson metric. The relevant equations have been virtually solved, but we have not been able to put them in a readable form.

Bibliography of Files: [1, 34]

3.2 The Alekseev Determinant Method

In this section we will consider the other problem highlighted in the introduction to this chapter: that of having an efficient algorithm to explicitly construct the soliton solutions.

Let us consider the BZ N -Soliton solution in the form given in (2.77b). It can be rewritten as:

$$g_{ab} = \prod_{k=1}^N \frac{\mu[k]}{\alpha} \left(g_{0ab} - \mathcal{K}_a \mathcal{K}_b H_{ab} \right), \quad H_{ab} = \frac{1}{\mathcal{K}_a} \frac{1}{\mathcal{K}_b} \sum_{k,j=1}^N (\tilde{\Gamma}_{[k,j]})^{-1} N_a^{(k)} N_b^{(j)}, \quad (3.7)$$

where \mathcal{K}_a are arbitrary functions. Let us introduce the matrix $G_{[k,l]}$ defined by:

$$G_{ab[k,l]} = \mathcal{K}_a \mathcal{K}_b \frac{1}{N_a^{(k)}} \frac{1}{N_b^{(l)}} \tilde{\Gamma}_{[k,l]}, \quad H_{ab} = \sum_{k,l=1}^n G_{ab}^{-1[k,l]}. \quad (3.8)$$

By using the determinant formula:

$$\det(1 + G_{[k,l]}) = \det G_{[k,l]} + \det G_{[k,l]} \sum_{r,s=1}^n G^{-1[r,s]}, \quad (3.9)$$

and by introducing the quantities:

$$\Theta_{ab} = \det \left(\frac{1}{\mathcal{K}_a} \frac{1}{\mathcal{K}_b} N_a^{(k)} N_b^{(l)} + \tilde{\Gamma}_{[k,l]} \right), \quad \Gamma = \det \tilde{\Gamma}_{[k,l]}, \quad (3.10)$$

the metric coefficients can be rewritten as:

$$g_{ab} = \prod_{k=1}^N \frac{\mu[k]}{\alpha} \left(g_{0ab} + \mathcal{K}_a \mathcal{K}_b - \mathcal{K}_a \mathcal{K}_b \frac{\Theta_{ab}}{\Gamma} \right). \quad (3.11)$$

The functions \mathcal{K}_a can be appropriately chosen as to simplify the explicit calculations of the relevant determinants.

Hereafter we will give a few example of the determinants Θ_{ab} and Γ for solitons solutions generated starting from a number of seeds. Cosmological and cylindrically symmetric coordinates – i.e. $\epsilon = 1$ – will be considered.

In [95] the explicit formulae are given for the Minkowski seed case with $\epsilon = -1$, $\alpha = \rho$ and $\beta = t$.

Bibliography of Files: [2]

3.2.1 Kasner Seed Metric

Let us introduce the Kasner metric:

$$\mathbf{g}_0 = \begin{pmatrix} \alpha^{1-2q} & 0 \\ 0 & \alpha^{1+2q} \end{pmatrix}, \quad (3.12)$$

generated by the following Ψ_0 [94]:

$$\Psi_0 = \begin{pmatrix} (\alpha^2 + 2\lambda\beta + \lambda^2)^{1/2-q} & 0 \\ 0 & (\alpha^2 + 2\lambda\beta + \lambda^2)^{1/2+q} \end{pmatrix}. \quad (3.13)$$

By using (3.12) as seed, and with the following choice of the arbitrary functions

$$\mathcal{K}_1 = \alpha^{\frac{1}{2}-q}, \quad \mathcal{K}_2 = \alpha^{\frac{1}{2}+q}, \quad (3.14)$$

the determinants in (3.10) are given by:

$$\Gamma = \det \left(A_{[k]} \ A_{[j]} \ \frac{c_{[k]} c_{[j]} + \lambda_{[k]}^{2q} \lambda_{[j]}^{2q}}{\lambda_{[k]} \lambda_{[j]} - 1} \right), \quad (3.15a)$$

$$\Theta_{11} = \det \left(A_{[k]} \ A_{[j]} \ \frac{c_{[k]} c_{[j]} + \lambda_{[k]}^{2q-1} \lambda_{[j]}^{2q-1}}{\lambda_{[k]} \lambda_{[j]} - 1} \right), \quad (3.15b)$$

$$\Theta_{12} = \det \left(A_{[k]} \ A_{[j]} \ \frac{c_{[k]} c_{[j]} + \lambda_{[k]}^{2q} \lambda_{[j]}^{2q} - c_{[k]} \lambda_{[k]}^{2q} (1 - \lambda_{[k]}^{-1} \lambda_{[j]}^{-1})}{\lambda_{[k]} \lambda_{[j]} - 1} \right), \quad (3.15c)$$

$$\Theta_{22} = \det \left(B_{[k]} \ B_{[j]} \ \frac{c_{[k]} c_{[j]} + \lambda_{[k]}^{2q+1} \lambda_{[j]}^{2q+1}}{\lambda_{[k]} \lambda_{[j]} - 1} \right), \quad (3.15d)$$

where $i, j = 1, \dots, N$ and:

$$A_{[k]} = 2^{q-\frac{1}{2}} w_{[k]}^{q-\frac{1}{2}} \lambda_{[k]}^{\frac{1}{2}-q}, \quad B_{[k]} = 2^{q-\frac{1}{2}} w_{[k]}^{q-\frac{1}{2}} \lambda_{[k]}^{-\frac{1}{2}-q}, \quad (3.16a)$$

$$\kappa_{2[k]} = 2^{2q} w_{[k]}^{2q} c_{[k]}, \quad \mu_{[k]} = \alpha \lambda_{[k]}. \quad (3.16b)$$

It is trivial to check that (3.15)'s, when specialized to the case $q = \frac{1}{2}$ (Minkowski metric), are compatible with the formulae given in [95], appropriately converted to the cosmological coordinates by the ansatz $\rho \rightarrow i \alpha$.

Bibliography of Files: [3]

3.2.2 Bianchi VI₀ Seed Metric

Let us introduce the Bianchi VI₀ metric [94]:

$$\mathbf{g}_0 = \begin{pmatrix} \alpha e^{+\beta} & 0 \\ 0 & \alpha e^{-\beta} \end{pmatrix}, \quad (3.17)$$

generated by the following Ψ_0 [94]:

$$\Psi_0 = \begin{pmatrix} (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} e^{\beta+\lambda/2} & 0 \\ 0 & (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} e^{-\beta-\lambda/2} \end{pmatrix}. \quad (3.18)$$

In passing, it is worth noticing that this metric has been used as seed by Belinski to generate his gravibreather solution – see [101, 102, 103, 104, 105] – and to study possible topological properties of soliton solutions.

By using (3.17) as seed, and with the following choice of the arbitrary functions

$$\mathcal{K}_1 = \alpha^{1/2} \Omega^{1/2}, \quad \mathcal{K}_2 = \alpha^{1/2} \Omega^{-1/2}, \quad \Omega = e^\beta, \quad (3.19)$$

the determinants in (3.10) are given by:

$$\Gamma = \det \left(A_{[k]} \ A_{[j]} \ \frac{\chi_{[k]} \ \chi_{[j]} + c_{[k]} \ c_{[j]}}{\lambda_{[k]} \ \lambda_{[j]} - 1} \right), \quad (3.20a)$$

$$\Theta_{11} = \det \left(A_{[k]} \ A_{[j]} \ \frac{\phi_{[k]} \ \phi_{[j]} + c_{[k]} \ c_{[j]}}{\lambda_{[k]} \ \lambda_{[j]} - 1} \right), \quad (3.20b)$$

$$\Theta_{12} = \det \left(A_{[k]} \ A_{[j]} \ \left(\frac{\chi_{[k]} \ \chi_{[j]} + c_{[k]} \ c_{[j]}}{\lambda_{[k]} \ \lambda_{[j]} - 1} - \frac{c_{[j]}}{\lambda_{[j]}} \phi_{[k]} \right) \right), \quad (3.20c)$$

$$\Theta_{22} = \det \left(B_{[k]} \ B_{[j]} \ \frac{\varphi_{[k]} \ \varphi_{[j]} + c_{[k]} \ c_{[j]}}{\lambda_{[k]} \ \lambda_{[j]} - 1} \right), \quad (3.20d)$$

where $i, j = 1, \dots, N$ and:

$$A_{[k]} = \frac{1}{\sqrt{2\Omega w_{0[k]}}} \frac{1}{\sqrt{\lambda_{[k]}}} e^{1/2 \alpha \lambda_{[k]}}, \quad B_{[k]} = \frac{1}{\sqrt{2\Omega w_{0[k]}}} \sqrt{\lambda_{[k]}} e^{1/2 \alpha \lambda_{[k]}}, \quad (3.21a)$$

$$\phi_{[k]} = \frac{1}{\lambda_{[k]}} e^{-\alpha \lambda_{[k]}}, \quad \varphi_{[k]} = \lambda_{[k]} e^{-\alpha \lambda_{[k]}}, \quad \chi_{[k]} = e^{-\alpha \lambda_{[k]}}, \quad (3.21b)$$

$$\kappa_2[k] = \frac{1}{\Omega} c_{[k]}, \quad \mu[k] = \alpha \lambda_{[k]}. \quad (3.21c)$$

Notice that, in this particular case, the $c_{[k]}$'s are not constant.

Bibliography of Files: [4]

3.2.3 General Diagonal Seed Metric

Let consider the general diagonal metric already introduced in section 3 1:

$$\mathbf{g}_0 = \begin{pmatrix} \alpha \Upsilon(\alpha, \beta, 0) & 0 \\ 0 & \alpha \frac{1}{\Upsilon(\alpha, \beta, 0)} \end{pmatrix}, \quad (3.22)$$

generated by the following Ψ_0 [94]:

$$\Psi_0 = \begin{pmatrix} (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} \Upsilon(\alpha, \beta, \lambda) & 0 \\ 0 & (\alpha^2 + 2\lambda\beta + \lambda^2)^{\frac{1}{2}} \frac{1}{\Upsilon(\alpha, \beta, \lambda)} \end{pmatrix}. \quad (3.23)$$

By using (3.22) as a seed, and with the following choice of the arbitrary functions

$$\mathcal{K}_1 = \alpha^{1/2} \Upsilon(\alpha, \beta, 0)^{1/2}, \quad \mathcal{K}_2 = \alpha^{1/2} \Upsilon(\alpha, \beta, 0)^{-1/2}, \quad (3.24)$$

the determinants in (3.10) are given by:

$$\Gamma = \det \left(A_{[k]} \ A_{[l]} \frac{B_{[k]} \ B_{[l]}}{\lambda_{[k]} \ \lambda_{[l]} - 1} \left(1 + c_{[k]} \ c_{[l]} \ \frac{\Upsilon_{[k]}^2 \ \Upsilon_{[l]}^2}{\Upsilon_{[0]}^2} \right) \right), \quad (3.25a)$$

$$\Theta_{11} = \det \left(A_{[k]} \ A_{[l]} \frac{C_{[k]} \ C_{[l]}}{\lambda_{[k]} \ \lambda_{[l]} - 1} \left(1 + c_{[k]} \ c_{[l]} \ \lambda_{[k]} \ \lambda_{[l]} \ \frac{\Upsilon_{[k]}^2 \ \Upsilon_{[l]}^2}{\Upsilon_{[0]}^2} \right) \right), \quad (3.25b)$$

$$\Theta_{12} = \det \left(A_{[k]} \ A_{[l]} \frac{B_{[k]} \ B_{[l]}}{\lambda_{[k]} \ \lambda_{[l]} - 1} \left(1 + c_{[k]} \ c_{[l]} \ \frac{\Upsilon_{[k]}^2 \ \Upsilon_{[l]}^2}{\Upsilon_{[0]}^2} \right) - \right. \\ \left. c_{[l]} \ \frac{A_{[k]} \ A_{[l]}}{\Upsilon(0)} \ C_{[k]} \ C_{[l]} \ \Upsilon_{[l]}^2 \right), \quad (3.25c)$$

$$\Theta_{22} = \det \left(A_{[k]} \ A_{[l]} \frac{B_{[k]} \ B_{[l]}}{\lambda_{[k]} \ \lambda_{[l]} - 1} \left(1 + c_{[k]} \ c_{[l]} \ \frac{\Upsilon_{[k]}^2 \ \Upsilon_{[l]}^2}{\Upsilon_{[0]}^2} \frac{1}{\lambda_{[k]} \ \lambda_{[l]}} \right) \right), \quad (3.25d)$$

where we introduced the shortened notation $\Upsilon_{[0]} = \Upsilon(\alpha, \beta, 0)$, $\Upsilon_{[k]} = \Upsilon(\alpha, \beta, \mu_{[k]})$. The $c_{[k]}$ s are just complex constants and the remaining functions appearing in the above formulae are given by:

$$A_{[k]} = \frac{1}{\sqrt{2} w_i} \sqrt{\Upsilon_{[0]}}, \quad B_{[k]} = \frac{\sqrt{\lambda_{[k]}}}{\Upsilon_{[k]}}, \quad C_{[k]} = \lambda_{[k]} B_{[k]}, \quad \mu_{[k]} = \alpha \lambda_{[k]}.$$

The $c_{[k]}$'s are arbitrary complex constants. The limit $c_{[k]} \rightarrow 0$ corresponds to the diagonal metric limit.

Bibliography of Files: [5]

3.2.4 Wainwright–Marshmann Seed Metric

Let us consider the Wainwright–Marshmann [96, 97, 98], [77, 119] metric:

$$ds^2 = t^{-3/8} e^n (d\alpha^2 - d\beta^2) - \sqrt{\alpha} (dx^2 + 2W dx dy + (\alpha + W^2) dy^2), \quad (3.26)$$

where W and n are functions of $v = \beta - \alpha$ only and the function $n(\alpha, \beta)$ is determined by the equation: $n' = W'^2$.

The relevant 2×2 metric is given by:

$$g_0 = \sqrt{\alpha} \begin{pmatrix} 1 & W \\ W & \alpha + W^2 \end{pmatrix}. \quad (3.27)$$

It can be shown that the generating function Ψ_0 reads [77, 119]:

$$\Psi_0 = \left(\frac{\lambda}{K(\alpha, \beta, \lambda)} \right)^{1/4} \begin{pmatrix} \cos Y & \frac{1}{\sqrt{K}} \sin Y \\ W \cos Y - \sqrt{\lambda} \sin Y & \sqrt{\frac{\lambda}{K}} (\cos Y + W \frac{1}{\sqrt{\lambda}} \sin Y) \end{pmatrix}, \quad (3.28)$$

where:

$$Y(\alpha, \beta, \lambda) = \sqrt{K(\alpha, \beta, \lambda)} I(\alpha, \beta, \lambda), \quad K = \frac{\lambda}{\lambda^2 + 2\beta \lambda + \alpha^2}, \quad (3.29a)$$

$$I = \int dv \frac{W'(v)}{\sqrt{1 - 2v K(\alpha, \beta, \lambda)}}. \quad (3.29b)$$

Notice that (3.28) is slightly different from that given in [119], which, in turn, is different from that in [77]¹

By using (3.27) as a seed, and with the following choice of the arbitrary functions

$$\mathcal{K}_1 = \alpha^{1/4}, \quad \mathcal{K}_2 = \alpha^{3/4}, \quad (3.30)$$

the determinants in (3.10) are given by:

$$\begin{aligned} \Gamma = \det \left(F[k] \ F[j] \frac{\lambda[k]^{1/4} \lambda[j]^{1/4}}{\lambda[k] \lambda[j] - 1} \left(\cos Y[k] \cos Y[j] (q[k] q[j] + \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \right. \\ \left. \cos Y[k] \sin Y[j] (q[k] + q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \\ \left. \sin Y[k] \cos Y[j] (q[j] + q[k] \lambda[k]^{1/2} \lambda[j]^{1/2}) + \right. \\ \left. \sin Y[k] \sin Y[j] (1 + q[k] q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) \right) \Bigg), \end{aligned} \quad (3.31a)$$

$$\begin{aligned} \Theta_{11} = \det \left(F[k] \ F[j] \frac{\lambda[k]^{-1/4} \lambda[j]^{-1/4}}{\lambda[k] \lambda[j] - 1} \left(\sin Y[k] \sin Y[j] (q[k] q[j] + \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \right. \\ \left. \sin Y[k] \cos Y[j] (q[k] + q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \\ \left. \cos Y[k] \sin Y[j] (q[j] + q[k] \lambda[k]^{1/2} \lambda[j]^{1/2}) + \right. \\ \left. \cos Y[k] \cos Y[j] (1 + q[k] q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) \right) \Bigg), \end{aligned} \quad (3.31b)$$

$$\begin{aligned} \Theta_{12} = \det \left(F[k] \ F[j] \frac{\lambda[k]^{1/4} \lambda[j]^{1/4}}{\lambda[k] \lambda[j] - 1} \left(\cos Y[k] \cos Y[j] (q[k] q[j] + \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \right. \\ \left. \cos Y[k] \sin Y[j] (q[k] + q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) - \right. \\ \left. \sin Y[k] \cos Y[j] (q[j] + q[k] \lambda[k]^{1/2} \lambda[j]^{1/2}) + \right. \\ \left. \sin Y[k] \sin Y[j] (1 + q[k] q[j] \lambda[k]^{1/2} \lambda[j]^{1/2}) \right) - \\ F[k] \ F[j] \left(\cos Y[k] \cos Y[j] G^+[j] - \right. \\ \left. \cos Y[k] \sin Y[j] G^-[j] q[j] - \right. \\ \left. \sin Y[k] \cos Y[j] G^+[j] q[k] + \right. \\ \left. \sin Y[k] \sin Y[j] G^-[j] q[k] q[j] \right) \Bigg), \end{aligned} \quad (3.31c)$$

where $i, j = 1, \dots, N$ and $\mu[k] = \alpha \lambda[k]$. The free parameters $q[k]$ are defined by the relation $\kappa_2[k] = 2^{1/2} w[k]^{1/2} q[k]$ and the other quantities are given by:

$$F[k] = 2^{-1/4} w[k]^{-1/4}, \quad G^\pm[k] = W(\alpha, \beta) \pm q[k]^\pm \frac{\sqrt{\alpha}}{\sqrt{\lambda[k]}}. \quad (3.32)$$

Bibliography of Files: [6]

¹In particular, formula (8a) in [119] corrects some misprints in formula (4.34) [77]. However, formula (8a) is not correct either. The exponent of K in Ψ_{012} is taken to be -1 . This is wrong as may be immediately checked by calculating the determinant and imposing that $\lim_{\lambda \rightarrow 0} \det \Psi_0 = \alpha^2$.

3.3 On the properties of the Pole-Trajectories

The fundamental bricks which all soliton solutions are built from, are the pole-trajectories (2.59):

$$\mu[k] = w[k] - \beta \pm \sqrt{(w[k] - \beta)^2 - \epsilon \alpha^2}.$$

They crucially determine the general properties of the solutions. It seems then appropriate to spend a few lines to recall their properties.

3.3.1 The case of complex poles

Let us first consider the case when the pole-trajectories assume complex values, i.e. the constants $w[k]$ are complex:

$$w[k] = z_0[k] + iw_0[k]. \quad (3.33)$$

Hereafter we will consider again $\epsilon = 1$. We will mainly consider $\alpha = t$ and $\beta = Z$. However, some formulae will be given in terms of α and β , as they may also apply to the cylindrically symmetric case.

The properties of pole-trajectories have been thoroughly investigated in [106]. We will here mention the main results for the cosmological case only. Following [106], the spacetime (t, Z) presents five regions of interest – figure (3.1):

1. **Causal Region:** $|z_0[k] - Z| < t \rightarrow \infty$
2. **Light-Cone Region:** $|z_0[k] - Z| \approx |Z| = t \rightarrow \infty$
3. **Far Region:** $t \ll |z_0[k] - Z| \approx Z \rightarrow \infty$
4. **Initial Region:** $|z_0[k] - z|, w_0[k], \gg t \rightarrow 0$
5. **Interaction Region:** $t \approx z_0[k], Z \approx 0$

With respect to the classification given in [106], we have also introduced the so-called “Interaction Region”. The importance of this will be evident in chapter 4.

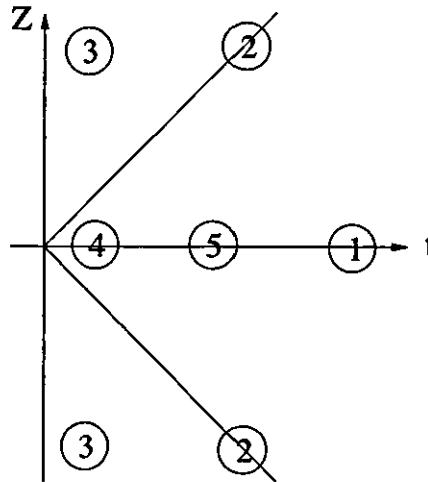


Figure 3.1: Spacetime regions of interest for cosmological soliton solutions with complex poles. Here, $\alpha = t$ and $\beta = Z$.

For complex-conjugate poles, we can introduce the function $\sigma[k]$ defined by:

$$\mu[k] = \alpha \sqrt{\sigma[k]} e^{i\phi[k]}. \quad (3.34)$$

Putting

$$A = \frac{w_0[k]^2 + (z_0[k] - \beta)^2}{\alpha^2}, \quad B = \frac{w_0[k]^2 - (z_0[k] - \beta)^2}{\alpha^2}, \quad (3.35)$$

it can be seen that $\sigma[k]$ must satisfy the fourth degree equation:

$$1 - 4A\sigma - 2(1 + 4B)\sigma^2 - 4A\sigma^3 + \sigma^4 = 0. \quad (3.36)$$

This can be factorised into the form

$$(1 - 2(A + C)\sigma + \sigma^2)(1 - 2(A - C)\sigma + \sigma^2) = 0, \quad (3.37)$$

where $C = \sqrt{1 + 2B + A^2}$. It may be assumed that $C > 0$ as the alternative sign is included by interchanging the factors. Then, since $1 + B \leq C < 1 + A$ with the equality occurring when $w_0[k] = 0$, it can be seen that $A + C \geq 1$, while $(A - C)^2 < 1$. Thus the quartic has just two real solutions which arise from the first factor and are given by:

$$\sigma_- = A + C - \sqrt{(A + C)^2 - 1} = A + C - \sqrt{2}\sqrt{B + A^2 + AC}, \quad (3.38a)$$

$$\sigma_+ = A + C + \sqrt{(A + C)^2 - 1} = A + C + \sqrt{2}\sqrt{B + A^2 + AC}. \quad (3.38b)$$

From these, it immediately follows that

$$\sigma_+ = 1/\sigma_- . \quad (3.39)$$

The freedom in the choice of this sign mimics the one in (2.59).

As we shall see, these solutions are located in the ranges $0 < \sigma_- < 1$ and $1 < \sigma_+ < \infty$. From this, it follows that the poles $\mu[k]$ and $\nu[k]$ are either always inside the circle $|\lambda| = \alpha$ or are always outside it.

The function $\sigma[k]$

The explicit form of function $\sigma[k]$ is

$$\sigma[k] = \frac{w_0[k]^2 + z[k]^2}{\alpha^2} + a[k] \pm \sqrt{2} \sqrt{\frac{w_0[k]^2 - z[k]^2}{\alpha^2} + \frac{(w_0[k]^2 + z[k]^2)^2}{\alpha^4} + a[k] \frac{w_0[k]^2 + z[k]^2}{\alpha^2}},$$

$$a[k] = \sqrt{1 + 2 \frac{w_0[k]^2 - z[k]^2}{\alpha^2} + \frac{(w_0[k]^2 + z[k]^2)^2}{\alpha^4}}, \quad z[k] = z_0[k] - \beta. \quad (3.40)$$

It may also be checked that the α -derivative $\dot{\sigma}[k]$ and the β -derivative $\sigma'[k]$ are given by:

$$\sigma'[k] = \frac{8z[k]}{\alpha^2} \frac{\sigma[k]}{H[k]} \frac{1 - \sigma[k]}{1 + \sigma[k]}, \quad \dot{\sigma}[k] = \frac{2}{\alpha} \frac{\sigma[k]}{H[k]} (1 - \sigma[k]^2), \quad (3.41a)$$

$$H[k] = (1 - \sigma[k])^2 + \frac{16 w_0[k]^2}{\alpha^2} \frac{\sigma[k]^2}{(1 - \sigma[k])^2}. \quad (3.41b)$$

Of course, by using these expressions, one can obtain the higher order derivatives in terms of $\sigma[k]$ and $H[k]$.

The function $\phi[k]$

The function $\phi[k]$ defined in (3.34) is given by:

$$\cos \phi[k] = 2 \frac{z[k]}{\alpha} \frac{\sqrt{\sigma[k]}}{1 + \sigma[k]}, \quad \sin \phi[k] = 2 \frac{w_0[k]}{\alpha} \frac{\sqrt{\sigma[k]}}{1 - \sigma[k]}. \quad (3.42)$$

Properties of $\sigma_+[k]$

Three are the main properties, enjoyed by $\sigma_+[k]$, that we want emphasize here:

1. $\sigma_+[k]^2 \geq 1$.
2. $\sigma_+[k]$ is a generally decreasing function.
3. $\sigma_+[k]$ is an even function.

$\sigma_+[k]^2 \geq 1$

This property follows immediately from the two trivial equations:

$$\sigma_+[k] \geq \sigma_-[k], \quad \sigma_+[k] = \frac{1}{\sigma_-[k]}.$$

Generally decreasing function

We wish to prove that for a fixed $\beta = \bar{\beta}$, the resulting function $\tilde{\sigma}[k] = \sigma_+[k]_{|\beta=\bar{\beta}}$ is a decreasing function.

Let us consider the α -derivative of $\sigma_+[k]$. From the α -derivative in (3.41a) and the from $\sigma_+[k]^2 \geq 1$ we can deduce that $\dot{\sigma}_+[k]$ is a negative function.

This ensures us that σ_+ is a monotonic and decreasing function (in the sense specified above).

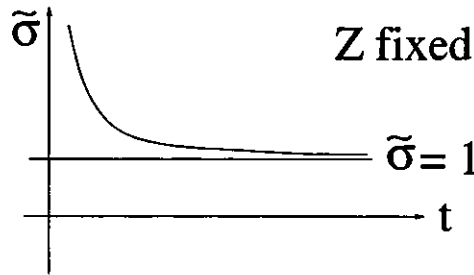


Figure 3.2: General behaviour of the function $\tilde{\sigma}_+[k]$. Here, $\alpha = t$ and $\beta = Z$.

Even Function

σ_+ depends only upon $z[k]^2$, therefore it is symmetric under reflections along the axis defined by $\beta = z_0[k]$ ($\beta \rightarrow -\beta$ if $z_0[k] = 0$ as in figure (3.2)).

By considering the sign of the first t -derivative, we can see that.

$$\begin{aligned} \dot{\sigma}_+ &< 0 & \beta < z_0[k], \\ \dot{\sigma}_+ &> 0 & \beta > z_0[k], \end{aligned}$$

thus the axis $Z = z_0[k]$ is a general minimum for the function

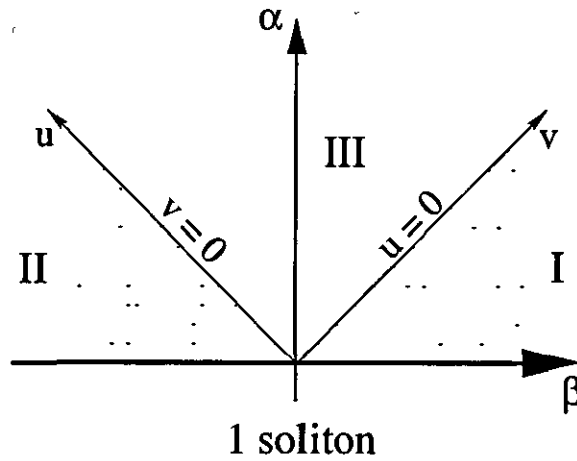


Figure 3.3: Spacetime regions of interest for cosmological soliton solutions with real poles

3.3.2 The case of real poles

When considering real pole solitons, we must set $w[k] \in \mathbb{R}$ in (2.59), i.e. $w_0[k] = 0$. This implies that the solution we are dealing with is not defined everywhere. With the help of figure (3.3) we will illustrate the various regions of the spacetime in which the soliton solution holds. It may also be emphasized that for axisymmetric spacetimes, where $\epsilon = -1$, the pole-trajectories are everywhere well defined even in the real pole case.

The null coordinates u, v in figure (3.3) are defined as:

$$\alpha = \frac{1}{\sqrt{2}} (u + v), \quad \beta_s = \frac{1}{\sqrt{2}} (v - u), \quad \beta_s = \beta - z_0[k], \quad (3.43a)$$

$$v = \frac{1}{\sqrt{2}} (\alpha + \beta), \quad u = \frac{1}{\sqrt{2}} (\alpha - \beta), \quad d\alpha^2 - d\beta^2 = 2 du dv. \quad (3.43b)$$

Region I

Region I is defined as:

$$\beta - z_0[k] \geq +\alpha, \quad \alpha > 0. \quad (3.44)$$

It is trivial to show that in this region $\sqrt{(\beta - z_0[k])^2 - \alpha^2}$ is real, so that the pole-trajectory are well defined. In terms of null coordinates u, v this region is defined by $u < 0, v > -u$.

If we introduce the variable $u = -|u|$ we get $v > |u|$, which shows as v is always positive. Finally, notice that in Region I the pole-trajectory $\mu[k]$ assumes negative values.

Region II

Region II is defined as:

$$\beta - z_0[k] \leq -\alpha, \quad \alpha > 0. \quad (3.45)$$

It is trivial to show that in this region $\sqrt{(\beta - z_0[k])^2 - \alpha^2}$ is real, so that the pole-trajectory are well defined. In terms of null coordinates u, v this region is defined by $v < 0, u > -v$.

If we introduce the variable $v = -|v|$ we get $u > |v|$, which shows as u is always positive. Finally, notice that in Region II the Pole-Trajectory $\mu[k]$ assumes positive values.

Region III

Region III is defined as:

$$-\alpha \leq \beta - z_0[k] \leq +\alpha, \quad \alpha > 0. \quad (3.46)$$

In this region $\sqrt{(\beta - z_0[k])^2 - \alpha^2}$ is imaginary, and therefore the pole-trajectory is not well defined. This rules out any possibility of constructing soliton solutions in such a region. In terms of null coordinates u, v this region is defined by $u > 0, v > 0$.

3.4 A few explicit BZ Soliton Solutions

It is now appropriate to give a few explicit solutions obtained by applying either the BZ soliton technique as illustrated in section 2.2 or its modified version sketched in section 3.2.

We will mostly consider known solutions. However, since we will use part of these in subsequent chapters, it seems useful to show them here, in order to fix notations and keep this work self-contained. Moreover, we will take the opportunity to see the Alekseev's determinant method at work.

3.4.1 BZ 1-Soliton Solution

Let us consider the Kasner metric (3.12) generated by the Ψ_0 given in (3.13). We also recall that, for the Kasner metric, the function f in (1.25) is given by [106, 81]:

$$f_0 = \alpha^{\frac{4q^2-1}{2}} \quad (3.47)$$

With this seed, we wish to construct a soliton solution with one (real) pole in cosmological coordinates. The appropriate pole-trajectory can be written in the form:

$$\mu[1] = w[1] - \beta \pm \sqrt{(w[1] - \beta)^2 - \alpha^2}, \quad w[1] = z_0 \in \mathbb{R}. \quad (3.48)$$

We will consider the plus sign only. This solution is defined in the regions $\beta - z_0 \geq \alpha$ (Region I) and $\beta - z_0 \leq -\alpha$ (Region II) only. The solution in Region I will take the form:

$$f = A \alpha^{2q^2} \frac{e^{(4q-1)\text{ArcCosh}(\beta-z_0)/\alpha}}{\sqrt{(\beta-z_0)^2 - \alpha^2}} (1 + q[1]^2 e^{4q\text{ArcCosh}(\beta-z_0)/\alpha}), \quad (3.49a)$$

$$g_{11} = \alpha^{1-2q} e^{(4q-1)\text{ArcCosh}(\beta-z_0)/\alpha} \frac{q[1]^2 + e^{(2-4q)\text{ArcCosh}(\beta-z_0)/\alpha}}{1 + q[1]^2 e^{4q\text{ArcCosh}(\beta-z_0)/\alpha}}, \quad (3.49b)$$

$$g_{12} = -2q[1] \sqrt{(\beta-z_0)^2 - \alpha^2} \frac{e^{2q\text{ArcCosh}(\beta-z_0)/\alpha}}{1 + q[1]^2 e^{4q\text{ArcCosh}(\beta-z_0)/\alpha}}, \quad (3.49c)$$

where f is defined up to an arbitrary constant. The constant q_1 is defined as $m_{[1]} = (2z_0)^{2q} q_1$. The diagonal limit is obtained when $q_1 \rightarrow 0$, yielding:

$$f = A \frac{\alpha^{2q^2}}{\sqrt{(\beta - z_0)^2 - \alpha^2}} e^{(4q-1)\text{ArcCosh}(\beta-z_0)/\alpha}, \quad (3.50a)$$

$$g_{11} = \alpha^{1-2q} e^{\text{ArcCosh}(\beta-z_0)/\alpha} = \alpha^{1-2q} \frac{\mu_{[1]}}{\alpha}. \quad (3.50b)$$

In this case, the expressions given in [106, 81] are recovered. We will consider again this solution in chapter 5.

A 1 Soliton solution generated from Friedmann metrics was studied by Belinski in [110]. Its relevance as cosmological model was discussed. Also, 1 soliton solutions with a Bianchi II seed have been considered in [99, 100].

Bibliography of Files: [7]

3.4.2 BZ 2-Soliton Solution

Let us now consider the Kasner metric (3.12) as seed and generate the soliton solution with two complex poles. We will use the formulae given in 3.2.1 with q_2 complex conjugate to q_1 . Furthermore, we will stick to the simpler case in which q_1 is taken to be real. Therefore:

$$q_2 = q_1 = c \in \mathbb{R}. \quad (3.51)$$

The resulting metric is then given by.

$$g_{11} = \frac{\mathcal{N}_{11}}{\mathcal{D}}, \quad g_{12} = \frac{\mathcal{N}_{12}}{\mathcal{D}}, \quad (3.52a)$$

$$\begin{aligned} \mathcal{N}_{11} = & 2\alpha^{1-2q} (\cos(2(1+2q)\phi_{[1]}) + i \sin(2(1+2q)\phi_{[1]})) \times \\ & \times \left(c \cos(2(1-2q)\phi_{[1]}) \sigma_{[1]}^{2q} (\sigma_{[1]} - 1)^2 + \right. \\ & \left. \cos(2\phi_{[1]}) (c \sigma_{[1]} + \sigma_{[1]}^{2q})^2 - (c^2 + \sigma_{[1]}^{2q}) (c^2 \sigma_{[1]}^2 + \sigma_{[1]}^{2q}) \right), \end{aligned} \quad (3.52b)$$

$$\begin{aligned} \mathcal{N}_{12} = & -\alpha c (\sigma_{[1]} - 1) \sigma_{[1]}^q \times \\ & \times (\cos(2\phi_{[1]}) + i \sin(2\phi_{[1]}) - 1) (\cos(2q\phi_{[1]}) + i \sin(2q\phi_{[1]})) \times \\ & \times \left((\sigma_{[1]} + c^2) (\cos(2\phi_{[1]}) - \cos(4q\phi_{[1]}) + i \sin(2\phi_{[1]}) - i \sin(4q\phi_{[1]})) + \right. \\ & \left. \sigma_{[1]} (\sigma_{[1]}^{2q-1} + 1) (\cos(2(1+2q)\phi_{[1]}) + i \sin(2(1+2q)\phi_{[1]}) - 1) \right), \end{aligned} \quad (3.52c)$$

$$\begin{aligned} \mathcal{D} = & \sigma_{[1]} (\sigma_{[1]}^{4q} + c^4) (e^{2i\phi_{[1]}} - 1)^2 e^{4q i \phi_{[1]}} + 2c^2 \sigma_{[1]}^{2q} e^{2i(1+2q)\phi_{[1]}} \times \\ & \times (-1 + \cos(4q\phi_{[1]}) (\sigma_{[1]} - 1)^2 - \sigma_{[1]}^2 + 2\sigma_{[1]} \cos(2\phi_{[1]})), \end{aligned} \quad (3.52d)$$

and the g_{22} component is obtained by using that $\det \mathbf{g} = \alpha^2$.

As it is, this metric is not very enlightening. However, it gives an opportunity to see the Alekseev's determinant method at work. We will come back to this metric when, in chapter 4, we will consider its limit $q = 1/2$ - Minkowski seed. We will also consider this solution in chapter 5.

The diagonal limit is recovered when $c \rightarrow 0$. It is easy to check that it is in agreement with the result we will give in section 3.4.4.

A 2 soliton solution with a Bianchi II seed has been considered in [99, 100].

Bibliography of Files: [12]

3.4.3 BZ 4-Soliton Solution

Let us now consider the Kasner metric (3.12) as seed and generate the soliton solution with four complex poles. Again, we will use the formulae given in 3.2.1 with q_2 complex conjugate to q_1 and q_4 complex conjugate to q_3 .

We will firstly present a solution obtained by considering a Kasner type D metric – i.e. $q = 0$ – and by letting the arbitrary parameters be:

$$q_4 = \overline{q_3} = -\frac{1}{c}, \quad q_2 = \overline{q_1} = c, \quad c \in \mathbb{R}. \quad (3.53)$$

The resulting metric is given by:

$$g_{11} = \frac{1}{1+c^2} \frac{c^2 \sigma_{[1]}^2 + \sigma_{[3]}^2}{\sigma_{[1]} \sigma_{[3]}}, \quad g_{12} = \alpha \frac{c}{1+c^2} \frac{\sigma_{[3]}^2 - \sigma_{[1]}^2}{\sigma_{[1]} \sigma_{[3]}}, \quad (3.54)$$

and the g_{22} component is obtained by using $\det g = \alpha^2$. The particularly simple form of this metric is due to the choice of the parameters made in (3.53).

We can now consider a Minkowski metric as seed by considering the special case of Kasner with $q = 1/2$. By using the same choice of parameters as in (3.53), we can obtain the solution:

$$g_{11} = (1+c^2) \frac{\sigma_{[1]} \sigma_{[3]}}{\sigma_{[1]}^2 + c^2 \sigma_{[3]}^2}, \quad (3.55a)$$

$$g_{12} = 2c \alpha \sqrt{\sigma_{[1]} \sigma_{[3]}} \frac{\sqrt{\sigma_{[1]}} (\sigma_{[3]} - 1) \cos \phi_{[3]} - \sqrt{\sigma_{[3]}} (\sigma_{[1]} - 1) \cos \phi_{[1]}}{\sigma_{[1]}^2 + c^2 \sigma_{[3]}^2}, \quad (3.55b)$$

and the g_{22} component is as usual obtained by using $\det g = \alpha^2$. Again, the simplicity of this solution is due to the choice (3.53).

In fact, the manner in which the parameters are chosen, crucially affects the structure of the matrices – 4×4 matrices in this case – whose determinants give the functions Γ and Θ_{ab} in (3.15). To give a flavour, we may report the form of two of these matrices for the metric (3.55):

$$\Gamma_{[k,j]} = \begin{pmatrix} \frac{c^2 + \lambda_{[1]}^2}{\lambda_{[1]}^2 - 1} & \frac{c^2 + \lambda_{[1]} \lambda_{[2]}}{\lambda_{[1]} \lambda_{[2]} - 1} & 1 & 1 \\ \frac{c^2 + \lambda_{[1]} \lambda_{[2]}}{\lambda_{[1]} \lambda_{[2]} - 1} & \frac{c^2 + \lambda_{[2]}^2}{\lambda_{[2]}^2 - 1} & 1 & 1 \\ 1 & 1 & \frac{c^{-2} + \lambda_{[3]}^2}{\lambda_{[3]}^2 - 1} & \frac{c^{-2} + \lambda_{[3]} \lambda_{[4]}}{\lambda_{[3]} \lambda_{[4]} - 1} \\ 1 & 1 & \frac{c^{-2} + \lambda_{[3]} \lambda_{[4]}}{\lambda_{[3]} \lambda_{[4]} - 1} & \frac{c^{-2} + \lambda_{[4]}^2}{\lambda_{[4]}^2 - 1} \end{pmatrix}, \quad \Gamma = \det \Gamma_{[k,j]}, \quad (3.56a)$$

$$\Theta_{11[k,j]} = \begin{pmatrix} \frac{1+c^2}{\lambda_{[1]}^2-1} & \frac{1+c^2}{\lambda_{[1]} \lambda_{[2]}-1} & 0 & 0 \\ \frac{1+c^2}{\lambda_{[1]} \lambda_{[2]}-1} & \frac{1+c^2}{\lambda_{[2]}^2-1} & 0 & 0 \\ 0 & 0 & \frac{1+c^{-2}}{\lambda_{[3]}^2-1} & \frac{1+c^{-2}}{\lambda_{[3]} \lambda_{[4]}-1} \\ 0 & 0 & \frac{1+c^{-2}}{\lambda_{[3]} \lambda_{[4]}-1} & \frac{1+c^{-2}}{\lambda_{[4]}^2-1} \end{pmatrix}, \quad \Theta_{11} = \det \Theta_{11[k,j]}, \quad (3.56b)$$

where the $\lambda_{[k]}$ functions are as usual defined as $\mu_{[k]} = \alpha \lambda_{[k]}$. Naturally, the presence of 1's and 0's considerably simplifies the explicit calculations of the determinants.

Bibliography of Files: [13]

3.4.4 BZ N-Soliton Diagonal Solution

Let us consider a generic diagonal metric (3.22) generated by the Ψ_0 given in (3.23). With this seed, the soliton solution with N poles may be generated using the formulae given in (3.25). Naturally, the higher the number of poles, the more cumbersome the solution will be. However, if we are interested in the diagonal limit only, the solution will enormously simplify.

For the case $N = 2$ by letting $\kappa_2[k] = 0$ in (3.25), we have

$$g_{11} = \alpha \frac{\alpha^2}{\mu[1] \mu[2]} \Upsilon_{[0]}, \quad g_{22} = \alpha \frac{\mu[1] \mu[2]}{\alpha^2} \frac{1}{\Upsilon_{[0]}}. \quad (3.57)$$

For the case $N = 4$, by letting again $\kappa_2[k] = 0$ in (3.25), we have:

$$g_{11} = \alpha \frac{\alpha^2}{\mu[1] \mu[2] \mu[3] \mu[4]} \Upsilon_{[0]}, \quad g_{22} = \alpha \frac{\mu[1] \mu[2] \mu[3] \mu[4]}{\alpha^2} \frac{1}{\Upsilon_{[0]}}. \quad (3.58)$$

Due to the linearity of the equations, this situation can obviously be generalized, yielding to the solution with N poles:

$$g_{11} = \alpha \Upsilon_{[0]} \prod_{k=1}^N \frac{\alpha}{\mu[k]}, \quad g_{22} = \alpha^2 \frac{1}{g_{11}}, \quad (3.59)$$

or in the alternative form:

$$g_{11} = \alpha e^V, \quad g_{22} = \alpha e^{-V}, \quad V = \log \Upsilon_{[0]} + \sum_{k=1}^N \log \left(\frac{\alpha}{\mu[k]} \right). \quad (3.60)$$

In the case of Kasner seed, the solution takes the form.

$$g_{11} = \alpha e^V, \quad g_{22} = \alpha e^{-V}, \quad V = -2q \log \alpha + \sum_{k=1}^N \log \frac{\alpha}{\mu[k]}. \quad (3.61)$$

This solutions have been extensively reviewed in [81]. In particular, diagonal solitons over a background of plane waves – given by Siklos (Bianchi VI) metrics – have been considered in [108].

Another application of these diagonal metrics has also been considered in [109]. The property that diagonal solutions can be linearly superposed, has been used to generalize the soliton construction as to include the case when the seed is not a vacuum metric. Various examples, that may be relevant as cosmological models, are discussed.

Bibliography of Files: [8]

3.4.5 BZ Degenerate N-Soliton Solution

In the diagonal case, due to the linearity of the main equations, the soliton solution (3.60) can be generalized as follow [81]:

$$g_{11} = \alpha e^V, \quad g_{22} = \alpha e^{-V}, \quad V = h_0 \log \Upsilon_{[0]} + \sum_{k=1}^N h_k \log \left(\frac{\alpha}{\mu[k]} \right) \quad (3.62)$$

where the h_k 's are real arbitrary constants.

In the case of a Kasner seed, we have [81]:

$$V = - \left(d \log \alpha + \sum_{k=1}^N h_k \log \frac{\mu_k}{\alpha} \right), \quad d = 2qh_0. \quad (3.63)$$

This situations would correspond to soliton solutions where the poles $\mu[k]$ occur h_k times in the S matrix, i.e. they are h_k -degenerate – although the h_k are not necessarily integers. The corresponding f coefficient may be obtained by considering the appropriate limit in the general expression [81]. In particular we have:

$$f = \alpha^{\frac{(d-H)^2-1}{2}} \prod_{i=1}^N \frac{\mu[k]^{h_i} (\mu_i + d - H)}{(\beta_i^2 - \alpha^2)^{h_i/2}} \prod_{i,j=1, i>j}^N (\mu[k] - \mu[j])^{2h_i h_j}, \quad (3.64a)$$

$$H = \sum_{i=1}^N h_i, \quad \beta_i = \beta - w[k]. \quad (3.64b)$$

Eventually, the generating function Ψ , associated with solutions (3.62), can be easily obtained by using formula (3.6). For the Kasner case (3.63), the corresponding $\Upsilon(\alpha, \beta, \lambda)$ is given by.

$$\Upsilon = (\lambda^2 + 2\beta\lambda + \alpha^2)^{H/2-q} \prod_{k=1}^N (\mu[k] - \lambda)^{-h_k}. \quad (3.65)$$

However, this approach cannot be extended to non-diagonal solutions, due the essential nonlinearity in this case.

Bibliography of Files: [9]

3.4.6 BZ (1 + 1)-Soliton Solution

Let us now consider the soliton solution with one pole $\mu[r]$ of degeneracy h and with Kasner seed. The appropriate Ψ_0 will be now given by (3.23) with the choice:

$$\Upsilon = (\lambda^2 + 2\beta\lambda + \alpha^2)^{h/2-q} (\mu[r] - \lambda)^{-h}. \quad (3.66)$$

Let us consider the above as a seed and construct a solution with one more soliton $\mu[s]$. The resulting metric will contain two solitons $\mu[r]$ and $\mu[s]$. By applying the usual formulae, we get the solution:

$$g_{11} = \alpha^{1-2q} \frac{1}{\lambda[r]^h \lambda[s]} \frac{c[s]^2 \lambda[r]^{2h} \lambda[s]^{2h+2} + \alpha^{2h} (\lambda[r] - \lambda[s])^{4h} \lambda[s]^{4q}}{c[s]^2 \lambda[r]^{2h} \lambda[s]^{2h} + \alpha^{2h} (\lambda[r] - \lambda[s])^{4h} \lambda[s]^{4q}}, \quad (3.67a)$$

$$g_{12} = -\alpha^{1+h} c[s] \frac{\lambda[r]^h \lambda[s]^{2q+h-1} (\lambda[r] - \lambda[s])^{2h} (\lambda[s]^2 - 1)}{c[s]^2 \lambda[r]^{2h} \lambda[s]^{2h} + \alpha^{2h} (\lambda[r] - \lambda[s])^{4h} \lambda[s]^{4q}}, \quad (3.67b)$$

$$g_{22} = \alpha^{1+2q} \lambda[r]^h \lambda[s] \frac{c[s]^2 \lambda[r]^{2h} \lambda[s]^{2h-2} + \alpha^{2h} (\lambda[r] - \lambda[s])^{4h} \lambda[s]^{4q}}{c[s]^2 \lambda[r]^{2h} \lambda[s]^{2h} + \alpha^{2h} (\lambda[r] - \lambda[s])^{4h} \lambda[s]^{4q}}, \quad (3.67c)$$

where $\mu[k] = \alpha \lambda[s]$ and the $c[s]$'s are arbitrary constants.

This is an explicit illustration of the “add-one-by-one” procedure to construct soliton solutions.

Bibliography of Files: [10]

Most of the solutions presented above are time-dependent solutions: $\epsilon = +1$. As an example of time-independent solutions, obtained by *directly* applying the BZ technique – and other than the solutions given in [48, 49] – we wish to quote here the paper by A. Tomimatzu [107]. The solutions presented in it may be supposed to describe rotating black holes surrounded by matter.

3.5 A few more remarks on BZ Solitons

3.5.1 On the “Permutation Theorem” of Bäcklund transformations

The Bianchi permutation theorem for a Bäcklund transformations states that the solution obtained by applying two subsequent Backlund transformations to a seed is independent from the order in which these are considered [82].

In the context of solitons, the content of this theorem is schematically illustrated in figure 3 4 (top)

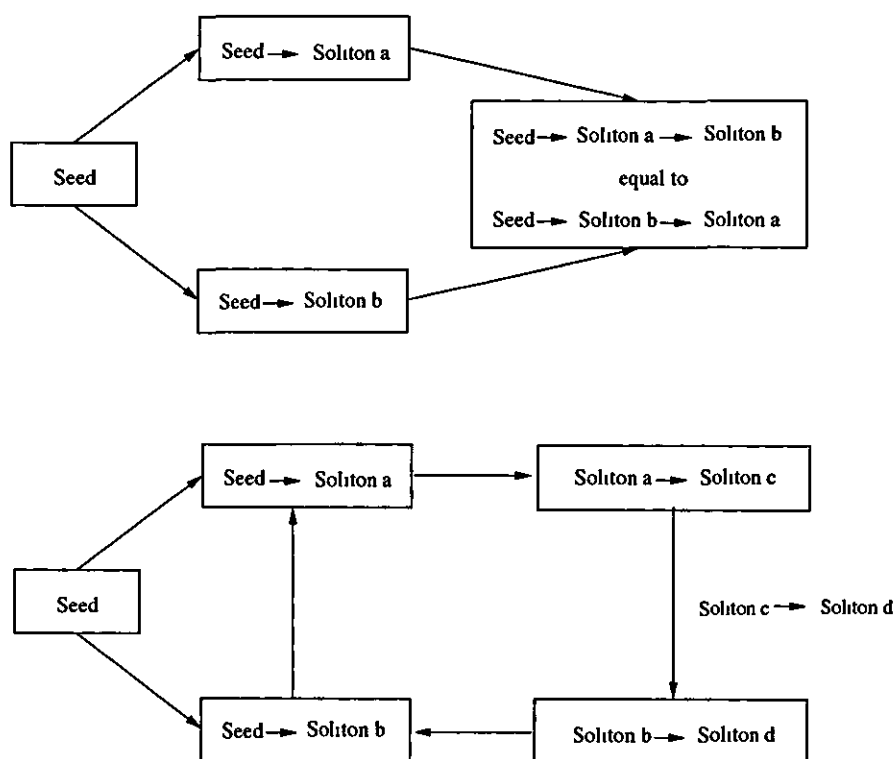


Figure 3 4: Schematic illustration of Bianchi’s “Permutation Theorem” (top) and of the analogue theorem for Backlund transformations of the Ernst equation (bottom).

This theorem applies for Backlund transformations of equations such as KdV or SG. Had this theorem be true for gravitational soliton, we would have expected that the metric (3 67) – with $h = 1$ – be symmetric in $[r]$ and $[s]$, i.e. by interchanging the two solitons. However this is not the case, as can be easily inspected.

This is in agreement with a known result obtained by Neugebauer in [37] – see also Neugebauer in [12] – where the analogue of the permutation theorem is given for Bäcklund transformations of the Ernst equations – see figure (3.4) (bottom). Indeed, Bäcklund transformations may be regarded as one-to-many mappings. Therefore, the relation that connect two solutions – obtained by applying to different transformations to the same seed – is not unique, in general.

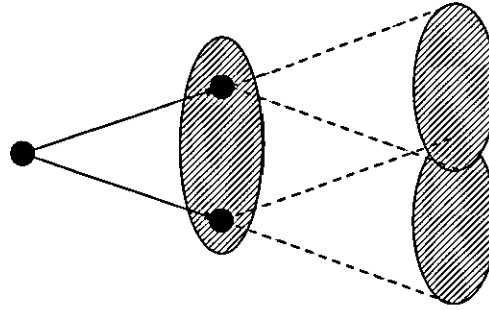


Figure 3.5: Bäcklund transformations are one-to-many mappings.

The real meaning of commutativity for Bäcklund transformations is that the intersection between the two shaded regions in figure (3.5) is different from zero. The situation recalled in (3.4) (top) corresponds to the case when the intersection is given by one element only.

3.5.2 On a possible interpretation of degenerate poles

Let us consider the 2-soliton solution in a Kasner background with real poles. Let us consider the limit when the poles coincide ($z_{0[1]} = z_{0[2]}$, see figure (3.6))

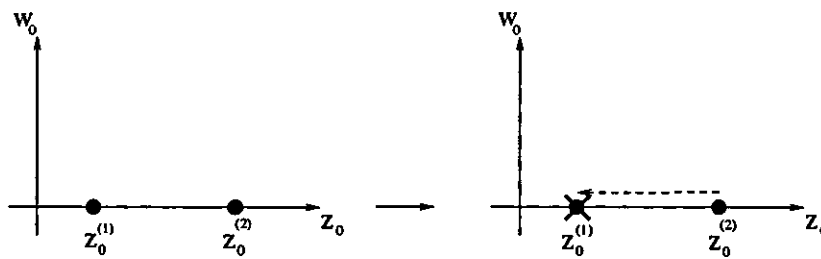


Figure 3.6: Two real poles in the limit when they coincide.

A simple calculation shows that this limit is well defined and it gives

$$g_{11}^{(RP \rightarrow CP)} = \frac{\alpha^{3-2q}}{(-\beta + \sqrt{\beta^2 - \alpha^2})^2}, \quad (3.68a)$$

$$g_{22}^{(RP \rightarrow CP)} = \alpha^{2q-1} (-\beta + \sqrt{\beta^2 - \alpha^2})^2. \quad (3.68b)$$

Let us also consider the 2-soliton solution in a Kasner background with a pair of complex poles, and consider the limit when the poles became real ($w_{0[1]} = 0$, see figure (3.7)).

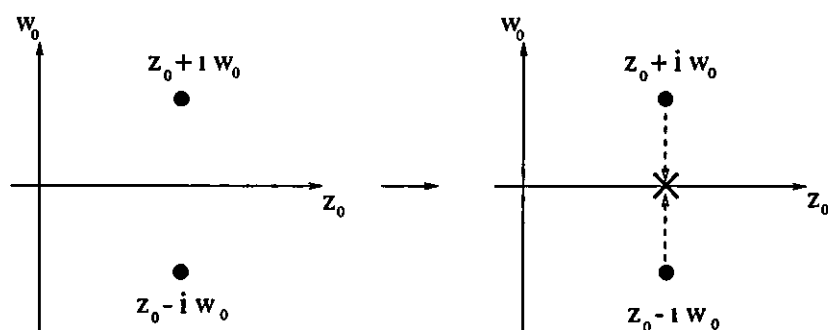


Figure 3.7: Two complex poles in the limit when they became real.

Again, this limit is well defined and gives

$$g_{11}^{(CCR)} = \frac{\alpha^{3-2q}}{(-\beta + \sqrt{\beta^2 - \alpha^2})^2}, \quad (3.69a)$$

$$g_{22}^{(CCR)} = \alpha^{2q-1} (-\beta + \sqrt{\beta^2 - \alpha^2})^2. \quad (3.69b)$$

It may be checked by direct inspection that (3.68) and (3.69) are identical and they are exactly the metric corresponding to 1 pole with degeneracy $h = 2$ given in (3.65). This illustrates the meaning associated with metrics containing degenerate poles. This is confirmed by the analysis of M. Berg and M. Bradley [111].

Bibliography of Files: [11]

Chapter 4

Time-Shift in Gravitational Solitons

As emphasized in chapter 2 the behaviour under interaction is what makes solitons different from solitary waves. The aim of this chapter is to analyze the interaction between gravitational solitons and see whether or not they display the same time-shift (T-S) effect as the solitons of ordinary nonlinear p.d.e.'s.

4.1 The Problem of Time-Shift

4.1.1 T-S in soliton/soliton interaction

The picture below, see figure (4.1), shows the interaction between two KdV solitons.

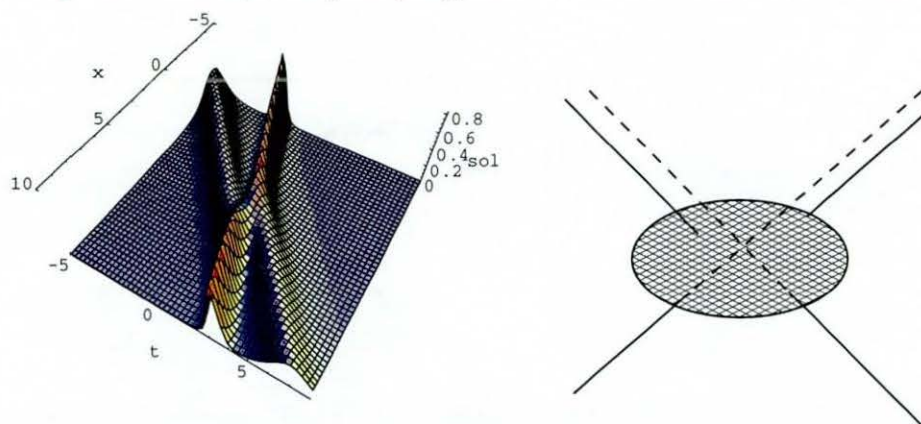


Figure 4.1: *Time-Shift in the 2 soliton solution of KdV equation.*

It describes two solitons travelling in the same direction but with the larger soliton catching up and overtaking the smaller one. The shapes of the two solitons after the interaction are unchanged with respect to their form before the interaction. However, it is quite evident that the lines along which they move apart are shifted with respect to their initial paths. The larger soliton has been shifted forwards, and the smaller soliton backwards. Roughly speaking, this is what we call Time-Shift!

More seriously, let us consider the two soliton solution given in [83], formulae

(1.4-13):

$$u(x, t) = \frac{\partial^2}{\partial x^2} \log f(x, t), \quad f(x, t) = 1 + e^{\theta_1} + e^{\theta_2} + A e^{\theta_1 + \theta_2}, \quad (4.1a)$$

$$\theta_i = a_i x - a_i^3 t + \delta_i, \quad A = \left(\frac{a_1 - a_2}{a_1 + a_2} \right)^2, \quad (4.1b)$$

where a_i and δ_i are real arbitrary constants. Figure (4.1) is nothing but a 3-D plot of this solution with parameters $a_1 = 1.2$, $\delta_1 = 0$, $a_2 = 1.8$, $\delta_2 = 5$.

By analyzing the asymptotic behaviour ($t \approx \pm\infty$) of this solution, it can be seen that – see [83], formulae (1.4-16):

$$u_{\pm}(x, t) \approx \sum_{i=1}^2 \frac{1}{4} a_i^2 \operatorname{sech}^2 \frac{1}{2}(\theta_i + \Delta_i^{\pm}), \quad \text{as } t \rightarrow \pm \infty \quad (4.2a)$$

$$\Delta_1^+ = \log A, \quad \Delta_1^- = 0, \quad \Delta_2^+ = 0, \quad \Delta_2^- = \log A, \quad (4.2b)$$

$$\Delta_1^+ - \Delta_2^+ = +\log A > 0, \quad \Delta_1^- - \Delta_2^- = -\log A < 0, \quad (4.2c)$$

where each term of the sum in (4.2a) is the expression for the 1-soliton solution. Thus, asymptotically, the 2-soliton solution is given by a “superposition” of two 1-soliton solutions. It may be remarked that the phases Δ_i^+ of the solitons at $t = +\infty$ are different from the phases Δ_i^- at $t = -\infty$. Consequently, the solitons after the interaction travel along directions which are not the same as before but just parallel: this is the T-S effect we are considering here. The above effect can be found also in an N -soliton solution. The relevant formulae have been given in [83] – formulae (4.3-18) and neighbouring.

4.1.2 T-S in soliton/non-soliton interaction

Further examples of T-S can be found by analyzing the interaction between solitons and waves. The first to derive a solution describing the non-linear superposition of a soliton and a wave was Wahlquist for the KdV equation [112]. He considered a Bäcklund transformation to a background of cnoidal waves. Figure (4.2) shows a typical case, when the height of the soliton is comparable with that of the background waves.

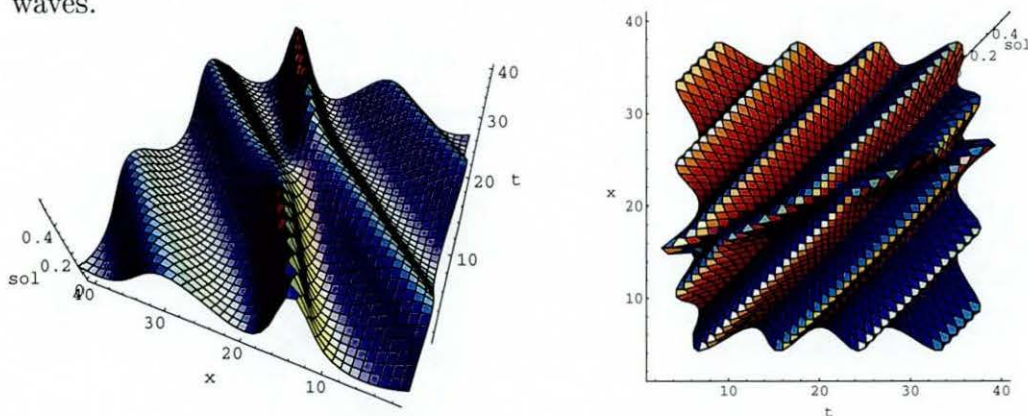


Figure 4.2: Examples of interaction of solitons and waves for the KdV equation. This picture shows the interaction of a small soliton with the background waves.

A phase-shift of the soliton and each wave is clearly visible. In figure (4.3) the situation is considered when the soliton is much bigger than the waves. Notice also the similarity between these pictures and that given in 4.1.

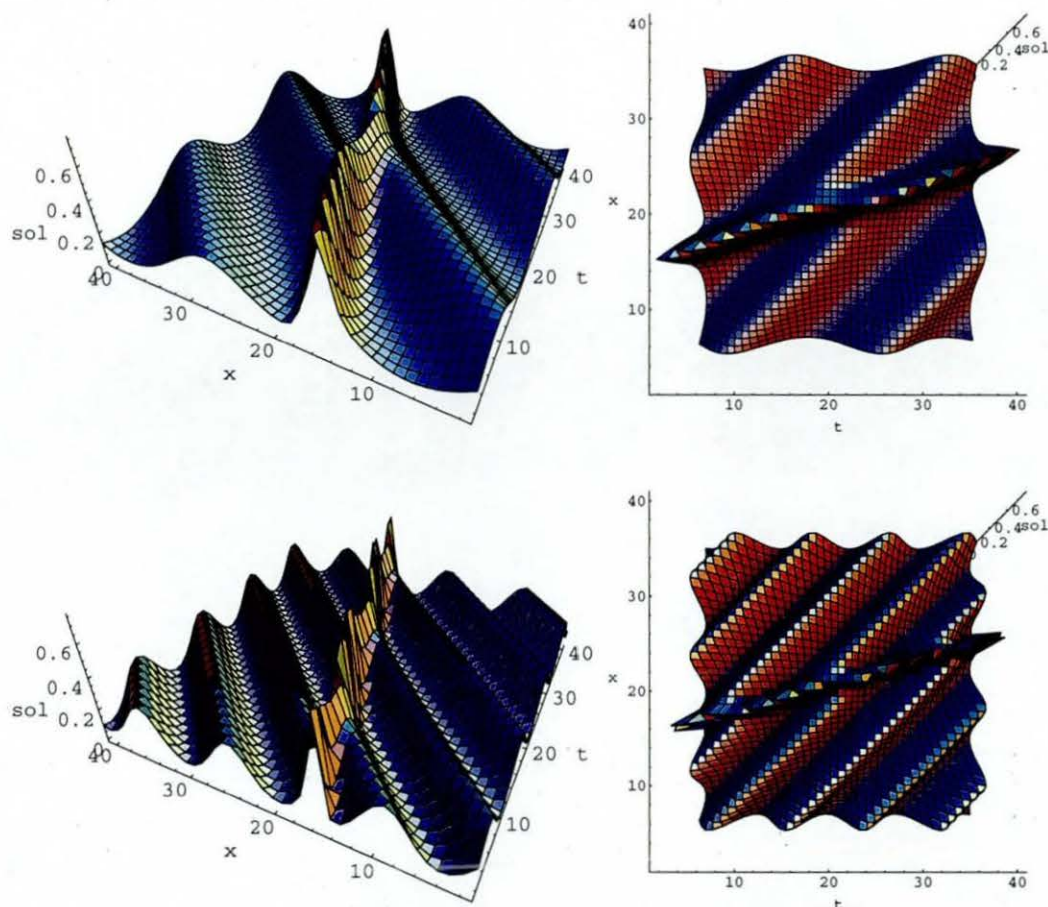


Figure 4.3: Examples of interaction of solitons and waves for the KdV equation. This picture shows the interaction of a rather large soliton with the background waves. Different ranges and viewpoints are shown for the same set of parameters.

The actual procedure given in [112] turns out to be rather involved. It has been considerably simplified by Hoenselaers and the present author, for the Sine-Gordon (SG) equation:

$$\phi_{xx} - \phi_{tt} = \sin \phi . \quad (4.3)$$

In [113, 114] a Bäcklund transformation to a background of steady progressing waves – i.e. functions of $x - vt$, $v \leq c$ – given in terms of Jacobi elliptic functions, has been considered. By using the galilean invariance enjoyed by SG, the seed can be chosen to be steady: this is tantamount to choosing a frame in which the waves are at rest. That considerably simplifies the Riccati equations associated to the Bäcklund transformation. The final solution is again given in terms of Jacobi elliptic functions. In figure (4.4) a few typical solutions are displayed.

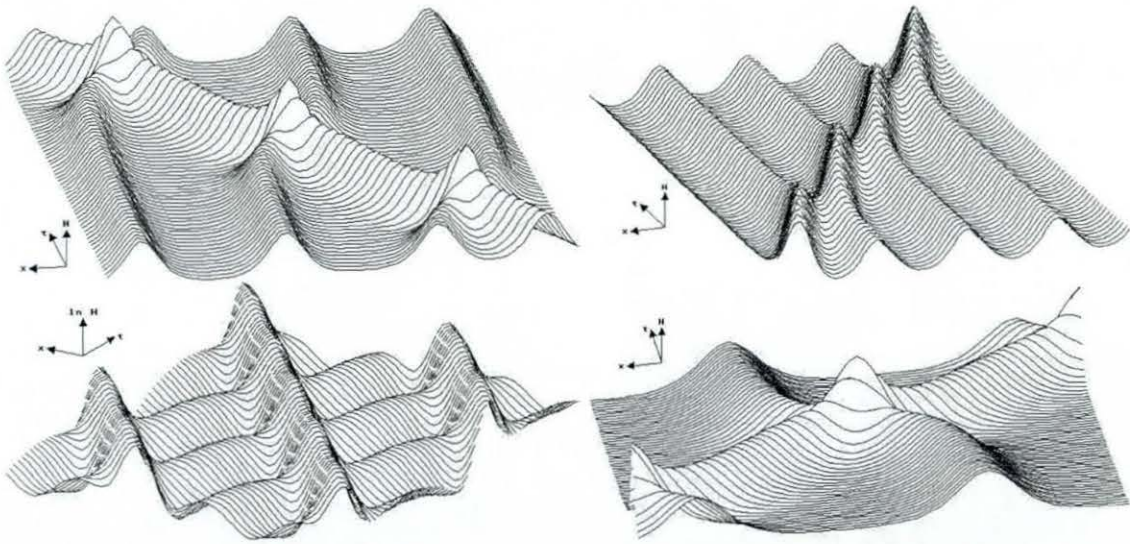


Figure 4.4: *Examples of interaction of solitons and waves for the SG equation. These solutions are obtained by way of a Bäcklund transformation.*

These pictures clearly show how both the soliton and the background waves are shifted when interacting. In passing, notice how the fourth picture – from the top left corner and moving clockwise – shows multiple solitons. This is a particular case in which some of the functions entering the solution become trigonometric.

Examples of T-S can also be obtained by generating solutions that apparently have no relation with solitons. In fact, in [113, 114] solutions of SG are also generated by way of a separation Ansatz: $\phi(x, t) = \alpha(x)/\beta(t)$. In figure (4.5) a few solutions are displayed.

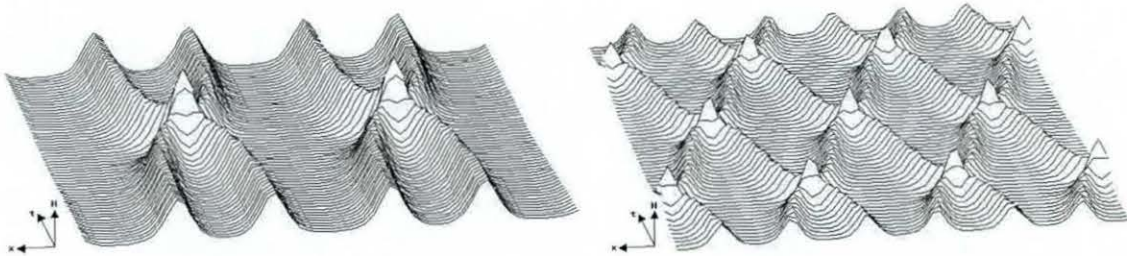


Figure 4.5: *Examples of interaction of solitons and waves for the SG equation. These solutions are obtained by way of a Separation Ansatz.*

The T-S effect is clearly visible. Analogous results for the non-linear Schrödinger equations have also been achieved although they have not yet been published.

4.1.3 T-S in Gravitational Solitons

Boyd–Centrella–Klasky Results

In [115] a 4-soliton diagonal solution with Kasner seed is considered. The solution is given in cosmological coordinates (1.28). Any evidence for T-S is looked for when the two inner solitons collide. The result is negative: the two solitons pass through each other without any modification at all, see figure (4.6).

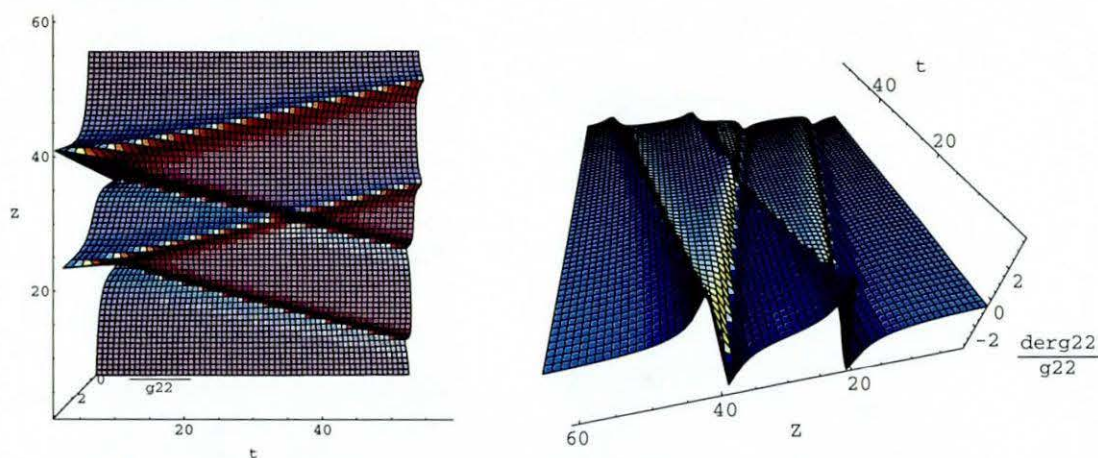


Figure 4.6: No Time Shift in the diagonal 4 soliton solution studied in [115]. The picture on the left is seen from above the (t, Z) plane. The Z -derivative of $\log(g_{11}/g_{011})$ is plotted.

It must be emphasized that the metric under consideration is a diagonal one. In particular it is nothing but the diagonal limit of (3.55) or, alternatively, the case $N = 4$, of solution (3.61). Since these are solutions to a linear equation – equation (1.44) – this negative result is not surprising: when interacting, the two solitons linearly superpose. This argument also apply to diagonal soliton solutions with any number of poles.

Dagotto–Gleiser–Nicasio Results

In [116] the 2-soliton solution obtained by using the Alekseev technique is considered. It is a cylindrically symmetric non diagonal solution. Therefore the field equations are truly non-linear. In this case, the soliton is considered to interact with itself as it passes through – or is reflected from – the axis of symmetry.

As a first step, the authors solve (asymptotically) the geodesic equation associated with the soliton metric. In particular, they write the Christoffel symbols by substituting for the metric coefficients their limit when $u, v \rightarrow \infty$, where u and v are the relevant null coordinates. Consequently, they are able to find an explicit expression for the velocity of a “geodesic” particle (i.e. a test particle like a photon) – see formulae (18) to (23).

Their second step is to study the Weyl tensor coefficients and the electromagnetic energy at null infinity – formulae (25) to (27) and (41) to (43). This simplification enables them in giving an explicit expression for the maxima of these physical quantities – formulae (47) and (48).

Finally, they consider two particles: the first travelling from $-\infty$ towards the axis – ingoing particle – and the second travelling from the axis to $+\infty$ – outgoing particle – both travelling unperturbed and along the maxima of the solitons as previously found. These two particles meet the axis $\rho = 0$ at different times t_1 and t_2 . A T-S is claimed to be seen, as the proper time $\Delta\tau$ measured between t_1 and t_2 by an observer at rest on the axis.

In a second paper [117], the same authors apply the previous technique when the metric is diagonal: i.e. it is an Einstein-Rosen one. Again a T-S is found. Clearly,

this last result seems to contradict that of [115].

In [118], the same analysis is applied to the case of multisoliton metrics.

Céspedes-Verdaguer Results

In [119] a 4-Soliton non-diagonal Solution is analyzed. It has been generated by considering the Wainwright-Marshmann metric (3.26). The function W is taken to be:

$$W = \begin{cases} H & v < v_F \\ H - A \left(1 - \cos \left(2\pi \frac{v-v_F}{v_B-v_F} \right) \right) & v_F < v < v_B \\ H & v > v_B \end{cases} \quad (4.4)$$

This represents a pulse wave propagating into a Kasner background. Consequently, the solution considered in [119] is supposed to describe the interaction between the solitons and a pulse wave.

Among the results they obtain, the one relevant to us is a small T-S – that seems observable in picture (3.c) in [119] – occurring when one of the solitons interacts with the pulse. This is analytically proved in that the asymptotic behaviour of the two pairs of solitons is not symmetric – formulae (19) in [119].

It is worth noticing here that the physical content in this metric is different from that in [115, 116, 117]. In the last situations, solitons freely propagate into a Minkowski or Kasner background, and eventually interact only amongst themselves. On the contrary, the one presented in [119] is an example for a soliton/non-soliton interaction.

Bibliography of Files: [14]

4.2 The case of Weber–Wheeler–Bonnor Solution

In [135] Griffiths and the present author pointed out that the effect found in [116, 117] can also be observed in the Weber–Wheeler–Bonnor (WWB) solution [120, 121]

4.2.1 The WWB Solution

Let us consider the Einstein–Rosen line element already considered in section 1.3:

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - (\rho^2 e^{-2\psi} d\phi^2 + e^{+2\psi} dz^2), \quad (4.5)$$

whose field equations were given in (1.38). We just recall that the equation for ψ is linear. As known, its solution is a (linear) superposition of Bessel functions J_0 and Y_0 . The WWB solution is a particular case of these solutions, given by a linear superposition of monochromatic waves with a cut-off in the frequency space.

$$\psi = 2c \int_0^\infty dk e^{-ak} \cos(kt) J_0(k\rho), \quad (4.6)$$

which gives explicitly.

$$\psi = \sqrt{2}c \left(\frac{[(a^2 + \rho^2 - t^2)^2 + 4a^2 t^2]^{1/2} + a^2 + \rho^2 - t^2}{(a^2 + \rho^2 - t^2)^2 + 4a^2 t^2} \right)^{1/2}, \quad (4.7a)$$

$$\gamma = \frac{c^2}{2a^2} \left(1 - \frac{2a^2 \rho^2 [(a^2 + \rho^2 - t^2)^2 - 4a^2 t^2]}{[(a^2 + \rho^2 - t^2)^2 + 4a^2 t^2]^2} + \frac{\rho^2 - a^2 - t^2}{[(a^2 + \rho^2 - t^2)^2 + 4a^2 t^2]^{1/2}} \right). \quad (4.7b)$$

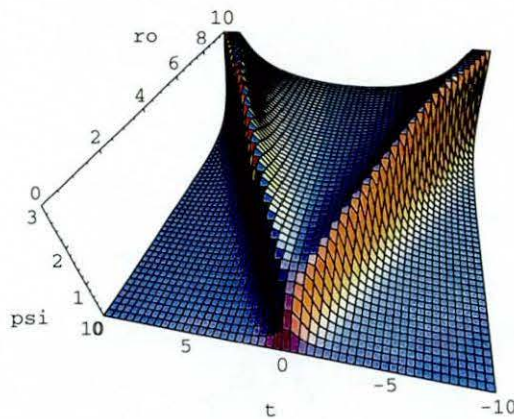


Figure 4.7: This picture illustrates the character of the WWB pulse, particularly the fact that the wave appears to be reflected before it meets the axis.

This solution describes an incoming and outgoing wave pulse. Its maxima are asymptotically given by:

$$t = +\rho - \frac{a}{\sqrt{3}} \quad \text{outgoing wave}, \quad t = -\rho + \frac{a}{\sqrt{3}} \quad \text{ingoing wave}, \quad (4.8)$$

which seems to show that the incoming wave is reflected at $\rho = \frac{a}{\sqrt{3}}$ – and not at $\rho = 0$. This is the same effect that has been interpreted as a solitonic T-S in [116, 117]. But, as the previous argument shows, this effect is also shared by non-solitonic solutions of the standard wave equations in cylindrical polar coordinates. Thus we can argue that this apparent shift must have another physical explanation.

4.2.2 The WWB Solution in Cosmological Coordinates

The above solution may be reinterpreted in Cosmological Coordinates simply by making the substitutions $\rho_{cyl} \rightarrow t_{cosm}$, $t_{cyl} \rightarrow Z_{cosm}$ in (4.7) and appropriately extending the coordinates domain to the full range $(-\infty, +\infty)$ as we pointed out in chapter 1:

$$ds^2 = e^{2(\tilde{\gamma}-\tilde{\psi})}(dt^2 - dZ^2) - (t^2 e^{-2\tilde{\psi}} dx^2 + e^{+2\tilde{\psi}} dy^2), \quad (4.9)$$

$$\tilde{\psi} = \sqrt{2}c \left(\frac{[(a^2 + t^2 - Z^2)^2 + 4a^2 Z^2]^{1/2} + a^2 + t^2 - Z^2}{(a^2 + t^2 - Z^2)^2 + 4a^2 Z^2} \right)^{1/2}. \quad (4.10)$$

Once again we can work out the asymptotic expressions for the maxima of $\tilde{\psi}$:

$$Z = +t - \frac{a}{\sqrt{3}} \quad \text{outgoing wave}, \quad Z = -t + \frac{a}{\sqrt{3}} \quad \text{ingoing wave}, \quad (4.11)$$

which, as expected, shows a shift: in this case a delay.

The considerations we made in the cylindrical case can be repeated here: this is a non-solitonic solution of a linear equation, thus this shift can not be interpreted as a solitonic effect. Moreover, in this case we have a delay and not an advance as typical for “classical” solitons.

One more remark: in the limit when c is small, WWB may be considered as a weak wave propagating on a Minkowski background (written in Kasner form):

$$ds^2 = dt^2 - dZ^2 - t^2 dx^2 - dy^2 .$$

By performing the following coordinate transformation:

$$t = \pm \sqrt{T^2 - X^2}, \quad Z = \mathcal{Z}, \quad x = \frac{1}{2} \log \frac{T - X}{T + X}, \quad y = Y, \quad (4.12)$$

the Minkowski metric becomes the usual one:

$$ds^2 = dT^2 - d\mathcal{Z}^2 - dX^2 - dY^2 .$$

Notice that (4.12) is defined only in the region $T^2 \geq X^2$, equivalent to $t \geq 0$.

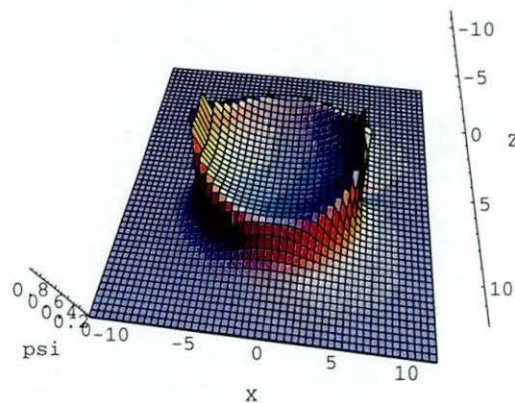


Figure 4.8: WWB pulse in cartesian coordinates T, X, Y, \mathcal{Z} . In this picture $T = 6a$.

Going back to (4.10), we can observe that the wave surfaces $t \pm Z = k$ are now mapped into null cones:

$$T^2 - (\mathcal{Z} - k)^2 - X^2 = 0 . \quad (4.13)$$

These represent cylindrical waves parallel to the Y axis, centered in $(X = 0, \mathcal{Z} = k)$, however the waveform itself is not cylindrically symmetric – see figure (4.8).

If we would extend the solution (4.10) to the region $T^2 < X^2$ (using the natural extension of the T, X coordinates), curvature singularities might appear when $\mathcal{Z} = 0$ and $T^2 = X^2 - a^2$. These singularities, however, can not be considered as the “physical” source of the shift found in the “good” region $T^2 \geq X^2$. Once again, the ones we are dealing with are solutions of linear equations!

Bibliography of Files: [15]

4.3 Time-Shift in cosmological context

The discussion put forward so far seems to indicate that the debate about T-S in gravitational solitons is still controversial – see [81], section 3.6.b. In this section we will try to give meaning to the pictures below – figure (4.9) – as the gravitational counterpart of the analogue picture in figure (4.1). We will here consider the time-dependent case only.

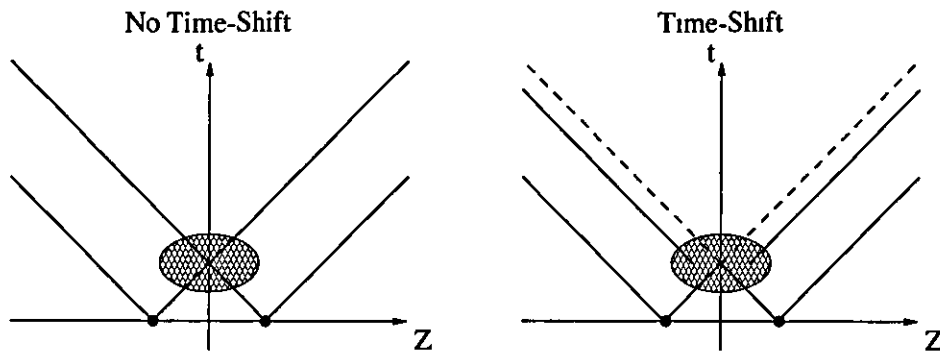


Figure 4.9: Schematic illustration of T-S for cosmological solitons. The “interaction region” is shaded, and there is an initial cosmological singularity at $t = 0$.

In this context, the problem of T-S can be posed in the following manner. Given two metrics, one diagonal (for which EFE’s are linear) and one nondiagonal (for which EFE’s are truly nonlinear), and both describing the interaction of two solitons, do they show any difference?

As it is, this statement is still vague. It has the merit to indicate that some information on a possible T-S must be looked for by comparing the diagonal and non-diagonal case. However a few more comments are due:

- An ideal situation would be that in which one may consider some physical quantity and study this both in the non-diagonal and diagonal case. We have in mind the energy associated with the metric or the Weyl Tensor, better the explicit components Ψ_0 , Ψ_2 and Ψ_4 . However, the first possibility is ruled out in this context, since no acceptable definition of energy can be given. The second possibility is theoretically welcome, but the explicit calculations are far too lengthy and we have not been able to give a reasonably compact form for the above Weyl tensor components.
- This situation can be overcome if the metric components themselves are considered. Naturally, the explicit form of the metric components is simpler than those of the Weyl tensor. In addition g_{11} , g_{22} and g_{12} have the important property of being respectively proportional to $|\xi_1|^2$, $|\xi_2|^2$ and $\xi_1 \cdot \xi_2$, where ξ_1 and ξ_2 are the Killing vectors given in chapter 1. However, it is only their derivatives which describe the gravitational field, and even then not in an invariant way.
- In figure (4.9) a schematic 4-soliton solution is pictured. In this case we have two pairs of solitons generated at different space-like points at the cosmological singularity $t = 0$. As they start to propagate along the background, the two external solitons travel apart unperturbed. Moreover, the two inner solitons will interact at some instant $t = t_0 > 0$ [133, 134].

The fact that the interaction occurs away from the cosmological singularity prevents any possible mixing of effects due to mere interaction with others due to the cosmological singularity itself (or the axis of symmetry in the cylindrically symmetric context). This possibility is obviously ruled out if a simpler two soliton solution is considered. In fact, in this case the two solitons were

generated at the same point (at $t = 0$) and would freely propagate along the background without ever interacting with each other. The 4-Soliton solution is then the simplest case in which the soliton/soliton interaction can be considered.

The possibility of having any T-S or not, will be dictated by the asymptotic $t \rightarrow \infty$ – behaviour of the metrics under consideration. In fact, if this is the same as the diagonal metric, then obviously no T-S would appear – as in the metric discussed in [115] – and that would correspond to the case shown in the left picture of figure (4.9)

To conclude, we just observe that the physical interpretation of the solutions we are going to study is that considered in [115], rather than that given in [116, 117]. This last case, in which the metrics are cylindrically symmetric, has been also interpreted as describing the interaction between (solitonic) gravitational waves and cosmic strings located on the axis of symmetry [122, 123, 124], [125, 126, 127, 128] and [129, 130, 131].

4.4 Interaction between gravitational solitons

In this section we will investigate the T-S in solitons generated from particular diagonal seeds, namely Minkowski and Kasner. In the case of Minkowski seed, we will therefore consider the interaction of solitons travelling on a “flat” background. As suggested in section 4.2.2, by performing an appropriate coordinate transformation, these spacetimes may also be interpreted as cylindrical waves. In the case of Kasner seed, it may be noticed that the seed, for arbitrary parameters, is not even asymptotically flat, and there exists an initial cosmological singularity.

The results described in this section should provide the gravitational counterpart of the cases dealt with in section 4.1.1. This problem has also been analyzed in [133, 134].

4.4.1 Solutions with Minkowski Seed

Let us consider a 4-soliton non-diagonal solution with complex poles propagating in a Minkowski background. The most general 4-soliton solution (3.15) would contain 2 arbitrary complex parameters q_1, q_3 and four other real parameters $z_0[1], z_0[3], w_0[1], w_0[3]$ that enter the solution through the expressions for the pole-trajectories.

Since the parameters $z_0[i]$ always are in the combination $z_0[i] - Z$ we can put, without loss of generality, $z_0[3] = -z_0[1]$. Moreover, to obtain an explicit solution we make the assumptions: $\kappa_2[1] \in \mathbb{R}, \kappa_2[3] \in \mathbb{R}$. In terms of the constants q_k defined in (3.16), we have $q_1, q_3 \in \mathbb{R}$. Furthermore we will set $q_3 = -1/q_1, q_1 = c$.

With these, the solution is given by (3.55), with the obvious replacements $\alpha \rightarrow t, \beta \rightarrow Z$.

$$g_{11} = (1 + c^2) \frac{\sigma[1] \sigma[3]}{\sigma[1]^2 + c^2 \sigma[3]^2}, \quad (4.14a)$$

$$g_{12} = 2c t \sqrt{\sigma[1] \sigma[3]} \frac{\sqrt{\sigma[1]} (\sigma[3] - 1) \cos \phi[3] - \sqrt{\sigma[3]} (\sigma[1] - 1) \cos \phi[1]}{\sigma[1]^2 + c^2 \sigma[3]^2}, \quad (4.14b)$$

where $\sigma_{[k]}$ are given by (3.40).

As we outlined in section 4.1.1 the T-S effect in the KdV equation can be shown by analyzing the asymptotic behaviour ($t \rightarrow \infty$) of the solution. This analysis is actually performed by looking at the maxima of the solitons in the relevant asymptotic regions. Hereafter we will apply the same procedure to the gravitational solitons [136]. As an example, we will firstly consider the 2-soliton case.

A preliminary step: the 2-Soliton non-diagonal Solution

Let us consider a 2-soliton non-diagonal solution. Now we only have one complex parameter $q_{[1]}$. With the simple choice $q_{[1]} = m \in \mathbb{R}$ we get the solution shown in (3.52). With $q = 1/2$, and by considering $\sigma_{[i]} = \sigma_{+[i]}$ this simplifies to:

$$g_{11}^{(2-sol)} = -(1 + m^2) \frac{\sigma_{+[1]}}{m^2 + \sigma_{+[1]}^2}, \quad g_{12}^{(2-sol)} = -4mZ \frac{\sigma_{+[1]} - 1}{\sigma_{+[1]} + 1} \frac{\sigma_{+[1]}}{m^2 + \sigma_{+[1]}^2}. \quad (4.15)$$

By looking at its first Z -derivative, we can see that the maxima of $g_{11}^{(2-sol)}$ can be given either by the same maxima of the diagonal solution or by the points where $\sigma_{+[1]} = \pm m$. In order to see if these new maxima actually occur, let us preliminarily consider the function $\sigma_{+[k]}$. We recall here the properties we described in chapter 3

- $\sigma_{+[k]}$ decays to 1 as $t \rightarrow \infty$, according to:

$$\sigma_{+[k]} \approx 1 + 2 \frac{w_0[k]}{t} \quad \text{as } t \rightarrow \infty.$$

- The function $\sigma_{+[k]}$ depends only upon $z[k]^2$. That means it is symmetric under reflections along the axis defined by $Z = z_0[k]$.
- $\sigma_{+[k]}$ is a generally decreasing function. With this expression we mean that for a fixed $Z = \bar{Z}$, the resulting function $\tilde{\sigma}[k] = \sigma_{+[k]}|_{Z=\bar{Z}}$ is a decreasing function.

From these considerations it can be pointed out that g_{11} has the new maxima only if $m^2 > 1$. Moreover, we can also argue that these new maxima do not exist in the *Causal Region* $|z_0[k] - Z| < t \rightarrow \infty$. In that, the maxima of the nondiagonal solution coincide with those of the diagonal solution.

In order to study the g_{12} component, it is useful to consider the following function:

$$\tilde{g}_{12} = \log \frac{g_{12}^{(2-sol)}}{Z}. \quad (4.16)$$

The \tilde{g}_{12} Z -derivative is given by:

$$\tilde{g}'_{12} = g_{11}'^{(2-sol-d)} \left(\frac{2\sigma_{+[1]}^2}{\sigma_{+[1]}^2 - 1} + \frac{m^2 - \sigma_{+[1]}^2}{m^2 + \sigma_{+[1]}^2} \right), \quad (4.17)$$

where a prime indicates a Z -derivative and $g_{11}^{(2-sol-d)}$ is the (11) component of the corresponding diagonal limit. Some simple algebra shows that, being $\sigma_{+[1]} > 1$, the expression in round brackets never vanishes, so that $\tilde{g}'_{12} = 0$ only if $g_{11}'^{(2-sol-d)} = 0$. That means \tilde{g}_{12} and the diagonal component $g_{11}^{(2-sol-d)}$ have the same stationary points. Of course, the same is true for $g_{12}^{(2-sol)}$.

Since both g_{11} and g_{12} asymptotically have the same maxima of a diagonal metric, no T-S can be expected

Two possible 4-Soliton Solutions

The freedom in choosing the \pm in (3.40) allows us to have 4 solutions: $\sigma_{[1]} = \sigma_{+[1]}$, $\sigma_{[3]} = \sigma_{+[3]}$ or $\sigma_{[1]} = \sigma_{+[1]}$, $\sigma_{[3]} = \sigma_{-[3]}$ or $\sigma_{[1]} = \sigma_{-[1]}$, $\sigma_{[3]} = \sigma_{+[3]}$ or $\sigma_{[1]} = \sigma_{-[1]}$, $\sigma_{[3]} = \sigma_{-[3]}$. Of course only the first two cases are of interest.

That being so, we can rewrite (4.14) in the two alternative forms:

$$g_{11}^{(++)} = (1 + c^2) \frac{\frac{\sigma_{+[1]}}{\sigma_{+[3]}}}{c^2 + \frac{\sigma_{+[1]}^2}{\sigma_{+[3]}^2}}, \quad g_{12}^{(++)} = g_{11}(t, Z) F_{++}(t, Z), \quad (4.18)$$

and

$$g_{11}^{(+-)} = (1 + c^2) \frac{\frac{\sigma_{+[1]} \sigma_{+[3]}}{c^2 + \sigma_{+[1]}^2 \sigma_{+[3]}^2}}, \quad g_{12}^{(+-)} = g_{11}^{(+-)}(t, Z) F_{+-}(t, Z), \quad (4.19)$$

where we introduced the two functions:

$$F_{++}(t, Z) = -\frac{4c}{1 + c^2} \left(z_{[1]} \frac{\sigma_{+[1]} - 1}{\sigma_{+[1]} + 1} - z_{[3]} \frac{\sigma_{+[3]} - 1}{\sigma_{+[3]} + 1} \right), \quad (4.20)$$

$$F_{+-}(t, Z) = -\frac{4c}{1 + c^2} \left(z_{[1]} \frac{\sigma_{+[1]} - 1}{\sigma_{+[1]} + 1} + z_{[3]} \frac{\sigma_{+[3]} - 1}{\sigma_{+[3]} + 1} \right). \quad (4.21)$$

Time-Shift in the 11 components

By introducing the quantities:

$$\gamma_{++} = \frac{\sigma_{+[1]}}{\sigma_{+[3]}}, \quad \gamma_{+-} = \sigma_{+[1]} \sigma_{+[3]}, \quad (4.22)$$

it can be shown that the first Z -derivatives of the metric components can be expressed as:

$$g_{11}'^{(++)} = g_{11}^{(++)} \frac{c^2 - \gamma_{++}^2}{c^2 + \gamma_{++}^2} \frac{\gamma_{++}'}{\gamma_{++}}, \quad g_{11}'^{(+-)} = g_{11}^{(+-)} \frac{c^2 - \gamma_{+-}^2}{c^2 + \gamma_{+-}^2} \frac{\gamma_{+-}'}{\gamma_{+-}}. \quad (4.23)$$

Equations (4.23) show that the maxima of the non diagonal metric components $g_{11}^{(++)}$ and $g_{11}^{(+-)}$ can be given by:

1. the same maxima of the diagonal solution;
2. the points where $\gamma_{++} = \pm c$ or $\gamma_{+-} = \pm c$.

Therefore, as a general feature, the non diagonal metrics may present the existence of new stationary points in addition to those of the diagonal solutions.

Solution (4.19) seems to have a behaviour quite similar to the one encountered for the 2-soliton non-diagonal solution. By just using the same arguments – now applied to the quantity γ_{+-} – we can deduce that actually these new maxima exist only if we choose $c > 1$ and, anyway, can not exist in the *Causal Region*. Then the picture is as follows. new maxima may appear when considering a non-diagonal solution, however, for late times they will merge with those of the diagonal metric. Thus the asymptotic behaviour of the solution is not modified with respect to the diagonal case – see figure (4.10).

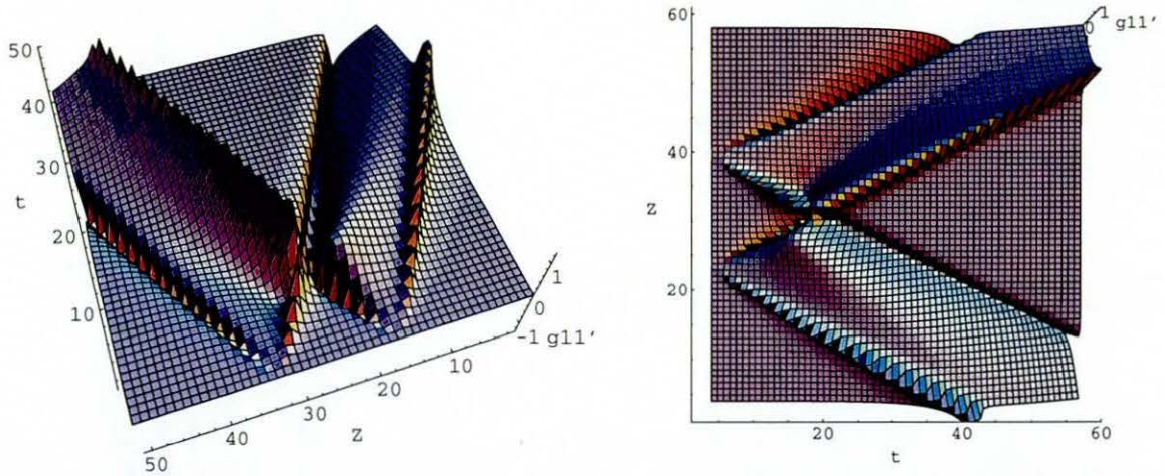


Figure 4.10: The pictures show the $g'_{11}^{(+ -)}$ component of the 4-Soliton Non-Diagonal Solution, from two different viewpoints.

Solution (4.18) seems to have a richer structure: the considerations made in the 2-Soliton case cannot apply here and in fact $\gamma_{++} = \pm c$ can be satisfied also when $c < 1$. Nevertheless we still have that $\gamma_{++} \rightarrow 1$ as $t \rightarrow \infty$, which implies that the previous equation can not be satisfied in the Causal Region. Once again, the asymptotic behaviour of the solution is similar to that of the diagonal one – see figure (4.11).

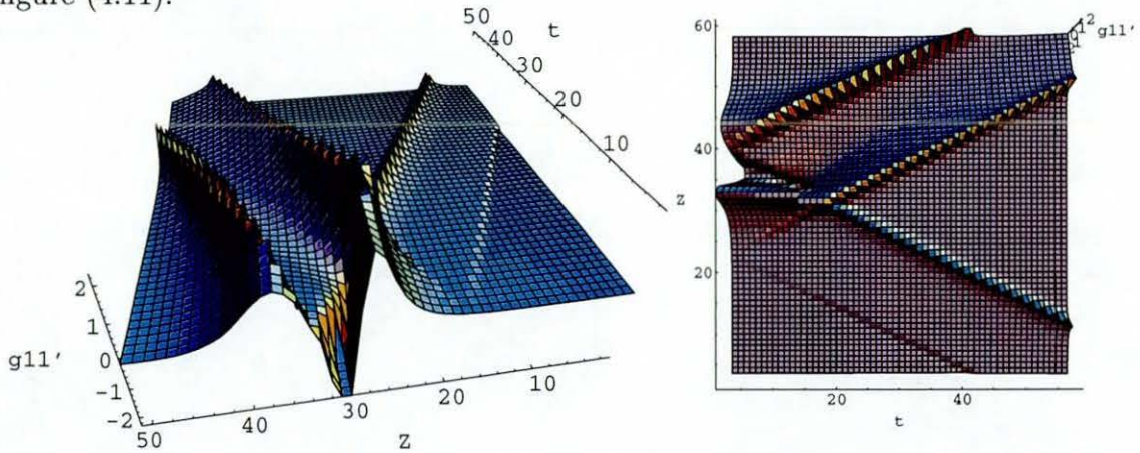


Figure 4.11: The pictures show the $g'_{11}^{(+ +)}$ component of the 4-Soliton Non-Diagonal Solution, from two different viewpoints.

Time-Shift in the 12 components

The two functions F_{++} and F_{+-} can be more properly rewritten as:

$$F_{++}(t, Z) = \frac{g_{12}^{(2-sol)}[1]}{g_{11}^{(2-sol)}[1]} - \frac{g_{12}^{(2-sol)}[3]}{g_{11}^{(2-sol)}[3]}, \quad F_{+-}(t, Z) = \frac{g_{12}^{(2-sol)}[1]}{g_{11}^{(2-sol)}[1]} + \frac{g_{12}^{(2-sol)}[3]}{g_{11}^{(2-sol)}[3]}, \quad (4.24)$$

where $g^{(2-s)}$ is the 2-Soliton non-diagonal Solution presented in section 4.4.1. Having that, $g_{12}^{(+ +)}$ and $g_{12}^{(+ -)}$ become:

$$g_{12}^{(+ +)} = g_{11}^{(+ +)} \left(\frac{g_{12}^{(2-sol)}[1]}{g_{11}^{(2-sol)}[1]} - \frac{g_{12}^{(2-sol)}[3]}{g_{11}^{(2-sol)}[3]} \right), \quad g_{12}^{(+ -)} = g_{11}^{(+ -)} \left(\frac{g_{12}^{(2-sol)}[1]}{g_{11}^{(2-sol)}[1]} + \frac{g_{12}^{(2-sol)}[3]}{g_{11}^{(2-sol)}[3]} \right). \quad (4.25)$$

Notice here how each function into the round brackets depends upon the variables of only one couple of solitons, which shows that neither $g_{12}^{(++)}$ nor $g_{12}^{(+-)}$ show any T-S – see figure (4.12).

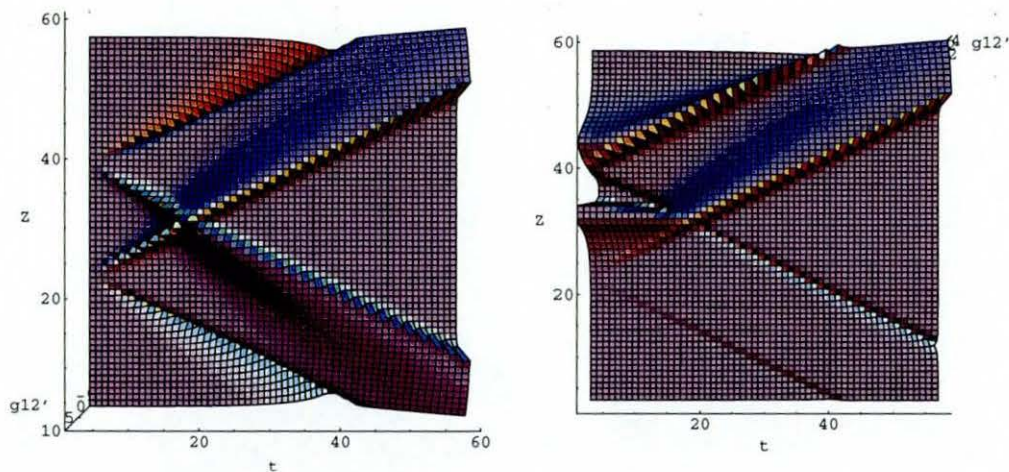


Figure 4.12: The pictures show the $g'_{12}^{(+ -)}$ (left) and $g'_{12}^{(+ +)}$ (right) component of the 4-Soliton Non-Diagonal Solution.

Below we give pictures of the metric components themselves, rather than of their derivatives.

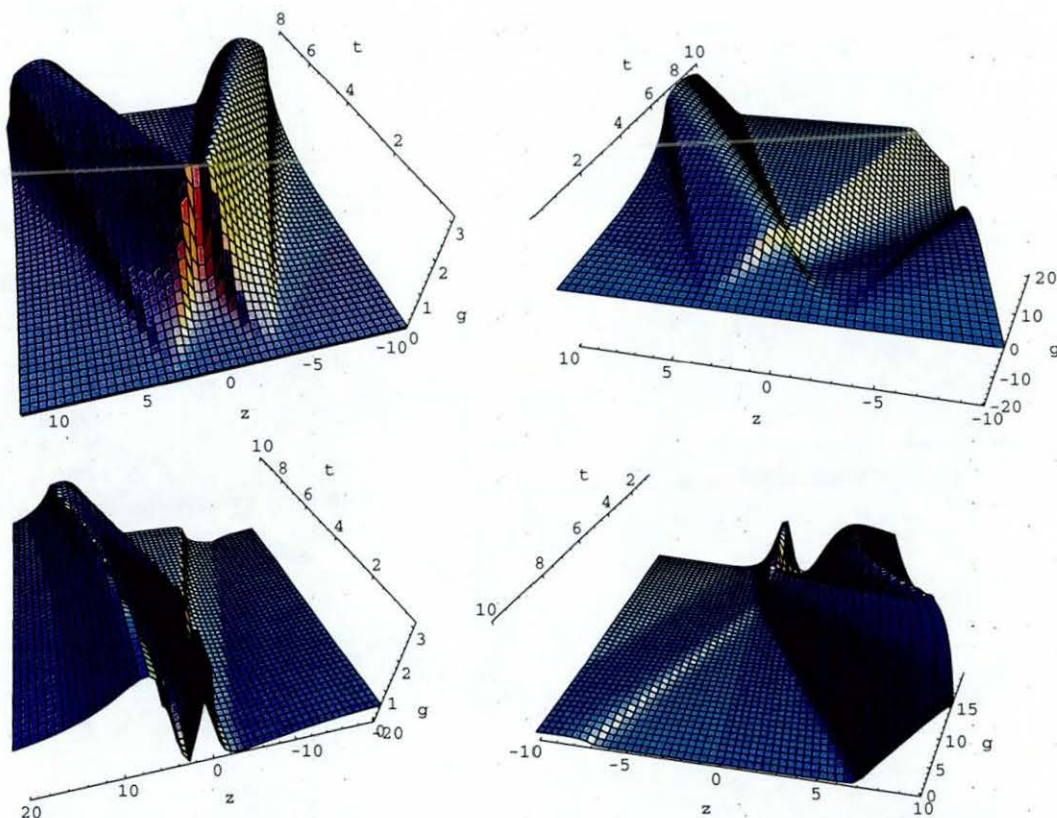


Figure 4.13: From top left and clockwise, the pictures show the $g_{11}^{(+ -)}$, $g_{12}^{(+ -)}$, $g_{12}^{(+ +)}$ and $g_{11}^{(+ +)}$ component of the 4-soliton nondiagonal solution presented in section 4.4.1.

4.4.2 Solutions with Kasner Seed: asymptotic behaviour

Much in the spirit of section 4.3, we want now to consider the asymptotic limit of the soliton metrics generated from a general Kasner seed. We wish to generalize the result of section 4.4.1 and check whether, even for a general Kasner seed, the asymptotic behaviour of these is still driven by their diagonal limit.

Let us consider the formulae (3.15). They can obviously be rewritten as:

$$\Gamma = \Gamma^{(d)} \det \left(1 + \frac{q_{[i]} q_{[j]}}{\lambda_{[i]}^{2q} \lambda_{[j]}^{2q}} \right), \quad (4.26a)$$

$$\Theta_{11} = \Theta_{11}^{(d)} \det \left(1 + \frac{q_{[i]} q_{[j]}}{\lambda_{[i]}^{2q-1} \lambda_{[j]}^{2q-1}} \right), \quad (4.26b)$$

$$\Theta_{12} = \Gamma^{(d)} \det \left(1 + \frac{q_{[i]} q_{[j]}}{\lambda_{[i]}^{2q} \lambda_{[j]}^{2q}} - \frac{q_{[i]} \lambda_{[i]}^{2q}}{\lambda_{[i]}^{2q+1} \lambda_{[j]}^{2q+1}} (\lambda_{[i]} \lambda_{[j]} - 1) \right), \quad (4.26c)$$

$$\Theta_{22} = \Theta_{22}^{(d)} \det \left(1 + \frac{q_{[i]} q_{[j]}}{\lambda_{[i]}^{2q+1} \lambda_{[j]}^{2q+1}} \right), \quad (4.26d)$$

where $\Gamma^{(d)}$ and $\Theta_{ab}^{(d)}$ are the usual Γ and Θ_{ab} in which $q_{[k]} = 0$. We also recall the functions \mathcal{K}_i :

$$\mathcal{K}_1 = t^{1/2-q}, \quad \mathcal{K}_2 = t^{1/2+q}. \quad (4.27)$$

Let us now consider equations (3.11). In particular, for the case under consideration they become:

$$g_{11} = \prod_{k=1}^N \frac{\mu[k]}{t} t^{1-2q} \left(2 - \frac{\Theta_{11}}{\Gamma} \right), \quad (4.28a)$$

$$g_{12} = \prod_{k=1}^N \frac{\mu[k]}{t} t \left(1 - \frac{\Theta_{12}}{\Gamma} \right), \quad (4.28b)$$

$$g_{22} = \prod_{k=1}^N \frac{\mu[k]}{t} t^{1+2q} \left(2 - \frac{\Theta_{22}}{\Gamma} \right). \quad (4.28c)$$

In the limit when $t \rightarrow \infty$, the functions $\lambda_{[k]}$ approach 1. Therefore, in that limit, from (4.26) we have:

$$\Gamma \rightarrow \Gamma^{(d)}, \quad \Theta_{11} \rightarrow \Theta_{11}^{(d)}, \quad \Theta_{12} \rightarrow \Gamma^{(d)}, \quad \Theta_{22} \rightarrow \Theta_{22}^{(d)}. \quad (4.29)$$

By inserting these expressions into the (4.28)'s, it becomes evident that the soliton metric approaches its diagonal limit

4.5 Interaction of gravitational solitons and waves

In this section we will investigate the T-S in solitons generated from the Wainwright–Marshmann (W&M) seed already considered in 3.2.4 – with $\alpha = t$ and $\beta = Z$. This solution may be supposed to describe the interaction between solitons and gravitational waves, thus providing the gravitational counterpart to the cases dealt with in section 4.1.2.

4.5.1 Asymptotic behaviour of Solutions with W&M Seed

The explicit form of the soliton solutions can be obtained by using the formulae given in (3.31). However, they are not very enlightening and we will thus refrain from giving them here. In figure (4.14) a solution with two complex poles is pictured.

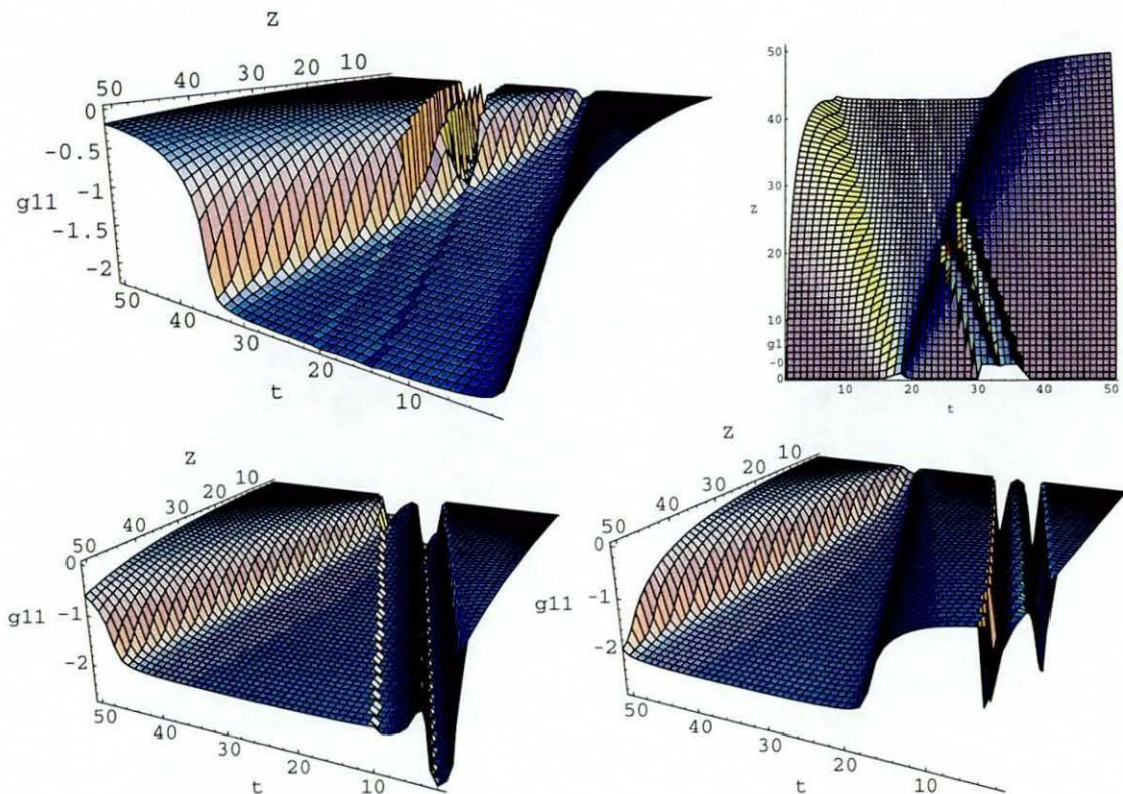


Figure 4.14: The pictures show the g_{11} (left) and g_{12} (right) component of the 2-soliton nondiagonal solution generated from a Wainwright–Marshmann seed. Different viewpoints are considered. Here: $H = 1$, $A = 1/2$, $w_0 = 0.1$, $u_B = 1$, $u_F = -1$. Moreover, $z_0 = -5$ in the two top pictures and $z_0 = 0$ (left), $z_0 = +5$ (right) in the lower ones.

These pictures seem to show that no T-S occurs when the gravitational wave interacts with the soliton, thus giving a different result from that found in [119].

To support our claim, we will consider now the asymptotic limit $-t \rightarrow \infty$ of a general N -soliton solution generated from a W&M metric as seed. The results we will obtain apply to a generic N -soliton solution and to a general $W(t, Z)$ for which:

$$\lim_{t \rightarrow \infty} \frac{W(t, Z)}{\sqrt{t}} = 0. \quad (4.30)$$

In particular, this condition is fulfilled by the particular W – formula (4.4) – considered in [119].

We can now show that the asymptotic limit $\alpha \rightarrow \infty$ of the above metrics is given by the soliton solution with the same number of poles and with Kasner seed. The Kasner parameter q has to be set as $q = 1/4$, yielding the following:

$$g^{(1/4)} = \sqrt{t} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}. \quad (4.31)$$

Incidentally, this also is the limit $W \rightarrow 0$ of the metric (3.27).

Let us consider the N -soliton solution in Kasner seed given in section 3.2.1. It can be rewritten as:

$$\Theta_{11}^{(q)} = \det \left(F_{q[i]} \ F_{q[j]} \ \lambda_{[i]}^{-1/2+q} \ \lambda_{[j]}^{-1/2+q} \ \frac{1 + c_{q[i]} \ c_{q[j]} \ \lambda_{[i]}^{2 \ q} \ \lambda_{[j]}^{2 \ q}}{\lambda_{[i]} \ \lambda_{[j]} - 1} \right), \quad (4.32a)$$

$$\Theta_{12}^{(q)} = \det \left(F_{q[i]} \ F_{q[j]} \ \lambda_{[i]}^{-1/2+q} \ \lambda_{[j]}^{-1/2+q} \ \frac{c_{q[i]} \ c_{q[j]} + \lambda_{[i]}^{2 \ q} \ \lambda_{[j]}^{2 \ q}}{\lambda_{[i]} \ \lambda_{[j]} - 1} + \right. \\ \left. - F_{q[i]} \ F_{q[j]} \ \frac{c_{q[j]}}{\lambda_{[j]}^{2q}} \ \frac{1}{\lambda_{[i]}^{1/2-q} \ \lambda_{[j]}^{1/2-q}} \right), \quad (4.32b)$$

$$\Gamma^{(q)} = \det \left(F_{q[i]} \ F_{q[j]} \ \lambda_{[i]}^{1/2-q} \ \lambda_{[j]}^{1/2-q} \ \frac{c_{q[i]} \ c_{q[j]} + \lambda_{[i]}^{2 \ q} \ \lambda_{[j]}^{2 \ q}}{\lambda_{[i]} \ \lambda_{[j]} - 1} \right), \quad (4.32c)$$

where we remember that $K_1 = \alpha^{1/2-q}$, $K_2 = \alpha^{1/2+q}$ and:

$$F_{q[k]} = 2^{q-\frac{1}{2}} w_{0[k]}^{q-\frac{1}{2}}, \quad m_2^{(0)[k]} = 2^{2 \ q} w_{[k]}^{2 \ q} c_{q[k]}, \quad \mu[k] = t \ \lambda_{[k]}. \quad (4.33)$$

By introducing the quantities:

$$d_{[k]} = \frac{\sin Y_{[k]} - c_{[k]} \cos[k]}{\cos Y_{[k]} + c_{[k]} \sin[k]}, \quad \tilde{F} = F_{[k]} (\cos Y_{[k]} + c_{[k]} \sin[k]), \quad (4.34)$$

the expressions associated to the W&M metric given in (3.31) become:

$$\Theta_{11} = \det \left(\tilde{F}_{[i]} \ \tilde{F}_{[j]} \ \lambda_{[i]}^{-1/4} \ \lambda_{[j]}^{-1/4} \ \frac{1 + d_{[i]} \ d_{[j]} \ \lambda_{[i]}^{1/2} \ \lambda_{[j]}^{1/2}}{\lambda_{[i]} \ \lambda_{[j]} - 1} \right), \quad (4.35a)$$

$$\Theta_{12} = \det \left(\tilde{F}_{[i]} \ \tilde{F}_{[j]} \ \lambda_{[i]}^{1/4} \ \lambda_{[j]}^{1/4} \ \frac{d_{[i]} \ d_{[j]} + \lambda_{[i]}^{1/2} \ \lambda_{[j]}^{1/2}}{\lambda_{[i]} \ \lambda_{[j]} - 1} + \right. \\ \left. - \tilde{F}_{[i]} \ \tilde{F}_{[j]} \ \frac{d_{[j]}}{\lambda_{[j]}^{1/2}} \ \frac{1}{\lambda_{[i]}^{1/4} \ \lambda_{[j]}^{1/4}} \ \mathcal{M}_{[j]} \right), \quad (4.35b)$$

$$\Gamma = \det \left(\tilde{F}_{[i]} \ \tilde{F}_{[j]} \ \lambda_{[i]}^{1/4} \ \lambda_{[j]}^{1/4} \ \frac{d_{[i]} \ d_{[j]} + \lambda_{[i]}^{1/2} \ \lambda_{[j]}^{1/2}}{\lambda_{[i]} \ \lambda_{[j]} - 1} \right), \quad (4.35c)$$

where:

$$\mathcal{M}_{[j]} = \left(1 - \frac{\sqrt{\lambda_{[j]}}}{d_{[j]}} \frac{W(t, Z)}{\sqrt{t}} \right). \quad (4.36)$$

It may then be noticed that (4.35a) and (4.35c) are formally identical to (4.32a) and (4.32c) in which $q = 1/4$. Moreover, in the limit when $t \rightarrow \infty$, it may be shown that $Y_{[i]}$ approaches constant values – see appendix B for a detailed account. Therefore, in that limit, the quantities $d_{[i]}$ and $\tilde{F}_{[i]}$ become constant. As a result, in that limit, (4.35a), (4.35c) and (4.32a) and (4.32c) may be let respectively coincide, with an appropriate choice of parameters. We may then conclude that the (11) component of the N -Soliton solution with W&M seed asymptotically approaches the (11) component of the N -Soliton solution with Kasner $q = 1/4$ seed.

Had not it be for the factor $\mathcal{M}_{[j]}$, (4.35b) would have been formally identical to (4.32b). However, in the limit when $t \rightarrow \infty$, we have that $\mathcal{M} \rightarrow 1$ and therefore, in that limit, (4.35b) and (4.32b) may be let coincide, with an appropriate choice of parameters. Again, the (12) component of the N -Soliton solution with W&M seed asymptotically approaches the (12) component of the N -Soliton solution with Kasner $q = 1/4$ seed.

As shown in section 4.4.2, the asymptotic behaviour of a soliton solution with a Kasner seed is in turn driven by its diagonal limit. Therefore, the asymptotic limit of the soliton metrics with a W&M seed is dictated by the diagonal soliton solution with the same number of poles and with a Kasner seed with $q = 1/4$. It seems then clear that a T-S effect can not be expected even in this case.

Bibliography of Files: [18]

4.6 Conclusion

The main issue which we have tried to demonstrate is that the soliton nondiagonal metrics, in the causal region, tend to diagonality and, precisely for that, no Time-Shift can be expected [136].

We have found it very useful to prove this by looking at the functions Γ and Θ_{ab} from the Alekseev determinant method.

However, it is compulsory to quote here two major papers where the same issue was discussed, though not in the context of T-S: that by Belinski and Fargion [132] and that by Carr and Verdaguer [106]. In [132] the asymptotic behaviour – in the causal region – of a two soliton solution with Kasner seed was considered and in [106] a generalization to the n -soliton case was provided. The results we have set out in this chapter, while obtained using a different approach, confirm these and extend them to the case when a nondiagonal seed is considered.

As we remarked in section 4.3, the problem of T-S for gravitational solitons is still controversial, in particular papers [115] and [125, 116] seem to propose contrasting results. It has also been pointed out in [81] – see section 3.6.b – that a possible way out of this problem might consists in a careful analysis of the f coefficient of the relevant metrics, which is also necessary for a complete description of any spacetime. However, if we forget about any general-relativity-implication and regard these solutions as purely mathematical solutions to a given nonlinear p.d.e., namely the Ernst equation, the problem still persists.

The other proposal, also given in [81], of looking at T-S in a non-diagonal metric, has seemed to us more sound. As explained in this chapter we have endeavoured to investigate it, with the result that no T-S occurs for gravitational solitons interacting with either gravitational solitons or gravitational waves.

We are aware that the analysis we have undertaken is not totally satisfactory, in that it has considered the metric components themselves rather than any more sound physical quantity as, possibly, the Weyl tensor. However, we trust that the comments on that, made in section 4.3, were sufficiently explicatory.

The question still remains of how and where the nonlinearity in the Einstein's equations shows up.

Undoubtedly, the pictures we have given for the diagonal – see figure (4.6) – and nondiagonal – see figure (4.13) – 4-soliton solution with Minkowski seed display some major differences. These are mainly concentrated in the interaction region and consist in small bumps that appear in the proximity of the interaction point and disappear for late times. These bumps are absent in the diagonal case. They correspond to the additional stationary points of $g_{11}^{(++)}$ and $g_{11}^{(+-)}$ that we found in section 4.4.1. However, we showed that these new stationary points cannot occur in the causal region. This is represented by the fact that the bumps are visible in the interaction region only.

All our results seem to indicate that gravitational solitons behave as solitary waves rather than true solitons.

Chapter 5

Real Pole Solitons and their extensions

In this chapter we are interested in solutions obtained by using the Belinski–Zakharov technique (BZ). In particular we will devote our attention to the time dependent case in which the trajectory-poles $\mu[k]$ are real [99, 100, 110, 105, 137, 138, 139, 140, 141, 142]. In this case the metric generated using BZ is defined only in particular regions of the spacetime. Moreover, on the null hypersurfaces that border these regions, the metric is singular. The nature of these singularities needs to be investigated. An extension would only be possible if the singularity is removable

Usually such an extension is easily performed by attaching the soliton metric to its seed [99, 100, 106]. One disadvantage of such an approach is that the Ricci tensor develops δ -like singularities on the null hypersurfaces along which the matching is performed [137, 141, 142]. This corresponds to the presence of sheets of null matter along these hypersurfaces.

The purpose of this chapter is to look for a possible extension of the soliton solutions without the occurrence of matter. In sections 5.1 and 5.2 we will introduce the soliton solutions and show that the singularities are removable by simply a coordinate transformation. In section 5.3 we will show how the matter-free and singularity-free extensions can be realized. In section 5.4 we will consider a first possible matter-free extension, following the lines pointed out in [137]. Moreover, for the diagonal case, we will give a general nonsolitonic extension in section 5.5. In section 5.6 we will consider another possible extension made by matching the soliton solution with a plane gravitational wave (PGW) and discuss its physical relevance. Finally, in section 5.7 we will draw our conclusions.

The basic results of this chapter have already been published [148].

5.1 Soliton solutions with Real Poles

Hereby we will refer to pole-trajectories (2.59) in which the plus sign has been chosen. Having that, we will present soliton solutions built up using the Kasner metric (3.12) as a seed.

$$f = \alpha^{2q^2-1/2}, \quad g_{11} = \alpha^{1-2q}, \quad g_{22} = \alpha^{1+2q}. \quad (5.1)$$

5.1.1 Soliton solutions with Real Poles: the domain

In section 3.3.2 we have already discussed the various regions of the spacetime in which soliton solutions with one real pole can be considered. In this section we want to extend that discussion to the case when two real poles are considered. In figure (5.1) the relevant regions are illustrated.

Hereafter we will set $z_{0[1]} = -\Delta$ and $z_{0[2]} = +\Delta$, with $\Delta > 0$ and $\Delta = \sqrt{2} \Theta$. The null coordinates u, v have already been introduced in (3.43):

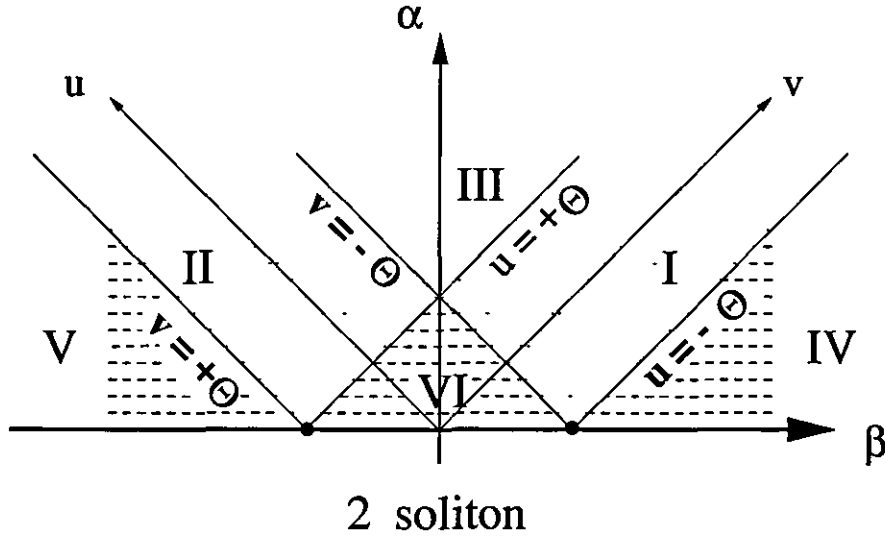


Figure 5.1: Spacetime regions of interest for cosmological soliton solutions with two real poles

Region I

Region I is defined as:

$$\beta + \Delta \geq +\alpha \geq \beta - \Delta \cup \beta - \Delta \geq -\alpha. \quad (5.2)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{0[1]})^2 - \alpha^2}$ is real, so that Region I contains the 1-soliton solution with pole $\mu_{[1]}$.

Also notice that in this region the pole $\mu_{[2]}$ is not well defined. Moreover, the 1-soliton solution with pole $\mu_{[1]}$ also holds in Region IV: in fact in Region IV $\sqrt{(\beta - z_{0[1]})^2 - \alpha^2}$ is still well defined.

In terms of null coordinates u, v this region is defined by

$$-\Theta \leq u \leq \Theta, \quad v > \Theta. \quad (5.3)$$

Notice that in Region I the Pole-Trajectory $\mu_{[1]}$ assumes negative values.

Region II

Region II is defined as:

$$\beta - \Delta \leq -\alpha \leq \beta + \Delta \cup \beta + \Delta \leq \alpha. \quad (5.4)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{0[2]})^2 - \alpha^2}$ is real, so that Region II contains the 1-soliton solution with pole $\mu_{[2]}$.

Also notice that in this region the pole $\mu_{[1]}$ is not defined. Moreover, the 1-soliton solution with pole $\mu_{[2]}$ also holds in Region V: in fact in Region V $\sqrt{(\beta - z_{[2]})^2 - \alpha^2}$ is still well defined.

In terms of null coordinates u, v this region is defined by

$$-\Theta \leq v \leq \Theta, \quad u \geq \Theta. \quad (5.5)$$

Notice that in Region II the Pole-Trajectory $\mu_{[2]}$ assumes positive values.

Region III

Region III is defined as:

$$\alpha - \Delta \geq \beta \cup \alpha - \Delta \geq -\beta. \quad (5.6)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{[i]})^2 - \alpha^2}$ is imaginary for both $i = 1$ and $i = 2$, so that Region III does not contain any soliton solution.

In terms of null coordinates u, v this region is defined by

$$u > \Theta, \quad v > \Theta. \quad (5.7)$$

Region IV

Region IV is defined as:

$$\beta - \Delta \geq +\alpha, \quad \alpha > 0. \quad (5.8)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{[i]})^2 - \alpha^2}$ is real both for $i = 1$ and $i = 2$, so that Region IV contains the 2-soliton solution.

In terms of null coordinates u, v this region is defined by

$$u < -\Theta, \quad v > -u. \quad (5.9)$$

If we introduce the variable $u = -|u|$ we get $v > |u|$, which shows as v is always positive. Finally, notice that in Region IV the Pole-Trajectories $\mu_{[1]}$ and $\mu_{[2]}$ assume negative values

Region V

Region V is defined as:

$$\beta + \Delta \leq -\alpha, \quad \alpha > 0. \quad (5.10)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{[i]})^2 - \alpha^2}$ is real both for $i = 1$ and $i = 2$, so that Region V contains the 2-soliton solution.

In terms of null coordinates u, v this region is defined by

$$v < -\Theta, \quad u > -v. \quad (5.11)$$

If we introduce the variable $v = -|v|$ we get $u > |v|$, which shows as u is always positive. Finally, notice that in Region V the Pole-Trajectories $\mu_{[1]}$ and $\mu_{[2]}$ assume positive values.

Region VI

Region VI is defined as:

$$\beta - \Delta \leq -\alpha \cup \beta + \Delta \geq \alpha, \quad \alpha > 0. \quad (5.12)$$

It is trivial to show that in this region $\sqrt{(\beta - z_{[i]})^2 - \alpha^2}$ is real for both $i = 1$ and $i = 2$, so that Region VI contains the 2-soliton solution. Also notice that in this region the two 1-soliton solutions with pole $\mu_{[1]}$ and $\mu_{[2]}$ are well defined.

5.1.2 1-Soliton solution

The explicit form of the 1-soliton solution has already been given in section 3.4.1. For convenience we repeat it here, with a slightly different notation:

$$f^{(I)} = \frac{2^{-2(q+1)} A_I}{z_0[1]^{1+2q}} \frac{\alpha_I^{2q^2}}{\sqrt{\beta_{Is}^2 - \alpha_I^2}} e^{2q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}} \left[2^{4q} z_0[1]^{4q} + m^2 e^{4q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}} \right], \quad (5.13a)$$

$$g_{11}^{(I)} = \alpha_I^{1-2q} e^{\operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}} \frac{2^{4q} z_0[1]^{4q} + m^2 e^{(4q-2) \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}}{2^{4q} z_0[1]^{4q} + m^2 e^{4q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}}, \quad (5.13b)$$

$$g_{12}^{(I)} = 2^{1+2q} z_0[1]^{2q} m \alpha_I^{2q} \sqrt{\beta_{Is}^2 - \alpha_I^2} \frac{e^{2q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}}{2^{4q} z_0[1]^{4q} + m^2 e^{4q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}}, \quad (5.13c)$$

where $\beta_{Is} = \beta_I - z_0[1]$. As we learnt in section 5.1.1, these 1-Soliton solutions are defined only in Region I and Region II of the spacetime – see figure (3.3).

Strictly speaking, solution (5.13) holds in Region I only. The expressions in Region II are the same but the signs of the expressions for the metric components g_{ab} must be changed: this takes into account the fact that the pole-trajectories are respectively negative and positive in Region I and Region II. The constant A_I may be chosen in order to obtain a positive expression for $f^{(I)}$.

The diagonal limit – i.e. $m \rightarrow 0$ – of the above metric is given by:

$$f^{(I)} = 4^{-1+q} A_I z_0[1]^{2q-1} \frac{\alpha_I^{2q^2}}{\sqrt{\beta_{Is}^2 - \alpha_I^2}} e^{-2q \operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}, \quad (5.14a)$$

$$g_{11}^{(I)} = \alpha_I^{1-2q} e^{\operatorname{ArcCosh} \frac{\beta_{Is}}{\alpha_I}}. \quad (5.14b)$$

5.1.3 2-Soliton solution

The explicit form of the 2-soliton solution with Kasner seed has already been given in section 3.4.2. Now we have to set $z_0[2] = \Delta = -z_0[1]$. As we learnt in section 5.1.1, this 2-Soliton solution is defined only in Region IV, Region V and Region VI of the spacetime – see figure (5.1).

The diagonal limit – i.e. $c \rightarrow 0$ – can be obtained from formulae in section 3.4.4. It can also be put in the form:

$$f^{(IV)} = 2^{4(q-1)} A_{IV} \Delta^{4q-2} \frac{\alpha_{IV}^{2q^2-1/2}}{\sqrt{(\beta_{IV} - \Delta)^2 - \alpha_{IV}^2} \sqrt{(\beta_{IV} + \Delta)^2 - \alpha_{IV}^2}} \times (5.15a)$$

$$\times \frac{e^{(1-2q) \operatorname{ArcCosh} \frac{(\beta_{IV}-\Delta)}{\alpha_{IV}}} e^{(1-2q) \operatorname{ArcCosh} \frac{(\beta_{IV}+\Delta)}{\alpha_{IV}}}}{\left(1 - e^{\operatorname{ArcCosh} \frac{(\beta_{IV}-\Delta)}{\alpha_{IV}}} e^{\operatorname{ArcCosh} \frac{(\beta_{IV}+\Delta)}{\alpha_{IV}}} \right)^2},$$

$$g_{11}^{(IV)} = \alpha_{IV}^{1-2q} e^{(1-2q) \operatorname{ArcCosh} \frac{(\beta_{IV}-\Delta)}{\alpha_{IV}}} e^{(1-2q) \operatorname{ArcCosh} \frac{(\beta_{IV}+\Delta)}{\alpha_{IV}}}. \quad (5.15b)$$

Again, solution (5.15) strictly holds only in Region IV, due to the fact the pole-trajectories assume different signs in the various regions of the spacetime.

Bibliography of Files: [19]

5.2 On the Nature of the Singularities

5.2.1 1-Soliton Diagonal Solution: Metric Components

Let us firstly introduce in Region I null coordinates (u_I, v_I) defined as:

$$\alpha_I = \frac{1}{\sqrt{2}}(u_I + v_I), \quad \beta_{I*} = \frac{1}{\sqrt{2}}(v_I - u_I), \quad ds_I^2 = 2f^{(I)} du_I dv_I - g_{ab}^{(I)} dx^a dx^b,$$

and consider the boundary between regions I and III. The singularity on the hypersurface $\alpha_I = \beta_I - z_0[1]$ is now replaced by the one on $u_I = 0$. In the proximity of this hypersurface the metric components (5.14) have the following behaviour:

$$g_{ab}^{(I)} \text{ finite as } |u_I| \rightarrow 0^+, \quad f^{(I)} \propto \frac{1}{\sqrt{|u_I|}} \text{ as } |u_I| \rightarrow 0^+. \quad (5.16)$$

Let us now consider the coordinate transformation:

$$\begin{aligned} (u_I, v_I) &\mapsto (u, v_I), & u_I &= -u^n, \\ (f^{(I)}, g_{ab}^{(I)}) &\mapsto (F^{(I)}, G_{ab}^{(I)}), & \alpha_I &= \frac{1}{\sqrt{2}}(u^n + v_I), \\ & & \beta_{I*} &= \frac{1}{\sqrt{2}}(v_I - u^n), \end{aligned} \quad (5.17)$$

$$F^{(I)} = \frac{|(u_I, v_I)|}{|(u, v_I)|} f^{(I)}, \quad \frac{|(u_I, v_I)|}{|(u, v_I)|} = n u^{n-1},$$

Such a coordinate transformation removes the singularity in $F^{(I)}$ provided $n = 2$. In fact we have.

$$G_{11}^{(I)} = 2^{-1/2+q} (\sqrt{v_I} + u^{n/2})^2 (v_I - u^n)^{-2q}, \quad (5.18a)$$

$$F^{(I)} = \frac{2^{-5/2+2q-q^2} A_I n (u^{n/2} - \sqrt{v_I})^{4q} (v_I - u^n)^{2q(q-1)}}{z_0[1]^{1-2q} \sqrt{v_I}} u^{(n-2)/2}. \quad (5.18b)$$

Notice here that for $n < 2$ $F^{(I)}$ is unbounded and for $n > 2$ it vanishes on $u = 0$. On the other hand, for $n = 2$, $F^{(I)}$ becomes a constant. Thus a coordinate singularity may only be avoided by putting $n = 2$. In the proximity of the line $u = 0$ (and with $n = 2$) the above solutions behave like:

$$G_{11}^{(I)} \approx 2^{-1/2+q} v_I^{1-2q}, \quad F^{(I)} \approx 2^{-3/2+2q-q^2} z_0[1]^{-1+2q} A_I v_I^{-1/2+2q^2}. \quad (5.19)$$

Analogously we can remove the singularity in Region II by making the coordinate transformation:

$$\begin{aligned} (u_{II}, v_{II}) &\mapsto (u_{II}, v), & v_{II} &= -v^2, \\ (f^{(II)}, g_{ab}^{(II)}) &\mapsto (F^{(II)}, G_{ab}^{(II)}), & \beta_{II*} &= \frac{1}{\sqrt{2}}(-v^2 - u_{II}), \\ & & \alpha_{II} &= \frac{1}{\sqrt{2}}(u_{II} - v^2), \end{aligned} \quad (5.20)$$

$$F^{(II)} = \frac{|(u_{II}, v_{II})|}{|(u_{II}, v)|} f^{(II)}, \quad \frac{|(u_{II}, v_{II})|}{|(u_{II}, v)|} = 2v,$$

giving the following metric components.

$$G_{11}^{(II)} = 2^{-1/2+q} (\sqrt{u_{II}} - v)^2 (u_{II} - v^2)^{-2q}, \quad (5.21a)$$

$$F^{(II)} = 2^{-3/2+2q-q^2} A_{II} z_{0[1]}^{-1+2q} \frac{(v + \sqrt{u_{II}})^{4q} (u_{II} - v^2)^{2q(q-1)}}{\sqrt{u_{II}}}. \quad (5.21b)$$

5.2.2 1-Soliton Diagonal Solution: Weyl Tensor

The nature of the above singularities as coordinate singularities only, can be also proved by inspecting the Weyl tensor components Ψ_0, Ψ_2, Ψ_4 [141, 142] and the Invariants I_1 and I_2 defined as [11]:

$$I_1 = \Psi_0 \Psi_4 + 3 \Psi_2^2, \quad I_2 = (\Psi_0 \Psi_4 - \Psi_2^2) \Psi_2. \quad (5.22)$$

Let us concentrate on Region I, the situation in Region II being analogue. An explicit calculation shows that on $|u_I| = 0$ the Weyl Tensor Components, as $|u_I| \rightarrow 0^+$, are given by:

$$\Psi_0^{(I)} \approx 0, \quad \Psi_2^{(I)} \approx -\frac{2^{5/2-2q+q^2} q z_{0[1]} v_I^{-1-2q^2}}{A_I}, \quad \Psi_4^{(I)} \propto \frac{1}{\sqrt{|u_I|}}. \quad (5.23)$$

However, the Weyl tensor invariants $I_1^{(I)}$ and $I_2^{(I)}$ remain bounded as $|u_I| \rightarrow 0^+$:

$$I_1^{(I)} \approx 3 \cdot 2^{7-4q+2q^2} \frac{q^4 z_{0[1]}^{2-4q}}{A_I^2} v_I^{-2-4q^2}, \quad (5.24a)$$

$$I_2^{(I)} \approx -2^{19/2-6q+3q^2} \frac{q^3 (3q^2 - 1) z_{0[1]}^{3-6q}}{A_I^3} v_I^{-3-6q^2}. \quad (5.24b)$$

When performing the coordinate transformation $u_I = -u^n$ - which remove the singularity in $f^{(I)}$ when $n = 2$ - the situation is as follows:

$$\begin{aligned} \Psi_0^{(I)} = & -\frac{2^{(5/2-2q+q^2)} q z_{0[1]}^{1-2q}}{A_I n} \frac{1}{\sqrt{v_I}} u^{(2-n)/2} \times \\ & \times \frac{(u^{n/2} - \sqrt{v_I})^{-2-4q} (v_I - u^n)^{-2q(q-1)}}{(u^{n/2} + \sqrt{v_I})^2} \left((4q^2 - 1)v_I + 6q\sqrt{v_I} u^{n/2} + 3u^n \right), \end{aligned} \quad (5.25a)$$

$$\begin{aligned} \Psi_2^{(I)} = & -\frac{2^{(5/2-2q+q^2)} q z_{0[1]}^{1-2q}}{A_I} \times \\ & \times \frac{(u^{n/2} - \sqrt{v_I})^{-2-4q}}{(u^{n/2} + \sqrt{v_I})^2 (v_I - u^n)^{-2q(q-1)}} \left(v_I + 2q\sqrt{v_I} u^{n/2} + u^n \right), \end{aligned} \quad (5.25b)$$

$$\begin{aligned} \Psi_4^{(I)} = & -\frac{2^{(5/2-2q+q^2)} n q z_{0[1]}^{1-2q}}{A_I} u^{(n-2)/2} \sqrt{v_I} \times \\ & \times \frac{(u^{n/2} - \sqrt{v_I})^{-2-4q} (v_I - u^n)^{-2q(q-1)}}{(u^{n/2} + \sqrt{v_I})^2} \left((4q^2 - 1)u^n + 6q\sqrt{v_I} u^{n/2} + 3v_I \right). \end{aligned} \quad (5.25c)$$

By inspection, these expressions are all bounded on the null hypersurfaces $u = 0^+$ when $n = 2$. In addition, the Weyl tensor invariants are bounded for any n . In fact, the possible singularities coming only from Ψ_0 and Ψ_4 , we may notice that Ψ_0 contains a term $u^{(2-n)/2}$ and Ψ_4 contains a term $u^{(n-2)/2}$. Since the invariants contain Ψ_0 and Ψ_4 only in the combination $\Psi_0 \Psi_4$, the two alternately divergent factors cancel each other.

5.2.3 2-Soliton Diagonal Solution

Even for the 2-soliton solution the singularities on the lines $\alpha = \pm(\beta - z_0[i])$, $i = 1, 2$ are coordinate singularities only.

Let us now consider solution (5.15). In particular, after introducing null coordinates u_{IV} and v_{IV} as stated in (3.43), we can introduce in Region IV the further coordinate transformation:

$$\begin{aligned} (u_{IV}, v_{IV}) &\mapsto (u, \bar{v}), & u_{IV} &= -\Theta - u^2, & v_{IV} &= +\Theta + \bar{v}, \\ (f^{(IV)}, g_{ab}^{(IV)}) &\mapsto (F^{(IV)}, G_{ab}^{(IV)}), & \alpha_{IV} &= \frac{1}{\sqrt{2}} (-u^2 + \bar{v}), \\ & & \beta_{IV} &= \frac{1}{\sqrt{2}} (2\Theta + \bar{v} - u^2), \end{aligned} \quad (5.26)$$

$$F^{(IV)} = \frac{|(u_{IV}, v_{IV})|}{|(u, \bar{v})|} f^{(IV)}, \quad \frac{|(u_{IV}, v_{IV})|}{|(u, \bar{v})|} = 2u,$$

which remove the singularity on $\alpha = \beta - z_0[2]$:

$$G_{11}^{(IV)} = 2^{q-1/2} \frac{(\bar{v} - u^2)^{3-2q}}{(u - \sqrt{\bar{v}})^2 (u^2 + \bar{v} + 4\Theta - 2\sqrt{u^2 + 2\Theta} \sqrt{\bar{v} + 2\Theta})}, \quad (5.27a)$$

$$\begin{aligned} F^{(IV)} &= -2^{-27/4+6q-q^2} A_{IV} \Theta^{4q-2} \frac{(u - \sqrt{\bar{v}})^{2(2q-1)}}{\sqrt{\bar{v}}} \times \\ &\times \frac{(u^2 + \bar{v} + 4\Theta - 2\sqrt{u^2 + 2\Theta} \sqrt{\bar{v} + 2\Theta})^{2(q+1)}}{(u\sqrt{\bar{v}} - 2\Theta + \sqrt{u^2 + 2\Theta} \sqrt{\bar{v} + 2\Theta})^2} \times \\ &\times \frac{1}{(\bar{v} + 2\Theta - \sqrt{u^2 + 2\Theta} \sqrt{\bar{v} + 2\Theta}) (u^2 + 2\Theta - \sqrt{u^2 + 2\Theta} \sqrt{\bar{v} + 2\Theta})}. \end{aligned} \quad (5.27b)$$

Near the line $u = 0$ the above solutions behave as:

$$G_{11}^{(IV)} \approx 2^{-1/2+q} \frac{\bar{v}^{2-2q}}{\bar{v} + 4\Theta - 2\sqrt{2} \sqrt{\Theta} \sqrt{\bar{v} + 2\Theta}}, \quad (5.28a)$$

$$\begin{aligned} F^{(IV)} &\approx -2^{-27/4+6q-q^2} A_{IV} \Theta^{4q-7/2} \bar{v}^{2q(q-1)} \times \\ &\times \frac{(\bar{v} + 4\Theta - 2\sqrt{2} \sqrt{\Theta} \sqrt{\bar{v} + 2\Theta})^{2(1+q)}}{(\bar{v} + 2\Theta - \sqrt{2} \sqrt{\Theta} \sqrt{\bar{v} + 2\Theta}) (2\sqrt{\Theta} - \sqrt{2} \sqrt{\bar{v} + 2\Theta})^3}, \end{aligned} \quad (5.28b)$$

which shows that the singularity is removed.

Analogously we can remove the singularities in Region V and Region VI. Below one can find summarized the transformations that remove the singularities in the three regions where the 2-soliton solution is defined:

$$\begin{aligned} \text{Region IV} \quad & u_{IV} = -\Theta - u^2 & v_{IV} &= +\Theta + (v - c_{II}) \\ \text{Region V} \quad & u_V = +\Theta + (u - c_I) & v_V &= -\Theta - v^2 \\ \text{Region VI} \quad & u_{VI} = +\Theta - \frac{2\Theta}{c_I^2} (c_I - u)^2 & v_{VI} &= +\Theta - \frac{2\Theta}{c_{II}^2} (c_{II} - v)^2 \end{aligned} \quad (5.29)$$

where c_I and c_{II} are positive real constants. Now the line $\alpha = \beta - z_0[2]$ is mapped on to $u = 0$ and the line $\alpha = \beta - z_0[1]$ is mapped on to $u = c_I$. Similarly the line $\alpha = -\beta + z_0[1]$ is mapped on to $v = 0$ and $\alpha = -\beta + z_0[2]$ becomes $v = c_{II}$.

5.2.4 Non-Diagonal Solitons

1-Soliton Solution

It may be checked by inspection that, on the null lines $\beta = z_{0[1]} + \alpha$ and $\beta = z_{0[1]} - \alpha$, the non-diagonal metric (5.13) and its diagonal limit (5.14) approaches the same values but the replacement

$$A_I^{(diag)} \mapsto A_I \frac{m^2 + 2^{4q} z_{0[1]}^{4q}}{2^{4q} z_{0[1]}^{4q}} \quad (5.30)$$

is required. Therefore we can follow the same procedure as for the diagonal case in order to remove the singularities in (5.13).

2-Soliton Solution

When considering non-diagonal solutions, some extra care has to be used in doing the matching. In fact, unlike in the 1-soliton case, the g_{12} coefficient does not vanish on the null lines where the singularities are located. Therefore we have to match the 2-soliton non-diagonal solutions with 1-soliton non-diagonal solutions as well. Moreover the parameters in the 1-soliton solutions have to be chosen in such a way to ensure a continuous matching.

To give a flavour we anticipate that, if one were interested in continuously matching the 2-soliton solution of region IV with a 1-soliton solution in region I, then:

$$m = 2^{3q} \Theta^{2q} m$$

where m and m are the arbitrary parameters in the 2-soliton and 1-soliton solution respectively.

5.2.5 Solitons with Degenerate Poles

It has already been clarified in section 3.5 that the limit when two real poles coincides, is equivalent to the 1-soliton solution with degeneracy $h = 2$. Moreover this solution also coincides with the limit in which two complex poles become real.

It is worth showing in here that in these cases, the singularities occurring on the null lines are not removable by a coordinate transformation.

In fact, let us introduce null coordinates (u_I, v_I) , according to (3.43), in the 1-soliton solution with degeneracy $h = 2$ introduced in section 3.4.5 – formulae (3.68) and (3.69). Having that, let us perform the further coordinate transformation:

$$\begin{aligned} (u_I, v_I) &\mapsto (u, v), & u_I &= -u^n, & v_I &= v^m, \\ (f, g_{ab}) &\mapsto (F^{(I)}, G_{ab}^{(I)}), & \alpha_I &= \frac{1}{\sqrt{2}} (-u^n + v^m), \\ & & \beta_{I,} &= \frac{1}{\sqrt{2}} (v^m + u^n), \end{aligned} \quad (5.31)$$

The metric components are now given by

$$F^{(I)} = \frac{2^{-39/4+6q-q^2} A_I n m}{\Theta^{2-4q}} \frac{(v^m - u^n)^{3/2-4q+2q^2} (u^{n/2} - v^{m/2})^{8q}}{u^{n+1} v^{m+1}}, \quad (5.32)$$

$$G_{11}^{(I)} = 2^{q-1/2} \frac{(-u^n + v^m)^{3-2q}}{(u^{n/2} - v^{m/2})^4}. \quad (5.33)$$

In order for $F^{(I)}$ to be bounded, we should consider negative values for n and m , which is not admissible.

Bibliography of Files [20]

5.3 On the Extensions of Soliton Solutions

We are now left with the problem of extending the soliton solutions through the (removed) singularity. In this section we will show how this can be done, both with and without the occurrence of matter.

5.3.1 1-soliton solution

Hereafter we will consider mostly diagonal solutions. In fact, we have already pointed out in section 5.2.4 that, on the null-lines $u = 0$ and $v = 0$, the diagonal and non-diagonal metrics coincide. Therefore, since we are interested in continuously extending the soliton metrics, the results we will present below for the diagonal metrics also apply for the non-diagonal ones.

Matter Extension

On the null lines $\alpha = \pm(\beta - z_{0[i]})$ the metric coefficients approach the seed (Kasner). Thus a possible extension from regions I and II to region III is to match the soliton solutions with a Kasner metric itself [106]. Thus consider in region III the metric:

$$ds_{III}^2 = f^{(III)}(d\alpha_{III}^2 - d\beta_{III*}^2) - g_{ab}^{(III)} dx^a dx^b, \quad (5.34a)$$

$$f^{(III)} = \alpha_{III}^{(4\delta^2-1)/2}, \quad g_{11}^{(III)} = \alpha_{III}^{1-2\delta}, \quad g_{22}^{(III)} = \alpha_{III}^{1+2\delta}. \quad (5.34b)$$

By introducing the null coordinates u_{III} and v_{III} defined as in (3.43), the metrics (5.18) and (5.21) can be continuously matched with (5.34) by identifying:

$$v_I = v, \quad u_{II} = u, \quad (5.35a)$$

$$u_{III} = u, \quad v_{III} = v, \quad \delta = q, \quad (5.35b)$$

and $A_I = 2^{7/4-2q} z_{0[1]}^{1-2q}$, $A_{II} = 2^{7/4-2q} z_{0[1]}^{1-2q}$. A complete matching can therefore be achieved with:

$$\text{Region I} \quad \alpha_I = \frac{1}{\sqrt{2}} (-u^2 + v), \quad \beta_{I*} = \frac{1}{\sqrt{2}} (u^2 + v),$$

$$\text{Region II} \quad \alpha_{II} = \frac{1}{\sqrt{2}} (u - v^2), \quad \beta_{II*} = \frac{1}{\sqrt{2}} (-v^2 - u), \quad (5.36)$$

$$\text{Region III} \quad \alpha_{III} = \frac{1}{\sqrt{2}} (u + v), \quad \beta_{III*} = \frac{1}{\sqrt{2}} (v - u).$$

The spacetime can thus be described in terms of the coordinates u and v as described in figure (5.2). The lines β_I and β_{II} are given by $u = -\sqrt{|v|}$. They describe the cosmological singularity $\alpha = 0$.

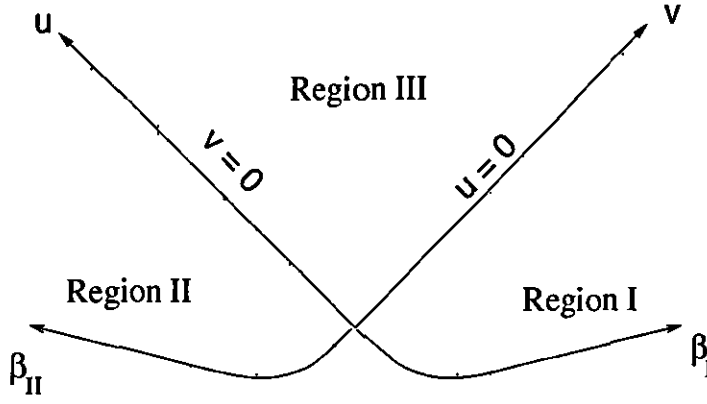


Figure 5 2: The picture shows the parametrization of the spacetime, containing a soliton with one real pole, in terms of the coordinates u, v . These are the coordinates that remove the singularities into the metric components of the soliton metric.

This simple result is anyway affected by the following problem: the u -derivative of α on the line $u' = 0$ suffer a jump, as well as the v -derivative of α on the line $v = 0$:

$$\partial_u \alpha_I = 0 \quad \partial_u \alpha_{III} = 1 \quad \text{as } u \rightarrow 0 \quad (5.37a)$$

$$\partial_v \alpha_{II} = 0 \quad \partial_v \alpha_{III} = 1 \quad \text{as } v \rightarrow 0 \quad (5.37b)$$

It may be recalled that discontinuities in the derivatives of α across a hypersurface induce nonzero components in the Ricci Tensor, and hence in the Energy-Momentum Tensor. In this case, these are given by:

$$T_{uu} \propto F^{(I)} \frac{[\alpha_u]}{\alpha} \quad T_{vv} \propto F^{(II)} \frac{[\alpha_v]}{\alpha} \quad (5.38)$$

Thus a discontinuity as in (5.37) gives rise to an impulsive component in the Energy-Momentum tensor corresponding to a thin sheet of null matter located on this hypersurface. That has been explicitly described in [137, 141]. Moreover, with this time orientation, the matter has *negative* energy density [142].

No-matter extension

As stated in section 1.3, the field equations imply that

$$\alpha = \mathcal{F}(u) + \mathcal{G}(v) \quad (5.39)$$

where $\mathcal{F}(u)$ and $\mathcal{G}(v)$ are two completely arbitrary functions of u and v respectively. For the Kasner metrics of previous sections we have considered the choice $\mathcal{F}(u) = u/\sqrt{2}$ and $\mathcal{G}(v) = v/\sqrt{2}$. However, this freedom can now be used to avoid the occurrence of matter along the relevant hypersurfaces. Let us consider:

$$\begin{aligned} \text{Region I} \quad \alpha_I &= \frac{1}{\sqrt{2}} (-u^2 + v^2), & \beta_{Is} &= \frac{1}{\sqrt{2}} (u^2 + v^2), \\ \text{Region II} \quad \alpha_{II} &= \frac{1}{\sqrt{2}} (u^2 - v^2), & \beta_{IIs} &= \frac{1}{\sqrt{2}} (-v^2 - u^2), \\ \text{Region III} \quad \alpha_{III} &= \frac{1}{\sqrt{2}} (u^2 + v^2), & \beta_{IIIs} &= \frac{1}{\sqrt{2}} (v^2 - u^2). \end{aligned} \quad (5.40)$$

In this case the u and v -derivatives of α across the lines $u = 0$ and $v = 0$ do not suffer any discontinuity. The problem is therefore to find a solution in region III compatible with the choice in (5.40), i.e. which satisfy the EFE's and is bounded on the null line $u = 0$.

More clearly, the EFE's associated with the line element (1.29) are given by [11].

$$\alpha = \frac{1}{\sqrt{2}} (a(u) + b(v)), \quad \beta = \frac{1}{\sqrt{2}} (b(v) - a(u)), \quad (5.41a)$$

$$f = \frac{a' b'}{\sqrt{a+b}} e^{-S}, \quad S_a = -\frac{1}{2} (a+b) \Phi_a^2, \quad S_b = -\frac{1}{2} (a+b) \Phi_b^2, \quad (5.41b)$$

$$g_{11} = \alpha e^\Phi, \quad 2(a+b) \Phi_{ab} + \Phi_a + \Phi_b = 0. \quad (5.41c)$$

Let us consider in Region III the coordinate transformation $a(u) = \varepsilon u^2$ and $b(v) = v^2$. Then the equations for S became:

$$S_u = -\frac{1}{4} \varepsilon (\varepsilon u^2 + v^2) \Phi_u^2, \quad S_v = -\frac{1}{4} \varepsilon (\varepsilon u^2 + v^2) \Phi_v^2. \quad (5.42)$$

Of course there are three relevant cases, respectively associated to $\varepsilon = +1$, $\varepsilon = 0$ and $\varepsilon = -1$.

In order for f to be bounded on the line $u = 0$ we must impose $S = \log u + o(u)$, so that the first of equations (5.42) reads:

$$1 = -\frac{1}{4} \varepsilon (\varepsilon u^2 + v^2) \Phi_u^2 \quad \text{as } u \rightarrow 0, \quad (5.43)$$

which admit solution only if ε is not positive. Thus, the matter-free prescriptions (5.40) cannot be satisfied.

The above argument can be generalized to the non-diagonal case. The EFE's in this case are given by (see [11] eqn. (7.9) without electromagnetic field):

$$S_b = -\frac{a+b}{2} [\Phi_b^2 \cosh^2 W + W_b^2], \quad S_a = -\frac{a+b}{2} [\Phi_a^2 \cosh^2 W + W_a^2], \quad (5.44)$$

where Φ and W are now defined by $g_{11} = \alpha e^\Phi \cosh W$ and $g_{12} = \alpha \sinh W$. The terms in the square brackets being positive, the argument given previously is not affected.

As a result, we cannot make a matter-free extension using the coordinates presented in (5.40), which matches two separate regions. Extensions can only be made if, after having removed the singularity, we can attach to Region I any region, call it \mathcal{A} , defined by one of the two following:

$$\begin{aligned} 1) \text{ Region } \mathcal{A} \quad (\varepsilon = -1) \quad & \alpha = \frac{1}{\sqrt{2}} (-u^2 + b(v)) \quad \beta = \frac{1}{\sqrt{2}} (b(v) + u^2) \\ 2) \text{ Region } \mathcal{A} \quad (\varepsilon = 0) \quad & \alpha = \frac{1}{\sqrt{2}} b(v) \quad \beta = \frac{1}{\sqrt{2}} b(v) \end{aligned} \quad (5.45)$$

A similar extension is required from Region II, although both regions need not be part of the same spacetime.

5.3.2 2-soliton solution

We are now interested in considering the possible extensions of the 2-soliton solution.

Matter extension

A matter extension is naturally achieved by matching the 2-soliton solution with the 1-Solitons in Regions I and II and the Kasner metric in Region III. The appropriate prescriptions – see figure (5.3) – are given by:

$$\begin{aligned}
 \text{Region IV} \quad a^{(IV)} &= -\Theta - u^2 & b^{(IV)} &= +\Theta + (v - c_{II}) \\
 \text{Region I} \quad a^{(I)} &= +\Theta - \frac{2\Theta}{c_I^2} (c_I - u)^2 & b^{(I)} &= +\Theta + (v - c_{II}) \\
 \text{Region III} \quad a^{(III)} &= +\Theta + (u - c_I) & b^{(III)} &= +\Theta + (v - c_{II}) \\
 \text{Region II} \quad a^{(II)} &= +\Theta + (u - c_I) & b^{(II)} &= +\Theta - \frac{2\Theta}{c_{II}^2} (c_{II} - v)^2 \\
 \text{Region V} \quad a^{(V)} &= +\Theta + (u - c_I) & b^{(V)} &= -\Theta - v^2 \\
 \text{Region VI} \quad a^{(VI)} &= +\Theta - \frac{2\Theta}{c_I^2} (c_I - u)^2 & b^{(VI)} &= +\Theta - \frac{2\Theta}{c_{II}^2} (c_{II} - v)^2
 \end{aligned} \tag{5.46}$$

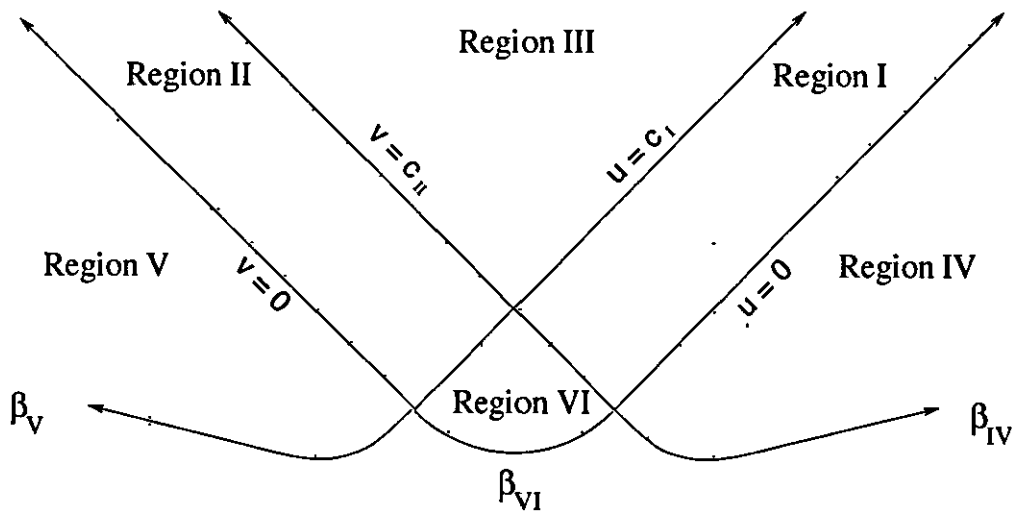


Figure 5.3: The picture shows the parametrization of the spacetime, containing a soliton with two real pole, in terms of the coordinates u, v . These are the coordinates that remove the singularities into the metric components of the soliton metric.

Clearly, the discontinuities in the derivatives of α across the boundaries of indicate the presence of sheets of null matter on those hypersurfaces.

No-matter extension

Let us now consider the possibility of obtaining a matter-free extension. The discussion put forward in section 5.3.1 tells us how to perform a matter free extension

between regions I and III. This is accommodated by the coordinate transformations in (5.47b), (5.47c):

$$\text{Region IV} \quad \begin{cases} a^{(IV)} = -\Theta - u^2 \\ b^{(IV)} = +\Theta + (v - c_{II})^2 \end{cases} \quad (5.47a)$$

$$\text{Region I} \quad \begin{cases} a^{(I)} = +\Theta - \frac{2\Theta r_I}{c_I^2} (c_I - u)^2 - \frac{2\Theta w_I}{c_I^2} (c_I - u)^{s_I} \\ b^{(I)} = +\Theta + (v - c_{II})^2 \end{cases} \quad (5.47b)$$

$$\text{Region III} \quad \begin{cases} a^{(III)} = +\Theta + (u - c_I)^2 \\ b^{(III)} = +\Theta + (v - c_{II})^2 \end{cases} \quad (5.47c)$$

$$\text{Region II} \quad \begin{cases} a^{(II)} = +\Theta + (u - c_I)^2 \\ b^{(II)} = +\Theta - \frac{2\Theta r_{II}}{c_{II}^2} (c_{II} - v)^2 - \frac{2\Theta w_{II}}{c_{II}^2} (c_{II} - v)^{s_{II}} \end{cases} \quad (5.47d)$$

$$\text{Region V} \quad \begin{cases} a^{(V)} = +\Theta + (u - c_I)^2 \\ b^{(V)} = -\Theta - v^2 \end{cases} \quad (5.47e)$$

$$\text{Region VI} \quad \begin{cases} a^{(VI)} = +\Theta - \frac{2\Theta r_I}{c_I^2} (c_I - u)^2 - \frac{2\Theta w_I}{c_I^2} (c_I - u)^{s_I} \\ b^{(VI)} = +\Theta - \frac{2\Theta r_{II}}{c_{II}^2} (c_{II} - v)^2 - \frac{2\Theta w_{II}}{c_{II}^2} (c_{II} - v)^{s_{II}} \end{cases} \quad (5.47f)$$

In (5.47) we will assume s_I and s_{II} to be arbitrary real numbers and:

$$r_I = 1 - (-1)^{s_I} w_I, \quad w_I = (-1)^{-s_I+1} \frac{2}{s_I - 2}, \quad s_I > 2, \quad (5.48a)$$

$$r_{II} = 1 - (-1)^{s_{II}} w_{II}, \quad w_{II} = (-1)^{-s_{II}+1} \frac{2}{s_{II} - 2}, \quad s_{II} > 2. \quad (5.48b)$$

Notice that, with respect of the prescriptions given in (5.40), an extra term, proportional to $(c_I - u)^{s_I}$, has been added to the expression for $a^{(I)}$. Naturally, that does not modify all the considerations made in section 5.3.1: the α derivatives across the boundary between regions I and III are still continuous. The reason for including such an extra term is that it might help in obtaining a matter-free extension between regions I and IV. Indeed, the relations in (5.48) provide the right parameters such that α_I and α_{IV} continuously match across the hypersurface $u = 0$. However, it must be noticed that the derivatives of α are not continuous across $u = 0$. Analogous considerations obviously apply for the extensions made from Region V. This sheets of null matter necessarily occur on the junctions between the 1-soliton and 2-soliton region. As a result, the 2-soliton solution cannot admit a matter-free extension. This result obviously generalizes to the N -Soliton solution.

Bibliography of Files: [21, 22, 23, 24], [25, 26]

5.4 A soliton extension

The choice $\varepsilon = -1$ discussed in (5.45) has been firstly analyzed by Gleiser in [137], although in that paper solitons were only generated from a Minkowski Seed. Let us

now consider:

$$\begin{aligned} \text{Region I} \quad \alpha_I &= \frac{1}{\sqrt{2}} (-u^2 + v^2), \quad \beta_{Is} = \frac{1}{\sqrt{2}} (v^2 + u^2), \\ \text{Region } \mathcal{A} \quad \alpha &= \frac{1}{\sqrt{2}} (-u^2 + v^2), \quad \beta = \frac{1}{\sqrt{2}} (v^2 + u^2), \end{aligned} \quad (5.49)$$

where $\beta_{Is} = \beta - z_{0[1]}$. Notice here that even in Region \mathcal{A} we have $\beta \geq \alpha$: therefore soliton solutions are well-defined in this region

It is now appropriate to consider the further coordinate transformation [137]:

$$\begin{aligned} (u, v) &\mapsto (\tau, \rho), \quad \tau = \frac{1}{\sqrt{2}} (u + v), \quad \rho = \frac{1}{\sqrt{2}} (v - u), \\ (F^{(I)}, G_{ab}^{(I)}) &\mapsto (\tilde{F}^{(I)}, \tilde{G}_{ab}^{(I)}), \quad \alpha_I = \sqrt{2} \rho \tau, \\ &\quad \beta_{Is} = \frac{1}{\sqrt{2}} (\tau^2 + \rho^2), \end{aligned} \quad (5.50)$$

$$\tilde{F}^{(I)} = \frac{|(u, v)|}{|(\tau, \rho)|} F^{(I)}, \quad \frac{|(u, v)|}{|(\tau, \rho)|} = -1.$$

Coordinates u and v are null-coordinates. For these solutions we have the remarkable simplification: $\mu = \tau^2$.

Soliton solutions (5.13) and (5.14) are well defined in Region I. For the diagonal case, from (5.18) we have:

$$F^{(I)} = 2^{-1/2+2q-q^2} A_I z_{0[1]}^{2q-1} (v-u)^{4q} (v^2-u^2)^{2q(q-1)}, \quad (5.51a)$$

$$G_{11}^{(I)} = 2^{q-1/2} (u+v)^2 (v^2-u^2)^{-2q}, \quad (5.51b)$$

$$\tilde{F}^{(I)} = 2^{-1/2+2q+q^2} A_I z_{0[1]}^{2q-1} \tau^{2q(q+1)} \rho^{2q(q-1)}, \quad (5.51c)$$

$$\tilde{G}_{11}^{(I)} = 2^{1/2-q} \rho^{2(1-q)} \tau^{-2q}. \quad (5.51d)$$

Notice that this solution suffers a singularity on $\tau = 0$. This is the usual cosmological-like singularity. It disappears in the case $q = 0$, which is a Minkowski space generated from a plane symmetric Kasner seed. The asymptotic behaviour of the Weyl tensor components near the null line $u = 0$ is given by:

$$\Psi_0^{(I)} \approx - \frac{2^{5/2-2q+q^2} z_{0[1]}^{1-2q}}{A_I} q (4q^2 - 1) v^{-2-4q^2}, \quad (5.52a)$$

$$\Psi_2^{(I)} \approx - \frac{2^{5/2-2q+q^2} z_{0[1]}^{1-2q}}{A_I} q v^{-2-4q^2}, \quad (5.52b)$$

$$\Psi_4^{(I)} \approx - \frac{2^{5/2-2q+q^2} z_{0[1]}^{1-2q}}{A_I} 3 q v^{-2-4q^2}. \quad (5.52c)$$

Incidentally, notice that for $q = 0$, the 1-soliton solution is just a (flat) Minkowski spacetime.

Region \mathcal{A} satisfies the condition $\beta \geq \alpha$, and we can consider in it another soliton solution, as pointed out in [137]. Let us consider the non-diagonal solution (5.13) in

which we now put $A_I = -A_A$, $\alpha_I = \alpha$ and $\beta_{Is} = \beta$. Then we apply to this solution the two successive transformations:

$$\begin{aligned}
 (\alpha, \beta) &\mapsto (u, v), & u, v &\geq 0 \\
 (f, g_{ab}) &\mapsto (F^{(A)}, G_{ab}^{(A)}), & \alpha &= \frac{1}{\sqrt{2}} (-u^2 + v^2), \\
 & & \beta &= \frac{1}{\sqrt{2}} (v^2 + u^2), \\
 (u, v) &\mapsto (\rho, \tau), & \tau &= \frac{1}{\sqrt{2}} (v - u), \\
 & & \rho &= \frac{1}{\sqrt{2}} (u + v), \\
 (F^{(A)}, G_{ab}^{(A)}) &\mapsto (\tilde{F}^{(A)}, \tilde{G}_{ab}^{(A)}), & \alpha &= \sqrt{2} \rho \tau, \\
 & & \beta &= \frac{1}{\sqrt{2}} (\tau^2 + \rho^2).
 \end{aligned} \tag{5.53}$$

Notice that in the above definition ρ and τ are interchanged with respect of (5.50). The new metric coefficients are given by:

$$\tilde{F}^{(A)} = \frac{2^{-1/2-2q+q^2} A_A}{z_0[1]^{1+2q}} \rho^{2q(q-1)} \tau^{2q(q-1)} \left(m_A^2 \tau^{4q} + 2^{4q} z_0[1]^{4q} \rho^{4q} \right), \tag{5.54a}$$

$$\tilde{G}_{11}^{(A)} = 2^{1/2-q} m_A z_0[1]^{2q} \rho^{-2q} \tau^{-2q} \frac{m_A^2 \tau^{4q} \rho^2 + 2^{4q} z_0[1]^{4q} \rho^{4q}}{m_A^2 \tau^{4q} + 2^{4q} z_0[1]^{4q} \rho^{4q}}, \tag{5.54b}$$

$$\tilde{G}_{12}^{(A)} = -2^{1/2+2q} m_A z_0[1]^{2q} \rho^{2q} \tau^{2q} \frac{(\rho^2 - \tau^2)}{m_A^2 \tau^{4q} + 2^{4q} z_0[1]^{4q} \rho^{4q}}. \tag{5.54c}$$

It is trivial to check that (5.54) matches continuously with solution (5.51) provided $A_A = A_A(1 + 2^{-4q} z_0[1]^{-4q} m_A^2)$. In this case, the Ricci tensor vanishes across the junction, although there may occur impulsive components in the Weyl tensor [147].

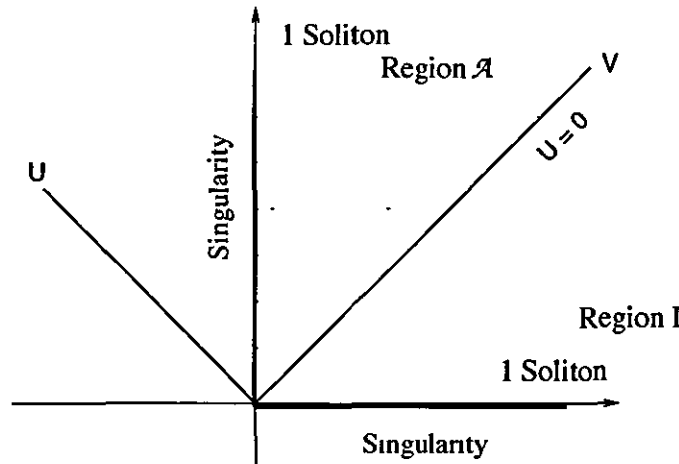


Figure 5 4: Matching of a 1-soliton solution with another soliton solution.

Finally, notice that solution (5.54) suffers a singularity on $\rho = 0$. This prevents us extending this compound solution to other regions of the spacetime. In fact, $\rho = 0$ corresponds to a singular axis of symmetry, as is evident in the case when $q = 0$.

5.4.1 A Physical Interpretation: impulsive spherical wave.

When the Seed is a plane symmetric (type D) Kasner metric ($q = 0$), a simple physical interpretation for the matter-extended solution can be given. Let us consider now the non diagonal solution (5.13) and the coordinate transformation (5.40). By introducing the new coordinates ρ and τ defined as:

$$\tau = \frac{1}{\sqrt{2}} (u + v), \quad \rho = \frac{1}{\sqrt{2}} (v - u), \quad (5.55)$$

so that $\alpha = \frac{1}{\sqrt{2}}(v^2 - u^2) = \frac{1}{\sqrt{2}} \rho \tau$, the 1-soliton solution (5.13) can finally be rewritten as:

$$F^{(I)} = \frac{A_I}{z_{0[I]}} \frac{1 + m^2}{\sqrt{2}}, \quad G_{12}^{(I)} = \frac{\sqrt{2}}{1 + m^2} m (\tau^2 - \rho^2), \quad (5.56a)$$

$$G_{11}^{(I)} = \frac{\sqrt{2}}{1 + m^2} (\rho^2 + m^2 \tau^2), \quad G_{22}^{(I)} = \frac{\sqrt{2}}{1 + m^2} (\tau^2 + m^2 \rho^2). \quad (5.56b)$$

A matter-extension may be achieved by matching the above solution with a Kasner metric. In terms of the (τ, ρ) coordinates this is given by:

$$ds_{III}^2 = d\tau^2 - d\rho^2 - \tau^2 dz^2 - \rho^2 d\phi^2, \quad x^1 = z, \quad x^2 = \phi. \quad (5.57)$$

After introducing the cartesian coordinates:

$$T = \tau \cosh z, \quad Z = \tau \sinh z, \quad X = \rho \cos \phi, \quad Y = \rho \sin \phi, \quad (5.58)$$

it is clear that (5.57) is the region of the Minkowski spacetime $ds^2 = dT^2 - dZ^2 - dX^2 - dY^2$ for which $|T| \geq |Z|$.

In this case it can be seen that the junction between the two regions, which is the null hypersurface $\tau = \rho$, is an expanding sphere given by $T^2 - Z^2 - X^2 - Y^2 = 0$. The solution therefore describes a gravitational wave with an exact spherical wavefront propagating into a Minkowski background. The axis $\rho = 0$ is now an axis of symmetry, so we do not need to consider the region $\rho < 0$.

It may also be observed that, by appropriately choosing A_I , the metric (5.56) may be rewritten in the form

$$ds_I^2 = \frac{A_I}{z_{0[I]}} \frac{2}{(1 + m^2)^2} (d\tau^2 - d\rho^2) - \left(\tau^2 (m dz - d\phi)^2 + \rho^2 (dz - m d\phi)^2 \right), \quad (5.59)$$

which is clearly simply a rotation of the flat metric (5.57), up to a rescaling of the variables. Thus, the spacetime both inside and outside the spherical wavefront is flat. The only possible nonzero components of the Weyl Tensor arise from the discontinuities in the derivatives of the metric components across the wavefront.

The solution described above therefore represents an impulsive spherical wave propagating in a Minkowski background. This is equivalent to the impulsive spherical wave that was obtained in [143] using a "cut-and-paste" method.

It may also be observed that $\tau = 0$ is simply a coordinate singularity. It is thus possible to add the Minkowski region in which $0 \leq T < |Z|$, and then to add the time-reverse of the solution for $T < 0$ and $\tau < 0$. The global solution then describes a contracting impulsive gravitational wave in Minkowski background.

which collapses to a point and then re-expands as an exact spherical impulsive wave. The only singularity occurs at the event at which the spherical wave has zero radius.

It may also be pointed out that a similar singularity-free solution – 1-soliton solution generated from a Bianchi VI₀ Seed – has been obtained in [105]. In that paper the soliton perturbation is described as “erasing” the “cosmological” singularity that occurs in the seed

Bibliography of Files [29, 30]

5.5 A General Diagonal Extension

Let us now devote our attention to a diagonal 1-Soliton in Region \mathcal{A} ($m_{\mathcal{A}} \rightarrow 0$). In this case the 1-soliton solution becomes:

$$g_{11} = \alpha e^{+\Phi^{(sol)}}, \quad g_{22} = \alpha e^{-\Phi^{(sol)}}, \quad \Phi^{(sol)} = -2q \log \alpha + \text{ArcCosh} \frac{\beta_s}{\alpha}.$$

For simplicity we have dropped any index. Now $\Phi^{(sol)}$ is solution for a linear equation. Then the natural question is: does there exist any other solution which may be linearly superposed to this one?

We can consider for our purposes another class of solutions presented in [145, 146, 147]. In these solutions Φ assumes the form $\Phi^{(AG)} = \sum_k c_k \alpha^k H_k(\xi)$, where $\xi = \beta/\alpha$. In terms of α and ξ the EFE's are given by:

$$f = \frac{1}{2\sqrt{\alpha}} e^{-S}, \quad S_{\alpha} = -\frac{\alpha}{2} \Phi_{\alpha}^2 - \frac{1}{2\alpha} (1 - \xi^2) \Phi_{\xi}^2, \quad S_{\xi} = -\alpha \Phi_{\alpha} \Phi_{\xi} + \xi \Phi_{\xi}^2 \quad (5.60)$$

$$g_{11} = \alpha e^{+\Phi}, \quad (1 - \xi^2) H_k'' + (2k - 1) \xi H_k' - k^2 H_k.$$

For integer k , all the solutions can be iteratively obtained starting from the H_0 solution and using the iterating formula

$$H_n(\xi) = \int_1^{\xi} d\xi' H_{n-1}(\xi'). \quad (5.61)$$

The f metric coefficient corresponding to the most general $\Phi^{(AG)}$ can be easily found by using a procedure illustrated, for example, in [147] – formula (4.16).

It is important here to emphasize that all the $\Phi^{(AG)}$'s vanish on the lines $\alpha = \pm \beta$ (or $\xi = \pm 1$), which makes them suitable for our purposes. Thus, we can generalize the extension (5.54) by considering the following solution:

$$\text{Region I} \quad \Phi^A = -2q \log \alpha_A + \text{ArcCosh} \frac{\beta_{As}}{\alpha_A}, \quad (5.62a)$$

$$\text{Region A} \quad \Phi^A = -2q \log \alpha + \sum_{k=0}^N c_k \alpha^k H_k. \quad (5.62b)$$

The choice of α and β_s we made in (5.49) forces us to choose the H_k series generated starting from $H_0(\xi) = \text{ArcCosh}(\xi)$, where $\xi > 1$.

By rewriting Φ^A in the form

$$\Phi^A = \Phi^{(sol)} + \Phi^{(AG)}, \quad (5.63a)$$

$$\Phi^{(sol)} = -2q \log \alpha + \text{ArcCosh} \frac{\beta_s}{\alpha}, \quad \Phi^{(AG)} = \sum_{k=1}^N c_k \alpha^k H_k, \quad (5.63b)$$

and substituting (5.63) into (5.60), we have that the function S in Region \mathcal{A} – call it S_A – can now be split into three components $S_A = S^{(1)} + S^{(2)} + S^{(3)}$ satisfying the following equations:

$$S_\alpha^{(1)} = -\frac{\alpha}{2} (\Phi_\alpha^{(sol)})^2 - \frac{1}{2\alpha} (1 - \xi^2) (\Phi_\xi^{(sol)})^2, \quad (5.64a)$$

$$S_\xi^{(1)} = -\alpha \Phi_\alpha^{(1)} \Phi_\xi^{(sol)} + \xi (\Phi_\xi^{(sol)})^2, \quad (5.64b)$$

$$S_\alpha^{(2)} = -\frac{\alpha}{2} (\Phi_\alpha^{(AG)})^2 - \frac{1}{2\alpha} (1 - \xi^2) (\Phi_\xi^{(AG)})^2, \quad (5.64c)$$

$$S_\xi^{(2)} = -\alpha \Phi_\alpha^{(AG)} \Phi_\xi^{(AG)} + \xi (\Phi_\xi^{(AG)})^2, \quad (5.64d)$$

$$S_\alpha^{(3)} = -\alpha \Phi_\alpha^{(sol)} \Phi_\alpha^{(AG)} - \frac{1}{\alpha} (1 - \xi^2) \Phi_\xi^{(sol)} \Phi_\xi^{(AG)}, \quad (5.64e)$$

$$S_\xi^{(3)} = -\alpha (\Phi_\alpha^{(sol)} \Phi_\xi^{(AG)} + \Phi_\alpha^{(AG)} \Phi_\xi^{(sol)}) + 2\xi \Phi_\xi^{(sol)} \Phi_\xi^{(AG)}. \quad (5.64f)$$

The $S^{(2)}$ component is obviously that associated with the bare $\Phi^{(AG)}$ and, as we mentioned above, can be found in [146]:

$$S^{(2)} = -\sum_{k=2}^{\infty} \frac{1}{2k} t^k K_k, \quad (5.65a)$$

$$K_k(\xi) = \sum_{j=1}^{k-1} c_j c_{k-j} [j(k-j)H_j H_{n-j} + (1 + \xi^2) H'_j H'_{n-j}]. \quad (5.65b)$$

The $S^{(1)}$ component is such that $F^{(I)}$ from (5.51) satisfies $F^{(I)} = 1/(2\sqrt{\alpha}) e^{-S^{(1)}}$. The $S^{(3)}$ component is new. Its evaluation can be performed in the general case with the result:

$$S^{(3)} = \bar{S} + \sum_{k=0}^{\infty} d_k \alpha^k G_k(\xi), \quad \bar{S} = c_0 \log \alpha, \quad (5.66a)$$

$$G_k(\xi) = 2q H_k(\xi) + \frac{1}{k} \sqrt{\xi^2 - 1} H_k(\xi)', \quad d_k = c_k \quad \forall k \geq 1, \quad (5.66b)$$

$$G_0(\xi) = \log(\xi^2 - 1) + 2q \log(\xi + \sqrt{\xi^2 - 1}), \quad d_0 = c_0 \quad (5.66c)$$

Some restrictions have to be imposed to the coefficients c_k in order to get bounded expressions both in the metric components and in the Weyl tensor. By using only the fact that all the $H_k(\xi)$ satisfy $H_k(1) = 0$, we can see that $S^{(2)}(\xi) \approx 0$ and $S^{(3)}(\xi) \approx \log(\xi - 1)$ as $\xi \rightarrow 1$. That implies:

$$e^{-S^{(2)}} \approx 1, \quad e^{-S^{(3)}} \approx (\xi - 1)^{-c_0}, \quad \text{as } \xi \rightarrow 1. \quad (5.67)$$

The f coefficient associated with (5.62b) is given by $f^{(A)} = F^{(I)} e^{-S^{(2)}} e^{-S^{(3)}}$ and therefore it is bounded only if $c_0 = 0$. This is the only restriction arising from the

new component $S^{(3)}$. Indeed, it is not a restriction, because we already knew [147] that $k \geq 1/2$ in order to prevent the appearance of step terms in the Weyl tensor components

Finally we mention that the solution presented here can be further generalized by introducing [147]:

$$\Phi^{(AG)} = \int_{1/2}^{\infty} \phi(k) \alpha^k H_k(\xi) \quad (5.68)$$

where now $\phi(k)$ is a real function of the real variable $k \geq 1/2$. Being $\phi(k)$ completely undetermined, this form of $\Phi^{(AG)}$ provides quite a general form for the solution in Region \mathcal{A} .

Bibliography of Files [31]

5.6 A Plane Wave Extension

The second extension we present here is that which originates from the choice $\varepsilon = 0$ discussed above. It is realized by attaching to the soliton region another one in which α does not depend upon u .

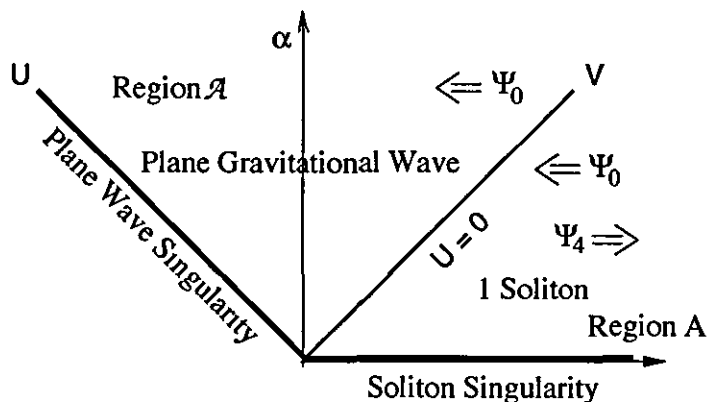


Figure 5.5: Matching of a 1-soliton solution with a Plane Gravitational Wave

Let us now consider:

$$\begin{aligned} \text{Region I} \quad \alpha_I &= \frac{1}{\sqrt{2}} (-u^2 + b(v)), & \beta_{I,} &= \frac{1}{\sqrt{2}} (b(v) + u^2), \\ \text{Region A} \quad \alpha &= \frac{1}{\sqrt{2}} b(v), & \beta &= \frac{1}{\sqrt{2}} b(v), \end{aligned} \quad (5.69)$$

where $b(v)$ may assume the general form $b(v) = v^k$. In terms of these coordinates the metric (5.14) becomes

$$F^{(I)} = \frac{2^{-3/2+2q-q^2} A_I k}{z_0[1]^{1-2q}} v^{1/2(k-2)} (v^{k/2} - u)^{4q} (v^k - u^2)^{2q(q-1)}, \quad (5.70a)$$

$$G_{11}^{(I)} = 2^{q-1/2} (u + v^{k/2})^2 (v^k - u^2)^{-2q}. \quad (5.70b)$$

The asymptotic behaviour of the Weyl tensor components near the line $u = 0$ is given by:

$$\Psi_0^{(I)} \approx - \frac{2^{3/2-2q+q^2} z_0[1]^{1-2q}}{A_I} k q (4q^2 - 1) v^{-1-k/2-2kq^2}, \quad (5.71a)$$

$$\Psi_2^{(I)} \approx - \frac{2^{5/2-2q+q^2} z_0[1]^{1-2q}}{A_I} q v^{-k(1+2q^2)}, \quad (5.71b)$$

$$\Psi_4^{(I)} \approx - \frac{2^{7/2-2q+q^2} z_0[1]^{1-2q}}{A_I} 3 q k^{-1} v^{1-k/2(3+4q^2)}. \quad (5.71c)$$

Let us consider the diagonal metric:

$$ds_A^2 = 2e^{-M_A} du dv - e^{-U_A} (e^{+\Phi_A} dx^2 + e^{-\Phi_A} dy^2), \quad (5.72)$$

in which all the coefficients depend only upon v . In order to continuously match (5.72) with (5.70), we have to choose:

$$\Phi_A = -2qk \log v + q \log 2, \quad e^{-U_A} = \frac{1}{\sqrt{2}} v^k, \quad (5.73a)$$

$$e^{-M_A} = 2^{-3/2+2q-q^2} A_1 k z_0[1]^{2q-1} v^{1/2(-2+k+4kq^2)}. \quad (5.73b)$$

The only non vanishing Weyl Tensor Component is:

$$\Psi_0^{(A)} = - \frac{2^{1/2-2q+q^2} z_0[1]^{1-2q}}{A_I} q (4q^2 - 1) k v^{-1-k/2-2kq^2}, \quad (5.74)$$

which is continuously matchable with (5.71a) for any k . Finally, notice that (5.73) suffer a singularity on $v = 0$, which prevents us from further extending this solution to the whole spacetime. The existence of this singularity is well known. It is the time inverse of that which appears in the plane wave regions of colliding plane wave spacetimes as described in [144] – see also [11], chapter 8.

Bibliography of Files: [32]

5.6.1 Reduction to the standard plane wave metric form

In order to show that solution (5.73) exists and to clarify its physical meaning, let us now perform the coordinate transformation defined by:

$$d\tilde{v} = e^{-M} dv, \quad \tilde{v} = \bar{\Gamma} v^{\frac{k}{2}(1+4q^2)}, \quad \bar{\Gamma} = \frac{2}{k(1+4q^2)} \Gamma, \quad \Gamma = \frac{2^{-3/2+2q-q^2} A_1 k}{z_0[1]^{1-2q}}.$$

Now the metric (5.73) becomes:

$$ds^2 = 2 du d\tilde{v} - (\tilde{g}_{11} dx^2 + \tilde{g}_{22} dy^2), \quad \tilde{g}_{11} = 2^{q-1/2} \bar{\Gamma}^{-\frac{2(1-2q)}{1+4q^2}} \tilde{v}^{\frac{2(1-2q)}{1+4q^2}}, \quad (5.75a)$$

$$\tilde{g}_{22} = 2^{-q-1/2} \bar{\Gamma}^{-\frac{2(1+2q)}{1+4q^2}} \tilde{v}^{\frac{2(1+2q)}{1+4q^2}}. \quad (5.75b)$$

Let us now make this second coordinate transformation:

$$x = \frac{1}{\sqrt{\gamma_1}} \tilde{v}^{-a} X, \quad y = \frac{1}{\sqrt{\gamma_2}} \tilde{v}^{-b} Y, \quad u = R - \frac{a}{2\tilde{v}} X^2 - \frac{b}{2\tilde{v}} Y^2, \quad (5.76)$$

$$a = \frac{(1-2q)}{1+4q^2}, \quad b = \frac{(1+2q)}{1+4q^2}, \quad \gamma_1 = 2^{q-1/2} \bar{\Gamma}^{-2a}, \quad \gamma_2 = 2^{-q-1/2} \bar{\Gamma}^{-2b}.$$

Now the metric (5.75) becomes:

$$ds^2 = 2 dR d\tilde{v} - (dX^2 + dY^2) - H(\tilde{v}) (X^2 - Y^2) d\tilde{v}^2, \quad (5.77a)$$

$$H(\tilde{v}) = 2 \frac{1}{\tilde{v}^2} \frac{q(1 - 4q^2)}{(1 + 4q^2)^2}. \quad (5.77b)$$

Formulae (5.77) represents the standard line element for a plane gravitational wave [11]. The function $H(\tilde{v})$ represents the Weyl tensor component Ψ_0 , i.e. a gravitational wave propagating in the negative- β direction. Moreover, in this case the amplitude profile function H becomes unbounded as $\tilde{v} \rightarrow 0$.

5.7 Conclusions

The discussion put forward so far, yields the conclusion that soliton solutions with real poles represent only part of a complete spacetime. Some extensions introduce the appearance of thin sheets of matter. According to the argument we gave in section 5.3.1, there exists a large family of possible matter-free extensions. The resulting spacetime is then a compound of two regions one of which contains the soliton solution and the other may contain either another soliton – with, possibly, different parameters – or a PGW, or a more general non-soliton solution.

The possible extensions can be characterized by their physical content as follows region I contains two gravitational waves described by Ψ_0 and Ψ_4 . They may be supposed to be both generated at the cosmological singularity $\alpha = 0$. The wave described by Ψ_4 propagates in the positive β -direction. The wave described by Ψ_0 keeps propagating in the negative β -direction and will necessarily pass out of the soliton region.

The particular extension to the region beyond $u = 0$ will depend on the possible Ψ_4 component in that region. If $\Psi_4 = 0$ here, the extension is simply that of a plane wave. If the soliton extension is taken, then a particular gravitational wave will occur that propagates behind the junction, which forms a wavefront for this component. A more general gravitational wave has also been given for the diagonal case, in which it may also be noted that the initial component H_0 is just the 1-soliton solution.

Chapter 6

Real Pole solitons in the Alekseev technique

The Alekseev soliton technique generates solutions of the Einstein–Maxwell equations. However, in the case in which the electromagnetic field vanishes, these electromagnetic solitons must be purely gravitational, and may be equivalent to those that can be obtained using the Belinski–Zakharov technique.

It is known that the 1-soliton solution in the Alekseev formalism is equivalent to the 2-soliton solution in the BZ formalism [15], and that the 2-soliton solution in the Alekseev formalism is equivalent to the 4-soliton solution in the BZ formalism [51]. No general explicit relation exists, as far as we know, for solutions with an arbitrary number of poles, unless diagonal metrics are considered. Moreover, the above correspondences are true only when complex poles, in both the formalisms, are chosen

This situation deserves a few comments:

- it is well known that the Alekseev technique can not provide any solution when $N = 1$ and the poles $w[k]$ and $\nu[k] = \bar{w}[k]$ are real. In fact, we would have a vanishing denominator in (2.155).
- given a 1-pole BZ-soliton it is not clear at all whether there is a corresponding solution in the Alekseev formalism. Moreover, an N -pole BZ-soliton, with N an odd number, does not seem to have any correspondence in the Alekseev formalism.

In order to complete the picture, we might here outline another problem that we have to deal with when interested in the relation between Alekseev's and Belinski–Zakharov's solitons. Let us consider an N -pole diagonal vacuum-soliton in the Alekseev formalism and also suppose that all the poles are complex ($N > 1$). As mentioned above, it can be shown that this solution is equivalent to the $2N$ -complex pole diagonal soliton in the BZ formalism. In the Alekseev formalism, the N -soliton solution is obtained by introducing N poles in the dressing matrix S . However, an additional N distinct poles have to be introduced into S^{-1} . In the BZ formalism, all the $2N$ poles occur in the dressing matrix. This explains the apparent doubling of poles.

Let us now consider the limit when all these poles become real. As we showed in chapter 3 for the diagonal case, the limit does not correspond to real (simple)-pole

BZ-solitons. We will have instead a $N/2$ soliton solution with real poles, each of which have degeneracy $h = 2$.

This simple argument seems to show that the class of Alekseev's vacuum-solitons is smaller than that of Belinski-Zakharov. As far as we can understand, the problems sketched above address two main questions:

1. is there a possible way to incorporate real-pole BZ-solitons within the framework of the Alekseev technique?
2. is it possible to obtain solitons with an odd number of poles by using (*a modified version of*) the Alekseev technique?

These are the questions we will try to answer in this chapter.

6.1 A Generalized Construction for Real Poles solitons

Let us consider the Alekseev linear pair (2.112). As we showed in section 2.3, a new solution for the EFE is given in terms of a previously-known solution by introducing the Dressing Ansatz (2.133). Equations (2.153), (2.155) and (2.163) provide us with an explicit N -poles solution for that.

In order to obtain (2.155) and (2.163), no assumption has been made on the nature of the poles $w_{[k]}$ in \mathbf{S} and $\nu_{[k]}$ in \mathbf{S}^{-1} . We only assumed that they are simple. However, in the Alekseev technique $\nu_{[k]}$ and $w_{[k]}$ are taken to be complex-conjugate to each other.

It is very simple to trace back where this condition came from: it is simply related to the fact that we assumed $\mathbf{K}(w) = \mathbf{K}_0(w)$ in (2.138), (2.139) – see *Proposition (2.3.17)* and *Proposition (2.3.18)*. In fact, let us consider the matrix function $\tilde{\mathbf{W}}$ defined as:

$$\begin{aligned} \mathbf{K}(w) &= \Psi^\dagger \mathbf{W} \Psi, \\ \tilde{\mathbf{W}} &\equiv \mathbf{S}^\dagger \mathbf{W} \mathbf{S}, \quad \mathbf{K}(w) = \Psi_0^\dagger \tilde{\mathbf{W}} \Psi_0. \quad (6.1) \\ \mathbf{K}_0(w) &= \Psi_0^\dagger \mathbf{W}_0 \Psi_0, \end{aligned}$$

Obviously, if we choose $\mathbf{K} = \mathbf{K}_0$ then $\tilde{\mathbf{W}} = \mathbf{W}_0$ and the above definition coincides with (2.139). Let us now consider (6.1) rewritten in the form $\tilde{\mathbf{W}} \mathbf{S}^{-1} = \mathbf{S}^\dagger \mathbf{W}$. We have:

$$\tilde{\mathbf{W}} + \sum_{k=1}^N \frac{1}{w - \nu_{[k]}} \tilde{\mathbf{W}} \mathbf{Q}_{[k]} = \mathbf{W} + \sum_{k=1}^N \frac{1}{w - \bar{w}_{[k]}} \mathbf{R}_{[k]}^\dagger \mathbf{W}. \quad (6.2)$$

If $\mathbf{K} = \mathbf{K}_0$ then both \mathbf{W} and $\tilde{\mathbf{W}}$ are linear in w . Therefore, in order to have the same pole structure in both members of the above equation, we have to set $\bar{w}_{[k]} = \nu_{[k]}$. However, if we relax that assumption, then $\tilde{\mathbf{W}}$ will no longer be linear in w and then it may contain poles. We might thus be able to retain $\bar{w}_{[k]} \neq \nu_{[k]}$.

In this section we will show that this second approach yields to a positive answer to question 1 above.

6.1.1 The 1-soliton case

Let us consider the dressing matrices (2.153) for a single real pole:

$$S = I + \frac{1}{w - w_{[1]}} R, \quad S^{-1} = I + \frac{1}{w - w_{[1]}} Q, \quad (6.3a)$$

$$R = n \otimes m, \quad Q = p \otimes q, \quad w_{[1]}, w_{[1]} \in \mathbb{R}, \quad w_{[1]} \neq w_{[1]}, \quad (6.3b)$$

where n, m, p, q are given in (2.155) and (2.163). In order to show that such a solution can exist we will have to prove that the function $K(w)$ defined above is a function of the spectral parameter only.

Proposition (6.1.1) Pole structure for \tilde{W}

Let us consider S and S^{-1} as in (6.3). Then \tilde{W} has at most simple poles.

Proof

Let us consider the w -derivative of \tilde{W} . By using the fact that W is linear in w , we can conclude that $\partial_w \tilde{W}$ may contain at most second order poles. Hence \tilde{W} may have only first order poles. \square q.e.d.

Proposition (6.1.2) Pole structure for \tilde{W}^{-1}

Let us consider S and S^{-1} as in (6.3). Then \tilde{W}^{-1} has at most simple poles.

Proof

Let us consider (6.1) rewritten in the form

$$\tilde{W}^{-1} S^\dagger W = S^{-1}. \quad (6.4)$$

The proposition is most easily proven by substituting (6.3) into (6.4) and by analyzing the pole structure of the two members of the resulting equation. \square q.e.d.

Proposition (6.1.3) Explicit expression for \tilde{W}

Let us consider S and S^{-1} as in (6.3). Then \tilde{W} is given by.

$$\tilde{W} = \tilde{W}^{(0)} + \frac{1}{w - w_{[1]}} \tilde{W}^{(1)}, \quad (6.5)$$

$$\tilde{W}^{(0)} = W_0 + 4i\beta_0 \Omega, \quad \tilde{W}^{(1)} = W(w_{[1]}) R + R^\dagger W(w_{[1]}) + 4i R^\dagger \Omega R.$$

Moreover, we also have $R^\dagger W(w_{[1]}) R = 0$

Proof

From (6.1) we have:

$$\tilde{W} = W + \frac{1}{w - w_{[1]}} [R^\dagger W + W R] + \frac{1}{(w - w_{[1]})^2} R^\dagger W R. \quad (6.6)$$

Being both \mathbf{W} and \mathbf{W}_0 linear in w , we can write them as

$$\mathbf{W} = \mathbf{W}(w_{[1]}) + 4i (w - w_{[1]}) \Omega, \quad \mathbf{W}_0 = \mathbf{W}_0(w_{[1]}) + 4i (w - w_{[1]}) \Omega. \quad (6.7)$$

By inserting (6.7) into (6.6) we get:

$$\begin{aligned} \tilde{\mathbf{W}} = \mathbf{W} + 4i [\mathbf{R}^\dagger \Omega + \Omega \mathbf{R}] + \frac{1}{(w - w_{[1]})^2} \mathbf{R}^\dagger \mathbf{W}(w_{[1]}) \mathbf{R} + \\ + \frac{1}{w - w_{[1]}} [\mathbf{R}^\dagger \mathbf{W}(w_{[1]}) + \mathbf{W}(w_{[1]}) \mathbf{R} + 4i \mathbf{R}^\dagger \Omega \mathbf{R}]. \end{aligned} \quad (6.8)$$

Finally, from (2.149) we have:

$$\begin{aligned} \tilde{\mathbf{W}} = \mathbf{W}_0 + 4i \beta_0 \Omega + \frac{1}{(w - w_{[1]})^2} \mathbf{R}^\dagger \mathbf{W}(w_{[1]}) \mathbf{R} + \\ + \frac{1}{w - w_{[1]}} [\mathbf{R}^\dagger \mathbf{W}(w_{[1]}) + \mathbf{W}(w_{[1]}) \mathbf{R} + 4i \mathbf{R}^\dagger \Omega \mathbf{R}]. \end{aligned} \quad (6.9)$$

This and the result of the previous proposition prove the assertion. \square q e d

In particular, the relation $\mathbf{R}^\dagger \mathbf{W}(w_{[1]}) \mathbf{R} = 0$ can be used to set constraints among the various parameters entering the solutions we are going to build:

$$\mathbf{R}^\dagger \mathbf{W}(w_{[1]}) \mathbf{R} = 0 \implies \bar{\mathbf{n}} \cdot \mathbf{W}(w_{[1]}) \cdot \mathbf{n} \implies \bar{\mathbf{p}} \cdot \mathbf{W}(w_{[1]}) \cdot \mathbf{p}. \quad (6.10)$$

Proposition (6.1.4) An expression for β_0

Let us consider \mathbf{S} and \mathbf{S}^{-1} as in (6.3). Then β_0 is given by:

$$\beta_0 = w_{[1]} - \nu_{[1]} - \operatorname{Re} (n_3 m_3) + \operatorname{Im} (\Phi^1 \bar{\Phi}^2 - \Phi_0^1 \bar{\Phi}_0^2). \quad (6.11)$$

Proof

Let us consider (2.145) and the matrix element G_{12} . We have:

$$-4 h^{12} + 4 \Phi^1 \bar{\Phi}^2 = -4 h_0^{12} + 4 \Phi_0^1 \bar{\Phi}_0^2 - 4i (n_2 m_2 + \bar{m}_1 \bar{n}_1) + 4i \beta_0. \quad (6.12)$$

By imposing $h^{12} = \bar{h}^{12}$ we get the following relations for β_0 :

$$\beta_0 = \operatorname{Re} (\mathbf{n} \cdot \mathbf{m}) - \operatorname{Re} (n_3 m_3) + \operatorname{Im} (\Phi^1 \bar{\Phi}^2 - \Phi_0^1 \bar{\Phi}_0^2). \quad (6.13)$$

Formulae (2.155) and (2.163) for the 1-soliton solution read

$$\mathbf{n} = + \frac{w_{[1]} - \nu_{[1]}}{\mathbf{m} \cdot \mathbf{p}} \mathbf{p}, \quad \mathbf{q} = - \frac{w_{[1]} - \nu_{[1]}}{\mathbf{m} \cdot \mathbf{p}} \mathbf{m}, \quad (6.14)$$

and thus it is easy to prove that

$$\mathbf{n} \cdot \mathbf{m} = \operatorname{Tr} \mathbf{R} = w_{[1]} - \nu_{[1]}. \quad (6.15)$$

The assertion is therefore proven. \square q e d.

In fact, as argued in *Proposition (2.3.11)*, β_0 is just a constant of integration. Therefore, any solution may be considered only after imposing – if possible – that the r.h.s. of equation (6.11) does not depend upon α and β . Indeed, the electromagnetic components Φ^a are determined by the 13 and 23 components of (2.145). Therefore, whether this condition will be satisfied or not, strictly depends on the form of the seed.

Proposition (6.1.5) An essential constraint

Let us consider S and S^{-1} as in (6.3). Then \tilde{W} must satisfy the constraint:

$$\tilde{W}(\nu_{[1]}) \cdot p = 0. \quad (6.16)$$

Proof

Let us now consider equation (6.2) for the case $N = 1$ and rewrite it in the form:

$$\tilde{W} + \frac{1}{w - \nu_{[1]}} \tilde{W} Q = W + \frac{1}{w - \bar{w}_{[1]}} R^\dagger W. \quad (6.17)$$

The limit $w \rightarrow \nu_{[1]}$ in both members of this equation is well defined only if we impose that $\tilde{W}(\nu_{[1]}) Q = 0$. The assertion is therefore proven by considering the explicit form of Q . \square q.e.d.

Proposition (6.1.6) \tilde{W} for the vacuum case

Let us consider S and S^{-1} as in (6.3). Let us consider a vacuum seed metric. Let us consider $\kappa_3 = \ell_3 = 0$. Then \tilde{W} can be put in the form:

$$\tilde{W}_{(2 \times 2)} = \frac{w - \nu_{[1]}}{w - w_{[1]}} \left(W_{0(2 \times 2)} + 4i (\beta_0 - (w_{[1]} - \nu_{[1]})) \Omega_{(2 \times 2)} \right), \quad (6.18a)$$

$$\tilde{W}_{15} = \tilde{W}_{15} = 0, \quad \tilde{W}_{55} = 1, \quad (6.18b)$$

where (2×2) denotes the upper left 2×2 corner of the relevant matrices and number 5 labels the electromagnetic components – the third row and column – of all matrices.

Proof

With $\kappa_3 = \ell_3 = 0$ and with a vacuum seed, we have that $n_3 = m_3 = 0 = p_3 = q_3$. Therefore the matrices R and Q are nonzero only in the upper left 2×2 components. Thus the pole structure indicated in *Proposition (6.1.1)* applies only to the upper left 2×2 components of \tilde{W} . Throughout this proof it will be understood that all the equations are restricted to this upper left 2×2 corner. The constraint in equation (6.16) will be satisfied if:

$$\tilde{W} = \frac{w - \nu_{[1]}}{w - w_{[1]}} (A w + B), \quad (6.19)$$

where the matrices A and B are independent of w . By comparing (6.19) with (6.9), it can be seen that A and B are given by:

$$A = 4i \Omega, \quad B = W_{0(w_{[1]})} + 4i (\beta_0 - (w_{[1]} - \nu_{[1]})) \Omega, \quad (6.20)$$

provided that the following relation holds:

$$\begin{aligned} \mathbf{R}^\dagger \mathbf{W}(w_{[1]}) + \mathbf{W}(w_{[1]}) \mathbf{R} + 4i \mathbf{R}^\dagger \boldsymbol{\Omega} \mathbf{R} = \\ (w_{[1]} - \nu_{[1]}) \mathbf{W}(w_{[1]}) + 4i (w_{[1]} - \nu_{[1]}) (\mathbf{R}^\dagger \boldsymbol{\Omega} + \boldsymbol{\Omega} \mathbf{R}) - 4i (w_{[1]} - \nu_{[1]})^2 \boldsymbol{\Omega} . \end{aligned} \quad (6.21)$$

By inserting (6.20) into (6.19), the assertion is proven \square q.e.d

Proposition (6.1.7) \mathbf{K} for the vacuum case

Let us consider \mathbf{S} and \mathbf{S}^{-1} as in (6.3). Let us consider a vacuum seed metric. Let us consider $\kappa_3 = \ell_3 = 0$. Then, with the ansatz (6.18), we also have:

$$\mathbf{K}_{(2 \times 2)} = \frac{w - \nu_{[1]}}{w - w_{[1]}} \mathbf{K}_{0(2 \times 2)} , \quad K_{13} = K_{23} = 0 , \quad K_{33} = K_{033} . \quad (6.22)$$

Proof

This simply follows from the previous proposition and by noticing that, in the vacuum case, $\beta_0 = w_{[1]} - \nu_{[1]}$. \square q.e.d.

The matrix \mathbf{K} defined in (6.22) is clearly a function of the spectral parameter w only. Therefore, the existence of the tentative solution (6.3) is demonstrated in the vacuum case for any arbitrary seed

The following two constraints

$$\mathbf{R}^\dagger \mathbf{W}(w_{[1]}) \mathbf{R} = 0 \quad (6.23)$$

and

$$\begin{aligned} \mathbf{R}^\dagger \mathbf{W}(w_{[1]}) + \mathbf{W}(w_{[1]}) \mathbf{R} + 4i \mathbf{R}^\dagger \boldsymbol{\Omega} \mathbf{R} = \\ (w_{[1]} - \nu_{[1]}) \mathbf{W}(w_{[1]}) + 4i (w_{[1]} - \nu_{[1]}) (\mathbf{R}^\dagger \boldsymbol{\Omega} + \boldsymbol{\Omega} \mathbf{R}) - 4i (w_{[1]} - \nu_{[1]})^2 \boldsymbol{\Omega} , \end{aligned} \quad (6.24)$$

appropriately restricted to the upper left 2×2 corner only, can be used to set restrictions among the various parameters which enter the solution.

The ansatz (6.18) may be generalized to the electrovacuum case by simply assuming that \mathbf{A} and \mathbf{B} are full 3×3 matrices. However, a number of problems will occur. In particular, an expression for $\tilde{\mathbf{W}}$ may be obtained by following the same lines of the proof given for *Proposition (6.1.6)*. With this, one might finally obtain:

$$\mathbf{K} = \frac{w - \nu_{[1]}}{w - w_{[1]}} \mathbf{K}_0 + 4i \frac{w - \nu_{[1]}}{w - w_{[1]}} (\beta_0 - (w_{[1]} - \nu_{[1]})) \Psi_0^\dagger \boldsymbol{\Omega} \Psi_0 . \quad (6.25)$$

In general, \mathbf{K} as defined in (6.25) is not a function of w only, due to the presence of the term proportional to $\Psi_0^\dagger \boldsymbol{\Omega} \Psi_0$. Therefore, the electrovacuum soliton solution with one real pole cannot normally exist. Nevertheless, there may exist special seeds for which the second term in the (6.25) vanishes: this might permit the generation of electrovacuum solitons with real poles.

6.1.2 The N -soliton case

Let us now consider the N -soliton case:

$$S = I + \sum_{k=1}^N \frac{1}{w - w[k]} R[k], \quad S^{-1} = I + \sum_{k=1}^N \frac{1}{w - \nu[k]} Q[k], \quad (6.26a)$$

$$R[k] = n[k] \otimes m[k], \quad Q[k] = p[k] \otimes q[k], \quad w[k], \nu[k] \in \mathbb{R}, \quad w[k] \neq \nu[k] \quad (6.26b)$$

where $n[k]$, $m[k]$, $p[k]$, $q[k]$ are given in (2.155) and (2.163). Since we are interest in vacuum solutions, we will consider vacuum seeds and will also set $\kappa_3[k] = \ell_3[k] = 0$. This will result in $n_3[k] = m_3[k] = p_3[k] = q_3[k] = 0$.

Again, the proof of the existence of such a N -soliton vacuum solution is based on a verification that $\Psi^\dagger W \Psi$ is a function of the spectral parameter only, where:

$$W = W_0 - 4i (R \Omega + R^\dagger \Omega) + 4i \beta_0 \Omega, \quad R = \sum_{k=1}^N R_k. \quad (6.27)$$

Proposition (6.1.8) An essential constraint

Let us consider S and S^{-1} as in (6.26). Then \tilde{W} must satisfy the constraint:

$$\tilde{W}(\nu[k]) \cdot p[k] = 0. \quad (6.28)$$

Proof

The proof is easily given by considering, as in the 1-soliton case, the limit $w \rightarrow \nu[k]$ in both members of equation (6.2). \square q.e.d.

Proposition (6.1.9) Property of β_0

Let us consider S and S^{-1} as in (6.26). Let us consider a vacuum seed. Let us put $\kappa_3[k] = \ell_3[k] = 0$. Then β_0 is given by:

$$\beta_0 = \sum_{k=1}^N w[k] - \nu[k]. \quad (6.29)$$

Proof

The proof goes along the same lines as in (6.11). We have only to keep in mind that R into (2.149) is now given by $R = \sum_{k=1}^N R[k]$. \square q.e.d.

Following the same lines as in the 1-soliton case, we may introduce the function $\tilde{W} = S^\dagger W S$.

$$\begin{aligned} \tilde{W} = & W + 4i [R^\dagger \Omega + \Omega R] + \\ & + \sum_{k=1}^N \frac{1}{w - w[k]} [R[k]^\dagger W(w[k]) + W(w[k]) R[k]] + \\ & + \sum_{k,j=1}^N \frac{1}{(w - w[k])(w - w[j])} R[k]^\dagger W(w) R[j]. \end{aligned} \quad (6.30)$$

Again, it can be proved that \tilde{W} contains simple poles only. Therefore we will have:

$$\mathbf{R}^{\dagger}_{[k]} \mathbf{W}(w_{[k]}) \mathbf{R}_{[k]} = 0, \quad \forall k = 1, \dots, N. \quad (6.31)$$

These relations can be used to set constraints between the free parameters entering the solution

Proposition (6.1.10) \mathbf{K} for the vacuum case

Let us consider \mathbf{S} and \mathbf{S}^{-1} as in (6.26). Let us consider a vacuum seed. Let us put $\kappa_{3[k]} = \ell_{3[k]} = 0$. Then the general expression for $\mathbf{K}(w)$ is given by:

$$\mathbf{K}_{(2 \times 2)} = \prod_{k=1}^N \left(\frac{w - \nu_{[k]}}{w - w_{[k]}} \right) \mathbf{K}_{0(2 \times 2)}, \quad K_{33} = K_{033}. \quad (6.32)$$

Proof

A direct proof of this assertion would be unnecessarily cumbersome. It is better to apply the following "add one-by-one" procedure.

Given the seed (and hence a certain \mathbf{K}_0), we can construct the 1-soliton solution and obtain some:

$$\mathbf{K}_1 = \frac{w - \nu_{[1]}}{w - w_{[1]}} \mathbf{K}_0. \quad (6.33)$$

We can now use this as a seed and dress it with another soliton. As a result we will have another.

$$\mathbf{K}_2 = \frac{w - \nu_{[2]}}{w - w_{[2]}} \mathbf{K}_1 = \frac{w - \nu_{[2]}}{w - w_{[2]}} \frac{w - \nu_{[1]}}{w - w_{[1]}} \mathbf{K}_0. \quad (6.34)$$

This procedure can obviously be iterated up to the N_{th} soliton, yielding to the formula given in (6.32). \square q.e.d.

The above proposition ensures that the tentative solution (6.26) can actually admit a \mathbf{K} matrix depending on the spectral parameter only. Therefore this concludes the proof of existence for the vacuum N -soliton solution with real poles.

As pointed out above, formulae (6.31) provide a first set of constraints for the free parameters entering the vacuum N -soliton solution. As for the 1 soliton case, one can derive a second set of constraints, analogous to those given in (6.24). They are most easily obtained by using the above procedure of adding one pole at a time.

6.2 Explicit 1-soliton Solutions

For a generic vacuum seed metric, the corresponding matrix function Ψ_0 will have the form:

$$\Psi_0(w) = \begin{pmatrix} \Psi_{11}(w) & \Psi_{12}(w) & 0 \\ \Psi_{21}(w) & \Psi_{22}(w) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.35)$$

To explicitly construct a general (electrovacuum) 1-soliton solution we may put:

$$\kappa = (1, \kappa_2, \kappa_3), \quad \ell = (1, \ell_2, \ell_3), \quad (6.36)$$

where the soliton suffix 1 has been suppressed. Then (2.155) and (2.163) take the form:

$$\mathbf{m} = (X_0, -Y_0, \kappa_3), \quad \mathbf{p} = (X_\delta, Y_\delta, \ell_3), \quad (6.37a)$$

$$\mathbf{q} = -\delta \frac{\det \Psi_0(z_0)}{\mathcal{D}} \mathbf{m}, \quad \mathbf{n} = \delta \frac{\det \Psi_0(z_0)}{\mathcal{D}} \mathbf{p}, \quad (6.37b)$$

where $w[1] = z_0$ and $\nu[1] = z_0 + \delta$, with $z_0, \delta \in \mathbb{R}$, and the following quantities have been introduced:

$$X_0 = \frac{\Psi_{22}(z_0) - k_2 \Psi_{21}(z_0)}{\det \Psi_0(z_0)}, \quad Y_0 = \frac{\Psi_{12}(z_0) - k_2 \Psi_{11}(z_0)}{\det \Psi_0(z_0)}, \quad (6.38a)$$

$$X_\delta = \Psi_{11}(z_0 + \delta) + \ell_2 \Psi_{12}(z_0 + \delta), \quad Y_\delta = \Psi_{21}(z_0 + \delta) + \ell_2 \Psi_{22}(z_0 + \delta), \quad (6.38b)$$

$$\mathcal{D} = X_0 X_\delta - Y_0 Y_\delta + \kappa_3 \ell_3 \det \Psi_0(z_0) \quad (6.38c)$$

It is worth noticing here that each component of the vectors \mathbf{n} and \mathbf{q} is proportional to δ . This confirms that a new solution can be obtained only if the poles in \mathbf{S} and \mathbf{S}^{-1} are distinct.

6.2.1 Minkowski Seed

Let us consider the particular case of the Minkowski seed metric, for which

$$\mathbf{g}_0 = \begin{pmatrix} -\epsilon & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \quad \Psi_0 = \begin{pmatrix} \frac{1}{\sqrt{(w-\beta)^2 - \epsilon \alpha^2}} & 0 & 0 \\ i\epsilon \frac{(w-\beta)}{\sqrt{(w-\beta)^2 - \epsilon \alpha^2}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.39)$$

By using formulae (6.37), it can be shown that $\Psi_0^\dagger \Omega \Psi_0$ does depend upon α and β . Therefore this seed only permit the generation of vacuum solutions. We will present here the vacuum soliton solution with one real pole.

We can therefore set $\kappa_3 = \ell_3 = 0$. We can also choose $z_0 = 0$. By specializing (6.37) to this seed, it can be shown that the constraints (6.23) and (6.24) give the following restrictions on the parameters:

$$|\kappa_2|^2 = 1, \quad |\ell_2|^2 = 1. \quad (6.40)$$

Diagonal Case

To construct a diagonal solution, we can take.

$$\kappa_2 = i, \quad \ell_2 = i. \quad (6.41)$$

With this, we obtain

$$g_{11} = \epsilon \frac{\delta + \epsilon \left(\sqrt{\beta^2 - \epsilon \alpha^2} - \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2} \right)}{\delta - \epsilon \left(\sqrt{\beta^2 - \epsilon \alpha^2} - \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2} \right)}, \quad (6.42a)$$

$$g_{22} = -\alpha^2 - 2 \epsilon \delta \frac{(\beta - \epsilon \sqrt{\beta^2 - \epsilon \alpha^2}) (\beta - \delta - \epsilon \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2})}{\delta - \epsilon \left(\sqrt{\beta^2 - \epsilon \alpha^2} - \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2} \right)}. \quad (6.42b)$$

It is worth mentioning that the parameter β_0 has been set as $\beta_0 = -\delta$, according to the results in (6.11). It can also be confirmed that $\det g = \epsilon \alpha^2$. Moreover the condition $\Psi^\dagger W \Psi = K(w)$ is fulfilled with

$$\Psi^\dagger W \Psi = \begin{pmatrix} 4 \epsilon \left(1 - \frac{\delta}{w} \right) & 0 & 0 \\ 0 & -4 \epsilon \left(1 - \frac{\delta}{w} \right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.43)$$

which is in agreement with (6.22).

Let us now introduce the functions μ_1^\pm and μ_2^\pm given by

$$\mu_1^\pm = -\beta \pm \sqrt{\beta^2 - \epsilon \alpha^2}, \quad \mu_2^\pm = -(\beta - \delta) \pm \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}, \quad (6.44)$$

which are the pole-trajectories seen in chapter 3. With these, the above metric (6.42) can be rewritten in the form:

$$\epsilon = +1: \quad g_{11} = \frac{\mu_1^- - \mu_2^-}{\mu_1^+ - \mu_2^+} = \frac{\frac{\alpha^2}{\mu_1^+} - \frac{\alpha^2}{\mu_2^+}}{\frac{\alpha^2}{\mu_1^+} - \frac{\alpha^2}{\mu_2^+}} = -\frac{\alpha^2}{\mu_1^+ \mu_2^+}, \quad (6.45a)$$

$$\epsilon = -1: \quad g_{11} = -\frac{\mu_1^+ - \mu_2^+}{\mu_1^- - \mu_2^-} = -\frac{\frac{-\alpha^2}{\mu_1^-} - \frac{-\alpha^2}{\mu_2^-}}{\frac{-\alpha^2}{\mu_1^-} - \frac{-\alpha^2}{\mu_2^-}} = -\frac{\alpha^2}{\mu_1^- \mu_2^-}, \quad (6.45b)$$

and the g_{22} components are easily obtained by using the condition $\det g = \epsilon \alpha^2$.

It may immediately be observed that this solution is identical to the BZ soliton solution with two real poles and the same seed [106].

Non-diagonal Case

Let us consider the following parameters

$$\kappa_2 = 1, \quad \ell_2 = 1. \quad (6.46)$$

This choice generates a nondiagonal solution given by:

$$g_{11} = -\epsilon \frac{\beta^2 - \beta \delta - \epsilon \alpha^2 + \sqrt{\beta^2 - \epsilon \alpha^2} \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}}{\beta^2 - \beta \delta + \delta^2 - \epsilon \alpha^2 + \sqrt{\beta^2 - \epsilon \alpha^2} \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}}, \quad (6.47a)$$

$$g_{12} = \delta \epsilon \frac{\sqrt{\beta^2 - \epsilon \alpha^2} (\beta - \delta) + \beta \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}}{\beta^2 - \beta \delta + \delta^2 - \epsilon \alpha^2 + \sqrt{\beta^2 - \epsilon \alpha^2} \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}}, \quad (6.47b)$$

$$g_{22} = -\alpha^2 - \frac{+2 \epsilon \beta (\beta - \delta) \delta^2 + \alpha^2 \delta^2}{\beta^2 - \beta \delta + \delta^2 - \epsilon \alpha^2 + \sqrt{\beta^2 - \epsilon \alpha^2} \sqrt{(\beta - \delta)^2 - \epsilon \alpha^2}}. \quad (6.47c)$$

It can again be confirmed that $\det \mathbf{g} = \epsilon \alpha^2$ and that the condition $\Psi^\dagger \mathbf{W} \Psi = \mathbf{K}(w)$ is fulfilled. Indeed, it is found that $\mathbf{K}(w)$ is identical to that for the diagonal case (6.42) – see formula (6.43). This is inevitable from the fact that the formula (6.22) does not contain the free parameters $\kappa_{[k]}$ and $\ell_{[k]}$ – it only contains the parameter δ that appears in the poles.

Using the poles-trajectories (6.44), the above metric components can be rewritten in the form:

$$g_{11} = -2\epsilon \frac{(\mu_1^+ - \mu_2^-)(\mu_2^+ - \mu_1^-)}{(\mu_1^+ - \mu_2^-)^2 + (\mu_2^+ - \mu_1^-)^2}, \quad (6.48a)$$

$$g_{12} = -2\epsilon \delta \frac{\mu_1^+ \mu_2^+ - \mu_1^- \mu_2^-}{(\mu_1^+ - \mu_2^-)^2 + (\mu_2^+ - \mu_1^-)^2}. \quad (6.48b)$$

The g_{22} component can be easily obtained by way of the usual condition $\det \mathbf{g} = \epsilon \alpha^2$.

6.2.2 Kasner type D Seed

Let us consider the particular case of the Kasner seed metric for which, when $q = 0$, we have:

$$\mathbf{g}_0 = \begin{pmatrix} \epsilon \alpha^{1-2q} & 0 \\ 0 & \alpha^{1+2q} \end{pmatrix}, \quad \Psi_0 = \begin{pmatrix} +2^{1/2} k_1 \frac{R^- - R^+}{R} & -2^{1/2} k_1 \frac{R^- + R^+}{R} & 0 \\ -2^{1/2} k_1 \frac{R^- + R^+}{R} & -2^{1/2} k_1 \frac{R^- - R^+}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^+ = \sqrt{\beta - w + \alpha} \quad R^- = \sqrt{\beta - w - \alpha} \quad R = R^+ R^-. \quad (6.49)$$

and construct the associated vacuum soliton solution with one real pole. We can therefore set $\kappa_3 = \ell_3 = 0$. We can also choose $z_0 = 0$. Furthermore, we will consider the case $q = 0$ only.

It may be firstly noticed that the matrix \mathbf{K}_0 is now given by:

$$\mathbf{K}_0 = \begin{pmatrix} 0 & -32i k_1^2 & 0 \\ -32i k_1^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.50)$$

This illustrates that the choice (2.152) is not necessary, and it is performed just because it may result convenient in certain situations.

We have found difficult to find out the implications of (6.23) and (6.24) for this particular seed. However a number of interesting features have been obtained. After rewriting the arbitrary parameters in the form:

$$\kappa_2 = M e^{i\theta_1}, \quad \ell_2 = P e^{i\theta_2}, \quad (6.51)$$

equation (6.23), for this seed, reads:

$$2 k_1^2 \alpha^2 (e^{2i\theta_2} - 1) P - 2 (e^{2i\theta_2} - 1) k_1^2 P (\beta - \delta)^2 = 0. \quad (6.52)$$

It is therefore evident that we have to make the choice:

$$\theta_2 = 0, \frac{\pi}{2}, \pi, \dots \quad (6.53)$$

The second constraint has not been given a readable form. We have checked that the choices $M = 0$ or $\theta_1 = 0$ are admissible, but we have not been able to prove that they are the only ones

Diagonal Case

To construct a diagonal solution, we can take

$$\kappa_2 = 0, \quad \ell_2 = 0. \quad (6.54)$$

With this, we obtain:

$$g_{11} = \frac{1}{\alpha} (\beta - \sqrt{\beta^2 - \alpha^2}) (\beta - \delta + \sqrt{(\beta - \delta)^2 - \alpha^2}), \quad (6.55a)$$

$$g_{22} = \frac{1}{\alpha} (\beta + \sqrt{\beta + \alpha} \sqrt{\beta - \alpha}) (\beta - \delta - \sqrt{\beta - \delta + \alpha} \sqrt{\beta - \delta - \alpha}). \quad (6.55b)$$

It is worth mentioning that the parameter β_0 has been set as $\beta_0 = -\delta$, according to the results in (6.11). It can also be confirmed that $\det g = \epsilon \alpha^2$. Moreover the condition $\Psi^\dagger \mathbf{W} \Psi = \mathbf{K}(w)$ is fulfilled with:

$$\Psi^\dagger \mathbf{W} \Psi = \begin{pmatrix} 0 & +32i k_1^2 \left(1 - \frac{\delta}{w}\right) & 0 \\ -32i k_1^2 \left(1 - \frac{\delta}{w}\right) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.56)$$

again in agreement with (6.22).

Using the poles-trajectories (6.44), the above metric components can be rewritten in the form:

$$g_{11} = -\frac{\mu_1^+ \mu_2^-}{\alpha}, \quad g_{22} = -\alpha^3 \frac{1}{\mu_1^+ \mu_2^-}. \quad (6.57)$$

It may immediately be observed that this solution is identical to the BZ soliton solution with two real poles and the same seed [106].

Non-diagonal Case

Let us consider the following parameters:

$$\kappa_2 = 1, \quad \ell_2 = 1. \quad (6.58)$$

This choice generates a nondiagonal solution given by:

$$g_{11} = \alpha - \delta \frac{\sqrt{\beta + \alpha} \sqrt{\beta - \delta - \alpha} - \sqrt{\beta - \alpha} \sqrt{\beta - \delta + \alpha}}{\sqrt{\beta + \alpha} \sqrt{\beta - \delta - \alpha} + \sqrt{\beta - \alpha} \sqrt{\beta - \delta + \alpha}}, \quad (6.59a)$$

$$g_{12} = -\delta \frac{\sqrt{\beta - \alpha} \sqrt{\beta - \delta - \alpha} - \sqrt{\beta + \alpha} \sqrt{\beta - \delta + \alpha}}{\sqrt{\beta + \alpha} \sqrt{\beta - \delta - \alpha} + \sqrt{\beta - \alpha} \sqrt{\beta - \delta + \alpha}}, \quad (6.59b)$$

$$g_{22} = \alpha - \delta \frac{\sqrt{\beta + \alpha} \sqrt{\beta - \delta - \alpha} - \sqrt{\beta - \alpha} \sqrt{\beta - \delta + \alpha}}{\sqrt{\beta + \alpha} \sqrt{\beta - \delta - \alpha} + \sqrt{\beta - \alpha} \sqrt{\beta - \delta + \alpha}}. \quad (6.59c)$$

It can again be confirmed that $\det g = \epsilon \alpha^2$ and that the condition $\Psi^\dagger \mathbf{W} \Psi = \mathbf{K}(w)$ is fulfilled. Again, it is found that $\mathbf{K}(w)$ is identical to that for the diagonal case (6.56).

Bibliography of Files. [33, 34]

6.3 Conclusions

Generally, the Alekseev N -soliton method for the construction of new solutions of the Einstein–Maxwell equations requires the addition of N distinct poles in the inverse of the scattering matrix. These are normally complex conjugate. In the vacuum case, the method is supposed to be equivalent to that of Belinski and Zakharov (with $2N$ poles) provided the poles occur as complex conjugate pairs.

The purpose of this chapter has been to modify the Alekseev inverse-scattering method to permit the use of real poles. This has been attempted by introducing distinct real poles in the inverse matrix. We have found that this construction is successful in the vacuum case in which it has shown to be equivalent to the BZ method with distinct real poles.

In principle, there might exist special seeds for which both $\Psi_0^\dagger \tilde{W} \Psi_0$ does not depend on α and β and the r.h.s. of (6.11) is a constant. If this is so, then an electrovacuum solution can be generated.

Chapter 7

Conclusions

In this thesis we have been involved with three main issues:

- **Time-Shift**

The problem was that of understanding whether or not gravitational solitons, after interacting, still travel along the initial directions or are somehow shifted. By analyzing the asymptotic behaviour ($t \rightarrow \infty$) of the soliton metrics, we concluded that such a time-shift does not occur for gravitational solitons, thus making a strong difference with other soliton solutions, such as for KdV or SG equations.

We also found that this effect similarly does not occur when the interaction between solitons and gravitational waves is considered.

- **Extensions of the real-pole BZ's solitons**

Given that soliton solutions may have singularities – along some null hypersurfaces – which are removable with a coordinate transformation, we were interested in possible extensions of the transformed metric such that also the Ricci tensor is everywhere singularity-free.

We have given a few examples of compound spacetimes that fulfill these requirements, including all possible extensions for the diagonal case.

- **Generation of the real-pole Alekseev's solitons**

The standard Alekseev formalism works perfectly when complex-pole solutions are considered. In this case the machinery provides both vacuum and electrovacuum solutions. However, problems occur when real poles solutions are considered, because the generating procedure develops singularities that prevent one from successfully produce solutions.

By modifying the pole-structure of the dressing matrix S and its inverse S^{-1} , we were able to develop a machinery that generates real pole solitons. In general, these solutions occur only for the vacuum case. For the electrovacuum case, solutions can be generated only if the seed satisfies certain restrictions.

We have also clarified that the vacuum solutions correspond to the case of BZ solitons with an even number of real poles.

As far as we can see, this third issues is worth of further investigation. We can summarize the work done in chapter 6 as follow. Given the dressing ansatz (2.133), there are two main freedoms we are left with in order to produce solutions. One is the nature of the poles in the dressing matrix S and its inverse. The second is the pole structure of matrix K (2.120).

In the standard Alekseev technique, both the dressing matrix and its inverse have meromorphic structure – i.e. they have just simple poles – and the choice $K = K_0$ is considered. This implies that the poles in S and S^{-1} must be complex conjugate to each other. In chapter 6, we have kept the meromorphic structure of S and S^{-1} but we have tried to modify the form of K . As a result, one possible modification – based upon the ansatz (6.18), and restricted to the upper left 2×2 corner of the relevant matrices – was $K = f(w) K_0$, with f a scalar function containing at most simple poles. Such a modification allowed for the generation of real pole solitons in the vacuum case. This ansatz can not be used to generate real pole electrovacuum solitons unless very special seeds are considered

In principle, a more general ansatz might be considered. In fact, the constant matrix K is only subject to the condition $K = K^\dagger$, regardless the pole structure of the dressing matrix S . Therefore it could be any linear superposition of twelve linearly independent hermitian matrices. The “windfall” of such an ansatz with respect to S and S^{-1} is – as yet – unknown.

A sound way-out to the problem of generating electrovacuum solutions with real poles seems to be given by the *linear integral equation* introduced by Alekseev himself in [57, 58]¹.

However, the Alekseev technique, even with the above generalization, still raises a few other questions. We have already addressed these in the introduction to chapter 6, namely: how can we generate solutions with an odd number of poles?

We believe that an answer to these questions strictly involves a rethink of the nature of poles in the dressing matrix, i.e. of the meromorphic structure of S and S^{-1} . We have tried, without success, the following:

$$S = I + \sum_{k=1}^N \frac{1}{w - w[k]} R[k], \quad S^{-1} = I + \sum_{k=1}^M \frac{1}{w - \nu[k]} Q[k], \quad M \neq N, \quad (7.1a)$$

or the other :

$$S = I + \sum_{k=1}^N \frac{1}{(w - w[k])^{h[k]}} R[k], \quad S^{-1} = I + \sum_{k=1}^N \frac{1}{(w - \nu[k])^{h[k]}} Q[k], \quad h[k] \neq 1, \quad (7.1b)$$

thus indicating that a more radical change is needed. Is the non-solitonic solution given by Alekseev himself in [55] useful in this context? With some stretch of imagination, we may say that the challenge finally results in trying to implement half pole solitons within the Alekseev formalism.

The issues considered in chapter 4 also need to be reconsidered in a wider context. We have recently become aware that soliton solutions that do not suffer Time-Shift can also be generated as solutions of the matrix-KdV equation [151]. In particular, 2×2 matrix solutions can be constructed by using a dressing ansatz. The situation

¹Private communication to J B Griffiths, (september 1999)

relevant to our discussion – no Time-Shift is observed – occurs when the dressing matrix has a vanishing determinant. This seems very intriguing because in the construction of gravitational soliton matrices with null determinant also appear – namely the $R[k]$ and $Q[k]$ matrices. If this is a key fact or just a fortuitous occurrence we do not know. However, we feel that this is worthy of further investigation. We suppose that in the context of matrix-KdV a physical quantity such as the hamiltonian of the system might be introduced without theoretical inconsistencies. Therefore one might be able to inspect whether the energy associated with these solitons can be propagated without dispersion or not.

Also, the problem of Time-Shift in GR should be considered in connection with some results obtained by Veselov in [152]. In that paper, it was shown that “rational” solitons of the Liouville equation do not suffer Time-Shift. The thing relevant to us is that certain solutions of the Ernst equation may have rational form if an appropriate system of coordinates is chosen.

In the context of stationary axisymmetric vacuum solutions, it is known that the calculation of the Weyl Tensor – and its components Ψ_0 , Ψ_2 and Ψ_4 – is simplified due to the existence of some factorization property – see, for example [161, 162] and references therein. As far as we know, no such a property has been looked for in the context of the solutions we have discussed in chapter 4. Obviously, we do not expect this factorization – if any – to take place in the same terms as in [161, 162]. However, it seems auspicious that some research in this direction be pursued. Any positive result would be of great importance even beyond the ambit of the Time-Shift problem we are discussing here.

In chapter 1 we quoted a sentence from [31] about a possible link between the soliton techniques and the group-like approach to the solution of the Einstein equations. Alekseev has proposed the linear integral equation [51, 58] as an alternative formulation to his “dressing” soliton technique. The relations between this and the HHP posed by Hauser and Ernst is surely another topic to be analyzed.

Also, the possibility of generating new solutions by solving the linear integral equation, in the case when rational “monodromy data” are considered, might be relevant to the issue of Time-Shift discussed above.

More ideas for future, possible research

A good job is such if some result is achieved. But a better job is produced if these results allow for new research to be performed.

Above we already introduced a few items worthy of additional investigation. However, given the title of this section, we want to go even further. And moreover we will feel free of speaking languages typical of other fields of Physics.

- Any physicist knows that, if a new particle cannot be created, this is so either because there is not enough energy or because symmetry considerations forbid it. Having that in mind, and thinking of solitons as particles, which one is the reason why the Alekseev formalism does not allow for the generation of solutions with an odd number of poles? What is the intrinsic symmetry hidden within the Alekseev formalism, such that a spare soliton is not allowed?

- The solitons techniques are based upon the dressing ansatz that a new solution is given by applying a certain operator to a seed metric. That reminds us of the procedure usually used to build up a Fock space of free particles in the second quantization scheme of relativistic quantum mechanics: given a vacuum state, states with a given number of particle/antiparticles are built by repeatedly applying the relevant creation/destruction operators. This is one of the basic bricks to construct a field theory. Thinking of the seed as a vacuum state and thinking of the dressing matrix as a creation operator, which one is the appropriate destruction operator? Is it then possible to write (canonical) commutation/anticommutation relations between these? Which one is the possible analogue of the antiparticle? Are the definitions of soliton and antisoliton introduced by Belinski in [101, 102, 104, 105] of any interest for this problem?

The aim is that of introducing some sort of quantization procedure “naturally” driven by the classical properties of the system [159, 160]. With the same aim, an attempt to a possible quantization of spacetimes with two Killing vectors has been made by D. Korotkin et al. in [153, 154, 155, 156, 157, 158]. There the starting point is precisely given by the Belinski–Zakharov linear pair. The procedure followed in these papers is obviously different from that sketched above. Nevertheless, we feel encouraged by the fact that the soliton technique is considered as a starting point.

In a sense – again we require some stretch of imagination – this is not surprising. In fact, the soliton techniques somehow provide a linearization of the nonlinear problem posed by the Einstein equations. And indeed the quantum systems must have an underlying linear structure, since the superposition of states/particles must be allowed.

One more issue. If the solitonic metrics are to provide a Fock-like space – in the sense described above – it means that an arbitrary state/metric must be described by an appropriate “linear” combination of these. The classical counterpart of this problem is posed as follows: do the soliton metrics represent the whole Geroch group [156]?

- It is known that a relation exists between the Ernst equation and the σ -models equation – see Hoenselaers and Gurses in [12]. If so, how is this related to the issue put forward above [163]?

It is very possible that a deeper investigation of these items would reveal them to be meaningless. However, on a pure hypothetical basis, we feel entitled to pose these questions

The two main approaches to Quantum Gravity (QG) may be outlined as follows [164]: on one hand we have the “hep-th” physicist who thinks about QG as a (string) theory beyond the *Standard Model*. On the other hand, we have the “gr-qc” physicist whose attempts are devoted to a quantization *of* the spacetime, rather than a quantization *over* the spacetime. We wish to embrace this second point of view since, as we recalled at page 1 of this thesis, *the spacetime must no longer be considered as a fixed frame*.

Appendix A

Construction of Alekseev's Linear Pair

Let us consider the operator:

$$\mathcal{D}_A = \partial_A + \mathcal{P}_A \frac{\partial}{\partial \omega}, \quad A = 1, 2, \quad (\text{A.1})$$

where ω is a complex parameter and \mathcal{P}_A is a numerical function depending upon x^A and ω . The requirement that

$$\epsilon^{AB} \mathcal{D}_A \mathcal{D}_B = 0 \quad (\text{A.2})$$

implies:

$$\mathcal{P}_A = -\partial_A \lambda \left(\frac{\partial \lambda}{\partial \omega} \right)^{-1}, \quad (\text{A.3})$$

where λ is an arbitrary function depending upon x^A and ω .

Let us introduce the complex matrix function

$$\Psi = \Psi(x^A, \omega), \quad (\text{A.4})$$

and consider the following matrix equation:

$$\mathcal{D}_A \Psi = \Lambda_A^B U_B \Psi, \quad (\text{A.5})$$

where Λ_A^B is a function depending upon x^A and ω and U_A is a 3×3 complex matrix depending only upon x^A . By substituting (A.5) into (A.2) we have:

$$\epsilon^{AB} U_A U_B \det \Lambda = \epsilon^{AB} \mathcal{D}_A \mathcal{D}_B. \quad (\text{A.6})$$

By substituting \mathcal{D}_B in the above expression, and using (A.5), we finally obtain:

$$\epsilon^{AB} \mathcal{D}_A \mathcal{D}_B = \epsilon^{AB} (\mathcal{D}_A \Lambda_B^C) U_C + \epsilon^{AB} \Lambda_B^C (\partial_A U_C), \quad (\text{A.7})$$

where the fact that U_C is independent of ω has been used. A comparison of (A.6) with (A.7) gives:

$$\left(\frac{\epsilon^{AB} \mathcal{D}_A \Lambda_B^C}{\det \Lambda} \right) U_C + \left(\frac{\epsilon^{AB} \Lambda_B^C}{\det \Lambda} \right) \partial_A U_C - \epsilon^{AB} U_A U_B = 0. \quad (\text{A.8})$$

Clearly, the set of equations (A.1), (A.2) and (A.5), is equivalent to the set (A.2) and (A.8). Moreover, \mathcal{P}_A is expressed by (A.3). With that, it is easy to verify that the set (A.2) + (A.8) is also equivalent to:

$$\epsilon^{AC} \Lambda_C^B = 2i\alpha \det \Lambda (\eta^{AB} + \mathcal{F} \epsilon^{AB}), \quad (\text{A.9a})$$

$$\epsilon^{AB} \mathcal{D}_A \Lambda_B^C = 0, \quad (\text{A.9b})$$

$$\epsilon^{AB} \mathcal{D}_A \mathcal{D}_B = 0, \quad (\text{A.9c})$$

where \mathcal{F} is an arbitrary function depending upon x^A and ω .

The solution of (A.9a) gives:

$$\Lambda_A^B = \frac{1}{2i} \frac{\epsilon \mathcal{F} \delta_A^B + \epsilon_A^B}{1 - \epsilon \mathcal{F}^2}, \quad (\Lambda^{-1})_A^B = 2i\alpha (\mathcal{F} \delta_A^B - \epsilon \epsilon_A^B). \quad (\text{A.10})$$

By considering (A.3), we have that the 3 equations in (A.9) can be substituted by.

$$\Lambda_A^B = \frac{1}{2i} \frac{\epsilon \mathcal{F} \delta_A^B + \epsilon_A^B}{1 - \epsilon \mathcal{F}^2}, \quad \epsilon^{AB} \mathcal{D}_A \Lambda_B^C = 0, \quad \mathcal{P}_A = -\partial_A \lambda \left(\frac{\partial \lambda}{\partial \omega} \right)^{-1}. \quad (\text{A.11})$$

By substituting the first and third into the second of the above equations we have:

$$\lambda = \alpha \mathcal{F} + \beta. \quad (\text{A.12})$$

The freedom in the definition of λ gives us the possibility of setting $\partial_A \lambda = 0$, therefore, from (A.3), we have $\mathcal{P}_A = 0$. From (A.12) we have:

$$\mathcal{F} = \frac{1}{\alpha} (\lambda - \beta). \quad (\text{A.13})$$

By substituting the above formula into (A.11) we have that (A.5) is equivalent to:

$$\partial_A \Psi = \Lambda_A^B U_B \Psi, \quad \Lambda_A^B = \frac{1}{2i} \frac{(\lambda - \beta) \delta_A^B + \epsilon \alpha \epsilon_A^B}{(\lambda - \beta)^2 - \epsilon \alpha^2}. \quad (\text{A.14})$$

Appendix B

More remarks on the asymptotic limit of the W-M metric

This appendix is devoted to a proof that the $Y_{[i]}$'s asymptotically approach a constant value, i.e. do not diverge.

The functions

$$Y_{[i]} = \lim_{\lambda \rightarrow \mu_{[i]}} Y(\lambda)$$

have been defined in section 3.2.4. We recall here that.

$$Y(\alpha, \beta, \lambda) = \sqrt{K(\alpha, \beta, \lambda)} I(\alpha, \beta, \lambda), \quad K = \frac{\lambda}{\lambda^2 + 2\beta\lambda + \alpha^2}, \quad (\text{B.1a})$$

$$I = \int dv \frac{W'(v)}{\sqrt{1 - 2vK(\alpha, \beta, \lambda)}}. \quad (\text{B.1b})$$

It must be firstly noticed that, in the limit when $\lambda \rightarrow \mu_{[i]}$, the function $K(\alpha, \beta, \lambda)$ becomes just a complex constant. In particular

$$\lim_{\lambda \rightarrow \mu_{[i]}} K(\alpha, \beta, \lambda) = 2(z_0[i] + iw_0[i])$$

As a result, the asymptotic behaviour of $Y_{[i]}$ is entirely determined by that of I . Moreover, the square root in the denominator of (B.1b) can never vanish

A little algebra shows that:

$$\begin{aligned} |I(\alpha, \beta)| &\leq (z_0[i]^2 + w_0[i]^2)^{1/4} \int dv \frac{1}{((z_0[i] - v)^2 + w_0[i]^2)^{1/4}} |W'(v)| \\ &\leq (z_0[i]^2 + w_0[i]^2)^{1/4} \int dv \frac{1}{w_0[i]^{1/2}} |W'(v)| \\ &\leq \frac{(z_0[i]^2 + w_0[i]^2)^{1/4}}{w_0[i]^{1/2}} \int dv |W'(v)|. \end{aligned} \quad (\text{B.2})$$

The last equation makes evident that, if W is a regular function, then $I(\alpha, \beta)$ – and therefore $Y_{[i]}$ – are bounded.

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