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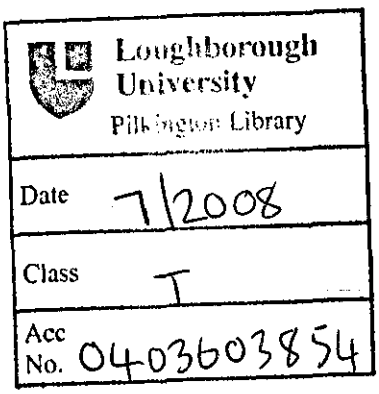
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Variation of Local Time and New Extensions to Itô's Formula

Chunrong Feng
(Department of Mathematical Sciences, Loughborough University, LE11 3TU, UK
and
School of Mathematics and System Sciences, Shandong University, Jinan 250100, China)


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Abstract

In this doctoral thesis, first we prove the continuous semimartingale local time L_t^i is of bounded p-varaition in the space variable in the classical sense for any $p > 2$ a.s., and based on this fact we define the integral of local time in the sense of Young integral, and in the sense of Lyons' rough path integral, so that we obtain the new extensions to Tanaka-Meyer's formula for more classes of f. We also give new conditions to two-parameter Young integral and extend Elworthy-Truman-Zhao's formula. In the final part we define a new integral, i.e. stochastic Lebesgue-Stieltjes integral and extend Tanaka-Meyer's formula to two dimensions.

Key Words: Young integral, two-parameter p, q-variation path integtal, local time, p-variation of local time, generalized Itô's formula, stochastic Lebesgue-Stieltjes integral, rough path.

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Chapter 1
One Parameter Integral of Local Time

§1.1 Introduction

The classical Itô's formula for twice differentiable functions has played a central role in stochastic analysis and almost all aspects of its applications and connection with analysis, PDEs, geometry, dynamical systems, finance and physics. It reads as follows.

(Itô(1944), Kunita & Watanabe(1967)) Let $f : R \to R$ be a function of class C^2 and let $X = \{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a continuous semimartingale. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d < X >_s . \tag{1.1.1}$$

But the restriction of Itô's formula to functions with twice differentiability often encounter difficulties in applications. Extensions to less smooth functions are useful in studying many problems such as partial differential equations with some singularities and mathematics of finance. Generally speaking, for any absolutely continuous function f and a continuous semi-martingale X_t, there exists A_t such that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + A_t. \tag{1.1.2}$$

To find A_t in both cases especially a pathwise formula becomes key to establish a useful extension to Itô's formula. In fact investigations already began in Tanaka [46] with a beautiful use of local times introduced in Lévy [29].

(Tanaka (1963)) For any real number a, there exists an increasing continuous process $L_\cdot^a$ called the local time of X in a such that,

$$|X_t - a| = |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + 2L_t^a$$
$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}}dX_s + L_t^a$$
$$(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{\{X_s \leq a\}}dX_s + L_t^a$$

The generalized Itô's formula in one-dimension for time-independent convex functions was developed in Meyer [36].

(Tanaka-Meyer(1976)) Let $f : R \to R$ be a convex function (or difference of two convex

functions) and μ its second derivative measure defined as $\mu([a,b)) := \nabla^- f(b) - \nabla^- f(a)$, $-\infty < a < b < \infty$. Then

$$f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s) dX_s + \int_{-\infty}^{\infty} L_t(x)\mu(dx) \ a.s., \tag{1.1.3}$$

where $\nabla^- f(x)$ is of bounded variation and $\int_{-\infty}^{\infty} L_t(x)\mu(dx)$ is a Lebesgue-Stieltjes integral associated with the measure $\mu(dx)$. L_t^x is the local time of semimartingale X_t at x.

An integral $\int_{-\infty}^{\infty} \nabla^- f(x) \mathrm{d}_x L_t(x)$ was introduced in [4] through the existence of the expression $f(X(t)) - f(X(0)) - \int_0^t \frac{\partial^-}{\partial x} f(X(s)) dX(s)$ in $L^2(\mathcal{F}, P)$, where $L_t(x)$ is the local time of the semi-martingale X_t. Generally speaking, one expects stronger conditions for the pathwise existence of the integrals of local times. However, in the framework of Lebesgue integrals, locally bounded variation in x for fixed t is the minimal condition on $\nabla^- f(x)$ to generate a measure, so it seems impossible to go beyond Tanaka-Meyer's formula. We remark that the striking fact that $L_t(x)$ is of bounded quadratic variation in x in the sense of Revuz and Yor [41] did not play a significant role in the proof of (1.1.3). It is therefore reasonable to conjecture that the conditions of defining the integrals of local times pathwise can be weakened. Inevitably, we have to go beyond Lebesgue integral. Here we use Young and Lyons' idea of integration (Lyons [30], [31], Lyons and Qian [32], Young [50], [51]) to define the integral of local time to go beyond the bounded variation condition.

We would like to remark that the quadratic variation in the sense of Revuz and Yor is not enough to define Young's integral for local times. So in Section 1.2, we prove local time $L_t(x)$ is of bounded p-variation in x for any $t \geq 0$, for any $p > 2$ almost surely. The main difficulty is overcome by using the idea of controlling the p-variation of continuous paths via the variations through dyadic partitions. This idea was originated by Lévy and used in [3], [18], [28] to prove the Brownian path is of bounded p-variation for $p > 2$.

In Section 1.3, using Young's integration of one parameter p-variation, we can immediately define $\int_{-\infty}^{\infty} \nabla^- f(x) d_x L_t(x)$ as a Young's integral if $\nabla^- f(x)$ is of bounded q-variation $(1 \leq q < 2)$. Then a new extension of Tanaka-Meyer's formula to f where $\nabla^- f(x)$ is of bounded q-variation $(1 \leq q < 2)$ follows immediately. And I also give an example to use our new extension of Tanaka-Meyer's formula.

§1.2 The p-variation of Local Time

First we recall the definition of p-variation path and its integration theory (see e.g. Young [50], Lyons and Qian [32]).

Definition 1.2.1 *We say a function $f : [x', x''] \to R$ is of bounded p-variation if*

$$\sup_{E} \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|^p < \infty, \tag{1.2.1}$$

where $E := \{x' = x_0 < x_1 < \cdots < x_m = x''\}$ is an arbitrary partition of $[x', x'']$. Here $p \geq 1$ is a fixed real number.

From Young [50], the integral $\int_{x'}^{x''} f(x)dg(x) = \lim_{m(E)\to 0} \sum_{i=1}^{m} f(\xi_i)(g(x_i) - g(x_{i-1}))$ is well defined if f is of bounded p-variation, g is of bounded q-variation, and f and g have no common discontinuities. Here $\xi_i \in [x_{i-1}, x_i]$, $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} > 1$, $m(E) = \sup_{1\leq i\leq m} (x_i - x_{i-1})$. And we also have:

(Theorem on term by term integration): Let $\{f_n\}$ be a W_p-sequence ($\{f_n\}$ is of bounded p-variation independent of n) converging densely to an f of W_p and converging uniformly to f at each point of a set A. Let $\{g_n\}$ be a W_q-sequence converging densely to a g of W_q, and converging uniformly at each point of a set B. Suppose further that $p, q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$, and that A includes the discontinuities of g, B those of f, $A \bigcup B$ all points of (x', x''). Then

$$\int_{x'}^{x''} f_n dg_n \to \int_{x'}^{x''} f dg.$$

Consider a continuous semimartingale X_t on a probability space $(\Omega, \mathcal{F}, P)$ with the decomposition

$$X_t = M_t + V_t, \tag{1.2.2}$$

where M_t is a local martingale, V_t is an adapted process of bounded variation. Then there exists semimartingale local time L_t^x of X_t as a nonnegative random field $L = \{L_t^x : (t, x) \in [0, \infty) \times R, \omega \in \Omega\}$ and

$$L(t, a) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{[a, a+\epsilon)}(X(s)) d <M>_s \quad a.s. \tag{1.2.3}$$

for each t and $a \in R$. Then it is well known for each fixed $a \in R$, $L(t, a, \omega)$ is continuous, and nondecreasing in t and right continuous with left limit (càdlàg) with respect to a ([25], [41]). Therefore we can define a Lebesgue-Stieltjes integral $\int_0^\infty \phi(s) dL(s, a, \omega)$ for each a for any Borel-measurable function ϕ. In particular

$$\int_0^\infty 1_{R\setminus\{a\}}(X(s)) dL(s, a, \omega) = 0 \quad a.s. \tag{1.2.4}$$

Furthermore if ϕ is differentiable, then we have the following integration by parts formula

$$\int_0^t \phi(s)dL(s,a,\omega) = \phi(t)L(t,a,\omega) - \int_0^t \phi'(s)L(s,a,\omega)\mathrm{d}s \quad a.s.. \tag{1.2.5}$$

Moreover, if $g(s,x,\omega)$ is measurable and bounded on $[0,t]\times R\times\Omega$, by the occupation times formula (e.g. see [25], [41]),

$$\int_0^t g(s,X(s))d<M>_s = 2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL(s,a,\omega)\mathrm{d}a \quad a.s..$$

If $g(\cdot,x)$ is absolutely continuous for each x, $\frac{\partial}{\partial s}g(s,x)$ is locally bounded and measurable in $[0,t]\times R$, then using the integration by parts formula, we have

$$\begin{aligned} & \int_0^t g(s,X(s))d<M>_s \\ = \; & 2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL(s,a,\omega)\mathrm{d}a \\ = \; & 2\int_{-\infty}^{\infty} g(t,a)L(t,a,\omega)\mathrm{d}a \; - 2\int_{-\infty}^{\infty}\int_0^t \frac{\partial}{\partial s}g(s,a)L(s,a,\omega)\mathrm{d}s\mathrm{d}a \; a.s.. \end{aligned} \tag{1.2.6}$$

On the other hand, by Tanaka formula

$$L(t,a) = (X(t)-a)^+ - (X(0)-a)^+ - \hat{M}(t,a) - \hat{V}(t,a),$$

where $\hat{Z}(t,a) = \int_0^t 1_{\{X(s)>a\}}dZ(s)$, $Z = M,V,X$. By a standard localizing argument, we may assume without loss of generality that there is a constant N for which

$$\sup_{0\le s\le t}|X(s)| \le N,\; <M>_t \le N,\; Var_t V \le N,$$

where $Var_t V$ is the total variation of V on $[0,t]$. From the property of local time (see Chapter 3 in [25]), for any $\gamma \ge 1$,

$$E|\hat{M}(t,a)-\hat{M}(t,b)|^{2\gamma} = E|\int_0^t 1_{\{a<X_s\le b\}}d<M>_s|^{\gamma} \le C(b-a)^{\gamma},\; a<b$$

where the constant C depends on γ and on the bound N. From Kolmogorov's tightness criterion (see [27]), we know that the sequence $Y_n(a) := \frac{1}{n}\hat{M}(t,a)$, $n = 1,2,\cdots$, is tight. Moreover for any $a_1,a_2,\cdots,a_k$,

$$\begin{aligned} & P(\sup_{a_i}|\frac{1}{n}\hat{M}(t,a_i)| \le 1) \\ = \; & P(|\frac{1}{n}\hat{M}(t,a_1)| \le 1, |\frac{1}{n}\hat{M}(t,a_2)| \le 1,\cdots,|\frac{1}{n}\hat{M}(t,a_k)| \le 1|) \\ \ge \; & 1-\sum_{i=1}^{k} P(|\frac{1}{n}\hat{M}(t,a_i)| > 1) \\ \ge \; & 1-\frac{1}{n^2}\sum_{i=1}^{k} E[\hat{M}^2(t,a_i)] \\ \ge \; & 1-\frac{k}{n^2}C(N), \end{aligned}$$

so by the weak convergence theorem of random fields (see Theorem 1.4.5 in [27]), we have

$$\lim_{n\to\infty} P(\sup_a |\hat{M}(t,a)| \leq n) = 1.$$

Furthermore it is easy to see that

$$\frac{1}{n}\hat{V}(t,a) \leq \frac{1}{n} Var_t V(t,a) \to 0, \ when\ n \to \infty,$$

so it follows that,

$$\lim_{n\to\infty} P(\sup_a |L(t,a)| \leq n) = 1.$$

Therefore in our localization argument, we can also assume $L(t,a)$ is bounded uniformly in a.

Note there is a different definition of variation established in Revuz and Yor [41] (see also Marcus and Rosen [34]) and the following result is known (Chapter VI, Theorem 1.21, [41]): Let (Δ_n) be a sequence of subdivisions of $[a,b]$ such that $|\Delta_n| \to 0$ as $n \to \infty$, for any nonnegative and finite random variable S,

$$\lim_{n\to\infty} \sum_{\Delta_n} (L_S^{a_{i+1}} - L_S^{a_i})^2 = 4\int_a^b L_S^x dx + \sum_{a<x\leq b} (L_S^x - L_S^{x-})^2 < \infty, \tag{1.2.7}$$

in probability. However this variation is not enough to enable us to apply Young's construction of integrals. We need the following new result to establish integrations of local times.

Lemma 1.2.1 *Continuous semimartingale local time L_t^x is of bounded p-variation in x for any $t \geq 0$, for any $p > 2$, almost surely.*

Proof: By the usual localization argument, we may first assume that there is a constant K for which $\sup_{0\leq s\leq t} |X_s|$, $\int_0^t |dV_s|$, $< M, M >_t \leq K$. By Tanaka's formula

$$L_t^x = (X_t - x)^+ - (X_0 - x)^+ - \widehat{M}_t^x - \widehat{V}_t^x, \tag{1.2.8}$$

where,

$$\widehat{M}_t^x = \int_0^t 1_{\{X_s > x\}} dM_s, \ \widehat{V}_t^x = \int_0^t 1_{\{X_s > x\}} dV_s.$$

First note the function $\varphi_t(x) := (X_t - x)^+ - (X_0 - x)^+$ is Lipschitz continuous in x with Lipschitz constant 2, which implies for any $p > 2$ and $a_i < a_{i+1}$

$$|\varphi_t(a_{i+1}) - \varphi_t(a_i)|^p \leq 2^p (a_{i+1} - a_i)^p. \tag{1.2.9}$$

Secondly, by Hölder's inequality, as V is of bounded variation, so

$$\begin{aligned} & |\widehat{V}_t^{a_{i+1}} - \widehat{V}_t^{a_i}|^p \\ \leq & \ |\int_0^t 1_{\{a_i<X_s\leq a_{i+1}\}}|dV_s||^p \\ \leq & \ c\int_0^t 1_{\{a_i<X_s\leq a_{i+1}\}}|dV_s|, \end{aligned} \tag{1.2.10}$$

where c is a generic constant. To treat $\widehat{M}_t^a$, we use the method in the proof of Lemma 3.7.5 in Karatzas and Shreve [25] or Theorem 6.1.7 in Revuz and Yor [41],

$$\begin{aligned} & E|\widehat{M}_t^{a_{i+1}} - \widehat{M}_t^{a_i}|^p \\ = & \ E|\int_0^t 1_{\{a_i<X_s\leq a_{i+1}\}}dM_s|^p \\ \leq & \ cE\left(\int_0^t 1_{\{a_i<X_s\leq a_{i+1}\}}d<M,M>_s\right)^{\frac{p}{2}} \\ = & \ cE(\int_{a_i}^{a_{i+1}} L_t^x dx)^{\frac{p}{2}} \\ = & \ c(a_{i+1}-a_i)^{\frac{p}{2}}E(\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}} L_t^x dx)^{\frac{p}{2}} \\ \leq & \ c(a_{i+1}-a_i)^{\frac{p}{2}}E\frac{1}{a_{i+1}-a_i}\int_{a_i}^{a_{i+1}}(L_t^x)^{\frac{p}{2}}dx \\ \leq & \ c(a_{i+1}-a_i)^{\frac{p}{2}}\sup_x E(L_t^x)^{\frac{p}{2}}. \end{aligned}$$

Here we used Burkholder-Davis-Gundy inequality, the occupation times formula, Jensen inequality and Fubini theorem. Now from (1.2.8) and using Burkholder-Davis-Gundy inequality again, we have

$$\begin{aligned} E(L_t^x)^{\frac{p}{2}} & \leq cE[(X_t-X_0)^{\frac{p}{2}} + (\int_0^t |dV_s|)^{\frac{p}{2}} + <M,M>_t^{\frac{p}{4}}] \\ & \leq cE<M,M>_t^{\frac{p}{4}} + cE(\int_0^t |dV_s|)^{\frac{p}{2}} + cE<M,M>_t^{\frac{p}{4}} < c_1(K,p). \end{aligned}$$

Therefore it follows that

$$E|\widehat{M}_t^{a_{i+1}} - \widehat{M}_t^{a_i}|^p \leq \ c(a_{i+1}-a_i)^{\frac{p}{2}}. \tag{1.2.11}$$

Here c is a constant depending on K,p. Now we use Proposition 4.1.1 in [32] ($i=1, \gamma > p-1$), for any partition $\{a_l\}$ of $[a,b]$

$$\sup_D \sum_l |\widehat{M}_t^{a_{l+1}} - \widehat{M}_t^{a_l}|^p \leq c(p,\gamma)\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n}|\widehat{M}_t^{a_k^n} - \widehat{M}_t^{a_{k-1}^n}|^p.$$

The crucial thing is that the right hand side does not depend on partition D, where

$$a_k^n = a + \frac{k}{2^n}(b-a), \ k=0,1,\cdots,2^n.$$

We take expectation

$$\begin{aligned} & E\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n}|\widehat{M}_t^{a_k^n}-\widehat{M}_t^{a_{k-1}^n}|^p \\ = & \sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n}E|\widehat{M}_t^{a_k^n}-\widehat{M}_t^{a_{k-1}^n}|^p \\ \leq & c\sum_{n=1}^{\infty} n^{\gamma}(\frac{b-a}{2^n})^{\frac{p}{2}-1} < \infty \ as \ p > 2. \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n}|\widehat{M}_t^{a_k^n}-\widehat{M}_t^{a_{k-1}^n}|^p < \infty \ a.s..$$

It turns out that for any interval $[a,b] \subset R$

$$\sup_D \sum_l |\widehat{M}_t^{a_{l+1}}-\widehat{M}_t^{a_l}|^p < \infty \ a.s..$$

But we know $L_t(a)$ has a compact support $[-K,K]$ in a. So for the partition $D := D_{-K,K} = \{-K = a_0 < a_1 < \cdots < a_r = K\}$, we obtain

$$\sup_D \sum_i |\widehat{M}_t^{a_{i+1}}-\widehat{M}_t^{a_i}|^p < \infty \ a.s.. \tag{1.2.12}$$

On the other hand, it is easy to see from (1.2.9) that

$$\begin{aligned} \sum_i |\varphi_t(a_{i+1})-\varphi_t(a_i)|^p & \leq 2^p \sum_i (a_{i+1}-a_i)^p \\ & \leq 2^p[\sum_i (a_{i+1}-a_i)]^p = 2^p(b-a)^p, \end{aligned} \tag{1.2.13}$$

and from (1.2.10) and bounded variation of V that

$$\sum_i |\widehat{V}_t^{a_{i+1}}-\widehat{V}_t^{a_i}|^p \leq c\int_0^t 1_{\{a<X_s\leq b\}}|dV_s| \leq c\int_0^t |dV_s| < \infty. \tag{1.2.14}$$

Then from (1.2.8), (1.2.12), (1.2.13), (1.2.14), we know that

$$\sup_D \sum_i |L_t^{a_{i+1}}-L_t^{a_i}|^p < \infty \qquad a.s..$$

Finally we can use the usual localization procedure to remove the assumption that $\sup_{0\leq s\leq t}|X_s|$, $\int_0^t |dV_s|$, $< M, M >_t \leq K$. For this, define a stopping time for an integer $K > 0$: $\tau_K = \inf\{s : \min\{|X_s|, \int_0^s |dV_r|, < M, M >_s\} > K\}$ if there exists s such that $\min\{|X_s|, \int_0^s |dV_r|, <$

$M, M >_s\} > K$ and $\tau_K = +\infty$ otherwise. Then the above result shows that there exists $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for each $\omega \in \Omega_1$ and each given integer $K > 0$,

$$\sup_D \sum_i |L^{a_{i+1}}_{t\wedge\tau_K} - L^{a_i}_{t\wedge\tau_K}|^p < \infty.$$

Since $\sup_{0\le s\le t} |X_s|(\omega)$, $\int_0^t |dV_s|(\omega)$ and $< M, M >_t (\omega)$ are finite almost surely so there exists $\Omega_2 \subset \Omega$ with $P(\Omega_2) = 1$ such that for each $\omega \in \Omega_2$, there exists an integer $K(\omega) > 0$ such that $\sup_{0\le s\le t} |X_s|(\omega)$, $\int_0^t |dV_s|(\omega)$, $< M, M >_t (\omega) \le K$. This leads to $\tau_K(\omega) > t$. So for each $\omega \in \Omega_1 \cap \Omega_2$,

$$\sup_D \sum_i |L^{a_{i+1}}_t - L^{a_i}_t|^p < \infty.$$

The result follows as $P(\Omega_1 \cap \Omega_2) = 1$. ◇

Recall the well-known result (see Revuz and Yor [41], P220) that for each t, the random function $x \to L^x_t$ is a càdlàg function hence only admits at most countably many discontinuous points. Denote $\widehat{L}^x_t = L^x_t - L^{x-}_t$. Then

$$\widehat{L}^x_t = \int_0^t 1_{\{x\}}(X_s)dV_s, \tag{1.2.15}$$

and for any $a < b$,

$$\sum_{a<x\le b} |\widehat{L}^x_t| = \int_0^t |dV_s| < \infty. \tag{1.2.16}$$

By Tanaka's formula

$$\begin{aligned}
L^x_t &= (X_t - x)^+ - (X_0 - x)^+ - \widehat{M}^x_t - \widehat{V}^x_t \\
&= (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s>x\}}dM_s - \int_0^t 1_{\{X_s>x\}}dV_s, \\
&= (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s>x\}}dM_s - \int_0^t dV_s + \int_0^t 1_{\{X_s\le x\}}dV_s \\
&= \tilde{L}^x_t + \sum_{x^*_k\le x} \int_0^t 1_{\{x^*_k\}}(X_s)dV_s \\
&= \tilde{L}^x_t + \sum_{x^*_k\le x} \widehat{L}^{x^*_k}_t,
\end{aligned} \tag{1.2.17}$$

where

$$\tilde{L}^x_t = (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s>x\}}dM_s - (V_t - V_0) \tag{1.2.18}$$

is continuous in x, and $\{x^*_k\}$ are the discontinuous points of L^x_t. Denote

$$h(t,x) := \sum_{x^*_k\le x} \widehat{L}^{x^*_k}_t. \tag{1.2.19}$$

Lemma 1.2.2 *Above defined $h(t,x)$ is of bounded variation in x for each t and of bounded variation in (t,x) for almost every $\omega \in \Omega$.*

Proof: Let $[-N,N]$ be the support of $L_t(x)$. To see that $h(t,x)$ is of locally bounded variation in x, consider any partition $D = \{-N = x_0 < x_1 < \cdots < x_{m-1} < x_m = N\}$, then from (1.2.16)

$$\begin{aligned}\sum_i |h(t,x_{i+1}) - h(t,x_i)| &= \sum_i |\sum_{x_i < x_k^* \le x_{i+1}} \widehat{L}_t^{x_k^*}| \\ &\le \sum_i \sum_{x_i < x_k^* \le x_{i+1}} |\widehat{L}_t^{x_k^*}| \\ &= \sum_{-N < x \le N} |\widehat{L}_t^x| < \infty.\end{aligned}$$

To see it is of bounded variation in (t,x), consider any partition $D' \times D$, where $D' = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}$, $D = \{-N = x_0 < x_1 < \cdots < x_{m-1} < x_m = N\}$,

$$\begin{aligned}&\sum_i |h(t_{j+1},x_{i+1}) - h(t_{j+1},x_i) - h(t_j,x_{i+1}) + h(t_j,x_i)| \\ =\ &\sum_i |\sum_{x_i < x_k^* \le x_{i+1}} (\widehat{L}_{t_{j+1}}^{x_k^*} - \widehat{L}_{t_j}^{x_k^*})| \\ \le\ &\sum_i \sum_{x_i < x_k^* \le x_{i+1}} |\widehat{L}_{t_{j+1}}^{x_k^*} - \widehat{L}_{t_j}^{x_k^*}| \\ =\ &\sum_{-N < x \le N} |\widehat{L}_{t_{j+1}}^x - \widehat{L}_{t_j}^x|. \end{aligned} \tag{1.2.20}$$

Now applying (5.3.3) leads to,

$$\begin{aligned}&\sum_{-N < x \le N} |\widehat{L}_{t_{j+1}}^x - \widehat{L}_{t_j}^x| = \sum_{-N < x \le N} |\int_{t_j}^{t_{j+1}} 1_{\{x\}}(X_s) dV_s| \\ \le\ &\int_{t_j}^{t_{j+1}} 1_{[-N,N]}(X_s)|dV_s|. \end{aligned} \tag{1.2.21}$$

From (1.2.20), (1.2.21) and the bounded variation assumption of V, we have

$$\begin{aligned}&\sum_j \sum_i |h(t_{j+1},x_{i+1}) - h(t_{j+1},x_i) - h(t_j,x_{i+1}) + h(t_j,x_i)| \\ \le\ &\int_0^t 1_{[-N,N]}(X_s)|dV_s| < \infty. \end{aligned} \tag{1.2.22}$$

◇

§1.3 One Parameter Integral of Local Time in the Sense of Young Integral

Due to the decomposition (1.2.17) of local time, the following integral is therefore defined by

$$\int_{-\infty}^{\infty} f(x)d_xL_t^x = \int_{-\infty}^{\infty} f(x)d_x\tilde{L}_t^x + \int_{-\infty}^{\infty} f(x)d_xh(t,x).$$

The last integral is a Lebesgue-Stieltjes integral, it doesn't matter whether or not f is continuous as long as it is measurable. If f is of finite p-variation $(1 \le p < 2)$, we know the integral $\int_{-\infty}^{\infty} f(x)d_x\tilde{L}_t^x$ is well defined by Young's integration theory.

Remark 1.3.1 *If f belongs to C^1, we have*

$$\int_{-\infty}^{\infty} f(x)d_xL_t^x = -\int_{-\infty}^{\infty} L_t^x df(x). \tag{1.3.1}$$

This is because L_t has a compact support for each t, so one can always add some points in the partition to make $L_t^{x_1} = 0$ and $L_t^{x_r} = 0$. So

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)d_xL_t^x \\
=\ &\lim_{m(D)\to 0}\sum_{k=1}^{r} f(x_{k-1})(L_t^{x_k} - L_t^{x_{k-1}}) \\
=\ &\lim_{m(D)\to 0}[\sum_{k=1}^{r} f(x_{k-1})L_t^{x_k} - \sum_{k=0}^{r-1} f(x_k)L_t^{x_k}] \\
=\ &-\lim_{m(D)\to 0}\sum_{k=1}^{r}(f(x_k) - f(x_{k-1}))L_t^{x_k} \\
=\ &-\int_{-\infty}^{\infty} L_t^x df(x).
\end{aligned}$$

Assume $g(x)$ is a left continuous function and locally bounded, we use the standard regularizing mollifiers to smoothrize g (e.g. see [25]). Define

$$\rho(x) = \begin{cases} ce^{\frac{1}{(x-1)^2-1}}, & \text{if } x \in (0,2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.3.2}$$

Here c is chosen such that $\int_0^2 \rho(x)dx = 1$. Take $\rho_n(x) = n\rho(nx)$ as mollifiers. Define

$$g_n(x) = \int_{-\infty}^{+\infty} \rho_n(x-y)g(y)dy, \quad n \ge 1.$$

Then $g_n(x)$ is smooth and

$$g_n(x) = \int_0^2 \rho(z) g(x - \frac{z}{n}) dz, \quad n \geq 1. \tag{1.3.3}$$

Using Lebesgue's dominated convergence theorem, one can prove that as $n \to \infty$, $g_n(x) \to g(x)$.

Theorem 1.3.1 *Let $g(x)$ be a left continuous function with finite p-variation in x, $1 \leq p < 2$, $g_n(x)$ be defined in (1.3.3). Then*

$$\int_{-\infty}^{\infty} g_n(x) d_x \tilde{L}_t^x \to \int_{-\infty}^{\infty} g(x) d_x \tilde{L}_t^x, \text{ as } n \to \infty. \tag{1.3.4}$$

Proof: Let $\delta > 0$ satisfy $\frac{1}{2+\delta} + \frac{1}{p} > 1$. From Lemma 1.2.1, $\tilde{L}_t^x$ is of bounded $(2+\delta)$-variation in x. From [51], $g(x)$ being of bounded p-variation, $1 \leq p < 2$, is equivalent to that for any partition $D := D_{-N,N} = \{-N = x_0 < x_1 < \cdots < x_r = N\}$ defined as before, there is an increasing function w such that

$$|g(x_{l+1}) - g(x_l)| \leq (w(x_{l+1}) - w(x_l))^{\frac{1}{p}}, \ \forall x_l, x_{l+1} \in D,$$

where $w(x)$ is the total p-variation of f in the interval $[-N-2, x]$. Using Hölder's inequality, we get

$$\begin{aligned}
& \sup_D \sum_{l=1}^{r} |g_n(x_l) - g_n(x_{l-1})|^p \\
= & \sup_D \sum_{l=1}^{r} \left| \int_0^2 \rho(z)[g(x_l - \frac{z}{n}) - g(x_{l-1} - \frac{z}{n})] dz \right|^p \\
\leq & M_1 \sup_D \sum_{l=1}^{r} \left(\int_0^2 |g(x_l - \frac{z}{n}) - g(x_{l-1} - \frac{z}{n})|^p dz \right) \\
\leq & M_1 \int_0^2 \sup_D \sum_{l=1}^{r} |g(x_l - \frac{z}{n}) - g(x_{l-1} - \frac{z}{n})|^p dz \\
\leq & M_1 \int_0^2 (w(N - \frac{z}{n}) - w(-N - \frac{z}{n})) dz,
\end{aligned}$$

where M_1 is a constant. As

$$w(N - \frac{z}{n}) - w(-N - \frac{z}{n}) \leq w(N),$$

so

$$\sup_D \sum_{l=1}^{r} |g_n(x_l) - g_n(x_{l-1})|^p \leq 2M_1 w(N) < \infty, \tag{1.3.5}$$

which means that $g_n(x)$ is of bounded p-variation in x uniformly in n. Then (1.3.4) follows from Young's ([50] or [51]) convergence theorem we can get the result directly. ◇

Remark 1.3.2 *From the Lebesgue's dominated convergence theorem, for g in the above theorem, we know*

$$\int_{-\infty}^{\infty} g_n(x)d_xh(t,x) \to \int_{-\infty}^{\infty} g(x)d_xh(t,x), \ as\ n \to \infty.$$

With Theorem 1.3.1, it follows that

$$\int_{-\infty}^{\infty} g_n(x)d_xL_t^x \to \int_{-\infty}^{\infty} g(x)d_xL_t^x, \ as\ n \to \infty. \tag{1.3.6}$$

Using the above theorem, we can get an extension of Itô's Formula.

Theorem 1.3.2 *Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale and $f : R \to R$ be an absolutely continuous function and have left derivative $\nabla^- f(x)$ being left continuous and locally bounded. Assume $\nabla^- f(x)$ is of bounded q-variation, where $1 \leq q < 2$. Then we have the following change-of-variable formula*

$$f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s)dX_s - \int_{-\infty}^{\infty} \nabla^- f(x)d_xL_t^x, \tag{1.3.7}$$

where L_t^x is the local time of X_t at x.

Proof: The integral $\int_{-\infty}^{\infty} \nabla^- f(x)d_xL_t^x$ is defined pathwise as a combination of rough path integral and Lebesgue-Stieltjes integral. We may quote the proof in [25] and define

$$f_n(x) = \int_{-\infty}^{+\infty} \rho_n(x-y)f(y)dy, \quad n \geq 1.$$

The convergence of all terms except the second order derivative term are the same as in the proof in [25]. By occupation times formula and Remark 1.3.1, the second order derivative term is

$$\begin{aligned}\frac{1}{2}\int_0^t \Delta f_n(X_s)d < M >_s &= \int_{-\infty}^{\infty} \Delta f_n(x)L_t^x dx \\ &= \int_{-\infty}^{\infty} L_t^x d\nabla f_n(x) \\ &= -\int_{-\infty}^{\infty} \nabla f_n(x)d_xL_t^x.\end{aligned}$$

It follows from (1.3.6) that,

$$\frac{1}{2}\int_0^t \Delta f_n(X_s)d < M >_s \to -\int_{-\infty}^{\infty} \nabla^- f(x)d_xL_t^x,$$

when $n \to \infty$. Our claim is asserted. ◇

Needless to say, there are many cases that Theorem 1.3.2 works, but other extensions of Itô's formula do not apply immediately. The following is an obvious example:

Example 1.3.1 *Consider a function* $f(x) = x^3 \cos\frac{1}{x}$ *for* $x \neq 0$ *and* $f(0) = 0$. *This function is* C^1 *and its derivative is* $f'(x) = 3x^2 \cos\frac{1}{x} + x \sin\frac{1}{x}$ *for* $x \neq 0$ *and* $f'(0) = 0$. *It is easy to see that* f' *is not of bounded variation, but of* p*-variation for any* $p > 1$ *(see Example 3.3.1 for a proof in a more complicated case). So Theorem 1.3.2 can be used, while Tanaka-Meyer's formula cannot apply to this situation.*

Chapter 2
Local Time as a Rough Path

§2.1 Introduction

In Chapter 1, Lemma 1.2.1 says that the semimartingale local time L_t^x is of bounded p-variation in x for any $t \geq 0$, $p > 2$ a.s.. So in Theorem 1.3.2, we gave a new condition for Tanaka-Meyer's formula and the integral $\int_{-\infty}^{\infty} \nabla^- f(x) d_x L_t^x$ is defined as Young integral, when $\nabla^- f(x)$ is of bounded q-variation ($1 \leq q < 2$). But how is about if $q \geq 2$? Can we still define such an integral $\int_{-\infty}^{\infty} \nabla^- f(x) d_x L_t^x$ pathwise? If we can, we will get Tanaka-Meyer's formula for wider class of functions. But Young's integration theory does not work here as the crucial condition $\frac{1}{p} + \frac{1}{q} > 1$ is no longer valid. So in this Chapter, we use rough path theory to extend q to any $2 \leq q < 3$.

§2.2 Brief Introduction to Lyons' Rough Path Theory

In this section, we introduce some basics of rough path theory, mainly from Lyons and Qian [32].

In Chapter 3 of their book, Lyons and Qian gave the main development of rough path. Here, I only list some which will be used in the next section.

For each $n \in N$, define the following (truncated) tensor algebra

$$T^{(n)}(V) := \sum_{k=0}^{n} \oplus V^{\otimes k}, \quad V^{\otimes 0} = R,$$

where V is a finite dimensional space (though it is also correct for Banach space). Its multiplication (also called tensor product) is the usual multiplication as polynomials, except that the higher-order (than degree n) terms are omitted. In other words, if $\xi = (\xi^0, \xi^1, \cdots, \xi^n)$, $\eta = (\eta^0, \eta^1, \cdots, \eta^n)$ are two vectors in $T^{(n)}(V)$, then $\zeta = \xi \otimes \eta \in T^{(n)}(V)$, where its kth component is

$$\zeta^k = \sum_{j=0}^{k} \xi^j \otimes \eta^{k-j}, \quad k = 0, 1, \cdots, n.$$

The norm $|\cdot|$ on $T^{(n)}(V)$ is defined by

$$|\xi| = \sum_{i=0}^{n} |\xi^i|, \quad if\ \xi = (\xi^0, \xi^1, \cdots, \xi^n).$$

We use Δ or Δ_T to denote the simplex $\{(s,t) : 0 \le s \le t \le T\}$. A control w is a continuous super-additive function on Δ with values in $[0,\infty)$ such that $w(t,t) = 0$. Therefore

$$w(s,t) + w(t,u) \le w(s,u), \quad \textit{for any } (s,t),(t,u) \in \Delta.$$

(Definition 3.1.1) A continuous map X from the simplex Δ into a truncated tensor algebra $T^{(n)}(V)$, and written as

$$X_{s,t} = (1, X^1_{s,t}, \cdots, X^n_{s,t}), \textit{ with } X^k_{s,t} \in V^{\otimes k}, \textit{ for any } (s,t) \in \Delta,$$

is called a multiplicative functional of degree n ($n \in N$, $n \ge 1$) if $X^0_{s,t} \equiv 1$ and

$$X_{s,t} \otimes X_{t,u} = X_{s,u}, \quad \textit{for any } (s,t),(t,u) \in \Delta, \tag{2.2.1}$$

where the tensor product $\otimes$ is taken in $T^{(n)}(V)$. Equality (2.2.1) is called the Chen identity.

Example 2.2.1 *Let $x : [0,T] \to V = R^d$ be a continuous path. Then its increment process $X : \Delta \to T^{(1)}(V)$ defined by $X_{s,t} = (1, X^1_{s,t})$, $X^1_{s,t} = x_t - x_s$ is a multiplicative functional of degree* 1. *In this case, Chen's identity is equivalent to the additive property of increments over different intervals. If, in addition, x is a Lipschitz path, we may build a sequence of iterated path integrals*

$$X^k_{s,t} = \int_{s<t_1<\cdots<t_k<t} dx_{t_1} \otimes \cdots \otimes dx_{t_k}.$$

Let's see the second iterated integral $X^2_{s,t}$. Let $e_1, \cdots, e_d$ be the basis of $V = R^d$, which implies that $x_t = \sum_{i=1}^{d} e_i x^i_t$, $t \in [0,T]$. Define $\int_{s<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2}$ as an element of $V \otimes V$ by

$$\int_{s<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2} = \sum_{i,j=1}^{d} e_i \otimes e_j \int_{s<t_1<t_2<t} dx^i_{t_1} dx^j_{t_2}.$$

It's easy to see for $0 \le s \le u \le t \le T$,

$$\begin{aligned}\int_{s<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2} \;=\; & \int_{s<t_1<t_2<u} dx_{t_1} \otimes dx_{t_2} + \int_{u<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2} \\ & + (x_u - x_s) \otimes (x_t - x_u),\end{aligned}$$

i.e.

$$X^2_{s,t} = X^2_{s,u} + X^2_{u,t} + X^1_{s,t} \otimes X^1_{u,t}$$

satisfying Chen's identity.

(Definition 3.1.2) Let $p \geq 1$ be a constant. We say that a map $X : \Delta \to T^{(n)}(V)$ possesses finite p-variation if

$$|X^i_{s,t}| \leq w(s,t)^{i/p}, \quad for\ any\ i = 1, \cdots, n,\ (s,t) \in \Delta, \tag{2.2.2}$$

for some control w.

Theorem 3.1.2 in [32] shows that the higher (than $[p]$) order terms X^k $(k > [p])$ are determined uniquely by X^i $(i \leq [p])$ among all possible extensions to a multiplicative functional which possess finite p-variations. Therefore we may give the following definition of rough path.

(Definition 3.1.3) A multiplicative functional with finite p-variation in $T^{([p])}(V)$ is called a rough path (of roughness p). We say that a rough path (of roughness p) is controlled by w if

$$|X^i_{s,t}| \leq w(s,t)^{i/p}, \quad for\ any\ i = 1, \cdots, [p],\ (s,t) \in \Delta. \tag{2.2.3}$$

The set of all rough paths with roughness p in $T^{([p])}(V)$ will be denoted by $\Omega_p(V)$.

Next, let's see a method of constructing rough path.

(Definition 3.2.1) Let $p \geq 1$ be a constant. A function $X : \Delta \to T^{([p])}(V)$ is called an almost rough path (of roughness p) if it is of finite p-variation, $X^0_{s,t} \equiv 1$, and for some control w and some constant $\theta > 1$,

$$|(X_{s,t} \otimes X_{t,u})^i - X^i_{s,u}| \leq w(s,u)^\theta,$$

for all $(s,t),(t,u) \in \Delta$ and $i = 1, \cdots, [p]$.

(Theorem 3.2.1) *If $X : \Delta \to T^{([p])}(V)$ is an almost rough path of roughness p, then there is a unique rough path $\hat{X}$ (with roughness p) in $T^{([p])}(V)$ such that*

$$|\hat{X}^i_{s,t} - X^i_{s,t}| \leq K_i w(s,t)^\theta, \quad for\ any\ 1 \leq i \leq [p],\ (s,t) \in \Delta,$$

for some control w, some constants K_i and $\theta > 1$.
Actually, $\hat{X}$ can be constructed like

$$\hat{X}^{k+1}_{s,t} = \lim_{m(D)\to 0} \sum_{l=1}^{r} \left(X^{k+1}_{t_{l-1},t_l} + \sum_{i=1}^{k} \hat{X}^i_{s,t_{l-1}} \otimes \hat{X}^{k+1-i}_{t_{l-1},t_l} \right), \quad for\ any\ (s,t) \in \Delta, \tag{2.2.4}$$

where D is a partition of $[s,t]$.

In the following, I will also cite some results about spaces of rough paths. Let $C_0(\Delta, T^{(n)}(V))$ denote the set of all continuous functions from simplex Δ into the truncated tensor algebra $T^{(n)}(V)$, with an appropriate norm, and $X^0_{s,t} \equiv 1$.

(Definition 3.3.1) A function $X \in C_0(\Delta, T^{(n)}(V))$ is said to have finite total p-variation if

$$\sup_D \sum_l |X^i_{t_{l-1},t_l}|^{p/i} < \infty, \quad i = 1, \cdots, n, \tag{2.2.5}$$

where sup_D runs over all finite divisions of $[0,T]$.

It is clear that if $X \in C_0(\Delta, T^{(n)}(V))$ is of finite p-variation, then X has finite total p-variation. Conversely we have the following proposition.

(Proposition 3.3.2) Let $p \geq 1$ be a constant, and let $X \in C_0(\Delta, T^{(n)}(V))$ satisfy Chen's identity. If X has finite total p-variation, then

$$w(s,t) = \sum_{i=1}^{n} \sup_{D_{[s,t]}} \sum_l |X^i_{t_{l-1},t_l}|^{p/i}, \quad \textit{for any } (s,t) \in \Delta \tag{2.2.6}$$

is a control function, and

$$|X^i_{s,t}| \leq w(s,t)^{i/p}, \quad \textit{for any } i = 1, \cdots, n, \ (s,t) \in \Delta.$$

Let $C_{0,p}(\Delta, T^{(n)}(V))$ denote the subspace of all $X \in C_0(\Delta, T^{(n)}(V))$ with finite total p-variation. The p-variation metric d_p on $C_{0,p}(\Delta, T^{([p])}(V))$ is defined by

$$d_p(X,Y) = \max_{1 \leq i \leq [p]} \sup_D \left(\sum_l |X^i_{t_{l-1},t_l} - Y^i_{t_{l-1},t_l}|^{p/i} \right)^{i/p}. \tag{2.2.7}$$

(Lemma 3.3.3) $(\Omega_p(V), d_p)$ is a complete metric space.

However, the distance function d_p is difficult to use in practice. Therefore we need the following definition.

(Definition 3.3.2) (i) We say a sequence $\{X(n)\}$ of $C_{0,p}(\Delta, T^{(N)}(V))$ converges to $X \in C_{0,p}(\Delta, T^{(N)}(V))$ in p-variation topology if there is a control w such that

$$|X(n)^i_{s,t}|, |X^i_{s,t}| \leq w(s,t)^{i/p}, \quad i = 1, \cdots, N, \ \textit{for any } (s,t) \in \Delta, \tag{2.2.8}$$

for any $n = 1, 2 \cdots$, and

$$|X(n)^i_{s,t} - X^i_{s,t}| \leq a(n) w(s,t)^{i/p}, \quad i = 1, \cdots, N, \ \textit{for any } (s,t) \in \Delta, \tag{2.2.9}$$

for some function $a(n)$ (which may depend on the sequence $X(n)$, X, and the control w) such that $\lim_{n\to\infty} a(n) = 0$.

(ii) Let $p, q \geq 1$ be two constants. We say a map $F : C_{0,p}(\Delta, T^{(N)}(V)) \to C_{0,q}(\Delta, T^{(N')}(W))$ is continuous in (p,q)-variation topology if, for any control w, there is a control w_1 and a function $\alpha : R^+ \to R^+$ satisfying the condition $\lim_{\varepsilon \downarrow 0} \alpha(\varepsilon) = 0$, such that, if $X, Y \in C_{0,p}(\Delta, T^{(N)}(V))$ and

$$|X^i_{s,t}|, |Y^i_{s,t}| \leq w(s,t)^{i/p}, \quad i = 1, \cdots, N, \ \textit{for any } (s,t) \in \Delta,$$
$$|X^i_{s,t} - Y^i_{s,t}| \leq \varepsilon w(s,t)^{i/p}, \quad i = 1, \cdots, N, \ \textit{for any } (s,t) \in \Delta,$$

then

$$|F(X)^j_{s,t} - F(Y)^j_{s,t}| \leq \alpha(\varepsilon) w_1(s,t)^{j/q}, \quad j = 1, \cdots, N', \ \textit{for any } (s,t) \in \Delta.$$

(Definition 3.3.3) A rough path $X \in \Omega_p(V)$ is called a smooth rough path if $t \to X_t \equiv X^1_{0,t}$ is a continuous path with finite variation and $X^i_{s,t}$ is the ith iterated path integral of the path X_t over the interval $[s,t]$ (for $i = 1, \cdots, [p]$), that is

$$X^i_{s,t} = \int_{s<t_1<\cdots<t_i<t} dX_{t_1} \otimes \cdots \otimes dX_{t_i}, \quad \textit{for any } (s,t) \in \Delta. \tag{2.2.10}$$

A rough path $X \in \Omega_p(V)$ is a geometric rough path if there is a sequence $X(n)$ of smooth rough paths in $\Omega_p(V)$ such that

$$d_p(X(n), X) \to 0, \quad \textit{as } n \to \infty.$$

In Chapter 4 [32], Lyons and Qian showed us how to construct a Brownian rough path. First they gave a key estimation. Let $[S,T]$ be any finite interval. Consider its dyadic decompositions $\{S = t^n_0 < t^n_1 \cdots < t^n_{2^n} = T\}$ of $[S,T]$, where

$$t^n_k = \frac{k}{2^n}(T - S) + S, \quad k = 0, \cdots, 2^n, \ n \in N.$$

In this chapter they let $[S,T]=[0,1]$, though the results can be applied to any bounded intercal, so $t_k^n = k/2^n (k=0,\cdots,2^n)$.

(Proposition 4.1.1) Let $X \in C_0(\Delta, T^{(N)}(V))$ (with a fixed running time interval, say $[0,1]$) be a multiplicative functional. Then for any $1 \le i \le N$, p satisfying $p/i > 1$, and any $\gamma > p/i - 1$, there exists a constant $C_i(p,\gamma)$ depending only on p, γ, and i, such that

$$\sup_D \sum_l |X^i_{t_{l-1},t_l}|^{p/i} \le C_i(p,\gamma) \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \sum_{j=1}^{i} |X^j_{t^n_{k-1},t^n_k}|^{p/j}, \tag{2.2.11}$$

Second, let's see how they construct a Brownian rough path. Let W be a continuous path in V and let $X^1_{s,t} = W_t - W_s$. For $m \in N$, we define a continuous and piecewise-linear path $W(m)$ by

$$W(m)_t = W_{t^m_{l-1}} + 2^m (t - t^m_{l-1}) \Delta^m_l W, \quad of\ t^m_{l-1} \le t \le t^m_l, \tag{2.2.12}$$

for $l = 1,\cdots,2^m$, where $n \in N$, $t^n_k = k/2^n$ $(k = 0,1,\cdots,2^n)$ are dyadic points, and $\Delta^n_k W = W_{t^n_k} - W_{t^n_{k-1}}$. The corresponding smooth rough path (of degree k) is denoted by $X(m)$ which is built by taking its iterated path integrals. That is,

$$X(m)^j_{s,t} = \int_{s<t_1<\cdots<t_j<t} dW(m)_{t_1} \otimes \cdots \otimes dW(m)_{t_j}. \tag{2.2.13}$$

(Definition 4.4.1) We say a real-valued, continuous stochastic process $(W_t)_{t\in[0,1]}$ on a completed probability space $(\Omega, \mathcal{F}, \mathcal{P})$ has (h,p)-long-time memory for some $h \in (0,1)$, $p > 1$, such that $hp > 1$, if there exists a constant C such that (W_t) satisfies the L^p Hölder condition,

$$E|W_t - W_s|^p \le C|t-s|^{hp}, \quad for\ any\ [s,t] \in [0,1], \tag{2.2.14}$$

and, for all $1 \ge t > s \ge 0$, $\tau > 0$ such that $(t-s)/\tau \le 1$, we have

$$|E(W_t - W_s)(W_{t+\tau} - W_{s+\tau})| \le C\tau^{2h} \left|\frac{t-s}{\tau}\right|^2. \tag{2.2.15}$$

A d-dimensional Brownian motion B_t satisfies

$$E|B_t - B_s|^p = C_{p,d}|t-s|^{p/2},$$

and therefore the Hölder condition is satisfied for $p > 0$ when $h = \frac{1}{2}$. However, the condition that $hp > 1$ forces $p > 2$.

(Theorem 4.4.1) Let $W_t = (w^1_t, \cdots, w^d_t)$ be d independent stochastic processes with

(h,p)-long-time memory for some $0 < h < \frac{3}{4}$ such that $ph > 1$. Then there is a unique function X^i on Δ which takes values in $(R^d)^{\otimes i}$ $(i = 1, 2)$ such that

$$\sum_{i=1}^{2} \sup_{D} \left(\sum_{l} \left| X(m)^i_{t_{l-1},t_l} - X^i_{t_{l-1},t_l} \right|^{p/i} \right)^{i/p} \to 0,$$

both almost surely and in $L^1(\Omega, \mathcal{F}, \mathcal{P})$, as $m \to \infty$.

The theory of rough paths provides a pathwise theory of integration, but pathwise with respect to X, not to stochastic process x. Chapter 5 mainly gives us how to construct path integration along rough paths. The aim of this chapter is to define path integral of the type: $\int \alpha(X)dX$ for a rough path X in $T^{([p])}(V)$, where $\alpha : V \to L(V, W)$. A special case is where $\alpha = df$ is the Fréchet differential of a W-valued smooth function f on V. We call such a function α a (W-valued) one-form on V.

From Chaper VI in [45], if $f : R^d \to R^m$, for $0 < \gamma \leq 1$, $Lip(\gamma, R^d)$ will be defined like

$$Lip(\gamma, R^d) := \{f : |f(x)| \leq M, |f(x) - f(y)| \leq M|x - y|^\gamma, x, y \in R^d\}.$$

But if $\gamma > 1$, $Lip(\gamma, R^d)$ consists constant only. So how is to define $Lip(\gamma, R^d)$ for any $\gamma > 0$?

Definition 2.2.1 *Let $k \geq 0$ be an integer, $\gamma \in (k, k+1]$ be a real number, $f : R^d \to R^m$. We say the collection $(f = f^{(0)}, f^{(1)}, \cdots, f^{(k)})$ is an element of $Lip(\gamma, R^d)$ if*

$$f^{(0)} = f, \quad f^{(j)}(x) = \sum_{l=0}^{k-j} \frac{f^{(j+l)}(y)}{l!}(x - y)^l + R_j(x, y) \tag{2.2.16}$$

and

$$|f^{(j)}(x)| \leq M, \quad |R_j(x, y)| \leq M|x - y|^{\gamma - j} \quad , 0 \leq j \leq k, \ for \ any \ x, y \in R^d.$$

Remark 2.2.1 *The above definition means f is continuous, bounded and has continuous bounded derivatives of order not greater than k, and $f^{(k)} \in Lip(\gamma - k, R^m)$. In fact, $f^{(k)} = d^k f$ and (2.2.16) is Taylor's expansion.*

(Definition 5.1.1) Let $p \geq 1$ and $p < \gamma \leq [p] + 1$. Let $\alpha : V \to L(V, W)$, where V and W are two finite dimensional spaces, and let $V^{\otimes 2}, \cdots, V^{\otimes [p]}$ and $W^{\otimes 2}, \cdots, W^{\otimes [p]}$ be their tensor spaces up to degree $[p]$. We say that the system $(\alpha, V^{\otimes j}, W^{\otimes j} : 1 \leq j \leq [p])$ is

admissible if

(i) α is a Lip(γ) one-form (with respect to p) in the sense that, for $j = 1, \cdots, [p]$, there exist functions $\alpha^j : V \to L(V^{\otimes j}, W)$ and $R_j : V \times V \to L(V^{\otimes j}, W)$ such that $\alpha^1 = \alpha$, and, for any Lipschitz path X in V, we have

$$\alpha^j(X_t) = \sum_{i=0}^{[p]-j} \alpha^{i+j}(X_s)(X_{s,t}^i) + R_j(X_s, X_t), \tag{2.2.17}$$

$$\int_s^t \alpha^{j+1}(X_u)(dX_u) = \alpha^j(X_t) - \alpha^j(X_s), \tag{2.2.18}$$

for all $t > s$, and

$$|\alpha^1(\xi)| \leq M(1 + |\xi|), \quad |\alpha^{j+1}(\xi)| \leq M, \quad \textit{for any } \xi \in V, \tag{2.2.19}$$

$$|R_j(\xi, \eta)| \leq M|\xi - \eta|^{\gamma - j}, \quad \textit{for any } \xi, \eta \in V, \tag{2.2.20}$$

for $j = 1, \cdots, [\gamma]$.

(ii) For all $j = (j_1, \cdots, j_k)$ (integers $j_i \geq 0$) such that $|j| = \sum_{i=1}^{k} j_i \leq [p]$, the linear operator $\alpha^{j_1}(\xi) \otimes \cdots \otimes \alpha^{j_k}(\xi)$ from $V^{\otimes |j|}$ to $W^{\otimes |j|}$ is bounded (with bound M), where

$$\begin{aligned} & \alpha^{j_1}(\xi) \otimes \cdots \otimes \alpha^{j_k}(\xi) \left(\sum_i v_i^{j_1} \otimes \cdots \otimes v_i^{j_k} \right) \\ = & \sum_i (\alpha^{j_1}(\xi)(v_i^{j_1})) \otimes \cdots \otimes (\alpha^{j_k}(\xi)(v_i^{j_k})), \end{aligned}$$

for all $v_i^{j_l} \in V^{\otimes j_l}$.

Remark 2.2.2 *Actually, $\alpha^{j+1} = d^j\alpha$ and (2.2.17) is Taylor's expansion. It's easy to see that if $\alpha : V \to L(V, W)$ possesses all bounded continuous derivatives up to degree $[p] + 1$, then α is a Lip(γ) one-form (w.r.t p such that $p < \gamma \leq [p] + 1$).*

(Condition 5.2.1) Assume that $(\alpha, V^{\otimes j}, W^{\otimes j} : j = 1, 2)$ is admissible with respect to $p < \gamma \leq [p] + 1$, where p and γ are fixed constants such that $[p] = 2$.

(Definition 5.2.1) (Under Condition 5.2.1) Let $X \in \Omega_p(V)$. Then the integral of the one-form α against the rough path X, denoted by $\int \alpha(X)dX$, is the unique rough path with roughness p in $T^{(2)}(W)$ associated with the almost rough path $Y \in C_0(\Delta, T^{(2)}(W))$, where

$$\begin{aligned} Y_{s,t}^1 &= \alpha^1(X_s)(X_{s,t}^1) + \alpha^2(X_s)(X_{s,t}^2), \\ Y_{s,t}^2 &= \alpha^1(X_s) \otimes \alpha^1(X_s)(X_{s,t}^2), \end{aligned}$$

for all $(s, t) \in \Delta$. The integration operator $\int \alpha$ is defined to be the map from $\Omega_p(V)$ into $\Omega_p(W)$ which sends a rough path X into $\int \alpha(X)dX$.

Remark 2.2.3 *The almost rough path $Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t})$ can be found by using Taylor's expansion. On the other hand, let $Z_{s,t} := \int_s^t \alpha(X_r)dX_r$, and according to (2.2.4), we can get*

$$Z^1_{s,t} = \lim_{m(D)\to 0} \sum_{l=1}^{r} Y^1_{t_{l-1},t_l} = \lim_{m(D)\to 0} \sum_{l=1}^{r} \left[\alpha^1(X_{t_{l-1}})(X^1_{t_{l-1},t_l}) + \alpha^2(X_{t_{l-1}})(X^2_{t_{l-1},t_l})\right]$$

$$Z^2_{s,t} = \lim_{m(D)\to 0} \sum_{l=1}^{r} \left[Y^2_{t_{l-1},t_l} + \sum_{i=1}^{k} Z^1_{s,t_{l-1}} \otimes Z^1_{t_{l-1},t_l}\right]$$

$$= \lim_{m(D)\to 0} \sum_{l=1}^{r} \left[\alpha^1(X_{t_{l-1}}) \otimes \alpha^1(X_{t_{l-1}})(X^2_{t_{l-1},t_l}) + \int_s^{t_{l-1}} \alpha(X)dX^1 \otimes \int_{t_{l-1}}^{t_l} \alpha(X)dX^1\right],$$

for any $(s,t) \in \Delta$, where $D := \{s = t_0 < t_1 < \cdots < t_r = t\}$ is a partition of $[s,t]$.

(Theorem 5.2.2) (Under Condition 5.2.1) The integration operator $\int \alpha$ is a continuous map from $\Omega_p(V)$ to $\Omega_p(W)$ in p-variation topology.

§2.3 Local Time as a Rough Path

In this section, we will prove the main results of this chapter. Let's first try to define the integral $\int_{-\infty}^{\infty} g(x)d_xL^x_t$ pathwise for a continuous $g(x)$ with bounded q-variation $(2 \le q < 3)$. And we also take $2 < p < 3$. We still decompose local time

$$L^x_t = \tilde{L}^x_t + \sum_{x^*_k \le x} \widehat{L}^{x^*_k}_t, \text{ where } \widehat{L}^x_t := L^x_t - L^{x-}_t. \tag{2.3.1}$$

Here $\tilde{L}^x_t$ is continuous in x, and $x^*_k, k = 1, 2\cdots$ are the countable discontinuous points of L^x_t. From Lemma 1.2.2, we know $h(t,x) := \sum_{x^*_k \le x} \widehat{L}^{x^*_k}_t$ is of bounded variation in x for each t. So the key point is to define $\int_{-\infty}^{\infty} g(x)d_x\tilde{L}^x_t$ pathwise for $g(x)$ with bounded q-variation $(2 \le q < 3)$. For this, we will use Lyons' rough path theory.

In fact, we will prove that $g(x)$ and $\tilde{L}^x_t$ can be regarded as rough paths. From [32], generally, we cannot expect to have an integration theory for defining integrals such as $\int_{-\infty}^{\infty} g(x)d_x\tilde{L}^x_t$. But using the method in Chapter 6 in [32], we can treat $Z_x := (\tilde{L}^x_t, g(x))$ together as a rough path and define $\hat{f}(x,y)(v,w) := (v, yv)$, so the integral will be the second element of $\int_{-\infty}^{\infty} \hat{f}(Z)dZ^1$. It's easy to know that Z_x is of bounded $\hat{q}$-variation in x, where $\hat{q} = q$, if $q > 2$, and $\hat{q} > 2$ can be taken as any number when $q = 2$. Most of the analysis in this section works for $2 \le q < 4$, especially we will establish the convergence of smooth rough path in θ-variation topology for any $\theta \in (q, 4)$ so to obtain $Z^1_{a,b}$ and $Z^2_{a,b}$. In particular, when $2 \le q < 3$, we obtain the existence of the geometric rough path

$X = (1, X^1, X^2)$ associated with $Z_\cdot$. In the following we consider $2 \leq q < 4$ otherwise we will explicitly say so.

Let $[x', x'']$ be any interval in R. From the proof of Lemma 1.2.1, for any $p \geq 2$, we know there exists a constant $c > 0$ such that

$$E|\tilde{L}_t(b) - \tilde{L}_t(a)|^p \leq c(b-a)^{\frac{p}{2}} \tag{2.3.2}$$

i.e. $\tilde{L}_t(x)$ satisfies Hölder condition (2.2.14) with exponent $\frac{1}{2}$. Denote by w the control of $g(x)$, i.e.

$$|g(b) - g(a)|^q \leq w(a,b),$$

for any $(a,b) \in \Delta := \{(a,b) : x' \leq a < b \leq x''\}$. It is obvious that $w_1(a,b) := w(a,b) + (b-a)$ is also a control of g. Set $h = \frac{1}{q} \in (\frac{1}{3}, \frac{1}{2}]$, it is trivial to see for any $\theta > q$ i.e. $h\theta > 1$ we have,

$$|g(b) - g(a)|^\theta \leq w_1(a,b)^{h\theta}, \ for\ any\ (a,b) \in \Delta. \tag{2.3.3}$$

Considering (2.3.2), we can get Z_x satisfies, for such $h = \frac{1}{q}$, and any $\theta > q$ i.e. $h\theta > 1$ there exists a constant c such that

$$E|Z_b - Z_a|^\theta \leq c w_1(a,b)^{h\theta}, \ for\ any\ (a,b) \in \Delta. \tag{2.3.4}$$

For any $m \in N$, define a continuous and bounded variation path $Z(m)$ by

$$Z(m)_x := Z_{x^m_{l-1}} + \frac{w_1(x) - w_1(x^m_{l-1})}{w_1(x^m_l) - w_1(x^m_{l-1})} \Delta^m_l Z, \tag{2.3.5}$$

if $x^m_{l-1} \leq x < x^m_l$, for $l = 1, \cdots, 2^m$, and $\Delta^m_l Z = Z_{x^m_l} - Z_{x^m_{l-1}}$. Here $D_m := \{x' = x^m_0 < x^m_1 < \cdots < x^m_{2^m} = x''\}$ is a partition of $[x', x'']$ such that $w_1(x^m_l) - w_1(x^m_{l-1}) = \frac{1}{2^m} w_1(x', x'')$, where $w_1(x) := w_1(x', x)$. It is obvious that $x^m_l - x^m_{l-1} \leq \frac{1}{2^m} w_1(x', x'')$. The corresponding smooth rough path $X(m)$ is built by taking its iterated path integrals, i.e.

$$X(m)^j_{a,b} = \int_{a<x_1<\cdots<x_j<b} dZ(m)_{x_1} \otimes \cdots \otimes dZ(m)_{x_j}. \tag{2.3.6}$$

In the following, we will prove $\{X(m)\}_{m \in N}$ converges to a geometric rough path X in θ-variation topology (2.2.7). We call X the canonical geometric rough path associated with Z.

Let's first look at the first level path $X(m)^1_{a,b}$. The method and results are similar to Chapter 4 in [32]. Similar to Proposition 4.2.1 in [32], we can prove

Proposition 2.3.1 *Let (Z_x) be a continuous path, $\theta \geq 1$, $X(m)$ defined as above. Then for all $n \in N$, $m \mapsto \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta$ is increasing. Therefore*

$$\sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta = \lim_{m\to\infty} \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta$$

Proof: By definition, $X(m)_{x^n_k} = Z_{x^n_k}$ for any $n \leq m$ and therefore

$$X(m)^1_{x^n_{k-1},x^n_k} = Z^1_{x^n_{k-1},x^n_k}, \quad k = 1, \cdots, 2^n, \quad \textit{for any } n \leq m.$$

Hence,

$$\sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta = \sum_{k=1}^{2^n} |\Delta^n_k Z|^\theta.$$

If $n > m$, then we may find a unique integer $0 < l \leq 2^m$ such that $x^m_{l-1} \leq x^n_{k-1} < x^n_k < x^m_l$, so that

$$X(m)_{x^n_j} = Z_{x^m_{l-1}} + \frac{w(x^n_j) - w(x^m_{l-1})}{w(x^m_l) - w(x^m_{l-1})} \Delta^m_l Z, \quad j = k,\ k-1.$$

Therefore,

$$X(m)^1_{x^n_{k-1},x^n_k} = 2^{m-n} \Delta^m_l Z, \quad \textit{for any } n > m.$$

Since for each l from 1 to 2^m, there are 2^{n-m} elements of $\{x^n_1, x^n_2, \cdots, x^n_{2^n}\}$ in $[x^m_{l-1}, x^m_l)$,

$$\sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta = (\frac{1}{2^n})^{\theta-1} (2^m)^{\theta-1} \sum_{l=1}^{2^m} |\Delta^m_l Z|^\theta$$

Observe that $\Delta^m_l Z = \Delta^{m+1}_{2l} Z + \Delta^{m+1}_{2l-1} Z$, so

$$\begin{aligned}
(2^m)^{\theta-1} \sum_{l=1}^{2^m} |\Delta^m_l Z|^\theta &= (2^{m+1})^{\theta-1} \sum_{l=1}^{2^m} (\frac{1}{2})^{\theta-1} |\Delta^{m+1}_{2l} Z + \Delta^{m+1}_{2l-1} Z|^\theta \\
&\leq (2^{m+1})^{\theta-1} \sum_{l=1}^{2^m} (|\Delta^{m+1}_{2l} Z|^\theta + |\Delta^{m+1}_{2l-1} Z|^\theta) \\
&= (2^{m+1})^{\theta-1} \sum_{l=1}^{2^{m+1}} |\Delta^{m+1}_l Z|^\theta
\end{aligned}$$

This ends the proof. ◇

Let $X^1_{a,b} = Z_b - Z_a$. Inequality (2.3.4) implies $E|X^1_{a,b}|^\theta \leq c w_1(a,b)^{h\theta}$. In particular, $E|X^1_{x^n_{k-1},x^n_k}|^\theta \leq c(\frac{1}{2^n})^{h\theta} w_1(x', x'')^{h\theta}$. Therefore for any $\gamma > \theta - 1$, there is a constant $C(\theta, c)$ such that

$$E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X^1_{x^n_{k-1},x^n_k}|^\theta \leq C(\theta, c) \sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x', x'')^{h\theta}.$$

For such points $\{x_k^n\}$, $k = 1, \cdots, 2^n$, $n = 1, 2, \cdots$, defined above we still have the inequality (2.2.11), for any $\gamma > \theta - 1$, there exists a constant $C_1(\theta, \gamma, c) > 0$ such that

$$\begin{aligned} E \sup_D \sum_l |X^1_{x_{l-1},x_l}|^\theta &\leq C(\theta,\gamma) E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C_1(\theta,\gamma,c) \sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x',x'')^{h\theta}. \end{aligned} \tag{2.3.7}$$

Since $h\theta - 1 > 0$, the series on the right-hand side of (2.3.7) is convergent, so that $\sup_D \sum_l |X^1_{x_{l-1},x_l}|^\theta < \infty$ almost surely. This shows that X^1 has finite θ-variation almost surely. Furthermore, by Proposition 2.3.1,

$$\begin{aligned} E \sup_m \sup_D \sum_l |X(m)^1_{x_{l-1},x_l}|^\theta &\leq C(\theta,\gamma) E \sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C(\theta,\gamma) E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X^1_{x^n_{k-1},x^n_k}|^\theta \\ &\leq C_1(\theta,\gamma,c) \sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x',x'')^{h\theta} < \infty. \end{aligned}$$

So

$$\sup_m \sup_D \sum_l |X(m)^1_{x_{l-1},x_l}|^\theta < \infty, \quad a.s.,$$

which means $X(m)^1_{a,b}$ have finite θ-variation uniformly in m.

Next we want to show that under (2.3.4), $X(m)^1_{a,b}$ converges to $X^1_{a,b}$ in θ-variation distance. Note that if $n \leq m$, $X(m)^1_{x^n_{k-1},x^n_k} = X^1_{x^n_{k-1},x^n_k}$; and
if $n > m$, then $|X(m)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^\theta \leq 2^{\theta-1}(|X(m)^1_{x^n_{k-1},x^n_k}|^\theta + |X^1_{x^n_{k-1},x^n_k}|^\theta)$.
Therefore,

$$\begin{aligned} & E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^\theta \\ = \; & E \sum_{n=m+1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^\theta \\ \leq \; & C \sum_{n=m+1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x',x'')^{h\theta} \\ \leq \; & C (\frac{1}{2^m})^{\frac{h\theta-1}{2}} \sum_{n=m+1}^{\infty} n^\gamma (\frac{1}{2^n})^{\frac{h\theta-1}{2}} \\ \leq \; & C (\frac{1}{2^m})^{\frac{h\theta-1}{2}}, \end{aligned}$$

where C depends on θ, h, $w_1(x', x'')$, and c in (2.3.4). By inequality (2.2.11),

$$\begin{aligned} & E \sup_D \sum_l |X(m)^1_{x_{l-1},x_l} - X^1_{x_{l-1},x_l}|^\theta \\ \leq\ & C(\theta,\gamma) E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^\theta \\ \leq\ & C(\frac{1}{2^m})^{\frac{h\theta-1}{2}}. \end{aligned}$$

By Hölder's inequality,

$$E \sup_D \Big(\sum_l |X(m)^1_{x_{l-1},x_l} - X^1_{x_{l-1},x_l}|^\theta \Big)^{\frac{1}{\theta}} \leq \Big(E \sup_D \sum_l |X(m)^1_{x_{l-1},x_l} - X^1_{x_{l-1},x_l}|^\theta \Big)^{\frac{1}{\theta}},$$

hence,

$$E \sum_{m=1}^{\infty} \sup_D \Big(\sum_l |X(m)^1_{x_{l-1},x_l} - X^1_{x_{l-1},x_l}|^\theta \Big)^{\frac{1}{\theta}} \leq C \sum_{m=1}^{\infty} (\frac{1}{2^m})^{\frac{h\theta-1}{2\theta}} < \infty, \tag{2.3.8}$$

for $h\theta > 1$, where C depends on θ, h, $w_1(x', x'')$, and c in (2.3.4). So we obtain

Theorem 2.3.1 *For a continuous path Z_x with (2.3.4), we have*

$$\sum_{m=1}^{\infty} \sup_D \Big(\sum_l |X(m)^1_{x_{l-1},x_l} - X^1_{x_{l-1},x_l}|^\theta \Big)^{\frac{1}{\theta}} < \infty \ a.s.. \tag{2.3.9}$$

In particular, $(X(m)^1_{a,b})$ converges to $(X^1_{a,b})$ in θ-variation distance a.s. for any $(a,b) \in \Delta$.

We next consider the second level path $X(m)^2_{a,b}$. From [32], we know if $n \geq m$, $X(m)^2_{x^n_{k-1},x^n_k} = 2^{2(m-n)-1}(\Delta^m_l Z)^{\otimes 2}$; if $n < m$,

$$X(m)^2_{x^n_{k-1},x^n_k} = \frac{1}{2}\Delta^n_k Z \otimes \Delta^n_k Z + \frac{1}{2} \sum_{r<l=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta^m_r Z \otimes \Delta^m_l Z - \Delta^m_l Z \otimes \Delta^m_r Z),$$

so

$$\begin{aligned} & X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k} \\ =\ & \frac{1}{2} \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta^{m+1}_{2l-1} Z \otimes \Delta^{m+1}_{2l} Z - \Delta^{m+1}_{2l} Z \otimes \Delta^{m+1}_{2l-1} Z), \end{aligned} \tag{2.3.10}$$

$k = 1, \cdots, 2^n$. Similar to the proof of Proposition 4.3.3 in [32], we have

Proposition 2.3.2 *Suppose Z_x is continuous in x and satisfies (2.3.4). Then for $n \geq m$,*

$$\sum_{k=1}^{2^n} E|X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \leq C(\frac{1}{2^{n+m}})^{\frac{\theta h-1}{2}}, \tag{2.3.11}$$

where C depends on θ, h, $w_1(x', x'')$, and c in (2.3.4).

Proof: If $n \geq m$, $X(m)^2_{x^n_{k-1},x^n_k} = 2^{2(m-n)-1}(\Delta^m_l Z)^{\otimes 2}$, and since for each l from 1 to 2^{m+1}, there are 2^{n-m-1} elements of $\{x^n_1, x^n_2, \cdots, x^n_{2^n}\}$ in $[x^{m+1}_{l-1}, x^{m+1}_l)$, we have

$$\begin{aligned}
&\sum_{k=1}^{2^n} E|X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \\
&= \sum_{l=1}^{2^{m+1}} 2^{n-m-1} E|2^{2(m-n)+1}(\Delta^{m+1}_l Z) \otimes (\Delta^{m+1}_l Z) - 2^{2(m-n)-1}(\Delta^m_l Z) \otimes (\Delta^m_l Z)|^{\frac{\theta}{2}} \\
&\leq C(\frac{2^m}{2^n})^\theta \sum_{l=1}^{2^{m+1}} 2^{n-m-1}(\frac{1}{2^m})^{h\theta} w_1(x', x'')^{h\theta} \\
&\leq C(\frac{2^m}{2^n})^\theta 2^{n-m-1} \cdot 2^{m+1}(\frac{1}{2^m})^{h\theta} \\
&= C(\frac{2^m}{2^n})^{\theta-h\theta}(\frac{1}{2^n})^{h\theta-1} \\
&\leq C(\frac{1}{2^n})^{h\theta-1} \\
&\leq C(\frac{1}{2^{m+n}})^{\frac{h\theta-1}{2}},
\end{aligned}$$

where C depends on θ, h, $w_1(x', x'')$, and c in (2.3.4). ◇

Proposition 2.3.3 *Assume $2 \leq \theta < 4$ and $q < \theta < 4$. Then for $m > n$, we have*

$$E|X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \leq C\Big[(\frac{1}{2^n})^{\frac{\theta}{4}}(\frac{1}{2^m})^{\frac{1}{2}h\theta} + (\frac{1}{2^n})^{\frac{\theta}{2}}(\frac{1}{2^m})^{\frac{1}{2}h\theta-\frac{1}{8}\theta}\Big], \quad (2.3.12)$$

where C is a generic constant and also depends on θ, $h(:=\frac{1}{q})$, $w_1(x', x'')$, and c in (2.3.4).

Proof: For $m > n$, we have

$$\begin{aligned}
&E|X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k}|^2 \\
= \;& \frac{1}{4}E\Big|\sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta^{m+1}_{2l-1}Z \otimes \Delta^{m+1}_{2l}Z - \Delta^{m+1}_{2l}Z \otimes \Delta^{m+1}_{2l-1}Z)\Big|^2 \\
= \;& \frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{2} E \sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta^{m+1}_{2l-1}Z^i\Delta^{m+1}_{2l}Z^j - \Delta^{m+1}_{2l}Z^i\Delta^{m+1}_{2l-1}Z^j) \\
&\qquad \cdot(\Delta^{m+1}_{2r-1}Z^i\Delta^{m+1}_{2r}Z^j - \Delta^{m+1}_{2r}Z^i\Delta^{m+1}_{2r-1}Z^j) \\
= \;& \frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{2} \sum_{l,r=2^{m-n}(k-1)+1}^{2^{(m-n)}k} \Big[E(\Delta^{m+1}_{2l-1}Z^i\Delta^{m+1}_{2r-1}Z^i)E(\Delta^{m+1}_{2l}Z^j\Delta^{m+1}_{2r}Z^j) \\
&\qquad + E(\Delta^{m+1}_{2l}Z^i\Delta^{m+1}_{2r}Z^i)E(\Delta^{m+1}_{2l-1}Z^j\Delta^{m+1}_{2r-1}Z^j)\Big] \\
&-\frac{1}{4}\sum_{\substack{i,j=1 \\ i\neq j}}^{2} \sum_{l,r=2^{m-n}(k-1)+1}^{2^{(m-n)}k} \Big[E(\Delta^{m+1}_{2l}Z^i\Delta^{m+1}_{2r-1}Z^i)E(\Delta^{m+1}_{2l-1}Z^j\Delta^{m+1}_{2r}Z^j)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +E(\Delta_{2l-1}^{m+1}Z^i\Delta_{2r}^{m+1}Z^i)E(\Delta_{2l}^{m+1}Z^j\Delta_{2r-1}^{m+1}Z^j)\Big] \\
&= \frac{1}{4}\sum_{l,r}\Big[E(\Delta_{2l-1}^{m+1}\tilde{L}_t^x\Delta_{2r-1}^{m+1}\tilde{L}_t^x)(\Delta_{2l}^{m+1}g(x)\Delta_{2r}^{m+1}g(x)) \\
&\qquad +E(\Delta_{2l}^{m+1}\tilde{L}_t^x\Delta_{2r}^{m+1}\tilde{L}_t^x)(\Delta_{2l-1}^{m+1}g(x)\Delta_{2r-1}^{m+1}g(x))\Big] \\
&\quad -\frac{1}{4}\sum_{l,r}\Big[E(\Delta_{2l}^{m+1}\tilde{L}_t^x\Delta_{2r-1}^{m+1}\tilde{L}_t^x)(\Delta_{2l-1}^{m+1}g(x)\Delta_{2r}^{m+1}g(x)) \\
&\qquad +(\Delta_{2l-1}^{m+1}\tilde{L}_t^x\Delta_{2r}^{m+1}\tilde{L}_t^x)(\Delta_{2l}^{m+1}g(x)\Delta_{2r-1}^{m+1}g(x))\Big] \\
&\quad +\frac{1}{4}\sum_{l,r}\Big[(\Delta_{2l-1}^{m+1}g(x)\Delta_{2r-1}^{m+1}g(x))E(\Delta_{2l}^{m+1}\tilde{L}_t^x\Delta_{2r}^{m+1}\tilde{L}_t^x) \\
&\qquad +(\Delta_{2l}^{m+1}g(x)\Delta_{2r}^{m+1}g(x))E(\Delta_{2l-1}^{m+1}\tilde{L}_t^x\Delta_{2r-1}^{m+1}\tilde{L}_t^x)\Big] \\
&\quad -\frac{1}{4}\sum_{l,r}\Big[(\Delta_{2l}^{m+1}g(x)\Delta_{2r-1}^{m+1}g(x))E(\Delta_{2l-1}^{m+1}\tilde{L}_t^x\Delta_{2r}^{m+1}\tilde{L}_t^x) \\
&\qquad +(\Delta_{2l-1}^{m+1}g(x)\Delta_{2r}^{m+1}g(x))E(\Delta_{2l}^{m+1}\tilde{L}_t^x\Delta_{2r-1}^{m+1}\tilde{L}_t^x)\Big]. \qquad (2.3.13)
\end{aligned}
$$

Let $X_t = M_t + V_t$, where M_t is a continuous local martingale, V_t is a continuous process with finite variation. So from [25] and [41], it's easy to know that

$$
\begin{aligned}
\tilde{L}_t^x &= (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s > x\}} dM_s - (V_t - V_0) \\
&:= \phi(x) - \int_0^t 1_{\{X_s > x\}} dM_s - (V_t - V_0),
\end{aligned}
$$

and using some estimate in the proof of Lemma 1.2.1, we have

$$
\begin{aligned}
&E\Big[\Delta_{2r-1}^{m+1}\tilde{L}_t^x\Delta_{2l-1}^{m+1}\tilde{L}_t^x\Big] \\
=\ &E\Big[(\tilde{L}_t(x_{2r-1}^{m+1}) - \tilde{L}_t(x_{2r-2}^{m+1}))(\tilde{L}_t(x_{2l-1}^{m+1}) - \tilde{L}_t(x_{2l-2}^{m+1}))\Big] \\
=\ &E\Big[(\phi(x_{2r-1}^{m+1}) - \phi(x_{2r-2}^{m+1}) - \int_0^t 1_{\{x_{2r-2}^{m+1} \le X_s < x_{2r-1}^{m+1}\}} dM_s) \\
&\quad \cdot(\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1}) - \int_0^t 1_{\{x_{2l-2}^{m+1} \le X_s < x_{2l-1}^{m+1}\}} dM_s)\Big] \\
\le\ &E|\phi(x_{2r-1}^{m+1}) - \phi(x_{2r-2}^{m+1})| \cdot |\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})| \\
&+E|\phi(x_{2r-1}^{m+1}) - \phi(x_{2r-2}^{m+1})| \cdot |\int_0^t 1_{\{x_{2l-2}^{m+1} \le X_s < x_{2l-1}^{m+1}\}} dM_s| \\
&+E|\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})| \cdot |\int_0^t 1_{\{x_{2r-2}^{m+1} \le X_s < x_{2r-1}^{m+1}\}} dM_s| \\
&+E|\int_0^t 1_{\{x_{2r-2}^{m+1} \le X_s < x_{2r-1}^{m+1}\}} 1_{\{x_{2l-2}^{m+1} \le X_s < x_{2l-1}^{m+1}\}} d < M >_s | \\
\le\ &C\Big[(x_{2r-1}^{m+1} - x_{2r-2}^{m+1})(x_{2l-1}^{m+1} - x_{2l-2}^{m+1}) + (x_{2r-1}^{m+1} - x_{2r-2}^{m+1})(x_{2l-1}^{m+1} - x_{2l-2}^{m+1})^{\frac{1}{2}} \\
&\quad +(x_{2l-1}^{m+1} - x_{2l-2}^{m+1})(x_{2r-1}^{m+1} - x_{2r-2}^{m+1})^{\frac{1}{2}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+E|\int_0^t 1_{\{x_{2r-2}^{m+1}\le X_s<x_{2r-1}^{m+1}\}}1_{\{x_{2l-2}^{m+1}\le X_s<x_{2l-1}^{m+1}\}}d<M>_s| \\
\le\ & C\Big[(\frac{1}{2^{m+1}})^2 w_1(x',x'')^2+2(\frac{1}{2^{m+1}})^{\frac{3}{2}}w_1(x',x'')^{\frac{3}{2}}\Big] \\
&+E|\int_0^t 1_{\{x_{2r-2}^{m+1}\le X_s<x_{2r-1}^{m+1}\}}1_{\{x_{2l-2}^{m+1}\le X_s<x_{2l-1}^{m+1}\}}d<M>_s| \\
\le\ & \begin{cases} C(\frac{1}{2^{m+1}})^{\frac{3}{2}}, & \text{if } r\neq l, \\ C\frac{1}{2^{m+1}}, & \text{if } r=l. \end{cases}
\end{aligned} \tag{2.3.14}
$$

Here C is a generic constant and also depends on $w_1(x',x'')$. So

$$
\begin{aligned}
&\sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k} E(\Delta_{2r-1}^{m+1}\tilde{L}_t^x\Delta_{2l-1}^{m+1}\tilde{L}_t^x)E(\Delta_{2r}^{m+1}g(x)\Delta_{2l}^{m+1}g(x)) \\
\le\ & C\Big[2^{m-n}(\frac{1}{2^{m+1}})^{1+2h}+2^{2(m-n)}(\frac{1}{2^{m+1}})^{\frac{3}{2}+2h}\Big].
\end{aligned}
$$

The other terms in (2.3.13) can be treated similarly, therefore

$$
E|X(m+1)^2_{x_{k-1}^n,x_k^n}-X(m)^2_{x_{k-1}^n,x_k^n}|^2\le C\Big[2^{m-n}(\frac{1}{2^{m+1}})^{1+2h}+2^{2(m-n)}(\frac{1}{2^{m+1}})^{\frac{3}{2}+2h}\Big].
$$

Hence, for $2\le\theta<4$, by Jensen's inequality,

$$
\begin{aligned}
&E|X(m+1)^2_{x_{k-1}^n,x_k^n}-X(m)^2_{x_{k-1}^n,x_k^n}|^{\frac{\theta}{2}} \\
\le\ & \Big(E|X(m+1)^2_{x_{k-1}^n,x_k^n}-X(m)^2_{x_{k-1}^n,x_k^n}|^2\Big)^{\frac{\theta}{4}} \\
\le\ & C\Big[2^{m-n}(\frac{1}{2^{m+1}})^{1+2h}+2^{2(m-n)}(\frac{1}{2^{m+1}})^{\frac{3}{2}+2h}\Big]^{\frac{\theta}{4}} \\
\le\ & C\Big[2^{(m-n)\frac{\theta}{4}}(\frac{1}{2^{m+1}})^{\frac{\theta}{4}+\frac{1}{2}h\theta}+2^{(m-n)\frac{\theta}{2}}(\frac{1}{2^{m+1}})^{\frac{3\theta}{8}+\frac{1}{2}h\theta}\Big] \\
\le\ & C\Big[(\frac{1}{2^n})^{\frac{\theta}{4}}(\frac{1}{2^m})^{\frac{1}{2}h\theta}+(\frac{1}{2^n})^{\frac{\theta}{2}}(\frac{1}{2^m})^{\frac{1}{2}h\theta-\frac{1}{8}\theta}\Big],
\end{aligned}
$$

where C is a generic constant and also depends on θ, h, $w_1(x',x'')$, and c. ◇

Theorem 2.3.2 *Assume $2\le q<4$. Let $q<\theta<4$ and $h\in(0,1)$ satisfy $h\theta>1$. Then there exists a unique X^2 such that $(1,X^1,X^2)$ is a rough path and there exist a sequence $\{X(m)\}_{m=1}^{\infty}$ of smooth rough path such that*

$$
\sum_{i=1}^{2}\sup_D\Big(\sum_l|X(m)^i_{x_{l-1},x_l}-X^i_{x_{l-1},x_l}|^{\frac{\theta}{i}}\Big)^{\frac{i}{\theta}}\to 0,
$$

both almost surely and in $L^1(\Omega,\mathcal{F},\mathcal{P})$ as $m\to\infty$. In particular, when $2\le q<3$, X is the canonical geometric rough path associated to $Z_\cdot$.

Proof: The convergence of $X(m)^1$ to X^1 is Theorem 2.3.1. In the following we will prove $X(m)^2_{a,b}$ converges in θ-variation distance. By Proposition 4.1.2 in [32],

$$\begin{aligned}
& E\sup_D \sum_l |X(m+1)^2_{x_{l-1},x_l} - X(m)^2_{x_{l-1},x_l}|^{\frac{\theta}{2}} \\
\leq\ & C(\theta,\gamma)E\left(\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n} |X(m+1)^1_{x^n_{k-1},x^n_k} - X(m)^1_{x^n_{k-1},x^n_k}|^{\theta}\right)^{\frac{1}{2}} \\
& \cdot\left(\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n}\left(|X(m+1)^1_{x^n_{k-1},x^n_k}|^{\theta} + |X(m)^1_{x^n_{k-1},x^n_k}|^{\theta}\right)\right)^{\frac{1}{2}} \\
& + C(\theta,\gamma)E\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n} |X(m+1)^2_{x^n_{k-1},x^n_k} - X(m)^2_{x^n_{k-1},x^n_k}|^{\frac{\theta}{2}} \\
:=\ & A + B.
\end{aligned}$$

We will estimate part A, B respectively. First from (2.3.8), we know

$$\begin{aligned}
A \leq\ & C\Big(E\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n} |X(m+1)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^{\theta} + |X(m)^1_{x^n_{k-1},x^n_k} - X^1_{x^n_{k-1},x^n_k}|^{\theta}\Big)^{\frac{1}{2}} \\
& \cdot\Big(E\sum_{n=1}^{\infty} n^{\gamma}\sum_{k=1}^{2^n} |X(m+1)^1_{x^n_{k-1},x^n_k}|^{\theta} + |X(m)^1_{x^n_{k-1},x^n_k}|^{\theta}\Big)^{\frac{1}{2}} \\
\leq\ & C(\frac{1}{2^m})^{\frac{h\theta-1}{4}}\sum_{n=1}^{\infty} n^{\gamma}(\frac{1}{2^n})^{\frac{h\theta-1}{2}}.
\end{aligned}$$

Secondly from Proposition 2.3.2 and Proposition 2.3.3, we know

$$\begin{aligned}
B \leq\ & C\sum_{n=m}^{\infty} n^{\gamma}(\frac{1}{2^{m+n}})^{\frac{h\theta-1}{2}} + C\Big[\sum_{n=1}^{m-1} n^{\gamma}(\frac{1}{2^n})^{\frac{\theta}{4}-1}(\frac{1}{2^m})^{\frac{h\theta}{2}} + \sum_{n=1}^{m-1} n^{\gamma}(\frac{1}{2^n})^{\frac{\theta}{2}-1}(\frac{1}{2^m})^{\frac{h\theta}{2}-\frac{\theta}{8}}\Big] \\
\leq\ & C\Big[(\frac{1}{2^m})^{\frac{h\theta-1}{2}} + (\frac{1}{2^m})^{\frac{h\theta-1}{2}} + (\frac{1}{2^m})^{\frac{h\theta}{2}-\frac{\theta}{8}}\Big],
\end{aligned}$$

as $q < \theta < 4$, and $h\theta > 1$. So

$$E\sup_D \sum_l |X(m+1)^2_{x_{l-1},x_l} - X(m)^2_{x_{l-1},x_l}|^{\frac{\theta}{2}} \leq C[(\frac{1}{2^m})^{\frac{h\theta-1}{4}} + (\frac{1}{2^m})^{\frac{h\theta}{2}-\frac{\theta}{8}}].$$

Similar to the proof of Theorem 2.3.1, we can easily deduce that $(X(m)^2)_{m\in N}$ is a Cauchy sequence in θ-variation distance. So when $m \to \infty$, it has a limit, denote it by X^2, and from completeness under θ-variation distance (Lemma 3.3.3 in [32]), X^2 is also of finite θ-variation. The theorem is asserted. ◇

Remark 2.3.1 *We would like to point out that the above method does not seem to work for two arbitrary functions f of p-variation and g of q-variation ($2 < p, q < 3$) to define a*

rough path $Z_x = (f(x), g(x))$. However the special property (2.3.14) of local times makes our analysis work. A similar method was used in [32] for fractional Brownian motion with the help of long-time memory. Here (2.3.14) serves a similar role of the long-time memory as in [32].

In the following, we will only consider the case that $2 \leq q < 3$ and take $q < \theta < 3$.

As local time L_t^x has a compact support in x for each ω and t, so we can define integral of local time directly in R. For this, we take $[x', x'']$ covering the support of L_t^x. For $2 < \theta < 3$, recall the definition of the one form $\hat{f} : R^2 \to L(R^2, R^2)$,

$$\hat{f}(z)\xi = (v, yv),$$

where $z = (x, y)$ and $\xi = (v, w)$. So for $\xi_1 = (v_1, w_1)$, $\xi_2 = (v_2, w_2)$,we have

$$\hat{f}(z)^2(\xi_1 \otimes \xi_2) = d\hat{f}(z)(\xi_2)\xi_1 = \begin{pmatrix} 0 \\ v_2 w_1 \end{pmatrix}.$$

Define

$$\begin{aligned} Y_{a,b}^1 &= \hat{f}(Z_a)Z_{a,b}^1 + \hat{f}^2(Z_a)Z_{a,b}^2 \\ Y_{a,b}^2 &= (\hat{f}(Z_a) \otimes \hat{f}(Z_a))Z_{a,b}^2. \end{aligned}$$

From Chapter 5 in [32], we know that $Y = (1, Y_{a,b}^1, Y_{a,b}^2)$ is an almost multiplicative functional of degree 2 and therefore one can use the almost rough path to construct the unique rough path $\int_{-\infty}^{\infty} \hat{f}(Z)dZ$ with roughness θ in $T^{(2)}(R^2)$. In particular,

$$\int_{-\infty}^{\infty} \hat{f}(Z)dZ^1 = \lim_{m(D)\to 0} \sum_i \left[\hat{f}(Z_{x_{i-1}})(Z_{x_{i-1},x_i}^1) + \hat{f}^2(Z_{x_{i-1}})(Z_{x_{i-1},x_i}^2)\right],$$

where the limit exists so the integral is well-defined. Note

$$\hat{f}(Z(a))(Z_{a,b}^1) + \hat{f}^2(Z(a))(Z_{a,b}^2) = \left(\tilde{L}_t^b - \tilde{L}_t^a, g(a)(\tilde{L}_t^b - \tilde{L}_t^a)\right) + \left(0, (Z_{a,b}^2)_{2,1}\right),$$

where $(Z_{a,b}^2)_{2,1}$ means the lower-left element of 2×2 matrix $Z_{a,b}^2$. Note in our case formally the rough path

$$\int_{-\infty}^{\infty} \hat{f}(Z)dZ^1 = \left(\int_{-\infty}^{\infty} d\tilde{L}_t^x, \int_{-\infty}^{\infty} g(x)d\tilde{L}_t^x\right).$$

In particular,

$$\int_{-\infty}^{\infty} g(x)d\tilde{L}_t^x = \lim_{m(D)\to 0} \left[\sum_i g(x_{i-1})(\tilde{L}_t^{x_i} - \tilde{L}_t^{x_{i-1}}) + (Z_{x_{i-1},x_i}^2)_{2,1}\right].$$

The limit exists. Note it is clear to us that the Riemann sum $\sum_i g(x_{i-1})(\tilde{L}_t^{x_i} - \tilde{L}_t^{x_{i-1}})$ itself does not have a limit as $m(D) \to 0$. This is the very reason we need to use Lyons' rough path integration theory. Here we still denote the integral by $\int_{-\infty}^{\infty} g(x)d\tilde{L}_t^x$.

Note from Theorem 5.2.2 in [32], $\int \hat{f}$ is a continuous map from $\Omega_\theta(R^2)$ to $\Omega_\theta(R^2)$ in θ-variation topology. Let $Z_n(x) := (\tilde{L}_t^x, g_n(x))$, $Z(x) := (\tilde{L}_t^x, g(x))$, where $g_n(\cdot)$ is of bounded q-variation uniformly in n, $2 \le q < 3$, and when $n \to \infty$, $g_n(x) \to g(x)$ for all $x \in R$. What we should prove is that rough path $Z_n(\cdot) \to Z(\cdot)$ in θ-variation distance. Repeating the above argument, we can find the canonical geometric rough path associated with Z_n is $X_n = (1, X_n^1, X_n^2)$, the smooth rough path is $X_n(m) = (1, X_n(m)^1, X_n(m)^2)$. Actually, in [32], it shows that $(X_n)_{a,b}^1 = (\tilde{L}_t^b - \tilde{L}_t^a, g_n(b) - g_n(a))$, $X_{a,b}^1 = (\tilde{L}_t^b - \tilde{L}_t^a, g(b) - g(a))$, so $(X_n)_{a,b}^1 \to X_{a,b}^1$ in the sense of uniform topology, and also in the sense of θ-variation topology. As for $(X_n)_{a,b}^2$, we can easily see that

$$|(X_n)_{a,b}^2 - X_{a,b}^2| \le |(X_n)_{a,b}^2 - (X_n(m))_{a,b}^2| + |(X_n(m))_{a,b}^2 - X(m)_{a,b}^2| + |X(m)_{a,b}^2 - X_{a,b}^2|.$$

From Theorem 2.3.2, we know that the first and the third term on the righthand side is smaller than $\varepsilon w_1(a,b)^{\frac{2}{\theta}}$, for any small $\varepsilon > 0$. The second term can be easily dealt with from the definition of $(X_n(m))_{a,b}^2$ and $X(m)_{a,b}^2$. It is convergent in the θ-variation topology uniformly in m. So $\int \hat{f}(Z_n)dZ_n \to \int \hat{f}(Z)dZ$ in θ-variation distance a.s.. Therefore $\int \hat{f}(Z_n)dZ_n^1 \to \int \hat{f}(Z)dZ^1$ a.s., i.e. $\int_{-\infty}^{\infty} g_n(x)d\tilde{L}_t^x \to \int_{-\infty}^{\infty} g(x)d\tilde{L}_t^x$ a.s., when $n \to \infty$. As for the jump part, from Lebesgue's dominated convergence theorem, $\int_{-\infty}^{\infty} g_n(x)dh(t,x) \to \int_{-\infty}^{\infty} g(x)dh(t,x)$, when $n \to \infty$. So we can get $\int_{-\infty}^{\infty} g_n(x)dL_t^x \to \int_{-\infty}^{\infty} g(x)dL_t^x$,when $n \to \infty$. If $g(x)$ has discontinuities, we can use the method in [49] to deal with. Finally, we deduce an extension of Tanaka-Meyer's formula. A similar smoothing procedure with [11] can be used and the above convergence is enough to make our proof work.

Theorem 2.3.3 *Let $X = (X_t)_{t\ge 0}$ be a continuous semimartingale and $f : R \to R$ be an absolutely continuous function and have left derivative $\nabla^- f(x)$ being left continuous and locally bounded. Assume $\nabla^- f(x)$ is of bounded q-variation, where $1 \le q < 3$, then*

$$f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s)dX_s - \int_{-\infty}^{\infty} \nabla^- f(x)d_x L_t^x. \tag{2.3.15}$$

Here the integral $\int_{-\infty}^{\infty} \nabla^- f(x)d_x L_t^x$ is a Lebesgue-Stieltjes integral when $q = 1$, a Young integral when $1 < q < 2$ and a Lyons' rough path integral when $2 \le q < 3$ respectively.

Proof: Similar to the proof of Theorem 1.3.2.

Remark 2.3.2 *This chapter is included in paper [13].*

Example 2.3.1 *Consider a function* $f(x) = x^{\frac{5}{2}} \cos \frac{1}{x}$ *for* $x \neq 0$ *and* $f(0) = 0$. *This function is* C^1 *and its derivative is* $f'(x) = \frac{5}{2} x^{\frac{3}{2}} \cos \frac{1}{x} + \sqrt{x} \sin \frac{1}{x}$ *for* $x \neq 0$ *and* $f'(0) = 0$. *It is easy to see that* f' *is not of quadratic variation, but of* p*-variation for any* $p > 2$. *So Theorem 2.3.3 can be used, while either Tanaka-Meyer formula or Theorem 1.3.2 cannot apply to this situation.*

Chapter 3
Two-parameter p, q-variation Path Integrals

§3.1 Introduction

In this chapter, we will consider how to define the integral of two-parameters, i.e. $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)$ just like Young integral without any measure. Young (1938) [51] considered this problem, but his conditions are strong and difficult to check. It seems to us that the theory of two-parameter Φ_1, Ψ_1-variation (p, q-variation as a special case) integration has not been investigated and developed well in the literature. Inspired by the work of Young [51] and Lyons and Qian [32], in Section 3.4, I give a new condition for the existence of two-parameter Young's integral (Theorem 3.4.1). In Section 3.2, I give some notations and introduce Young's two-parameter integral. In Section 3.3, I give an example of $p,1$-variation ($p>1$) function.

§3.2 Definition of Two Parameter p, q-variation Path and Young's Theorem

In this chapter, the following notation is used: Φ, Ψ, Φ_1, Ψ_1 denote continuous real valued convex functions on $[0,\infty)$ increasing and vanishing at 0; φ, ψ, φ_1, ψ_1 denote the inverse functions of Φ, Ψ, Φ_1, Ψ_1, respectively; ω, χ are continuous increasing functions of one variable.

Before we proceed, we need the following definition.

Definition 3.2.1 *We say $F(x,y)$ is of bounded Φ- and Ψ-bivariation, if*

$$\sup_{E}\sup_{\gamma,\delta\in[y',y'']}\sum_{k=1}^{N}\Phi(|F(x_k,\gamma)-F(x_{k-1},\gamma)-F(x_k,\delta)+F(x_{k-1},\delta)|)<\infty,$$

$$\sup_{E'}\sup_{\alpha,\beta\in[x',x'']}\sum_{l=1}^{N'}\Psi(|F(\alpha,y_l)-F(\alpha,y_{l-1})-F(\beta,y_l)+F(\beta,y_{l-1})|)<\infty,$$

where $\sup_E$ runs over all finite partitions of $[x',x'']$, $\sup_{E'}$ runs over all finite partitions of $[y',y'']$, namely $E:=\{x'=x_0<x_1<\cdots<x_N=x''\}$, $E':=\{y'=y_0<y_1<\cdots<y_{N'}=y''\}$.

We say $G(x,y)$ is of bounded Φ_1, Ψ_1-variation in (x,y), if

$$\sup_{E\times E'}\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\Phi_1(|\Delta_i\Delta_j G|)\right)<\infty, \tag{3.2.1}$$

where

$$\Delta_i\Delta_j G := G(x_i, y_j) - G(x_{i-1}, y_j) - G(x_i, y_{j-1}) + G(x_{i-1}, y_{j-1}),$$

and $\sup_{E\times E'}$ *runs over all finite partitions of* $[x', x''] \times [y', y'']$.
If $\Psi_1(u) = u$, *we call* $G(x,y)$ *is of bounded* Φ_1*-variation in* (x,y). *If* $\Phi_1(u) = u^p$, $\Psi_1(u) = u^q$, $p, q \geq 1$, *we call* $G(x,y)$ *is of bounded* p,q*-variation in* (x,y). *If* $p = q = 1$, $G(x,y)$ *is of bounded variation in* (x,y).

We quote the theorem of two-parameter integral of Young (1938) ([51]) here.

(6.3 Theorem): Suppose given a convex Φ and a convex Ψ with the inverse ϕ, ψ (all continuous and strictly increasing as usual); monotone increasing ϱ and σ subject to $\varrho(u)\sigma(u) = u$; monotone increasing λ, μ such that

$$\sum_n \varrho[\varphi(1/n)]\lambda(1/n) < \infty \quad \textit{and} \quad \sum_n \sigma[\psi(1/n)]\mu(1/n) < \infty \tag{3.2.2}$$

and monotone increasing ω and χ given (monotone increasing). Then given $\varepsilon > 0$, we can determine finite sets E and E' of values of x and y on $[x', x'']$ and $[y', y'']$ respectively so that for every function $F(x,y)$ whose total Φ- and Ψ-bivariationsin x and y repectively, are less than fixed constants P and Q, and for every function $G(x,y)$ which satisfies the condition

$$|\Delta_i\Delta_j G| \leq \lambda(\Delta_i\omega)\mu(\Delta_j\chi), \tag{3.2.3}$$

we have

$$\left|\int_{x'}^{x''}\int_{y'}^{y''} \{F(x,y) - F_{Z,Z'}(x,y)\} d_{x,y}G(x,y)\right| < \varepsilon,$$

as soon as Z and Z' include respectively the points of E and of E' among their points of division. Here $\Delta_i\omega = \omega(x_i) - \omega(x_{i-1})$, $\Delta_j\chi = \chi(y_j) - \chi(y_{j-1})$, $F_{Z,Z'}(x,y)$ is the step function of F on Z, Z'.

Remark 3.2.1 *Young's condition is very strong and the class of functions that satisfy Young's condition is restricted. In particular, Young's condition does not seem to include the class of functions of bounded variation and many important examples. We give a new and weaker condition for the integration in Section 3.4. We will use Lyons' idea of control functions to simplify our proof. One can see our condition is a natural extension of locally bounded multi-dimensional L-S measure.*

§3.3 An Example

In the following, we will give an example of $p,1$-variation ($p>1$) function..It seems difficult, if not impossible, to check Young's condition (3.2.3) for this example.

Example 3.3.1 *Consider*

$$f(x,y)=xysin(\frac{1}{x}+\frac{1}{y}),\ 0<x,y\leq 1,\ f(0,y)=f(x,0)=f(0,0)=0.$$

This is a continuous function of unbounded variation but of bounded $p,1$-variation ($p>1$). To see it is of unbounded variation, we take the partition $E_1\times E_2=\{0<\frac{1}{n\pi+\frac{\pi}{2}-1}<\frac{1}{n\pi-1}<\cdots<\frac{1}{\pi-1}<1,0<1\}$,

$$\begin{aligned}
&\sum_{i,j}|\Delta_i\Delta_j f|=\sum_i|x_isin(\frac{1}{x_i}+1)-x_{i-1}sin(\frac{1}{x_{i-1}}+1)|\\
=&\sum_i|\frac{1}{i\pi+\frac{\pi}{2}-1}sin(i\pi+\frac{\pi}{2})-\frac{1}{i\pi-1}sin(i\pi)|\\
=&\sum_i\frac{1}{i\pi+\frac{\pi}{2}-1}\\
=&\infty.
\end{aligned}$$

To see it is of bounded $p,1$-variation for any $p>1$, consider any partition $E\times E'$

$$\begin{aligned}
&\sum_{i,j}|\Delta_i\Delta_j f|^p\\
=&\sum_{i,j}\Big|x_iy_jsin(\frac{1}{x_i}+\frac{1}{y_j})-x_{i-1}y_jsin(\frac{1}{x_{i-1}}+\frac{1}{y_j})\\
&\quad -x_iy_{j-1}sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})+x_{i-1}y_{j-1}sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}})\Big|^p\\
=&\sum_{i,j}\Big|y_j\Big[x_isin(\frac{1}{x_i}+\frac{1}{y_j})-x_{i-1}sin(\frac{1}{x_{i-1}}+\frac{1}{y_j})\Big]\\
&\quad -y_{j-1}\Big[x_isin(\frac{1}{x_i}+\frac{1}{y_{j-1}})-x_{i-1}sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}})\Big]\Big|^p\\
=&\sum_{i,j}\Big|y_j(x_i-x_{i-1})sin(\frac{1}{x_i}+\frac{1}{y_j})+y_jx_{i-1}\Big[sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j})\Big]\\
&\quad -y_{j-1}(x_i-x_{i-1})sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})\\
&\quad -y_{j-1}x_{i-1}\Big[sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}})\Big]\Big|^p\\
=&\sum_{i,j}\Big|(x_i-x_{i-1})\Big[y_jsin(\frac{1}{x_i}+\frac{1}{y_j})-y_{j-1}sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})\Big]
\end{aligned}$$

$$
\begin{aligned}
&+x_{i-1}\Big[y_j((sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j}))\\
&\qquad -y_{j-1}(sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}}))\Big]\Big|^p\\
=\ &\sum_{i,j}\Big|(x_i-x_{i-1})\Big[(y_j-y_{j-1})sin(\frac{1}{x_i}+\frac{1}{y_j})\\
&\qquad +y_{j-1}(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_i}+\frac{1}{y_{j-1}}))\Big]\\
&+x_{i-1}\Big[(y_j-y_{j-1})(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j}))\\
&\qquad +y_{j-1}(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j})\\
&\qquad\qquad -sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})+sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}}))\Big]\Big|^p\\
\le\ & c_p\Big\{\sum_{i,j}\Big|(x_i-x_{i-1})(y_j-y_{j-1})sin(\frac{1}{x_i}+\frac{1}{y_j})\Big|^p\\
&+\sum_{i,j}\Big|y_{j-1}(x_i-x_{i-1})(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_i}+\frac{1}{y_{j-1}}))\Big|^p\\
&+\sum_{i,j}\Big|x_{i-1}(y_j-y_{j-1})(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j}))\Big|^p\\
&+\sum_{i,j}\Big|x_{i-1}y_{j-1}(sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_{i-1}}+\frac{1}{y_j})\\
&\qquad\qquad -sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})+sin(\frac{1}{x_{i-1}}+\frac{1}{y_{j-1}}))\Big|^p\Big\}\\
:=\ & c_p(I+II+III+IV),
\end{aligned}
\tag{3.3.1}
$$

where c_p is a constant. It's easy to see that

$$
I\le\sum_{i,j}(x_i-x_{i-1})^p(y_j-y_{j-1})^p\le 1. \tag{3.3.2}
$$

For II, as $|sinx|\le x$, so

$$
\begin{aligned}
II &\le 2^{p-1}\sum_{i,j}y_{j-1}^p(x_i-x_{i-1})^p\Big|sin(\frac{1}{x_i}+\frac{1}{y_j})-sin(\frac{1}{x_i}+\frac{1}{y_{j-1}})\Big|\\
&= 2^{p-1}\sum_{i,j}y_{j-1}^p(x_i-x_{i-1})^p\cdot\Big|2cos\frac{\frac{2}{x_i}+\frac{1}{y_j}+\frac{1}{y_{j-1}}}{2}sin\frac{\frac{1}{y_j}-\frac{1}{y_{j-1}}}{2}\Big|\\
&\le 2^{p-1}\sum_{i,j}y_{j-1}^p(x_i-x_{i-1})^p\left(\frac{1}{y_{j-1}}-\frac{1}{y_j}\right)\\
&= 2^{p-1}\sum_i(x_i-x_{i-1})^p\sum_j y_{j-1}^p\left(\frac{1}{y_{j-1}}-\frac{1}{y_j}\right).
\end{aligned}
$$

It is obvious that

$$\sum_i (x_i - x_{i-1})^p < \infty.$$

And also because

$$\begin{aligned}\sum_j y_{j-1}^p \left(\frac{1}{y_{j-1}} - \frac{1}{y_j}\right) &\leq \sum_j y_{j-1}^p \frac{y_j - y_{j-1}}{y_{j-1}^2} \\ &= \sum_j y_{j-1}^{p-2}(y_j - y_{j-1}) \\ &\leq \int_0^1 y^{p-2} dy = \frac{1}{p-1}. \end{aligned} \tag{3.3.3}$$

So we get $II < \infty$. Similar to the discussion of II, we can also prove that $III < \infty$. About IV,

$$\begin{aligned} & IV \\ = & \sum_{i,j} x_{i-1}^p y_{j-1}^p \Big| 2cos\frac{\frac{1}{x_i} + \frac{1}{x_{i-1}} + \frac{2}{y_j}}{2} sin\frac{\frac{1}{x_i} - \frac{1}{x_{i-1}}}{2} \\ & \qquad -2cos\frac{\frac{1}{x_i} + \frac{1}{x_{i-1}} + \frac{2}{y_{j-1}}}{2} sin\frac{\frac{1}{x_i} - \frac{1}{x_{i-1}}}{2} \Big|^p \\ = & 2^p \sum_{i,j} x_{i-1}^p y_{j-1}^p \Big| sin\frac{\frac{1}{x_i} - \frac{1}{x_{i-1}}}{2} \Big|^p \cdot \Big| cos\frac{\frac{1}{x_i} + \frac{1}{x_{i-1}} + \frac{2}{y_j}}{2} - cos\frac{\frac{1}{x_i} + \frac{1}{x_{i-1}} + \frac{2}{y_{j-1}}}{2} \Big|^p \\ = & 2^p \sum_{i,j} x_{i-1}^p y_{j-1}^p \Big| sin\frac{\frac{1}{x_i} - \frac{1}{x_{i-1}}}{2} \Big|^p \\ & \qquad \cdot \Big| -2sin\frac{\frac{1}{x_i} + \frac{1}{x_{i-1}} + \frac{1}{y_j} + \frac{1}{y_{j-1}}}{2} sin\frac{\frac{1}{y_j} - \frac{1}{y_{j-1}}}{2} \Big|^p \\ \leq & 2^p \cdot 2^p \sum_{i,j} x_{i-1}^p y_{j-1}^p \frac{1}{2}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) \cdot \frac{1}{2}\left(\frac{1}{y_{j-1}} - \frac{1}{y_j}\right) \\ = & 2^{2p-2} \sum_i x_{i-1}^p \left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) \sum_j y_{j-1}^p \left(\frac{1}{y_{j-1}} - \frac{1}{y_j}\right) \\ \leq & \frac{2^{2p-2}}{(p-1)^2}, \end{aligned}$$

following from a similar argument as in (3.3.3). So the function $f(x,y) = xysin(\frac{1}{x} + \frac{1}{y})$, $0 < x, y \leq 1$, $f(0,y) = f(x,0) = f(0,0) = 0$, is of bounded $p,1$-variation for any $p > 1$. Moreover, from the above proof, we can see for this function $f(x,y)$ on $(x,y) \in [0,\delta_1] \times [0,\delta_2]$, its $p,1$-variation tends to 0 when either δ_1 or δ_2 decreases to 0. ◇

§3.4 Two-parameter p,q-variation Path Integrals

We say a function $f(x,y)$ has a jump at (x_1,y_1) if there exists an $\varepsilon>0$ such that for any $\delta>0$, there exists (x_2,y_2) satisfying $\max\{|x_1-x_2|,|y_1-y_2|\}<\delta$ and $|f(x_2,y_2)-f(x_1,y_2)-f(x_2,y_1)+f(x_1,y_1)|>\varepsilon$. For a function $G(x,y)$ of bounded Φ_1,Ψ_1-variation, for any given $\varepsilon>0$, it is easy to see that there exists a $\delta(\varepsilon)>0$ and a finite number of jump points $\{(x_1,y_1),\cdots,(x_{n_0},y_{m_0})\}$ such that $|G(x,y)-G(x,\tilde{y})-G(\tilde{x},y)+G(\tilde{x},\tilde{y})|<\varepsilon$ whenever $\max\{|\tilde{x}-x|,|\tilde{y}-y|\}<\delta(\varepsilon)$, $[\tilde{x},x]\cap\{x_1,\cdots,x_{n_0}\}=\emptyset$ and $[\tilde{y},y]\cap\{y_1,\cdots,y_{m_0}\}=\emptyset$. Denote $H_0\times H_0':=\{x_1,\cdots,x_{n_0}\}\times\{y_1,\cdots,y_{m_0}\}$. In the following, we assume the following finite large jump condition: for any $\varepsilon>0$, there exists at most finite many points $\{x_1,\cdots,x_{n_1}\}$, $\{y_1,\cdots,y_{m_1}\}$ and a constant $\delta(\varepsilon)>0$ such that the total Φ_1,Ψ_1-variation of G on $[x,x+\delta]\times[y',y'']$ is smaller than ε if $[x,x+\delta]\cap\{x_1,\cdots,x_{n_1}\}=\emptyset$, and the total Φ_1,Ψ_1-variation of G on $[x',x'']\times[y,y+\delta]$ is smaller than ε if $[y,y+\delta]\cap\{y_1,\cdots,y_{m_1}\}=\emptyset$. Denote $H\times H':=\{x_1,\cdots,x_{n_1}\}\times\{y_1,\cdots,y_{m_1}\}$. It is obviously that $H\times H'\supset H_0\times H_0'$. There are many examples of bounded Φ_1,Ψ_1-variation functions that satisfy the finite large jump condition. But it is not clear whether or not the bounded Φ_1,Ψ_1-variation condition implies automatically the finite large jump condition in the two parameter case although this is true in the one parameter case.

For the partition $E\times E'$, denote by $m(E\times E')$ the mesh of the partition.

We need the following simple inequalities: Let f be a nonnegative and nondecreasing function, then

$$\sum_{p=0}^{\infty}2^{p-1}f(\frac{1}{2^p})\le\sum_{m=1}^{\infty}f(\frac{1}{m})\le\sum_{p=0}^{\infty}2^pf(\frac{1}{2^p}),\tag{3.4.1}$$

and for any $v\ge1$,

$$\sum_{p=v}^{\infty}2^{p-1}f(\frac{1}{2^p})\le\sum_{m=2^{v-1}+1}^{\infty}f(\frac{1}{m})\le\sum_{p=v-1}^{\infty}2^pf(\frac{1}{2^p}),\tag{3.4.2}$$

if the series $\sum_{m=1}^{\infty}f(\frac{1}{m})$ is convergent. These inequalities were also used in the proof of Young's main results. We listed them here only for the purpose to make the proof of the following theorem easier to understand. The proof is elementary and omitted.

First, if $F(x,y)$ is a simple function, say

$$F(x,y)=\sum_{i=1}^{M}\sum_{j=1}^{M'}F(x_{i-1},y_{j-1})1_{\{x_{i-1}<x\le x_i,y_{j-1}<y\le y_j\}},$$

as normal we can see that the integral of the simple function can be defined as

$$\int_{x'}^{x''}\int_{y'}^{y''}F(x,y)d_{x,y}G(x,y)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N'} F(x_{i-1},y_{j-1})\left(G(x_i,y_j)-G(x_{i-1},y_j)-G(x_i,y_{j-1})+G(x_{i-1},y_{j-1})\right).$$

Theorem 3.4.1 *Let $F(x,y)$ be a continuous function for which there exist continuous increasing functions $\omega(x)$, $\chi(y)$ and such that for any $x_1,x_2\in[x',x'']$, $y_1,y_2\in[y',y'']$,*

$$|F(x_1,y_1)-F(x_2,y_2)|\le\varphi(\omega(x_2)-\omega(x_1))+\psi(\chi(y_2)-\chi(y_1)). \tag{3.4.3}$$

Let $G(x,y)$ be of bounded Φ_1,Ψ_1-variation in (x,y) and satisfy the finite large jump condition, where Ψ_1, Φ_1 are as at the beginning of Section 3.2. If there exist increasing concave functions ϱ and σ subject to $\varrho(u)\sigma(u)=u$ such that

$$\sum_{m,n}\varrho[\varphi(\frac{1}{n})]\sigma[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})]<\infty, \tag{3.4.4}$$

then the integral

$$\begin{aligned}&\int_{x'}^{x''}\int_{y'}^{y''}F(x,y)d_{x,y}G(x,y)\\ =\ &\lim_{m(E\times E')\to 0}\sum_{i=1}^{N}\sum_{j=1}^{N'}F(x_{i-1},y_{j-1})\Delta_i\Delta_jG\end{aligned} \tag{3.4.5}$$

is well defined using the partitions $E\times E'$ of $[x',x'']\times[y',y'']$ which include the finite sets $H\times H'$ defined above, i.e. for any given $\varepsilon>0$, we can determine finite sets H and H' of variables x and y respectively such that

$$\left|\int_{x'}^{x''}\int_{y'}^{y''}F(x,y)d_{x,y}G(x,y)-\sum_{i=1}^{N}\sum_{j=1}^{N'}F(x_{i-1},y_{j-1})\Delta_i\Delta_jG\right|<\varepsilon.$$

Proof: For any partition $E\times E':=\{x'=x_0<x_1<\cdots<x_N=x'',y'=y_0<y_1<\cdots<y_{N'}=y''\}$, consider

$$F_{E,E'}(x,y):=\sum_{i=1}^{N}\sum_{j=1}^{N'}F(x_{i-1},y_{j-1})1_{\{x_{i-1}\le x<x_i,y_{j-1}\le y<y_j\}},$$

then

$$\begin{aligned}S(E,E')\ &:=\ S_F(E,E'):=\int_{x'}^{x''}\int_{y'}^{y''}F_{E,E'}(x,y)d_{x,y}G(x,y)\\ &=\ \sum_{i=1}^{N}\sum_{j=1}^{N'}F(x_{i-1},y_{j-1})\Delta_i\Delta_jG.\end{aligned}$$

From the assumption of F,

$$\begin{aligned}|F(x_k,y)-F(x_{k-1},y)|&\le\varphi(\omega(x_k)-\omega(x_{k-1})),\ k=1,2,\cdots,N,\\ |F(x,y_l)-F(x,y_{l-1})|&\le\psi(\chi(y_l)-\chi(y_{l-1})),\ l=1,2,\cdots,N'.\end{aligned}$$

Obviously, if $y_{j-1} \le y < y_j$, $j = 1, \cdots, N'$,

$$\begin{aligned} & |F_{E,E'}(x_k, y) - F_{E,E'}(x_{k-1}, y)| \\ = \; & |F(x_k, y_{j-1}) - F(x_{k-1}, y_{j-1})| \\ \le \; & \varphi(\omega(x_k) - \omega(x_{k-1})), \quad k = 1, 2, \cdots, N, \end{aligned}$$

and if $x_{i-1} \le x < x_i$, $i = 1, \cdots, N$,

$$\begin{aligned} & |F_{E,E'}(x, y_l) - F_{E,E'}(x, y_{l-1})| \\ = \; & |F(x_{i-1}, y_l) - F(x_{i-1}, y_{l-1})| \\ \le \; & \psi(\chi(y_l) - \chi(y_{l-1})), \quad l = 1, 2, \cdots, N'. \end{aligned}$$

Let $P := \omega(x'')$, $Q := \chi(y'')$. Because ω and χ are both continuous increasing functions, we can define a sequence of finite sets $E_p := \{x' = x_0 < x_1 < \cdots < x_{2^p} = x''\}$ such that $\omega(x_{i+1}) - \omega(x_i) = 2^{-p}P$, $i = 0, \cdots, 2^p$; $E'_q := \{y' = y_0 < y_1 < \cdots < y_{2^q} = y''\}$ such that $\chi(y_{j+1}) - \chi(y_j) = 2^{-q}Q$, $j = 0, \cdots, 2^q$. It's easy to see $E_p \subset E_{p+1}$, $E'_q \subset E'_{q+1}$. We will prove our theorem in four steps.

Step 1: Note

$$\begin{aligned} & S(E_{p+1}, E'_{q+1}) - S(E_p, E'_{q+1}) - S(E_{p+1}, E'_q) + S(E_p, E'_q) \\ = \; & \sum_{i=1,3,5\cdots,2^{p+1}-1} \; \sum_{j=1,3,5\cdots,2^{q+1}-1} \Big[F(x_{i-1}, y_{j-1}) \Delta_i \Delta_j G \\ & + F(x_{i-1}, y_j) \Delta_i \Delta_{j+1} G + F(x_i, y_j) \Delta_{i+1} \Delta_{j+1} G + F(x_i, y_{j-1}) \Delta_{i+1} \Delta_j G \\ & - F(x_{i-1}, y_{j-1}) \Big(G(x_{i+1}, y_j) - G(x_{i-1}, y_j) \\ & \qquad - G(x_{i+1}, y_{j-1}) + G(x_{i-1}, y_{j-1}) \Big) \\ & - F(x_{i-1}, y_j) \Big(G(x_{i+1}, y_{j+1}) - G(x_{i+1}, y_j) \\ & \qquad - G(x_{i-1}, y_{j+1}) + G(x_{i-1}, y_j) \Big) \\ & - F(x_{i-1}, y_{j-1}) \Big(G(x_i, y_{j+1}) - G(x_{i-1}, y_{j+1}) \\ & \qquad - G(x_i, y_{j-1}) + G(x_{i-1}, y_{j-1}) \Big) \\ & - F(x_i, y_{j-1}) \Big(G(x_{i+1}, y_{j+1}) - G(x_{i+1}, y_{j-1}) - G(x_i, y_{j+1}) + G(x_i, y_{j-1}) \Big) \\ & + F(x_{i-1}, y_{j-1}) \Big(G(x_{i+1}, y_{j+1}) - G(x_{i+1}, y_{j-1}) \\ & \qquad - G(x_{i-1}, y_{j+1}) + G(x_{i-1}, y_{j-1}) \Big) \Big] \end{aligned}$$

$$= \sum_{i=1,3,5\cdots,2^{p+1}-1} \sum_{j=1,3,5\cdots,2^{q+1}-1} \Big(\Delta_i\Delta_j F\Big)\Big(\Delta_{i+1}\Delta_{j+1} G\Big). \tag{3.4.6}$$

Because

$$\begin{aligned} |\Delta_i\Delta_j F| &\leq |F(x_i,y_j)-F(x_{i-1},y_j)|+|F(x_i,y_{j-1})-F(x_{i-1},y_{j-1})| \\ &\leq 2\varphi(2^{-(p+1)}P) \leq 2C\varphi(2^{-p}P), \end{aligned}$$

and also

$$\begin{aligned} |\Delta_i\Delta_j F| &\leq |F(x_i,y_j)-F(x_i,y_{j-1})|+|F(x_{i-1},y_j)-F(x_{i-1},y_{j-1})| \\ &\leq 2\psi(2^{-(q+1)}Q) \leq 2C\psi(2^{-q}Q), \end{aligned}$$

it is easy to see

$$|\Delta_i\Delta_j F| \leq 2\varrho[\varphi(2^{-p}P)]\sigma[\psi(2^{-q}Q)] \tag{3.4.7}$$

for any increasing concave functions ϱ, σ satisfying $\varrho(u)\sigma(u)=u$.

For the function G, let M be its total Φ_1,Ψ_1-variation, then

$$\sum_{j=1}^{2^q}\Psi_1\left(\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|)\right) \leq M.$$

It is trivial to see that,

$$2^{-q}\sum_{j=1}^{2^q}\Psi_1\left(\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|)\right) \leq 2^{-q}M. \tag{3.4.8}$$

As Ψ_1 is convex, so

$$2^{-q}\sum_{j=1}^{2^q}\Psi_1\left(\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|)\right) \geq \Psi_1\left(2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|)\right). \tag{3.4.9}$$

It turns out from (3.4.8) and (3.4.9) that

$$\Psi_1\left(2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|)\right) \leq 2^{-q}M. \tag{3.4.10}$$

This leads to

$$2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|) \leq \psi_1(2^{-q}M). \tag{3.4.11}$$

This is equivalent to

$$2^{-p}2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|) \leq 2^{-p}\psi_1(2^{-q}M). \tag{3.4.12}$$

But, by the convexity of Φ_1, we have

$$2^{-p}2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|) \;\geq\; \Phi_1\left(2^{-p}2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}|\Delta_i\Delta_j G|\right). \tag{3.4.13}$$

So it follows from (3.4.12) and (3.4.13) that

$$\Phi_1\left(2^{-p}2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}|\Delta_i\Delta_j G|\right) \;\leq\; 2^{-p}\psi_1(2^{-q}M). \tag{3.4.14}$$

Therefore,

$$2^{-p}2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}|\Delta_i\Delta_j G| \;\leq\; \varphi_1(2^{-p}\psi_1(2^{-q}M)). \tag{3.4.15}$$

So

$$\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}|\Delta_i\Delta_j G| \;\leq\; 2^{p+q}\varphi_1(2^{-p}\psi_1(2^{-q}M)). \tag{3.4.16}$$

By the same method, one can see that

$$\sum_{j=1,3,5,\cdots 2^{q+1}-1}\;\sum_{i=1,3,5,\cdots,2^{p+1}-1}|\Delta_{i+1}\Delta_{j+1}G| \leq 2^{p+q}\varphi_1(2^{-p}\psi_1(2^{-q}M)). \tag{3.4.17}$$

Therefore, it follows from (3.4.6), (3.4.7) and (3.4.17) that there exists $K>0$ such that

$$\begin{aligned}&\left|S(E_{p+1},E'_{q+1})-S(E_{p+1},E'_q)-S(E_p,E'_{q+1})+S(E_p,E'_q)\right|\\ \leq\;& K2^{p+q}\varphi_1(2^{-p}\psi_1(2^{-q}M))\varrho[\varphi(P2^{-p})]\sigma[\psi(Q2^{-q})].\end{aligned}$$

Step 2: Let's prove that

$$\lim_{p,q\to\infty}S(E+E_p,E'+E'_q)-S(E_p,E'_q)=0. \tag{3.4.18}$$

Denoting by x_l, $l=0,1,\cdots,L$ (y_n, $n=0,1,\cdots,L'$) the distinct points of E_p (E'_q) in increasing order, and by $x_{l-1,i}$, $i=0,1,\cdots,M_l$ ($y_{n-1,j}$, $j=0,1,\cdots,M'_n$) those of $E+E_p$ ($E'+E'_q$) lying in the interval $x_{l-1}\leq x\leq x_l$ ($y_{n-1}\leq y\leq y_n$) with $x_{l-1,0}=x_{l-1}, x_{l-1,M_l}=x_l$ ($y_{n-1,0}=y_{n-1}, y_{n-1,M'_n}=y_n$), we have

$$\begin{aligned}&S(E+E_p,E'+E'_q)-S(E_p,E'_q)\\ =\;&(S(E+E_p,E'+E'_q)-S(E+E_p,E'_q))+(S(E+E_p,E'_q)-S(E_p,E'_q))\\ =\;&\sum_{l=1}^{L}\sum_{i=1}^{M_l}\sum_{n=1}^{L'}\sum_{j=1}^{M'_n}\Big\{[F(x_{l-1,i-1},y_{n-1,j-1})-F(x_{l-1,i-1},y_{n-1})]\end{aligned}$$

$$
\begin{aligned}
&+[F(x_{l-1,i-1},y_{n-1})-F(x_{l-1},y_{n-1})]\Big\}\Big\{G(x_{l-1,i},y_{n-1,j})\\
&-G(x_{l-1,i-1},y_{n-1,j})-G(x_{l-1,i},y_{n-1,j-1})+G(x_{l-1,i-1},y_{n-1,j-1})\Big\}\\
\leq\ & 4N_1N_2[\psi(2^{-q}Q)+\varphi(2^{-p}P)]\cdot\max|G|\\
\to\ & 0,\ as\ p,\ q\to\infty.
\end{aligned}
$$

Here N_1, N_2 denote the number of points of $E+E_0$, $E'+E_0'$, respectively.

Step 3: Let $F(x,y)$ vanish for $x=x'$ identically in y, and for $y=y'$ identically in x, so

$$
\begin{aligned}
&F_{E_0,E'}(x,y)=F(x',y)=0,\quad F_{E,E_0'}(x,y)=F(x,y')=0,\\
&F_{E_0,E_0'}(x,y)=F(x',y')=0.
\end{aligned}
$$

If this is so, note that $S(E,E')=S_{F_{E,E'}}(E+E_p,E'+E_q')$, then from Step 2, Step 1 and (3.4.1),

$$
\begin{aligned}
&|S(E,E')|\\
=\ &|S(E,E')-S(E_0,E')-S(E,E_0')+S(E_0,E_0')|\\
=\ &\Big|\lim_{p,q\to\infty}\Big[S_{F_{E,E'}}(E+E_p,E'+E_q')-S_{F_{E,E'}}(E_p,E_q')+S_{F_{E,E'}}(E_p,E_q')\\
&\qquad -S_{F_{E,E'}}(E_0,E_q')-S_{F_{E,E'}}(E_p,E_0')+S_{F_{E,E'}}(E_0,E_0')\Big]\Big|\\
=\ &\Big|\lim_{p,q\to\infty}\Big[S_{F_{E,E'}}(E_p,E_q')-S_{F_{E,E'}}(E_0,E_q')-S_{F_{E,E'}}(E_p,E_0')\\
&\qquad +S_{F_{E,E'}}(E_0,E_0')\Big]\Big|\\
=\ &\Big|\sum_{p,q=0}^{\infty}\Big[S_{F_{E,E'}}(E_{p+1},E_{q+1}')-S_{F_{E,E'}}(E_{p+1},E_q')\\
&\qquad -S_{F_{E,E'}}(E_p,E_{q+1}')+S_{F_{E,E'}}(E_p,E_q')\Big]\Big|\\
\leq\ &\sum_{p,q=0}^{\infty}K2^{p+q}\varrho[\varphi(P2^{-p})]\sigma[\psi(Q2^{-q})]\varphi_1(2^{-p}\psi_1(2^{-q}M))\\
\leq\ &4K\sum_{m,n=1}^{\infty}\varrho[\varphi(\frac{P}{n})]\sigma[\psi(\frac{Q}{m})]\varphi_1(\frac{1}{n}\psi_1(\frac{1}{m}M)).
\end{aligned}
\tag{3.4.19}
$$

Let $F_{x',y'}(x,y):=F(x,y)-F(x',y)-F(x,y')+F(x',y')$ and replace $F(x,y)$ by $F_{x',y'}(x,y)$ for $x'\leq x\leq x''$, $y'\leq y\leq y''$. This alteration doesn't affect double difference of F. Therefore we may suppose that $F(x,y)$ vanishes identically on the lines $x=x'$ and $y=y'$ as above.

Step 4: We determine a set of finite points $H_v\times H_{v'}:=\{x'=x_0<x_1<\cdots<x_L=x'', y'=y_0<y_1<\cdots<y_{L'}=y''\}$, where $L\leq 2\cdot 2^v$, $L'\leq 2\cdot 2^{v'}$, such that in the rectangle

$[x_{l-1}+\delta, x_l-\delta]\times[y_{k-1}+\delta, y_k-\delta]$, $|\Delta_i\Delta_j G| < \varepsilon(v,v')$ for any $0<\delta\le\frac{1}{4}\min\{\min_{1\le l\le L}\{x_l - x_{l-1}\}, \min_{1\le l\le L'}\{y_l - y_{l-1}\}\}$. Moreover, in the interval $[x', x'']\times[y_{k-1}+\delta, y_k-\delta]$, $\chi(y_k-\delta)-\chi(y_{k-1}+\delta)\le Q\cdot 2^{-v'}$, the total Φ_1, Ψ_1- variation of G is at most $M\cdot 2^{-v'}$; in the interval $[x_{l-1}+\delta, x_l-\delta]\times[y', y'']$, $\omega(x_l-\delta)-\omega(x_{l-1}+\delta)\le P\cdot 2^{-v}$ and the total Φ_1-variation of G in x for the given partition of H'_v of $[y', y'']$ is at most $2^{-v}L'\psi_1(\frac{1}{L'}M)$. Here, the first and the third statements are obvious, and the second statement follows from the finite large jump condition. The last one can be seen by observing that $\sum_{j=1}^{L'}\Psi_1\left(\sum_{i=1}^{L}\Phi_1(|\Delta_i\Delta_j G|)\right)\le M$ is equivalent to $\sum_{j=1}^{L'}\sum_{i=1}^{L}\Phi_1(|\Delta_i\Delta_j G|)\le L'\psi_1(\frac{1}{L'}M)$. More generally, for any partition $E=\{x'_1, x'_2, \cdots, x'_N\}$, we have $\sum_{i=1}^{N}\sum_{j=1}^{L'}\Phi_1(|\Delta'_i\Delta_j G|)\le L'\psi_1(\frac{1}{L'}M)$. Here $\Delta'_i\Delta_j G$ is the double increment of G on $(x'_{i-1}, x'_i)\times(y_{j-1}, y_j)$. We can make E include H_v among their points of divisions and let $E_{l-1}=\{x_{l-1,1}, x_{l-1,2}, \cdots, x_{l-1,N_{l-1}}\}$ denote all the points in E falling into the interval (x_{l-1}, x_l) $(l=1,2,\cdots,L)$. We can certainly make

$$\sum_{i=1}^{N_{l-1}}\sum_{j=1}^{L'}\Phi_1(|\Delta'_{l-1,i}\Delta_j G|)\le 2^{-v}L'\psi_1(\frac{1}{L'}M), \tag{3.4.20}$$

where $\Delta'_{l-1,i}\Delta_j G$ is the double increment of G on $(x'_{l-1,i-1}, x'_{l-1,i})\times(y_{j-1}, y_j)$. In fact E_{l-1} can be any partition of $[x_{l-1}+\delta, x_l-\delta]$ for any sufficiently small $\delta>0$.

We need to prove that for any $\varepsilon>0$,

$$|S(\tilde{D},\tilde{D}')-S(D,D')|<\varepsilon, \tag{3.4.21}$$

as long as $\tilde{D}\times\tilde{D}'$, and $D\times D'$ include $H_v\times H_{v'}$. Observe that

$$\begin{aligned}
& |S(\tilde{D},\tilde{D}')-S(D,D')| \\
\le\ & |S(\tilde{D},\tilde{D}')-S(\tilde{D},D')|+|S(\tilde{D},D')-S(D,D')| \\
\le\ & \left|\int_{x'}^{x''}\int_{y'}^{y''}(F_{\tilde{D},\tilde{D}'}-F_{\tilde{D},D'})d_{x,y}G(x,y)\right| \\
& +\left|\int_{x'}^{x''}\int_{y'}^{y''}(F_{\tilde{D},D'}-F_{D,D'})d_{x,y}G(x,y)\right|.
\end{aligned} \tag{3.4.22}$$

First, since $F_{\tilde{D},\tilde{D}'}-F_{\tilde{D},D'}$ vanishes identically in x, when $y=y_{k-1}$, from Step 3 and (3.4.1), (3.4.2), and the concavity of ϱ, σ, φ, ψ, φ_1 and ψ_1, we obtain for any sufficiently small $\delta>0$,

$$\begin{aligned}
& |S(\tilde{D},\tilde{D}')-S(\tilde{D},D')| \\
\le\ & \sum_{k=1}^{L'}\left|\int_{x'}^{x''}\int_{y_{k-1}+\delta}^{y_k-\delta}(F_{\tilde{D},\tilde{D}'}-F_{\tilde{D},D'})d_{x,y}G(x,y)\right| \\
\le\ & \sum_{k=1}^{L'}4K\sum_{m,n=1}^{\infty}\varrho[\varphi(\frac{P}{n})]\sigma[\psi(\frac{2^{-v'}Q}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m}2^{-v'}M)]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{L'} 4K \sum_{n=1}^{\infty}\sum_{q=0}^{\infty} \varrho[\varphi(\frac{P}{n})]2^q\sigma[\psi(2^{-(v'+q)}Q)]\varphi_1[\frac{1}{n}\psi_1(2^{-(v'+q)}M)] \\
&= L'4K \sum_{n=1}^{\infty}\sum_{q=v'}^{\infty} \varrho[\varphi(\frac{P}{n})]2^{-v'+q}\sigma[\psi(2^{-q}Q)]\varphi_1[\frac{1}{n}\psi_1(2^{-q}M)] \\
&\leq 16K \sum_{n=1}^{\infty}\sum_{m=2^{v'-1}+1}^{\infty} \varrho[\varphi(\frac{P}{n})]\sigma[\psi(\frac{Q}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m}M)] \\
&\leq 16K(P\vee 1)(Q\vee 1))(M\vee 1)\sum_{n=1}^{\infty}\sum_{m=2^{v'-1}+1}^{\infty} \varrho[\varphi(\frac{1}{n})]\sigma[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] \\
&\leq \varepsilon_{v'}, \qquad (3.4.23)
\end{aligned}
$$

where $\varepsilon_{v'} \to 0$, as $v' \to \infty$.

Second, since $F_{\tilde{D},D'} - F_{D,D'}$ vanishes identically in y, when $x = x_{l-1}$. From the discussion above and (3.4.20), we know for any partition $E_p = \{x_1, x_2, \cdots, x_{2^p}\}$ of $[x_{l-1} + \delta, x_l - \delta]$, and any partition $E_{q'} = \{y_1, y_2, \cdots, y_{2^q}\}$ of $[y', y'']$, (3.4.11) bacomes

$$
2^{-q}\sum_{j=1}^{2^q}\sum_{i=1}^{2^p}\Phi_1(|\Delta_i\Delta_j G|) \leq 2^{-v}\psi_1(2^{-q}M). \qquad (3.4.24)
$$

So from Step 3 and (3.4.1), (3.4.2),

$$
\begin{aligned}
&\left|\int_{x_{l-1}+\delta}^{x_l-\delta}\int_{y'}^{y''}(F_{\tilde{D},D'} - F_{D,D'})d_{x,y}G(x,y)\right| \\
&\leq 4K \sum_{m,n=1}^{\infty} \varrho[\varphi(\frac{2^{-v}P}{n})]\sigma[\psi(\frac{Q}{m})]\varphi_1[\frac{2^{-v}}{n}\psi_1(\frac{1}{m}M)] \\
&\leq 4K \sum_{m=1}^{\infty}\sum_{q=0}^{\infty} 2^q\varrho[\varphi(2^{-(v+q)}P)]\sigma[\psi(\frac{Q}{m})]\varphi_1[2^{-(v+q)}\psi_1(\frac{1}{m}M)] \\
&= 4K2^{-v} \sum_{m=1}^{\infty}\sum_{q=v}^{\infty} 2^q\varrho[\varphi(2^{-q}P)]\sigma[\psi(\frac{Q}{m})]\varphi_1[2^{-q}\psi_1(\frac{1}{m}M)] \\
&\leq 8K2^{-v} \sum_{m=1}^{\infty}\sum_{n=2^{v-1}+1}^{\infty} \varrho[\varphi(\frac{P}{n})]\sigma[\psi(\frac{Q}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m}M)].
\end{aligned}
$$

And also from the concavity of ϱ, σ, φ, ψ, φ_1 and ψ_1, it turns out that

$$
\begin{aligned}
&|S(\tilde{D},D') - S(D,D')| \\
&\leq \sum_{l=1}^{L}\left|\int_{x_{l-1}+\delta}^{x_l-\delta}\int_{y'}^{y''}(F_{\tilde{D},D'} - F_{D,D'})d_{x,y}G(x,y)\right| \\
&\leq 16K \sum_{m=1}^{\infty}\sum_{n=2^{v-1}+1}^{\infty} \varrho[\varphi(\frac{P}{n})]\sigma[\psi(\frac{Q}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m}M)]
\end{aligned}
$$

$$\leq 16K(P\vee 1)(Q\vee 1))(M\vee 1)\sum_{m=1}^{\infty}\sum_{n=2^{v-1}+1}^{\infty}\varrho[\varphi(\frac{1}{n})]\sigma[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})]$$

$$\leq \varepsilon_v, \tag{3.4.25}$$

where $\varepsilon_v \to 0$, as $v \to \infty$.

Thus we can get (3.4.21) from (3.4.22), (3.4.23) and (3.4.25), as $v, v' \to \infty$, which means $S(D, D')$ is a Cauchy sequence, so $\lim_{m(D\times D')\to 0} S(D, D')$ exists. In the following, we show the limit is unique. For this, let $D_1 \times D_1'$, $D_2 \times D_2'$ be two arbitrary partitions of $[x', x''] \times [y', y'']$ including $H \times H'$. From the above we know,

$$|S(D_1 \cup D_2, D_1' \cup D_2') - S(D_1, D_1')| \to 0, \ as\ m(D_1 \times D_1') \to 0,$$
$$|S(D_1 \cup D_2, D_1' \cup D_2') - S(D_2, D_2')| \to 0, \ as\ m(D_2 \times D_2') \to 0.$$

Therefore,

$$\lim_{m(D_1\times D_1')\to 0} S(D_1, D_1') = \lim_{m(D_2\times D_2')\to 0} S(D_2, D_2')$$
$$= \lim_{m(D_1\times D_1'),m(D_2\times D_2')\to 0} S(D_1 \cup D_2, D_1' \cup D_2'),$$

that is to say, $\lim_{m(D\times D')\to 0} S(D, D')$ is unique, and we define it as $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)$. So we proved our theorem. ◇

In the following when we say an integral is well defined if it is in the sense of Theorem 3.4.1. The following convergence theorem plays an important role in establishing Itô's formula:

Theorem 3.4.2 *Assume $F_k(x,y)$ and $F(x,y)$ are continuous functions and satisfy (3.4.3) and for F_k uniformly in k; $G(x,y)$ and $G_k(x,y)$ are of bounded Φ_1, Ψ_1-variation in (x,y) uniformly in k and satisfy the finite large jump condition, where Φ_1, Ψ_1 are convex functions. If there exist increasing concave functions ϱ_i and σ_i subject to $\varrho_i(u)\sigma_i(u) = u$, i=1,2, and a positive number $\delta > 0$ such that*

$$\sum_{m,n}\varrho_1[\varphi(\frac{1}{n})]\sigma_1[\psi(\frac{1}{m})]\varphi_1^{\frac{1}{1+\delta}}[\frac{1}{n}\psi_1(\frac{1}{m})]$$
$$+\sum_{m,n}\varrho_2[\varphi((\frac{1}{n})^{\frac{1}{1+\delta}})]\sigma_2[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty, \tag{3.4.26}$$

or

$$\sum_{m,n}\varrho_1[\varphi(\frac{1}{n})]\sigma_1[\psi(\frac{1}{m})]\varphi_1^{\frac{1}{1+\delta}}[\frac{1}{n}\psi_1(\frac{1}{m})]$$
$$+\sum_{m,n}\varrho_2[\varphi(\frac{1}{n})]\sigma_2[\psi((\frac{1}{m})^{\frac{1}{1+\delta}})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty, \tag{3.4.27}$$

and let $F_k(x,y) \to F(x,y)$, $G_k(x,y) \to G(x,y)$ *as* $k \to \infty$ *uniformly in* (x,y). *Then we have*

$$\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}G_k(x,y) \to \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y), \tag{3.4.28}$$

when $k \to \infty$.

Proof: First note that from Theorem 3.4.1 under the above assumptions, the integral $\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}G_k(x,y)$ and $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)$ are all well defined. It's easy to see that

$$\begin{aligned}
&\frac{1}{2}\left(\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}G_k(x,y) - \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)\right)\\
=\ &\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}\frac{1}{2}(G_k(x,y)-G(x,y))\\
&+\int_{x'}^{x''}\int_{y'}^{y''} \frac{1}{2}(F_k(x,y)-F(x,y))d_{x,y}G(x,y).
\end{aligned}$$

We study $\frac{1}{2}$ of the integral only for convenience in what follows. First consider the integral $\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}(G_k(x,y)-G(x,y))$. Note there exist constant $P_1, Q_1, M_1, M_2 > 0$, which are independent of k such that for any partition $E \times E'$defined before

$$\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\Phi_1(|\Delta_i\Delta_j G_k|)\right) \le M_1, \tag{3.4.29}$$

$$\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\Phi_1(|\Delta_i\Delta_j G|)\right) \le M_2. \tag{3.4.30}$$

For the small $\delta > 0$ given in condition (3.4.26), from the convexity of Φ_1 and Ψ_1 and $G_k \to G$ when $k \to \infty$, we have

$$\begin{aligned}
&\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\Phi_1\Big(|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|^{1+\delta}\Big)\right)\\
=\ &\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\Phi_1\Big(|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|^{\delta}\cdot|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|\Big)\right)\\
\le\ &\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|^{\delta}\Phi_1\Big(|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|\Big)\right)\\
\le\ &\sum_{j=1}^{N'}\Psi_1\left(\max_i|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|^{\delta}\sum_{i=1}^{N}\Phi_1\Big(|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|\Big)\right)\\
\le\ &\sum_{j=1}^{N'}\max_i|\Delta_i\Delta_j\frac{1}{2}(G_k-G)|^{\delta}\Psi_1\left(\sum_{i=1}^{N}\Phi_1\Big(\frac{1}{2}|\Delta_i\Delta_j G_k|+\frac{1}{2}|\Delta_i\Delta_j G|\Big)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{i,j}|\Delta_i\Delta_j(G_k-G)|^{\delta}\sum_{j=1}^{N'}\Psi_1\left(\sum_{i=1}^{N}\left(\frac{1}{2}\Phi_1\Big(|\Delta_i\Delta_j G_k|\Big)+\frac{1}{2}\Phi_1\Big(|\Delta_i\Delta_j G|\Big)\right)\right)\\
&\leq \varepsilon_1(k)M,
\end{aligned}$$

where $\varepsilon_1(k)\to 0$ as $k\to\infty$, and M is a constant independent of k. If we define

$$S(E,E')=\sum_{i=1}^{N}\sum_{j=1}^{N'}F_k(x_{i-1},y_{j-1})\Big(\Delta_i\Delta_j(G_k-G)\Big),$$

and similar to (3.4.19), let $P_1:=\omega(x'')$, $Q_1:=\chi(y'')$, by dominated convergence theorem to the infinite series,

$$|S(E,E')|\leq 4K\sum_{m,n}\varrho_1[\varphi(\frac{P_1}{n})]\sigma_1[\psi(\frac{Q_1}{m})]\varphi_1^{\frac{1}{1+\delta}}[\frac{1}{n}\psi_1(\frac{2\varepsilon_1(k)}{m}M)]\to 0,\ as\ k\to\infty,$$

as the series $\sum_{m,n}\varrho_1[\varphi(\frac{1}{n})]\sigma_1[\psi(\frac{1}{m})]\varphi_1^{\frac{1}{1+\delta}}[\frac{1}{n}\psi_1(\frac{1}{m})]<\infty$. This implies as $k\to\infty$,

$$\lim_{k\to\infty}\int_{x'}^{x''}\int_{y'}^{y''}F_k(x,y)d_{x,y}(G_k(x,y)-G(x,y))=0. \tag{3.4.31}$$

For the second integral $\int_{x'}^{x''}\int_{y'}^{y''}(F_k(x,y)-F(x,y))d_{x,y}G(x,y)$, we can use a similar method to prove

$$\lim_{k\to\infty}\int_{x'}^{x''}\int_{y'}^{y''}(F_k(x,y)-F(x,y))d_{x,y}G(x,y)=0. \tag{3.4.32}$$

For this, we note from the assumption there is a $\delta>0$ such that,

$$\begin{aligned}
&\Phi^{1+\delta}\Big(|\frac{1}{2}(F_k-F)(x_i,y)-\frac{1}{2}(F_k-F)(x_{i-1},y)|\Big)\\
\leq\ &\max_i\Phi^{\delta}\Big(|\frac{1}{2}(F_k-F)(x_i,y)-\frac{1}{2}(F_k-F)(x_{i-1},y)|\Big)\cdot\\
&\Phi\Big(\frac{1}{2}|F_k(x_i,y)-F_k(x_{i-1},y)|+\frac{1}{2}|F(x_i,y)-F(x_{i-1},y)|\Big)\\
\leq\ &\varepsilon_2(k)(\omega(x_i)-\omega(x_{i-1}))\\
\leq\ &\varepsilon_2(k)P_1,
\end{aligned}$$

where $\varepsilon_2(k)\to 0$, as $k\to\infty$, and P_1 is a constant independent of k. So under the assumption $\sum_{m,n}\varrho_2[\varphi((\frac{1}{n})^{\frac{1}{1+\delta}})]\sigma_2[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})]<\infty$, we can prove (3.4.32) using the same argument in proving (3.4.31). Therefore under assumption (3.4.26), we prove the desired result. The proof is similar under the assumption (3.4.27). ◇

Remark 3.4.1 *From the proof we can easily see that under the condition that there exist two functions ϱ and σ subject to $\varrho(u)\sigma(u)=u$ and a small number $\delta>0$ such that*

$$\sum_{m,n}\varrho[\varphi(\frac{1}{n})]\sigma[\psi(\frac{1}{m})]\varphi_1^{\frac{1}{1+\delta}}[\frac{1}{n}\psi_1(\frac{1}{m})]<\infty. \tag{3.4.33}$$

Then as $k \to \infty$,

$$\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G_k(x,y) \to \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y). \tag{3.4.34}$$

Similarly, under the condition that there exist two functions ϱ *and* σ *subject to* $\varrho(u)\sigma(u) = u$ *and a small number* $\delta > 0$ *such that*

$$\sum_{m,n} \varrho[\varphi((\frac{1}{n})^{\frac{1}{1+\delta}})]\sigma[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty, \tag{3.4.35}$$

or

$$\sum_{m,n} \varrho[\varphi(\frac{1}{n})]\sigma[\psi((\frac{1}{m})^{\frac{1}{1+\delta}})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty. \tag{3.4.36}$$

Then as $k \to \infty$,

$$\int_{x'}^{x''}\int_{y'}^{y''} F_k(x,y)d_{x,y}G(x,y) \to \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y). \tag{3.4.37}$$

It is easy to see that in the definition of $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)$, one can take $F(x_i,y_j)$ instead of $F(x_{i-1},y_{j-1})$ in (3.4.5). One can also prove the convergence of (3.4.5) in this case and denote the integral by $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d^*_{x,y}G(x,y)$, the backward integral. In general, this should be different from $\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y)$. But under slightly stronger conditions than those in Theorem 3.4.1, as in the one-parameter case, these two integrals equal. This result is proved in the following proposition.

Proposition 3.4.1 *Assume* $F_k(x,y)$ *and* $F(x,y)$ *are continuous functions and satisfy (3.4.3) and for* F_k *uniformly in* k; $G(x,y)$ *and* $G_k(x,y)$ *are of bounded* Φ_1, Ψ_1-*variation in* (x,y) *uniformly in* k *and satisfy the finite large jump condition, where* Φ_1, Ψ_1 *are convex functions as above. If there exist increasing concave functions* ϱ *and* σ *subject to* $\varrho(u)\sigma(u) = u$ *and a positive* $\delta > 0$ *such that one of the following two conditions is satisfied*

(i) $F(x,y)$ *is continuous in* x *and*

$$\sum_{m,n} \varrho[\varphi((\frac{1}{n})^{\frac{1}{1+\delta}})]\sigma[\psi(\frac{1}{m})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty,$$

(ii) $F(x,y)$ *is continuous in* y *and*

$$\sum_{m,n} \varrho[\varphi(\frac{1}{n})]\sigma[\psi((\frac{1}{m})^{\frac{1}{1+\delta}})]\varphi_1[\frac{1}{n}\psi_1(\frac{1}{m})] < \infty.$$

Then

$$\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y) = \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d^*_{x,y}G(x,y).$$

Proof: We only prove the result when condition (i) is satisfied. Write

$$S(E,E') = \sum_{i=1}^{N}\sum_{j=1}^{N'} F(x_{i-1},y_{j-1})\Delta_i\Delta_j G,$$

$$S^*(E,E') = \sum_{i=1}^{N}\sum_{j=1}^{N'} F(x_i,y_j)\Delta_i\Delta_j G.$$

Here E and E' are the same as before. Denote

$$\tilde{F}_{\delta^1,\delta^2}(x,y) = F(x+\delta^1,y+\delta^2) - F(x,y).$$

Set $\delta_{x_{i-1}} = x_i - x_{i-1}$, $\delta_{y_{j-1}} = y_j - y_{j-1}$. Then

$$S^*(E,E') - S(E,E') = 2\sum_{i=1}^{N}\sum_{j=1}^{N'}\frac{1}{2}\tilde{F}_{\delta_{x_{i-1}},\delta_{y_{j-1}}}(x_{i-1},y_{j-1})\Delta_i\Delta_j G.$$

Note from the assumptions, there is a $\delta > 0$ such that

$$\begin{aligned}
& \Phi^{1+\delta}\Big(\Big|\frac{1}{2}[\tilde{F}_{\delta_{x_i},\delta_{y_{j-1}}}(x_i,y_{j-1}) - \tilde{F}_{\delta_{x_{i-1}},\delta_{y_{j-1}}}(x_{i-1},y_{j-1})]\Big|\Big) \\
\le\ & \max_{i,j}\Phi^{\delta}\Big(\frac{1}{2}\Big|[F(x_{i+1},y_j) - F(x_i,y_{j-1})] - [F(x_i,y_j) - F(x_{i-1},y_{j-1})]\Big|\Big) \\
& \Phi\Big(\Big|\frac{1}{2}[F(x_{i+1},y_j) - F(x_i,y_j)] - \frac{1}{2}[F(x_i,y_{j-1}) - F(x_{i-1},y_{j-1})]\Big|\Big) \\
\le\ & \max_{i,j}\Phi^{\delta}\Big(\frac{1}{2}\Big|[F(x_{i+1},y_j) - F(x_i,y_j)] - [F(x_i,y_{j-1}) - F(x_{i-1},y_{j-1})]\Big|\Big) \\
& \Phi\Big(\frac{1}{2}|F(x_{i+1},y_j) - F(x_i,y_j)| + \frac{1}{2}|F(x_i,y_{j-1}) - F(x_{i-1},y_{j-1})|\Big) \\
\le\ & \varepsilon(E,E')(\omega(x_{i+1}) - \omega(x_{i-1})) \\
\le\ & \varepsilon(E,E')P,
\end{aligned}$$

where $\varepsilon(E,E') \to 0$, when $m(E,E') \to 0$ and $P := \omega(x'')$ is a constant. Therefore following (3.4.19), we see that

$$\begin{aligned}
& |S^*(E,E') - S(E,E')| \\
=\ & |\sum_{i=1}^{N}\sum_{j=1}^{N'}\tilde{F}_{\delta_{x_{i-1}},\delta_{y_{j-1}}}(x_{i-1},y_{j-1})\Delta_i\Delta_j G| \\
\le\ & 8K\sum_{m,n=1}^{\infty}\varrho[\varphi\Big((\frac{2\varepsilon(E,E')P}{n})^{\frac{1}{1+\delta}}\Big)]\sigma[\psi(\frac{Q}{m})]\varphi_1(\frac{1}{n}\psi_1(\frac{1}{m}M)) \\
\to\ & 0,\ as\ \varepsilon(E,E') \to 0,
\end{aligned}$$

where $Q := \chi(y'')$. Therefore

$$S^*(E,E') - S(E,E') \to 0\ as\ \varepsilon(E,E') \to 0.$$

That is to say,

$$\int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d_{x,y}G(x,y) = \int_{x'}^{x''}\int_{y'}^{y''} F(x,y)d^*_{x,y}G(x,y).$$

◇

From Theorem 3.4.1 we can easily generalize it to the multi-parameter integral.

Definition 3.4.1 *Let* $E_1 \times \cdots \times E_n = \{a_1 = x_1^0 < x_1^1 < \cdots < x_1^{N_1} = b_1, \cdots, a_n = x_n^0 < x_n^1 < \cdots < x_n^{N_n} = b_n\}$ *be an arbitrary partition of* $[a_1, b_1] \times \cdots [a_n, b_n]$. *We call* $G(x_1, \cdots, x_n)$ *is of bounded* $\Psi_1, \cdots, \Psi_n$*-variation in* $(x_1, \cdots, x_n)$, *if*

$$\sup_{E_1\times\cdots\times E_n} \sum_{k_n=1}^{N_n} \Psi_n\Big(\cdots\big(\sum_{k_1=1}^{N_1} \Psi_1(|\Delta_{x_n^{k_n-1},x_n^{k_n}} \cdots \Delta_{x_1^{k_1-1},x_1^{k_1}} G|)\big)\cdots\Big) < \infty. \tag{3.4.38}$$

We say a function $f(x_1, \cdots, x_n)$ has a jump at $(x_1^0, \cdots, x_n^0)$ if there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there exists $(x_1^1, \cdots, x_n^1)$ satisfying $\max\{|x_1^0 - x_1^1|, \cdots, |x_n^0 - x_n^1|\} < \delta$ and $|\Delta_{x_n^0,x_n^1}, \cdots, \Delta_{x_1^0,x_1^1} f| > \varepsilon$. For a function $G(x_1, \cdots, x_n)$ of bounded $\Psi_1, \cdots, \Psi_n$-variation, for any given $\varepsilon > 0$, it is easy to see that there exists a $\delta(\varepsilon) > 0$ and a finite number of jump points $\{(x_1^1, \cdots, x_n^1), \cdots, (x_1^{m_1}, \cdots, x_n^{m_n})\}$ such that $|\Delta_{\tilde{x}_n,x_n}, \cdots, \Delta_{\tilde{x}_1,x_1} G| < \varepsilon$ whenever $\max\{|\tilde{x}_1 - x_1|, \cdots, |\tilde{x}_n - x_n|\} < \delta(\varepsilon)$, $[\tilde{x}_i, x_i] \cap \{x_1^i, \cdots, x_n^{m_i}\} = \emptyset$ for all $i = 1, 2, \cdots, n$. Denote $H_{10} \times \cdots \times H_{n0} := \{x_1^1, \cdots, x_1^{m_1}\} \times \cdots \times \{x_n^1, \cdots, x_n^{m_n}\}$. In the following, we assume the following finite large jump condition: for any $\varepsilon > 0$, there exists at most finite many points $\{x_1^1, \cdots, x_1^{m_1'}\}, \cdots, \{x_n^1, \cdots, x_n^{m_n'}\}$ and a constant $\delta(\varepsilon) > 0$ such that for each $i = 1, 2, \cdots, n$, the total $\Psi_1, \cdots, \Psi_n$-variation of G on $[x_1', x_1''] \times \cdots [x_i, x_i + \delta] \times \cdots \times [x_n', x_n'']$ is smaller than ε if $[x_i, x_i + \delta] \cap \{x_i^1, \cdots, x_i^{m_i'}\} = \emptyset$. Denote $H_1 \times \cdots \times H_n := \{x_1^1, \cdots, x_1^{m_1'}\} \times \cdots \times \{x_n^1, \cdots, x_n^{m_n'}\}$. It is obvious that $H_1 \times \cdots \times H_n \supset H_{10} \times \cdots \times H_{n0}$.

Similarly we can define $m(E_1 \times E_2 \times \cdots \times E_n)$ as in Theorem 3.4.1 and get the theorem for multi-parameter integral.

Theorem 3.4.3 *Let* $F(x_1, \cdots, x_n)$ *be a continuous function and there exists continuous functions* $\omega_i(x)$ *such that,*

$$\Phi_i(|\Delta_{x_i^{k_i-1},x_i^{k_i}} F|) \le \omega_i(x_i^{k_i}) - \omega(x_i^{k_i-1}), \tag{3.4.39}$$

Here Δ *is the difference operator (see [1]) as follows,*

$$\begin{aligned}\Delta_{x_i^{k_i-1},x_i^{k_i}} F \;=\; & F(x_1, \cdots, x_{i-1}, x_i^{k_i}, x_{i+1}, \cdots, x_n) \\ & - F(x_1, \cdots, x_{i-1}, x_i^{k_i-1}, x_{i+1}, \cdots, x_n).\end{aligned}$$

Let $G(x_1,\cdots,x_n)$ be of bounded $\Psi_1,\cdots,\Psi_n$-variation in $(x_1,\cdots,x_n)$ and satisfy the finite large jump condition, where $\Psi_1,\cdots,\Psi_n$ are convex functions. If there exist monotone increasing concave functions $\varrho_1,\cdots,\varrho_n$ subject to $\varrho_1(u)\cdots\varrho_n(u)=u$ such that

$$\sum_{k_n=1}^{\infty}\cdots\sum_{k_1=1}^{\infty}\varrho_1[\varphi_1(\frac{1}{k_1})]\cdots\varrho_n[\varphi_n(\frac{1}{k_n})]\psi_1[\frac{1}{k_1}[\cdots\psi_n(\frac{1}{k_n})\cdots]]<\infty, \tag{3.4.40}$$

then the integral

$$\begin{aligned}&\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}F(x_1,\cdots,x_n)d_{x_1,\cdots,x_n}G(x_1,\cdots,x_n)\\&=\lim_{m(E_1\times\cdots\times E_n)\to 0}\sum_{k_n=1}^{N_n}\cdots\sum_{k_1=1}^{N_1}F(x_1^{k_1-1},\cdots,x_n^{k_n-1})(\Delta_{x_n^{k_n-1},x_n^{k_n}}\cdots\Delta_{x_1^{k_1-1},x_1^{k_1}}G)\end{aligned}$$

is well defined, as long as $E_1\times E_2\times\cdots\times E_n$ include $H_1\times H_2\times\cdots\times H_n$.

Chapter 4
Two-parameter Integrals of Local Times

§4.1 Introduction

In Chapter 1 and 2, I gave the extension of Tanaka-Meyer's formula, which is a time-independent case. So how is about the time dependent case? Elworthy, Truman and Zhao [7] proved if $f(t,x) = f_h(t,x) + f_v(t,x)$, where $\Delta^- f_h(t,x)$ and $\nabla^- f(t,x)$ exist and are left continuous, and $\nabla^- f_v(t,x)$ is of locally bounded variation in x for a fixed t and of locally bounded variation in (t,x), then

$$\begin{aligned} & f(t,X(t)) - f(0,X(0)) \\ = & \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))\mathrm{d}s + \int_0^t \nabla^- f(s,X(s))dX_s \\ & + \frac{1}{2}\int_0^t \Delta^- f_h(s,X(s))d<X>_s + \int_{-\infty}^{\infty} L_t(x)\mathrm{d}_x \nabla^- f_v(t,x) \\ & - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f_v(s,x) \quad a.s. \end{aligned} \tag{4.1.1}$$

where $\int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f_v(s,x)$ is a space-time Lebesgue-Stieltjes integral and needless to say, defined pathwise. Elworthy-Truman-Zhao's formula was given in a very general form. It includes as special cases classical Itô's formula, Tanaka's formula, Meyer's formula, Azéma-Jeulin-Knight-Yor's formula [2]. A special and earlier version of Elworthy-Truman-Zhao's formula was obtained by Peskir [38] independently.

On the other hand, there are some works which define $\int_{-\infty}^{\infty}\int_0^t \nabla^- f(s,x)d_{s,x}L_s(x)$ for a time dependent function $f(s,x)$ using forward and backward integrals for Brownian motion in [9] and for semi-martingales other than Brownian motion in [10]. This integral was also defined in [42] as a stochastic integral with excursion fields, and in [?] through Itô's formula without assuming the reversibility of the semi-martingale which was required in [9]. Generally speaking, one expects stronger conditions for the pathwise existence of the integrals of local times. However, in the framework of Lebesgue integrals, locally bounded variation in x for fixed t and locally bounded variation in (t,x) are minimal conditions on $\nabla^- f(t,x)$ to generate a measure, so it seems impossible to go beyond Elworthy-Truman-Zhao's formula. Chapter 3 gives a new condition on integral of two-parameters, in this chapter, I will define the integral $\int_{-\infty}^{\infty}\int_0^t \nabla^- f(s,x)d_{s,x}L_s(x)$ and then give an extension of Elworthy-Truman-Zhao's formula. We also give an example to use this formula.

§4.2 Two-parameter Integral of Local Times

Assume that $B = (B_t)_{t\geq 0}$ is a one-dimensional standard Brownian Motion, L_t^x is the local time of B_t at x. From [25] (Section 3.6, Page 208), the local time of Brownian Motion L_t^x has the property of locally Hölder continuous: for any exponent $\gamma \in (0, \frac{1}{2})$, $T > 0$, $K > 0$, there exists a P^0-a.s. positive random variable $h(\omega)$ and a constant $\delta > 0$ such that

$$P^0\left[\omega \in \Omega; \sup_{\substack{0<|t-s|+|a-b|<h(\omega) \\ 0\leq s,t\leq T, -K\leq a,b\leq K}} \frac{|L_t(a,\omega) - L_s(b,\omega)|}{|t-s|^\gamma + |a-b|^\gamma} \leq \delta\right] = 1. \tag{4.2.1}$$

Therefore immediately we can apply Theorem 3.4.1 to Brownian local time.

In the following we can prove the continuous part of continuous semimartingale local time satisfies condition (3.4.3) in Theorem 3.4.1.

Lemma 4.2.1 *Let $X_t = M_t + V_t$ be a continuous semimartingale, where M_t is a local martingale and V_t is a process of locally bounded variation; L_t^x be the local time of X_t at x and $\tilde{L}_t(x)$ be the continuous part (i.e. in (1.2.18)) of L_t^x. Then for any exponent $\gamma \in (0, \frac{1}{2})$, $T > 0$, $N > 0$ and almost all $\omega \in \Omega$, there exist positive random variable $h(\omega)$ and a constant $\delta > 0$ such that*

$$|\tilde{L}_t^x(\omega) - \tilde{L}_s^y(\omega)| \leq \delta\left(\left|(<M>_t + Var_{[0,t]}(V)) - (<M>_s + Var_{[0,s]}(V))\right|^\gamma + |x-y|^\gamma\right), \tag{4.2.2}$$

for $0 < |x-y| + |\ <M>_t - <M>_s\ | < h(\omega)$, $0 \leq s,t \leq T$, $-N \leq x,y \leq N$. Here $Var_{[0,t]}(V)$ means the total variation of $V_\cdot$ in $[0,t]$.

Proof: First we recall (1.2.18),

$$\begin{aligned} \tilde{L}_t^x &= (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s > x\}} dM_s - (V_t - V_0) \\ &:= \varphi(t,x) + I_t(x) - (V_t - V_0). \end{aligned}$$

From the proof of Lemma 3.7.5 in [25] (Page 221), we may choose a Brownian motion B for which we have the equations

$$\begin{aligned} I_t(x) &:= \int_0^t 1_{\{X_v > x\}} dM_v = \int_0^{<M>_t} 1_{\{Y_u > x\}} dB_u \\ H_s(x) &:= \int_0^s 1_{\{Y_u > x\}} dB_u = \int_0^{T(s)} 1_{\{X_v > x\}} dM_v, \end{aligned}$$

where $T(s) := \inf\{t \geq 0; <M>_t > s\}$ is given in Theorem 3.4.6 (time-change for martingales) in [25], $Y_s := X_{T(s)}$, for $0 \leq s < <M>_\infty$. We know that for an arbitrary constant k, there exists contant C such that,

$$E|H_{s_2}(x) - H_{s_1}(y)|^{2k} \leq C[|s_2 - s_1|^k + |x-y|^k].$$

So $H_s(x)$ is jointly Hölder continuous in (s,x) with exponent γ for any $\gamma \in (0,\frac{1}{2})$, i.e. for almost all $\omega \in \Omega$, there exist positive random variable $h(\omega)$ and a constant $\delta > 0$ such that

$$|H_{s_2}(x) - H_{s_1}(y)| \leq \delta(|s_2 - s_1|^\gamma + |x-y|^\gamma),$$

for $0 < |s_2 - s_1| + |x-y| < h(\omega)$. It's easy to see $I_t(x) = H_{<M>_t}(x)$, which leads to that for almost all $\omega \in \Omega$,

$$|I_{s_2}(x) - I_{s_1}(y)| \leq \delta(|<M>_{s_2} - <M>_{s_1}|^\gamma + |x-y|^\gamma),$$

for $0 < |<M>_{s_2} - <M>_{s_1}| + |x-y| < h(\omega)$. Moreover,

$$\begin{aligned} |\varphi(s_2,x) - \varphi(s_1,y)| &\leq 2(|(M_{s_2}+V_{s_2}) - (M_{s_1}+V_{s_1})| + |x-y|) \\ &\leq 2(|M_{s_2} - M_{s_1}| + |V_{s_2} - V_{s_1}| + |x-y|) \\ &\leq \delta(|<M>_{s_2} - <M>_{s_1}|^\gamma + |Var_{[0,s_2]}V - Var_{[0,s_1]}V|^\gamma + |x-y|^\gamma), \end{aligned}$$

and

$$|V_{s_2} - V_{s_1}| \leq |Var_{[0,s_2]}V - Var_{[0,s_1]}V| \leq \delta |Var_{[0,s_2]}V - Var_{[0,s_1]}V|^\gamma.$$

Here $Var_{[0,s]}V$ means the total variation of V on $[0,s]$. Therefore we proved the desired result. ◇

In this section we will define $\int_{-\infty}^{\infty}\int_0^t g(s,x)d_{s,x}L_s^x$. First we can use Theorem 3.4.1 to define the integral $\int_{-\infty}^{\infty}\int_0^t \tilde{L}_s^x d_{s,x}g(s,x)$ directly, for in fact in condition (3.4.3), $\omega(s) = \delta^{\frac{1}{\gamma}}(<M>_s + Var_{[0,s]}V)$ and $\chi(x) = \delta^{\frac{1}{\gamma}}x$ are both increasing functions.

Theorem 4.2.1 *Assume $g : [0,t]\times R \to R$ is of bounded Φ_1, Ψ_1-variation in (s,x), i.e. $\sup_{E\times E'}\sum_{i=0}^{l-1}\Psi_1(\sum_{j=0}^{m-1}\Phi_1(|\Delta_j\Delta_i g|)) < \infty$ for the partition we defined as before and satisfies the finite large jump condition. Then if there exist increasing concave functions ϱ and σ subject to $\varrho(u)\sigma(u) = u$ such that for $\gamma \in (0,\frac{1}{2})$*

$$\sum_{n,m}\varrho[(\frac{1}{n})^\gamma]\sigma[(\frac{1}{m})^\gamma]\varphi_1(\frac{1}{n}\psi_1(\frac{1}{m})) < \infty, \tag{4.2.3}$$

the integral

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^t \tilde{L}_s^x d_{s,x}g(s,x) \\ =\ &\lim_{m(E\times E')\to 0}\sum_{i=0}^{l-1}\sum_{j=0}^{m-1}\tilde{L}(s_j,x_i)\Big(g(s_{j+1},x_{i+1}) - g(s_{j+1},x_i) \\ &\qquad -g(s_j,x_{i+1}) + g(s_j,x_i)\Big) \end{aligned} \tag{4.2.4}$$

is well defined for almost all $\omega \in \Omega$ in the sense of Theorem 3.4.1.

Proposition 4.2.1 *Assume $g : [0,t] \times R \to R$ is of bounded p,q-variation, i.e.* $\sup_{E\times E'} \sum_{i=0}^{l-1} \left(\sum_{j=0}^{m-1} |\Delta_j \Delta_i g|^p \right)^q < \infty$, *where $p, q \geq 1$, $2q+2 > 3pq$ and satisfies the finite large jump condition, then the integral*

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_0^t \tilde{L}_s^x d_{s,x} g(s,x) \\ = & \lim_{m(E\times E')\to 0} \sum_{i,j} \tilde{L}(s_j, x_i) \Big(g(s_{j+1}, x_{i+1}) - g(s_{j+1}, x_i) \\ & - g(s_j, x_{i+1}) + g(s_j, x_i) \Big) \end{aligned} \tag{4.2.5}$$

is well defined in the sense of Theorem 3.4.1.

Proof: For any $p, q \geq 1$ satisfying $2q+2 > 3pq$, we have $2(1-\frac{1}{p}) < \frac{2}{pq} - 1$. Therefore there exists a number α such that $2(1-\frac{1}{p}) < \alpha < \frac{2}{pq} - 1$. This implies that $\frac{\alpha}{2} + \frac{1}{p} > 1$ and $\frac{1-\alpha}{2} + \frac{1}{pq} > 1$. So there exists $0 < \gamma < \frac{1}{2}$ such that $\alpha\gamma + \frac{1}{p} > 1$ and $(1-\alpha)\gamma + \frac{1}{pq} > 1$. Take $\varrho(u) = u^\alpha$ and $\sigma(u) = u^{1-\alpha}$, then it is easy to see that

$$\sum_{n,m} \varrho[(\frac{1}{n})^\gamma] \sigma[(\frac{1}{m})^\gamma] (\frac{1}{n})^{\frac{1}{p}} (\frac{1}{m})^{\frac{1}{pq}} < \infty. \tag{4.2.6}$$

Therefore the integral (4.2.5) is well defined. ◇

After defining the integral $\int_{-\infty}^{\infty} \int_0^t \tilde{L}_s(x) d_{s,x} g(s,x)$, let's study the integral $\int_{-\infty}^{\infty} \int_0^t g(s,x) d_{s,x} \tilde{L}_s^x$. Note

$$\begin{aligned} & \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} g(s_j, x_i) \left[\tilde{L}_{s_{j+1}}(x_{i+1}) - \tilde{L}_{s_j}(x_{i+1}) - \tilde{L}_{s_{j+1}}(x_i) + \tilde{L}_{s_j}(x_i) \right] \\ = & \sum_{i=1}^{l} \sum_{j=1}^{m} g(s_{j-1}, x_{i-1}) \tilde{L}_{s_j}(x_i) - \sum_{i=1}^{l} \sum_{j=0}^{m-1} g(s_j, x_{i-1}) \tilde{L}_{s_j}(x_i) \\ & - \sum_{i=0}^{l-1} \sum_{j=1}^{m} g(s_{j-1}, x_i) \tilde{L}_{s_j}(x_i) + \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} g(s_j, x_i) \tilde{L}_{s_j}(x_i) \\ = & \sum_{i=1}^{l} \sum_{j=1}^{m} \tilde{L}_{s_j}(x_i) \left[g(s_j, x_i) - g(s_j, x_{i-1}) - g(s_{j-1}, x_i) + g(s_{j-1}, x_{i-1}) \right] \\ & - \sum_{i=1}^{l} \left[g(0, x_{i-1}) \tilde{L}_0(x_i) - g(t, x_{i-1}) \tilde{L}_t(x_i) \right] \\ & - \sum_{j=1}^{m} \left[g(s_{j-1}, -N) \tilde{L}_{s_j}(-N) - g(s_{j-1}, N) \tilde{L}_{s_j}(N) \right] \\ & + \sum_{j=0}^{m-1} \left[g(s_j, -N) \tilde{L}_{s_j}(-N) - g(s_j, N) \tilde{L}_{s_j}(N) \right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{l-1}\Big[g(0,x_i)\tilde{L}_0(x_i)-g(t,x_i)\tilde{L}_t(x_i)\Big]\\
=\;&\sum_{i=1}^{l}\sum_{j=1}^{m}\tilde{L}_{s_j}(x_i)\left[g(s_j,x_i)-g(s_j,x_{i-1})-g(s_{j-1},x_i)+g(s_{j-1},x_{i-1})\right]\\
&-\sum_{i=1}^{l}\tilde{L}_t(x_i)(g(t,x_i)-g(t,x_{i-1})).
\end{aligned}
\tag{4.2.7}
$$

Under the conditions of Theorem 4.2.1 and Proposition 3.4.1 and noticing that $\tilde{L}_t(x)$ is continuous in t, we know that the first term of (4.2.7) converges to $\int_{-\infty}^{\infty}\int_0^t \tilde{L}_s(x)d_{s,x}g(s,x)$, and from rough path integration of one parameter, we know that the second term converges to $\int_{-\infty}^{\infty}\tilde{L}_t(x)d_xg(t,x)$ if further $g(s,x)$ is of bounded $\theta-$variation $(1\le\theta<3)$ in x. So the sum

$$
\sum_{i=0}^{l-1}\sum_{j=0}^{m-1}g(s_j,x_i)\Big[\tilde{L}_{s_{j+1}}(x_{i+1})-\tilde{L}_{s_j}(x_{i+1})-\tilde{L}_{s_{j+1}}(x_i)+\tilde{L}_{s_j}(x_i)\Big]
$$

converges. We denote its limit by

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t g(s,x)d_{s,x}\tilde{L}_s^x\\
=\;&\lim_{m(E\times E')\to 0}\sum_{i=0}^{l-1}\sum_{j=0}^{m-1}g(s_j,x_i)\Big[\tilde{L}_{s_{j+1}}(x_{i+1})-\tilde{L}_{s_j}(x_{i+1})\\
&\qquad -\tilde{L}_{s_{j+1}}(x_i)+\tilde{L}_{s_j}(x_i)\Big],
\end{aligned}
\tag{4.2.8}
$$

and

$$
\int_{-\infty}^{\infty}\int_0^t g(s,x)d_{s,x}\tilde{L}_s^x=\int_{-\infty}^{\infty}\int_0^t\tilde{L}_s^xd_{s,x}g(s,x)-\int_{-\infty}^{\infty}\tilde{L}_t(x)d_xg(t,x).
\tag{4.2.9}
$$

Now recall the decomposition (1.2.17) and (1.2.19) and Lemma 1.2.2. As in Elworthy, Truman and Zhao [7], the integral $\int_0^t\int_{-\infty}^{\infty}g(s,x)d_{s,x}h(s,x)$ is defined as a two-parameter Lebesgue-Stieltjes integral. Therefore we can define

$$
\int_0^t\int_{-\infty}^{\infty}g(s,x)d_{s,x}L(s,x)=\int_0^t\int_{-\infty}^{\infty}g(s,x)d_{s,x}\tilde{L}(s,x)+\int_0^t\int_{-\infty}^{\infty}g(s,x)d_{s,x}h(s,x).
$$

Remark 4.2.1 *It is worth mentioning here that Young's theorem does not apply to define the integral* $\int_{-\infty}^{\infty}\int_0^t g(s,x)d_{s,x}\tilde{L}_s^x$. *To see this, first from the Hölder continuity of* $\tilde{L}$, *for* $\gamma\in(0,\frac{1}{2})$, *there exists a constant* C *such that*

$$
|\Delta_i\Delta_j\tilde{L}|\le C(t_{j+1}-t_j)^{\gamma\beta}(x_{i+1}-x_i)^{\gamma(1-\beta)},a.s.,
$$

for any $\beta \in (0,1)$. So in Young's theorem, $\lambda(x) = x^{\gamma\beta}, \mu(x) = x^{\gamma(1-\beta)}$. And if we define $\Phi(x) = x^p$, $\Psi(x) = x^q$, $\varrho(x) = x^\alpha$, $\sigma(x) = x^{1-\alpha}$, $\alpha \in (0,1)$, then according to Young's condition, we should require the series

$$\sum_n (\frac{1}{n})^{\frac{\alpha}{p}} (\frac{1}{n})^{\gamma\beta} < \infty \quad and \quad \sum_m (\frac{1}{m})^{\frac{1-\alpha}{q}} (\frac{1}{m})^{\gamma(1-\beta)} < \infty,$$

i.e.

$$\frac{\alpha}{p} + \gamma\beta > 1 \quad and \quad \frac{1-\alpha}{q} + \gamma(1-\beta) > 1.$$

We deduce from above that

$$\frac{\alpha}{p} + \frac{1-\alpha}{q} > \frac{3}{2}.$$

This is impossible since $p \geq 1$ and $q \geq 1$.

Remark 4.2.2 *If $g(s,x)$ is C^1 in x, we have*

$$\int_{-\infty}^{\infty} \int_0^t g(s,x) d_{s,x} L_s^x = -\int_{-\infty}^{\infty} \int_0^t \nabla g(s,x) d_s L_s(x) dx.$$

This can be seen from the following. As one can always add some points in the partition to make $L_{s_j}^{x_1} = 0$ and $\tilde{L}_{s_j}^{x_{l+1}} = 0$ for all $j = 1, 2, \cdots, m$, as L has a compact support in x, therefore

$$\begin{aligned}
& \lim_{m(E\times E')\to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} g(s_j, x_i) \Big[L_{s_j}^{s_{j+1}}(x_{i+1}) - L_{s_j}^{s_{j+1}}(x_i) \Big] \\
= & \lim_{m(E\times E')\to 0} \left(\sum_{i=1}^{l} \sum_{j=1}^{m} g(s_j, x_i) L_{s_j}^{s_{j+1}}(x_{i+1}) - \sum_{i=0}^{l-1} \sum_{j=1}^{m} g(s_j, x_{i+1}) L_{s_j}^{s_{j+1}}(x_{i+1}) \right) \\
= & -\lim_{m(E\times E')\to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \Big[g(s_j, x_{i+1}) - g(s_j, x_i) \Big] L_{s_j}^{s_{j+1}}(x_{i+1}) \\
= & -\lim_{m(E\times E')\to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \Big[\int_0^1 \nabla g(s_j, x_i + h(x_{i+1} - x_i)) dh \Big] L_{s_j}^{s_{j+1}}(x_{i+1})(x_{i+1} - x_i) \\
= & -\lim_{m(E'_{[-N,N]})\to 0} \sum_i \int_0^t \nabla g(s, x_{i+1}) d_s L_s(x_{i+1})(x_{i+1} - x_i) \\
& -\lim_{m(E'_{[-N,N]})\to 0} \sum_i \int_0^t \Big[\int_0^1 (\nabla g(s, x_i + h(x_{i+1} - x_i)) - \nabla g(s, x_{i+1})) dh \Big] d_s L_s(x_{i+1})(x_{i+1} - x_i) \\
= & -\int_{-\infty}^{\infty} \int_0^t \nabla g(s,x) d_s L_s(x) dx.
\end{aligned}$$

Theorem 4.2.2 *Let $f : [0,t] \times R \to R$ be of bounded θ-variation in x and of bounded p,q-variation in (s,x) and satisfy the finite large jump condition, where $1 \le \theta < 3$ and $p,q \ge 1$, $2q+2 > 3pq$, and*

$$f_n(s,x) := \int_0^2 \int_0^2 \rho(r)\rho(z) f(s-\frac{r}{n}, x-\frac{z}{n}) dr dz, \quad n \ge 1, \tag{4.2.10}$$

where ρ is the mollifier defined in (1.3.2). Then

$$\int_{-\infty}^{\infty}\int_0^t f_n(s,x) d_{s,x}L_s^x \to \int_{-\infty}^{\infty}\int_0^t f(s,x) d_{s,x}L_s^x, \text{ as } n \to \infty.$$

Proof: First we can easily verify that f_n are also of bounded p,q-variation. We extend f to $s < 0$ by defining $f(s,x) \equiv 0$, for $s < 0$, and denote an arbitrary partition of $[0,t] \times [-N-2, N]$ by

$$E \times E_1' := \{0 = s_0 < s_1 < \cdots < s_m = t, -N-2 = x_0 < x_1 < \cdots < x_{l'} = N\}.$$

Because $[-N-2, N]$ also covers the compact support of local time, we have

$$\sup_{E \times E_1'} \sum_{i=1}^{l'} \left(\sum_{j=1}^{m} |\Delta_j \Delta_i f|^p \right)^q = M,$$

and

$$\sup_{s\in[0,t]} \sup_{E_1'} \sum_{i=1}^{l'} |f(s,x_i) - f(s,x_{i-1})|^\theta = M',$$

where M and M' are constants. So by Hölder's inequality,

$$\begin{aligned}
&\sum_{i=1}^{l} \left(\sum_{j=1}^{m} |\Delta_j \Delta_i f_n|^p \right)^q \\
=\ & \sum_{i=1}^{l} \left(\sum_{j=1}^{m} \left| \int_0^2 \int_0^2 \rho(r)\rho(z) \Delta_j \Delta_i f(\cdot - \frac{r}{n}, \cdot - \frac{z}{n}) dr dz \right|^p \right)^q \\
\le\ & A \sum_{i=1}^{l} \left(\int_0^2 \int_0^2 \sum_{j=1}^{m} \left| \Delta_j \Delta_i f(\cdot - \frac{r}{n}, \cdot - \frac{z}{n}) \right|^p dr dz \right)^q \\
\le\ & B \int_0^2 \int_0^2 \sum_{i=1}^{l} \left(\sum_{j=1}^{m} \left| \Delta_j \Delta_i f(\cdot - \frac{r}{n}, \cdot - \frac{z}{n}) \right|^p \right)^q dz dr \\
\le\ & B \int_0^2 \int_0^2 \sup_{E \times E_1'} \sum_{i=1}^{l'} \left(\sum_{j=1}^{m} |\Delta_j \Delta_i f|^p \right)^q dr dz \\
\le\ & M_1,
\end{aligned}$$

where A, B and M_1 (independent of n) are constants. Also from the above estimate, the finite large jump condition for f_n when n is sufficiently large follows from the finite large jump assumption of f. Similarly,

$$\begin{aligned}
&\sum_{i=1}^{l} |f_n(s,x_i) - f_n(s,x_{i-1})|^\theta \\
= &\sum_{i=1}^{l} |\int_0^2 \int_0^2 \rho(r)\rho(z) \left(f(s-\frac{r}{n}, x_i - \frac{z}{n}) - f(s-\frac{r}{n}, x_{i-1} - \frac{z}{n} \right) dr dz|^\theta \\
\leq & C \int_0^2 \int_0^2 \sum_{i=1}^{l} |f(s-\frac{r}{n}, x_i - \frac{z}{n}) - f(s-\frac{r}{n}, x_{i-1} - \frac{z}{n})|^\theta dr dz \\
\leq & C \int_0^2 \int_0^2 \sup_{E'} \sum_{i=1}^{l'} |f(s,x_i) - f(s,x_{i-1})|^\theta dr dz \\
\leq & M_2
\end{aligned}$$

where C and M_2 (independent of n) are constants. So the integral $\int_{-\infty}^{\infty} \int_0^t f_n(s,x) d_{s,x} L_s^x$ is well defined, by argument we discussed before,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_0^t f_n(s,x) d_{s,x} L_s^x \\
= &\int_{-\infty}^{\infty} \int_0^t \tilde{L}_s^x d_{s,x} f_n(s,x) - \int_{-\infty}^{\infty} \tilde{L}_t^x d_x f_n(t,x) \\
&+ \int_{-\infty}^{\infty} \int_0^t f_n(s,x) d_{s,x} h(s,x). \qquad (4.2.11)
\end{aligned}$$

For such p, q satisfying $p, q \geq 1$, and $2q + 2 > 3pq$, there exist a small positive number $\delta > 0$ such that $2q + 2 > 3(p+\delta)q$, so

$$\sum_{n,m} \varrho[(\frac{1}{n})^\gamma] \sigma[(\frac{1}{m})^\gamma] (\frac{1}{n})^{\frac{1}{p+\delta}} (\frac{1}{m})^{\frac{1}{(p+\delta)q}} < \infty$$

still holds for $\rho(u) = u^\alpha$, $\sigma(u) = u^{1-\alpha}$, where $2(1 - \frac{1}{p+\delta}) < \alpha < \frac{2}{(p+\delta)q} - 1$. By Theorem 3.4.2 and Proposition 3.4.1, we can pass the limit to get

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \int_0^t \tilde{L}_s^x d_{s,x} f_n(s,x) = \int_{-\infty}^{\infty} \int_0^t \tilde{L}_s^x d_{s,x} f(s,x).$$

Using a similar method as in the proof of Theorem 1.3.1, we can prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \tilde{L}_t^x d_x f_n(t,x) = \int_{-\infty}^{\infty} \tilde{L}_t^x d_x f(t,x).$$

The convergence of the last term $\int_{-\infty}^{\infty} \int_0^t f_n(s,x) d_{s,x} h(s,x)$ in (4.2.11) follows from Lebesgue's dominated convergence theorem. So we proved the desired result. ◇

Theorem 4.2.3 *Let $X = (X_s)_{s\geq 0}$ be a continuous semimartingale and assume $f : [0,\infty)\times R \to R$ satisfy*

(i) f is absolutely continuous in t, x respectively,

(ii) the left derivatives $\frac{\partial^-}{\partial t}f$ and $\nabla^- f$ exist at all points of $(0,\infty)\times R$ and $[0,\infty)\times R$ respectively,

(iii) $\frac{\partial^-}{\partial t}f$ and $\nabla^- f$ are left continuous and locally bounded,

(iv) $\nabla^- f(t,x)$ is of bounded θ-variation in x and of bounded p,q-variation in (t,x) and satisfies the finite large jump condition, where $1 \leq \theta < 3$, and $p, q \geq 1$, $2q+2 > 3pq$. Then we have:

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial^-}{\partial s} f(s,X_s)ds + \int_0^t \nabla^- f(s,X_s)dX_s \\ &\quad - \int_0^t \int_{-\infty}^{\infty} \nabla^- f(s,x) d_{s,x} L_s^x, \end{aligned} \tag{4.2.12}$$

where L_t^x is the local time of X_t at x, the last integral is defined in (4.2.8).

Proof: Similar to the proof in [7], we can use smoothing procedure and take the limit to prove our result. The main different key point is the following : by Remark 4.2.2 and Theorem 4.2.2,

$$\begin{aligned} & \frac{1}{2}\int_0^t \Delta f_n(s,X_s) d < X >_s \\ = & \int_{-\infty}^{\infty}\int_0^t \Delta f_n(s,x) dL_s^x dx \\ = & -\int_{-\infty}^{\infty}\int_0^t \nabla f_n(s,x) d_{s,x} L_s^x \\ \to & -\int_{-\infty}^{\infty}\int_0^t \nabla^- f(s,x) d_{s,x} L_s^x, \end{aligned}$$

when $n \to \infty$. ◇

Example 4.2.1 *Consider a function $f(t,x) = x^3 t^3 \cos(\frac{1}{t}+\frac{1}{x})$ for $t,x \neq 0$ and $f(t,0) = f(0,x) = f(0,0) = 0$. This function is $C^{1,1}$ and its derivative about x is $\frac{\partial}{\partial x}f(t,x) = 3t^3x^2\cos(\frac{1}{t}+\frac{1}{x}) + xt^3\sin(\frac{1}{t}+\frac{1}{x})$ for $t,x \neq 0$ and $\frac{\partial}{\partial x}f(t,0) = \frac{\partial}{\partial x}f(0,x) = \frac{\partial}{\partial x}f(0,0) = 0$. It is easy to see that $\frac{\partial}{\partial x}f(t,x)$ is of unbounded variation in x and in (t,x), but of θ-variation in x for any $\theta > 1$, $p,1$-variation in (t,x) for any $p > 1$(similar to Example 3.1). So Theorem 4.2.3 can be used.*

Remark 4.2.3 *Chaper 1,3,4 are included in paper [11] which is published in* Potential Analysis.

Chapter 5
Stochastic Lebesgue-Stieltjes Integrals and A Generalized Itô's Formula in Two-Dimensions

§5.1 Introduction

Extensions of the classical Itô's formula for twice differentiable functions to less smooth functions have been made mainly in one-dimension beginning with Tanaka's pioneering work [46] for $|X_t|$ to which the local time was beautifully linked. Further extensions were made to a time independent convex function $f(x)$ in [36] and [48] as the following Tanaka-Meyer formula:

$$f(X(t)) = f(X(0)) + \int_0^t f'_-(X(s))dX(s) + \int_{-\infty}^{\infty} L_t(x)d(f'_-(x)), \tag{5.1.1}$$

where the left derivative f'_- exists and is increasing due to the convex assumption. This can be generalized easily to include the case when f'_- is of bounded variation where the integral $\int_{-\infty}^{\infty} L_t(x)d(f'_-(x))$ is a Lebesgue-Stieltjes integral. The extension to the time dependent case was given in [7].

The purpose of this chapter is to extend formula (5.1.1) to two dimensions. This is a nontrivial extension as the local time in two-dimensions does not exist. But we observe for a smooth function f, formally by the occupation times formula and the property that $\int_0^{\infty} 1_{R\setminus\{a\}}X_1(s,\omega)d_sL_1(s,\omega) = 0$ a.s., "formal integration by parts formula",

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Delta_1 f(X_1(s), X_2(s))d <X_1>_s \\
=\ &\int_{-\infty}^{+\infty}\int_0^t \Delta_1 f(X_1(s), X_2(s))\mathrm{d}_sL_1(s,a)\mathrm{d}a \\
=\ &\int_{-\infty}^{+\infty}\int_0^t \Delta_1 f(a, X_2(s))\mathrm{d}_sL_1(s,a)\mathrm{d}a \\
=\ &\int_{-\infty}^{+\infty} L_1(t,a)\mathrm{d}_a\nabla_1 f(a, X_2(t)) - \int_{-\infty}^{+\infty}\int_0^t L_1(s,a)\mathrm{d}_{s,a}\nabla_1 f(a, X_2(s)).
\end{aligned} \tag{5.1.2}$$

Here the last equality needs to be justified, and the integral $\int_{-\infty}^{+\infty}\int_0^t L_1(s,a)\mathrm{d}_{s,a}\nabla_1 f(a, X_2(s))$ needs to be properly defined. It is worth noting that the right hand side does not include any second order derivative of f explicitly. Here $\nabla_1 f(a, X_2(s))$ is a semimartingale for any fixed a, following Tanaka-Meyer formula. For this, we study this kind of the integral $\int_{-\infty}^{+\infty}\int_0^t g(s,a)\mathrm{d}_{s,a}h(s,a)$ in Section 5.2. Here $h(s,x)$ is a continuous martingale with cross variation $< h(\cdot,a), h(\cdot,b) >_s$ of locally bounded variation in (s,a,b), and

$E\left[\int_0^t \int_{R^2} |g(s,a)g(s,b)||\mathrm{d}_{a,b,s} < h(\cdot,a), h(\cdot,b) >_s |\right] < \infty$. The integral is different from the Lebesgue-Stieltjes integral and Itô's stochastic integral. But it is a natural extension to the two-parameter stochastic case and therefore called a stochastic Lebesgue-Stieltjes integral. According to our knowledge, this integral is new. It's different from integration with Brownian sheet defined by Walsh ([47]) and integration w.r.t. Poisson random measure (see [19]). A generalized Itô's formula in two dimensions is proved in Section 5.3. It is noted that Peskir recently gave a generalized Itô's formula in multi-dimensions using local times on surfaces where the first order derivative might be discontinuous under the condition their second derivative has limit from both sides of the surfaces in [39]. We will give an example to demonstrate that Peskir's formula can not be used while our formula can. Our formula does not need conditions on the existence of limits of second order derivatives when x goes to the surface. There are numerous examples that classical Itô's formula and Peskir's formula may not work immediately, but our formula can be used (see Example 5.3.1 and 5.3.2).

Other kinds of relevant results include the work for absolutely continuous function with their first derivative being locally bounded in [41]; for $W^{1,2}_{loc}$ functions of a Brownian motion for one dimension in [15] and [16] for multi-dimensions. It was proved in [15] that $f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}[f(B), B]_t$, where $[f(B), B]_t$ is the covariation of the processes $f(B)$ and B and is equal to $\int_0^t f(B_s)d^*B_s - \int_0^t f(B_s)dB_s$ as a difference of backward and forward integrals. See [44] for the case of continuous semi-martingale. The multi-dimensional case was considered by [16], [44] and [37]. But our results here are new.

§5.2 The Definition of Stochastic Lebesgue-Stieltjes Integrals and the Integration by Parts Formula

For a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, denote by $\mathcal{M}_2$ the Hilbert space of all processes $X = (X_t)_{0\leq t\leq T}$ such that $(X_t)_{0\leq t\leq T}$ is a $(\mathcal{F}_t)_{0\leq t\leq T}$ right continuous square integrable martingale with inner product $(X, Y) = E(X_T Y_T)$. A three-variable function $f(s,x,y)$ is called left continuous iff it is left continuous in all three variables together i.e. for any sequence $(s_1,x_1,y_1) \leq (s_2,x_2,y_2) \leq \cdots \leq (s_k,x_k,y_k) \leq (s,x,y)$ and $(s_k,x_k,y_k) \to (s,x,y)$, as $k \to \infty$, we have $f(s_k,x_k,y_k) \to f(s,x,y)$ as $k \to \infty$. Here $(s_1,x_1,y_1) \leq (s_2,x_2,y_2)$ means $s_1 \leq s_2$, $x_1 \leq x_2$ and $y_1 \leq y_2$. Define

$$\mathcal{V}_1 := \Big\{h: \quad [0,t]\times(-\infty,\infty)\times\Omega \to R \text{ } s.t. \text{ } (s,x,\omega) \mapsto h(s,x,\omega) \text{ } is \text{ } \mathcal{B}([0,s]\times R)\times\mathcal{F}_s\text{-}measurable, \text{ } and \text{ } h(s,x) \text{ } is \text{ } \mathcal{F}_s\text{-}adapted \text{ } for \text{ } any \text{ } x \in R\Big\},$$

$$\mathcal{V}_2 := \Big\{h: \quad h \in \mathcal{V}_1 \textit{ is a continuous (in } s) \textit{ } \mathcal{M}_2 - \textit{martingale for each } x,$$
$$\textit{and the crossvariation } < h(\cdot,x), h(\cdot,y) >_s \textit{ is left continuous}$$
$$\textit{and of locally bounded variation in } (s,x,y)\Big\}.$$

In the following, we will always denote $< h(\cdot,x), h(\cdot,y) >_s$ by $< h(x), h(y) >_s$.

We now recall some classical results (see [1] and [35]). A three-variable function $f(s,x,y)$ is called monotonically increasing if whenever $(s_2,x_2,y_2) \geq (s_1,x_1,y_1)$, then

$$f(s_2,x_2,y_2) - f(s_2,x_1,y_2) - f(s_2,x_2,y_1) + f(s_2,x_1,y_1)$$
$$-f(s_1,x_2,y_2) + f(s_1,x_1,y_2) + f(s_1,x_2,y_1) - f(s_1,x_1,y_1) \geq 0.$$

For a left-continuous and monotonically increasing function $f(s,x,y)$, one can define a Lebesgue-Stieltjes measure by setting

$$\begin{aligned} & \nu([s_1,s_2) \times [x_1,x_2) \times [y_1,y_2)) \\ = \; & f(s_2,x_2,y_2) - f(s_2,x_1,y_2) - f(s_2,x_2,y_1) + f(s_2,x_1,y_1) \\ & -f(s_1,x_2,y_2) + f(s_1,x_1,y_2) + f(s_1,x_2,y_1) - f(s_1,x_1,y_1). \end{aligned}$$

For $h \in \mathcal{V}_2$, define

$$< h(x), h(y) >_{t_1}^{t_2} := < h(x), h(y) >_{t_2} - < h(x), h(y) >_{t_1}, \; t_2 \geq t_1.$$

Note as $< h(x), h(y) >_s$ is left continuous and of locally bounded variation in (s,x,y), so it can be decomposed to the difference of two increasing and left continuous functions $f_1(s,x,y)$ and $f_2(s,x,y)$ (see McShane [35] or Proposition 2.2 in Elworthy, Truman and Zhao [7] which also holds for multi-parameter functions). Note each of f_1 and f_2 generates a measure, so for any measurable function $g(s,x,y)$, we can define

$$\begin{aligned} & \int_{t_1}^{t_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} < h(x), h(y) >_s \\ = \; & \int_{t_1}^{t_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} f_1(s,x,y) \\ & - \int_{t_1}^{t_2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} f_2(s,x,y). \end{aligned}$$

In particular, a signed product measure in the space $[0,T] \times R^2$ can be defined as follows: for any $[t_1,t_2) \times [x_1,x_2) \times [y_1,y_2) \subset [0,T] \times R^2$

$$\begin{aligned} & \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{d}_{x,y,s} < h(x), h(y) >_s \\ = \; & \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{d}_{x,y,s} f_1(s,x,y) - \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{d}_{x,y,s} f_2(s,x,y) \end{aligned}$$

$$
\begin{aligned}
&= < h(x_2), h(y_2) >_{t_1}^{t_2} - < h(x_2), h(y_1) >_{t_1}^{t_2} \\
&\quad - < h(x_1), h(y_2) >_{t_1}^{t_2} + < h(x_1), h(y_1) >_{t_1}^{t_2} \\
&= < h(x_2) - h(x_1), h(y_2) - h(y_1) >_{t_1}^{t_2} . \qquad (5.2.1)
\end{aligned}
$$

Define

$$
|\mathrm{d}_{x,y,s} < h(x), h(y) >_s | = \mathrm{d}_{x,y,s} f_1(s,x,y) + \mathrm{d}_{x,y,s} f_2(s,x,y). \qquad (5.2.2)
$$

Moreover, for $h \in \mathcal{V}_2$, define:

$$
\begin{aligned}
\mathcal{V}_3(h) := \Big\{ g \quad : \quad & g \in \mathcal{V}_1, \textit{ and there exists } N \textit{ such that } (-N, N) \textit{ covers} \\
& \textit{the compact support of } g(s,\cdot,\omega) \textit{ for a.a. } \omega, \textit{ and } s \in [0,T] \textit{ and} \\
& E\left[\int_0^t \int_{R^2} |g(s,x)g(s,y)| |\mathrm{d}_{x,y,s} < h(x), h(y) >_s | \right] < \infty \Big\}. \\
\mathcal{V}_4(h) := \Big\{ g \quad : \quad & g \in \mathcal{V}_1 \textit{ has a compact support in } x \textit{ for a.a. } \omega, \textit{ and} \\
, \quad & E\left[\int_0^t \int_{R^2} |g(s,x)g(s,y)| |\mathrm{d}_{x,y,s} < h(x), h(y) >_s | \right] < \infty \Big\}.
\end{aligned}
$$

Consider now a simple function in $\mathcal{V}_3$, and always assume for any $s > 0$, $g(s,-N) = g(s,N) = 0$,

$$
g(s,x,\omega) = \sum_{i=0}^{n-1} e_{0,i} 1_{\{0\}}(s) 1_{(x_i, x_{i+1}]}(x) + \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} e_{j,i} 1_{(t_j, t_{j+1}]}(s) 1_{(x_i, x_{i+1}]}(x) \qquad (5.2.3)
$$

where $\{t_n\}_{m=0}^{\infty}$ with $t_0 = 0$ and $\lim_{m\to\infty} t_m = \infty$, $-N = x_0 < x_1 < x_2 < \cdots < x_n = N$, $e_{j,i}$ are $\mathcal{F}_{t_j}$-measurable. For $h \in \mathcal{V}_2$, define an integral as:

$$
\begin{aligned}
I_t(g) &:= \int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} e_{j,i} \Big[h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \\
&\qquad - h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \Big]. \qquad (5.2.4)
\end{aligned}
$$

This integral is called the stochastic Lebesgue-Stieltjes integral of the simple function g. It's easy to see for simple functions $g_1, g_2 \in \mathcal{V}_3(h)$,

$$
I_t(\alpha g_1 + \beta g_2) = \alpha I_t(g_1) + \beta I_t(g_2), \qquad (5.2.5)
$$

for any $\alpha, \beta \in R$. The following lemma plays a key role in extending the integral of simple functions to functions in $\mathcal{V}_3(h)$. It is equivalent to the Itô's isometry formula in the case of the stochastic integral.

Lemma 5.2.1 *If $h \in \mathcal{V}_2$, $g \in \mathcal{V}_3(h)$ is simple, then $I_t(g)$ is a continuous martingale with respect to $(\mathcal{F}_t)_{0\le t\le T}$ and*

$$E\Big(\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x)\Big)^2 = E\int_0^t \int_{R^2} g(s,x)g(s,y)\mathrm{d}_{x,y,s} < h(x),h(y) >_s . \tag{5.2.6}$$

Proof. From the definition of $\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x)$, it is easy to see that I_t is a continuous martingale with respect to $(\mathcal{F}_t)_{0\le t\le T}$. As $h(s,x,\omega)$ is a continuous martingale in $\mathcal{M}_2$, using a standard conditional expectation argument to remove the cross product parts, we get:

$$\begin{aligned}
& E\left[\left(\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x)\right)^2\right] \\
= \; & E\sum_{j=0}^{\infty}\left(\sum_{i=0}^{n-1} e_{j,i}\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}) \right. \\
& \qquad \left. -h(t_{j+1}\wedge t, x_i) + h(t_j\wedge t, x_i)\Big]\right)^2 \\
= \; & E\sum_{j=0}^{\infty}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\cdot \right. \\
& \quad \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}) - h(t_{j+1}\wedge t, x_i) + h(t_j\wedge t, x_i)\Big]\cdot \\
& \quad \left.\Big[h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1}) - h(t_{j+1}\wedge t, x_k) + h(t_j\wedge t, x_k)\Big]\right) \\
= \; & E\sum_{j=0}^{\infty}\left\{\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\cdot \right. \\
& \quad \Big[(h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}))(h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1})) \\
& \quad -(h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}))(h(t_{j+1}\wedge t, x_k) - h(t_j\wedge t, x_k)) \\
& \quad -(h(t_{j+1}\wedge t, x_i) - h(t_j\wedge t, x_i))(h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1})) \\
& \quad \left.+(h(t_{j+1}\wedge t, x_i) - h(t_j\wedge t, x_i))(h(t_{j+1}\wedge t, x_k) - h(t_j\wedge t, x_k))\Big]\right\} \\
= \; & E\int_0^t \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} g(s,x_{i+1})g(s,x_{k+1})\Big[\mathrm{d}_s < h(x_{i+1}),h(x_{k+1}) >_s - \mathrm{d}_s < h(x_{i+1}),h(x_k) >_s \\
& \qquad -\mathrm{d}_s < h(x_i),h(x_{k+1}) >_s + \mathrm{d}_s < h(x_i),h(x_k) >_s \Big] \\
= \; & E\sum_{j=0}^{\infty}\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\Big[< h(x_{i+1}),h(x_{k+1}) >_{t_j\wedge t}^{t_{j+1}\wedge t} - < h(x_{i+1}),h(x_k) >_{t_j\wedge t}^{t_{j+1}\wedge t} \\
& \qquad - < h(x_i),h(x_{k+1}) >_{t_j\wedge t}^{t_{j+1}\wedge t} + < h(x_i),h(x_k) >_{t_j\wedge t}^{t_{j+1}\wedge t} \Big]
\end{aligned}$$

$$= E\left[\int_0^t \int_{R^2} g(s,x)g(s,y)\mathrm{d}_{x,y,s} < h(x), h(y) >_s\right].$$

So we proved the desired result. ◇

The idea is to use (5.2.6) to extend the definition of the integrals of simple functions to integrals of functions in $\mathcal{V}_3(h)$ and finally in $\mathcal{V}_4(h)$, for any $h \in \mathcal{V}_2$. We achieve this goal in several steps:

Lemma 5.2.2 *Let $h \in \mathcal{V}_2$, $f \in \mathcal{V}_3(h)$ be bounded uniformly in ω, $f(\cdot,\cdot,\omega)$ be continuous for each ω on its compact support. Then there exist a sequence of bounded simple functions $\varphi_{m,n} \in \mathcal{V}_3(h)$ such that*

$$E\int_0^t \int_{R^2} \mid (f-\varphi_{m,n})(s,x)(f-\varphi_{m',n'})(s,y)\mid \mid \mathrm{d}_{x,y,s} < h(x), h(y) >_s \mid \to 0,$$

as $m, n, m', n' \to \infty$.

Proof. Let $0 = t_0 < t_1 < \cdots < t_m = t$, and $-N = x_0 < x_1 < \cdots < x_n = N$ be a partition of $[0,t]\times[-N,N]$. Assume when $n, m \to \infty$, $\max\limits_{0\le j\le m-1}(t_{j+1}-t_j) \to 0$, $\max\limits_{0\le i\le n-1}(x_{i+1}-x_i) \to 0$. Define

$$\varphi_{m,n}(s,x) := \sum_{i=0}^{n-1} f(0,x_i)1_{\{0\}}(s)1_{(x_i,x_{i+1}]}(x) + \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} f(t_j,x_i)1_{(t_j,t_{j+1}]}(s)1_{(x_i,x_{i+1}]}(x). \quad (5.2.7)$$

Then $\varphi_{m,n}(s,x)$ are simple and $\varphi_{m,n}(s,x) \to f(s,x)$ *a.s. as* $m, n \to \infty$. The result follows from applying Lebesgue's dominated convergence theorem. ◇

Lemma 5.2.3 *Let $h \in \mathcal{V}_2$ and $k \in \mathcal{V}_3(h)$ be bounded uniformly in ω. Then there exist functions $f_n \in \mathcal{V}_3(h)$ such that $f_n(\cdot,\cdot,\omega)$ are continuous for all ω and n, and*

$$E\int_0^t \int_{R^2} \mid (k-f_n)(s,x)(k-f_{n'})(s,y)\mid \mid \mathrm{d}_{x,y,s} < h(x), h(y) >_s \mid \to 0,$$

as $n, n' \to \infty$.

Proof. Define

$$f_n(s,x) = n^2 \int_{x-\frac{1}{n}}^{x}\int_{s-\frac{1}{n}}^{s} k(\tau,y)d\tau dy.$$

Then $f_n(s,x)$ is continuous in s,x, and when $n \to \infty$, $f_n(s,x) \to k(s,x)$ a.s.. So for sufficiently large n, $f_n(s,x)$ also has compact support in $(-N,N)$ for all $s \in [0,T]$. The desired convergence follows from applying Lebesgue's dominated convergence theorem. ◇

Lemma 5.2.4 *Let $h \in \mathcal{V}_2$ and $g \in \mathcal{V}_3(h)$. Then there exist functions $k_n \in \mathcal{V}_3(h)$, bounded uniformly in ω for each n, and*

$$E\int_0^t \int_{R^2} \mid (g-k_n)(s,x)(g-k_{n'})(s,y)\mid \mid \mathrm{d}_{x,y,s} < h(x), h(y) >_s \mid \to 0,$$

as $n, n' \to \infty$.

Proof. Define

$$k_n(t,x,\omega) := \begin{cases} -n & \text{if } g(t,x,\omega) < -n \\ g(t,x,\omega) & \text{if } -n \le g(t,x,\omega) \le n \\ n & \text{if } g(t,x,\omega) > n. \end{cases} \tag{5.2.8}$$

Then as $n \to \infty$, $k_n(t,x,\omega) \to g(t,x,\omega)$ for each (t,x,ω). Note $|k_n(t,x,\omega)| \le |g(t,x,\omega)|$ and $k_n \in \mathcal{V}_3(h)$. So applying Lebesgue's dominated convergence theorem, we obtain the desired result. ◇

Lemma 5.2.5 *Let $h \in \mathcal{V}_2$ and $g \in \mathcal{V}_4(h)$. Then there exist functions $g_N \in \mathcal{V}_3(h)$ such that*

$$E\int_0^t \int_{R^2} \mid (g-g_N)(s,x)(g-g_{N'})(s,y)\mid \mid \mathrm{d}_{x,y,s} < h(x), h(y) >_s \mid \to 0, \tag{5.2.9}$$

as $N, N' \to \infty$.

Proof. Define

$$g_N(s,x,\omega) := g(s,x,\omega)1_{[-N+1,N-1]}(x). \tag{5.2.10}$$

Then $|g_N| \le |g|$ and $g_N \to g$ a.s., as $N \to \infty$. So applying Lebesgue's dominated convergence theorem, we obtain the desired result. ◇

From Lemmas 5.2.4, 5.2.3, 5.2.2, for each $h \in \mathcal{V}_2$, $g \in \mathcal{V}_3(h)$, we can construct a sequence of simple functions $\{\varphi_{m,n}\}$ in $\mathcal{V}_3(h)$ such that,

$$E\int_0^t \int_{R^2} \mid (g-\varphi_{m,n})(s,x)(g-\varphi_{m',n'})(s,y)\mid \mid \mathrm{d}_{x,y,s} < h(x), h(y) >_s \mid \to 0,$$

as $m, n, m', n' \to \infty$. For $\varphi_{m,n}$ and $\varphi_{m',n'}$, we can define stochastic Lebesgue-Stieltjes integrals $I_t(\varphi_{m,n})$ and $I_t(\varphi_{m',n'})$. From Lemma 5.2.1 and (5.2.5), it is easy to see that

$$\begin{aligned} & E\left[I_T(\varphi_{m,n}) - I_T(\varphi_{m',n'})\right]^2 \\ = \; & E\left[I_T(\varphi_{m,n} - \varphi_{m',n'})\right]^2 \\ = \; & E\int_0^T \int_{R^2} (\varphi_{m,n} - \varphi_{m',n'})(s,x)(\varphi_{m,n} - \varphi_{m',n'})(s,y) d_{x,y,s} < h(x), h(y) >_s \end{aligned}$$

$$
\begin{aligned}
&= E\int_0^T\int_{R^2}[(\varphi_{m,n}-g)-(\varphi_{m',n'}-g)](s,x)\cdot \\
&\qquad [(\varphi_{m,n}-g)-(\varphi_{m',n'}-g)](s,y)d_{x,y,s}<h(x),h(y)>_s \\
&= E\int_0^T\int_{R^2}(\varphi_{m,n}-g)(s,x)(\varphi_{m,n}-g)(s,y)d_{x,y,s}<h(x),h(y)>_s \\
&\quad -E\int_0^T\int_{R^2}(\varphi_{m,n}-g)(s,x)(\varphi_{m',n'}-g)(s,y)d_{x,y,s}<h(x),h(y)>_s \\
&\quad -E\int_0^T\int_{R^2}(\varphi_{m',n'}-g)(s,x)(\varphi_{m,n}-g)(s,y)d_{x,y,s}<h(x),h(y)>_s \\
&\quad +E\int_0^T\int_{R^2}(\varphi_{m',n'}-g)(s,x)(\varphi_{m',n'}-g)(s,y)d_{x,y,s}<h(x),h(y)>_s \\
&\le E\int_0^T\int_{R^2}|\,(\varphi_{m,n}-g)(s,x)(\varphi_{m,n}-g)(s,y)\,||\,d_{x,y,s}<h(x),h(y)>_s| \\
&\quad +E\int_0^T\int_{R^2}|\,(\varphi_{m,n}-g)(s,x)(\varphi_{m',n'}-g)(s,y)\,||\,d_{x,y,s}<h(x),h(y)>_s| \\
&\quad +E\int_0^T\int_{R^2}|\,(\varphi_{m',n'}-g)(s,x)(\varphi_{m,n}-g)(s,y)\,||\,d_{x,y,s}<h(x),h(y)>_s| \\
&\quad +E\int_0^T\int_{R^2}|\,(\varphi_{m',n'}-g)(s,x)(\varphi_{m',n'}-g)(s,y)\,||\,d_{x,y,s}<h(x),h(y)>_s| \\
&\to 0,
\end{aligned}
$$

as $m,n,m',n'\to\infty$. Therefore $\{I_\cdot(\varphi_{m,n})\}_{m,n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_2$ whose norm is denoted by $\|\cdot\|$. So there exists a process $I(g)=\{I_t(g),0\le t\le T\}$ in $\mathcal{M}_2$, defined modulo indistinguishability, such that

$$\| I(\varphi_{m,n})-I(g)\|\to 0,\ as\ m,n\to\infty.$$

By the same argument as for the stochastic integral, one can easily prove that $I(g)$ is well-defined (independent of the choice of the simple functions), and (5.2.6) is true for $I(g)$. We now can have the following definition.

Definition 5.2.1 *Let $h\in\mathcal{V}_2$, $g\in\mathcal{V}_3(h)$. Then the integral of g with respect to h can be defined in $\mathcal{M}_2$ as:*

$$\int_0^t\int_{-\infty}^{\infty}g(s,x)\mathrm{d}_{s,x}h(s,x)=\lim_{m,n\to\infty}\int_0^t\int_{-\infty}^{\infty}\varphi_{m,n}(s,x)\mathrm{d}_{s,x}h(s,x).$$

Here $\{\varphi_{m,n}\}$ is a sequence of simple functions in $\mathcal{V}_3(h)$, s.t.

$$E\int_0^t\int_{R^2}|\,(g-\varphi_{m,n})(s,x)(g-\varphi_{m',n'})(s,y)|\,|\,\mathrm{d}_{x,y,s}<h(x),h(y)>_s\,|\to 0,$$

as $m,n,m',n'\to\infty$. Note $\varphi_{m,n}$ may be constructed by combining the three approximation procedures in Lemmas 5.2.4, 5.2.3, 5.2.2. For $g\in\mathcal{V}_4(h)$, we can then define the integral

in $\mathcal{M}_2$ *as:*

$$\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x) \;=\; \lim_{N\to\infty} \int_0^t \int_{-\infty}^{\infty} g(s,x)1_{[-N+1,N-1]}(x)\mathrm{d}_{s,x}h(s,x).$$

It is a continuous martingale with respect to $(\mathcal{F}_t)_{0\le t\le T}$ *and for each* $0 \le t \le T$,

$$E\Big(\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x)\Big)^2$$
$$= E\int_0^t \int_{R^2} g(s,x)g(s,y)\mathrm{d}_{x,y,s} < h(x), h(y) >_s . \qquad (5.2.11)$$

The following integration by parts formula will be useful in the proof of our main theorem in the next section.

Proposition 5.2.1 *If* $h \in \mathcal{V}_2$, $g \in \mathcal{V}_4(h)$, *and* $g(t,x)$ *is* C^2 *in* x, $\Delta g(t,x)$ *is bounded uniformly in* t, *then a.s.*

$$-\int_{-\infty}^{+\infty} \int_0^t \nabla g(s,x)\mathrm{d}_s h(s,x)dx = \int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathbf{d}_{s,x}h(s,x). \qquad (5.2.12)$$

Moreover, for any $g \in \mathcal{V}_4(h)$, $h \in \mathcal{V}_2$ *and* C^1 *in* x, $\nabla h \in \mathcal{M}_2$,

$$\int_{-\infty}^{+\infty} \int_0^t g(s,x)\mathrm{d}_s \nabla h(s,x)dx = \int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathbf{d}_{s,x}h(s,x). \qquad (5.2.13)$$

Proof: If g is a simple function in $\mathcal{V}_3(h)$ as given in (5.2.3), and note that $e_{j,0} = e_{j,n} = 0$, we have

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x) \\
=\;& \sum_{i=0}^{n-1}\sum_{j=0}^{\infty} e_{j,i}\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \\
&\qquad -h(t_{j+1}\wedge t, x_i) + h(t_j \wedge t, x_i)\Big] \\
=\;& -\sum_{i=0}^{n-1}\sum_{j=0}^{\infty} e_{j,i+1}\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1})\Big] \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{\infty} e_{j,i}\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1})\Big] \\
=\;& -\sum_{i=0}^{n-1}\sum_{j=0}^{\infty} \Big[e_{j,i+1} - e_{j,i}\Big]\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1})\Big].
\end{aligned}$$

If $g(t,x)$ is C^2 in x, let

$$\varphi_{m,n}(s,x) := \sum_{i=0}^{n-1} g(0,x_i)1_{\{0\}}(s)1_{(x_i,x_{i+1}]}(x) + \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} g(t_j,x_i)1_{(t_j,t_{j+1}]}(s)1_{(x_i,x_{i+1}]}(x),$$

then

$$\varphi_{m,n}(s,x) \to g(s,x) \ a.s. \ as \ m,n \to \infty.$$

Moreover, by the intermediate value theorem,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^t g(s,x)\mathrm{d}_{s,x}h(s,x) \\
=\ & -\lim_{\delta_t,\delta_x\to 0}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\Big[g(t_j\wedge t,x_{i+1})-g(t_j\wedge t,x_i)\Big] \\
&\qquad\Big[h(t_{j+1}\wedge t,x_{i+1})-h(t_j\wedge t,x_{i+1})\Big] \qquad (limit\ in\ \mathcal{M}_2) \\
=\ & -\lim_{\delta_t,\delta_x\to 0}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\Big[\int_0^1\nabla g(t_j\wedge t,x_i+\alpha(x_{i+1}-x_i))d\alpha\Big]\Big[h(t_{j+1}\wedge t,x_{i+1})-h(t_j\wedge t,x_{i+1})\Big]\cdot \\
&\qquad (x_{i+1}-x_i) \\
=\ & -\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\Big[\int_0^1\nabla g(s,x_i+\alpha(x_{i+1}-x_i))d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) \quad (limit\ in\ \mathcal{M}_2) \\
=\ & -\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\nabla g(s,x_{i+1})\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) \\
&-\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\Big[\int_0^1(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1}))d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) \\
=\ & -\int_{-\infty}^{+\infty}\int_0^t\nabla g(s,x)\mathrm{d}_s h(s,x)dx. \qquad (limit\ in\ \mathcal{M}_2)
\end{aligned}$$

Here $\delta_t = \max\limits_{1\le j\le m}|t_{j+1}-t_j|$, $\delta_x = \max\limits_{1\le i\le m}|x_{i+1}-x_i|$. To prove the last equality, first notice that

$$\begin{aligned}
&\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\nabla g(s,x_{i+1})\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) \\
=\ & \int_{-\infty}^{+\infty}\int_0^t\nabla g(s,x)\mathrm{d}_s h(s,x)dx.
\end{aligned}$$

Second, by the intermediate value theorem again, and from the assumption that $\Delta g(s,x)$ is bounded uniformly in s, the second term can be estimated as:

$$\begin{aligned}
&E\left[\sum_{i=0}^{n-1}\int_0^t\Big[\int_0^1(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1}))d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i)\right]^2 \\
=\ & E\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\Big[\int_0^t\Big[\int_0^1(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1}))d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i)\cdot \\
&\qquad\int_0^t\Big[\int_0^1(\nabla g(s,x_k+\alpha(x_{k+1}-x_k))-\nabla g(s,x_{k+1}))d\alpha\Big]\mathrm{d}_s h(s,x_{k+1})(x_{k+1}-x_k)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} E\int_0^t \Big[\int_0^1 (\nabla g(s, x_i + \alpha(x_{i+1} - x_i)) - \nabla g(s, x_{i+1}))d\alpha\Big] \cdot \\
&\qquad \Big[\int_0^1 (\nabla g(s, x_k + \alpha(x_{k+1} - x_k)) - \nabla g(s, x_{k+1}))d\alpha\Big] \\
&\qquad \mathrm{d}_s < h(x_{i+1}), h(x_{k+1}) >_s (x_{i+1} - x_i)(x_{k+1} - x_k) \\
&\leq E\Big[\sup_i \sup_{s \atop \eta\in(x_i,x_{i+1})} |\Delta g(s,\eta)||((1-\alpha)(x_{i+1} - x_i))| \cdot \\
&\qquad \sup_k \sup_{s \atop \eta\in(x_k,x_{k+1})} |\Delta g(s,\eta)||((1-\alpha)(x_{k+1} - x_k))| \cdot \\
&\qquad | < h(x_{i+1}) >_t < h(x_{k+1}) >_t |^{\frac{1}{2}}\Big] \cdot \left(\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}(x_{i+1} - x_i)(x_{k+1} - x_k)\right) \\
&\to \ 0, \ as \ \delta_x \to 0,
\end{aligned}
$$

So (5.2.12) is proved.

For (5.2.13), first consider $g \in \mathcal{V}_3(h)$ and sufficiently smooth jointly in (s, x), by (5.2.12) and integration by parts formula,

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathrm{d}_{s,x}h(s,x) \\
&= -\int_{-\infty}^{+\infty}\int_0^t \nabla g(s,x)\mathrm{d}_s h(s,x)dx \\
&= -\int_{-\infty}^{+\infty} [\nabla g(s,x)h(s,x)]_0^t dx + \int_{-\infty}^{+\infty}\int_0^t \left(\frac{\partial}{\partial s}\nabla g(s,x)\right) h(s,x)dsdx. \quad (5.2.14)
\end{aligned}
$$

But by integration by parts formula and Fubini theorem,

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^t \left(\frac{\partial}{\partial s}\nabla g(s,x)\right) h(s,x)dsdx \\
&= \int_0^t\int_{-\infty}^{+\infty} \left(\nabla\frac{\partial}{\partial s} g(s,x)\right) h(s,x)dxds \\
&= -\int_0^t\int_{-\infty}^{+\infty} \frac{\partial}{\partial s} g(s,x)\nabla h(s,x)dxds \\
&= -\int_{-\infty}^{+\infty}\int_0^t \frac{\partial}{\partial s} g(s,x)\nabla h(s,x)dsdx \\
&= -\int_{-\infty}^{+\infty} [g(s,x)\nabla h(s,x)]_0^t dx + \int_{-\infty}^{+\infty}\int_0^t g(s,x)d_s\nabla h(s,x)dx. \qquad (5.2.15)
\end{aligned}
$$

By (5.2.14), (5.2.15) and integration by parts formula, it follows that for g being sufficiently smooth

$$
\int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathrm{d}_{s,x}h(s,x) = \int_{-\infty}^{+\infty}\int_0^t g(s,x)d_s\nabla h(s,x)dx.
$$

But any bounded function $g \in \mathcal{V}_3(h)$ can be approximated by a sequence of smooth

functions $g_n \in \mathcal{V}_3(h)$, the desired result for $g \in \mathcal{V}_3(h)$ follows from (5.2.11) and

$$\begin{aligned} & E|\int_{-\infty}^{+\infty}\int_0^t (g_n(s,x)-g(s,x))d_s\nabla h(s,x)dx|^2 \\ \leq\ & 2N\int_{-\infty}^{+\infty} E|\int_0^t (g_n(s,x)-g(s,x))d_s\nabla h(s,x)|^2dx \\ =\ & 2N\int_{-\infty}^{+\infty} E\int_0^t |g_n(s,x)-g(s,x)|^2 d_s < \nabla h(x) >_s dx \\ \to\ & 0, \end{aligned}$$

when $n \to \infty$. From Lemma 5.2.4, 5.2.5, we can get (5.2.12) and (5.2.13) also hold for $g \in \mathcal{V}_4(h)$. ◇

§5.3 A Generalized Itô's Formula in Two-dimensional Space

Let $X(s) = (X_1(s), X_2(s))$ be a two-dimensional continuous semi-martingale with $X_i(s) = X_i(0) + M_i(s) + V_i(s)(i = 1, 2)$ on a probability space $(\Omega, \mathcal{F}, P)$. Here $M_i(s)$ is a continuous local martingale and $V_i(s)$ is an adapted continuous process of locally bounded variation (in s). Let $L_i(t, a)$ be the local time of $X_i(t)$ (i=1,2). From localization argument in Section 1.2, we can assume $L_1(t, a)$ and $L_2(t, a)$ are bounded uniformly in a.

In the following we assume some conditions on $f : R \times R \to R$:

Condition (i) the function $f(\cdot,\cdot) : R \times R \to R$ is jointly continuous and absolutely continuous in x_1, x_2 respectively;

Condition (ii) the left derivative $\nabla_i^- f(x_1, x_2)$ is locally bounded, jointly left continuous, and of locally bounded variation in x_i $(i = 1, 2)$;

Condition (iii) the left derivaties $\nabla_1^- f(x_1, x_2)$ is absolutely continuous in x_2, and $\nabla_2^- f(x_1, x_2)$ is absolutely continuous in x_1;

Condition (iv) the derivatives $\nabla_i^-\nabla_j^- f(x_1, x_2)$ $(i, j = 1, 2,\ i \neq j)$ are jointly left continuous, and of locally bounded variation in x_1, x_2 respectively and also in (x_1, x_2).

From the assumption of $\nabla_1^- f$, we can use Tanaka-Meyer formula to have,

$$\begin{aligned} \nabla_1^- f(a, X_2(t)) \ -\ & \nabla_1^- f(a, X_2(0)) = \int_0^t \nabla_2^-\nabla_1^- f(a, X_2(s))dX_2(s) \\ & + \int_{-\infty}^{\infty} L_2(t, x_2)\mathrm{d}_{x_2}\nabla_2^-\nabla_1^- f(a, x_2)\ \ a.s.. \end{aligned} \tag{5.3.1}$$

Therefore $\nabla_1^- f(a, X_2(t))$ is a continuous semimartingale, and can be decomposed as

$$\nabla_1^- f(a, X_2(t)) = \nabla_1^- f(a, X_2(0)) + h(t, a) + v(t, a), \tag{5.3.2}$$

where h is a continuous local martingale and v is a continuous process of locally bounded variation (in t). In fact $h(t,a) = \int_0^t \nabla_2^- \nabla_1^- f(a, X_2(s)) dM_2(s)$. Define

$$\begin{aligned} F_s(a,b) &:= < h(a), h(b) >_s = < \nabla_1^- f(a), \nabla_1^- f(b) >_s \\ &= \int_0^s \nabla_2^- \nabla_1^- f(a, X_2(r)) \nabla_2^- \nabla_1^- f(b, X_2(r)) \mathrm{d} <M_2>_r, \\ F(a,b)_{s_k}^{s_{k+1}} &:= < h(a), h(b) >_{s_k}^{s_{k+1}} = < \nabla_1^- f(a), \nabla_1^- f(b) >_{s_k}^{s_{k+1}} \\ &= \int_{s_k}^{s_{k+1}} \nabla_2^- \nabla_1^- f(a, X_2(r)) \nabla_2^- \nabla_1^- f(b, X_2(r)) \mathrm{d} <M_2>_r . \end{aligned}$$

We need to prove $h \in \mathcal{V}_2$. To see this, as $\nabla_2^- \nabla_1^- f(x_1, x_2)$ is of locally bounded variation in x_1, so for any compact set $[-N, N]$, $\nabla_2^- \nabla_1^- f(x_1, x_2)$ is of bounded variation in x_1 for $x_1 \in [-N, N]$. Let $\mathcal{P}$ be the partition on $[-N, N]^2 \times [0, t]$, $\mathcal{P}_i$ be a partition on $[-N, N]$ $(i = 1, 2)$, $\mathcal{P}_3$ be a partition on $[0, t]$ such that $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$. Then we have:

$$\begin{aligned} & \mathrm{Var}_{s,a,b}(F_s(a,b)) \\ = & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| F(a_{i+1}, b_{j+1})_{s_k}^{s_{k+1}} - F(a_{i+1}, b_j)_{s_k}^{s_{k+1}} - F(a_i, b_{j+1})_{s_k}^{s_{k+1}} \\ & \qquad + F(a_i, b_j)_{s_k}^{s_{k+1}} \Big| \\ = & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| \int_{s_k}^{s_{k+1}} \nabla_2^- \nabla_1^- f(a_{i+1}, X_2(r)) \nabla_2^- \nabla_1^- f(b_{j+1}, X_2(r)) \mathrm{d} <M_2>_r \\ & \qquad - \int_{s_k}^{s_{k+1}} \nabla_2^- \nabla_1^- f(a_{i+1}, X_2(r)) \nabla_2^- \nabla_1^- f(b_j, X_2(r)) \mathrm{d} <M_2>_r \\ & \qquad - \int_{s_k}^{s_{k+1}} \nabla_2^- \nabla_1^- f(a_i, X_2(r)) \nabla_2^- \nabla_1^- f(b_{j+1}, X_2(r)) \mathrm{d} <M_2>_r \\ & \qquad + \int_{s_k}^{s_{k+1}} \nabla_2^- \nabla_1^- f(a_i, X_2(r)) \nabla_2^- \nabla_1^- f(b_j, X_2(r)) \mathrm{d} <M_2>_r \Big| \\ = & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| \int_{s_k}^{s_{k+1}} \Big(\nabla_2^- \nabla_1^- f(a_{i+1}, X_2(r)) - \nabla_2^- \nabla_1^- f(a_i, X_2(r)) \Big) \\ & \qquad \Big(\nabla_2^- \nabla_1^- f(b_{j+1}, X_2(r)) - \nabla_2^- \nabla_1^- f(b_j, X_2(r)) \Big) \mathrm{d} <M_2>_r \Big| \\ \leq & \int_0^s \sup_{\mathcal{P}_1} \sum_i \Big| \nabla_2^- \nabla_1^- f(a_{i+1}, X_2(r)) - \nabla_2^- \nabla_1^- f(a_i, X_2(r)) \Big| \\ & \qquad \sup_{\mathcal{P}_2} \sum_j \Big| \nabla_2^- \nabla_1^- f(b_{j+1}, X_2(r)) - \nabla_2^- \nabla_1^- f(b_j, X_2(r)) \Big| \mathrm{d} <M_2>_r \\ = & \int_0^s \Big(\mathrm{Var}_a(\nabla_2^- \nabla_1^- f(a, X_2(r))) \Big)^2 \mathrm{d} <M_2>_r < \infty. \end{aligned}$$

Therefore under the localization assumption, $\int_{-\infty}^{\infty} \int_0^t L_1(s,a) d_{s,a} h(s,a)$ can be defined by Definition 5.2.1, i.e. it is a stochastic Lebesgue-Stieltjes integral. On the other hand, under the localization assumption and condition (iii) and (iv), let's prove that

$$v(s,a) = \int_0^s \nabla_2^- \nabla_1^- f(a, X_2(r)) dV_2(r) + \int_{-\infty}^{\infty} L_2(s, x_2) \mathrm{d}_{x_2} \nabla_2^- \nabla_1^- f(a, x_2) := v_1(s,a) + v_2(s,a)$$

is of bounded variation in (s,a) for $s \in [0,t]$, $a \in [-N,N]$. In fact,

$$\begin{aligned} Var_{s,a}v_1(s,a) &= \sup_{\mathcal{P}_1\times\mathcal{P}_3} \sum_k \sum_i |v_1(s_{k+1},a_{i+1}) - v_1(s_k,a_{i+1}) - v_1(s_{k+1},a_i) + v_1(s_k,a_i)| \\ &= \sup_{\mathcal{P}_1\times\mathcal{P}_3} \sum_k \sum_i | \int_{s_{k+1}}^{s_k} \Big[\nabla_2^- \nabla_1^- f(a_{i+1},X_2(r)) - \nabla_2^- \nabla_1^- f(a_i,X_2(r))\Big] dV_2(r)| \\ &\le \int_0^t \sup_{\mathcal{P}_1} \sum_i |\nabla_2^- \nabla_1^- f(a_{i+1},X_2(r)) - \nabla_2^- \nabla_1^- f(a_i,X_2(r))| |dV_2(r)| \\ &< \infty, \end{aligned}$$

as $\nabla_2^- \nabla_1^- f(x_1,x_2)$ is locally bounded and of bounded variation in x_1. Moreover, in the case when $\nabla_1^- \nabla_2^- f(x_1,x_2)$ is increasing in (x_1,x_2),

$$\begin{aligned} Var_{s,a}v_2(s,a) &= \sup_{\mathcal{P}_1\times\mathcal{P}_3} \sum_k \sum_i |v_2(s_{k+1},a_{i+1}) - v_2(s_k,a_{i+1}) - v_2(s_{k+1},a_i) + v_2(s_k,a_i)| \\ &= \sup_{\mathcal{P}_1\times\mathcal{P}_3} \sum_k \sum_i \int_{-\infty}^{\infty} \Big(L_2(s_{k+1},x_2) - L_2(s_k,x_2)\Big) \\ &\qquad \mathrm{d}_{x_2}\Big(\nabla_2^- \nabla_1^- f(a_{i+1},x_2) - \nabla_2^- \nabla_1^- f(a_i,x_2)\Big) \\ &\le \sum_i \int_{-\infty}^{\infty} L_2(t,x_2) \mathrm{d}_{x_2}\Big(\nabla_2^- \nabla_1^- f(a_{i+1},x_2) - \nabla_2^- \nabla_1^- f(a_i,x_2)\Big) \\ &\le \max_{x_2} L_2(t,x_2)(\nabla_2^- \nabla_1^- f(N,N) - \nabla_2^- \nabla_1^- f(N,-N) \\ &\qquad -\nabla_2^- \nabla_1^- f(-N,N) + \nabla_2^- \nabla_1^- f(-N,-N)) \\ &< \infty. \end{aligned}$$

In the general case when $\nabla_2^- \nabla_1^- f(x_1,x_2)$ is of bounded variation in (x_1,x_2), we can assert that $v_2(s,a)$ is also of bounded variation in (s,a) by applying the above result to the difference of two increasing functions. So $\int_0^t \int_{-\infty}^{\infty} L_1(s,a) d_{s,a} v(s,a)$ is a Lebesgue-Stieltjes integral. Hence, $\int_0^t \int_{-\infty}^{\infty} L_1(s,a) d_{s,a} \nabla_1^- f(a,X_2(s))$ can be well defined. A localization argument implies it is a semimartingale. Now we recall that the local time $L_1(s,a)$ can be decomposed

$$L_1(s,a) = \tilde{L}_1(s,a) + \sum_{x_k^* \le a} \widehat{L}_1(s,x_k^*) := \tilde{L}_1(s,a) + \bar{L}_1(s,a),$$

where $\tilde{L}_1(s,a)$ is jointly continuous in s, a, and $\{x_k^*\}$ are the discontinuous points of $L_1(s,a)$. From [41],

$$\widehat{L}_1(t,x) = L_1(t,x) - L_1(t,x-) = \int_0^t 1_{\{x\}}(X_s) dV_s. \tag{5.3.3}$$

Again we use the localization argument and assume the support of the local time is included in $(-N,N)$. Let $g_1(s,a) := \nabla_1^- f(a,X_2(s))$, by a computation in (4.2.7) in Section 4.2, for

any partition $\{0 = t_0 < t_1 < \cdots < t_m = t, -N = a_0 < a_1 < a_2 < \cdots < a_l = N\}$,

$$\begin{aligned}
&\sum_{i=0}^{l-1}\sum_{j=0}^{m-1} g_1(t_{j+1}, a_{i+1})\Big[\tilde{L}_1(t_{j+1}, a_{i+1}) - \tilde{L}_1(t_j, a_{i+1}) - \tilde{L}_1(t_{j+1}, a_i) + \tilde{L}_1(t_j, a_i)\Big] \\
= &\sum_{i=0}^{l-1}\sum_{j=0}^{m-1} \tilde{L}_1(t_j, a_i)\Big[g_1(t_{j+1}, a_{i+1}) - g_1(t_j, a_{i+1}) - g_1(t_{j+1}, a_i) + g_1(s_j, a_i)\Big] \\
&-\sum_{i=0}^{l-1} \tilde{L}_1(t, a_i)\Big[g_1(t, a_{i+1}) - g_1(t, a_i)\Big]. \qquad (5.3.4)
\end{aligned}$$

Note the first Riemann sum of the right hand side has a limit that is $\int_0^t \int_{-N}^N \tilde{L}_1(s,a) d_{s,a} g_1(s,a)$, the second Riemann sum of the right hand side has a limit that is $\int_{-N}^N \tilde{L}_1(s,a) d_a g_1(s,a)$, when $\delta_t = \max_j (t_{j+1} - t_j) \to 0$ and $\delta_x = \max_i (x_{i+1} - x_i) \to 0$. Therefore the left hand side converges as well when $\delta_t \to 0$, $\delta_x \to 0$, denote the limit by $\int_0^t \int_{-N}^N g_1(s,a) d_{s,a} \tilde{L}_1(s,a)$ on $\{\omega\colon\ L_1(t,a)$ has support which is included in $(-N, N)\}$. Taking the limit as $N \to \infty$ we can define $\int_0^t \int_{-\infty}^{\infty} g_1(s,a) d_{s,a} \tilde{L}_1(s,a)$ for almost all $\omega \in \Omega$ and it is easy to see that

$$\begin{aligned}
\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a} \tilde{L}_1(s,a) &= \int_0^t \int_{-\infty}^{\infty} \tilde{L}_1(s,a) d_{s,a} \nabla_1^- f(a, X_2(s)) \\
&\quad - \int_{-\infty}^{\infty} \tilde{L}_1(t,a) d_a \nabla_1^- f(a, X_2(t)). \qquad (5.3.5)
\end{aligned}$$

From Lemma 1.2.2, we know that $\bar{L}_1(t,a)$ is of bounded variation in (t,a) for almost every $\omega \in \Omega$. So $\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a} \bar{L}_1(s,a)$ is a Lebesgue-Stieltjes integral. Therefore the integral

$$\begin{aligned}
\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a} L_1(s,a) &= \int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a} \tilde{L}_1(s,a) \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a} \bar{L}_1(s,a)
\end{aligned}$$

can be well defined.

We will prove the following generalized Itô's formula in two-dimensional space.

Theorem 5.3.1 *Under conditions (i)-(iv), for any continuous two-dimensional semi-martingale $X(t) = (X_1(t), X_2(t))$, we have almost surely*

$$\begin{aligned}
&f(X(t)) - f(X(0)) \\
= &\sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f(a, X_2(s)) \mathrm{d}_{s,a} L_1(s,a) \qquad (5.3.6) \\
&- \int_{-\infty}^{+\infty} \int_0^t \nabla_2^- f(X_1(s), a) \mathrm{d}_{s,a} L_2(s,a) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^t \nabla_i^- \nabla_j^- f(X(s)) d <M_1, M_2>_s .
\end{aligned}$$

Proof: By a standard localization argument, we can assume $X_1(t)$, $X_2(t)$, their quadratic variations $<X_1>_t$, $<X_2>_t$, $<X_1, X_2>_t$ and the local times L_1, L_2 are bounded processes and f, $\nabla_i^- f$, $Var_{x_i}\nabla_i^- f$ $(i = 1, 2)$, $\nabla_i^-\nabla_j^- f$, $Var_{x_k}\nabla_i^-\nabla_j^- f$, $Var_{(x_1,x_2)}\nabla_i^-\nabla_j^- f$ $(i, j, k = 1, 2, i \neq j)$ are bounded.

We divide the proof into several steps:

(A) Define

$$\rho(x) = \begin{cases} ce^{\frac{1}{(x-1)^2-1}}, & \text{if } x \in (0,2), \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.7}$$

Here c is chosen such that $\int_0^2 \rho(x)dx = 1$. Take $\rho_n(x) = n\rho(nx)$ as mollifiers. Define

$$f_n(x_1, x_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_n(x_1 - y)\rho_n(x_2 - z)f(y, z)dydz.\ n \geq 1,$$

Then $f_n(x_1, x_2)$ are smooth and

$$f_n(x_1, x_2) = \int_0^2\int_0^2 \rho(y)\rho(z)f(x_1 - \frac{y}{n}, x_2 - \frac{z}{n})dydz,\ n \geq 1. \tag{5.3.8}$$

Because of the absolute continuity assumption, we can differentiate under the integral (5.3.8) to see f, $\nabla_i f_n$, $Var_{x_i}\nabla_i f_n$ $(i = 1, 2)$, $\nabla_i\nabla_j f_n$, $Var_{x_k}\nabla_i\nabla_j f_n$, $Var_{(x_1,x_2)}\nabla_i\nabla_j f_n$ $(i, j, k = 1, 2, i \neq j)$ are bounded. Furthermore using Lebesgue's dominated convergence theorem, one can prove that as $n \to \infty$,

$$f_n(x_1, x_2) \to f(x_1, x_2), \tag{5.3.9}$$

$$\nabla_1 f_n(x_1, x_2) \to \nabla_1^- f(x_1, x_2), \tag{5.3.10}$$

$$\nabla_2 f_n(x_1, x_2) \to \nabla_2^- f(x_1, x_2), \tag{5.3.11}$$

$$\nabla_i\nabla_j f_n(x_1, x_2) \to \nabla_i^-\nabla_j^- f(x_1, x_2),\ i, j = 1, 2,\ i \neq j, \tag{5.3.12}$$

and each $(x_1, x_2) \in R^2$.

(B) It turns out for any $g(t, x_1)$ being continuous in t and C^1 in x_1 and having a compact support, using the integration by parts formula and Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned} & \lim_{n\to+\infty}\int_{-\infty}^{+\infty} g(t, x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1, X_2(t)) \\ = & -\lim_{n\to+\infty}\int_{-\infty}^{\infty} \nabla g(t, x_1)\nabla_1 f_n(x_1, X_2(t))dx_1 \\ = & -\int_{-\infty}^{\infty} \nabla g(t, x_1)\nabla_1^- f(x_1, X_2(t))dx_1\ a.s.. \end{aligned} \tag{5.3.13}$$

Note $\nabla_1^- f(x_1, x_2)$ is of locally bounded variation in x_1 and $g(t, x_1)$ has a compact support in x_1 and Riemann-Stieltjes integrable with respect to $\nabla^- f$, so

$$-\int_{-\infty}^{+\infty} \nabla g(t, x_1) \nabla_1^- f(x_1, X_2(t)) dx_1 = \int_{-\infty}^{+\infty} g(t, x_1) \mathrm{d}_{x_1} \nabla_1^- f(x_1, X_2(t)).$$

Thus

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} g(t, x_1) \mathrm{d}_{x_1} \nabla_1 f_n(x_1, X_2(t)) = \int_{-\infty}^{\infty} g(t, x_1) \mathrm{d}_{x_1} \nabla_1^- f(x_1, X_2(t)). \quad (5.3.14)$$

(C) If $g(s, x_1)$ is C^2 in x_1, $\Delta g(s, x_1)$ is bounded uniformly in s, $\frac{\partial}{\partial s} \nabla g(s, x_1)$ is continuous in s and has a compact support in x_1, and $E\left[\int_0^t \int_{R^2} |g(s, x) g(s, y)| |\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\right] < \infty$, where $h \in \mathcal{V}_2$, then applying Lebesgue's dominated convergence theorem and Proposition 5.2.1 and the integration by parts formula,

$$\begin{aligned}
& \lim_{n \to +\infty} \int_{-\infty}^{+\infty} \int_0^t g(s, x_1) \mathbf{d}_{s, x_1} \nabla_1 f_n(x_1, X_2(s)) \\
= & -\lim_{n \to +\infty} \int_{-\infty}^{\infty} \int_0^t \nabla g(s, x_1) \mathrm{d}_s \nabla_1 f_n(x_1, X_2(s)) dx_1 \\
= & -\lim_{n \to +\infty} \Big(\int_{-\infty}^{\infty} \nabla g(s, x_1) \nabla_1 f_n(x_1, X_2(s)) \Big|_0^t dx_1 \\
& \qquad - \int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \nabla g(s, x_1) \nabla_1 f_n(x_1, X_2(s)) dx_1 \mathrm{d}s \Big) \\
= & -\int_{-\infty}^{\infty} \nabla g(s, x_1) \nabla_1^- f(x_1, X_2(s)) \Big|_0^t dx_1 \\
& + \int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \nabla g(s, x_1) \nabla_1^- f(x_1, X_2(s)) dx_1 \mathrm{d}s \\
= & -\int_{-\infty}^{+\infty} \int_0^t \nabla g(s, x_1) \mathrm{d}_s \nabla_1^- f(x_1, X_2(s)) dx_1 \\
= & \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \mathbf{d}_{s, x_1} \nabla_1^- f(x_1, X_2(s)) \ \ a.s.,
\end{aligned}$$

i.e.

$$\begin{aligned}
& \lim_{n \to +\infty} \int_{-\infty}^{+\infty} \int_0^t g(s, x_1) \mathrm{d}_{s, x_1} \nabla_1 f_n(x_1, X_2(s)) \\
= & \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \mathrm{d}_{s, x_1} \nabla_1^- f(x_1, X_2(s)) \ \ a.s.. \qquad (5.3.15)
\end{aligned}$$

(D) In the following we will prove that (5.3.14) also holds for any continuous function $g(t, x_1)$ with a compact support in x_1. Moreover, if $g \in \mathcal{V}_3$ and continuous, (5.3.15) also holds.

To see (5.3.14), first note any continuous function with a compact support can be approximated by smooth functions with a compact support uniformly by the following

standard smoothing procedure

$$g_m(t,x_1)=\int_{-\infty}^{\infty}\rho_m(y-x_1)g(t,y)dy=\int_0^2\rho(z)g(t,x_1+\frac{z}{m})dz.$$

Note that there is a compact set $G\subset R^1$ such that

$$\begin{aligned}\max_{x_1\in G}|g_m(t,x_1)-g(t,x_1)|\to 0 \quad &\text{as} \quad m\to+\infty,\\ g_m(t,x_1)=g(t,x_1)=0 \quad &\text{for} \quad x_1\notin G.\end{aligned}$$

Note

$$\begin{aligned}\int_{-\infty}^{+\infty}g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) &= \int_{-\infty}^{+\infty}g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \\ &\quad+\int_{-\infty}^{+\infty}(g(t,x_1)-g_m(t,x_1))\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)).\end{aligned}\tag{5.3.16}$$

It is easy to see from (5.3.14) and Lebesgue's dominated convergence theorem, that

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}\int_{-\infty}^{+\infty}g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\\ =\ &\lim_{m\to\infty}\int_{-\infty}^{\infty}g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t))\\ =\ &\int_{-\infty}^{\infty}g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t))\ a.s..\end{aligned}\tag{5.3.17}$$

Moreover,

$$\begin{aligned}&\Big|\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\Big|\\ \le\ &\Big(\max_{x_1\in G}|g(t,x_1)-g_m(t,x_1)|\Big)Var_{x_1\in G}\nabla_1 f_n(x_1,X_2(t)).\end{aligned}\tag{5.3.18}$$

But,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\Big(\max_{x_1\in G}|g(t,x_1)-g_m(t,x_1)|\Big)Var_{x_1\in G}\nabla_1 f_n(x_1,X_2(t))=0\ a.s..$$

So inequality (5.3.18) leads to

$$\lim_{m\to\infty}\limsup_{n\to\infty}\Big|\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\Big|=0\ a.s..\tag{5.3.19}$$

Now we use (5.3.16), (5.3.17) and (5.3.19)

$$\begin{aligned}&\limsup_{n\to\infty}\int_{-\infty}^{+\infty}g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\\ =\ &\lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty}g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\\ &+\lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))\\ =\ &\int_{-\infty}^{\infty}g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t))\ a.s..\end{aligned}$$

Similarly we also have

$$\liminf_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) = \int_{-\infty}^{\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t))\ a.s.. \quad (5.3.20)$$

So (5.3.14) holds for a continuous function g with a compact support in x_1.

Now we prove that (5.3.15) also holds for a continuous function $g \in \mathcal{V}_3$. Define

$$g_m(s,x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_m(y-x_1)\rho_m(\tau-s)g(\tau,y)d\tau dy.$$

Then there is a compact $G \subset R^1$ such that

$$\begin{aligned} \max_{0\le s\le t, x_1\in G} |g_m(s,x_1) - g(s,x_1)| &\to 0 \quad \text{as} \quad m\to+\infty, \\ g_m(s,x_1) = g(s,x_1) &= 0 \qquad\qquad \text{for} \quad x_1 \notin G. \end{aligned}$$

Then it is trivial to see

$$\begin{aligned} &\int_0^t\int_{-\infty}^{+\infty} g(s,x_1)\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ =\ &\int_0^t\int_{-\infty}^{+\infty} g_m(s,x_1)\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ &+\int_0^t\int_{-\infty}^{+\infty} (g(s,x_1)-g_m(s,x_1))\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)). \end{aligned}$$

But from (5.3.15), we can see that

$$\begin{aligned} &\lim_{m\to\infty}\lim_{n\to\infty}\int_0^t\int_{-\infty}^{+\infty} g_m(s,x_1)\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ =\ &\lim_{m\to\infty}\int_0^t\int_{-\infty}^{+\infty} g_m(s,x_1)\mathrm{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s))\ \ a.s. \\ =\ &\int_0^t\int_{-\infty}^{+\infty} g(s,x_1)\mathrm{d}_{s,x_1}\nabla_1 f(x_1,X_2(s)). \qquad (\textit{limit in } \mathcal{M}_2) \end{aligned} \quad (5.3.21)$$

The last limit holds because of the following:

$$\begin{aligned} &E\Big[\int_{-\infty}^{+\infty}(g_m(s,x_1)-g(s,x_1))\mathrm{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s))\Big]^2 \\ =\ &E\Big[\int_0^t\int_{R^2}(g_m-g)(s,a)(g_m-g)(s,b)\mathrm{d}_{a,b,s} < \nabla_1^- f(a),\nabla_1^- f(b) >_s\Big] \\ =\ &E\Big[\int_0^t\int_{-\infty}^{+\infty}(g_m-g)(s,a)(g_m-g)(s,b) \\ &\quad \mathrm{d}_{a,b}\nabla_1^-\nabla_2^- f(a,X_2(s))\nabla_1^-\nabla_2^- f(b,X_2(s))\Big]\mathrm{d} <M_2>_s \\ =\ &E\int_0^t\Big[\int_{-\infty}^{+\infty}(g_m-g)(s,a)\mathrm{d}_a\nabla_1^-\nabla_2^- f(a,X_2(s))\Big]^2\mathrm{d} <M_2>_s \\ \to\ &0,\ as\ m\to\infty. \end{aligned}$$

On the other hand, in $\mathcal{M}_2$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_0^t\int_{-\infty}^{+\infty}(g(s,x_1)-g_m(s,x_1))\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s))=0. \tag{5.3.22}$$

In fact,

$$\begin{aligned} & E\Big[\int_0^t\int_{-\infty}^{+\infty}(g(s,x_1)-g_m(s,x_1))\mathrm{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s))\Big]^2 \\ = \; & E\int_0^t\Big[\int_{-\infty}^{+\infty}(g-g_m)(s,a)\mathrm{d}_a\nabla_1\nabla_2 f_n(a,X_2(s))\Big]^2 d<M_2>_s . \end{aligned}$$

Noting that $\nabla_1\nabla_2 f_n(a,X_2(s))$ is of bounded variation in a, we can use an argument similar to the one in the proof of (5.3.19) and (5.3.20) to prove (5.3.22).

(E) Now we use the multi-dimensional Itô's formula to the function $f_n(X(s))$, then a.s.

$$\begin{aligned} & f_n(X(t))-f_n(X(0)) \\ = \; & \sum_{i=1}^{2}\int_0^t\nabla_i f_n(X(s))dX_i(s)+\frac{1}{2}\int_0^t\Delta_1 f_n(X(s))d<M_1>_s \\ & +\frac{1}{2}\int_0^t\Delta_2 f_n(X(s))d<M_2>_s+\int_0^t\nabla_1\nabla_2 f_n(X(s))d<M_1,M_2>_s . \end{aligned} \tag{5.3.23}$$

As $n\to\infty$, it is easy to see from Lebesgue's dominated convergence theorem and (5.3.9), (5.3.10), (5.3.11), (5.3.12) that, $(i=1,2)$

$$\begin{aligned} f_n(X(t))-f_n(X(0)) &\to f(X(t))-f(X(0)) \;\; a.s., \\ \int_0^t\nabla_i f_n(X(s))dV_i(s) &\to \int_0^t\nabla_i^- f(X(s))dV_i(s) \;\; a.s., \\ \int_0^t\nabla_i\nabla_j f_n(X(s))d<M_1,M_2>_s &\to \int_0^t\nabla_i^-\nabla_j^- f(X(s))d<M_1,M_2>_s \;\; a.s.\ (i,j=1,2,i\neq j) \end{aligned}$$

and

$$E\int_0^t(\nabla_i f_n(X(s)))^2 d<M_i>_s\to E\int_0^t(\nabla_i^- f(X(s))^2 d<M_i>_s .$$

Therefore in $\mathcal{M}_2$,

$$\int_0^t\nabla_i f_n(X(s))dM_i(s)\to\int_0^t\nabla_i^- f(X(s))dM_i(s),(i=1,2).$$

To see the convergence of $\frac{1}{2}\int_0^t\Delta_1 f_n(X(s))d<M_1>_s$, first from integration by parts formula and (5.2.13), we have

$$\begin{aligned} \frac{1}{2}\int_0^t\Delta_1 f_n(X(s))d<M_1>_s &= \int_{-\infty}^{+\infty}\int_0^t\Delta_1 f_n(a,X_2(s))\mathrm{d}_s L_1(s,a)\mathrm{d}a \\ &= \int_{-\infty}^{+\infty}L_1(t,a)\mathrm{d}_a\nabla_1 f_n(a,X_2(t)) \\ &\quad -\int_{-\infty}^{+\infty}\int_0^t L_1(s,a)\mathrm{d}_{s,a}\nabla_1 f_n(a,X_2(s)). \end{aligned}$$

But local time $L_1(s,a)$ can be decomposed as

$$L_1(s,a) = \tilde{L}_1(s,a) + \sum_{x_k^* \le a} \hat{L}_1(s,x_k^*) := \tilde{L}_1(s,a) + \bar{L}_1(s,a), \tag{5.3.24}$$

where $\tilde{L}_1(s,a)$ is jointly continuous in s, a, and $\{x_k^*\}$ are the discontinuous points of $L_1(s,a)$. From (D) and (5.3.5), we have as $n \to \infty$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \tilde{L}_1(t,a)\mathrm{d}_a \nabla_1 f_n(a, X_2(t)) - \int_{-\infty}^{+\infty}\int_0^t \tilde{L}_1(s,a)\mathbf{d}_{s,a}\nabla_1 f_n(a,X_2(s)) \\
\to\ &\int_{-\infty}^{\infty} \tilde{L}_1(t,a)\mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty}\int_0^t \tilde{L}_1(s,a)\mathbf{d}_{s,a}\nabla_1^- f(a,X_2(s)) \quad (\textit{limit in } \mathcal{M}_2) \\
=\ &-\int_{-\infty}^{\infty}\int_0^t \nabla_1^- f(a,X_2(s))\mathrm{d}_{s,a}\tilde{L}_1(s,a).
\end{aligned} \tag{5.3.25}$$

On the other hand, from Lemma 1.2.2, we know that $\bar{L}_1(s,a)$ is of bounded variation in a for each s and of bounded variation in (s,a) for almost every $\omega \in \Omega$. And also because $\nabla_1 f_n(a,X_2(s))$ is continuous in (s,a), $\int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a,X_2(s))\mathbf{d}_{s,a}\bar{L}_1(s,a)$ is Riemann-Stieltjes integral. Hence in (5.3.4), replacing $\tilde{L}_1(s,a)$ by $\bar{L}_1(s,a)$, $g_1(s,a)$ by $\nabla_1 f_n(a,X_2(s))$, we still can obtain an integration by parts formula as follows

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a)\mathbf{d}_{s,a}\nabla_1 f_n(a,X_2(s)) \\
=&\int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a,X_2(s))\mathbf{d}_{s,a}\bar{L}_1(s,a) + \int_{-\infty}^{\infty} \bar{L}_1(t,a) d_a \nabla_1 f_n(a,X_2(t))
\end{aligned}$$

Note here the integral $\int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a)\mathbf{d}_{s,a}\nabla_1 f_n(a,X_2(s))$ is also a Riemann-Stieltjes integral though it is stochastic. Therefore

$$\begin{aligned}
&\int_{-\infty}^{\infty} \bar{L}_1(t,a) d_a \nabla_1 f_n(a,X_2(t)) - \int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a)\mathbf{d}_{s,a}\nabla_1 f_n(a,X_2(s)) \\
=\ &-\int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a,X_2(s))\mathbf{d}_{s,a}\bar{L}_1(s,a) \\
\to\ &-\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a,X_2(s))\mathbf{d}_{s,a}\bar{L}_1(s,a)
\end{aligned} \tag{5.3.26}$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. So by (5.3.25) and (5.3.26),

$$\frac{1}{2}\int_0^t \Delta_1 f_n(X(s)) d <M_1>_s \to -\int_{-\infty}^{\infty}\int_0^t \nabla_1^- f(x_1, X_2(t))\mathbf{d}_{s,x_1} L_1(s,x_1),$$

as $n \to \infty$. The term $\frac{1}{2}\int_0^t \Delta_2 f_n(s,X(s)) d <M_2>_s$ can be treated similarly. So we proved the desired formula. ◇

The following theorem gives the new representation of $f(X_t)$, which leads to integration by parts formula for integrations of local times.

Theorem 5.3.2 *Under conditions (i)-(iv), for any continuous two-dimensional semi-martingale $X(t) = (X_1(t), X_2(t))$, we have almost surely*

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) \\ &\quad + \int_{-\infty}^{\infty} L_1(t,a) \mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t L_1(s,a) \mathrm{d}_{s,a} \nabla_1^- f(a, X_2(s)) \\ &\quad + \int_{-\infty}^{\infty} L_2(t,a) \mathrm{d}_a \nabla_2^- f(X_1(t), a) - \int_{-\infty}^{+\infty} \int_0^t L_2(s,a) \mathrm{d}_{s,a} \nabla_2^- f(X_1(s), a) \\ &\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^t \nabla_i^- \nabla_j^- f(X(s)) d < M_1, M_2 >_s . \end{aligned} \tag{5.3.27}$$

In particular, from (5.3.5), (5.3.6), we have the integration by parts formulae

$$\begin{aligned} & \int_{-\infty}^{\infty} g(t,a) \mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t g(s,a) \mathrm{d}_{s,a} \nabla_1^- f(a, X_2(s)) \\ = & - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f(a, X_2(s)) \mathrm{d}_{s,a} g(s,a), \end{aligned}$$

for $g(s,a) = L_1(s,a), \tilde{L}_1(s,a), \bar{L}_1(s,a)$ respectively.

Proof: For (5.3.27), we only need to prove the convergence in (5.3.25) holds for $\bar{L}_1(s,x)$. First let's prove, when $n \to \infty$, in $\mathcal{M}_2$,

$$\int_{-\infty}^{+\infty} \int_0^t \tilde{L}_1(s,a) \mathrm{d}_{s,a} \nabla_1 f_n(a, X_2(s)) \to \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathrm{d}_{s,a} \nabla_1^- f(a, X_2(s)).$$

From the assumption of $\nabla_1^- f$ and the definition of f_n, recall (5.3.2) and from Itô's formula we have $\nabla_1^- f(a, X_2(t)) = \nabla_1^- f(a, X_2(0)) + h(t,a) + v(t,a)$, $\nabla_1 f_n(a, X_2(t)) = \nabla_1 f_n(a, X_2(0)) + h_n(t,a) + v_n(t,a)$, where h_n, h are continuous local martingales and v_n, v are continuous processes with locally bounded variation (in t). From previous computations, we know that $h_n, h \in \mathcal{V}_2$, i.e. $< (h_n - h)(a), (h_n - h)(b) >_s$ is of bounded variation in (s,a,b) and $v_n(s,a)$, $v(s,a)$ are of bounded variation in (s,a). So

$$\begin{aligned} & E \Big| \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathrm{d}_{s,a} h_n(s,a) - \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathrm{d}_{s,a} h(s,a) \Big|^2 \\ = & \; E \int_0^t \int_{R^2} \bar{L}_1(s,a) \bar{L}_1(s,b) \mathrm{d}_{a,b,s} < h_n(a) - h(a), h_n(b) - h(b) >_s . \end{aligned}$$

Let $(-N, N)$ covers the compact support of local time $L_1(t, \cdot)$, N is fixed for each ω, and

$$\begin{aligned} & G(s,a,b) := \bar{L}_1(s,a) \bar{L}_1(s,b) \\ & G(a,b)_{s_k}^{s_{k+1}} := \bar{L}_1(s_{k+1}, a) \bar{L}_1(s_{k+1}, b) - \bar{L}_1(s_k, a) \bar{L}_1(s_k, b) \\ & H_n(s,a,b) := < h_n(a) - h(a), h_n(b) - h(b) >_s . \end{aligned}$$

We can show that $G(s,a,b)$ is of bounded variation in (s,a,b). In fact, let $\mathcal{P}$ be a partition on $[-N,N]^2\times[0,t]$, where $\mathcal{P}_i$ is a partition on $[-N,N]$ $(i=1,2)$, $\mathcal{P}_3$ is a partition on $[0,t]$ such that $\mathcal{P}=\mathcal{P}_1\times\mathcal{P}_2\times\mathcal{P}_3$, then

$$
\begin{aligned}
&\mathrm{Var}_{s,a,b}G(s,a,b)\\
=\ &\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big|G(a_{i+1},b_{j+1})_{s_k}^{s_{k+1}}-G(a_{i+1},b_j)_{s_k}^{s_{k+1}}-G(a_i,b_{j+1})_{s_k}^{s_{k+1}}+G(a_i,b_j)_{s_k}^{s_{k+1}}\Big|\\
=\ &\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big|\bar{L}_1(s_{k+1},a_{i+1})\bar{L}_1(s_{k+1},b_{j+1})-\bar{L}_1(s_k,a_{i+1})\bar{L}_1(s_k,b_{j+1})\\
&\qquad-\bar{L}_1(s_{k+1},a_i)\bar{L}_1(s_{k+1},b_{j+1})+\bar{L}_1(s_k,a_i)\bar{L}_1(s_k,b_{j+1})\\
&\qquad-\bar{L}_1(s_{k+1},a_{i+1})\bar{L}_1(s_{k+1},b_j)+\bar{L}_1(s_k,a_{i+1})\bar{L}_1(s_k,b_j)\\
&\qquad+\bar{L}_1(s_{k+1},a_i)\bar{L}_1(s_{k+1},b_j)-\bar{L}_1(s_k,a_i)\bar{L}_1(s_k,b_j)\Big|\\
=\ &\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big|(\bar{L}_1(s_{k+1},a_{i+1})-\bar{L}_1(s_{k+1},a_i))(\bar{L}_1(s_{k+1},b_{j+1})-\bar{L}_1(s_{k+1},b_j))\\
&\qquad-(\bar{L}_1(s_k,a_{i+1})-\bar{L}_1(s_k,a_i))(\bar{L}_1(s_k,b_{j+1})-\bar{L}_1(s_k,b_j))\Big|\\
=\ &\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big|[(\bar{L}_1(s_{k+1},a_{i+1})-\bar{L}_1(s_{k+1},a_i))-(\bar{L}_1(s_k,a_{i+1})-\bar{L}_1(s_k,a_i))]\\
&\qquad\cdot(\bar{L}_1(s_{k+1},b_{j+1})-\bar{L}_1(s_{k+1},b_j))\\
&\qquad+(\bar{L}_1(s_k,a_{i+1})-\bar{L}_1(s_k,a_i))\\
&\qquad\cdot[(\bar{L}_1(s_{k+1},b_{j+1})-\bar{L}_1(s_{k+1},b_j))-(\bar{L}_1(s_k,b_{j+1})-\bar{L}_1(s_k,b_j))]\Big|\\
\le\ &\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big[\sum_{a_i<x_m^*\le a_{i+1}}\int_{s_k}^{s_{k+1}}1_{\{x_m^*\}}(X_s)dV_s\Big]\cdot\Big[\sum_{b_j<x_m^*\le b_{j+1}}\int_0^t1_{\{x_m^*\}}(X_s)dV_s\Big]\\
&+\sup_{\mathcal{P}}\sum_k\sum_i\sum_j\Big[\sum_{a_i<x_m^*\le a_{i+1}}\int_0^t1_{\{x_m^*\}}(X_s)dV_s\Big]\cdot\Big[\sum_{b_j<x_m^*\le b_{j+1}}\int_{s_k}^{s_{k+1}}1_{\{x_m^*\}}(X_s)dV_s\Big]\\
=\ &2\Big(\sum_{-N<x_m^*\le N}\int_0^t1_{\{x_m^*\}}(X_s)dV_s\Big)^2\\
\le\ &2\Big(\int_0^t1_{(-N,N]}(X_s)|dV_s|\Big)^2\\
<\ &\infty.
\end{aligned}
$$

Define

$$
\begin{aligned}
\tilde{G}_1(s,a,b)&:=V_G([0,s]\times[-N,a]\times[-N,b])+G(s,a,b),\\
\tilde{G}_2(s,a,b)&:=V_G([0,s]\times[-N,a]\times[-N,b])-G(s,a,b),
\end{aligned}
$$

where $V_G([0,s]\times[-N,a]\times[-N,b])$ denotes the total variation of G on $[0,s]\times[-N,a]\times[-N,b]$. Then it's easy to see that $G(s,a,b)=\frac{1}{2}[\tilde{G}_1(s,a,b)-\tilde{G}_2(s,a,b)]$, and $\tilde{G}_1$, $\tilde{G}_2$ are nondecreasing in (s,a,b). Moreover, by additivity of variation, one can see that for

$s_2 \geq s_1$,

$$\begin{aligned} & \tilde{G}_1(s_2,a,b) - \tilde{G}_1(s_1,a,b) \\ = \; & V_G([s_1,s_2]\times[-N,x]\times[-N,y]) + G(s_2,a,b) - G(s_1,a,b) - G(s_2,a,-N) \\ & + G(s_1,a,-N) - G(s_2,-N,b) + G(s_1,a,-N) - G(s_2,-N,-N) + G(s_1,-N,-N) \\ \geq \; & 0. \end{aligned}$$

That is to say, $\tilde{G}_1(s,a,b)$ is increasing in s for each a and b. Also for any $a_2 \leq a_1$,

$$\begin{aligned} & \tilde{G}_1(s,a_2,b) - \tilde{G}_1(s,a_1,b) \\ = \; & V_G([0,s]\times[a_1,a_2]\times[-N,y]) + G(s,a_2,b) - G(s,a_1,b) - G(0,a_2,b) + G(0,a_1,b) \\ & - G(s,a_2,-N) + G(s,a_1,-N) + G(0,a_2,-N) - G(0,a_1,-N) \\ \geq \; & 0. \end{aligned}$$

So $\tilde{G}_1(s,a,b)$ is nondecreasing in a for each s and b. In the same way, $\tilde{G}_1(s,a,b)$ is nondecreasing in b for each s and a. Therefore $\tilde{G}_1(s,a,b)$ is nondecreasing in s, a, b respectively. Similarly, $\tilde{G}_2(s,a,b)$ is also nondecreasing in s, a, b respectively. Define

$$\begin{aligned} G_1(s,a,b) &= \lim_{s'\downarrow s, a'\downarrow a, b'\downarrow b} \tilde{G}_1(s',a',b') \\ G_2(s,a,b) &= \lim_{s'\downarrow s, a'\downarrow a, b'\downarrow b} \tilde{G}_2(s',a',b'). \end{aligned}$$

So G_1 and G_2 are right continuous in (s,a,b), and nondecreasing in s, a, b separately, and $G(s,a,b) = \frac{1}{2}[G_1(s,a,b) - G_2(s,a,b)]$. Now we claim for any $c > 0$,

$$A = \{(s,a,b) : G_1(s,a,b) < c\}$$

is an open set. To see this, for any $(s,a,b) \in A$, take $\varepsilon = \frac{1}{2}(c - G_1(s,a,b)) > 0$. First as $G(s,a,b)$ is right continuous in (s,a,b), so there exists $\delta > 0$ such that

$$|G_1(s',a',b') - G_1(s,a,b)| < \varepsilon,$$

when $s \leq s' < s+\delta$, $a \leq a' < a+\delta$, $b \leq b' < b+\delta$. That is to say, $[s,s+\delta)\times[a,a+\delta)\times[b,b+\delta) \subset A$. But for any $s' \leq s$, $a' \leq a$, $b' \leq b$,

$$G_1(s',a',b') \leq G_1(s,a',b') \leq G_1(s,a,b') \leq G_1(s,a,b) < c.$$

Therefore, $(-\infty,s+\delta)\times(-\infty,a+\delta)\times(-\infty,b+\delta) \in A$. This implies that A is an open set. Thus for any $c \geq 0$,

$$\{(s,a,b) : G_1(s,a,b) \geq c\}$$

is a closed set.

From the assumption, we know $H_n(s,a,b)$ is of bounded variation in (s,a,b) and when $n \to \infty$, $H_n \to 0$. We only consider the increasing part of H_n, still denote it by H_n. $H_n(s,a,b)$ is left continuous and increasing, so it generates Lebesgue-Stieltjes measure, denote it by μ_n. It's easy to see that $\mu_n([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2)) \to 0$, as $n \to \infty$, for any $[s_1,s_2)\times[a_1,a_2)\times[b_1,b_2) \subset [0,t]\times[-N,N]^2$. So $\mu_n \xrightarrow{W} 0$, as $n \to \infty$. Let P be a probability measure on $[0,t]\times[-N,N]^2$ and

$$P_n([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2)) = \frac{(P+\mu_n)([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2))}{(P+\mu_n)([0,t]\times[-N,N]\times[-N,N])}.$$

Then $P_n \xrightarrow{W} P$. Therefore, by the equivalent condition of weak convergence (cf. Proposition 1.2.4 in [19]), for any closed set E, $\limsup_{n\to\infty} P_n(E) \le P(E)$. Now without losing generality, we assume $0 \le G_1(s,a,b) \le 1$. Using the method of Proposition 1.2.4 in [19], we have for either $Q = P_n$ or P,

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{i-1}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\} \\
\le\ & \int_0^t \int_{-N}^{N} \int_{-N}^{N} G_1(s,a,b) Q(dsdadb) \\
\le\ & \sum_{i=1}^{k} \frac{i}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{i}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\} \\
=\ & \sum_{i=0}^{k-1} \frac{1}{k} Q\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\}.
\end{aligned}$$

But $E_i := \{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\}$ is closed, so

$$\limsup_{n\to\infty} P_n(E_i) \le P(E_i), \quad i = 0,1,\cdots,k-1.$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^t \int_{-N}^{N} \int_{-N}^{N} G_1(s,a,b) P_n(dsdadb) \\
\le\ & \limsup_{n\to\infty} \sum_{i=0}^{k-1} \frac{1}{k} P_n\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\} \\
\le\ & \sum_{i=0}^{k-1} \frac{1}{k} P\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\} \\
\le\ & \frac{1}{k} + \int_0^t \int_{-N}^{N} \int_{-N}^{N} G_1(s,a,b) P(dsdadb).
\end{aligned}$$

As k is arbitrary, so

$$\begin{aligned}&\limsup_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P_n(dsdadb)\\ \leq\ &\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P(dsdadb).\end{aligned}$$

Applying above to $1-G_1(s,a,b)$, we can prove

$$\begin{aligned}&\liminf_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P_n(dsdadb)\\ \geq\ &\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P(dsdadb).\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P_n(dsdadb)\\ =\ &\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)P(dsdadb).\end{aligned}$$

So,

$$\lim_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G_1(s,a,b)\mu_n(dsdadb)=0.$$

We can do the same thing to $G_2(s,a,b)$, and get

$$\lim_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G_2(s,a,b)\mu_n(dsdadb)=0.$$

Thus,

$$\lim_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G(s,a,b)\mu_n(dsdadb)=0.$$

But when $H_n(s,a,b)$ is of bounded variation in (s,a,b), it can be decomposed to two increasing functions. Therefore, we have

$$\lim_{n\to\infty}\int_0^t\int_{-N}^N\int_{-N}^N G(s,a,b)\mathbf{d}_{a,b,s}H_n(s,a,b)=0.$$

Hence, when $n\to\infty$, in $\mathcal{M}_2$

$$\int_{-\infty}^{+\infty}\int_0^t \bar{L}_1(s,a)\mathbf{d}_{s,a}h_n(s,a)\to\int_{-\infty}^{+\infty}\int_0^t \bar{L}_1(s,a)\mathbf{d}_{s,a}h(s,a).$$

We can also easily prove that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_0^t \bar{L}_1(s,a)\mathbf{d}_{s,a}v_n(s,a)\to\int_{-\infty}^{+\infty}\int_0^t \bar{L}_1(s,a)\mathbf{d}_{s,a}v(s,a),\\ &\int_{-\infty}^{+\infty}\bar{L}_1(t,a)d_a\nabla_1 f_n(a,X_2(t))\to\int_{-\infty}^{+\infty}\bar{L}_1(t,a)d_a\nabla_1^- f(a,X_2(t)).\end{aligned}$$

Similarly we can deal with the terms with $\bar{L}_2(s,a)$. So (5.3.27) is proved and the integration by parts formulae follow easily. ◇

The smoothing procedure in Theorem 5.3.1 can be used to prove that if $f : R\times R \to R$ is absolutely continuous in x_1, x_2 respectively and locally bounded, C^1 in x_1 and x_2, and the left derivatives $\frac{\partial^{2-}}{\partial x_i \partial x_j} f(x_1,x_2)$, $(i,j = 1,2)$ exist and are locally bounded and left continuous, then

$$\begin{aligned} & f(X(t)) - f(X(0)) \\ = & \sum_{i=1}^{2} \int_0^t \nabla_i f(X(s)) dX_i(s) + \frac{1}{2} \sum_{i,j=1}^{2} \int_0^t \frac{\partial^{2-}}{\partial x_i \partial x_j} f(X(s)) d <X_i, X_j>_s . \quad (5.3.28) \end{aligned}$$

This can be seen from the convergence in the proof of Theorem 5.3.1 and the fact that $\frac{\partial^2}{\partial x_i \partial x_j} f_n(x_1,x_2) \to \frac{\partial^{2-}}{\partial x_i \partial x_j} f(x_1,x_2)$ under the stronger condition on $\frac{\partial^{2-}}{\partial x_i \partial x_j} f$.

The next theorem is an easy consequence of the methods of the proofs of Theorem 5.3.1 and (5.3.28).

Theorem 5.3.3 *Let $f : R \times R \to R$ satisfy conditions (i) and $f(x_1,x_2) = f_h(x_1,x_2) + f_v(x_1,x_2)$. Assume f_h is C^1 in x_1, x_2 and the left derivatives $\frac{\partial^{2-}}{\partial x_i \partial x_j} f_h(x_1,x_2)(i,j = 1,2)$ exist and are left continuous and locally bounded; f_v satisfies conditions (ii)-(iv). Then*

$$\begin{aligned} & f(X(t)) - f(X(0)) \\ = & \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) + \frac{1}{2} \sum_{i=1}^{2} \int_0^t \Delta_i^- f_h(X(s)) d <X_i>_s \\ & - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f_v(a, X_2(s)) \mathrm{d}_{s,a} L_1(s,a) - \int_{-\infty}^{+\infty} \int_0^t \nabla_2^- f_v(X_1(s), a) \mathrm{d}_{s,a} L_2(s,a) \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^t \nabla_i^- \nabla_j^- f(X(s)) d <M_1, M_2>_s \\ = & \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) + \frac{1}{2} \sum_{i=1}^{2} \int_0^t \Delta_i^- f_h(X(s)) d <X_i>_s \\ & + \int_{-\infty}^{\infty} L_1(t,a) \mathrm{d}_a \nabla_1^- f_v(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t L_1(s,a) \mathrm{d}_{s,a} \nabla_1^- f_v(a, X_2(s)) \\ & + \int_{-\infty}^{\infty} L_2(t,a) \mathrm{d}_a \nabla_2^- f_v(X_1(t), a) - \int_{-\infty}^{+\infty} \int_0^t L_2(s,a) \mathrm{d}_{s,a} \nabla_2^- f_v(X_1(s), a) \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{2} \int_0^t \nabla_i^- \nabla_j^- f(X(s)) d <M_1, M_2>_s \quad a.s.. \end{aligned} \quad (5.3.29)$$

Example 5.3.1 *Consider*

$$f(x_1,x_2)=(x_1x_2)^+.$$

It's easy to see that

$$\begin{aligned}\nabla_1^- f(x_1,x_2) &= x_2 1_{\{x_1x_2>0\}}1_{\{x_2>0\}}+x_2 1_{\{x_1x_2\le 0\}}1_{\{x_2\le 0\}}\\ &= x_2 1_{\{x_1>0\}}1_{\{x_2>0\}}+x_2 1_{\{x_1\le 0\}}1_{\{x_2\le 0\}}\\ &= x_2^+ 1_{\{x_1>0\}}-x_2^- 1_{\{x_1\le 0\}},\end{aligned}$$

so $\Delta_1^- f(0,x_2)=\infty$, *which means that the classical Itô's formula doesn't work. But*

$$\nabla_1^-\nabla_2^- f(x_1,x_2)=1_{\{x_1>0\}}1_{\{x_2>0\}}+1_{\{x_1\le 0\}}1_{\{x_2\le 0\}}.$$

This suggests our generalized Itô's formula can be used.

Example 5.3.2 *Consider*

$$f(x_1,x_2)=x_2^{\frac{1}{3}}(x_1x_2)^+.$$

It's easy to see that

$$\begin{aligned}\nabla_1^- f(x_1,x_2) &= x_2^{\frac{1}{3}}x_2^+ 1_{\{x_1>0\}}-x_2^{\frac{1}{3}}x_2^- 1_{\{x_1\le 0\}},\\ \nabla_2^- f(x_1,x_2) &= \frac{1}{3}x_2^{-\frac{2}{3}}(x_1x_2)^+ + x_2^{\frac{1}{3}}x_1^+ 1_{\{x_2>0\}}-x_2^{\frac{1}{3}}x_1^- 1_{\{x_2\le 0\}}\\ &= \frac{4}{3}x_2^{-\frac{2}{3}}(x_1^+x_2^+ + x_1^-x_2^-),\\ \Delta_2^- f(x_1,x_2) &= -\frac{8}{9}x_2^{-\frac{2}{3}}(x_1^+1_{\{x_2>0\}}-x_1^-1_{\{x_2<0\}})\\ &\quad +\frac{4}{3}x_2^{-\frac{2}{3}}(x_1^+1_{\{x_2>0\}}-x_1^-1_{\{x_2\le 0\}}),\\ \nabla_i^-\nabla_j^- f(x_1,x_2) &= \frac{4}{3}x_2^{\frac{1}{3}}1_{\{x_1x_2>0\}}+\frac{4}{3}x_2^{\frac{1}{3}}1_{\{x_1=0\}}1_{\{x_2<0\}},\ (i,j=1,2,i\neq j).\end{aligned}$$

So $\Delta_2^- f(x_1,0)=-\infty$ *when* $x_1<0$, *and* $\lim_{x_2\to 0-}\Delta_2^- f(x_1,x_2)=-\infty$ *when* $x_1<0$, $\lim_{x_2\to 0+}\Delta_2^- f(x_1,x_2)=\infty$ *when* $x_1>0$. *These calculations suggest that neither the classical Itô's formula, nor the formula in [39] can be applied immediately. But our generalized Itô's formula can be used here.*

Remark 5.3.1 *This Chapter is in the paper [12], which is submitted to Stochastic Processes and Their Applications. Applications e.g. in the study of the asymptotics of the solutions of heat equations with caustics in two dimensions, will be considered in future publications.*

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