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MATHEMATICAL MODELLING PROCESSES: IMPLICATIONS FOR TEACHING AND LEARNING

By

K H OKE, BSc, MTech, FIMA

A doctoral thesis submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of the Loughborough University of Technology, 1984

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ABSTRACT

Mathematical Modelling Processes: Implications for Teaching and Learning

by K H Oke

CAMET

(Centre for Advancement of Mathematical Education in Technology)

Loughborough University of Technology

The principal aim of the project has been to investigate formulation-solution processes and the extent to which these processes lead to better guidance and understanding of teaching, learning, and assessment in mathematical modelling. The following main activities have been carried out in support of this aim: the development of case studies of the mathematical modelling approaches that may be used in the solution of practical problems; the design of teaching and learning experiments carried out mainly with undergraduates with some knowledge of physics and teachers on an MSc course in mathematical education; the theoretical development of formulation-solution processes by means of a concept matrix and a relationship level graph; the analysis of a selection of students' modelling attempts; an investigation of assessment methods and the implications of the theoretical development of formulation-solution processes for these methods.

The case studies were based on possible modelling approaches to practical problems which are connected in some way with every-day reality. These studies were used in seventeen

experiments with students working in a genuine educational environment under the usual time constraints. Most of the students involved had little or no modelling experience. Results have shown that students have a common set of difficulties, and a set of learning heuristics has been devised in an attempt to overcome these.

The theoretical development of formulation-solution processes has identified the following main characteristics in early model development: distribution of features from global (difficult to quantify) to specific (easily quantified) concepts; basic relationships are often generated as solution proceeds; relationships can occur in either general or specific forms; general progress is gauged by relationship 'level'; most variables and constants are generated with relationships; partitioning a problem into sub-problems may be possible initially, but such break-down into distinct parts is often only possible after having seen a pattern of linkages in a relationship level graph.

Finally, the implications for assessment methods are examined, and suggestions for further research investigations are made.

Key words: Mathematical Modelling, Models, Formulation Processes, Problem Solving, Real Problems, Concept Matrix

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During the course of this project I have benefited considerably from the students who have taken part in numerous experiments and I thank them for their co-operation. I should also like to acknowledge the assistance I received from two colleagues, Mr A L Jones and Mr G A Wright, in connection with the simultaneous observation of groups in one experiment.

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CHAPTER 1

INTRODUCTION

1.1 Background

Mathematical modelling, as with all forms of modelling, is concerned with representations of reality. Its main purpose is to shed light, by mathematical means, on the myriad problems which confront us in science, engineering, economics, business and many other aspects of the world around us. It involves recognition of a problem, its formulation into mathematical terms, solving the resulting mathematical equations, and, finally, the interpretation of the mathematical solution in the context of the original problem. In short, mathematical modelling is about the creative activities involved in using mathematics to gain a better understanding of the problems of the real world.

The usefulness of mathematics has long been an important justification for the teaching of the subject, but, because of the complexity of the real world, most applications of mathematics have proved difficult to teach. The traditional approach has been to simplify any real practical problem and to offer students a simpler world to consider. In effect, the traditional 'applied mathematics' approach has been to present a problem 'on-a-plate' ready for immediate mathematical treatment without any of the difficulties of translating the original problem into mathematical form. Once a solution was obtained, it was either deemed to be correct or incorrect. Even if correct, no demands were placed on students to interpret the solution, to test its usefulness or in what sense was it useful.

In recent years there has been a growing realisation that the traditional approach has been inappropriate in providing a mathematical training for industry and commerce. McLone (1973) in

his report 'The training of mathematicians' was amongst the first to identify the need for change. In a survey of employers of graduate mathematicians, he found that whilst the young graduate was technically competent at solving mathematical equations if they were provided in a form ready for solution, the graduate was at a loss if, instead, only the original practical problem was posed. Furthermore, that graduates having obtained a solution were then unable to communicate their results to a non-mathematical specialist, for example an engineer, and consequently the solution proved to be of little value and was allowed to collect dust.

The modelling approach to the teaching of applied mathematics differs from the traditional approach in that is always begins, and ends, in the real world situations that are modelled. There is no unique way in tackling a real problem and several papers and books illustrate the various attempts that have been made to develop a common approach to modelling and ways of teaching modelling. In simple terms, the activities of mathematical modelling are now quite widely accepted as consisting of the stages of:

Formulation:	Identification of a problem (or problems) relating to a real situation, and then posing the problem(s) in mathematical form.
Solution:	Attempts to solve the resulting mathematical equations.
Interpretation:	Relating the mathematical solution to the original problem context.
Validation:	Checking the predictive or other uses of the model against a fairly wide range of circumstances in the original context. Data is often provided for this purpose or observation and experimentation is involved.

Very little research has been carried out into identifying the essential ingredients of each stage and the way these stages interact, either iteratively or otherwise. What little has been reported in the literature shows that problem solving research has provided the strongest initial guidance. The processes involved are usually portrayed in a flow-diagram, or similar representation, implying linear or linear with looping sequencing of stages. Only recently, drawing on the still pioneering work of systems analysis (in information processing), have non-linear approaches been suggested, and then only in the very broadest of terms.

Closely coupled with the problem of gaining a deeper understanding of modelling processes is the problem of how to teach modelling. Examples of initial attempts at developing a methodology of teaching at a variety of levels (mainly sixth form and undergraduate) are reported in the literature. Naturally enough, in these early days, descriptions of experiences gained in teaching modelling cover a broad canvas, but each identifies the formulation stage as the most difficult one to come to grips with for both student and lecturer alike. It would appear that this is the most creative part of modelling; most seem able, after a little practice, to identify variables but the real difficulty comes in selecting key variables and in considering relationships which connect these variables.

Since an initial attempt at finding a mathematical solution will depend very largely on an initial formulation then the formulation-solution interface poses a particular challenge. Although it is recognised in the literature that the linkage between formulation and solution is highly oscillatory, little or no known work has been published on the details of the linkage which could help the student in modelling.

The author of this thesis first gained an interest in mathematical modelling in the early 1970's by working with a colleague, P B Taylor, on stock control problems in the field of operational research. The work was aimed at developing more realistic stock control models which were both accurate over a wide range of parameter values, as well as being easy to implement in a variety

of management contexts. Soon after this work was completed, the author became involved in several pieces of consultancy work in engineering fields - mainly heat exchange processes and thermal stressing. The latter was a more natural development in view of a background training in mathematics, physics, and in particular, numerical analysis. Throughout this period, in the mid-1970's, the author, at the Polytechnic of the South Bank, developed a two-year part-time M.Sc. degree course in Mathematical Education for graduate teachers of mathematics in secondary schools and colleges of further education. The course has been running since 1977 and the author has supervised ten dissertation projects in mathematical modelling so far. This course is the only one of its kind in the public sector of higher education, although there are now a few broadly similar courses in universities. The first M.Sc. course in mathematical education in the UK was developed by Professor A C Bajpai at Loughborough University of Technology in the early 1970's. The Loughborough course distinguishes itself, inter alia, as being the first at postgraduate level to include an option in mathematical modelling. The South Bank course identifies with mathematical modelling more strongly by making it a major compulsory component in highlighting its importance as a link between mathematics and education.

Subsequently, the author has also been devising and teaching on modelling courses for undergraduates, mainly in the physics and engineering areas. This information is provided in order to indicate the breadth of personal interest involved.

1.2 Purpose and Scope of the Research Project

The main purpose of this research is to investigate formulation-solution processes in mathematical modelling and the extent to which knowledge of these processes leads to better guidance and understanding of teaching, learning, and assessment. Chapter 2 discusses the background to the growth of interest in mathematical modelling, and Chapter 3 identifies some of the most recent and significant research which is related to this project. Both Chapters 2 and 3 are intended to provide a perspective to Chapter 4 which provides full details of the rationale and philosophy of approach to this research project.

In order to be able to carry out such a project, the following main activities were chosen:

1 The development of case studies of the mathematical modelling approaches that may be used in the solution of practical problems.

2 The design of teaching and learning experiments with students, largely in higher education, working under genuine conditions of the classroom and workshop.

3 The development of two theoretical constructs:

<u>A concept matrix</u>

<u>A relationship level graph</u>

which are used in the analysis of formulation-solution processes.

4 The study of various assessment modes and the construction of marking schemes.

Although a study of the formulation-solution interface is fundamental in investigating mathematical modelling, it is stressed that it is not possible to separate this entirely from the other stages of modelling. Consequently, most of the case studies (introduced in Chapter 5) also involve the important interpretation and validation stages.

1 All the investigations have been carried out using 'deterministic' and 'analytical' approaches. Most of the case studies are based on problems in the areas of physical sciences and engineering rather than in the social and organisational sciences. The purpose of the case studies was to provide a set of problems with sufficient modelling potential for students in the teaching and learning experiments. The case studies were based on the following design features:

Motivation

Each problem is practical and is connected in some way with every-day reality (e.g. central-heating of a house).

Level of difficulty

Each problem has sufficient scope for simple initial approaches to give good insights, and also for the more advanced students in modelling to produce more comprehensive solutions.

Scope

Each cast study provides an opportunity for formulation-solution, interpretation, and as often as possible, validation.

Content

Each case study has a problem statement and possible model development. Sometimes data is provided, on other occasions students are encouraged to ask for data.

Duration of modelling exercise

Most case studies are appropriate for short-duration introductory work or for extended project development.

2 The overall approach to the teaching and learning experiments is summarised in Chapter 4 where the key interactions between lecturer (teacher) and modeller (student) are illustrated. The case studies referred to earlier were used in studies of seventeen different groups of students, and details of the investigations are provided in Chapter 6. The purpose of the teaching and learning experiments was essentially three-fold:

(i) To determine the level of difficulty of modelling problems for different student types.

(ii) To observe how students tackle modelling activities under a variety of working conditions:

Interactive (working with lecturer)

Group ('short' and 'long' duration)

Individual work

or, Combination of above.

(iii) To develop learning heuristics for the student inexperienced in modelling.

The experiments were carried out mainly with undergraduates with mathematics, physics/engineering backgrounds and with teachers on the M.Sc. course in mathematical education. Some common difficulties of students in 'short' to 'medium' duration modelling activities are identified, and a set of learning heuristics designed to help overcome some of these difficulties is developed. Student opinion has also been canvassed on the usefulness of the heuristics; as experience is gained, some heuristics are considered to be more useful than is considered to be the case by untrained students in modelling.

3 In order to try and understand more fully the highly complex processes involved in formulation-solution, the main focus of this project, two theoretical constructs have been devised: a concept matrix (CM), and a relationship level graph (RLG). The ideas involved are introduced in Chapter 4 and they are developed fully in Chapter 7. The CM is designed to show which features, or concepts, are used in different stages. The matrix is also intended to provide information on the type of each concept. Since the features which arise in the development of a mathematical model are extremely varied, both in clarity and in complexity, it was considered to be inappropriate to develop a simple hierarchy. The RLG is designed to show that mathematical solution and formulation are interwoven; additional ideas on the nature of the problem are generated as a mathematical solution is developed. Initial, and more or less obvious simple

relationships are denoted by the level 0 (zero). Mathematics is shown to be based on level 0 and is used to derive level 1, 2 ... relationships as well as prompt the need for further level 0 types. A selection of the teaching and learning experiments referred to earlier is analysed by means of the CM and RLG, and the following main points are identified and illustrated in each case:

Distribution of features

Basic relationship generation

Forms of relationships

Relationship 'level' as goal seeking (measure of progress made)

Generation of variables and constants

Sub-problem identification

The extent to which this new work is in agreement with some published ideas, and the extent to which it disagrees with some developments is also examined.

4 Finally, the implications of the work on formulation-solution processes for assessment are studied in Chapter 8. Comparisons are made between written examinations and course-work modes of assessment, as well as informal (impression) marking and the use of formal marking schemes. A credit list of modelling attributes which is based on the work of Chapter 7 has been devised, and its use in contributing towards the assessment of both examination papers and course-work assignments is illustrated.

1.3 Summary

This chapter has briefly described the background, purpose, and scope of the project. A more comprehensive discussion which overviews published developments on the nature of models, modelling methodologies, and recent interests in the teaching of mathematical modelling in schools, polytechnics and

universities is provided in Chapter 2. Some of the most recent significant research which is related to this project is identified in Chapter 3; problem solving and modelling processes and the way in which such processes underpin the development of teaching and learning styles is examined. The contents of Chapters 2 and 3 are also intended to provide a wider perspective as well as identify key research needs in the development of mathematical modelling. One of the main research needs, which is adopted as a focus for this thesis, is for a better understanding of the complex linkages which exist between formulation and solution of a practical problem. The investigations that have been carried out in this connection together with the development of new theoretical ideas are introduced in Chapter 4.

Case studies consisting of possible modelling approaches to nine practical problems are provided in Chapter 5. The case studies have been used in seventeen experiments with students and details are reported in Chapter 6. The theoretical ideas of a concept matrix (CM), and of a relationship level graph (RLG), are developed in Chapter 7. An analysis of a selection of students' attempts at modelling using the CM and RLG has provided new insights into formulation-solution processes. The work of Chapter 7 in its support for the learning heuristics developed in Chapter 6 is also examined, and the implications for assessment are developed and illustrated in Chapter 8.

Conclusions and suggestions for further research are provided in Chapter 9.

CHAPTER 2

BACKGROUND AND OVERVIEW

2.1 Introduction

Since the earliest records began, we see abundant evidence of models. In this sense, the term model is all-embracing. It is a man-made pattern, concrete or abstract, which attempts to represent some aspect of reality. Concrete or tangible models may be as varied as a statue of Venus, a train set, or a model aircraft in a wind-tunnel. Abstract models, on the other hand, may cover representation of the mind in psychology, the structure of a molecule in chemistry, or the mathematical equations governing the spread of a rumour. These patterns, or models, are manifestations of man's endeavour to understand the diverse and confusing matters which make up the universe.

Mathematics has had a profound influence on man's understanding of what is going on around him. It has also been, and continues to be, an invaluable aid in predicting and guiding our actions. The contribution that mathematics has made to physics from the time of Aristotle to present-day is well recognised. The influence physics has had on mathematics is equally well recognised. So intimate, in fact, have mathematics and physics become, that the phrase abstract model could easily be taken to mean mathematical model or physics model. Newton's inverse square law of gravitation is a good illustration

(law being interpreted as a universally accepted model). The relationship

$$F = \frac{Gm_1m_2}{r^2}$$

where F is the force of attraction experienced by two bodies of masses m_1 and m_2 separated by a distance r, and G is the gravitational constant, is certainly mathematical and represents an aspect of reality. It is, therefore, a mathematical model. The relationship also expresses a physical phenomenon, and so could be termed a 'physical model'. However, Newton made great leaps in physics concepts in arriving at his model. French (1971) recounts how Newton, in about 1666, generalised the idea of a falling body to "explain" the movement of the moon about the earth, and how he began to think of the earth's gravity as extending out as far as the moon's orbit. Newton then imagined the sun's gravity extending out to the orbits of the planets in the same way. It is generally felt today, and certainly by the author, that a mathematician would use physics models as a starting point, and then apply mathematical techniques to construct a mathematical model. The deeper physics insights, and the task of carrying out laboratory experiments in data collection, are not normally carried out by the mathematical modeller. Very often in mathematical modelling, simplifying assumptions are made just to get started in the construction of a model. See Oke (1981a), for instance, in the design of pick-up arms in attempts to minimise sound distortion in a record-player. These simplifying assumptions can, on further investigation, be shown to have little or no physics basis. Yet, the mathematical model produced, though crude, can provide valuable insights into an overall process. A lofty and profound parallel is Einstein's modification to Newton's gravitational law.

More recently, in the historical perspective, mathematics has made major contributions to subjects other than physics,

and engineering. Economics, medicine, biology, organisational sciences, and many others have increasingly been influenced by a mathematical approach. Perhaps the biggest influence felt is in the field of applications of operational research (OR). Statistics has, of course, also made enormous contributions, but may be included under the general heading of OR as far as modelling methodology is concerned. The applications of OR, and the development of the theory and methods involved received tremendous impetus from the challenge of logistics problems in the second world war, Rivett (1980). Perhaps because of the particularly difficult challenge posed by problems where cause-and-effect are much less well understood than in the physical sciences, the methodological issues of modelling are better established in OR, Oke (1979). High on the list of contributions to the methodology of modelling in OR are Ackoff (1962) and with Sasieni in 1967, Morris (1967), White (1975), and Rivett (1980). Nearly all these methodologies stem from the original six stages of Ackoff (1962):

1 Formulating the problem

2 Constructing the model

3 Testing the model

4 Deriving the solution

5 Testing and controlling the solution

6 Implementing the solution

It should be pointed out at this stage, that the discussion started with the concept of a <u>model</u>, whereas we are now considering the <u>processes of modelling</u>. This is a most important distinction. Up until about ten years ago, most textbooks, and papers in journals, concentrated very little on the <u>constructing of a model</u> (Ackoff's phase 2) and spent most of their time on <u>deriving the solution</u> (Ackoff's phase 4).

Rarely, if ever, was consideration given to the other phases of the modelling activity (processes of modelling). The most difficult phase, formulating the problem (Ackoff's phase 1), was barely mentioned or touched upon. This is hardly surprising, as will be shown later, since the formulation stage and particularly the formulation/solution interface are the most difficult to achieve. Making sense of a practical problem and then making appropriate assumptions which lead to a set of tractable mathematical equations is a highly intuitive process (see Morris (1967) for example).

It is axiomatic in this thesis that everyone should possess some knowledge of mathematics, no matter how little, and that everyone should be able to apply their mathematical knowledge to the solution of practical problems. This aim is certainly reflected in the Cockcroft report: *Mathematics Counts* (1982), where, inter alia, the foreword states:

> The Committee's findings point to the need for teachers to devote more time to the use of mathematics in applications taken from real life.

Long before the Cockcroft report, of course, most have agreed on the importance of being able to apply mathematics to the solution of real-life problems. With the rapidly increasing needs of commerce, industry, surveying and navigation, numerous schools and academies in the USA and Europe were established during the eighteenth century to provide formal and practical training in mathematics and science (Howson, Keitel & Kilpatrick (1981)). Numerous curriculum changes naturally have taken place since then, and in recent times one has seen the development of the 'New-Math' approach, inspired by the work of the Bourbaki group. Howson, Keitel, and Kilpatrick provide an excellent account of these developments and refer to the impressive impact of Dieudonné (1959) in the early stages. The 'New-Math' approaches, however, relate mainly to the teaching of pure mathematics. The School Mathematics Project (SMP), which was firmly established at the University of Southampton under the directorship of Dr Bryan Thwaites in 1972, is perhaps one of the largest and

best known developments in the UK. The SMP concentrates mainly on the unifying aspects of pure mathematics, with some consideration of applications.

The end of the 1960's and the beginning of the 1970's saw a revival of interest in teaching the applications of mathematics. Although the importance of applying mathematics has long since been acknowledged, there was a great concern with the way applied mathematics was being taught. One of the first noteworthy figures to express this concern openly was H O Pollak (1968, 1969). He recommends an open-ended approach:

"Here is a situation. Think about it."

Rather than the narrow or closed approach of:

"Here is a problem. Solve it."

or

"Here is a theorem. Prove it."

Pollak suggests mathematical modelling as an instructive method of teaching, where the formulation of the problem is emphasised.

In the UK much concern was beginning to be shown on the nature of traditional applied mathematics courses in higher education. The content was dry and usually amounted to formal courses in mechanics and hydrodynamics where artificial assumptions were made. The problems bore little or no resemblance to genuine practical problems, and they were posed ready for immediate solution. No formulation of a real problem was needed. It is perhaps not surprising that there has been a decline in numbers of students taking GCE 'A' levels in applied mathematics since about 1970 (Ford & Hall (1970)). Clearly, a re-appraisal of the position was called for.

Apart from Pollak, who first aired his views at the Colloquium on 'How to Teach Mathematics So As To Be Useful', held in

Utrecht in 1967, Freudenthal and others also expounded the view that mathematics and real-life situations should be treated together in teaching. In 1969, Ormell (1969) started the School's Council project 'Mathematics Applicable'. The project, intended for the non-specialist mathematician in the sixth form, led to the development of problems based on 'Selected, Simplified, Projective' (SSP) applications of mathematics, although formulation stages are omitted. Bajpai, Director of the Centre for Advancement of Mathematical Education in Technology at the University of Technology, Loughborough, founded the International Journal of Mathematical Education in Science and Technology in 1970. The contribution that Bajpai has made by the establishment of such a journal, the first of its kind, in providing a forum for the wider debate of mathematical education and for the development of mathematical modelling in particular constitutes a major landmark. Bajpai was one of the first noteworthy figures to identify the importance of a modelling approach, and 'mathematical models arising from real situations' was emphasised in the 'Aims and Scope' of the first issue of the Journal (and continues to be emphasised in current issues). Ford & Hall (1970) were amongst the first to make an important contribution in the first issue of the Journal where they develop the case for mathematical model building as a unifying theme for applied mathematics. Margaret Brown (1972) reports on the attempts of the Chelsea Centre for Science Education in drawing up 'real' problems for teachers and sixth formers alike. Bajpai, Mustoe and Walker (1974, 1975, 1976) made a significant contribution to university teaching of mathematics by developing a modelling approach in the teaching of mathematics to engineers. James in a paper entitled 'How should the mathematical training of an engineering undergraduate be conducted?' to be published in the International Journal of Mathematical Education in Science and Technology, makes out a strong case for the inclusion of mathematical modelling in the engineering curriculum. James, who has made considerable contributions to the teaching of both mathematicians and engineers, argues his case in view of the increasingly complex

systems and computer technology that the modern engineer is being asked to tackle. The work of Burkhardt (1978, 1979, 1981), has made a significant contribution in identifying the processes and skills involved in teaching mathematical modelling in schools and other institutions. Burghes (1980) recommends a modelling approach and illustrates it with several problems for sixth form use. The Spode Group (1981-1983), under the guidance of Burghes, have published three books containing 'real' problems with solutions suitable for middle school to sixth form. A major course on mathematical modelling for teachers has been presented by the author's team since 1977: (Oke (1980, 1984)), as part of an MSc degree in mathematical education. This course concentrates on modelling problems for teachers and their students, as well as on the development of a greater understanding of the processes involved.

McLone (1973) in the conclusions of his report on 'The training of mathematicians' emphasises the importance of mathematicians being able to formulate real problems into mathematical terms and subsequently being able to express the results of mathematical analysis in a form readily understood by non-mathematicians. Andrews & McLone (1976) edited one of the first books which provided a collection of mathematical models covering a wide area of applications. Recently McLone edited with Howson a book on a wide variety of practical problems ranging from the elementary applications of mathematics in nursing to quite advanced industrial problems: Howson and McLone (1983). The book is designed to appeal to sixth formers and undergraduates. James, McDonald and Huntley created the National Mathematical Modelling Workshop in 1978, whereby a number of contributors from polytechnics and universities developed case studies in mathematical modelling suitable for undergraduates. These case studies have now been, or are about to be, published: James & McDonald (1981), James & Huntley (to be published). In 1980 a special workshop on mathematical modelling was organised by Bradley, Gibson and Cross which culminated in the publication of leading contributors' models, their development, and workshop participants'

efforts in attempting to construct these models from scratch: Bradley, Gibson, Cross (1981).

Mathematical Modelling activities are increasingly taking place in a number of countries and Kapur (1982) provides an outline of the breadth of the interests involved. Kapur mentions, amongst other things, that at the fourth International Congress on Mathematics Education (ICME) held in 1980, at least four sessions were devoted to mathematical modelling. A whole section, which was chaired by Professor A C Bajpai, was devoted to mathematical modelling at the college and university level at the fifth ICME held in Adelaide, Australia, in August 1984.

In the late 1970's a number of new journals on mathematical modelling had started to appear, but most of these are aimed at the professional modeller who is tackling complex problems. Such journals, for example 'Mathematical Modelling' (published by Pergamon Press) or 'Applied Mathematical Modelling' (published by Butterworth Scientific) may provide some inspiration for simplified problems but generally they are not appropriate for undergraduates.

So, there has been a tremendous burst of activity in the development of mathematical modelling. Although numerous workers and their activities since 1970 have been briefly mentioned, many more have in fact reported the results of their experiments on teaching modelling. However, in spite of this tremendous burst of enthusiasm, modelling in the classroom has taken place in a very short time span, namely some thirteen or fourteen years. The time span is short in the sense that a very different mode of thinking and operating in problem solving is being considered, compared with the

teaching and learning styles of traditional applied mathematics. It is not surprising, therefore, that there is a paucity of research into the complex processes involved in modelling: Burkhardt (1979); Treilibs (1979); Burkhardt (1981). Such research needs to be done however, in order to guide the teacher and student alike; the demands placed on both are considerable, as the processes involved are creative - particularly at the formulation/solution interface, Oke & Bajpai (1982).

The following sections of this chapter detail the major developments to date in mathematical modelling - both from an expert point of view (modelling in industry, commerce, research institution) and teachers'/students' points of view. It is the intention that such a review of current and past work should help to explain and provide an appropriate perspective to the research carried out by the author and reported on in later chapters.

2.2 The Nature of Mathematical Models

The essential characteristics of a mathematical model depend upon the purposes to which it will be used. The characteristics are also highly dependent on the aspects of a problem which the model is supposed to represent. Although attempts at providing a general definition of a model, as well as the provision of a list of model types is helpful, it is felt that a description of the processes of constructing and testing a model is the most illuminative. Consequently, this section will be relatively brief, leaving to subsequent sections of this chapter discussions on the nature of modelling to date.

Definitions of a model are numerous and varied. Ackoff (1962), for instance states that models are

> ... representations of states, objects, and events. They are idealized in the sense that they are less complicated than reality and hence easier to use for research purposes.

He is actually referring to scientific models, but implies that his definition applies equally well to mathematical models, the latter being symbolic and involving equations.

The following are further examples on the definition of a model:

> ... the model ... is a set of logical relationships, either qualitative or quantitative, which will link together the relevant features of the reality with which we are concerned.
>
> (Rivett (1972))

> ... the representation of our so-called 'real world' in mathematical terms so that we may gain a more precise understanding of its significant properties, and which ... (might) ... allow some form of prediction of future events.
>
> (Andrews & McLone (1976))

> ... an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose.
>
> (Bender (1978))

> A mathematical model is a collection of statements about a set of variables from which the truth or falsity of other statements can be deduced.
>
> (White (1975))

Mathematical modelling is such a widely encompassing activity that precise definitions which cover all aspects are extremely difficult, if not impossible, to provide. However, all the above definitions, with the exception of White's, refer to a mathematical model as being some representation of <u>reality</u>. Ackoff and Bender also emphasise that a model is a <u>simplification</u> of reality. For the purposes of this study, the following definition of a <u>mathematical model</u> will be adopted:

A simplified and solvable mathematical representation of an aspect of a practical problem.

This definition emphasises that a model is an imperfect reflection of some aspect of the real world in the following ways:

(a) Simplification ignores some details

(b) Mathematical representation 'forces' a particular kind of abstraction

(c) To make the mathematics solvable (tractable) further simplifications, or distortions of the 'truth' are often required

The phrase 'practical problem' has been used in preference to 'reality' to indicate a more homely and thus familiar set of experiences for teachers and students to concentrate on when applying mathematics. Topical problems, such as those found in alternative technology for instance, often provide better motivation for learning modelling rather than highly complex and abstruse situations: Burghes, Huntley, McDonald (1982); Burkhardt (1979, 1981); Oke (1983).

Operational research workers have carried out most of the work to date on the classification of model types. Often quoted in management science is the work of Ackoff and Rivett (1963). They have devised a classification system according to the form and content of problem areas in OR, rather than concentrate on solution techniques:

1 Queueing

2 Inventory

3 Allocation

4 Scheduling and routing

5 Replacement and maintenance

6 Search

7 Completion

Ackoff and Sasieni (1967) on the other hand have concentrated on a classification according to the difficulty of formulating structure:

1 Logical structure apparent, easy to solve

2 General structure apparent, not easy to symbolise

3 General structure not apparent, data analysis required

4 Experimentation needed to isolate effects of variables

5 Experimentation not possible, hypotheses formed

Clearly, a parallel to Ackoff's and Rivett's classification in traditional applied mathematics could be obtained by distinguishing the following problem areas:

1 Mechanics

2 Hydrostatics

3 Hydrodynamics

4 Electromagnetics

5 Thermodynamics

This list, however, looks very much like traditional syllabus headings and could equally well apply to physics. Little is indicated of the nature of models or of the processes of constructing a model.

Ackoff's and Sasieni's list is much more general and applies not only to problems in the OR, or social/organisational sciences, but to problems in the physical sciences/engineering, life and medical sciences. Examples of these model types illustrating the difficulties of formulating structure can be found in many books and journals, for example in Andrews & McLone (1976); Bender (1978); Bradley, Gibson, Cross (1981);

Burghes, Huntley, McDonald (1982); James & McDonald (1981); Haberman (1977). It should be understood, though, that they provide little guidance on how to actually set about the formulation of problems in mathematical terms. There are one or two noteworthy exceptions, however, and these will be dealt with later.

An important distinction to make is between deterministic and stochastic models. Deterministic models are those where the variables are definite and not random. Stochastic models contain random or probabilistic variables. For example, in the investigation of speed-wobble in motorcycles, the returning couple about the steering-axis is related to the moment of inertia and angular acceleration of the wheel and forks (Oke (1981)). There is no doubt (apart from accuracy of measurement) about the values of the variables (angular acceleration, etc), and so the mathematical equations define a deterministic model. On the other hand, models which represent arrival times of patients at a doctor's surgery for instance are best modelled stochastically, using an appropriate probability distribution for inter-arrival times.

Very often only a qualitative solution is sought to a practical problem. In fact, at a given initial stage, it is sometimes the only solution possible for the modeller. The approach is to make observations and codify them by constructing a table and/or fitting them to a 'sensible' graph form. The type of model so formed is called descriptive (Burkhardt (1979)) and relates closely to types (3) and (4) of the Ackoff-Sasieni classification. The technique of drawing a graph and/or producing an empirical formula as part of a modelling strategy has also been reported in Bajpai et al (1974, 1975).

Models of the type governed by some form of equation or equations are referred to by Burkhardt (1979) as analytical. They may, therefore, be either deterministic or stochastic.

All the reported modelling activities carried out by the author involve the construction of analytical and deterministic models.(Oke 1979, 1981a, 1981b, 1984a). Furthermore, the models arise from concentrating on problems in the physical sciences and technology. This may seem unduly restrictive, however the approaches used can be extended to a wide range of activities outside the well-defined areas of physics and engineering. For example, putting a shot in golf (Burghes (1981)), athletics events (The Journal of Sports Sciences (1983)) (in particular, pole vaulting (Sheridan (1980) and in a later chapter of this thesis), in biology and medicine (Burley (1979), and in many other instances mathematics and physics can be used to solve real problems.

2.3 Review of Modelling Methodologies

It is essential to provide a set of guide-lines on mathematical modelling processes in order that mathematics may be effectively applied to the solution of practical problems. Attempts to encapsulate the essential features of the stages or 'spectrum of activities' that actually take place when a real-life problem is being solved have resulted in the development of various methodologies. They range from descriptions of the modelling processes used by professional modellers in industry and commerce to detailed guidance for teachers and students.

The subject of mathematical modelling as an academic discipline is new as has been indicated earlier in this chapter. The evolving character of the subject, however, has much in common with other new disciplines. In order to develop a methodology one first has to observe the activities of modellers in various practical situations. There are a number of difficulties experienced here, and research workers are still endeavouring to devise more effective techniques. Morris (1967) has this to say when referring to the intuitive ways in which management scientists arrive at their models:

> The term "intuitive" refers here to thinking which the subject (management scientist) is unable or unwilling to verbalize. Indeed, really effective experienced persons in any field typically operate in a largely intuitive manner and view with impatience attempts to make their methods explicit.

This observational difficulty is not new of course, and is experienced by researchers in the fields of education, sociology, psychology, and more recently in the fields of systems analysis and computerised expert systems. Efforts to overcome the problem include getting the subject to 'talk through' his/her experiences (in spite of Morris' statement), a-posteriori analysis or reflection on what one has done after having solved a problem, and independent 'unobtrusive' observation and recording of perceived details whilst the modeller is actually working. Morris' methodology in common with many others is based largely on a-posteriori or reflective analysis.

The modelling methodologies developed to date are based on the traditional approaches used in science. An early example of this is by Ackoff (1962) and with Sasieni (1967) mentioned in section 2.1. It is basically linear and sequential in nature, although he acknowledges that the processes are usually cyclic and the steps are overlapping. The scientific, or linear paradigm, has naturally been adopted in other fields of study in attempts to make 'order out of chaos'. There is a gradually emerging view, however, that complex processes can with advantage be understood by non-linear approaches. In the field of systems analysis Checkland (1975) is concerned with the inadequacy of the science paradigm when applied to living systems and particularly human activity systems. Checkland recommends a holistic, or Gestalt approach, 'because systems are more than the addition of their individual parts'. In the field of curriculum design, some innovators hold a non-linear view of mathematics itself, as approached by the learner. Howson, Keitel and Kilpatrick (1981), whilst pointing out that there is insufficient evidence either to disprove or

to substantiate the claims of the 'non-linearists', have this to say about learning mathematics:

> Elaborate flow-diagrams are often supplied to which materials conform. Yet is this a valid model of mathematical learning? Is mathematics analogous to climbing a tree (first a common trunk, then a variety of branches to be tackled in a directed manner), or to solving a monster jigsaw puzzle (building isolated groups of pieces and then combining these by means of well-chosen links to form even bigger aggregates)?

There is merit in both the linear and non-linear approaches in studying complex processes. Clements (1982) suggests that students could initially be introduced to a linear methodology, and then as they gain maturity and experience in their modelling studies they could be given a more complex description. However, there are no details provided by Clements, or by anyone else, which give a deeper understanding specifically related to mathematical modelling processes. One of the best 'non-linear' guides to date seems to come largely from very general descriptions by systems analysts.

In a later chapter a linear/non-linear methodology, which is new, is developed. In the meantime, the main features of <u>linear</u> approaches will be discussed.

Morris (1967), appears to be one of the first professional modellers to attempt to articulate what actually happens in the modelling process. From his experiences, he offers some specific hypotheses for the guidance of the inexperienced modeller:

1 Factor the system problem into simpler problems

2 Establish a clear statement of the deductive objectives

3 Seek analogies

4 Consider a specific numerical instance of the problem

5 Establish some symbols

6 Write down the obvious

7 If a tractable model is obtained, enrich it. Otherwise simplify

All of the above would seem to be relevant to teaching, and in this connection, Morris provides a useful list of actions to carry out if simplification, phases (1) or (7), is necessary:

Simplify by:

making variables into constants

eliminating variables

using linear relations

adding stronger assumptions and restrictions

suppressing randomness

The latter, 'suppressing randomness' would not apply to a deterministic problem. However, this suppression could convert a difficult stochastic problem to a more manageable deterministic one. Enrichment, on the other hand, could involve the opposite of some of the actions, eg, turning constants into variables.

Rivett (1972, 1980) has provided a flow-chart of the model-building process as shown in Figure 1.

Rivett's description is very general and therefore has less to offer to the inexperienced modeller. Stages A to H correspond roughly to Ackoff's phases (1) to (3), with the added component of testing the objectives of the original problem. Both Ackoff and Rivett have testing stages, and Ackoff's final stage (6) 'Implementing the solution' corresponds

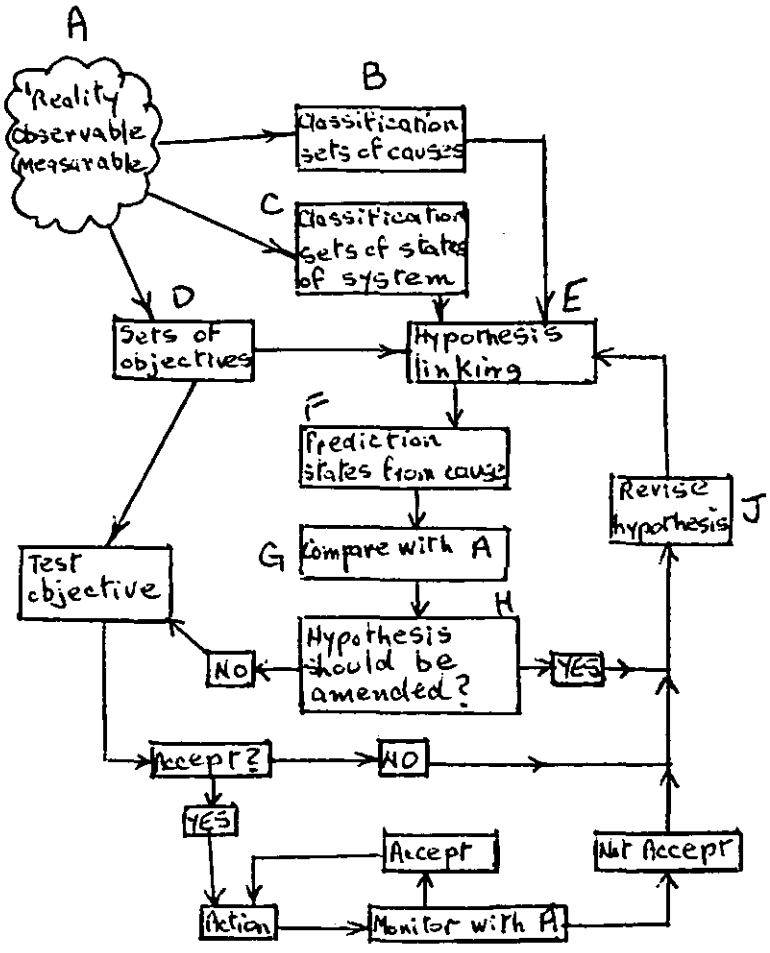


Fig 1

The model-building process (Rivett, 1972 and 1980)

to Rivett's 'Action'. In a teaching situation it is rarely possible to introduce the implementation stage, since there is little opportunity to try out a model in industry or commerce. Note that neither Ackoff nor Rivett suggest, unlike Morris, how to develop the actual mathematical equations in the formulation stage.

White (1980) in his book 'Decision Methodology', discusses several other modelling methodologies in addition to Ackoff's. Chief amongst these are Tocher's (1961), which is very similar to Ackoff's five stages, and Bonder's (1970). Bonder divides the modelling process into qualitative and quantitative stages which correspond, in general terms, to Burkhardt's descriptive and analytical model types respectively as mentioned in section 2.2. Several publications to date, for example Collier (1982), discuss mathematical modelling

strategies for the professional in a number of different application areas. These methodologies all tend to be in very general terms and stem largely from the OR methodologies discussed earlier.

Methodologies more closely related to educational requirements have also benefitted from OR approaches. However, the structure of modelling strategies appropriate to teaching and learning situations naturally emphasise these aspects. Bajpai, et al (1974, 1975) were amongst the first in the early 1970's to draw attention to the benefits to be gained from a modelling approach in teaching. Although this work concerns the teaching of mathematics to undergraduate engineers, much of the methodology is relevant to the teaching of mathematical modelling in general. The methodology is represented in a block diagram as shown in Figure 2.

An important feature appears in Figure 2, namely an inner loop regarding simplification. If the mathematics involved in the construction of the model is too difficult to solve, either because of the time available or because of the complexity of the equations (or both), then it is necessary to make simplifying assumptions. This emphasis on simplification was first mentioned in the earlier description of Morris' methodology. It is not mentioned by Rivett. Another important feature in Figure 2 is the outer loop corresponding to improving or changing the mathematical model. Rivett also has this loop and it contains boxes A, H, J and E as shown in Figure 1. The loop is travelled round successively until a mathematical solution is obtained which accords sufficiently well with observation (data obtained by measurement).

Burkhardt (1979, 1981) describes mathematical modelling processes, called problem solving processes by him, by referring to a flow-chart as shown in Figure 3.

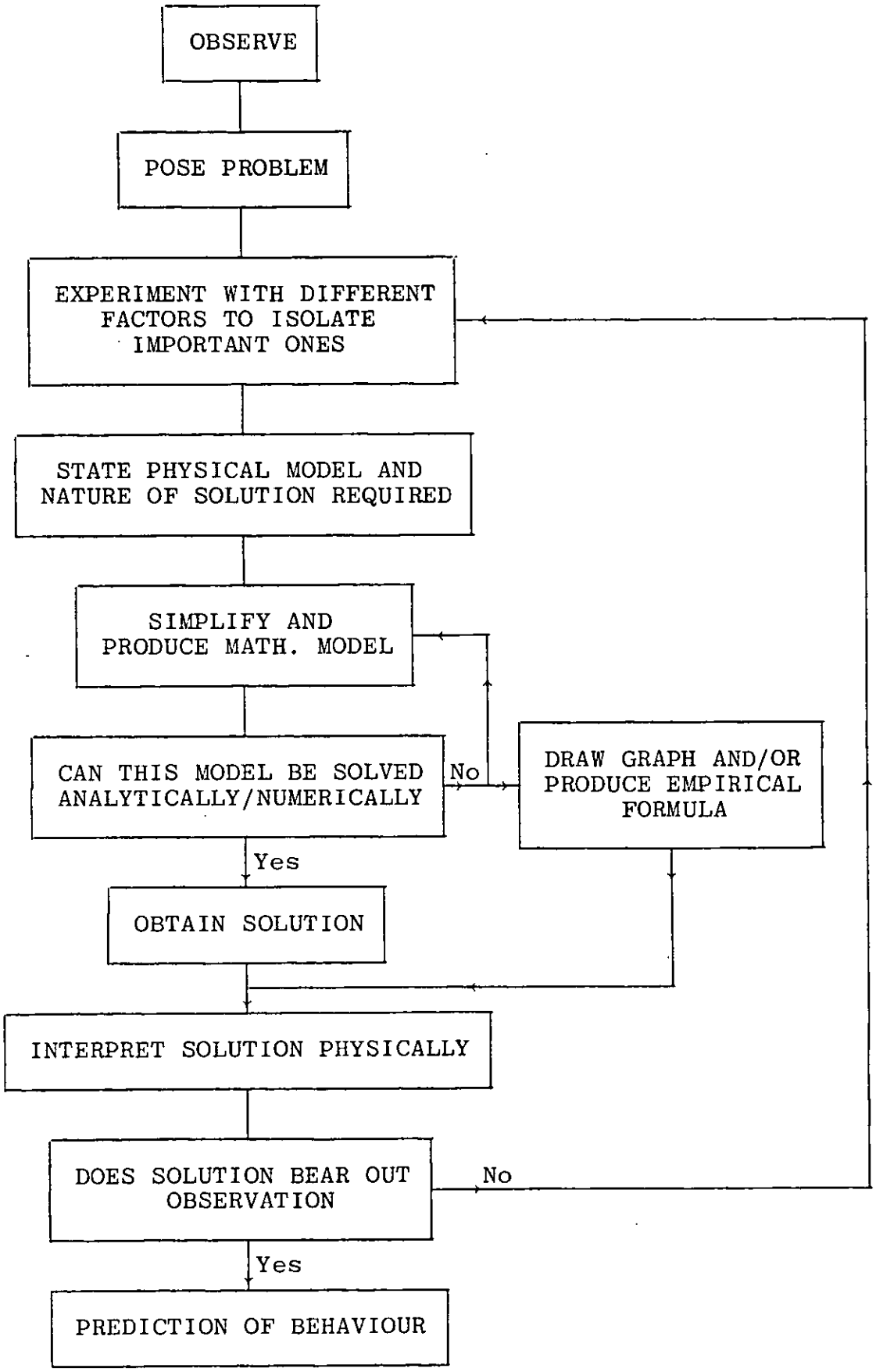


Fig 2

Methodology of Bajpai, et al (1974, 1975)

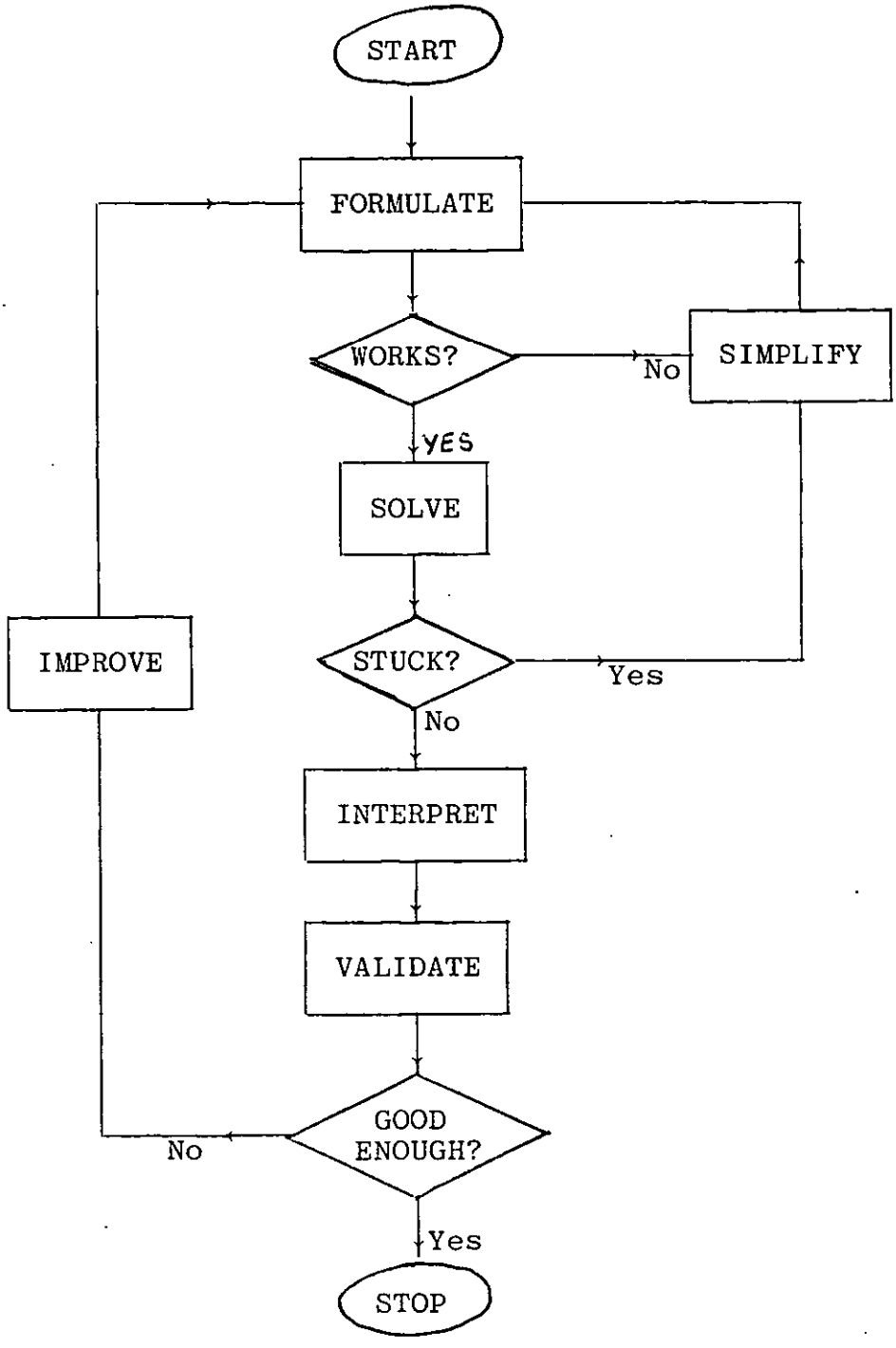


Fig 3

Problem Solving Processes (Burkhardt, 1979 and 1981)

Burkhardt has the inner 'simplify' loop and the outer 'improve' loop that occurs in Bajpai's block diagram (Figure 2). The 'formulate' box in Figure 3 corresponds approximately to boxes 2-5 (counting from the top) in Figure 2. Burkhardt represents formulation in more detail as shown in Figure 4.

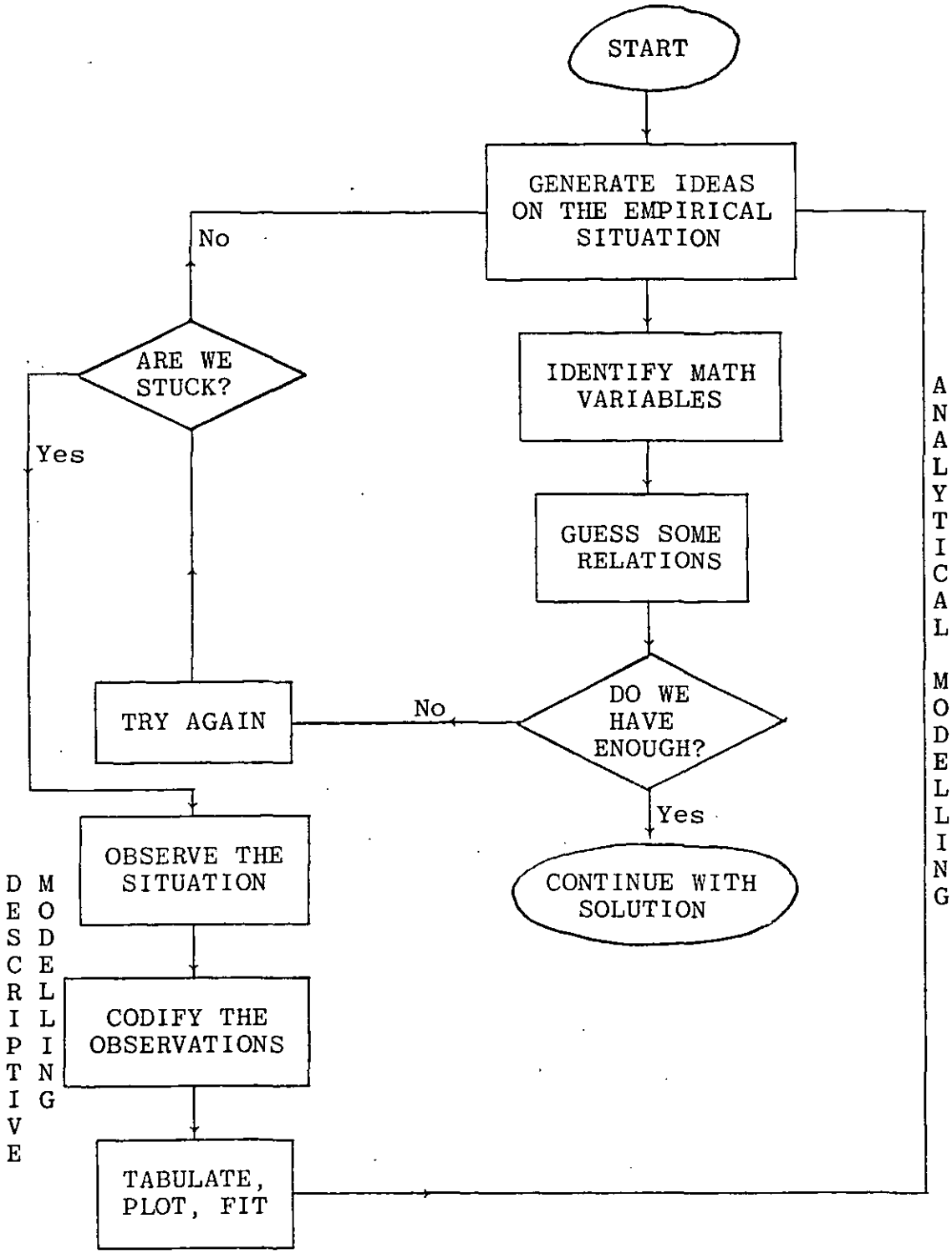


Fig 4

Formulation Processes (Burkhardt, 1979 and 1981)

In thinking about a new problem Burkhardt claims that we always start from our prejudices, which are rough analytical models. He goes on to say:

> We regard this as self-evident, since the decision on what observations to make from the myriad possibilities needs a model to guide it.
>
> Burkhardt (1981)

Burkhardt highlights important activities associated with the formulation stage in analytical modelling, based on the work of Treilibs (1979):

Generation of variables

Selection of variables

Generation of relationships

Selection of relationships

Identification of specific questions to be answered

Both Burkhardt and Treilibs point out that the relative importance and difficulty of these activities is not well understood, although it seems likely that the generation of useful relationships is one of the most demanding. This is a vital matter and is pursued in detail later on in this thesis.

Several other authors have published methodologies of modelling, but in the main they are either variants of established OR approaches or broad canvas descriptions which emphasise the movement from reality to mathematics (formulation) and back again (interpretation/validation/implementation).

2.4 Review of the Teaching of Mathematical Modelling

In section 2.1 reference was made to the large number of teaching experiments in modelling that have been carried out in the last ten years or so. All this activity has taken place in spite of the processes of modelling being ill-understood and relevant teaching skills still being in their infancy. These experiments have been carried out with students in school and at university, and they range in size from the endeavours of individuals to, occasionally, full-scale projects. Apart from treating simpler modelling exercises in school compared with university, there seems to be a consensus view that most students experience the same set of difficulties, particularly in the formulation stage. It is not surprising, therefore, that some experiments ignore this difficult stage altogether.

One of the largest projects, both in funding and the numbers of students involved, is USMES (Unified Sciences and Mathematics for Elementary Schools) formed in 1970 and based in Boston, USA, directed by Earle Lomon of MIT (Lomon (1980); Burkhardt (1981)). It is an interdisciplinary project that challenges students aged 6-11 to solve real problems from their school and community environment. In this way it aims to develop the cognitive strategies of problem solving and decision making in the students and to provide a possible bridge between the abstractions of the school curriculum and the world of the (young) student. As Lomon (1980) reports on the founding of USMES, the work of Gagné (1965) on cognitive strategies provided valuable guidance in the construction of their aims. The project is organised as a series of 26 "challenges" which students can tackle in any order. The level to which the students take each challenge and the investigations they pursue vary according to the abilities and interests of the students. The challenges

include problems such as:

1 Find ways of making bicycle riding a safe and convenient way to travel

2 How can we improve the daily class schedule?

3 Find ways of influencing rules and the decision-making process in the school

The method of presentation of each challenge is highly interactive with the teacher guiding and structuring classroom discussion. Extensive measuring exercises are carried out by the students in data collection and so the emphasis is very much on descriptive rather than analytical modelling. Lomon reports favourably on students gaining problem-solving skills after having been exposed to the USMES approach. Although the emphasis on such skills is largely on data collection and interpretation, opportunities for implementation are encouraged (eg, informing a school principal what pencils and erasers to purchase). The latter activity, as mentioned in section 2.3, is very difficult to arrange for students in higher education due to the more complex models involved and the difficulty in getting industrial and commercial firms to try out experiments.

At the undergraduate level, UMAP (Undergraduate Mathematics and its Applications Project) formed in 1976 and based in Newton, MA, in the USA (directed by Ross L Finney) provides valuable source material for modelling covering a very wide range of applications. UMAP produces modules which are contributed by lecturers in colleges and universities, each being reviewed, revised, and field-tested before being published by Birkhäuser Boston, Inc. By and large the modules are lesson-length booklets or units from which

students 'may learn professional applications of mathematics and statistics to such fields as biomedical sciences, business, economics, food management, American politics, harvesting, international relations, numerical methods, computer science, seismology, and traffic control'. The modules are intended to supplement existing courses and texts, and some present mathematical or statistical topics as theory rather than as applications. It is difficult to do justice by providing examples of some of the hundreds of modules that have now been produced, but titles of just a few chosen at random are:

1 The Relationship Between Directional Heading of An Automobile and Steering Wheel Deflection.

(Know what assumptions lead to the equation that relates $\theta(t)$ (compass heading) and $\phi(t)$ (steering wheel deflection) and how the equation is derived. Find automobile headings for given wheel deflection functions and initial conditions.)

2 The Digestive Process of Sheep.

(This unit introduces a mathematical model for the digestive processes of sheep. The model involves simple differential equations. There is a discussion of the assumptions of the model, support for its validity, and conclusions which can be drawn.)

3 Algorithms for Finding Zeros of Functions.

(Understanding of standard bisection, secant and Newton-Raphson root finding methods, and appreciation of their strong points and limitations. Introduction to more recent root finding methods.)

Example 1 provides an interesting application of mathematics without attempting to get the student very involved with initial formulation of the problem. Example 2 is a good example of a modelling approach where the student is encouraged to formulate (with guidance). Example 3 is an example without applications being mentioned (but very well done, nevertheless).

Although UMAP does not emphasise a modelling approach in most modules, there are so many well developed applications that experienced lecturers in modelling could well avail themselves of the units involved.

The earliest major attempt to introduce modelling in schools in the UK was made by Ormell in the creation of the School's Council project 'Mathematics Applicable' in 1969. It consists of a series of texts (The School's Council (1975 - 1978)), which aim to teach the skills of 'applied mathematics' in a modelling context. The material is designed for the non-specialist sixth former and examinations at AO level have recently been set by the University of London on behalf of several examination boards (Ormell (1983)). The applied mathematics skills are developed in the course of solving problems arising from concrete situations. The problems are often of a whimsical kind, eg, is 3-D knitting worth learning? Was the 'Star of Bethlehem' a possible phenomenon?, rather than of a more realistic kind. The texts of problems make no pretence of taking the student through the complete modelling process. Problems are well-defined, and the formulation stage only requires students to construct equations from clearly defined statements. Interpretation of mathematical solutions is required although no part is played by

validation. Each of the texts concentrates on a particular topic and the development of mathematics (eg linear functions, exponential growth) is exemplified by modelling problems. Mathematics Applicable has been adopted by many schools since 1970, and it has made a significant contribution to making mathematics more meaningful and interesting to school students. A review of the project and of SSP(Selective, Simplified, Projective) applications of mathematics can be found in an article by Ormell (1980).

Burkhardt, Director of the Shell Centre for Mathematical Education at the University of Nottingham, and his team have made a number of valuable contributions to the teaching of mathematical modelling mainly at school level. The first publication to detail this work is the project report "The Real World and Mathematics" (Burkhardt (1978)), now up-dated and available in book form (1981). These publications survey the work done at school level in mathematical modelling, provide many examples of models, and include suggestions for a methodology of modelling (see section 2.3 of this chapter) as well as approaches that can be used in the classroom. The range of activities - formulation, solution, interpretation, validation - is covered in the modelling approaches. Burkhardt has also devised a classification of interest level of problems, denoted by the acronym ABCDE, Burkhardt (1981):

> Action problems are those whose answers may directly affect decisions in our everyday lives
>
> 'How can I fit in my homework with the TV and going out?'
>
> Believable problems are those that we can recognise as Action problems either for ourselves in the future or for someone we care about.
>
> 'Should I get a job at 18 or go to college?'

Curious problems are those which intrigue us either because the phenomenon being studied is itself intriguing or because the analysis is.

'Why are there two high tides each day?'

Dubious problems are there simply to provide exercise in mathematical technique.

'See any traditional mathematics exam paper'

Educational problems are a rather special category - they are essentially 'Dubious' but make an important point of mathematical (or physical or economic) principle so clearly and beautifully that no-one would want to get rid of them.

'If I invested 1p at 5% compound interest in 512AD, what is it worth now?'

In facing up to the challenge of finding problems at various interest levels that could be used for modelling in the classroom, Burkhardt has set up a scheme called PAMELA (Problems in Applied Mathematics from Everyday Life Applications). Teachers are invited to contribute to a list of problems by providing information on a problem area such as title, short description of problem situation, and occasionally provide a mathematical model in their analysis. In return, teachers get an up-to-date copy of the list.

Burkhardt and his team have developed a 'Starter Pack' for those teachers who wish to try out modelling experiments with their students. The material is provided by the Shell Centre, and concentrates on providing guidance for the teacher by including detailed notes on how the gradual development of solutions to problems may be achieved. The pack is based on the experiences of a number of enthusiastic teachers as well as on the research work of Treilibs (1979).

By abstracting Burkhardt's (1978, 1981) observations on teaching styles in the classroom, the following list of points emerges:

1 No widely established methods of teaching mathematical modelling exist

2 Only mathematics very well absorbed is usable in modelling exercises (eg, arithmetic and simple algebra for GCE A/L students)

3 Problem situation must be simple (far simpler than if model is provided by the teacher)

4 Students left to carry out most of the modelling activities themselves - teacher provides minimum support (eg, help in creative ideas)

5 List initial student suggestions on blackboard - encourage class to 'thin-down' list before attempting solution

6 Formulation seems easier if class split into groups rather than left to work individually (unlike solution stage). Group size a matter of judgement

7 'Crunch' point reached, after initial stages of formulation, in generating relations. Needs teacher guidance, then leave students to solve mathematically.

8 Assessment of modelling skills not well developed

Several items in the above list are mentioned in one form or another by a number of authors in reports on their experiences of teaching modelling in schools, colleges, polytechnics and universities. Burkhardt, like many others, has pointed out that teaching modelling is not easy. Certainly, in the initial stages, an inexperienced teacher finds the less-structured student-centred classroom activities quite difficult. However, as with modelling itself or any other creative activity, experience gradually instils confidence, and there is a growing

number of teachers in schools and in higher education who are trying out modelling experiments of their own design.

The Open University started to introduce mathematical modelling components in 1975 by developing some topics in mathematics and illustrating them in simple but realistic applications. This approach, led by Penrose (1978), was presented in the foundation course M101. The course has had several up-dates since 1975, and the Open University has also produced other courses which include units on modelling. The second level course, TM281 'Modelling by Mathematics', was introduced in 1977. The emphasis of the course is on those aspects of mathematical modelling which require the ability to interpret well-posed problem statements (with hints on formulation) in mathematical terms and, subsequently, to show competence in techniques in acquiring a solution. Some challenging opportunities are also provided in the interpretation of a solution. Since the course is examined by a fixed-time three hour examination paper, the harder parts of formulation are not assessed. However, the course does present some refreshingly good ideas of the applications of mathematics and thus represents a significant improvement on the usual 'applications-oriented' approach. In 1980 the course PME 233, Units 5-9 was published, Open University (1980). This course, entitled 'Mathematics Across the Curriculum', is a second-level 'Post Experience Mathematics/Education' course. The approach builds on the experience of USMES and considers the nature of real problem solving and how this may be introduced to the curriculum. Unlike USMES, however, a list of problems is not provided and the teacher is urged to find his/her own in consultation with their students (middle secondary school). The solution to a problem is considered best as an extended project, each student taking several weeks to complete an assignment. In 1982 the Open University produced the course MST 204, an inter-faculty second-level unit in Mathematics, Science and Technology. Half this latter course consists of a development of mathematical methods

and the other half is on mathematical modelling. The approaches recommended are based on the work of Penrose, which is very similar to the methodology of Burkhardt. Berry and O'Shea (1982), who wrote the project guide for MST 204, report on the assessment procedures that are being used in grading student performances in modelling exercises. An early report on the effectiveness of these procedures, resulting from the first batch of students' attempts on this course, is reported by Berry and Le Masurier (1984). These two articles and the paper by Hall (1984), are amongst the more important recent reports on assessment of students' efforts on modelling exercises. They highlight the difficulties in applying a formal marking scheme, and compare this with 'impression' marking. The problem of assessing teacher's attempts in modelling on an MSc course in Mathematical Education is reported by Oke (1980).

Burghes has made significant contributions to the teaching of mathematical modelling at school and undergraduate levels as well as designing short courses for teachers. He has suggested problems that could be presented at school level; for example:

Depot location (analysis and mechanical analogue)

Drug concentration (differential equation on dosage)

Planetary motion (curve fitting of data)

(Burghes (1980))

These seem pretty difficult for all except the most able sixth former, and recently Burghes has led a team which has produced simpler problems for middle school average ability students as well as some more manageable problems for the sixth form: The Spode Group books (1981-1983). In furtherance of the provision of problems which can be tackled in a modelling way, he established the Journal of Mathematical Modelling for Teachers with Read (Open University) in 1978; this has now been superceded by the Institute of Mathematics and Its Applications Journal of Teaching Mathematics and Its Applications. Examples of Burghes' earlier work on mathematical modelling for teachers can be seen in the short course notes given by Cranfield Institute of Technology, Burghes (1980). In the latter notes, however, the models suggested for the school teacher's students tend to be for the most able in the sixth form. His work with undergraduates has been reported in Burghes and Huntley (1982). In this publication he describes with Huntley their experiences of teaching modelling on several courses in higher education. Some of the points made on teaching methods are very similar to Burkhardt's observations and, in particular, a list of DO's and DONT's is provided as guidance for the teacher:

DO

1 Use 'real world' problems extensively

2 Design all problems so that a definite answer is required

3 Choose problems which have an intrinsic interest for everyone

4 Encourage work in small groups and accept group reports

5 Encourage discussion and communication

DON'T

6 Expect your students to use relatively new mathematical ideas

7 Try to teach mathematics and modelling at the same time

8 Use fixed time examination in assessment

9 Interfere too much, too soon

10 Impose your solution on the class

Several other authors have also reported on their experiences of teaching mathematical modelling to undergraduates and HND students and the details relate mainly to the types of problems set and the modes of project working. Some noteworthy examples may be found in: McLone (1979), Clements (1978), Oke & Bajpai (1982), Burley & Trowbridge (1984), Gadian, Hudson, O'Carroll & Williams (1984). Further discussion and analysis of work done in higher education, coupled with comparisons of recent research into related problem solving strategies, is left to later chapters.

2.5 Summary and Conclusions

Mathematical modelling as carried out by scientists has been in existence since antiquity. However, not until the more theoretical aspects of economics were founded in the nineteenth

century, and the full-scale developments of operational research methods since the second-world war, have attempts been made to understand more fully what amounts to a highly intuitive and creative activity. Operational research workers, in particular, have attempted not only to classify 'real-world' problems and models of them, but they have also striven to analyse what activities are actually carried out in the construction and testing of such models.

Serious interest in the teaching of mathematical modelling first came into evidence in the late 1960's in attempts to make applied mathematics more meaningful and realistic. Much initial inspiration on the processes of modelling was gained from operational research methodologies. The extent to which methodologies of modelling are helpful in teaching is still an open question and subject to further research. It would certainly appear, however, that many mathematical educators would acknowledge the value of some aspects of a 'model' of modelling (meta-model or methodology) in that it provides some guide lines for the teacher and student alike. This is not to suggest that students should first be taught methodological issues and then be exposed to solving real-problems, on the contrary, but that a judicious introduction to formal aspects of procedure might help in the students' appreciation of modelling as experience and maturity are gradually gained. There is certainly a need for a better understanding of the processes involved, especially in the early stages of <u>formulation</u> (of the practical problem into mathematical terms) and <u>solution</u> (of the resulting mathematical equations). Following on from this, there is then the need for a 'bank' of problems and skilled teacher guidance to provide, as far as possible, graduated exercises in modelling that build-up student confidence and expertise. As mentioned in section 2.4, Burkhardt has started a collection of problems, PAMELA, for school level; the Spode group, under the direction of Burghes, have published problems with possible solutions for middle to upper school students. In higher education, the two volumes of case studies - James & McDonald

(1981), James & Huntley (to be published) make a contribution for undergraduates and HND students.

In section 2.2, a working definition of mathematical model is proposed:

> A simplified and solvable mathematical representation of an aspect of a practical problem

The reasons for this choice of definition are explained; it is broad enough to encompass a wide range of problems and yet it is specifically designed to cater for educational use (teaching and learning) in that it emphasises: simplification (both of the problem and of the mathematics) and solvability (it must be possible to find a mathematical solution or solutions). The definition would seem to be appropriate at all levels, from school to higher education, although it does emphasise analytical models rather than descriptive or empirical models (Burkhardt's terms). Also mentioned in 2.2 is that the investigations into teaching and learning in this thesis will concentrate mainly on analytical and deterministic models which arise from problems in the physical sciences and technological applications areas. The results of such investigations, however, have wider reaching implications and apply to some aspects of problem solving strategies generally.

In section 2.3 a survey of the leading methodologies of modelling is made. These methodologies represent the processes of modelling either as a linear sequence of activities as shown in Figure 5; or, the processes are represented by a linear sequence with looping, as shown in Figure 6.

Both Figures 5 and 6 are simplifications of the actual methodologies and are intended to show overall features only. For details see, for example, Figures 1(Rivett), 2(Bajpai, et al), 3(Burkhardt). Burkhardt and Treilibs have further analysed the formulation/solution activities and these are portrayed in Figure 4.

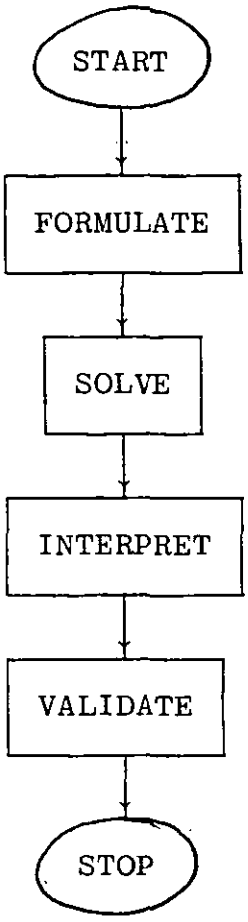


Fig 5

Processes of Modelling: Linear Sequence

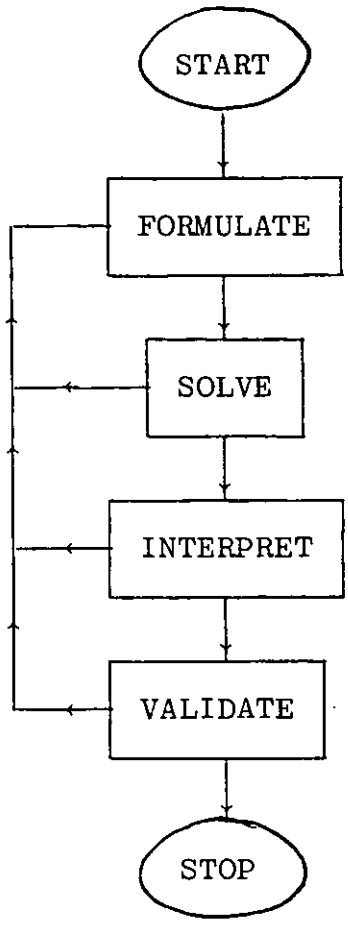


Fig 6

Processes of Modelling: Linear Sequence with Looping

Clements (1982) drawing on the work of systems analysts has suggested that modelling processes might better be portrayed as non-linear or holistic. Neither he, nor any other author to date however, has developed a non-linear methodology for mathematical modelling. Oke (1984) has reported on initial studies in this connection though and further developments, with important implications for teaching are reported later on in this thesis.

In section 2.4 a review has been made of some of the leading projects and individual efforts on investigations and experiments in teaching mathematical modelling. Activities have taken place at all levels from school to establishments in higher education. Inevitably there are different emphases placed in these experiments, notably in USMES and the work of Ormell in that of neglecting formulation of a practical problem. The emphasis in these projects is in finding the mathematical solution, perhaps empirical, to a well-defined problem. Most authors, however, emphasise the importance of students carrying out all the four stages: formulation, solution, interpretation, and (some at least of) validation. The work of USMES and Ormell also have the distinguishing feature of using modelling activities as a means of introducing a new mathematical topic (the Open University course M 101 is also similar in this sense). Whereas most would strongly advise against this, pointing out that modelling is difficult enough anyway, and so there is a considerable risk of confusing students by introducing new mathematical ideas at the same time. This no doubt goes a long way in explaining why USMES and Ormell, for example, omit the difficult formulation stage in their work.

Most reported work, however, does show a common consensus on the following points in connection with the teaching and learning of mathematical modelling:

1 Need of problems for modelling exercises

2 Formulation stage is most difficult, particularly in the generation of mathematical relationships

3 Once students have got started, work is best done in groups

4 Assessment of modelling attempts is very difficult

Burkhardt with his PAMELA list, Burghes with his Spode books, and James & McDonald and Huntley with their 'Case Studies' books, have made a valuable start to (1). Burkhardt and Treilibs are probably the best known for work on formulation processes (2). Practically all authors agree with (3), although the initial value of interactive teaching is also emphasised by some, eg, in Burkhardt (1981); Oke (1984). All agree with (4), and it is still an open question as to whether 'impression' marking or formal marking scheme, or a combination of both, is best.

To sum up, much more research needs to be done in developing an understanding of modelling processes in general, and in formulation processes in particular, that will provide help and guidance to both teachers and students. In the next and subsequent chapters this identified research need will be more fully investigated. Reference will be made to related research in problem solving processes as well as the development of a more detailed analysis on teaching and learning mathematical modelling - covering experiments to date in addition to proposals for the future.

CHAPTER 3

RELATED RESEARCH

3.1 Introduction

As pointed out in the last chapter, the teaching of mathematical modelling and of problem soving generally is still in the very early stages of development. In the last decade enthusiastic teachers, who have some experience of modelling themselves, have tried out various classroom experiments on tackling realistic problems. In order for mathematical modelling to have a wider impact on the curriculum more needs to be done for the majority of teachers and students. Attempts are currently being made in several directions and they may be briefly described as relating to:

1 Teaching styles

2 Learning styles

3 Assessment methods

4 Modelling processes

Naturally, these endeavours are being carried out concurrently, with different emphases being placed by various investigators. In order to make significant progress from the present 'state-of-the-art' a considerable amount of work needs to be carried out in each of 1 - 4. However, at the time of writing, hardly any research has been done. Some would argue that progress will best be made by using intuitive approaches in the classroom, whilst others would argue the case for the development of a

greater understanding of modelling processes. The author adopts a compromise position. In order to make mathematical modelling activities more widely available to institutions at all levels, it is proposed that a fuller understanding of modelling processes as related to the classroom is a valuable way forward. The aim, then, is to try and understand better what actually happens when a simplified but realistic problem is solved by mathematical means, and to use this understanding as a basis to developing modelling skills in students.

What little research has been carried out in mathematical modelling has been in the field of processes. Approaches compare with, and draw their inspiration from research in problem solving (not necessarily in mathematics). As pointed out in the last chapter, the processes involved are usually portrayed in a flow-diagram implying linear or linear with looping sequencing of stages. Only recently, drawing on the still pioneering work of systems analysis (in information processing), have non-linear approaches been suggested. Whilst there is very little reported research in modelling, there is a fairly large body of work done on problem solving (in its widest sense).

Both problem-solving and mathematical modelling are concerned with the study of creative processes. There is, therefore, bound to be much in common between the two approaches. The chief difference between problem solving and modelling can be found in the types of problems being tackled. In the former case, attention is focused on the methods used to find a mathematical solution of a well-defined and specific problem. Whereas, in the case of modelling, one seeks to make sense and gain a better understanding of an often ill-defined practical problem. Once the practical problem is better understood, then a mathematical solution, which helps in this understanding, can be developed. There is usually only one or at most two correct solutions to a 'problem solving' exercise, whereas in 'modelling' there is no such thing as a 'correct' solution; there are only 'good' or 'bad' solutions. The whole set of activities, then, in problem-solving tend to be much more

structured than in modelling. It is, therefore, often easier to get a sense of direction in which to proceed in problem-solving than in modelling. Two simple problem statements, by way of example, serve partly to highlight these differences:

Problem-solving problem

> The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides. Polya (1957)

Mathematical modelling problem

> Discuss the basic design features of a bicycle gear system. Try to formulate in your answer, in mathematical and physical terms, the speed ranges for each gear, and the number of gears for a given bicycle.

The difference in question styles is striking. The first is well-posed and it is quite clear what the final answer should look like. The modelling problem is typically vaguely posed, and one has to determine firstly what are the specific mathematical problems to be solved.

What problem-solving and modelling have in common, however, are certain aspects of the formulation stage. Although formulation is an even more complex process in modelling than in problem solving, there are nevertheless similarities; in both activities, the identification of variables and constants is needed as well as the construction of mathematical relationships which connect these. It is well known that students, at all levels, find these activities difficult. Problem solving research has, over a number of years, attempted to identify the skills required for carrying out such processes, as well as attempting to devise heuristics which help in skill acquisition.

In the next sections, the results of the main problem-solving and modelling research activities which are related to this thesis are presented. Problem-solving research covers a very wide area, not often concentrating on mathematics, and consequently less space will be devoted to these aspects than to modelling processes.

3.2 Problem-solving processes

In order to solve a problem one must be able to explore, manipulate and search for features of the problem area that will provide the desired outcome. Polya (1957) in his famed 'How to Solve It' has suggested certain procedures and maxims to facilitate the acquisition of problem-solving skills in mathematics. It is generally recognised that these procedures seldom provide infallible guidance but, approached from a practical teacher-oriented standpoint, they may help by giving the solver a general course of action to take. Polya, like many others, calls these procedures and maxims heuristics. Heuristics may be taken to mean imperfect but useful knowledge employed in many reasoning tasks such as plausible inference, discovery, and so on, where precise knowledge is lacking. This definition of heuristics is a little broader than Polya's but it does seem to fit most researchers' use of the term. Polya has outlined four phases in problem solving:

1 Understanding the Problem

What is the unknown? What are the data?
What is the condition?

2 Devising a Plan

Do you know a related problem? Look at the unknown.
Here is a problem related to yours and solved before, could you use it?

3 Carrying out the Plan

Carry out your plan of the solution and check each step.

4 Looking back

Examine the solution obtained.
Can you check the argument?
Can you derive the result differently?

Hatfield (1976) would refer to Polya's four phases collectively as a 'planning heuristic', whereas the four headings are themselves heuristics. The prompts provided under each heading may be viewed as the most detailed of Polya's heuristics. So, in a sense, one has a hierarchy of heuristics. This notion of a hierarchy has a parallel in the development of concepts in general, see for example Skemp (1979).

Perhaps the most crucial phases are (2) and (3). Phase (2) may be compared with Morris' 'Seek analogies' referred to in 2.3 (Chapter 2). Phase (3) involves the execution of the plan of attack. If it does not complete the solution to the problem but only reduces the difference between the data and that which is sought, then only a partial solution is obtained. Even so, the problem is closer to solution. With partial solutions, Polya recommends either returning to phase (1), or looking at what is required and then 'work backwards'. The former is looping, whereas the latter amounts to a 'means-end' heuristic (find a means of closing the gap between where you are and where you should be).

Recent studies in problem-solving have looked at the twin issues of how it is learned and how it can be taught. Unfortunately, there is little evidence to indicate that this area is being studied systematically. Few investigations follow on from previous research. Kilpatrick (1969) has reviewed studies on various aspects of problem-solving which were conducted in the period 1963-1969. He discovered that investigations were being carried out in the following areas:

1 Problem solving ability
2 Problem solving tasks
3 Problem solving processes
4 Instructional programmes.

Most of the work in (1) concentrates on the translation skills required where sentences of posed questions are written in mathematical notation. In (2), investigators asked subjects to choose between problem alternatives, and in one study it was found that students generally preferred problems that were closely related to their interests and experiences. Kilpatrick notes that mathematical problems are seldom used in the pursuit of (3); however, some valuable work has been carried out in more recent times and this is discussed later. Instructional programmes, (4), have concentrated on heuristic methods. In one experiment the subjects were taught to use one of three kinds of heuristic related to two theorem-proving tasks:

Task specific heuristic (applicable to the training task only)

Means-end heuristic ('bridge the gap')

General planning heuristic (similar to Polya's)

From the results of the experiment it was suggested that:

(a) Task-specific heuristics did not facilitate performance on the training tasks; in fact, the more general heuristic was found to be more effective than the others in several tasks

(b) The planning heuristic was superior to the others on the dissimilar transfer task

(c) From significant interactions, general heuristics learned in the first training task were practised on the second task, thereby facilitating transfer.

Hatfield (1976) has also reviewed several studies. One study revealed that achievement in mathematics had a large effect on successful mathematical problem solving ability, although the use of heuristic strategies did have some relation to this ability not accounted for by mathematics attainment.

Gagné (1966) identifies problem-solving processes as a linear sequence of stages. His development follows from attempts to investigate the intervening processes between 'stimulus' (posed problem) and 'response' (action taken).

His four stages are:

1 Recall of subordinate rules

2 Search and selection

3 Combining subordinate rules

4 Verification

Treilibs (1979) has constructed a table comparing problem solving processes (based on Gagné's work) with modelling stages. It is a useful comparison to make, although Treilibs admits that "the problem solving processes have been 'forced' under the same headings as the modelling processes". Essentially, the table may be summarised as shown on the following page:

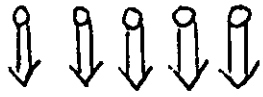
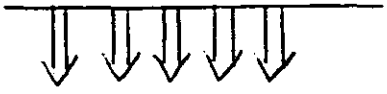
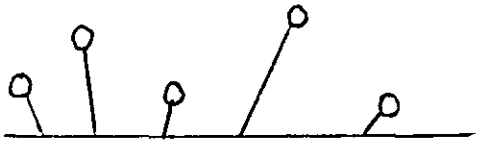
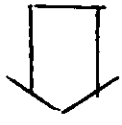
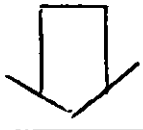
Problem solving (As per Gagné (1966)	Mathematical Modelling As per Burkhardt (1978)
 Recall of Subordinate Rules	
 Search and Selection	Formulation
 Combining Subordinate Rules	
Provisional Rule	Solution
 Verification	Interpretation
 Solution Rule	Validation

Table 1: Comparison of problem solving and modelling (Treilibs, 1979)

Table 1 is based on 'The Flowchart Phases' of Treilib's table.

Gagné's recall of subordinate rules depends on the store of previously learned rules and will be more successful if the solver has a good memory. Search and selection requires the solver to distinguish relevant from irrelevant aspects of the problem. Combining subordinate rules is a difficult process, for only certain combinations will lead to a successful solution. Verification requires the solver to try specific numerical instances in his solution as a checking procedure.

Although Table 1 provides useful insights into problem solving processes, it does seem to relate only to those mathematical modelling exercises which are themselves based on well-structured and well-posed practical problems. As mentioned earlier in Section 3.1, mathematical modelling activities are usually based on tackling realistic, and by their very nature, ill-posed problems.

Most researchers agree that prior experience in both problem-solving activities and in particular content areas is a very important ingredient for success. This also applies to mathematical modelling, and hence the need for more problems which provide practice in any given application domain. One of the chief difficulties in providing such problems is in graduating them in order of level of sophistication, particularly in modelling.

3.3 Mathematical modelling processes

To the extent that modelling processes are similar to problem-solving, the points discussed in the last section are relevant to both types of activity. The chief differences appear in the formulation and validation stages, the former presenting the greater conceptual challenge in modelling. With regard to the validation stage, data is required. In problem-solving sufficient data is usually provided in the problem statement, whereas the modeller has often to collect his/her own. In higher education, especially, data is either provided to save

time - in which case the modelling exercise is made somewhat easier, or for project work, students are expected to search for and collect their own. This section concentrates on formulation processes.

What little has been done has either been closely related to problem-solving processes, eg. Burkhardt (1978, 1979, 1981), Treilibs (1979), or has been tackled from a broad methodological point of view, eg. Clements (1982).

Figures 3 and 4 in Chapter 2 show Burkhardt's modelling and formulation processes in flowchart form. The dangers of over-simplification of this method of portrayal are emphasised by Burkhardt and, in particular, he points out that the highly oscillatory nature of the formulation and solution stages is hidden. However, even with this caveat, one is still left with the impression that the processes are carried out one after another (linear sequencing) or are topologically equivalent (linear with looping). Thus, in the case of formulation (F) and solution (S), we have the following representations:

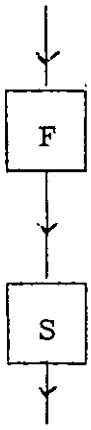


Figure 7: Formulation/Solution: Linear sequence

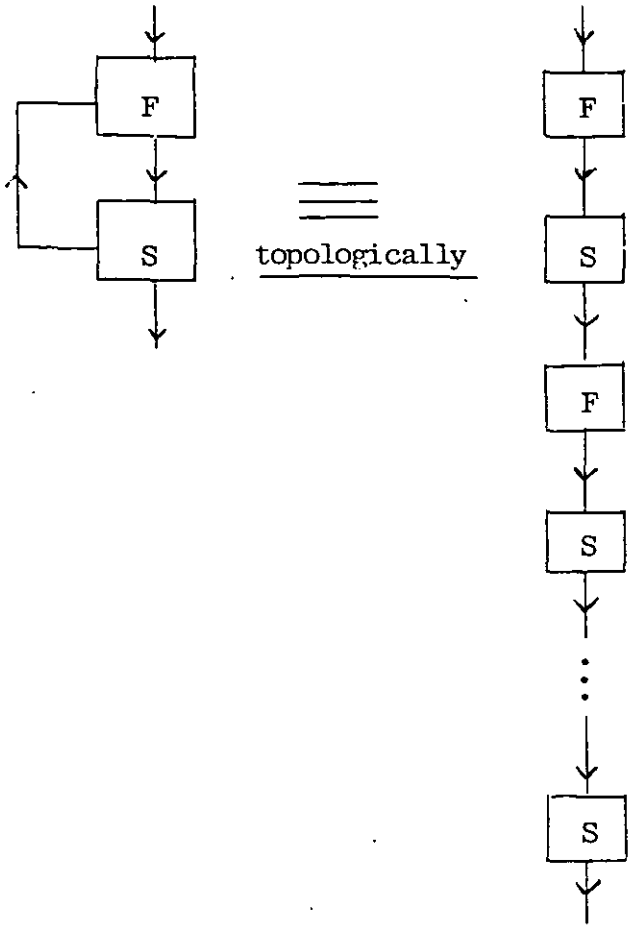


Figure 8: Formulation/Solution: Linear sequence with looping and topological equivalence

No research has been reported on a possible non-linear approach which shows more realistically the links between formulation and solution. Such an approach would require a suitable breakdown of both formulation and solution stages into smaller components, with a corresponding development of the complex linkages joining these smaller components.

Burkhardt has, however, identified some key features of formulation which could serve as the 'smaller components'. From Figure 4, Chapter 2, these features are:

Generate ideas on the empirical situation
Identify mathematical variables
Guess some relations

These features are prescriptive and so may be viewed as heuristics. They relate to analytical rather than descriptive modelling, which is relevant to the investigations carried out in this

thesis. Treilibs has provided a further breakdown of formulation and refers to his list as a set of skills:

GV: generating variables
SV: selecting variables
Q: identifying the specific questions
GR: generating relationships
SR: selecting relationships

Treilibs devised a set of tests, for each skill, and a set of problems on modelling skills generally, and administered these to a group of sixth formers who had no previous experience of modelling but who had above average ability in mathematics (predicted grade A or B potential at GCE 'A' level). In order for these tests to be carried out under controlled conditions, it was found necessary to restrict testing to well-defined problems that could be tackled by students working under examination conditions. Treilibs did not have the opportunity of pursuing more realistic and more complicated and time-consuming project type problems (the latter being more usual in higher education). He found that 'conventional' mathematics ability correlated significantly with scores on the characteristics tests on Q and SR only. SV was found to correlate significantly with neither modelling ability (as measured by a screening test) nor ability in mathematics.

What is additionally required is a better understanding of what may be termed the 'reality-mathematics interface', especially for the more complicated modelling exercises that are carried out in higher education. Many authors have addressed themselves in general terms to the problems of translating genuinely practical situations into more precise terms (not necessarily only mathematical), but hardly any have related this translation in such a way that gives a deeper understanding of formulating and solving the ensuing mathematics. These matters are very difficult to identify and so analysis of the activities involved is consequently even more difficult. Morris (1967), as discussed in Chapter 2, was one of the early authors to provide significant guidance in this connection. More recently, for example, Rubin

(1982), and Oke and Bajpai (1982), report on their experiences of teaching model formulation to undergraduates.

Rubin suggests the following general procedure, abbreviated here, as an aid in the initial stages of formulation ('system realisation'):

1 Identify the three basic components of the modelling problem: information, questions, evaluation criteria

2 Formulate the objectives

3 Make a list of variables used in the statement of the objective

4 Determine types of information required. Introduce new variables if necessary

5 Identify components which variables describe

6 Simulate phenomenon in a diagram. Add new variables if necessary

7 Continuation of step (6)

8 Examine list of variables for inconsistencies and redundancies

9 Remove inconsistencies and redundancies

10 Eliminate inconsistent or redundant interactions.

Having carried out the ten initial steps of formulation, Rubin then suggests various types of manipulation, similar to Morris' list, as a prelude to mathematical solution, eg: making variables into constants. Rubin does not comment on student feed-back on these procedures, and one suspects that it is impossible to carry out the steps in the order suggested. For example in step (8), although some inconsistencies might be spotted early on, it is highly unlikely that redundancies will be identified - these are usually only noticed at the solution stage (Oke and Bajpai (1982)). However, Rubin's list does give

a flavour of the sort of difficulties that are encountered in trying to analyse complex formulation processes that are encountered in higher education.

Oke and Bajpai (1982) emphasise the teaching aspects of problem formulation based on their experiences of teaching undergraduates in physics and engineering. They emphasise the importance of building up gradually in formulation, particularly with students previously inexperienced in modelling. They present typical lecturer/student interactive responses to the following range of activities: perceptions of real problem (breakdown into simpler problems if necessary); abstraction of perceptions (level of detail kept to a minimum, simplifying assumptions made); obtain initially only a crude representation. They report that students, like professional modellers, are able to see what are relevant variables only once a solution is obtained (the mathematics helps to 'eliminate'). They also report, that even the crudest of formulations and solutions often provide valuable insights to the original problem, and, in turn, provide further guidance on how to proceed to a more complicated formulation and solution.

Clements (1982) suggests an alternative to linear sequencing, or linear sequencing with looping, in developing a framework of modelling processes. His development relates to the whole range of modelling activities, and draws its inspiration from the system movement and Checkland's (1975) 'soft' system methodology. Although his discussion is in very general terms, it does offer some insight into systems approaches in tackling complex processes that could be helpful in providing a better understanding of the formulation/solution interface in mathematical modelling. The emphasis is on holistic rather than reductionist approaches. The former relates to viewing a system as a whole, even if one does not understand each of the component parts. Reductionist approaches, on the other hand, refer to the scientific paradigm where each component is reduced to the simplest level of understanding before any analysis is performed. Systems methodologies, however, do not conflict with the scientific approach, but complement it. As Checkland says:

> Hence there is an incentive to examine alternative paradigms to those of natural science, while continuing to build on the scientific bedrock: rationality applied to the findings of experience.

Clements quotes Checkland again in referring to the distinctive, and non-linear, features of the systems approach:

> ...although the methodology is most easily described as a sequence of phases, it is not necessary to move from phase 1 to phase 7: what is important is the content of the individual phases and the relationship between them. With that pattern established, the good systems thinker will use them <u>in any order</u>, <u>will iterate frequently</u>, and may well <u>work simultaneously on more than one phase</u>.
>
> (my underlining)

So, Checkland is discussing a much more complex linkage of phases (or stages) than is suggested in the usual descriptions of modelling processes. Since the formulation/solution interface is the most difficult aspect of modelling, it would seem that this systems approach could lead to better understanding. This is taken up further in later chapters of this thesis.

3.4 Teaching, learning and assessment

As pointed out in Chapter 2, there are no widely established methods of teaching and assessing mathematical modelling activities. Learning styles, and factors affecting them, are also little understood although some guidance may be obtained from work done in problem solving experiments as indicated in section 3.2. Even the latter, though, relate only to the simplest modelling problems that are well-posed.

Teaching and learning styles are closely connected but the chief difference between them is that the lecturer or teacher plays a much less active part in the latter. In learning situations, students work either individually or in groups

with a minimum of teacher guidance. Treilibs (1979), in experiments on group and individual working of modelling problems, found no evidence to support the notion that group performance is superior to that of individuals. He points out, however, that his use of ad hoc groups of students was a contributory factor; the student sample was obtained from several schools, and there was insufficient time for significant social relationships to develop, which in turn could have led to more natural and co-operative groupings. Problem-solving research generally points to the benefits of group working, although a number of disadvantages are also pointed out. For example, there is a difficulty if a group individual has a strong sense of direction and wishes to pursue a particular solution path, whilst other members of the group still wish to consider the possibility of a number of alternatives. The resolution of resulting tensions requires maturity and experience of group working.

Several authors in higher education recommend group working, see for instance Burghes and Huntley (1982), whilst others, for example Burkhardt (1981), Oke (1984), recommend a mixture of interactive teaching and group work. Certainly for inexperienced students, there seems much to commend the 'interactive' approach initially in order to get students started on a modelling problem. The teacher/lecturer lists initial suggestions on the blackboard, guides sensitively in creating one or two mathematical relationships (Burkhardt's crunch point) and then leaves students to continue working in groups. At the more detailed stages of a solution, however, it is often found that individuals are best left to work alone. As experience increases, extensive modelling projects are often set, particularly in higher education. These projects are often undertaken by groups, rather than by individuals, and it is left to each group to organise its mode of working.

Irrespective of teaching and learning styles, students inexperienced in modelling all tend to suffer with a common set of difficulties. Many authors have reported on some of

these, see for example Treilibs (1979), McLone (1979), Burkhardt (1979), James and Wilson (1983), Berry and Le Masurier (1984) amongst others. Some of these key difficulties are:

General lack of confidence

Loathness to simplify

Lack of skills in approximating and estimating

Inability to generate mathematical relationships

Knowing when to stop

Weakness in report writing

With skilled guidance from the teacher and as students gain experience, some of these problems are gradually overcome.

One way of overcoming students' difficulties is to provide a sufficient number of graded modelling exercises for them to carry out before setting, say, an end-of-term project. As mentioned in Chapter 2, there are beginning to appear a few published papers and books which make a contribution in this connection. The Spode group books (1981-1983) provide a number of modelling problems with hints for teachers at secondary school level. Books containing case studies in modelling aimed at undergraduates are also appearing; for example, Volumes 1 and 2 of the National Mathematical Modelling Workshop, James & McDonald (1981); James & Huntley (1984), and the North-East England Polytechnics' publication, Bradley, Gibson & Cross (1981). These publications are to be distinguished from those that only present models, for example Andrews & McLone (1976), in that hints for the lecturer are provided in James & McDonald (1981); James & Huntley (1984), and actual individuals' attempts at modelling some problems are provided in Bradley, Gibson & Cross (1981). However, these latest publications make no attempt to grade the modelling problems in order of difficulty. Most of the problems presented would be suitable for extended project work, perhaps taking several weeks to complete. It should be mentioned though, that with

less ambitious aims being incorporated, many of these case studies could be modified for use in introductory modelling sessions appropriate for the classroom.

Finally, there remains the question of assessment. Although most would agree that mathematical modelling is a much more complex activity than solving the traditionally well-defined mathematics problem, nevertheless some form of assessment is needed. Without such grading, it is difficult for students and others to gain an impression of their modelling performance. After all, other subjects are assessed in all types of courses, and so it would seem unrealistic to refuse to assess students in mathematical modelling. Furthermore, colleagues in other fields, such as fine art, manage somehow to form an opinion and attribute some mark or grade in accordance with that opinion. So, it may be argued, it should be possible to assess mathematical modelling.

The three main forms of assessment relate to:

Homework/Course-work	(small assignment)
Project	(major assignment)
Written examination	(formal, fixed-time)

Most authors agree that a formal written examination is the most inappropriate method. Occasionally it has to be used, see for instance the comments of Burley & Trowbridge (1984), in view of the large number (fifty or sixty) of students involved. With large numbers of students, staff resources usually do not stretch to the much more time-consuming process of reading and marking the more appropriate project type of assignment.

The marking of projects (small or large) is difficult because of knowing what criteria to use. Such criteria depend largely on one's understanding of the modelling process and on what students find most difficult in this process. Berry and O'Shea (1982) report on their experiences of assessing the mathematical

modelling project set in the Open University's MST 204 unit. This unit was presented to students for the first time in 1982, and it is the results for this year which are analysed by Berry and O'Shea, with some further discussion presented by Berry and Le Masurier (1984). The modelling project is a compulsory part of the course (unit) and occupies the students for about 40 hours of their time. The project is marked in two stages: the first, after 20 hours of work spread over two weeks, is where the student should have chosen a topic and written approximately one thousand words on the formulation stage. An abbreviated form of the marking scheme used is shown in Table 2.

Your task		Marks
1	Provide a statement of the problem	5
2	State variables and simplifying assumptions	5
3	Outline model to be used	5
4	Explain the mathematical formulation	5
	Total marks	20

Table 2

First Stage Assessment of Modelling Project in OU Course MST 204 (Berry and O'Shea, 1982)

The second stage of assessment, required the production of a final report based on an additional 20 hours of work spread over two weeks. An abbreviated form of the marking scheme is shown in Table 3.

Main section headings	Marks
Abstract	5
Formulation	5
Initial model	30
Data	10
Revisions to the model	20
Conclusions	5
Presentation	5
Total marks	80

Table 3

Second Stage Assessment of Modelling Project in OU Course MST 204 (Berry and O'Shea, 1982)

Thus, the first stage represents one fifth (20 marks) of the total assessment. It should be noted how few marks (5) are awarded to 'formulation' in the second stage, in view of the difficulties associated with this. However, if one combines 'initial model' marks with those for 'formulation', then one obtains 35 possible marks of the total. Relatively few marks (10) are given to 'data', since the experience of the OU with other projects has shown 'that students in difficulty may attempt to accumulate marks by amassing vast amounts of data'. A relatively high mark (20) is given to 'revisions to the model', thus encouraging students to be critical of their first attempts and to make some improvements. Berry and O'Shea report favourably on the consistency of project markers, quoting in one instance the set of marks out of 80 that were produced by 12 tutors on one student's project. The mean score, in this instance, was 55 with a standard deviation of 5.8, although caution is advised on statistical interpretation.

Berry and O'Shea's article is one of the most detailed recently published on assessment procedures in modelling. Their marking scheme represents an additive model, like most marking schemes, but Hall (1984) suggests that a product model of marking might be more realistic and also provide a more uniform method of marking different individual projects. He illustrates his method by referring to three main components in assessment, representing students' skills in modelling:

Content
Presentation
Drive

He provides a detailed list of sub-skills under each heading, and then proposes that a marking model should be both homogeneous and produce a zero condition. By homogeneous, is meant that if each component is given the same mark, as a fraction of the maximum for that component, then the total should be the same fraction of 100%. The zero condition means that if any component is given a zero mark, then the total should be zero; Hall argues, that no credit should be given if a vital component of modelling (content, presentation, or drive) is absent or is very badly done. The additive model and the product model are both shown to be homogeneous, whereas only the product model has the property imposed by the zero condition. If marks x, y, z expressed as percentages are awarded respectively to each of the components, then according to the additive model the project mark will be

$$A = (n_1x + n_2y + n_3z) / (n_1 + n_2 + n_3)$$

and according to the product model:

$$P = \sqrt[(n_1+n_2+n_3)]{x^{n_1} \times y^{n_2} \times z^{n_3}}$$

where n_1, n_2, n_3 are respectively the weights attached to each of the components.

So, Hall is definitely recommending strict standards according to a formal marking scheme in forming an assessment of modelling projects. In this connection, he further recommends that 'double-blind' marking is used (two markers neither of whom has seen the other's marks, mark each project). Several authors would seriously question the advocacy of formal marking, and would rather argue for 'impression marking'; for example, Burghes & Huntley (1982) recommend 'marking by interview' where groups of students discuss their report with a lecturer and a mark is jointly agreed. Berry & Le Masurier (1984) have even found that 'impression marking' has led to marks rarely differing by about 5 or 8 out of a total of 100 from those marks obtained by rigidly applying a marking scheme. It would appear that formal markers use some judgement by increasing the marks that they would originally have given to a section in order to make the total agree with their overall impression.

3.5 Summary and Conclusions

The last sections have reported on some of the most significant recent research and other investigations that have been carried out concerning the teaching styles, learning modes and assessment methods used in mathematical modelling. In order to be more effective in each of these areas, a research need has been identified which investigates more fully the processes of modelling and, in particular, the complex nature of the formulation/solution interface.

Sections 3.2 and 3.3 discuss problem solving and modelling processes, highlighting common features as well as differences. Both processes are creative and consequently what can be learnt in the one will also be of some relevance to the other. Formulation has been identified as the most difficult stage to carry out and the possibility of investigating the complex linkages between formulation and solution stages has been mentioned, drawing on the work of Burkhardt and Treilibs who

have broken formulation down to smaller steps, and Clements who suggests a 'soft systems' approach generally in modelling. Most of the reported research relates to problem-solving or the rather better posed, and hence more structured, modelling problems. In higher education, modelling problems tend to be more complicated and so the difficulties of analysing processes are even more pronounced.

In section 3.4, concerning teaching and learning styles, the two main approaches of interactive teaching and group working, or a combination of both, have been identified. Some of the key difficulties experienced by students have also been mentioned, and it is still an open question as to how best to remove or alleviate these difficulties. Clearly more needs to be done on investigations on various teaching and learning methods for all levels of student. Some research questions have been identified in this connection, and they relate to the construction of heuristic methods and of graduated modelling material suitable for classroom/workshop activities. Underpinning these requirements is the need for a better understanding of modelling processes, both in the sense of how an experienced mathematical modeller solves practical problems and how less experienced students tackle such activities. Without such additional understanding, assessment methods will also largely remain a matter of informed guesswork.

Not surprisingly then, even less has been published on the assessment of mathematical modelling. Some authors are against any form of marking and grading, relying instead on the informal opinions of lecturers. Others recommend 'impression' marking only, with a grade letter indicating performance, whilst a few strongly suggest that a formal marking scheme should be used with a detailed break-down showing how marks are awarded for each section of a report. The protagonists of formal marking make out their case for fear of undue bias affecting the final assessment if only impression methods are used. Yet, in one reported case, namely that of Berry & Le Masurier, marks formed by overall impression were very

close most of the time to those marks obtained by following a marking scheme. There is, thus, a need for more reported experiences of modelling teachers in this connection before a more balanced view may be formed.

CHAPTER 4

AIMS AND SCOPE OF THE RESEARCH PROJECT

4.1 Introduction

The principal aim of the project is to investigate formulation-solution processes in mathematical modelling. The extent to which these processes lead to better guidance and understanding of teaching, learning and assessment in mathematical modelling is also investigated.

In order to be able to carry out such investigations, the following main activities were chosen:

1 The development of case studies of the mathematical modelling approaches used in the solution of practical problems

2 The development of courses in mathematical modelling for students at a variety of levels

3 The design of teaching and learning experiments

4 The study of various assessment modes and the construction of marking schemes

Case studies and teaching experiments related to these have been carried out with students at a variety of levels, namely:

(i) Undergraduates, mainly engineers and physicists

(ii) Postgraduates, mainly teachers attending an MSc course in mathematical education

(iii) Secondary school students with MSc teachers

A secondary aim, but a very important one, is to investigate to what extent the activities outlined in (1) - (4) above are affected by student type. In particular, how general maturity, intelligence, and level of attainment in mathematics affects modelling ability. Two main experiments have been carried out in this connection; the first concerned one case study, namely 'Minimisation of sound distortion in a record-player', being presented to one sample in each of the student categories (i) - (iii). The second experiment involved an analysis of formulation-solution processes of a variety of case studies presented to secondary school students, undergraduates, and others, in an attempt to find common features.

All the investigations have been carried out using deterministic and analytical modelling problems. Most of the case studies are based on problems involving applications of mathematics in the areas of the physical sciences and engineering. This is so largely because of author interest and experience, yet, as pointed out in Chapter 2, very little attention to mathematical modelling has been paid in these areas. Most attention has focused on operational research applications in the social and organisational sciences. However, for the purposes of carrying out investigations with secondary school students, certain organisational problems, as well as some problems in athletics, have also been presented.

Although the study of formulation-solution processes is fundamental in investigating mathematical modelling, it is stressed that a fuller understanding can only be gained by considering modelling as covering a whole range of complex activities. Consequently, most of the case studies considered in this thesis also involve the important interpretation and

validation stages of a model. Just as it has been argued in Chapters 2 and 3 that formulation and solution activities are so interwoven that they should be treated holistically, so it may be argued that all stages in mathematical modelling should be viewed as a single whole.

Mathematical modelling in education is still in its pioneering stages and little is known of the relevant parameters which are involved in its processes. Consequently, teaching mathematical modelling is still in its early experimental stages. Any investigation in this area of work must therefore choose a balance between focusing on certain features and providing a broad portrayal. The philosophy of approach in this thesis is an attempt to achieve such a balance. The focal point is a study of formulation-solution processes, and the broad portrayal is provided by a description of the observations from experiments on how such processes are related to teaching, learning and assessment in mathematical modelling. The scope of the project is therefore limited by being primarily concerned with description and interpretation rather than by measurement and prediction. The theoretical analysis of the complex linkages between formulation and solution is deemed to be the most important and creative part of the project.

The teaching experiments were conducted under genuine working conditions in the classroom, with the usual constraints of fixed-time periods in operation. The students involved were either taking mathematical modelling as part of the curriculum of their course, or were introduced to modelling by a specially constructed series of lessons, lectures or workshops. The difficulties of observation involved in teaching and learning situations were mentioned in Chapter 2, for example see Morris' (1967) account of a subject's refusal or inability to verbalize what the person is doing. Such difficulties and attempts to overcome them are discussed in later chapters.

4.2 The Development of Case Studies

In an endeavour to provide modelling approaches to practical problems a set of case studies was developed and used in the investigations. The case studies cover the following problems:

1 U-tube accelerometer (Oke & Bajpai (1982))

2 Modelling the heating of a baby's milk bottle (Oke (1979))

3 Speed-wobble in motorcycles (Oke (1981))

4 Minimisation of sound distortion in a record-player (Oke (1981))

5 Windmill power (Oke (1983))

6 Pole-vaulting (Sheridan (1980))

7 Central-heating (Oke, Internal Report)

Several of the above problems have been tackled by students at all levels. Additional case studies were also devised for work with secondary school students:

8 Evacuation of a school (Wilson (1983))

9 Motorway and 'A' road travel costs (Wilson (1983))

By referring to secondary school students by S, to undergraduates by U, and to postgraduates by P, the following table shows which case studies were used with which type of student:

		Student type		
		S	U	P
Case Studies	1		✓	✓
	2			✓
	3		✓	✓
	4	✓	✓	✓
	5		✓	✓
	6	✓		✓
	7		✓	✓
	8	✓		
	9	✓		

S: Secondary school

U: Undergraduate

P: Postgraduate

Table 4: Modelling case studies used with different types of student

The numbers used in Table 4 identify the case studies listed earlier.

Secondary school students were either in the fourth, fifth or sixth forms covering a wide range of ability from potential CSE grade 4 to GCE A/L grade A or B. The undergraduates were all in the second year of degree courses with strong mathematics and physics/engineering components. The postgraduates were largely those graduate secondary school teachers following a two-year part-time MSc (CNAA) course in mathematical education at the Polytechnic of the South Bank. However, several workshops were also arranged with the AIMEC* Project group of graduate teachers from India at the University of Technology, Loughborough. One case study, as mentioned in section 4.1, was presented to a sample of students of each of the three categories (4: Minimisation of sound distortion in a record player); in the case of the secondary school students, this case study was presented by O'Hare (1980).

4.3 The Development of Courses

The major development concerns the construction of the mathematical modelling component, one of four, in the part-time MSc course in mathematical education referred to above, and

* (All India Mathematics Education at CAMET)

also in Chapters 2 and 3. The author is team leader of a group of three which teach on this course, and the structure and content of the modelling component, together with teaching, learning and assessment methods have been reported in Oke (1980, 1984b). The first year of the course concentrates on the development of fairly elementary models, covering a wide range of applications in the physical sciences, life sciences and social/organisational sciences. The second year concentrates on methodological issues in modelling and on the teaching of modelling to students at a level familiar to the teachers. Several assessment modes have been experimented with on this course and details are provided in a later chapter.

One of the MSc students, Jones (1980), designed a short course on modelling for undergraduates on an engineering product design degree. Brief details of the design are also provided in a later chapter.

The author has supervised nine mathematical modelling projects on the MSc course, and three of them, namely Sheridan (1980), O'Hare (1980), and Wilson (1983), were specifically designed to investigate aspects of teaching modelling in a secondary school which provide further evidence for this thesis.

4.4 Teaching and Learning Experiments

This section is closely related to section 4.2 on the development of case studies and to the following section 4.5 on formulation-solution processes. The design of case studies is based on requirements for teaching and learning in the classroom and modelling workshop, and formulation-solution processes affect teaching, learning and assessment styles. The key areas investigated, and their mutual interactions, may be summarised as shown in the influence diagram in Figure 9.

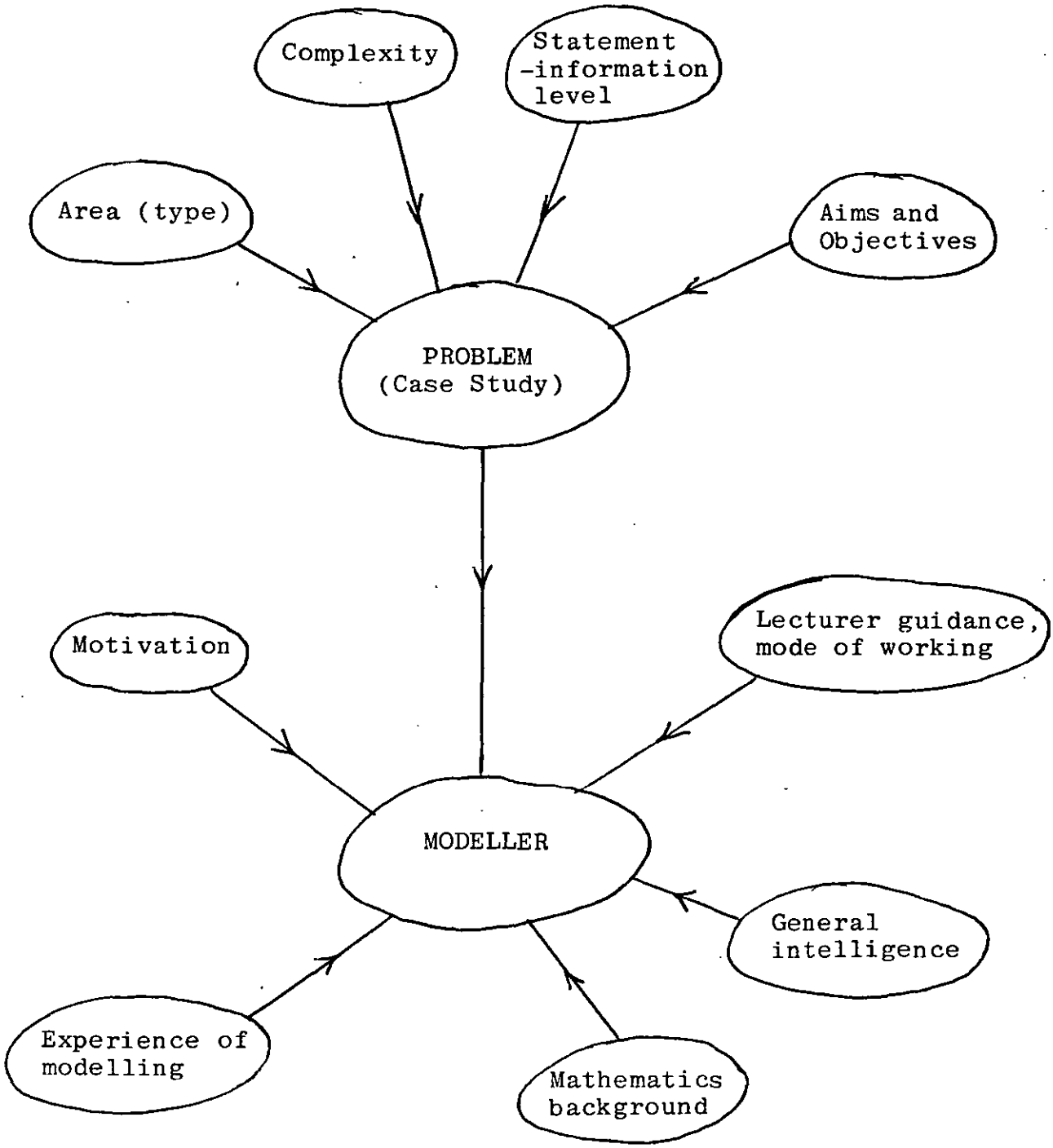


Figure 9: Influence diagram of problem and modeller

The two foci in Figure 9 are 'PROBLEM (Case Study)' and 'MODELLER', and the connection between them is indicated by the arrow from the former to the latter. Features influencing 'PROBLEM' and 'MODELLER' are also shown with one-direction arrows. The use of one-direction arrows is to emphasise what is affected and by what, although two-direction arrows could be used in some instances; for example, when the (student) modeller simplifies his/her assumptions and subsequent mathematics, this has implications for the problem. So, an arrow from 'MODELLER' to 'PROBLEM' could also be drawn in to indicate this two-way interaction.

The teaching and learning modes which have been observed are shown by the 'Lecturer guidance, mode of working' and 'MODELLER' link in Figure 9. Further details of this link are shown in Figure 10.

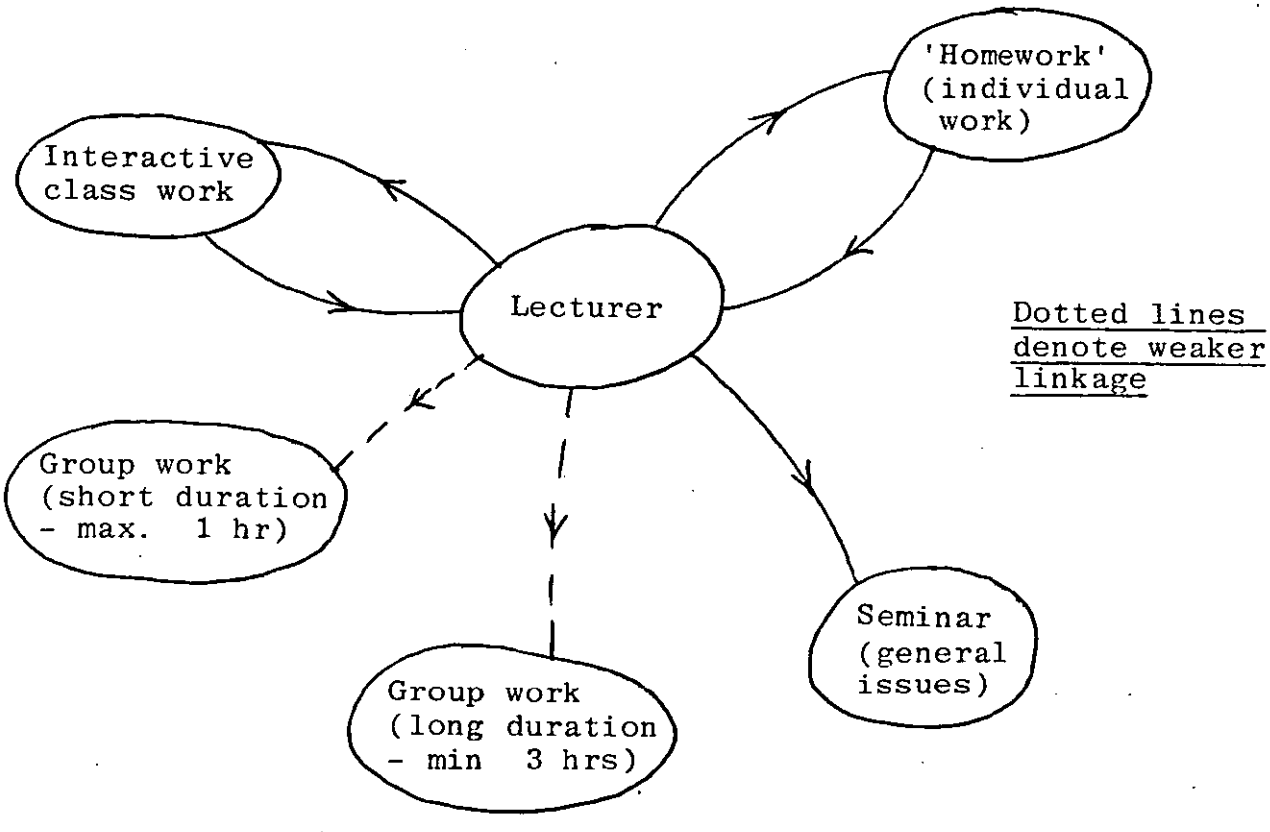


Figure 10: Influence diagram on teaching and learning modes

Figure 10 is intended to show the different roles played by the lecturer in various teaching, learning modes. The roles played in this investigation may be summarised as follows, referring to Figure 10:

1 Presenter of modelling problems

Lecturer interacting with students: lecturer and students start developing a model together.

2 Consultant

Lecturer poses practical problem, students model it. Lecturer guides, providing a few hints if students get stuck, in short duration group work. The very minimum or no hints are provided, and lecturer acts only as 'expert' (eg, engineer) asking group to model and provide mathematical solution in long duration group work. Group size: usually four students. Several groups formed from one class.

3 Assessor

Lecturer evaluates individual homeworks, group course-works, projects, and examination papers (if set).

In the investigations, the author acted as unobtrusively as possible during group working and maintained a log. During interactive teaching experiments, audio and audio-visual recordings were made. In some cases, physical apparatus or a film was shown as part of the presentation of a practical problem.

4.5 Formulation-Solution Processes

In an attempt to gain a fuller understanding of the complex nature of formulation-solution processes, two theoretical ideas were developed:

1 Concept matrix (CM)

2 Relationship level graph (RLG)

The concept matrix (1) arises from analyses of modelling activities and is designed to show which features, or concepts, are used in different modelling stages. The matrix is also intended to provide information on the type of each concept. Since the features or concepts which arise in the development of a mathematical model are extremely varied, both in clarity and in complexity, it was considered inappropriate to attempt to develop a simple hierarchy of concepts as discussed, for example, by Skemp (1979). Initial attempts at classifying concepts by their relevance to the model were abandoned, since relevance only becomes clear in an a posteriori sense, that is after the model has been constructed and interpreted. The matrix finally adopted is two-dimensional and is represented in Figure 11.

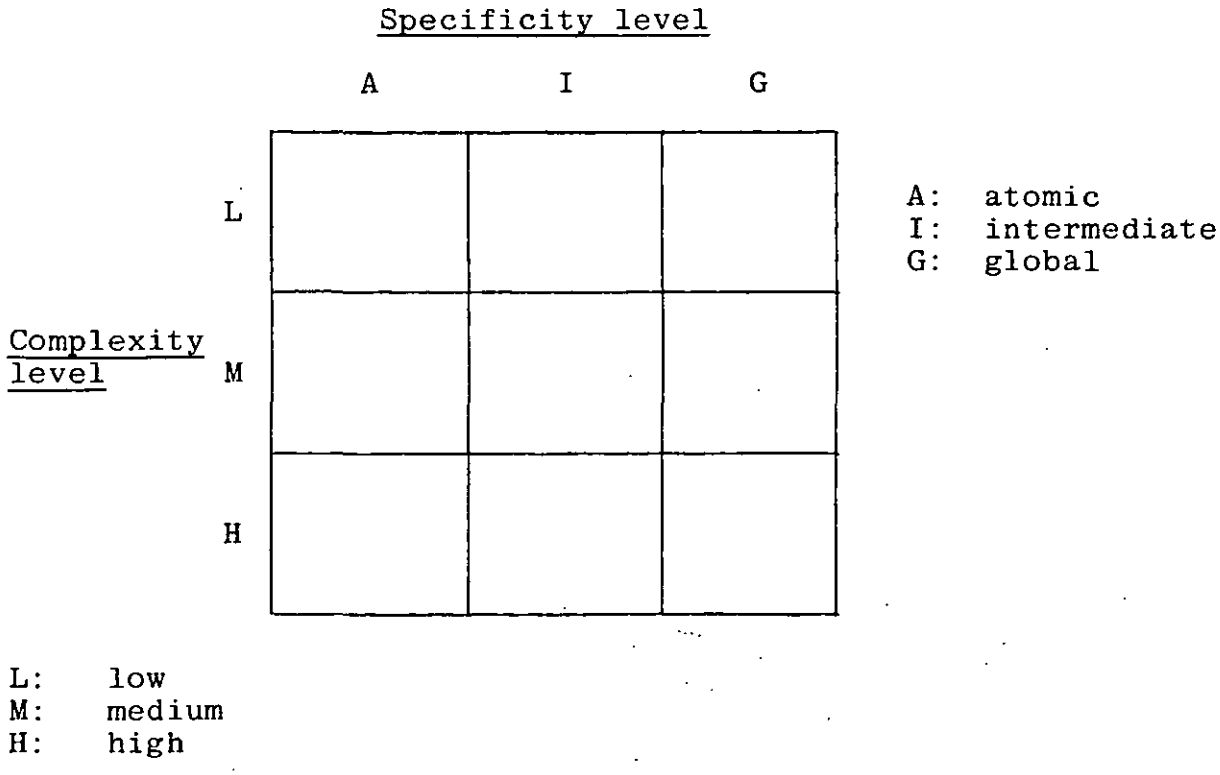


Figure 11: Concept matrix

Later chapters show that initial formulation takes place by identifying features that tend to fit at or near the bottom right-hand corner of the matrix. Early, and subsequent,

solution activities involve features tending towards the upper left-hand corner of the matrix. 'Global' features tend to be those that are only broadly related to the problem in hand, whereas 'atomic' features are those in the most simple form, eg, variables, constants, which are immediately amenable to mathematical treatment. A 'high' complexity level denotes a feature, that may be highly specific to the problem, which may not be easily quantified. 'Low' complexity level indicates fairly easy quantification.

The relationship level graph (2) was developed to show which relationships were formed and at what stage in the complete formulation-solution activities. Initial, and more or less obvious simple relationships are denoted by the level 0 (zero). These relationships, although usually mathematical in nature, require no mathematical solution techniques to derive or form; they are mathematical representations of one variable and its dependence on another or others, written down from an initial understanding of the problem. This initial understanding, which might well arise from inspired guessing, is often related to knowledge of a non-mathematical type, eg, of physics, biology, or medicine, depending on the problem. Usually one, two, or at most three, of level 0 relationships need to be formed in order to be able to use mathematical techniques to form new relationships. The first simple relationships deduced mathematically from level 0 types are then referred to as level 1 types. After further mathematical solution work, and frequently the need of forming another level 0 relationship, level 2 types may be derived, and so on. Many modelling problems carried out by undergraduates, and others, reach a very significant stage by the time levels 6-8 are reached. Quite often an acceptable solution is obtained without further improvement being required. A typical graph showing relationship levels and their generation is shown in Figure 12.

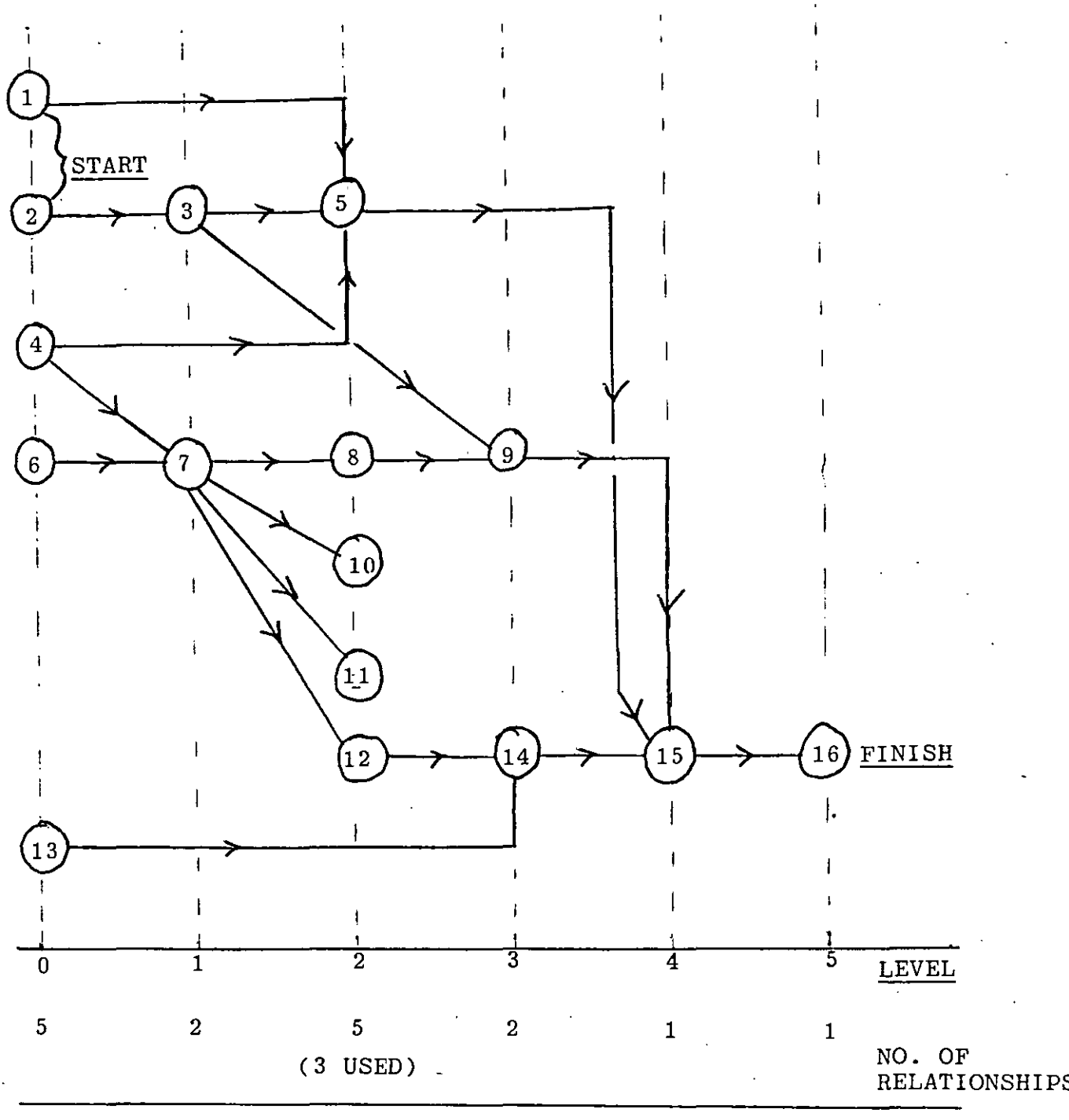


Figure 12: Relationship level graph

The number in each circle indicates the order in which each relationship is formed. A glance at Figure 12 shows that there is no particular order in which relationships are formed, but that level numbers 0, 1, 2, ..., indicate overall progress from starting a model (relationship (1), (2) and level 0) to finish (relationship (16), level 5). Note also, that not all relationships generated are used in obtaining a final solution; for instance, relationships (10) and (11), level 2, are not used in obtaining (16). One of the most important features illustrated in a relationship level graph, is that the mathematical solution stage is intimately interwoven with the formulation stage; mathematical techniques are themselves used in the generation of relationships. Most of the reported literature emphasises the need to formulate (generate features and relationships) before attempting a mathematical solution, although Burkhardt (1981), Treilibs (1979), and others have made the point that movement between formulation and solution is highly oscillatory. The relationship level graph shows, however, that formulation-solution processes are more complicated than a linear sequence of steps followed by oscillations. The numerical ordering in the circles shows an almost random order of events in some instances, whilst the generation of some relationships, eg, (15) in Figure 12, take place simultaneously working at a variety of levels ((5), level 2, (9) and (14), level 3). The latter phenomenon is a clear illustration of Checkland's (1975) reference to the distinctive and non-linear nature of the systems approach mentioned in section 3.3 in Chapter 3. It should be pointed out, however, that this non-linear interpretation of modelling is new, as to the author's best knowledge, there are no published detailed accounts of such processes in either 'systems' or in 'mathematical modelling'.

In later chapters, students' attempts at modelling selected problems will be analysed in terms of the concept matrix and relationship level graph. It should be emphasised, however, that it is the relationship level graph (RLG) that provides the deeper insights into modelling processes. The RLG is more 'dynamic' that the concept matrix (CM) in that it illustrates progress through relationship generation towards some goal. The CM is essentially only an aid in classifying features that are identified in a model development.

The implications of this new analysis for teaching, learning and assessment in mathematical modelling are quite fundamental. For instance, one form of guidance for the student modeller is to get started as soon as possible with mathematical techniques, since the mathematical mode of working will help in focusing on the original practical problem and also help in the identification of new features and the formation of new relationships. In the case of assessment, it could be recommended that formulation and solutions are lumped together, and not treated separately, for the purposes of marking.

4.6 Summary and Conclusions

The principal aim of the project is to investigate formulation-solution processes in mathematical modelling. The extent to which these processes lead to better guidance and understanding of teaching, learning and assessment in mathematical modelling is also investigated.

The philosophy of approach is to achieve a balance between focusing on certain features and providing a wider perspective. The focal point is the theoretical development of formulation-solution processes by means of:

A concept matrix

A relationship level graph

The wider perspective is provided by description and interpretation of various teaching and learning experiments based on selected case studies of the mathematical modelling activities involved in the solution of deterministic and analytical practical problems. The theoretical analysis of the complex linkages between formulation and solution is deemed to be the most important and creative part of the project.

In view of the early 'state-of-the-art' stage of mathematical modelling in education, it was felt to be inappropriate to carry out any statistical analysis on the parameters identified

in the teaching and learning experiments. Consequently, in the spirit of 'illuminative evaluation', see Parlett and Hamilton (1977) for example, the observations of students modelling in real working conditions are based on complex situations in which little or no attempt is made to control, manipulate, or eliminate factors pertaining to the classroom or workshop. Notwithstanding the latter comment, the author has been able to choose in his capacity as a lecturer, how a modelling session would progress:

(i) Interactively with students

(ii) Students working in groups

(iii) Students working individually

or (iv) A combination of the above.

CHAPTER 5

THE CASE STUDIES

5.1 Choice and Design

This chapter presents a selection of practical problems and possible modelling approaches. The emphasis is on the problem and a mathematical model of it, rather than on the modelling processes themselves. Observations and analysis of some of the problems tackled in a variety of teaching and learning environments is left to subsequent chapters. In view of the fairly large number of case studies involved, namely nine, only abbreviated modelling solutions are presented. Key features of initial formulation-solution activities are emphasised, and details of interpretation and validation are included where appropriate. The modelling approaches used have been devised by the author, with the exception of two case studies which were developed by an MSc teacher, Wilson (1983), for a dissertation under the author's supervision. One other case study was initially developed by Sheridan (1980), also under the author's supervision for an MSc, but this has subsequently been extended by the author. Several of the case studies have been published in their entirety, details already having been mentioned in section 4.2 of chapter 4. Since most of the students who have tackled the problems have some physics background, most of the case studies involve problems in the physical sciences and technology areas. Each model produced is deterministic and analytical.

The development of the case studies was based on the following design features:

1 Motivation

Each problem, as far as possible, is practical and is connected in some way with every-day reality. Thus, it is hoped that students have some intrinsic interest in the background to a problem.

2 Level of difficulty

This is largely determined by the students' background, level of maturity, and previous experience of modelling. Each problem has sufficient scope for simple approaches to give good insights, and also for the more advanced students to produce more sophisticated solutions.

3 Scope

Each case study provides an opportunity for formulation-solution, interpretation, and as often as possible, validation. Most case studies also provide the opportunity for sub-problem identification (breaking down into smaller and related problems). Treatment is often hierarchical, ie, the end of one sub-model leads naturally to the beginning of another sub-model, or is linked in the sense that one sub-model is related to another but not following end-on.

4 Content

Each case study has a problem statement and model development. Sometimes data is provided, on other occasions students are encouraged to ask for data (which is then provided as far as possible in the form they want). Some case studies have 'follow-up' questions which either test understanding of a given model, or ask for extensions, or pose slightly different problems to be modelled from scratch.

5 Duration of modelling exercise

Most case studies involve problems which are appropriate for modelling in either of the following modes:

(a) Interactive class work (~ 1 hr duration)

(b) Short duration group work (~ 1 hr duration)

(c) Long duration group work (~ min. 3 hrs duration)

(d) Extended project (~ 2/3 months in own time)

(See section 4.4, chapter 4, and subsequent chapters on teaching and learning.)

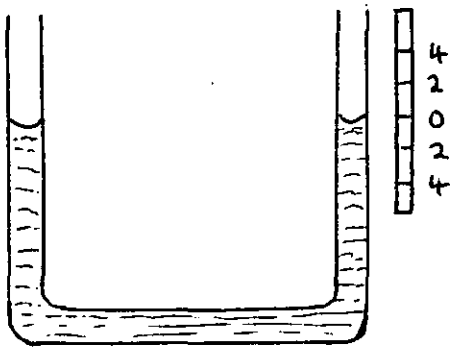
5.2 U-Tube Accelerometer

This is a simple problem and was first posed by Crank (1962) and again by D'Inverno & McLone (1977), although neither of these offer a solution. The initial formulation-solution experiences with undergraduates by Oke were reported in Oke & Bajpai (1982). The problem is simple in the sense that it is well-defined, requires only the minimum of physics, and only the most trivial mathematics is needed for an initial solution. It is one of the first problems presented to students for these very reasons, and yet it still seems to be quite a challenge to the uninitiated in modelling.

Problem statement

A U-tube accelerometer is fitted with its vertical limbs fore and aft in a car. The U-tube is partly filled with a liquid, and a graduated scale is provided to measure the difference in levels of the fluid as the car accelerates. Consider the design features of the accelerometer for various conditions.

Accelerating car



U-tube accelerometer

Fig 13 U-tube accelerometer

Model construction

Assumptions

1 U-tube of uniform circular x-section (bore: radius r)

2 Car does not jerk, therefore uniform acceleration (a)

3 Limbs of U-tube are vertical, with one horizontal tube joining them: distance between vertical limbs = ℓ. Vertical difference in fluid heights = h

4 Intensity due to gravity: g (assumed constant)

5 Density of fluid constant: ρ

6 Surface tension unimportant

7 Viscosity of fluid unimportant

8 Fluid is incompressible

9 Car accelerates on horizontal surface

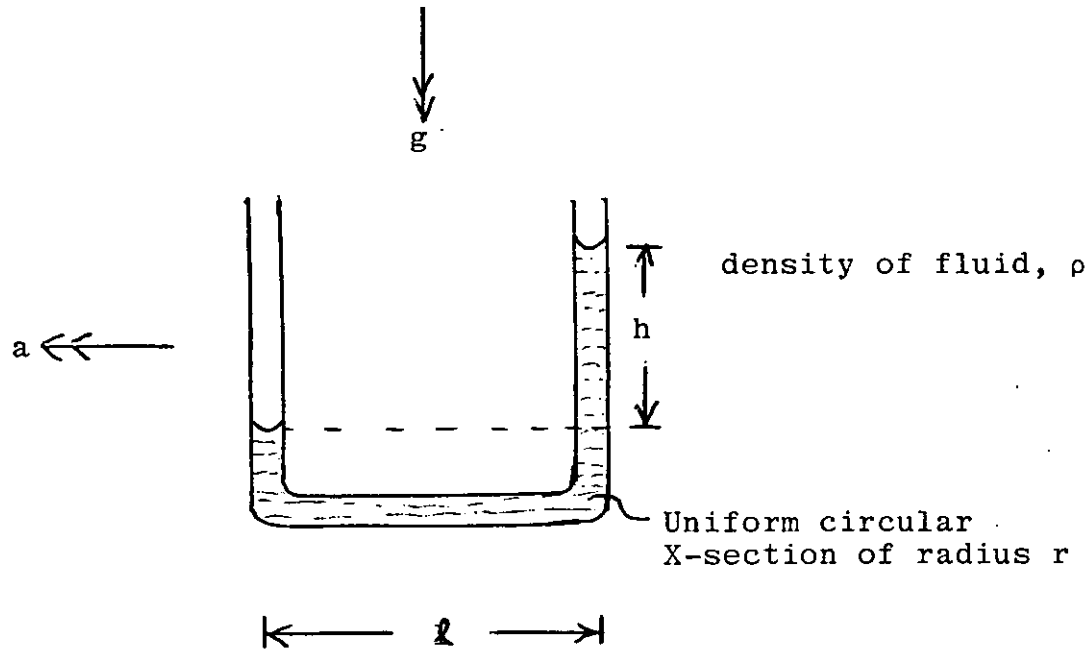


Fig 14

Variables identified for U-tube accelerometer

Accelerating pressure of fluid in horizontal limb is balanced by the pressure due to the vertical difference in levels of the fluid, hence

$$\frac{\text{mass of fluid in horizontal limb}}{\text{x-section area}} \times a$$

$$= \frac{\text{weight of fluid of ht. h}}{\text{x-section area}}$$

ie, $$\frac{\pi r^2 \rho \ell a}{\pi r^2} = \frac{\pi r^2 \rho g h}{\pi r^2}$$

hence $$a = \frac{gh}{\ell}$$

Note that neither r nor ρ appear in the expression for the acceleration 'a'. Relevance of variables in early formulation is often shown in modelling to be unfounded; the mathematics determines relevance by elimination or otherwise.

A further implicit assumption relates to the end-points in the measurement of ℓ. It is assumed that $\ell >> r$, and consequently the precise locations of the end-points become unnecessary.

This model has natural extensions which show:

(a) The limb connecting the vertical limbs need not be horizontal

(b) The vertical limbs need not be vertical, but difference in fluid levels must be measured vertically

(c) The x-section need not be circular, although for practical purposes it should be uniform in shape

Follow-up problems

1 Consider problem where horizontal limb has a different uniform x-sectional area to the vertical limbs

2 Consider fluid initially (a = 0) with fluid of different density in horizontal limb to that in the vertical limbs

3 How does analysis in (1) & (2) affect sensitivity of instrument? (Consider an old Ford Escort versus a Porsche)

5.3 Modelling the Heating of a Baby's Milk Bottle

The inspiration for this development came from a suggestion by Pollak (1968). The treatment consists of four models developed hierarchically and full details may be found in Oke (1979). The intention is to provide an opportunity to model a heat-

exchanger problem, which is typical of problems with a definite theme. It is homely, everyone feels that they can understand easily what is involved, although some may argue that it borders on the whimsical. A related problem in the home concerns the heating of a hot water cylinder.

Problem statement

Imagine a sleepy parent removing a baby's milk bottle, full of milk, from a refrigerator, placing it in a saucepan of water and preparing to heat the milk to a comfortable temperature. The saucepan would be heated by either a gas or electric ring. How much water should there be in order that the milk is heated as quickly as possible?

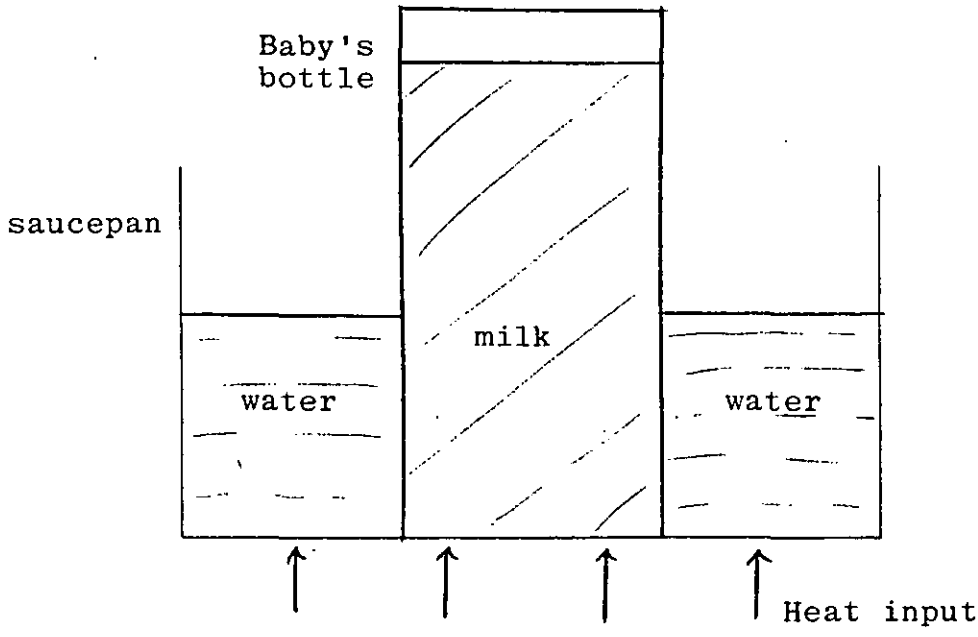


Fig 15

Simplified illustration of a baby's bottle being heated in a saucepan containing water

The author, with the help of Mr Jones of the School of Physics at the Polytechnic of the South Bank, carried out a series of experiments in order to collect data for validation purposes. Complete results from these experiments may be found in Oke (1979). When the case study is presented to students, either they ask for data or if time is short, data is provided.

Simple model

The first approach consists of treating the system (bottle, milk, saucepan, water) in a lumped fashion. Thus, the system requires heating energy to raise its temperature to a certain point (eg, blood temperature). The time required to heat the system is simply obtained by dividing the heating energy required by the rate of heat input from the gas or electric ring.

The heating energy required = thermal capacity of system × temperature rise

Thermal capacity = sum of thermal capacities of bottle, milk, saucepan and water

The major assumption made here is of instantaneous transfer of heat from the water to the bottle to the milk. Other assumptions are also implied, and to keep the working as simple as possible these assumptions are:

Assumptions

1 The water in the saucepan does not boil

2 Rate of heat input to saucepan (m kW) is constant

3 Milk and water are well stirred

4 Both milk and water have specific heats of 4.2 kJ kg^{-1} K^{-1}

5 Both milk and water have densities of 1000 kg m^{-3}

6 The initial temperatures of the milk, bottle, water and saucepan are the same (θ_o ^oC)

7 The bottle and saucepan are both circular cylinders (cross-sectional areas 'a' and A m^2 respectively)

8 There is no heat loss to surroundings

9 There is an instantaneous transfer of heat from the water to the bottle to the milk. This implies that the bottle material is a perfect conductor (infinite thermal conductivity Hence, $\theta_w = \theta_m$ for all t, where θ_w and θ_m are respectively the temperatures of the water and milk at any instant of time t (in seconds).

Let the height of water in the saucepan be H and the height of milk be h (it is further assumed that the bottle is full of milk).

Since thermal capacity = specific heat × mass, and in view of assumptions (4) and (5):

Thermal capacity of water = $H(A - a) \times 1000 \times 4.2$ kJ K^{-1}

Thermal capacity of milk = $ha \times 1000 \times 4.2$ kJ K^{-1}

Let thermal capacities of saucepan and bottle be c_1 and c_2 respectively, then total thermal capacity C of system is given by:

$$C = c_1 + c_2 + 4200[H(A - a) + ha] \quad \text{kJ K}^{-1}$$

If θ_f $(= \theta_w = \theta_m)$ is the final temperature of the system, then the heat required to reach this temperature is given by:

$$\text{Heat required} = C(\theta_f - \theta_o) \quad \text{kJ}$$

Since the heat input is at the constant rate of m kW (m kJ s^{-1}), then the time t (in seconds) required to increase the temperature of the system from θ_o ^oC to θ_f ^oC is given by

$$t = C(\theta_f - \theta_o)/m$$

ie,

$$t = \{c_1 + c_2 + 4200[H(A - a) + ha]\}(\theta_f - \theta_o)/m$$

For a given saucepan, bottle, quantity of milk and final temperature θ_f, t is clearly linearly dependent on H. The time is a minimum for this model when H is just sufficiently large to prevent boiling. The problem of estimating this critical value of H is left to a later model.

Validation of simple model

Using the data from the original paper, Oke (1979), one obtains

$$t = 50(1 + 40H)$$

A few values of t for corresponding H are shown in Table 5.

H (mm)	t(s)
50	150
100	250
150	350

Table 5

Heating times of the system for various heights of water

The predicted heating times for various values of H are all approximately half the observed experimental values. For instance, when H = 50 mm, the experimental value for t is 310 seconds (compared with 150 seconds from Table 5). This discrepancy is hardly surprising in view of the crude assumptions made in the development of the model. However, even this simple model has valuable uses. Since extra time will be needed for the heat from the water to conduct through the wall of the bottle (not to mention heat loss), then the calculated times in Table 5 all represent lower bounds for the actual heating times. One has, therefore, measures of the right order of magnitude which can be used in checking more sophisticated models. It should also be noted that for the values of H considered, namely 50, 100, 150 mm, that the water did not boil in the experiments carried out - thus satisfying assumption (1); in fact, the water was observed to start boiling for $H \leqslant 30$ mm before the milk reached blood temperature (taken to be 35 ^{o}C).

Modelling Heat Losses

An attempt is made to model the heat loss from the exposed curved cylindrical surface (ie, above water) of the bottle (see Figure 15).

Assumptions

1 Assumptions (1) - (7) and (9) of simple model

2 The heat loss from the top (teat end) of the bottle is negligible

3 The heat losses from the water and saucepan are negligible

4 The temperature of the air above the water can be approximated by a linear function

Assumption (3) can easily be accommodated by measuring the heat input to the system by timing a certain rise of temperature of a measured amount of water in the saucepan (without the milk bottle). The heat input so calculated from these experimental observations will thus largely be net of heat loss from the water and the saucepan.

The chief difficulty with this model is knowing how to find the temperature of the air above the water in the saucepan (air convection currents that develop as the water heats up are very complex). Assumption (4) leads to a simple linear expression for the temperature of the air, and by subtracting this from the temperature of the water (and hence of the system), Newton's law of cooling is used for the exposed curved cylindrical part of the bottle. The simple model is then modified accordingly by calculating a new net heat gained. Calculated values for heating times for H = 50, 100, 150 mm are about 10% higher than for the simple model. Consequently the heat loss model was abandoned.

Taking into account the thermal conductivity of the bottle

Here good progress is made with predicted heating times agreeing with experimental values to within ±15%. The assumptions made are (1) - (8) of the simple model. Two differential equations arise from considerations of the net rate at which the water is heated (input from gas or electric ring less heat conducted through bottle to milk), and the rate at which the milk heats up.

Thus, rate of flow of heat into the bottle and milk is given by

$$(c_2 + 4200\ ha)\frac{d\theta_m}{dt} = \frac{\kappa A_s}{d}(\theta_\omega - \theta_m)$$

The net rate of flow of heat into the saucepan and water is given by

$$[c_1 + 4200H(A - a)]\frac{d\theta_\omega}{dt} = m - \frac{\kappa A_s}{d}(\theta_\omega - \theta_m)$$

where θ_m, θ_ω are the temperatures of the milk and water respectively at time t, κ is the thermal conductivity of the material of the bottle (eg of plastic), A_s is the curved cylindrical surface area of the bottle immersed in water. These two differential equations are simply solved, by substituting one into the other as they are uncoupled, to find θ_m and θ_ω. By choosing a final temperature for the milk ($\theta_m = 35^oC$), the time taken to reach this temperature can be found; also, to make sure that the water does not boil, θ_ω can be calculated for this value of time. These calculations require the Newton-Raphson rule, which fortunately is rapidly converging given the crude starting values for times from the simple model.

Taking into account the thermal conductivity of the bottle and the possibility of having the water boiling

The solution strategy is to find the heating times for various H (it turns out that $H \leqslant 30$ mm for boiling water before the milk has reached 35^oC) for the water to boil and then the additional time for the milk to reach 35^oC. The calculations predict an optimum H (~ 22 mm) for minimum time for milk to reach 35^oC; however, this optimum cannot be realised in practice, because with so little water in the saucepan it tends to 'boil away' quite rapidly with risk of burning the bottle. However, for $H \geqslant 20$ mm, the calculated total heating times again agree with experimental values to within ±15%.

5.3 Speed-Wobble in Motorcycles

The inspiration for this problem was gained from an article by Pickering and Burley (1977). The complete modelling treatment may be found in Oke (1981).

Problem statement

Many of us have experienced wobbling of the front wheel, felt through the handle-bar, of a motorcycle (or of an ordinary bicycle) when travelling at certain speeds. What causes this wobble, or oscillation, of the steered wheel?

(Note: The above paragraph could serve as the complete problem statement, although the following additional background information and data (if asked for) is considered helpful to a slightly less than familiar situation.)

The wobbling phenomenon is not confined to motorcycles and bicycles but it is also known to occur in the front wheels of cars, supermarket or tea trolleys, and in aircraft nose wheels. These wide-ranging situations all have something in common, namely that the steered wheel is designed as a castor. A castor is defined as a steered rolling wheel, whose point of contact with the ground lies behind the point of intersection of the steering axis and the ground. Figure 16 illustrates the basic configuration of two typical castors (not drawn to scale).

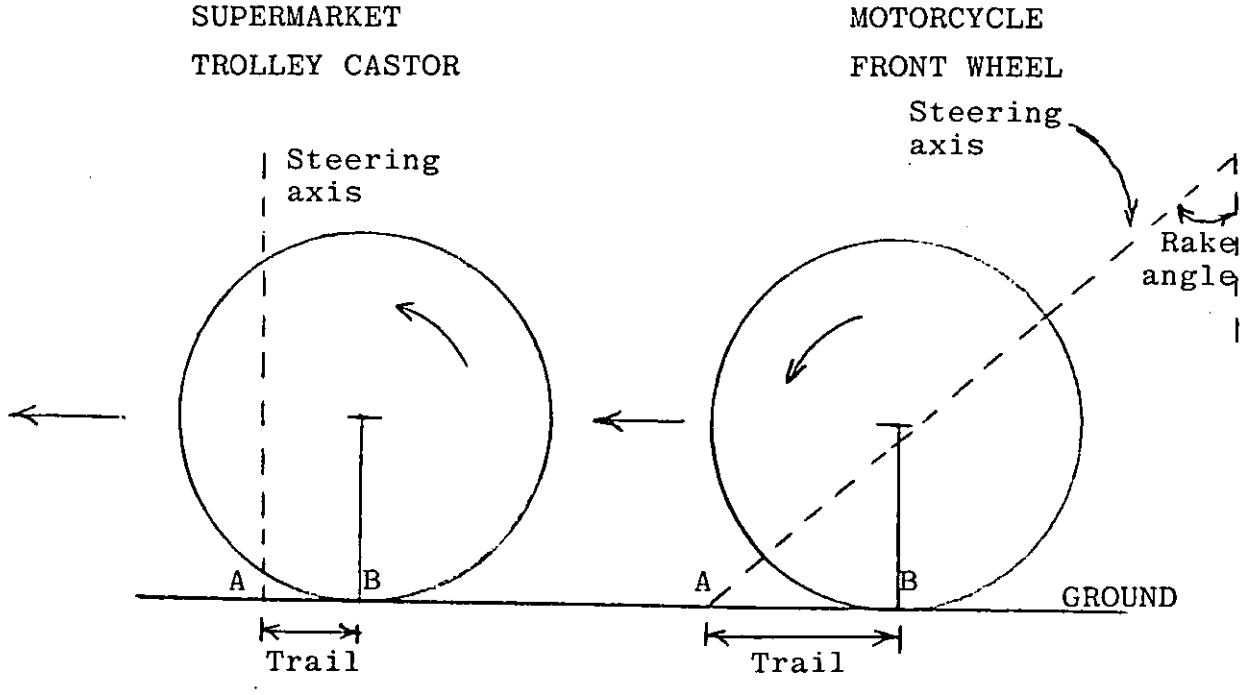


Fig 16
Two typical castors

For motorcycles, the tyre of the steered wheel and also the suspension in the front forks will obviously affect steering stability. Consequently there will not be hard (point) contact between the wheel and ground. The area of the 'contact patch' depends on tyre pressure, forward speed of the motorcycle, and whether the cycle is banking on a bend. The flexibility of the tyre also permits lateral movement of the wheel without slipping.

It has been found in practice that the front wheel, even in the 'wheel-locked' case (ie, brakes jammed on hard), can in the case of motorcycles oscillate with a frequency somewhere in the range of 6-8 Hz. In stable cases (which hopefully form the vast majority!), these oscillations rapidly decay to zero.

The problem, then, is to formulate a model which explains some or all of these observations.

Data

Typical motorcycle values, for front wheel oscillations of 6-8 Hz are:

Speed of motorcycle = 30 ms^{-1}

Moment of inertia of wheel about steering axis = 0.27 kg m^2

Trail = 0.12 m

Coefficient of friction between wheel and ground = 1

Normal reaction between wheel and ground = 700 N

Angular velocity of front wheel about steering axis = ±12 rad s^{-1} when wheel not turned, ie, when turning angle is 0.

Model construction

The problem as it stands is quite complex. It seems necessary to make a number of simplifying assumptions in order to define a conceptually easier problem. It is to be hoped that the 'easier problem' will lead to some manageable mathematics, and that some deductions can be made which will provide useful insights to the original situation of motorcycles.

Clearly, it would be easier to consider a supermarket trolley castor, which has a tyreless wheel and a rigid vertical steering axis. Gyroscopic couples can then be ignored as well as tyre flexibility and suspension in the steering axis. As a tyreless wheel is now being considered, hard (point) contact with the ground might be assumed also. In practice, however even with a tyreless wheel, some frictional resistance is felt when the wheel is steered; this is mainly due to resistance at the small, but finite, area of contact between the wheel and the ground. A smaller resistance, which will be ignored, is due to a frictional torque in the steering column bearings.

For this much simplified problem, consideration must now be given to a representation of the frictional forces acting at the point of contact of the wheel with the ground. It seems reasonable to assume that there can be (must be?) side-slip when the wheel oscillates; that is to say, that for the wheel to oscillate about its steering axis, there will be a tendency for the instantaneous point of contact with the ground to slip in a direction perpendicular to the plane of the wheel. Limiting friction would operate and a frictional force ($F = \mu R$) would act in a direction opposing motion. The following assumptions concerning friction are therefore made:

1 Simple Coulomb friction applies, with frictional force μR proportional to normal reaction R, with μ the dynamical coefficient of friction

2 This Coulomb friction applies irrespective of the forward speed v of the castor

3 Coulomb friction applies both when the wheel is locked and when the wheel is rolling

4 The direction of the frictional force, μR, is directly opposite to the direction of motion of the contact point of the castor with the ground

A further assumption is that the forward speed v is constant.

Figure 17 shows the basic geometry of a plan view of a vertical steering axis castor.

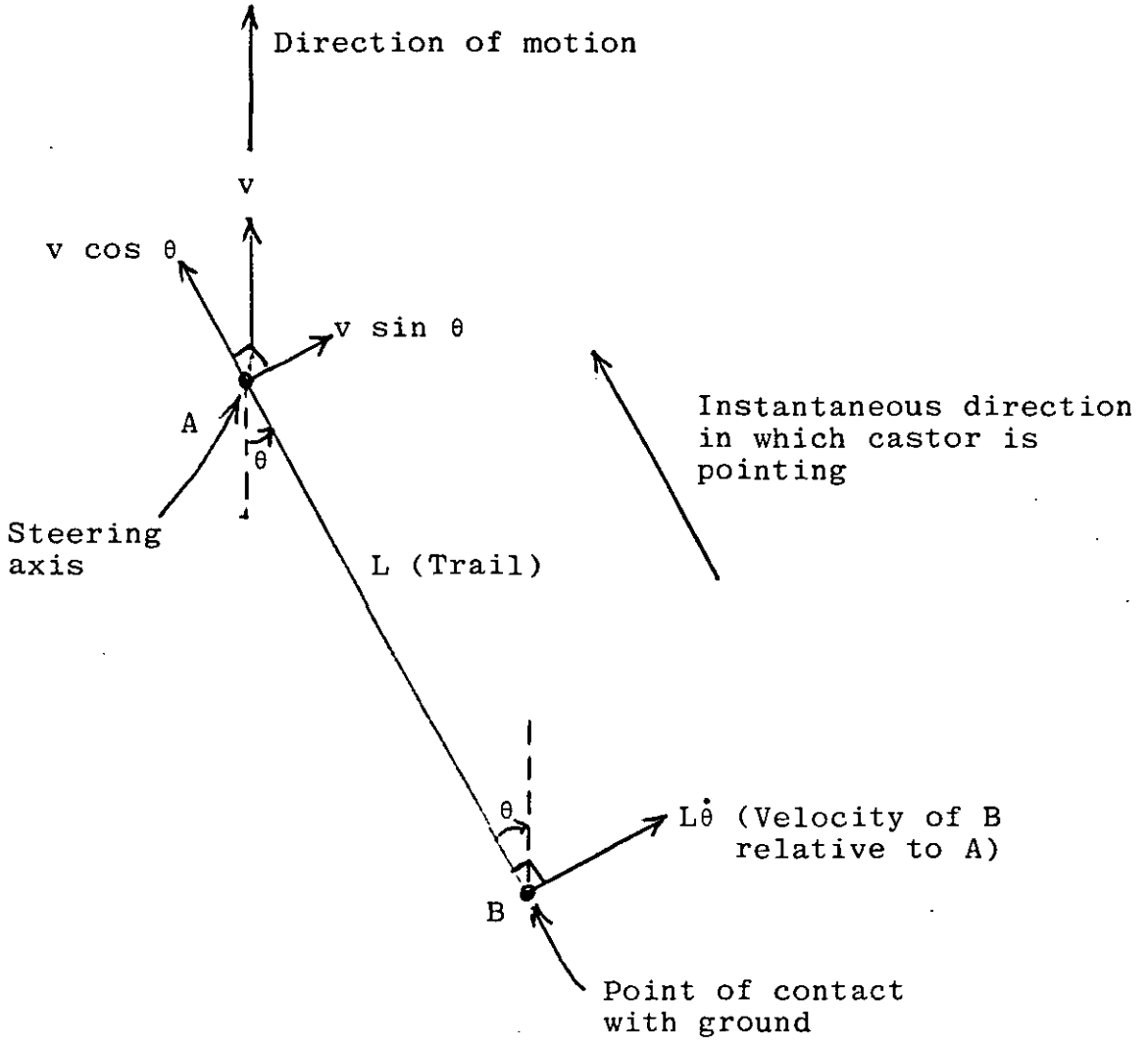


Fig 17

Plan view of vertical steering axis castor

The velocity of B relative to A, namely $L\dot{\theta}$, would be zero if no oscillations took place (since $\dot{\theta}$ would then be zero; also θ would be zero, incidently). The velocity of B relative to the ground depends on whether the wheel is locked (brakes jammed on hard in the case of a motorcycle) or whether the wheel is rolling. Both cases are considered in Oke (1981). Since a vertical steering axis castor is now being considered, which

therefore has no brakes, the rolling wheel case is investigated. Since the wheel is rolling, the velocity of B relative to the ground in the direction of BA is zero; however, the velocity of B relative to the groundin a direction perpendicular to BA will be $L\dot{\theta}$ + a component of v in this direction, namely v sin θ. The frictional force μR acting at B will therefore be in the opposite direction to $L\dot{\theta}$ + v sin θ, as shown in Figure 18.

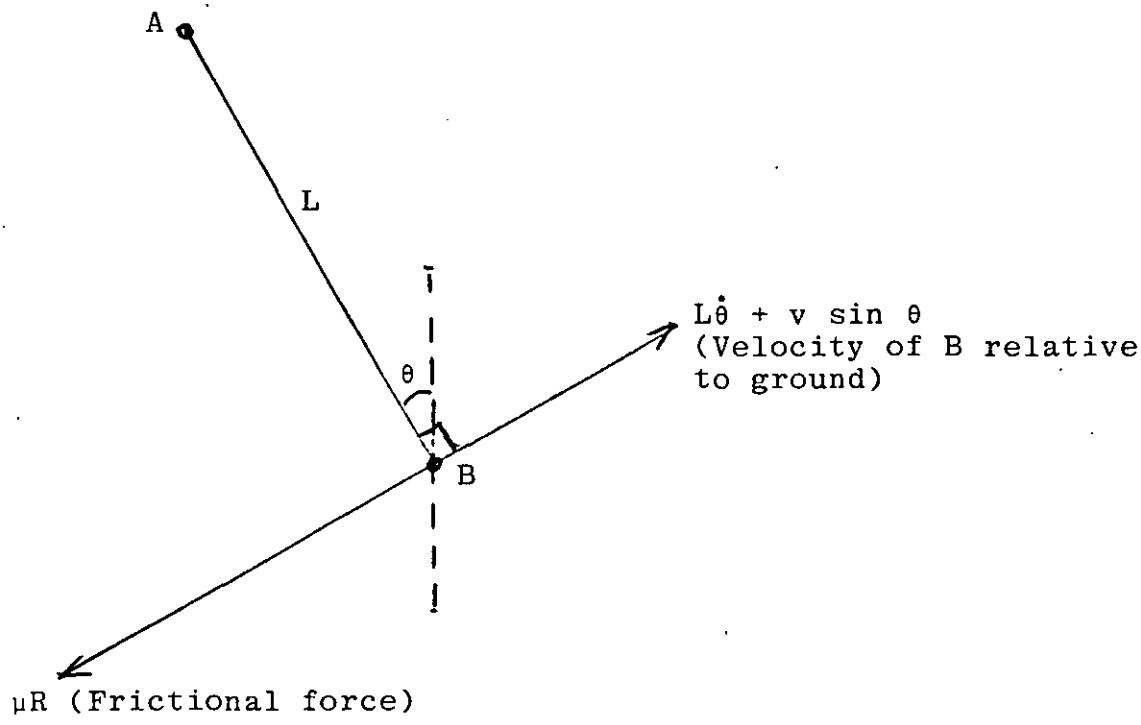


Fig 18

Velocity of B relative to the ground and frictional force acting at B, wheel rolling

Referring to Figures 17 and 18, and taking moments about A, one obtains the following equation of motion for small oscillations:

$$I\ddot{\theta} = -\mu RL, \quad L\dot{\theta} + v\theta > 0$$
$$= +\mu RL, \quad L\dot{\theta} + v\theta < 0$$

where θ is written for sin θ, I is the moment of inertia of the wheel and attachments about the steering axis. This is a second-order nonlinear differential equation and needs numerical

techniques to solve it. The solution turns out to be oscillatory with a finite number of cycles, and then θ decays exponentially to zero (no slide-slip in this motion). Using the data given in the problem statement leads to $\theta_{max} = 0.2$ rad and a frequency range for oscillations of 7-13 Hz which compares well with the experimental values of 6-8 Hz (no doubt fortuitously in view of the simplicity of the model).

A number of follow-up questions are provided in Oke (1981) which relate to the mathematical development of the model. One of the questions, however, tests basic understanding by asking for a different (but related) modelling approach to be adopted:

> Investigate whether it is possible or not for a castor to oscillate if there is no side-slip. What assumptions will you make about the frictional force acting at the point of contact of the castor with the ground? Are the predictions made by your model realistic?

Analysis shows, in the case of the follow-up question above, that for no side-slip, $L\dot{\theta} + v \sin \theta = 0$. Therefore no frictional force acts at B, and straightforward integration leads to

$$\theta = \theta_o \exp(- vt/L)$$

where θ_o is the value of θ when t (time) = 0. This is an exponential decay, and hence oscillations are not possible. This situation also prevails after the last cycle where oscillations (with side-slip) have taken place.

The mathematics involved in the whole development requires two integrations (simple analytical) to obtain θ from the equation of motion, and a simple numerical procedure of 'binary chopping' to find the times when $L\dot{\theta} + v\theta < 0$ and $L\dot{\theta} + v\theta > 0$.

5.5 Minimisation of Sound Distortion in a Record Player

A comprehensive modelling treatment of this problem may be found in Oke (1981).

Problem statement

We are concerned with the shape and size of a pick-up arm and its proximity to the turntable in our efforts to minimise sound distortion. We will, therefore, disregard other factors which affect quality in the reproduction of sound, eg, stylus characteristics, tracking weight, dynamics of pick-up arm, etc. The most common discs today are 12" in diameter and run at $33\frac{1}{3}$ r.p.m. with laterally recorded groove(wave form of signal 'horizontal' and transverse to the groove.

The two most common types of pick-up arm in use are straight arms and off-set arms. In the case of the latter, the arm has a bend in it or the cartridge is aligned towards the centre of the turntable in relation to the line of the arm.

Data that might be found useful

The distance from the centre of the disc to the innermost part of the groove is typically 1.875", and to the outermost part 5.75" for a 12" diam. $33\frac{1}{3}$ r.p.m disc.

Typical length for pick-up arm is about 8" and cannot be much larger in view of the desirability of keeping the record playing deck to within reasonably compact dimensions.

The best range of hearing for an individual is 20 - 20,000 Hz; middle - C on a piano is 256 Hz.

Maximum signal amplitude is typically 0.002".

A simplified representation is shown in Figure 19.

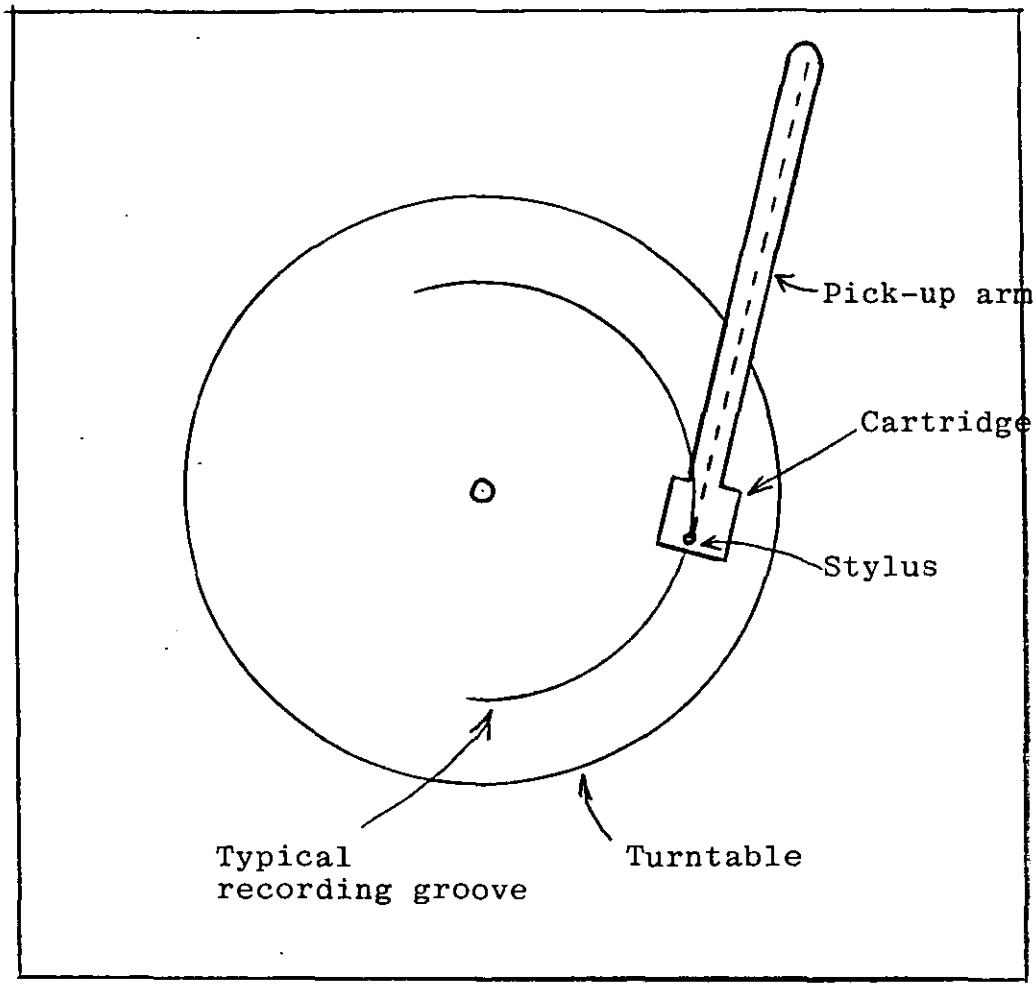


Fig 19

Simplified illustration of pick-up arm and turntable

Model construction

There is quite a lot of scope with this problem for developing a number of different approaches, each of which provide good ideas for further and more complicated developments.

The first simple approach is to consider a straight arm pick-up; the off-set arm follows on without much further difficulty. Next stage is to treat the recording groove as a system of concentric circles, concentrate on the basic geometry and ignore for the time being the precise form of the recorded signal. Referring to one particular circle of radius r, as shown in Figure 20, the initial model evolves quite naturally. By drawing a number of circles of various radii on a scale diagram, and using L ~ 8", one sees that in general the pick-up arm is

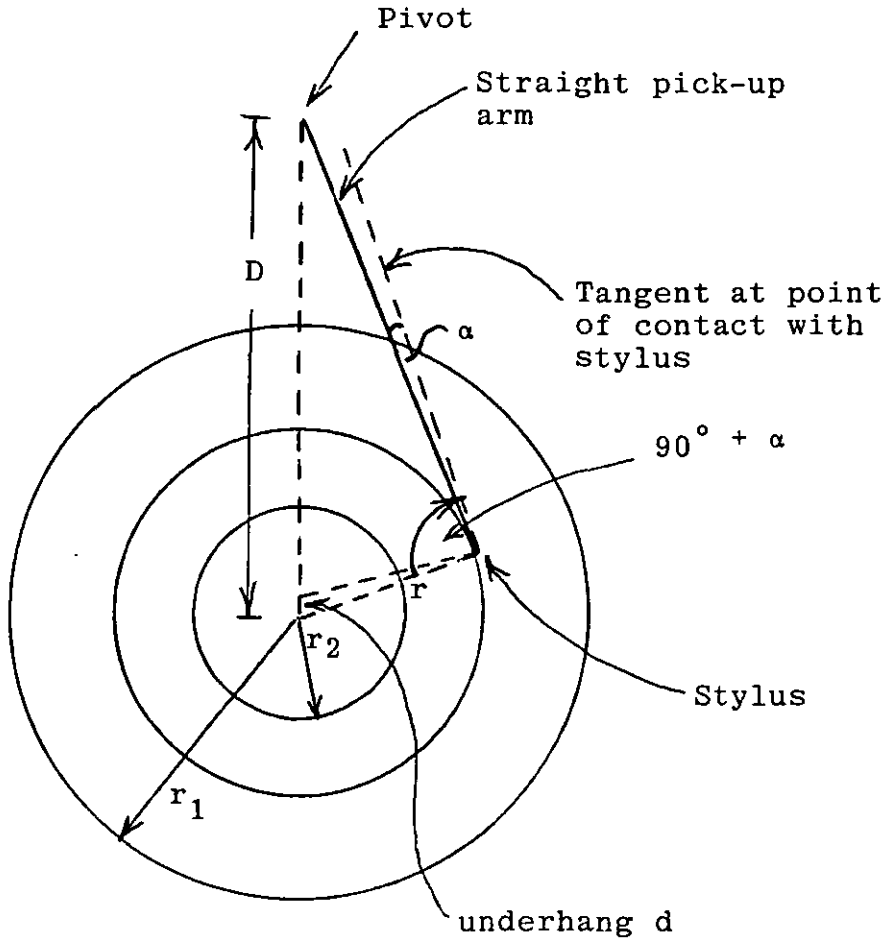


Fig 20

Geometry of straight pick-up arm in relation to recording groove

not tangential to most circles. The angle between the arm and the tangent at the point of contact with the stylus is denoted by α. Assuming that no distortion occurs when α = O, then a reasonable first solution to the problem is to try and minimise α throughout playing time, ie, for $r_2 \leq r \leq r_1$, where r_1 and r_2 are the radii of the outer and inner grooves respectively. Scale drawing shows that if the stylus is 'underhung' (shortfall of stylus at centre of turntable), α can be reduced for some trial values of r. So D, the distance of pivot from centre of turntable, is given by D = L + d, where d is the underhang. For an off-set arm, 'overhang' minimises α.

Applying the cosine rule enables one to obtain a relationship between r, d, L and α:

$$(L + d)^2 = L^2 + r^2 - 2Lr \sin \alpha$$

leading to

$$\sin \alpha = \frac{r}{2L} - \frac{d}{r} - \frac{d^2}{2Lr}$$

A reasonable upper bound for d (from scale diagram) is 0.3", and taking $r = r_{min} = r_2 = 1.875"$, $L = 8"$:

$$\frac{r}{2L} = 0.1172, \quad \frac{d}{r} = 0.1600, \quad \frac{d^2}{2Lr} = 0.0030$$

One is thus encouraged to write

$$\alpha = \frac{r}{2L} - \frac{d}{r}$$

as a good approximation. The sketch graph shown in Figure 21 illustrates how α varies with r for given L and d. It is evident that α will be kept as small as possible in numerical value throughout the range $r_2 \leq r \leq r_1$ if the maximum values of $|\alpha|$ which occur when $r = r_2$ and when $r = r_1$, are set equal to each other. Thus,

$$\frac{r_1}{2L} - \frac{d}{r_1} = - \left(\frac{r_2}{2L} - \frac{d}{r_2} \right)$$

or

$$d = \frac{r_1 r_2}{2L}$$

For a 12" disc, $r_2 = 1.875"$, $r_1 = 5.75"$, $L = 8"$ and so d is calculated to be 0.67", and the maximum numerical value of α is given by:

$$\alpha = \frac{r_1}{2L} - \frac{d}{r_1} = \frac{5.75}{16} - \frac{0.67}{5.75} = 0.24 \text{ rad}$$

ie, $\alpha = 13.9^{o}$

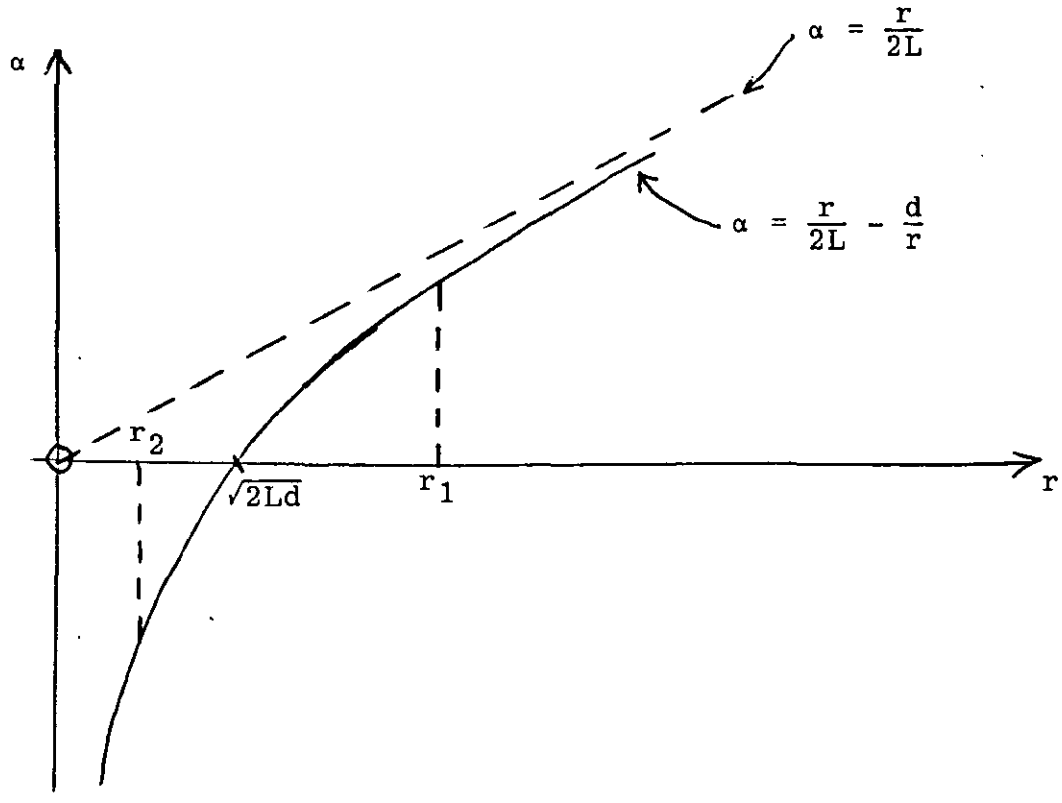


Fig 21

Graph of α against r for an underhung straight arm pick-up

The value of 0.67" seems intuitively to be rather a high value for d (underhang) and this is confirmed when a more detailed analysis involving the nature of the distorted signal is undertaken. However, a good start has already been made in the modelling of the problem and this encourages one to continue.

In Oke (1981) a sine wave signal and its distorted wave form ($\alpha \neq 0$) is analysed. The distorted form is interpreted as the original sine-wave as fundamental with harmonics superimposed. The ratio of the amplitude of the first harmonic to the amplitude of the fundamental provides a measure of distortion. With this analysis, it is discovered that what should be minimised over the range $r_2 \leq r \leq r_1$ is the function α/r rather than α. In this case, using the same values as before for a 12" disc, d is calculated to be 0.40", a much more realistic value than the value of 0.67" calculated by optimising α.

The analysis for an arm with an off-set angle of β follows quite naturally, where sound distortion is obtained by studying the function $(\alpha - \beta)/r$. The optimum case leads to a 'overhang' of 0.57", an off-set of 23.3^{o} and maximum sound distortion of one-fifth of the corresponding value for straight arms.

5.6 Windmill Power

The modelling treatment in this case study involves a novel momentum approach when considering the effects of air striking the blades of a windmill. Comprehensive details may be found in Oke (1983) and subsequently a number of wind-tunnel experiments have been carried out for validation purposes.

Problem statement

The building and testing of a flat-bladed windmill is just one step in many in trying to understand how windmills work. The mathematics and physics for flat blades (stationary or moving) is expected to be simpler than for conventional blades with aerofoil cross section ('flat' or twisted). Surprisingly enough, the design of windmills is still largely an empirical process. Is it possible to find a simple mathematical model which will greatly simplify the design of windmills? What quantities are likely to be involved in determining the power developed by the windmill, for example, in the generation of electricity?

Figure 22 illustrates a simple horizontal axis, two flat-bladed windmill. Once set, the pitch of the blades remains fixed whilst the windmill is working. To alter the pitch, the blades must be made stationary again before allowing the wind to rotate them. Figure 23 shows the pitch angle for one blade.

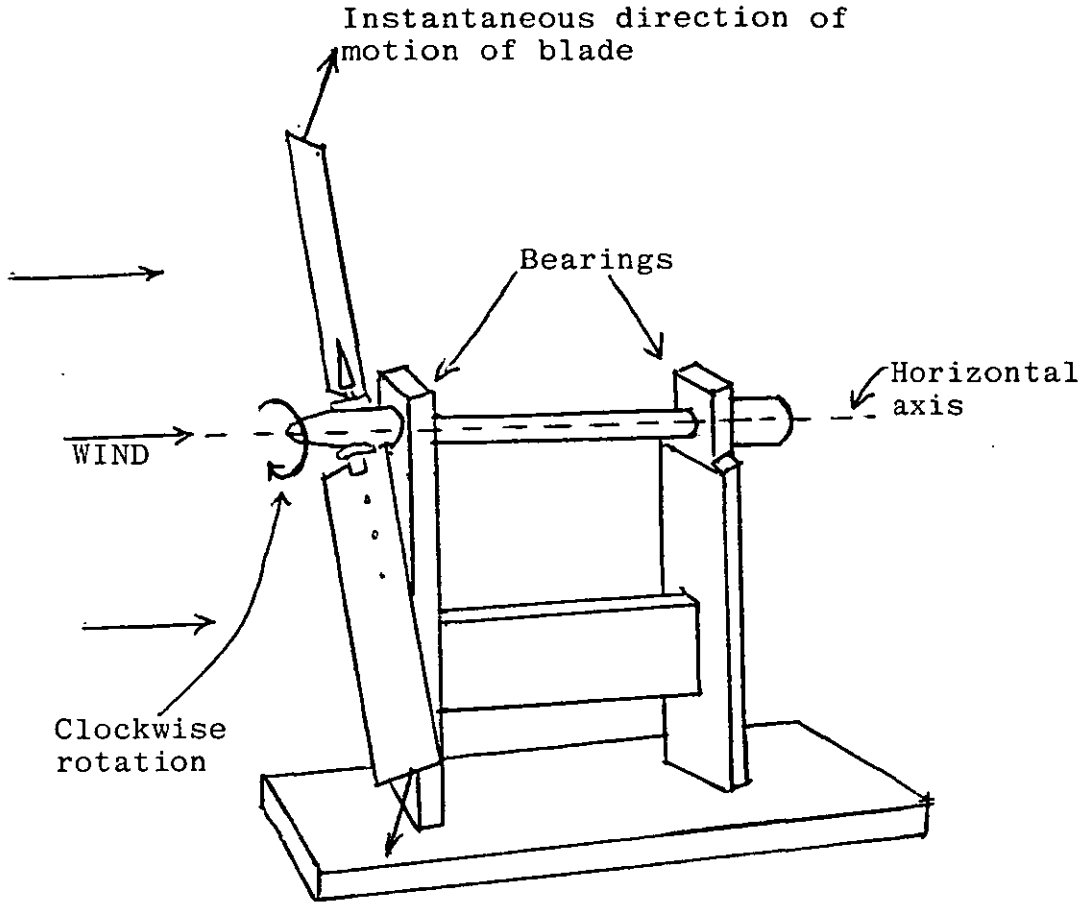


Fig 22

Simple horizontal axis, two flat-bladed windmill

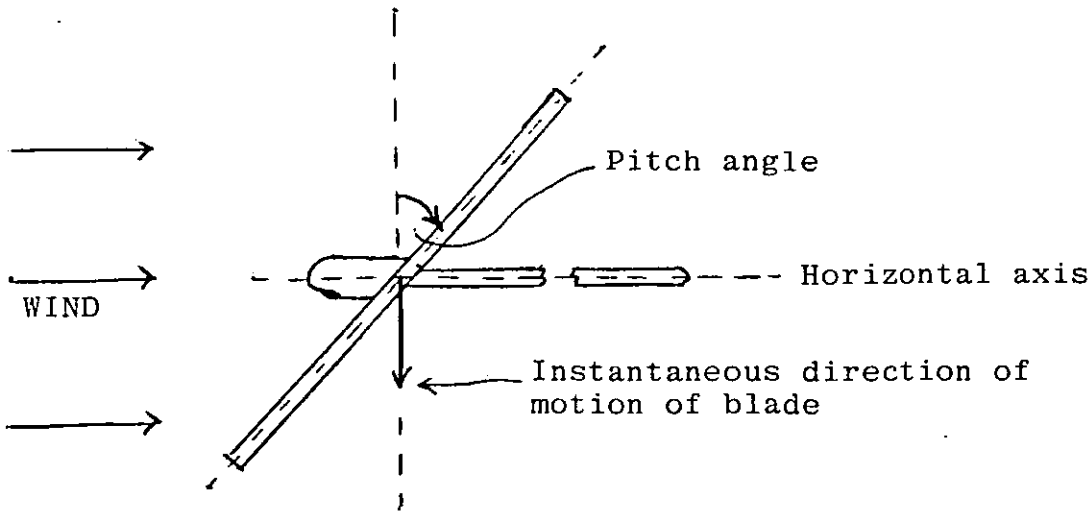


Fig 23

Plan view of one blade showing pitch angle

Model construction

The approach briefly described here follows from the analogy of a water jet being directed on to a flat sheet of metal. So one of the first things to calculate is the force acting on a blade due to the momentum change of the air impinging upon it. The problem may be split up into the following parts:

(a) Force on a fixed blade

(b) Torque produced by a fixed blade about horizontal axis

(c) Force on a moving blade

(d) Torque produced by a moving blade about horizontal axis

(e) Power produced = Torque × Angular velocity

The following quantities are likely to be involved:

Surface area of blades
Blade angle (pitch)
Speed of wind
Rotational speed
Mass of blades
Friction in bearings
Mechanical load on windmill (eg due to electric generator)
Density of air

In order to keep the development as simple as possible, the following major assumptions are made:

(i) The windmill blades are smooth

(ii) The air hitting the blades has no viscosity

(iii) The mass of the blades and the horizontal shaft may be neglected

(iv) The speed of the wind is constant

From assumptions (i) and (ii), one notes that the air is assumed to strike a blade and then move off along the blade surface without causing a tangential frictional force.

Force on a fixed blade

Force, by Newton's second law, is considered to be the rate of change of momentum of the air at the blade surface.

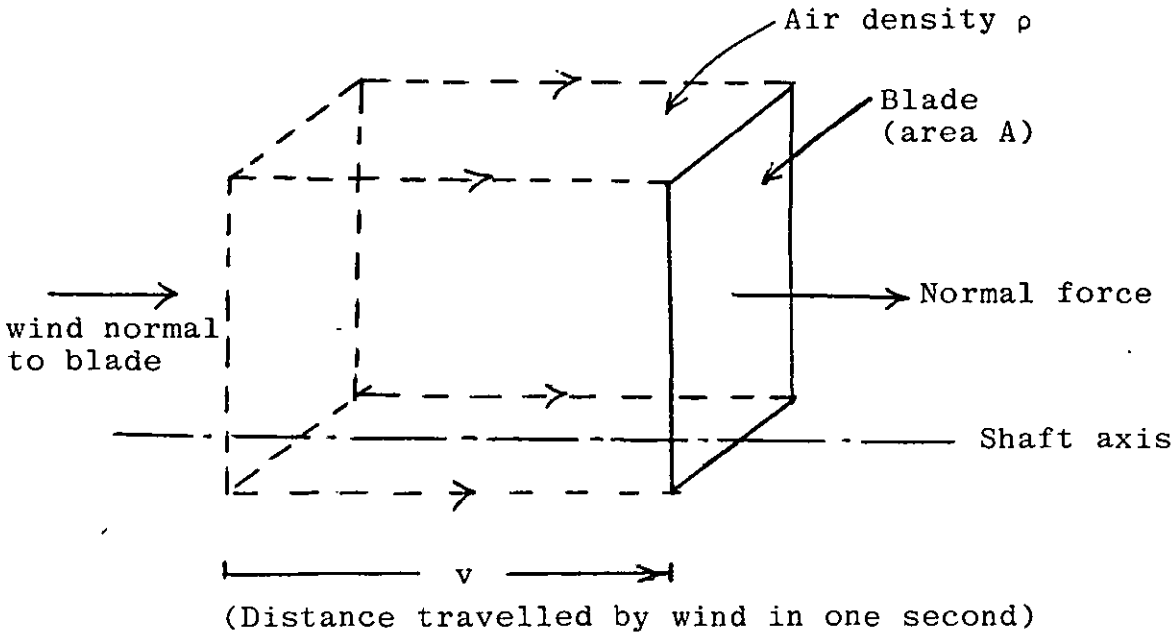


Fig 24

Element of air moving with speed v; pitch angle zero; blade fixed

Referring to Figure 24, which shows an element of air moving with speed v, the mass flow-rate of air, $\dot{m}$, is given by

$$\dot{m} = \rho A v$$

Normal force on blade due to air impinging normally upon it is given by

$$\frac{d}{dt}\,(\text{momentum}) = \frac{d}{dt}\,(mv) = \dot{m}v = \rho A v^2$$

where v is constant (assumption (iv)).

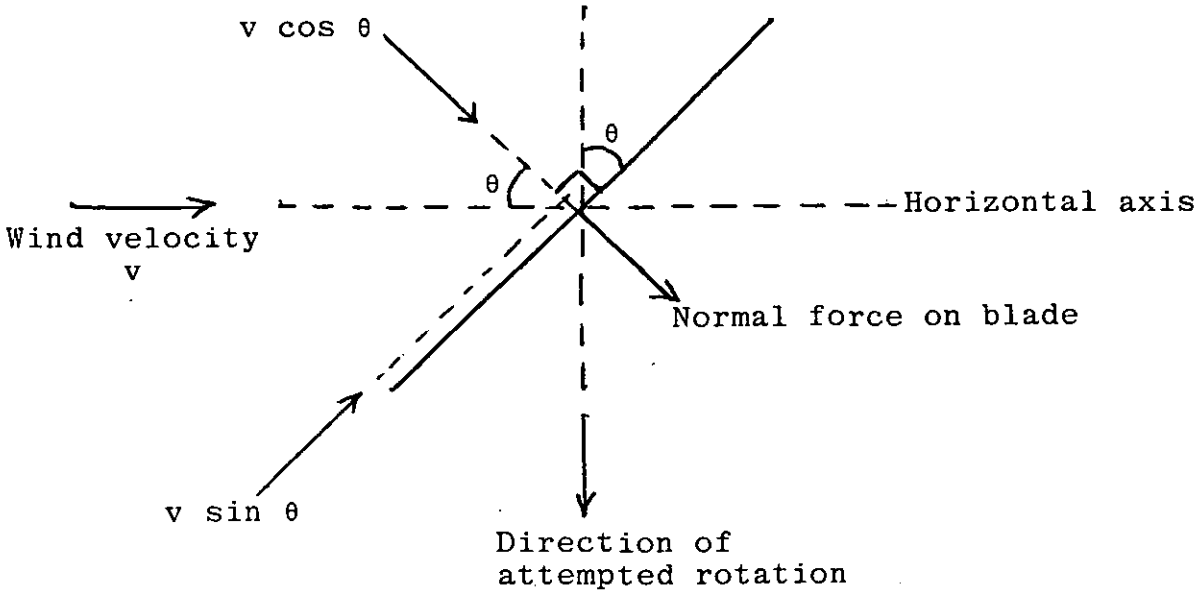


Fig 25

Velocity diagram; pitch angle θ; blade fixed, plan view

Referring to Figures 23 and 25, a result is now derived for the normal force on a fixed blade with non-zero pitch angle θ. Since the blade is assumed to be smooth and that the 'spent' air moves off tangentially along the blade surface (with speed v sin θ), the normal force is given by

$$\frac{d}{dt}\,(\text{momentum}) = \frac{d}{dt}(mv\cos\theta) = \dot{m}v\cos\theta$$

where v and θ are both constant. Since the mass flow-rate, $\dot{m}$, is given by

$$\dot{m} = \rho A v \cos\theta$$

where A is the blade area, one obtains

$$\rho A v^2 \cos^2\theta$$

for the normal force, and its component F perpendicular to the windmill shaft is given by

$$F = \rho A v^2 \cos^2\theta \sin\theta$$

See Figure 26. Note that F is the force producing motion, and is identically zero for $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$; this fits surprisingly well with the known behaviour of windmills, in spite of the somewhat unrealistic air-flow pattern adopted for this early stage of the modelling.

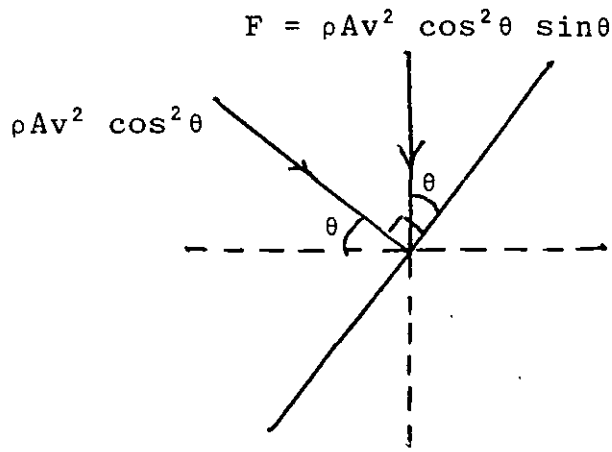


Fig 26

Force component F acting on a fixed blade

In contrast with the flow pattern of this model, an aerofoil approach incorporating the Kutta-Joukowski law would have suggested

$$F = \pi\rho A v^2 \cos\theta \qquad \text{for } \theta \to \pi/2$$

However, this latter approach is inappropriate for windmill blades. This is because the pitch angle θ for real windmills is often as little as 4°. On the other hand, the pitch angle for an aircraft wing or the sail of a close-hauled dinghy often approaches 90°, ie, with a small 'angle-of-attack'.

To determine the starting torque T on a windmill blade, it is necessary to divide the blade into elementary areas; integration over the whole area gives

$$T = \tfrac{1}{2} A L \rho v^2 \cos^2\theta \sin\theta$$

where L is the length of a blade. In experiments carried out by the author, and Mr A L Jones of the Polytechnic of the South

Bank, the calculated torques consistently predicted maximum 'starting torques' as measured experimentally with wind-speed range $25.3 \leqslant v \leqslant 33.3\ ms^{-1}$. The calculated torques agreed with the measured ones to within 12%, although the calculated pitch angle for maximum torque is 35.3^{o} compared with the experimental value of 47.5^{o}.

The rest of the model development entails determining the velocity of oncoming air relative to the front surface and the effect of the air on the rear surface of a moving blade. A modified expression for the torque is now obtained and the power delivered by the windmill is given by power = torque × angular velocity. Comparison of calculated power values with experimental results is still in its early stages, but there is an indication that more modelling needs to be done on the effects of the air on the rear surface of a blade.

The case study has follow-up questions which can be found in Oke (1983). For example, how does stress vary along the length of a blade?

5.7 Pole-Vaulting

This case study was first presented by Sheridan (1980), a teacher supervised by the author for the dissertation of the MSc (CNAA) in Mathematical Education. The modelling treatment has subsequently been extended by the author.

Problem statement

The pole vault is an event which requires the athlete to clear a high bar with the aid of a pole. Prior to the 1960's the pole was made of tubular metal (main constituent steel or alloy), but subsequent technological advances have seen the adoption of hollow fibre-glass poles by club and international calibre athletes.

The technique employed in vaulting clear of the bar is one of the most complex in track and field athletics; for the vaulter initiates a rotational moment about the base of the pole and sets in motion what appear to be two pendulums - one is the pole and the other is the athlete who rotates about his hands. See Figure 27.

The rules of the competition recognise the winner as that man who clears the greatest height (without 'pole-climbing') before he records three consecutive failures (ie, dislodges the bar from the upright on three successive vaults) regardless of the height at which any such failure occurs.

Attempt to identify those features of the event which characterize good pole vaulting.

Data

The following data for a typical pole-vaulter (fibre-glass pole) may (or may not) be useful. All quantities quoted are approximate.

Height of bar above horizontal ground = 5 m

Time taken from take-off to landing = 2.5 s

Time taken from take-off to pole release = 1.5 s

Horizontal distance from take-off point to bar = 4.5 m

Length of pole = 4 m

Mass of pole = 3 kg

Sprint speed with pole = 9 ms^{-1}
(World class sprinters = 10.3 ms^{-1})

$g = 9.8\ ms^{-1}$

Height of centre of mass of sprinter above ground at take-off = 1 m

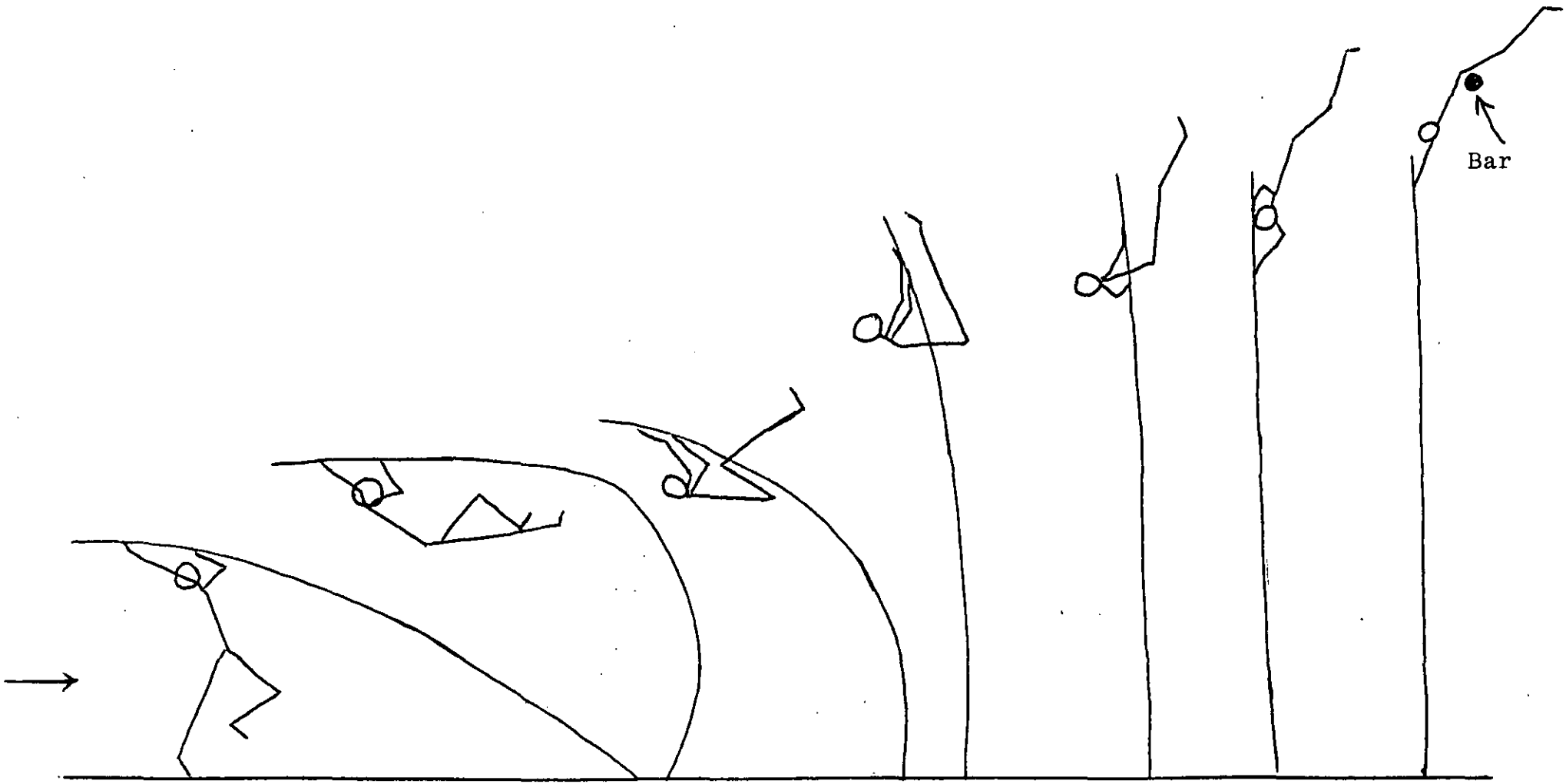


Fig. 27

Various stages of a fibre-glass pole vault viewed at right angles to the direction of motion; scaled sketches from closed-loop film

Model construction

Clearly, a comprehensive approach using mechanics will be very complicated indeed. Useful insights can in fact be gained by considering only very simple models. The approaches adopted here are based on considerations of:

(a) Energy

(b) Kinematics

Energy

The major assumption to be made is that the athlete converts all his kinetic energy developed in the approach run to potential energy gained in raising his centre of mass sufficient to clear the bar. This assumption, together with some secondary ones, are listed below:

1 The pole moves in a vertical plane in the direction of the vaulter's approach run

2 The lowest point of the vaulter's body relative to the pole at the instant of release is sufficiently high to allow clearance of the bar

3 The pole does not knock the bar off its stands

4 The combined mass of the pole and vaulter is considered to act through the top end of the pole

5 The vaulter's kinetic energy developed in the approach run is used solely in raising his centre of gravity

From the above assumptions,

$$\tfrac{1}{2}mv^2 = mgh$$

where m is the combined mass of vaulter and pole, v is the approach speed at take-off, and h is the final height of the vaulter's centre of mass (hopefully at least equal to the bar height). Consequently, h is given by

$$h = \frac{v^2}{2g} ,$$

which is independent of both the mass of the pole and of the vaulter.

If the height of the vaulter's centre of mass above the ground at take-off is 'c', then the final height above the ground of the centre of mass is given by

$$h + c = \frac{v^2}{2g} + c$$

From assumption (2), and from Fig 27, one assumes that the rotation of the vaulter at the bar is such that no further increase in height is involved. Consequently, using the data provided in the problem statement:

$$h + c = \frac{9^2}{2 \times 9.8} + 1 = 5.1 \text{ m}$$

A fortuitous result, given the crudity of approach, since a typical bar height from the data provided is 5 m.

Kinematics

The simple approaches outlined are based on treating the vaulter as a projectile (without wind resistance), or on running a closed-loop film for the purposes of sketching the displacement of the vaulter's centre of gravity and velocity components. Projectile treatment produces graphs as shown in Figure 28. The vertical distance risen by the vaulter is denoted by y after having travelled a horizontal distance x. Corresponding velocity components are denoted by $\dot{y}$ and $\dot{x}$.

From a closed-loop film the corresponding graphs are shown in Fig 29.

Clearly the projectile model bears little resemblance to the closed-loop film sketches. A least-squares cubic fit to the first of the graphs shown in Figure 29 could provide some information on where the athlete should position himself relative to the pole at various stages of a vault.

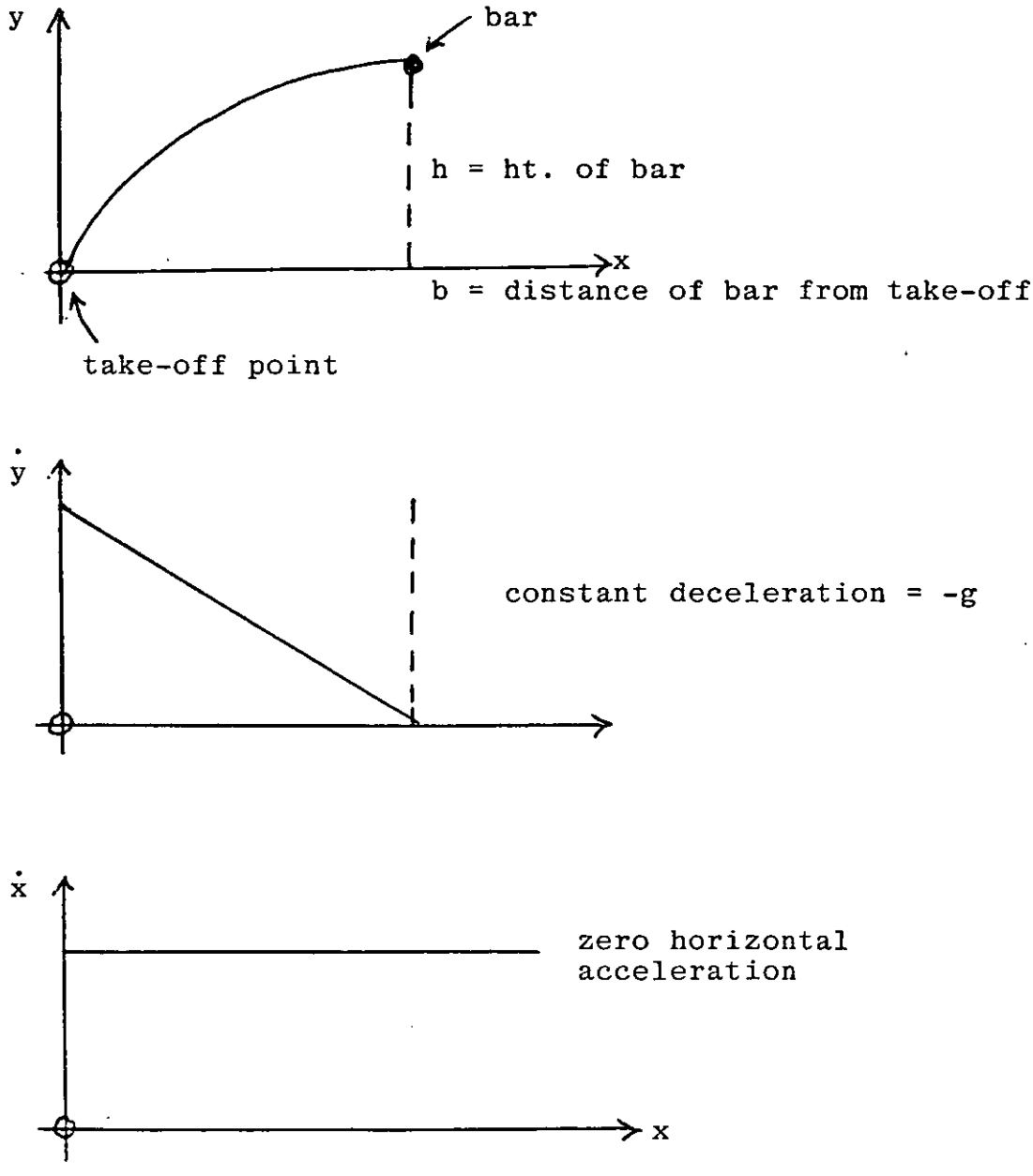


Fig 28

Displacement of vaulter, with velocity components, using a projectile model

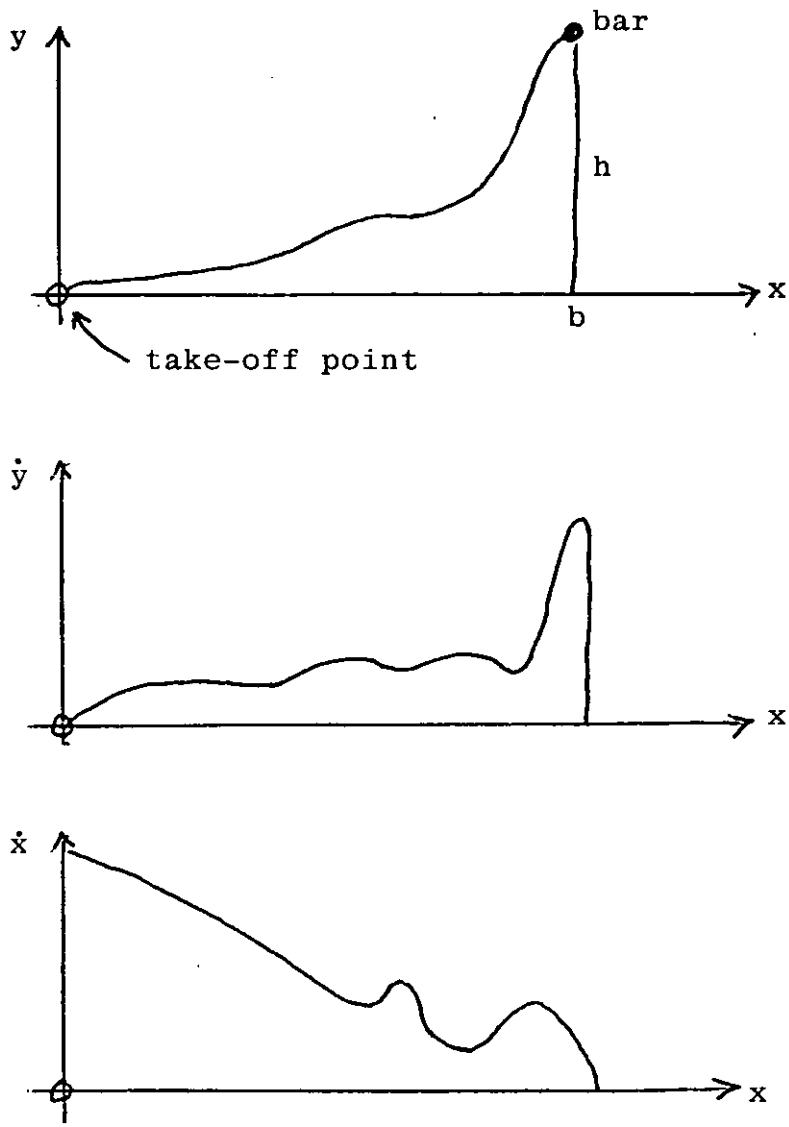


Fig 29

Displacement of vaulter, with velocity components, from closed-loop film

5.8 Central-heating

Problem statement

In using a central heating system in a house, which is the best strategy for minimising heating costs:

(a) Let house cool down naturally, central heating switched off when warmth not required

(b) Set thermostat to a certain value so that house cools less when warmth not required

Strategy required for any 24-hour period in winter.

Model construction

To simplify the problem it is assumed that there is only one warmth period, during the day, and that there is only one cooling period, during the night, throughout any 24-hour interval. Furthermore, a very rapid response is assumed, so that the instant the C-H boiler (central heating boiler) lights up, the heat it generates is immediately imparted to the house via the radiators.

The problem may be viewed from three main aspects:

(i) Heat required during the day to maintain a steady temperature

(ii) Heat lost from house during cooling at night

(iii) Heat required to raise temperature reached at night to steady temperature required during the day

A sketch diagram, shown in Figure 30, is useful in illustrating temperature variations of the house throughout a 24-hour period. For convenience, time (t) has as origin the instant the C-H system is switched-off at the start of cooling (eg at 11.00 pm).

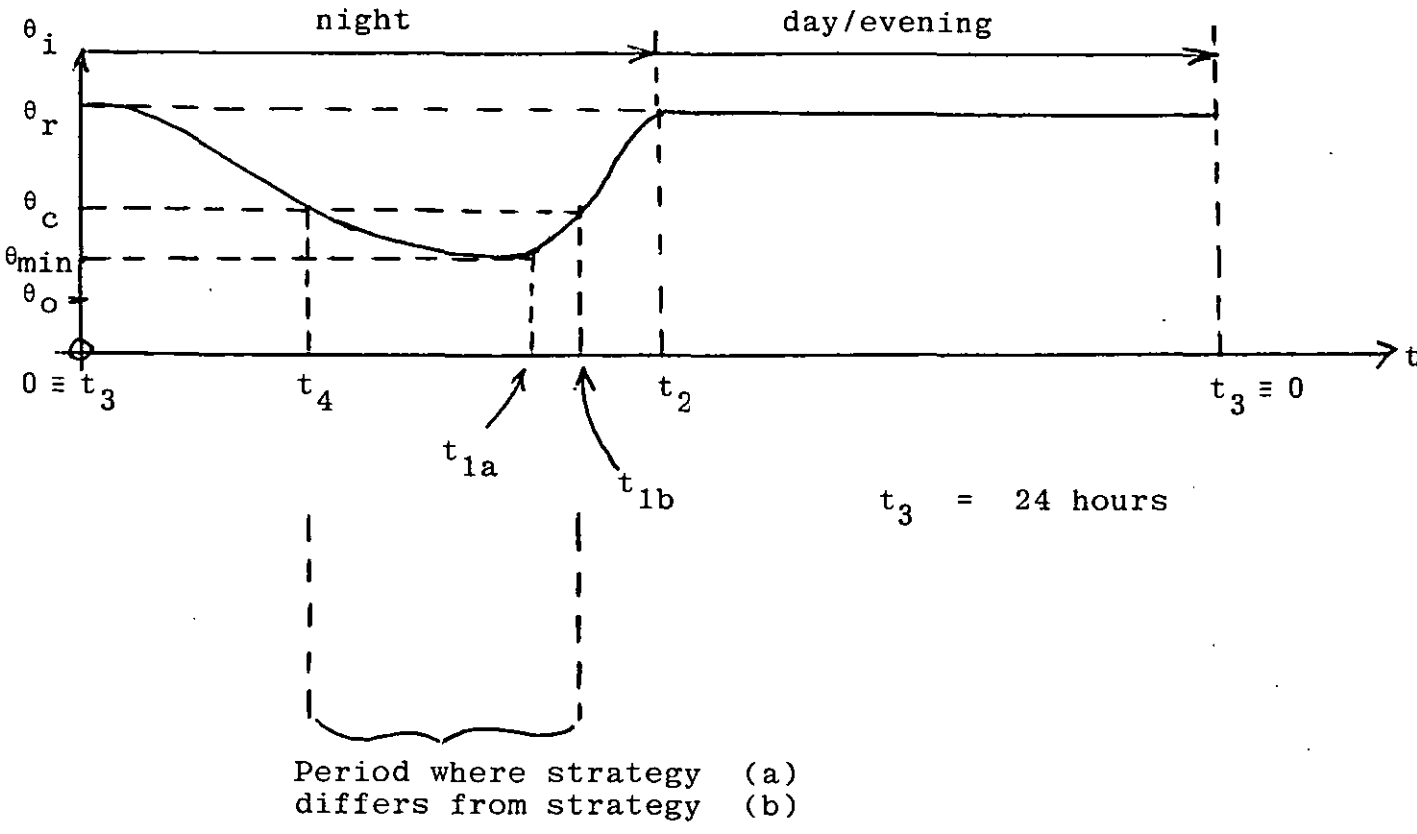


Fig 30

Temperature variations in house over 24-hour period

Referring to Figure 30, the symbols have the following meanings:

θ_i	temperature inside house at any time t
θ_r	required steady temperature at day-time
θ_c	temperature reached at night with strategy (b). $t_4 \leq t \leq t_{1b}$
θ_{min}	lowest temperature reached at night with strategy (a). $t=t_{1a}$
θ_o	outside temperature (assumed constant) (all temperatures in ^oC)

Note also that at time $t = t_4$, strategy (b) is being used where thermostat is set at temperature θ_c. At times $t = t_{1a}$ and $t = t_{1b}$, the thermostat is set at temperature θ_r for strategies (a) and (b) respectively. It is assumed that the required day temperature, with either strategy, is θ_r for the time interval $t_2 \leq t \leq t_3$.

At this stage, a clear insight into the problem has already been gained. The difference in heating costs is due to the difference in the amounts of heat required as follows:

Strategy (a): Heat required to raise temperature from θ_{min} to θ_c in time $(t_{1b} - t_{1a})$

Strategy (b): Heat required to maintain temperature of θ_c in time $(t_{1b} - t_4)$

The difference in heating costs can then be calculated, in principle, for various θ_c and an optimum policy decided.

It is further assumed that only two thermostat settings are required for each strategy: θ_r (eg, 68°C) for both, θ_{min} (eg, 55°C) for (a) and θ_c (eg, 62°C) for (b). θ_c or θ_{min} are set at time t_3 (eg, 11.00 pm), and θ_r is set at time t_{1a} (eg, 6.45 am) for strategy (a) or at time t_{1b} (eg, 7.15 am) for strategy (b). The programmer clock will be set for continuous running - the thermostat settings determining the times t_4, t_{1a}, t_{1b} given θ_c, t_2, t_3. So, in practice, a householder will need to know t_{1a} or t_{1b} only, in addition to thermostat settings.

To find the times t_4, t_{1a} and t_{1b}, cooling and heating of the house needs to be investigated. For cooling, Newton's law of cooling is assumed:

$$T \frac{d\theta_i}{dt} = - K(\theta_i - \theta_o)$$

and by simple integration this leads to

$$t_4 = \frac{1}{A} \left\{ \ln \left(\frac{\theta_r - \theta_o}{\theta_c - \theta_o} \right) \right\} \qquad (1)$$

and

$$t_{1a} = \frac{1}{A} \left\{ \ln \left(\frac{\theta_r - \theta_o}{\theta_{min} - \theta_o} \right) \right\} \qquad (2)$$

where T is the thermal capacity of the house, and K is a lumped constant for heat loss through walls, windows, and roof. $A = K/T$.

For heating up in the interval $t_{1a} \leq t \leq t_{1b}$, one has

$$T \frac{d\theta_i}{dt} = H_G - K(\theta_i - \theta_o)$$

and integrating

$$t_{1b} - t_{1a} = \frac{1}{A} \ln \left[\frac{B - A\theta_{min}}{B - A\theta_c} \right] \qquad (3)$$

where $B = (H_G + K\theta_c)/T$, and H_G is the rate at which the radiators impart heat to the house. Also,

$$t_2 - t_{1a} = \frac{1}{A} \ln \left[\frac{B - A\theta_{min}}{B - A\theta_r} \right] \qquad (4)$$

For given H_G, K and T, A and B can be calculated and then substituted into (2) and (4); the latter are then solved iteratively for t_{1a} and θ_{min}, since t_2 and θ_r are chosen for a given household. t_{1a} and θ_{min} are then subsituted into (3) and t_{1b} is found.

Hence, difference in amounts of heat required when comparing strategy (a) with strategy (b) is given by:

$$K(\theta_c - \theta_o)(t_{1b} - t_4) - H_G(t_{1b} - t_{1a}) \qquad (5)$$

where t_4 is obtained from (1).

If £C is the cost of heating per kilojoule, and t is measured in seconds, then multiplying expression (5) by C will provide the cost difference in pounds sterling.

In order to validate this model, several experiments with various thermostat settings θ_c would need to be carried out. In addition, further information would be required on outside temperature θ^o (easily obtained), H_G, K and T (not so easily obtained).

5.9 Evacuation of a School

This problem was devised and presented by Wilson (1983), a teacher supervised by the author for the dissertation of the MSc (CNAA) in Mathematical Education.

Problem statement

This model is designed to provide insight into the phenomenon of crowding during the evacuation of a building (here a school) and to provide some means whereby we might be able to predict how this could be prevented or minimised. It will thus be required to predict the fastest, safe evacuation time having taken into account the number of exists available (here two). (The problem relates to a particular building, namely 'Neave Comprehensive School', although the study adopted may be generalised to cover other types of building).

Within Neave school there exists two exits which serve the major proportion of the classes in the building in the event of an emergency such as a fire. There is on occasion considerable confusion at a particular juncture point as a result of crowding. The problem is to examine whether by ordering the exits of the children firstly from the classroom and then from the building, bearing in mind time and direction, we might arrive at a sensible procedure which would help to eliminate confusion yet lead to the evacuation of the building in the minimum possible time and thus reduce the risk of danger. Observation shows that children take less time to travel the distance between classes than it takes an average class to evacuate a classroom. Thus before all the children from one particular class have left a room, others from an adjoining class begin entering their exit space. The problem is heightened when several streams of children combine at a particular point. Thus a certain time must elapse before the children are allowed to begin their exit in order to allow for those before, ie, nearer to the exit, to have left their room.

Model construction

Wilson (1983) lists sixteen assumptions, some of the chief ones (based on observations) are as follows:

The best means of evacuating a building is by way of an orderly procession

No matter where an emergency occurs, all children will be moving out of the building

The children will travel at a constant speed along the corridors

The children will leave the classroom at a constant rate c

The emergency will not affect the routes or the exits that the children take

Stairs will not affect the flow of children, and a distance can be assigned to stairs

The best means of ensuring an orderly procession and avoiding crowding is to introduce a time delay into following classes

A good start can be made by simply considering a row of i + 1 classes shown in the simplified diagram in Figure 31.

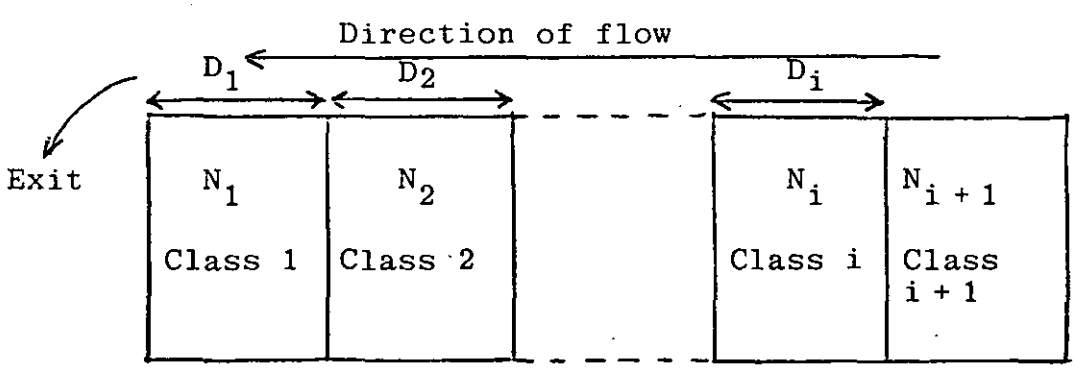


Fig 31

Children evacuating a row of i + 1 classes

Thus, $I_i = D_i/s$ and $t_i = N_i/c$, where I_i is the time taken for children to travel the distance D_i between classes i and i + 1, s is the (assumed constant) speed that the children travel along the corridor in an orderly manner, t_i is the time for N_i

children to leave class i in an orderly manner at the rate of c children per second.

Now observation shows that $I_i < t_i$, ie, $\frac{D_i}{s} < \frac{N_i}{c}$ (one of the assumptions) and consequently a time delay d_i for class i + 1 is introduced:

$$d_i = \frac{N_i}{c} - \frac{D_i}{s}$$

Consequently, the instant class i + 1 starts to leave classroom i + 1 is d_i seconds <u>after</u> class i have started to evacuate class i. This should ensure no crowding in the corridor and hence produce an orderly flow.

Using data obtained from observation for just two classes:

$c = 2$, $N = 24$, hence $t_1 = 12$ seconds

$D_1 = 8$ m, $s = \frac{4}{3}$ ms^{-1}, hence $I_1 = 6$ seconds

=> $d_1 = 6$ seconds. Thus a time delay of 6 seconds is introduced to class 2 (which seems reasonable)

Generalising to i + 1 classes leaving from <u>one</u> exit, the total evacuation time T is given by:

$$T = \frac{N_{i+1}}{c} + \sum_{j=1}^{i} \left(\frac{N_j}{c} - \frac{D_j}{s} \right) + \sum_{j=1}^{i} \frac{D_j}{s} + \frac{D_\ell}{s}$$

where time for children to leave class i + 1 is N_{i+1}/c, delay time for class i + 1 is given by

$$\sum_{j=1}^{i} \left(\frac{N_j}{c} - \frac{D_j}{s} \right)$$

time taken for last child to walk total distance from class i + 1 to class 1 is

$$\sum_{j=1}^{i} \frac{D_j}{s} \quad ;$$

and time taken to walk the final distance D_ℓ from class 1 to the exit is D_ℓ/s. The expression for T clearly simplifies to

$$T = \sum_{j=1}^{i+1} \frac{N_j}{c} + \frac{D_\ell}{s} \ .$$

To check the reasonableness of the result so far, consider ten classes all with 24 pupils, all 8m between the doors and class 1 being 20 m from the exit. Then the total evacuation time T is given by

$$T = \sum_{1}^{10} \frac{24}{2} + \frac{20}{4/3} = 135 \text{ seconds}$$

This result was closely confirmed by actual observation (140 seconds).

So the modelling activity thus far has already provided valuable insights into the original problem, by considering a single row of classrooms with just one exit. Wilson (1983) provides a topological map of the location of ten classrooms in part of his school, which shows the location of two exits and the distances involved between classes and the exits. By considering a natural generalisation of the early approach to the two exits, and by calculating the evacuation times depending on which classes use which exit, a minimum overall time is found to evacuate the building as well as the ordering of classes using each exit to achieve this minimum.

5.10 Motorway and 'A' Road Travel Costs

This problem and its modelling development was also first presented by Wilson (1983).

Problem statement

It is often said that it is better to travel by a motorway than by a normal road even if it does require travelling a greater distance. A large trucking company have asked you to investigate this with a view to re-examining their existing routes. The company provides the following basic data in the first instance:

1 A map of Britain containing the main towns that the company deliver to, and the routes available to these towns, differentiating between motorway and A-route.(Wilson (1983) provides a map of England and Wales. The sketch map shown

in Figure 32 shows only the South of England, and is provided to show a sample of the level of detail involved).

2 The main depot is in London

3 The hourly cost of a truck, excluding petrol consumption, has been calculated at £30

4 The average speed of a truck on the motorway is 50 mph, on an A-route it is 35 mph

5 Petrol consumption on a motorway is 12 mpg, on an A-route it is 15 mpg

6 Cost of petrol £1.60 per gallon

The company have requested that, if possible, you provide some means whereby they might predict how a change in data would affect their choice of routes. Further, they point out that at this stage they have only provided you with the main delivery points from London. They allow of course delivery between depots, and any further information that could be discovered concerning this would be greatly appreciated.

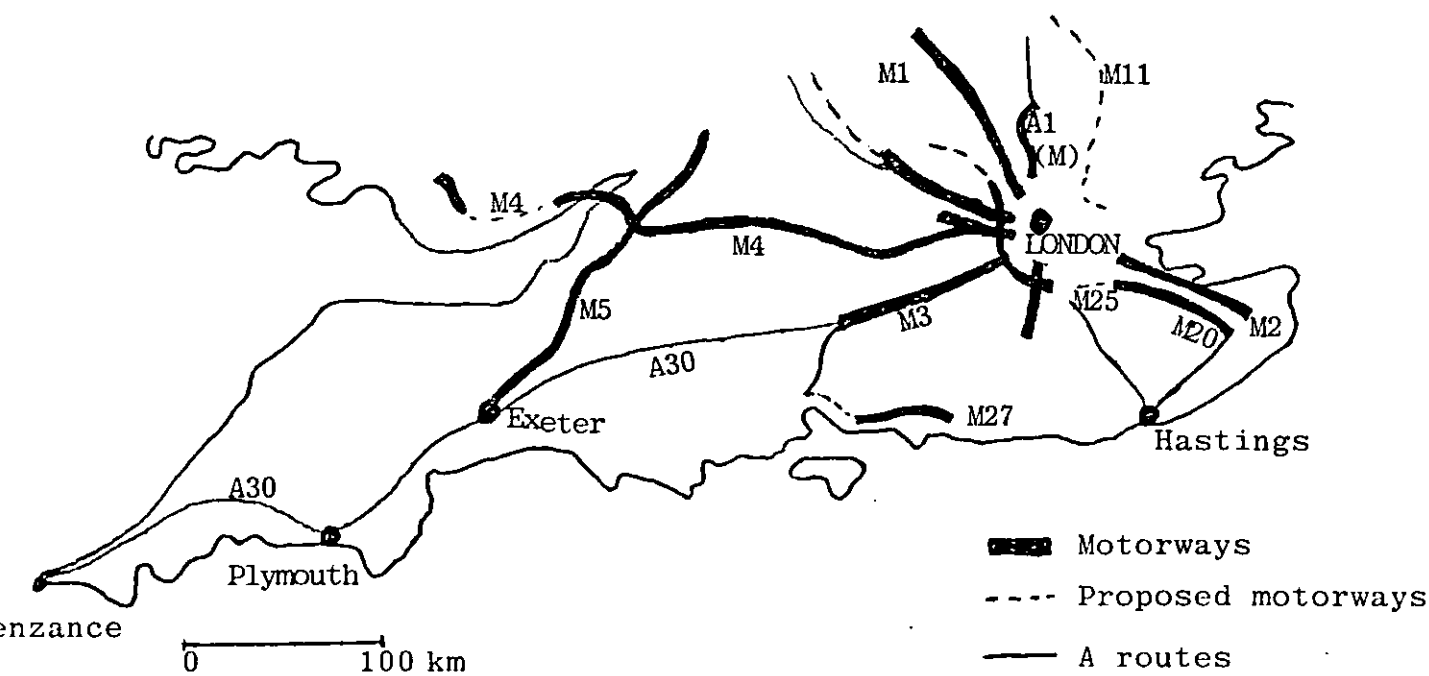


Fig 32

Sketch map of part of Southern England

Model construction

The cost of travel on a given journey may be viewed as (cost per mile) × number of miles. The (cost per mile) will depend on fixed costs (assumption (3)) and petrol consumed on the journey.

From the data provided, one can rapidly get a feel for the calculations needed. Thus, considering a motorway journey for instance:

Distance per hour on motorway = 50 miles

Petrol used per hour $= \frac{50}{12}$ gallons

Cost of petrol per hour $= \frac{50}{12} \times 1.60 =$ £6.70

Total cost per hour = 30 + 6·70 = £36.70

Hence, total cost per mile $= \frac{36.70}{50} \approx$ £0.73

Similarly for an A-route:

Total cost per mile ≈ £0.96

Clearly, it will be cheaper to travel on the motorway if the distance by motorway is less than 0.96/0.73 = 1.31 times the alternative distance by A-route.

Again, based on the data provided, one can choose the most economic routes and cost them accordingly. For example, consider the two possible routes from London to Exeter:

London to Exeter

M4 followed by M5:	300 km :	Alternative 1
M3 followed by A30:	85 km + 178 km:	Alternative 2

Equating motorway mileage with A-route mileage one need only compare

215 km with 178 km

so $1.31 \times 178 = 233.2 > 215$

and consequently it is more economical to choose alternative 1. Note from the map, from which the above distances have been obtained by scaling, that alternative 1 will also be the best start for a route to either Plymouth or Penzance.

The costs for the two routes, London to Exeter, are easily obtained as follows:

Alternative 1: Now $\frac{5}{8}$ km = 1 mile

$\therefore$ 300 km $= \frac{5}{8} \times 300 = 187.5$ miles

Cost per mile on motorway = £0.73

$\therefore$ Total cost $= 187.5 \times 0.73 =$ £137.00

Alternative 2: Total cost $= \frac{5}{8} \times 85 \times 0.73 + \frac{5}{8} \times 178 \times 0.96$

$=$ £145.60

Hence, saving of £8.60 ≈ 6%.

So, using the data provided, minimum cost routes may be found and a minimum cost matrix constructed. If the company were to use this cost matrix, then overall minimum costs should be obtained - the company may however decide to reject some or all of the proposed routes due to considerations (eg, driver fatigue, etc) not originally provided.

If the company bought a new fleet of trucks, then new data would be needed to cost the routes. In which case it would be advantageous to work out general expressions for the costs per mile:

Let hourly cost (fixed) = £h

petrol consumption = g mpg

average speed = s mph

cost of petrol = £p per gallon

Then cost per mile = $c = \left(\frac{sp}{g} + h \right) / s$

$$= \frac{p}{g} + \frac{h}{s}$$

Letting the subscript m apply to the motorway, and a to the A-route, the difference in costs/mile, D, is given by:

$$D = c_a - c_m = \left(\frac{p}{g_a} + \frac{h}{s_a} \right) - \left(\frac{p}{g_m} + \frac{h}{s_m} \right)$$

This saving can now be related to the extra mileage which could be covered on a motorway compared with the mileage on an A-route for an equivalent cost. Thus extra mileage on motorway per mile on A-route, E, is given by:

$$E = \frac{c_a - c_m}{c_m} = \frac{h\left(\frac{1}{s_a} - \frac{1}{s_m} \right) + p\left(\frac{1}{g_a} - \frac{1}{g_m} \right)}{\frac{h}{s_m} + \frac{p}{g_m}}$$

By way of illustration, one can again use the data provided:

$$E = \frac{30 \left(\frac{1}{35} - \frac{1}{50} \right) + 1.6\left(\frac{1}{15} - \frac{1}{12} \right)}{\frac{30}{50} + \frac{1.6}{12}} \approx 0.31 \text{ miles}$$

Thus, for every mile on an A-route the company can afford to travel 1.31 miles on a motorway, a result deduced arithmetically earlier on.

CHAPTER 6

TEACHING AND LEARNING EXPERIMENTS

6.1 Introduction

This chapter concentrates on the teaching and learning styles involved with students on a variety of courses from school to postgraduate level. Most of the results reported are based on observations of students who are taking mathematical modelling as a normal part of their course. However, a significant number of observations are also based on trial lessons and workshops where mathematical modelling is an entirely new experience for the students involved. Most of the work has been done with physical sciences/engineering undergraduates and graduate mathematics teachers attending the part-time MSc degree course in mathematical education at the Polytechnic of the South Bank. Where experiments have been carried out at school level, they were designed and tried out in the classroom by MSc teachers as part of their dissertation under the author's supervision.

The overall approach is summarised in Figures 9 and 10 in Chapter 4, where the key interactions between lecturer (teacher) and modeller (student) are illustrated. A total of seventeen different groups of students have been observed and a log has been kept or a transcription has been made of an audio recording of each experiment. The nine case studies presented in Chapter 5 have been used, and Table 4 in Chapter 4 indicates which have been presented to which level of student group.

The purpose of these investigations is essentially three-fold:

(a) To determine the level of difficulty of modelling problems for different student types

(b) To observe how students tackle modelling activities under a variety of working conditions:

Interactive (working with lecturer)
Group ('short' and 'long' duration)
Individual work
or, Combination of above

(c) To develop learning heuristics for the student inexperienced in modelling

To a large extent (a), (b) and (c) are interrelated. The level of difficulty of modelling problems is investigated by interpreting the following considerations based on observation of all seventeen different groups of students:

(i) Previous experience of modelling

(ii) Mathematics ability

(iii) General maturity of student (school, HE)

(iv) Extent of teacher/lecturer interaction - considerable in the case of school students whose basic mathematical skills are lacking

(v) Amount of information provided in problem statement

(vi) How well-posed the problem is

(vii) How much time is available to tackle a given problem

Put another way, a problem statement in itself does not determine the level of difficulty of a modelling problem. For instance, the statement 'determine how to evacuate a building in the event of a fire' could be treated as a modelling problem for 12-year olds, for undergraduates, or for specialists in

operational research. The way any group of individuals tackles the problem helps to set standards (of difficulty) for that group. Of course, there are natural bounds to this approach. If, for example, the problem statement is 'design and cost a nuclear reactor (of a given type) to produce 50 megawatts of electricity', then only a team of the most highly experienced and specialist personnel could 'solve the problem'. Published literature on mathematical modelling does not take this into account.

In order to illustrate in more detail the above points and also to identify students' strengths and weaknesses in modelling, the following case studies and modes of presentation to various groups is developed more fully with samples of logs and transcripts. Table 6 summarises the experiments involved.

Case study	Type of student	Mode of working
Modelling the heating of a baby's milk bottle	MSc Math. Ed.	Group work
Speed-wobble in motorcycles (castors)	MSc Math. Ed. BSc 2 Appd. Physics	Group work Interactive
Minimisation of sound distortion in record player	CSE 4th form BSc 2 Appd. Physics Polymodel 3 (Postgrad)	Interactive Group work Group work
Evacuation of a school	GCE A/L VIth form	Interactive/ group work

Table 6

Teaching and learning experiments reported in detail

Finally, a set of learning heuristics has been devised and an attempt has been made to gauge the usefulness of this. Observations of students working on problems as well as canvassing student opinion on the early stages of modelling, particularly in the initial formulation-solution stages, have contributed to the construction of each heuristic.

6.2 Level of Difficulty of Modelling Problems

The results of each of the seventeen experiments in which the nine case studies (modelling problems) were used were analysed according to the following format:

Course, or type of student

Mathematics background of student

General knowledge of problem area
(eg, specific topics in physics)

Previous experience of modelling
(usually little or none)

How well-posed is problem?
(eg, is sub-problem identification involved?)

Amount of information in problem statement
(eg, how much background information is provided? any data provided?)

How was problem tackled?
time available (eg, 1 hr/wk for 3 wks)
time spent in between 'sessions'
interactive or group or combination
extent of lecturer intervention

Student performance
(eg, how far do they get with problem?
chief difficulties experienced?)

Each experiment is reported on briefly in this section, concentrating on key features only. More detailed observation notes and discussion is left to a sample of experiments covered in subsequent sections.

In the case of interactive experiments, or where a significant amount of discussion took place with the lecturer (author) during group working, an audio recording and transcript was made. In the case of group experiments the author kept a diary and notes on key observations, and the students were asked to keep a careful log of all their work and to include this in a group report. Copies of all the transcripts, diaries, and reports have been kept by the author for reference. The observation notes and further discussions which are presented in this and subsequent sections, and Chapter 7, are summarised from the author's transcripts and diaries and from students' group reports.

Where the author was asked by students for some help with the physics background to a problem, the author (depending on the nature of the question asked) acted as a reference text-book on physics, or helped students answer their own questions by modelling the physics involved by encouraging intuitive approaches. At no time did the author provide a physics law or other information that was in a form specific to the problem being modelled; the interpretation of the physics in a form appropriate to the model development of the students at any given stage was left to the students.

U-tube accelerometer

Two experiments were involved: one with MSc Math. Ed. teachers (all graduates of mathematics or have taken mathematics as a major component of a first degree, twenty in class of whom about half had studied physics to at least GCE A/L standard), the other with fifteen BSc 2 Applied Physics undergraduates (all of whom have studied both mathematics and physics for at least a year beyond GCE A/L). Neither group of students had any previous experience of modelling. Both groups were presented with a well-posed problem statement, without data, as given in section 5.2 of Chapter 5. The problem was treated interactively - author modelling with students for one hour (each group). The students in each group were required to find an expression for acceleration in terms of the difference in fluid levels and to interpret this for use in different cars - one week later their efforts were discussed in class. With my suggestion, some students tried different shapes for the U-tube (differing radii of limbs, inclined limbs) and also the effect of initially having the horizontal limb filled with a liquid of density different to that in the vertical limbs.

Student performance

Most students were able to derive $a = gh/\ell$. Some derived this by considering the analogous problem of a bowl of water and the angle of the free surface to the horizontal - this also helped in answering problems about the shape of the U-tube (radii of limbs irrelevant). The general difference between the two groups was that the BSc 2 physics students were quicker in listing many possible factors but both groups initially made the mistake of considering weights and forces rather than pressure in their arguments. MSc teachers were better at extending the model to include liquids of two different densities. Both groups needed some help from me, initially, in considering realistic accelerations for cars - 'mini' to 'porsche'. Both groups agreed that mathematics was needed, albeit very simple in this case, to see that density of a uniform liquid was irrelevant. Some MSc students had difficulty

in deciding on the 'start' and 'end-point' for measuring the length of the horizontal limb. Those students in the MSc group with very little physics background appreciated some help from me - intuitive approaches were suggested and they generally responded well once a bowl of water was mentioned (by one of their colleagues). For future problems either I acted as a physics text-book ('state law if needed', not, how to use a law in problem) or teachers asked more knowledgeable colleagues. This problem was the first tackled by modelling by each group as part of a one-term session on modelling.

Modelling the heating of a baby's milk bottle

The experiment considered here is reported on in more detail in a subsequent section of this chapter. The problem has been presented both interactively and for student group work with MSc Math. Ed. teachers. The class who tackled the problem interactively made initial progress more rapidly than did the class who worked in groups (of four teachers in each, on average). The group experiment is concentrated on here. The class of twenty-two teachers, who had eight weeks previous experience of interactive and group modelling in a one-term's course, were split into five groups (of their own choosing). Most teachers had GCE A/L background in physics (somewhat rusty in several cases). The groups were presented with a broadly posed statement of the problem, without data, as given in section 5.3 of Chapter 5. I acted as a physics text-book and tried to avoid giving any hints. I told the groups that if they wanted any data, then they should specify precisely what it was they needed - I would then endeavour to provide it in a form as near to their requirements as possible. The groups had one hour followed by a week (in their own time, where there was little opportunity unfortunately for group working) followed by an additional hour working together as groups.

Student performance

Some initial clarifications were needed - some groups wanted to change the problem (eg, put lid on saucepan, design an insulated container with immersion heater). Many features were listed initially (in spite of my advice) and requests for data were made, in some instances, before any clear ideas were formed about how to proceed. I suggested a simplification for all groups: temperatures of milk, saucepan, bottle, water, all initially the same (cold 'fridge!). Some groups considered heat loss, and others tried to incorporate the time taken for heat to conduct through the bottle wall. I suggested at the end of the first hour that they concentrate on some simpler approach - heat required, rate of heat supplied, hence heating time - by working on a lumped mass system with instantaneous heat transfer. Some groups, in the following week, had managed this simplification with the deduction that minimum heating time is achieved with no water in the saucepan - hence burnt bottle. No group appreciated that they now had a lower-bound to heating time. In the second hour (second week), I was frequently asked for data - in the case of one group, where they asked for the rate of heat input from a gas or electric ring, I gave them the quantities involved in heating a given volume of water in a saucepan in a given time for a $15^{o}C$ to $35^{o}C$ temperature rise; the group wanted to know the answer for rate of heat-input and not have to calculate it (with heat loss from ring and saucepan automatically taken into account with my data. The group didn't realise this.).

Speed-wobble in motorcycles

Two experiments were involved, which are reported in more detail in a subsequent section of this chapter. One experiment involved a group activity with MSc Math. Ed. (the same class who had attempted the baby's milk bottle problem) - again one hour per week for two weeks. The other experiment involved a BSc 2 Applied Physics class taught interactively for 1½ hrs only. The undergraduate physicists had previously had two introductory modelling sessions in previous weeks. Both classes

were presented with a fairly well-posed problem statement, without data, as given by the full statement in section 5.4 of Chapter 5. I acted as a physics text-book and tried to avoid giving any hints to the MSc groups. The purpose of the interactive work with BSc 2 physicists was introductory, some students continuing with their work in their own time on a voluntary basis.

Student performance

The main difference between the BSc 2 physicists and the MSc Math. Ed. teachers was that the former wanted to concentrate on the design of motorcycles - forks, suspension, etc, and was not too keen on getting down to some mathematics. However, the physicists did start to concentrate on a simple castor after about an hour, with my prompting, and generally made some progress in what was after all a very short time. I advised the MSc groups to consider a vertical axis castor from the outset and that I had some data related to motorcycles if they wanted it - they had to specify exactly what was wanted, however. The MSc groups spent most of their first hour concentrating on the rolling (no braking) castor and what was happening to the point of contact on the ground. Moment of inertia about the steering axis was identified as important, but I had to suggest that the effect of the motorcyclist would be to modify the value for inertia - he/she would, in effect, have no damping effect in view of the high frequency of oscillations observed ("thought experiment - imagine riding an ordinary bicycle with speed-wobble"). Towards the end of the first hour, some groups had arrived at a SHM solution involving a force acting in the opposite sense to v (direction of motion of steering-axis). No group had carefully considered the direction of motion of the point of contact of the castor with the ground (see Figure 18, Chapter 5). Consequently, groups had difficulty relating their frictional force with normal reaction. I intervened at this stage by clarifying matters about rolling wheels in general - no slipping, rolling and without oscillations. The following week, all groups were clear about the point of contact of an oscillating castor, but SHM solutions still prevailed.

Minimisation of sound distortion in a record player

Three main experiments were carried out: one by O'Hare (1980) interactively with an average ability fourth form at school, one by group work with a BSc 2 Applied Physics class (the same one that tried earlier the speed-wobble problem), and one by a postgraduate group of academic and industrial mathematicians working on a group basis at the Polymodel 3 workshop, Oke (1981). Apart from the BSc physics groups who had very little modelling experience, the participants in the other two experiments had no previous experience of modelling. The school class spent a total of 4½ hours on the problem, spread over four interactive sessions. The BSc physicists spent two weeks, with intervening "own time", at the rate of 3 hours per week. The Polymodel 3 group consisted of two polytechnic mathematics lecturers and two mathematicians from industry (recently graduated) and spent a total of 5 hours on the problem, spread over an afternoon - "own time" in the evening - and the first part of the following morning. The interactive experiment with the school fourth formers started with the teacher showing the students how a pick-up arm was approximately tangential to the recording groove with an actual record player. The BSc physics and Polymodel groups were both given a quite well-posed problem statement with data as in section 5.5 of Chapter 5; the lecturer, in both cases, role-played as an engineering designer - trying to avoid giving hints.

Student performance

The three experiments are reported in more detail in a subsequent section of this chapter. The key difference between the three very different student types, quite predictably, was that the school students required much more help from the teacher. O'Hare skillfully drew out as many suggestions as possible from the students, but even when tangents were appreciated he had to show them how to draw scale-diagrams. The BSc groups made good progress, one of them starting on signal analysis in simple terms, once design problems had been cleared up. Some advice had to be given, though, it getting most groups

off many scaled-diagrams - they had to be urged to do some maths (even the cosine rule). Polymodel 3 made the best initial progress recognising 'underhang' as an improvement and 'overhang' a worsening of distortion for the straight arm case; they needed some help, however, in distinguishing parameters from variables.

Windmill power

This case study was presented interactively to two classes: BSc 2 Engineering Product Design (South Bank), and BSc 2 Engineering Mathematics (Loughborough University). Neither class had any previous modelling experience. The engineering design students studied some mathematics and physics in the first year of their course (mainly of GCE O/L entry in both subjects). The Loughborough students, mainly of GCE A/L entry in both mathematics and physics, had studied mathematics, physics, and engineering science in their first year. Both classes were presented with a physical model (but not wind-tunnel demonstration) similar to that shown in Figure 22, Chapter 5, together with the broadly-posed problem statement as in section 5.6. Both experiments were of short duration and intended only as introductory to modelling. BSc Eng. Prod. Des. spent two one-hour sessions, and BSc Eng. Maths. spent one two-hour session (with break) on the problem.

Student performance

The interactive treatment concentrated mainly on getting to grips with the problem and developing some simplified approaches. In both cases, Eng. Design and Eng. Maths., attention was focussed on a fixed blade with qualitative discussion on moving blades. Both classes were able to identify many features (velocity of wind, area of blades, angular velocity of blades,...) but (both) had difficulty in seeing how the power developed depended on the load on the windmill (eg, mechanical input needed to drive an electric generator). At my suggestion in each case, students thought out what would happen if no loading was applied (apart from frictional couples in the shaft bearings).

Some form of 'back-resistance' of air due to high-speed rotation of blades was arrived at - I then suggested applying a brake to the shaft and this helped to clarify matters regarding torque and power (all students knew that power = torque × angular velocity). Regarding the initial, and most time-consuming, initial formulation stages, students were prompted to consider a fixed blade (none suggested this). The eng. design students, with weaker maths and physics backgrounds, wanted to discuss design features such as strength of materials used (especially the blades). Both classes needed help with a simple approach to finding the force acting on a blade - although the eng. maths. group soon found $F = \rho A v^2$ for fluid impinging normally on a flat sheet. The eng. design group were encouraged to try a dimensional argument - force and velocity in dimensional terms led to $F \alpha v^2$. For a blade with non-zero pitch angle, there were again difficulties with both groups - physical intuition seemed to be lacking (I suggested that they imagine running a wet finger across the width of the blade - this helped considerably by getting them off the idea of 'pellets' of air). The final expression derived by most members in each class, although much more readily with the eng. maths. students, was the starting torque for the windmill.

Pole-vaulting

Two experiments were carried out: one with a school upper sixth form taking both GCE A/L pure mathematics and applied mathematics as separate subjects, by Sheridan (1980) starting off interactively and then splitting the class up into groups. The other experiment was with a postgraduate class of Indian school and college teachers on a group basis; these teachers were attending an AIMEC course at Loughborough University on a one-year leave of absence programme. Neither class had any modelling experience. The sixth formers each had GCE O/L physics as background, and the AIMEC teachers each had at least GCE A/L or equivalent in physics. Both classes were presented with a broadly-posed problem statement with data and 'match-stick' diagram as in section 5.7 in Chapter 5. Additionally,

both classes were shown a closed-loop film of a pole vaulter in action. The sixth formers worked over two 1½ hr periods, in groups but with frequent teacher interventions, and the AIMEC 'class' worked in groups with minimum intervention over a 2 hr period (with a small break).

Student performance

Both classes were able to identify many features - although the sixth formers were more vague than the AIMEC participants. The AIMEC groups quickly got involved with several velocity and force considerations but found difficulty in simplifying. The sixth formers needed frequent help, but once this was given ('gentle' hints), they worked quite well in groups. Both classes appreciated the closed-loop film - it generated considerable interest. An analogy with a long jumper was used with the sixth formers to help them understand better the initial flight path of the vaulter. The brighter sixth formers were better at 'guessing' relationships, one of them producing $\frac{1}{2}I\omega^2 = mgh$ leading to $v^2/2g$ for the height reached. At the teacher's suggestion, two sixth formers arrived at a graph of the form shown for the flight path in the top sketch in Figure 29, Chapter 5. The AIMEC groups had effectively much less time in which to work on the problem, and eventually conservation of energy of the vaulter was suggested as a (gross) simplification. One AIMEC group wanted to incorporate bending beam theory for the pole - I suggested they 'guess' a reasonable relationship for the restoring force on the vaulter in terms of the displacement of the bent pole from a straight line. This led to some interesting further development, but was eventually abandoned due to shortage of time.

Central-heating

Two experiments were involved with this case-study. One with a BSc 2 Applied Physics class, who had only one previous introductory session on modelling, and the other with a MSc Math. Ed. class who had two introductory modelling sessions. Both experiments were carried out interactively, each lasting

one hour. The purpose was to investigate early formulation-solution stages. Each class was presented with a brief but fairly well-posed problem statement as in section 5.8, Chapter 5. Approximately one third of the MSc group had at least GCE A/L physics as background, the rest about GCE O/L in physics.

Student performance

After initial discussion with each class, it was agreed that only one warmth period and only one cooling down period would be considered in any 24-hour interval. Both the BSc physicists and the MSc classes had some initial difficulties in identifying simplified aspects of the problem and I suggested that they concentrated on the warmth period during the day. Further prompting was required to work on the simplification of 'heat in = heat out' on a lumped system for the steady day-time temperature required; the physicists were even more concerned than the MSc teachers on separate components of the system - boiler, radiator, different thermal conductivities for walls, windows, roof. Eventually, both classes got to grips with 'lumping' variables (overall heat loss factor, etc) and arrived at an expression (in various forms) for heat loss = heat gained.

An interesting experiment was carried out however by the author and a small group of MSc teachers, by working independently, in seeing how long it would take to get a 'solution' as far as that developed in section 5.8, Chapter 5. It took the author a total of six hours (one two-hour, and one four-hour attempt). The MSc group spent about three hours of their private time and considered cooling at night as well as heating up in the early morning, but had not related their results in such a way that any clear conclusions could be drawn.

Evacuation of a school

This experiment was carried out by Wilson (1983) with a small lower sixth form consisting of six students preparing for the GCE A/L in mathematics. The results are reported in more detail

in a subsequent section of this chapter. The students had no previous experience of modelling. The students were initially given the following broad and imprecise statement:

> After a recent fire-drill at your school the head-master expressed some concern about the crowding and confusion that resulted in the corridors and the excessive time taken to evacuate the building. As a result he has asked the sixth year if they could come up with some suggestions as to how to improve the situation.

The students worked alternately: interactively, and as a single group on the problem, for a total of seven hours in six sessions spread over a three-week period.

Student performance

In view of the imprecise, or ill-posed, problem statement, teacher-student discussion started with identifying particular questions (time to evacuate a classroom for example). After the interactive start, students were left to work in a group and were free to leave the classroom to take any measurements they may want. Students were also invited to work in between sessions as homework. Students quickly identified and (from their own observations, guided by the teacher) obtained estimates to evacuate a typical class and the time taken to walk from one classroom to another (it having been suggested that they concentrate on one exit initially). The students took some time in appreciating that a 'time delay' could be introduced to stagger class exit times in order to produce a smooth and orderly flow. Students preferred to work in arithmetic rather than algebraically, and frequently went into the corridor to check their work. Eventually, after a little help with sigma notation (which they had not used before), the students arrived at an algebraic expression for the total evacuation time of n classes, all in one row, using one exit. Measurements taken were used to interpret their results. The teacher led the group finally by considering a topological map of ten classrooms and two exits of part of the school.

Motorway and 'A' road travel costs

This experiment was also carried out by Wilson (1983) with the same lower sixth form group. However, the sessions were carried out on a (single) group basis with the teacher role-playing as a company consultant. A total time of six hours was spent in five sessions over a two-week period (which followed straight on from the 'Evacuation of a school' problem). A quite well-posed and detailed problem statement as in section 5.10, Chapter 5 was handed out.

Student performance

Initially discussion took place with the teacher on clarification of the problem ('jamming' in towns could be assumed to be taken into account when referring to the company's figures for average speeds, for example). Students soon identified that costings were required, although they preferred to work on several routes (distances taken from the map) which had either 'A' routes or motorways but initially they were loathe to consider a combination. Generally the students still preferred to work with actual routes arithmetically, even after some of them had derived algebraic expressions. Although most students arrived arithmetically at the savings per mile on a motorway, and several had an algebraic expression for this, no one stated a maximum distance that would be travelled before an A road became more economic. Consequently, the students had difficulty in arriving at a strategy that could help the transport firm cost any route - with a combination of motorway and A road travelling.

6.3 Modelling the heating of a baby's milk bottle

As pointed out in section 6.2 the discussion centres on observations of an MSc Math. Ed. class working for two one-hour sessions (separated by a week) in five groups. The physics background of each group is as follows:

Group 1	4 members, 3 of whom with GCE A/L physics
Group 2	4 members, 2 of whom with GCE A/L physics
Group 3	4 members, 3 of whom with GCE A/L physics
Group 4	4 members, 1 of whom with GCE A/L physics
Group 5	4 members, 4 of whom with GCE A/L physics - out of which, 2 had taken extra physics in their degree

All the MSc Math. Ed. class are teachers (mainly in secondary school, but occasionally in FE) who either have a degree in mathematics or have a degree in which mathematics is a major subject. The teachers chose their own grouping after I had suggested no more than four per group. This was the first group modelling experience for them after eight weeks part-time of introductory modelling - interactive and individual 'homeworks'. Each group was asked to keep a record of working consisting of initial thoughts, lecturer hints (if any), data requested (from lecturer), 'scrap' working, and any conclusions reached. In view of the shortage of time involved, only one or two initial models with interpretation was expected. The overall performance of the groups is reported on in section 6.2.

Before group work commenced, a brief refresher was provided on SI units of heat (kJ, kW, ^{o}C , ^{o}K), together with the ideas of specific heat and thermal capacity (for the benefit of the 'non-physicists' in the class, these concepts were introduced intuitively - eg, large block of material needs more heat than small block of same material for a given temperature rise). At the end of the first hour (first week), all groups felt they needed a refresher on thermal conductivity ('how long will heat take to pass through bottle to reach milk?') and also on

heat losses (Newton's law of cooling was intuitively developed). No hints were provided on how to use these concepts, although I suggested that before using them they might like to consider a simple lumped system (instantaneous heat transfer throughout all components in the system); this suggestion was made becuase each group had difficulty in simplifying their original ideas. Each group worked co-operatively together, except Group 3; in the case of the latter, especially in the second hour (second week), members disagreed on the way forward and continued on an individual basis - this occasional difficulty with group working was first discussed in section 3.4, Chapter 3.

The following descriptions of performance of groups 1 and 4 are provided to give samples of the modelling activities carried out in the time available (2 hrs). Group 1 is chosen because it is one of the best attempts (3 out of 4 members with GCE A/L physics background, albeit 'rusty') and group 4 is also chosen because it has only one member with comparable physics background. The descriptions are based on group reports (rough work - unpolished and including initial thoughts) as well as my own observations.

Group 1

Symbols clearly defined and assumptions initially made are clear. Water assumed not to boil, although there is some uncertainty here. After hint (treat as lumped system), mathematical expression is written down: heat lost by water = heat gained by milk and bottle. The expression does not involve time and seems to relate to a different problem, namely that of placing the bottle in pre-heated water. Some attempt is made at finding heat lost from the system but areas involved are not clear. Thermal conductivity of bottle considered but soon dropped. Many factors now considered and there is general 'drifting' - losing sight of simple objectives. Eventually they get back to the lumped system and introduce rate at which gas (or electric ring) supplies heat and arrive at a general expression for time in terms of height of water in saucepan.

Data requested at end of first hour: specific heats, densities, temperatures of water and milk ('treat milk like water' - hint). No numerical values for time were calculated, although an expression showing linear dependence of time with height of water was derived; the group were not impressed with this latter expression and were unable to interpret its usefulness in providing a lower bound for heating time. They had made a good start to the problem, by eventually reaching a simplification, but were unable to proceed further in the time available.

Group 4

Symbols generally not defined, although one can guess their meaning. Initial assumptions are clear (including pouring the milk straight into the saucepan - different problem, but soon discarded). After my hint (to all groups after one hour), the group soon derived an expression for the lumped system in terms of the mass of water and other variables (consistent with their assumptions):

$$\frac{d\theta}{dt} = \frac{Q}{m_w s_w + m_m s_m}$$

where $\frac{d\theta}{dt}$ = rate of temperature rise of system

Q = heat needed by system

m_w, m_m: masses of water and milk respectively

s_w, s_m: specific heats of water and milk respectively

The group wanted to maximise $\frac{d\theta}{dt}$ and noted that other things remaining constant, the value for m_w should be zero. They realised that this would lead to a burnt bottle, but did not appreciate that they would have obtained an upper bound for $\frac{d\theta}{dt}$ leading to a lower bound for heating time. Private individual time plus the second hour of group time concentrated on attempting to model heat losses. This led to a sensible differential equation based on heat supplied = lumped system gain + heat lost; some initial attempt was made to solve this differential equation, but (although mathematically consistent) a very general result was obtained and then dropped. No

attempt was made to take thermal conductivity into consideration. No request for data was made.

Conclusions

Overall, the groups were more appreciative of the necessity to simplify than they were eight weeks previously when they had no modelling experience. However, they still had considerable difficulty in making simplifications without my help (eg treat as lumped system). Having obtained a lumped-system result, which I thought was initially very useful, the groups were unable to interpret it - or, at best, realised that no water in the saucepan would burn the bottle. No realisation of lower bound for heating time was apparent. Some initially good attempts at modelling heat losses led to solutions which were very general - no attempts were made to check if constants introduced were measurable (in principle at least). It would appear that 'lumping' has the disadvantage of making 'un-lumping' difficult once a solution has been obtained. Generally, the groups had little difficulty in identifying many features and found that by expressing their ideas (no matter how 'half-baked') mathematically, they could more easily see what variables and constants they additionally required at any stage. For instance, even if only incomplete relationships are considered, mathematical expressions for them help to complete the sought after relationship.

There was no discernible difference between group performance that could easily be accounted for by differences in physics background; group 4 performance was one of the best organised in that it had a better sense of direction and kept to initial objectives without wandering, yet members had very little GCE A/L or equivalent in physics background knowledge.

6.4 Speed-wobble in motorcycles

The main experiment reported on is based on observations of the same MSc Math. Ed. class working in groups of four who had just completed the baby's milk bottle problem. The same grouping took place and each group was asked to keep a log of its progress. The time allocated was the same as for the baby's milk bottle problem, namely two hours for group time (one hour per week for two weeks) with intervening individual time. The overall performance of the groups is reported on in section 6.2.

Before the groups started working, brief revision was provided on the basic friction law $F \leqslant \mu R$, and the result $I\ddot{\theta}$ = couple acting on a rigid body was intuitively developed. Most teachers seemed to know these results anyway. It was suggested at the outset that a vertical-axis castor with solid wheel should be considered. The groups as a whole found this problem difficult and towards the end of the first hour I chose to help them visualise better what was happening with a rolling wheel without slipping and initially without oscillations. I then showed the class how to find the velocity of the point of contact of the castor with the ground, and this then enabled them to obtain the correct direction of the frictional force. At the end of the first hour, then, all groups understood Figures 17 and 18 of Chapter 5. Each of the group's attempts is described below.

Group 1

Unlike with the baby's milk bottle problem, this group made less progress. Some initial considerations were mentioned, eg, 'wobble - to do with C of G'; 'suspect - larger the trail, greater the wobble'; 'friction causes wobble'. A muddled force diagram involving mass (of what is unclear), 'pushing force' (acting on what is unclear), friction and reaction acting at point of contact of wheel with the ground is shown. Friction force is in direction opposing motion of steering axis. Angular momentum (factors involved not defined), and moment

of inertia (again vague) are considered, but no mathematical relationship was written down. Not even after my intervention with the whole class on motion of point of contact was any further progress made.

Group 2

Symbols poorly defined, if at all. After initial considerations, the equation $I\ddot{\theta} = -Fr$ is written down; F is a force acting on the steering axis in its direction of motion, r is the radius of the wheel, I is not defined. Clearly, the equation is nonsense. Variations of $I\ddot{\theta} = -Fr$ are tried and a SHM equation is derived, but which unfortunately is meaningless in the context of the problem. No further progress was made even after my intervention regarding point of contact.

Group 3

Most symbols are defined, although the axis about which the moment of inertia relates is not mentioned. The equation of motion considered initially is

$$mT^2\ddot{\theta} = - F_s \sin \theta . T$$

where T is trail, F_s is a frictional force in a direction opposing motion of the steering axis, m is the mass of the wheel, θ is the angular displacement of the plane of the wheel from the direction of motion of the steering wheel (ie, θ is the same as that shown in Figures 17 and 18 of Chapter 5). Although incorrect, since F_s at the point of contact is in the wrong direction and $I \neq mT^2$, the equation shows some intuitive understanding. A SHM solution is then derived using $F_s = \mu W$, W weight (of castor), μ coefficient of friction. No interpretation of this solution is provided. After my intervention on motion of point of contact, some elaborate attempts were made by considering accelerations and their components. The working is muddled and very difficult to follow.

Group 4

Factors assumed to affect the castor are clearly stated, eg, 'trail', 'velocity', 'material of castor', 'surfaces in contact - frictional forces'. The initial equation considered is

$$F \sin \theta . x = -I\ddot{\theta}$$

where x is trail, F is friction (direction not clear), I is moment of inertia (axis not specified), θ is angular displacement from forward motion of steering axis. A SHM solution is derived. After my intervention about the motion of the point of contact, the following almost correct (ie, consistent with assumptions) equation of motion is considered:

$$F.L = I\ddot{\theta}$$

where F (friction force) is in the correct direction (Figure 18, Chapter 5), L is trail - but pity about the minus sign missing. The attempt finishes with:

$$\text{'integrate: } \int FLdt = \text{'}$$

Writing $F = \mu R$, this could have led somewhere. This report was the most readable and clear of all the groups.

Group 5

The only group to get the moment of inertia 'I_A' correctly defined to be about the steering axis (A in a plan sketch). The equation of motion considered is

$$- I_A\ddot{\theta} = F\ell \sin \theta$$

where θ is angular displacement from forward motion of steering axis, F is friction in a direction opposite to the motion of the steering axis, ℓ is the trail. A SHM solution is obtained, which the group could not interpret. After my intervention on the point of contact, no further progress was made.

Conclusions

In spite of the good physics background of the teachers, each group found this to be a particularly difficult problem. Conceptually the hardest part is understanding how the point of contact of the wheel moves, assuming limiting friction, in relation to the forward motion of the steering axis. It is surprising, therefore, that once this had been developed by myself at the end of the first hour, that practically no further progress was made. Apart from group 5, groupswere vague about moments of inertia - even to the extent about which axis to refer to. No requests for data were made. Most groups quickly settled for the security of SHM, and seemed to lack experience of any other possible type of oscillatory problems. The problem might better be presented as the culmination of a set of graded problems starting, perhaps, with a problem on a supermarket trolley castor that did not suffer with 'wobble' (eg, how is 'ease' of steering affected by design?).

6.5 Minimisation of sound distortion in a record player

As mentioned in section 6.2, three experiments were carried out with this problem with the following type of participant:

Average ability 4th form, secondary school (O'Hare, 1980)

BSc 2 Applied Physics (South Bank Polytechnic)

Polymodel 3, postgraduate (Oke, 1981a)

The results of these experiments, taken from observation notes and student reports, are presented below under separate headings.

Average ability 4th form, secondary school

The class spent a total of 4½ hours on the problem, spread over four interactive sessions.

After initial discussion on record players in general, most students owning or having access to one in their homes, the teacher (O'Hare) asked the class if the design of the pick-up arm might affect distortion ('quality of sound being affected'). Apart from noting that the arm should be balanced, no suggestions were forthcoming. After pointing out that some arms are straight and some 'bent', the teacher drew some sketches on the blackboard as illustrations (choosing a circle to represent a typical part of the recording groove). At this juncture, several students saw that the arm should be tangential to the groove at the point of contact with the stylus; this was accepted as the problem to consider (the teacher did not suggest, nor did any student, that the speed of the recording groove might be important). The problem as perceived was reinforced at the next lesson by the teacher demonstrating with a school record player how an (offset) arm approximated to a tangent to each groove at the stylus. The teacher then asked the students if they thought straight arm pick-ups would be unsatisfactory. One student suggested that although they would not be as good, straight arm pickups must give reasonable reproduction as they are still in use. When questioned on how 'good' straight arms were, several students suggested that they could measure how near the arm was to the tangent at the stylus. The teacher found that he had to translate the problem into finding the angle between the (straight) pickup arm and the tangent to the groove (ie, measure tracking angle). Scale-diagrams were then drawn using a straight arm of initial length of 20 cm (~ 8" as per data in section 5.5, Chapter 5) and measurements using a protractor were taken of the tracking angle at the inner and outermost circles of the recording groove. The students needed help initially in scaling their measurements correctly, and also needed some revision on how to draw a tangent to a circle from an external point (arm pivot). Student averages for these tracking angles

were found to be 13° and 20° respectively for the inner and outer grooves - no underhang being suggested. Further discussion with the students led to the suggestion of trying lengths of 15, 25 and 30 cm.for the arm, effectively amounting to underhang and overhang measurements (pivot point remaining fixed) - this suggestion was, in fact, eventually made by the teacher.

Scale drawings were also made by using different arm lengths without underhang, and the students soon noticed that the longer the arm the smaller tracking angles became. It required some more help from the teacher, in view of inaccuracies in scale-drawings, to see that underhang produced more useful answers. The students readily appreciated that having a very long arm (without underhang) would be impracticable however, and this helped in their persistence with the underhang diagrams. Eventually, some students (for homework) found a reasonable value for underhang (~ 2 cm) and noted (correctly) that this occurred when tracking angles were equal in size at the inner and outer grooves.

Generally, the students found the problem difficult but interesting (they had no previous modelling experience apart from an introductory session on the location of a school). They considered it a good way of learning constructions, but found that their interest was waning towards the end of the exercise. The teacher had hoped for more student suggestions in directing the modelling activity, but perhaps because of the problem being conceptually difficult or due to the lack of modelling experience of the students (or both), the teacher found that he had to take the lead most of the time.

BSc 2 Applied Physics (South Bank Polytechnic)

In section 6.2 it is pointed out that the students worked on this problem in groups, spending 3 hours (2 sessions: 2 hours, 1 hour respectively) per week over a two-week period with intervening "own time". Four groups were formed by the students

themselves, each consisting of four members (except group 2), and the author was assisted by two other members of staff who observed group working:

Group 1 4 members, observed by author

Group 2 5 members, observed by Mr Wright

Group 3 4 members, observed by author

Group 4 4 members, observed by Mr Jones

All observers agreed to provide no hints if possible, but if any hints or clarifications were given then a note of these was to be made. The author also observed all four groups on several occasions.

Before group work started, the author spent one hour with the class discussing general points and clarifying certain matters. This work was assessed and the marks counted towards the end of session profile for the course. Details of the assessment are discussed in Chapter 8.

In the initial clarifying discussion with the class it was pointed out that students would be required to keep a careful log, including rough work, and that the staff involved would act as engineering designers who would not be able to help much with physics and certainly not with any mathematics. Students asked many design questions (typically of physicists and engineers) regarding type of discs used, quality of apparatus, type of recording head, type of stylus, nature of recording groove. I reminded the class that the problem concerned the geometrical shape and size of the pickup arm in an effort to reduce noise distortion and that they should concentrate on that (as an 'engineering designer' I pointed out that I had other experts working on the other aspects of the problem, eg, balancing the arm). Considerable discussion then followed on the nature of the recorded signal. We all eventually agreed on the approach that, whether mono or stereo, the signal was picked up transversely (perpendicular to the

recording groove and in the plane of the record). I left the students to consider whether or not the width of the groove was important. All groups were initially advised to draw a few scale diagrams, using the data provided in the problem statement (see section 5.5, Chapter 5) in order to get a better 'feel' for the problem.

Each group started by continuing their attention on design features, eg, Is arm rigid? Shape of stylus? Depth and width of groove relevant? It was agreed amongst the observers to encourage each group to concentrate on basic geometry (but details left to students to decide). All groups took approximately 1-2 hours (out of a total of 6 hours) to identify tracking angle and the need to minimise this in some way; the tracking angle was usually taken as that angle between a radius (normal) and a straight line segment moved by the stylus as it transversed across grooves - this is the same angle between the arm (straight) and a tangent, but initially this was not considered. Group 2 had the greatest difficulties; angles were measured between circular arcs (loci of stylus) and radii (tangential to such arcs); later on clarifications of the tracking angle were made, and an initial assumption was made in setting this angle to zero half-way across the disc (record); some elaborate trigonometry then followed, introducing many variables, and then the group got stuck in trying to find a solution - observer help was needed in suggesting some simplifications. Groups 1, 3, 4 made better progress, with some exceptionally good insights shown in signal distortion by group 4. By considering a sinusoidal signal, the latter group were able to recognise the advance and retard effect when tracking angle was non-zero. Group 4 wrote the distorted signal in the form $y = A \sin(\omega t + \phi)$, where $y = A \sin \omega t$ was the original signal. By considering the stylus, on a magnified portion gleaned from a scale drawing, moving in a non-perpendicular direction to the recording groove, they were able to get an expression for ϕ.

Although most groups considered the possibility of tracking angle α (and occasionally off-set angle for a bent arm) changing

sign: +, O, -, throughout the duration of playing a record, this approach was soon dropped. Group 3 obtained a triangle, similar to Figure 20 in section 5.5, Chapter 5, which was consistent with their assumptions, and then applied the cosine rule - wrongly (forgetful of GCE O/L mathematics?). Group 1 applied the cosine rule correctly and then made a careless mistake in its interpretation.

Having initially made the suggestion of drawing a few scale diagrams, groups (except group 4) had to be encouraged to carry out some mathematics. Left to their own devices, scaled diagrams would seem to lead to empirical solutions only.

The observers of the groups decided to withdraw for two sessions because students kept asking for confirmation about their ideas. However, even with group 4, it was decided to 'visit' the groups regularly until the end of the two-week period in order to 'nudge along' and break fixations and mental blockages. As mentioned earlier, group 2 became quite frustrated with their trigonometry and some guidance seemed necessary. Overall though, the opinion of the observers was that the students who had practically no previous modelling experience had achieved quite a lot in the time available. The students found the problem very interesting and were anxious to know our views of their performance.

Polymodel 3, postgraduate

As mentioned in section 6.2, this group consisted of two polytechnic lecturers in mathematics and two recently graduated mathematicians in industry. The group spent a total of five hours tackling the problem (an afternoon and the following morning). A full report of the group's performance may be found in Oke (1981), Pentech Press. No member of the group had specialist knowledge of the problem and none had any previous modelling experience. The author played the part of an engineering designer, as with the BSc Applied Physicists reported previously, with the intention of providing no help beyond being a source of technical information and practical (design) advice. Each group member had GCE A/L in physics or equivalent background.

Instead of initiating discussion, I invited the group to 'interrogate' me on background information. So, in this way, just under an hour was spent going through the nature of a sound recording, how such a recording was made, and what were the design constraints involved in an average record player. The group soon decided that it would concentrate on a straight pickup arm rather than the bent or off-set type. The geometrical aspects of the problem were then soon identified, and the length of the arm (a), the distance of the pivot of the arm to the centre of the turn-table (ℓ), the radius (r) of a typical groove, the tracking angle (ϕ), and a couple of other angles were considered as important. A triangle was drawn showing these quantities (essentially the same as that shown in Figure 20 in section 5.5, Chapter 5) and the cosine rule was used (correctly) to obtain.

$$\ell^2 = r^2 + a^2 - 2ar \sin \phi$$

ie, $$\sin \phi = \frac{r}{2a} + \frac{a^2 - \ell^2}{2ar}$$

The group wished to minimise ϕ (and hence $\sin \phi$) in some way. They had considerable difficulty in deciding what to hold constant and what to allow to vary. Eventually, it was decided to draw a sketch graph of $\sin \phi$ against a; this implies that ℓ and r would be kept constant, but ℓ <u>depends</u> on a (which is varying) and when a disc is played, r <u>must</u> vary. Approximately one hour was spent on this, drawing various graphs for different ℓ and r. I felt that some help was needed at this stage, and so I discussed with the group the various quantities involved and suggested that it might be easier if they considered sketching a graph of $\sin \phi$ against r (for given a and ℓ). I further reminded the group that a (arm length) could not be much different from ℓ (design considerations). This intervention seems to have been the catalyst required, for the group quickly made progress by sketching two graphs as shown in Figure 33.

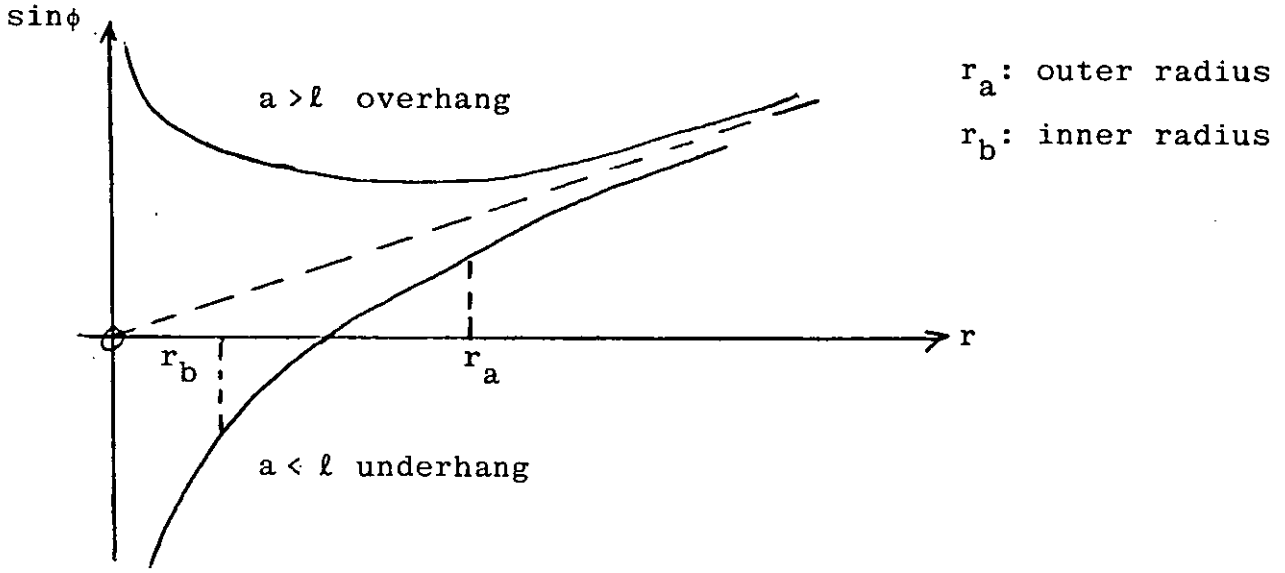


Fig 33

Graph of sin ϕ against r: Polymodel 3 group

Underhang (a < ℓ) was readily identified and an optimum condition for sin ϕ, and hence a (arm length), was derived by setting

$$|\sin \phi|_{r=r_b} = |\sin \phi|_{r=r_a} .$$

The group easily obtained the result for underhang, very similar to that obtained in section 5.5, Chapter 5.

The group then went on to consider the mechanisms by which distortion is introduced at the pickup, rather than be satisfied with merely a geometric result. A few sketches of the situation where the stylus was not tracking the groove in a perpendicular manner ($\phi \neq 0$) convinced the group that while tracking a signal the stylus would also move backwards and forwards, advancing some parts of the signal and retarding others. The group then embarked on the task of quantifying this advance and retard effect for a pure sine wave signal, but in the time remaining were unable to make much progress.

Conclusions

The three experimental groups worked hard and seemed to enjoy the problem. The 4th formers needed most help, in spite of the problem form eventually agreed with their teacher being

well-structured. A lack of modelling experience coupled with weak mathematics ability were contributing factors to the difficulties found by the students. Although progress was made with the (restricted) problem, it is doubtful if further work would have produced much more in view of the waning interest of the students after 4½ hours.

The BSc 2 Applied Physics students and the Polymodel 3 group both made considerable progress towards obtaining results which could be of practical use. However, in both experiments it was found necessary to intervene occasionally in order to prevent frustration setting in or to prevent wasteful activities (in terms of time) from obscuring the objectives of the problem. In both cases, participants were quite happy to expend considerable effort on tabulation (using the data and derived intermediate results) and drawing diagrams and graphs. Considerable initial difficulty was experienced, even with the Polymodel 3 group, in distinguishing between variables which could be controlled (eg, arm length) and those which could not (eg, radius of recording groove). Physically these things are obvious, but the mathematical interpretation of them seems to be more subtle. It is felt that if the BSc and Polymodel groups had more time, then once over their initial hurdles, they would have made more progress with the problem.

6.6 Evacuation of a school

This experiment, as pointed out in section 6.2, was carried out by Wilson (1983) with a small lower sixth form consisting of six students preparing for the GCE A/L in mathematics. The class spent a total of seven hours on the problem, in six sessions spread over a three-week period. The class either worked interactively with the teacher, or alone as a single group, or on an individual basis. The students had no previous experience of modelling.

The session started interactively with the broadly specified problem statement as given in section 6.2. Wilson wrote on the blackboard:

'What is the question/problem?'

In answer came the student replies:

"Why is there crowding? - What causes delay?"

"How can we shorten evacuation time?"

These questions were written on the blackboard. In response to this, ideas began to emerge very quickly:

<u>Students</u> "Let's take each class separately"

"We've got to find out how long it takes to get out of a classroom"

"How many children in a class?"

Discussion amongst the students led to the suggestion of taking a maximum number of 30 students in a class. The class was then asked to continue working as a group, being encouraged to take any measurements or record any information that they wanted. The second session started by the students reporting that they had found the time to cover the longest routes (along corridors) and had identified a bottleneck (confusion point) on the main stairway - all based on observation. They were, however, unclear on how to make use of this information; the teacher asked the class what they were trying to achieve, and after some brief discussion the students said they would like to find out how long it would take to evacuate a classroom. The students then timed how long it took for five of them to exit in single file: 4 seconds for 5 to exit, leading to 24 seconds for 30 students (student deduction). One student suggested $T = \frac{4}{5} N$ for time (T) of evacuation for N students. The class quickly agreed on $T = \frac{4}{5}N + \frac{d}{s}$ for the time to evacuate a class and then walk a distance d at speed s - the latter being measured. One of the students suggested a 'delay time':

"You've got 24 seconds for this one. Then you've got to find the time it takes to travel from their class to this one and take it off 24 seconds"

However, this was not pursued further at this stage. Instead, the students wanted to introduce the time taken to descend the stairs - they had difficulty however in deciding on the 'distance down the stairs'. The teacher suggested that the speed down the stairs would be the same as that along corridors (s) in an orderly flow - hence diagonal distance (measured) divided by s would provide the required time.

In subsequent sessions, after much pacing up and down in the corridors and many arithmetical calculations, the students decided on double rather than single file flow (still orderly) leading to $T = \frac{2}{5}N$ for one class and $T = \frac{2}{5}N + (x - 1)D$ for x classes where D is delay time. The students took a long time to realise that they had already in effect found D $(= \frac{d}{s})$ - still more extensive arithmetical calculation was carried out. Once the teacher had introduced sigma notation, the students were able to derive the following expression for total evacuation time for x classes:

$$T = \frac{D_\ell}{s} + \sum_{i=1}^{x} \frac{2}{5}N_i - \sum_{i=1}^{x-1} \frac{d_i}{s}$$

where D_ℓ/s is the time for the last pair to walk from the door of the last class to the (single) exit. The expression was tested in practice, with the cooperation of other teachers, for 3 classes in a given corridor.

Finally, the teacher presented a topological map to cover classes that are not all in one corridor and which showed two exits. Considerable time was spent pouring over the map and the teacher had to intervene by suggesting that the time taken by each class to a bottleneck was first calculated. Eventually, after much further arithmetical calculation and teacher guidance in encouraging the students to write out their ideas algebraically, each class was assigned an exit for minimum evacuation time.

The students are reported as having enjoyed the modelling activities and they felt that, in spite of frequent teacher guidance, they had achieved significant results by themselves.

6.7 Learning heuristics

In an attempt to help inexperienced students in modelling, in the spirit of offering general guidance and in the hope of providing some confidence in what is an unfamiliar activity, a list of heuristics ('rules of thumb') was devised. The construction of the heuristics was based on published literature in problem solving and mathematical modelling (c.f. Chapter 3), and on the results of the teaching and learning experiments reported in earlier sections of this chapter. Some aspects of the heuristics, and their implications for learning modelling, are developed further in the next chapter on formulation-solution processes.

The number of heuristics in the list has been kept deliberately low. The reasons for this are:

(a) Too many considerations serve only to confuse when considering any one problem

(b) A large list would tend to make each heuristic highly specialised and so dependent on a specific problem being considered

Reason (a) is almost synonymous with one of the heuristics: 'Don't write a vast list of features'. Reason (b) implies that each heuristic is couched in fairly broad terms in the hope that it has general applicability and is, therefore, not heavily problem dependent.

The list, which aims to cover most initial stages of the modelling activity is as follows:

Establish a clear statement of objectives

State obvious or natural objectives - but try and be fairly precise

Don't write a vast list of features

A large list of everything you can think of only serves to confuse. However, do write down any features considered important - then decide which to consider in detail

Simplify

Build up very gradually.
Make guesses, make assumptions, add restrictions.
Lump components (attributes) together and treat as single component or if original highly complex, break-down into simpler problems and treat each separately

Get started with maths, as soon as possible

Identify a few variables, parameters, constants.
Write down one or two obvious mathematical relationships.
Keep mathematics as simple as possible.

Carry out some mathematics on initial relationships

This itself generates more variables, constants and relationships

Got a solution yet?

If relationships so far do not satisfy objectives then create more obvious mathematical relationships and combine (mathematically) with those you already have

Know when to stop

Do not seek perfection. A mathematical model is only an approximation to reality. There are no right or wrong answers, only good or bad

Interpret your solution

Use common-sense.
Test for self-consistency (ie, no mathematical or logical mistakes, insert a few reasonable looking numerical values).

Validate your solution

Use known data to test model's ability to predict or verify over a wide range

If stuck

Observe practical situation if possible or carry out a 'thought experiment' and imagine what is happening.
Plot measured values, form empirical relationships

Have frequent rests

The modelling activity is difficult - it is a creative and intuitive act (even for quite simple models using only elementary mathematics). Do not spend more than about one hour at a time when starting a new problem

The order in which the heuristics have been listed is not necessarily the order in which they may be recommended for use. The results of the teaching and learning experiments reported in this chapter, together with the ideas developed in Chapters 3, 4 and 7, show that modellers (experienced as well as inexperienced) move forwards, recap, then move forwards again often carrying out several modelling activities simultaneously. However, the main intention of the heuristics is to provide some sort of guidance for the inexperienced when a new problem is starting to be tackled. In which case, the first few heuristics might with advantage be carried out in the order listed (eg, down to 'Carry out some mathematics on initial relationships').

In an attempt to gauge student opinion of the usefulness of the heuristics as initial guidance in modelling, a questionnaire containing the list was issued. Students were asked to rank the usefulness of each heuristic according to the following

numerical scale:

Essential	Very useful	Useful	Not very useful	Useless
1	2	3	4	5

In addition, students were invited to make comments:

> Any comments? For example, what helpful advice would you give to someone trying out a modelling activity for the first time? Can you add to the list of heuristics? Would you like to delete any heuristics? Are the statements explaining what is meant by each heuristic clear?

Several undergraduate and MSc Math. Ed. classes, including some of the groups who took part in the teaching and learning experiments reported earlier, were issued with the questionnaire. The list of heuristics was in each case handed out at the start of a modelling session of a new problem and students were requested to complete the questionnaire after the problem had been tackled and not during the modelling activities taking place. Depending on the way in which students worked, either group or individual replies were collected. Since this investigation was basically exploratory, an experimental design with control groups with consequent statistical analysis was considered inappropriate. Instead, general students' views were sought.

For each class, the average rank for each heuristic was calculated (see examples in the case of MSc Math. Ed. shown in Tables 7, 8 and 9). Overall, students found the heuristics useful as measured by the grand average rank for all eleven heuristics (2.6). A rank of 3 for any heuristic is considered as critical. There is considerable variation amongst individuals and groups of individuals in the ranking value given to each heuristic, but the most popular (useful) heuristics are chosen by most as the following:

Establish a clear statement of objectives

Simplify

Get started with maths as soon as possible

Generally, the least useful heuristic was found to be:

Don't write a vast list of features

No new heuristics were suggested by students but a number of varied and general comments were made. The most important general comment, that was made by several classes both undergraduate and MSc, was that given more time they could concentrate more in interpretation and validation and hence would probably give those heuristics concerned with these aspects a better (lower) ranking value. This latter point is borne out in Table 9, where teachers who had completed the first year of the MSc in modelling and who had also completed a course-work (average time spent: 52 hours) have had much more opportunity and experience in modelling. Table 9 shows average ranks of 1.7 and 1.5 for interpretation and validation respectively.

The rankings given in Tables 7 and 8 refer to those given by the MSc Math. Ed. teachers who had tackled problems on a group basis as reported in sections 6.3 and 6.4. The average rank per group has not changed significantly from one problem to the next, except in the case of group 2: Table 7, average rank is 1.8; Table 8, average rank is 3.8. The latter value (3.8) is high due to the fact that group 2 made little progress with the problem (in the time available) on 'speed-wobble in motorcycles'; the last six heuristics are given the value of 5 ('useless'). Group 2 may, of course, have made more progress with the problem given more help from the author and also more time in which to work. On the other hand, the problem may not have sufficiently interested this group (observation suggests this) and consequently they 'threw in the towel' at an early stage.

Heuristic	Groups					Average
	1	2	3	4	5	
Establish a clear...	1	1	3	1	1	1.4*
Don't write a...	5	3	2	3	3	3.2
Simplify..	2	1	2	1	3	1.8*
Get started...	3	1/2	2	2	2	2.1*
Carry out...	4	1/2	2	3	1	2.3*
Got a...	3	2/3	5	3	5	3.7
Know when...	4	3	4	3	5	3.8
Interpret...	2	2	2	2	2	2.0*
Validate...	3	1	4	3	2	2.6*
If stuck...	3	1	3	3	3	2.6*
Have frequent...	1	2	1	2	1	1.4*
Average	2.8	1.8	2.7	2.4	2.5	2.5

* Denotes most useful (score less than 3)

Table 7

MSc Math. Ed. rankings after problem 'Modelling the heating of a baby's milk bottle'

Heuristic	Groups					Average
	1	2	3	4	5	
Establish a clear...	1	1	4	1	2	1.8*
Don't write a...	3	2	2	3	3	2.6*
Simplify	1/2	3	4	1	2	2.3*
Get started...	1	2	1	1	2	1.4*
Carry out...	3	4	1	2	3	2.6*
Got a...	5	5	3	3	4	4.0
Know when...	4	5	2	3	3	3.4
Interpret...	2	5	4	3	4	3.6
Validate...	2	5	5	2	2	3.2
If stuck...	1	5	5	4	1	3.2
Have frequent...	1	5	2	2	1	2.2
Average	2.2	3.8	2.3	2.3	2.5	2.7

* Denotes most useful (score less than 3)

Table 8

MSc Math. Ed. rankings after problem 'Speed-wobble in motorcycles'

Heuristic	Teachers						Average
	A	B	C	D	E	F	
Establish a clear...	3	4	2	2	3	1	2.5*
Don't write a...	5	1	4	3	3	2	3.0
Simplify	1	1	2	2	2	2	1.7*
Get started...	3	1	2	1	2	2	1.8*
Carry out...	3	3	2	2	2	1	2.2*
Got a...	3	3	3	2	1	2	2.3*
Know when...	1	3	5	2	1	1	2.2*
Interpret...	2	1	3	1	1	2	1.7*
Validate...	2	2	1	1	2	1	1.5*
If stuck...	2	3	4	2	2	1	2.3*
Have frequent...	1	3	5	2	3	4	3.0
Average	2.4	2.3	3.0	1.8	2.0	1.7	2.2

* Denotes most useful (score less than 3)

Table 9

MSc Math. Ed. rankings, more experienced in modelling, general opinions

6.8 Summary and Conclusions

This chapter has concentrated on teaching and learning experiments mainly with undergraduates with mathematics, physics/engineering backgrounds, with teachers on the MSc in Mathematical Education course, and on occasions with secondary school students. It was stated at the outset that the purposes of the investigations were essentially:

(a) To determine the level of difficulty of modelling problems

(b) To observe how students tackle modelling activities under a variety of working conditions

(c) To develop learning heuristics

It has been found that (a), (b) and (c) are largely inter-related. The level of difficulty of a modelling problem is determined by how students are enabled to work and what are the lecturer's expectations of them in the time available. Student performance is governed by the extent of lecturer guidance, both specific to the problem in hand as well as in general (eg, provision of heuristics).

All the experiments considered in this chapter have been concerned with students who have little or no modelling experience. Furthermore, all experiments have been based on short to medium duration activities, that is students spending time ranging from one hour to ten hours on a given problem. Long duration project type work is usually given to students who have some experience of modelling, and the assessment of such projects in the case of MSc Math. Ed. teachers is left to Chapter 8.

Section 6.2 reports on the observation of seventeen experiments based on the nine case studies presented in Chapter 5. Each experiment provides information on the following:

Course, or type of student

Mathematics background of student

General knowledge of problem area

Previous experience of modelling

How well-posed the problem is

Amount of information in problem statement

How problem was tackled

Student performance

Further details are provided for a sub-set of experiments in sections 6.3 - 6.6.

Level of difficulty of modelling problems

The emphasis throughout, in the time available to the students, was in providing an opportunity to concentrate on the initial formulation-solution activities of modelling. In order for these activities to be meaningful, interpretation and some crude validation was also carried out. Since most of the problems were based on applications areas in the physical sciences, the two fundamental challenges for the students were:

(a) The ability to recognise the basic physical ideas inherent in the problem

(b) The ability to interpret the physical ideas in a mathematical form amenable to some initial analysis

Most of the students had at least GCE A/L physics as background knowledge, although this was somewhat 'rusty' in the case of some MSc Math. Ed. teachers. Once the difficulty of recognising what sort of physics was involved, and this itself was encouraged by 'guessing' and the development of intuitive ideas, the difficulties experienced were observed to be essentially the same for each problem - even also in the cases of the 'non-physics' problems: 'Evacuation of a school' and 'Motorway and 'A' road travel costs'. These 'common' difficulties, which are exemplified in sections 6.2 - 6.6, may be summarised as below:

Students' difficulties in short to medium duration modelling activities

- Tendency to want to work on problem other than that posed

- Variables and constants: which to choose as dependent, independent, parameters (particular difficulty for school students)

- Relationships and variables: level of detail (too much detail leads to confusion, too little or excessive 'lumping' leads to general mathematical solutions which are difficult to interpret)

- Tendency to: keep listing features, draw many diagrams/ graphs, carry out large amounts of computation rather than use analytical techniques (even elementary ones). School students particularly prefer arithmetic to algebraic or other methods

- Lack of confidence in making simplifications - 'bears no resemblance to reality'. Even when simplifications are made, difficulties are experienced in interpreting mathematical solutions arising from them

- Tendency to drift and lose sight of objectives. Fixations formed (unwilling to try other more fruitful paths)

Not all the above difficulties are experienced in each case although they do tend to occur quite often. Experience in modelling does help to overcome the extent of the difficulties, as mentioned in one of the earlier sections of this chapter in the case of the second year teachers on the MSc in Mathematical Education. However, it appears that even experienced modellers continue to find some of the difficulties. As experience is gained, confidence in perservering with a problem leads to a better chance of reaching meaningful solutions.

In choosing the case studies for the teaching and learning experiments, it is true to say with hindsight that the author of this thesis did not appreciate fully the potential difficulties of some of the modelling exercises involved, eg: 'Speed-wobble in motorcycles'. The reasons for choosing such problems largely arose from the following considerations:

(i) Apart from the organisational problems presented by Wilson at sixth form level, all the students had a good background in physics (usually at least to GCE 'A' level).

(ii) The applications areas, namely those involving some knowledge and appreciation of physics, partly arose from the work of providing modelling opportunities for undergraduates in applied physics and engineering which the author teaches at South Bank.

(iii) In order to provide a broad spectrum of experience in modelling, it was decided at the outset to provide problems involving physics to the teachers on the M.Sc. course in mathematical education (at South Bank) in addition to problems in the life sciences, and the social and organisational sciences. Neither external examiners nor the students (teachers) have objected to physics-based problems to date, although more help with initial concepts, has been requested (in the form of physics background sheets).

(iv) In order to make the physics-based problems as widely appealing as possible, two main themes have been chosen: mechanics (including fluid flow) and heat transfer. The physics needed to make good progress is illustrated in each case in Chapter 5 and it is at most GCE 'A' level. The more specialised areas of optics, electrical circuits and electronics have been avoided, again with the intention of reaching as potentially wide an audience as possible.

In the short time that was available for the experiments, it is the considered view of the author that the students involved had made good progress and had gained significantly in confidence as a result of most of the modelling exercises. As reported in section 6.7, the teachers on the M.Sc. course themselves felt that they had gained in knowledge and ability in modelling from following the one year programme on modelling. The teachers also felt that experience with the more difficult modelling problems was necessary for their development.

An important deduction can be made from the above discussion, namely that for students inexperienced in modelling, the level to be expected of another subject (eg: physics) should be considerably lower in a modelling context than that gained by formal study. This is well recognised in the case of mathematics knowledge, that the amount of mathematics one should expect to be used in a modelling exercise will be several (2 or 3) years below that gained on more formal courses.

Teaching/learning styles

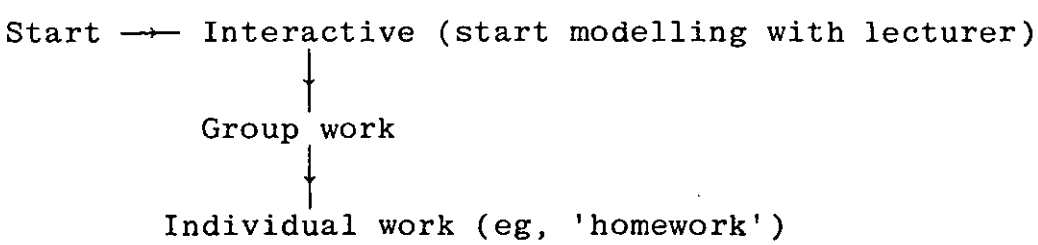
The experiments reported on in sections 6.3 - 6.6 cover the two basic styles:

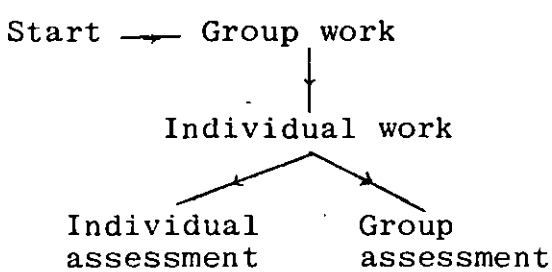
Interactive

Group

It has been observed that the interactive approach is suitable for modelling activities that are being tackled for the first time, especially in the case of school students, but that group work enables students to gain confidence and ability once the first one or two interactive sessions have been experienced. It has also been illustrated that lecturer intervention is needed at certain key points in order to prevent 'frustration', 'fixation', and other difficulties from taking over. Lecturer help and guidance is needed no matter what mode or style is followed and once it has taken place, it is important not to reduce the overall sense of achievement of the students.

Although two basic styles have been emphasised, the experiments have also illustrated variations and combinations of these styles:





Research generally has shown (see section 3.4, Chapter 3) that work done in groups is useful in the early stages of feature identification, but that the solution stage is best done on an individual basis. This has been confirmed in the experiments conducted by the author, and furthermore that much of the formulation of a problem is still being carried out at the solution stage. Generation of relationships, working on these mathematically and so generating further relationships seems to be achieved by individual momentum and is thus better carried on outside the (original) group context. There is no research recommending group size, where group working is carried out, but judgement of the author and of others is that 4 seems to be optimum; fewer implies lack of flow of initial ideas, more than 4 can lead to organisational problems and consequent splitting into sub-groups.

Learning heuristics

A set of heuristics has been devised in an attempt to provide some 'rules-of-thumb' for the student inexperienced in modelling. The heuristics are described together with student opinion in section 6.7. The most popular (useful) heuristics were deemed to be:

Establish a clear statement of the objectives

Simplify

Get started with maths as soon as possible

The choice of the last heuristic chosen needs a word or two of explanation in view of the fact that one of the key difficulties experienced by students is a reluctance to use even elementary analytical methods. Lecturer guidance is often needed in order to prevent seemingly endless computation, graph/diagram drawing, by suggesting that an elementary piece of algebra (trigonometry, etc) can not only tie up loose-ends but can actually predict for a whole range of values what is happening. Possibly as a result of such emphasis being given, students realise the benefits of the advice given and so give

a high priority to this type of action. The more experienced modellers in MSc 2 have also given this heuristic a good (low) ranking and have shown a marked improvement in this respect.

The most unpopular heuristic, with experienced and inexperienced alike is:

> Don't write a vast list of features

Possibly the explanation accompanying this heuristic should be changed to:

> Do write down any features considered important and then decide which to consider in detail

which is now almost identical to the form used as one piece of advice given to students of the Open University, Berry and O'Shea (1982).

Observations on how relationships are formed, and how the formulation and solution stages are related is investigated in the next chapter.

CHAPTER 7

FORMULATION - SOLUTION PROCESSES

7.1 Introduction

As pointed out in section 3.3, Chapter 3, mathematical modelling processes are usually portrayed as a linear sequence or linear sequence with looping. The limitations of this portrayal have been examined and in order to try and understand more fully the highly complex processes involved in formulation - solution activities, two theoretical constructs have been devised. These two constructs, namely a concept matrix and relationship level graph, were first introduced in section 4.5, Chapter 4. The next section of this chapter defines and illustrates in detail the nature of the ideas involved.

Subsequent sections of this chapter analyse students' attempts at modelling in short to medium duration activities in a selection of experiments taken from Chapter 6. The analyses are based on notes of the author's observations and students' logs.

The implications for teaching and learning mathematical modelling are discussed as well as the implications for assessing students' attempts.

7.2 Concept matrix and relationship level graph

Figures 11 and 12 in Chapter 4 show the general form of a concept matrix and of a relationship level graph.

The purpose of the concept matrix is to show the nature of the features or ideas involved ranging from the initial

thoughts on a problem to the final stages of solution and interpretation. As explained in Chapter 4, the notion of relevance of features is not considered as this can only be determined once a solution is obtained. All key considerations as a model is being developed are entered in the matrix, their position being determined by how specific they appear to be (specificity level) and by their complexity (complexity level). These features or considerations are defined to be those statements, sketches and diagrams that consist of:

Questions
Assumptions
Variables and constants
Relationships between variables and constants

The relationship level graph is designed to show that mathematical solution and formulation are interwoven; additional ideas on the nature of the problem are generated as a mathematical solution is developed. Initial understanding of the problem leads to simple relationships based on knowledge, guessing or both on the background to the problem. These first relationships are defined to be at level 0 (zero). The relationships deduced mathematically from level 0 are defined to be at level 1. Since several relationships each at a variety of levels are often used simultaneously to derive any new relationship, then the level of the latter is defined to be at one more than the highest level of the preceding. Relationships are numbered in the order in which they are generated. Thus, for any relationship numbered n, meaning it is the nth generated, its level is defined to be i where it is derived from relationships of level ℓ and $\ell \leqslant i - 1$; $i \geqslant 1$. For example, referring to Figure 12 in Chapter 4, relationship 15 is derived from relationship 5 at level 2 and relationships 9 and 14 both at level 3; relationship 15 is therefore defined to be at level 4. In this example, $i=4$ and $\ell=2$, 3 respectively. Although relationships are generated in no discernible order, relationship level gives a good indication of how a solution (with formulation) is proceeding; relationship level therefore provides a linear sequence as a guide to goal seeking from a randomly generated set of relationships.

The main link between the concept matrix and relationship level graph is in showing how and where relationships are generated amongst all the other features that are identified. Early, but by no means all, formulation stages indicate that the more global and complex features are identified. Towards the 'solution' of a problem (or sub-problem), features tend to be more specific and less complex; the latter usually implies that features are more readily quantified and hence amenable to mathematical treatment.

The relationship level graph (RLG) has the greater potential for illustrating the structure of a model development rather than the concept matrix (CM). RLG shows how formulation and solution are interwoven and, in particular, shows how model development is achieved by the linked generation of relationships. CM is intended mainly as an aid in classifying features that are identified.

In order to exemplify the characteristics discussed above, an analysis of the author's attempts at tackling the central-heating problem will now be provided. The problem was presented in section 5.8, Chapter 5, although it was in 'polished' form. In order to illustrate as closely and as accurately as possible what happened in the author's first crude attempts (of six hours duration: first 'stab' 2 hours, second 'stab' 4 hours), the list below shows the features considered in the order in which they occurred. All 'blind-alleys' (eg: solution paths dropped at intermediate stages) and 'groping around' (what relationships to use, or derive, to do what and next?) are included.

Central-Heating Problem

Author's initial modelling attempts

Feature list in order of occurrence:

Feature	Order of Occurrence
Thermal capacity of system(house) (T)	A
Heat generated by boiler and radiators	B
Internal temperature of house (θ_i)	C

External temperature (θ_o)	D
Heat loss is involved	E
Cost of heating (per unit volume?)	F
Sub-problem: Cost of maintaining a particular temperature	G
Areas of walls, roof, windows	H
Assume steady-state: heat loss = heat gained	I
Relationship 1	J
H_G = heat generated/unit time	K
Relationship 2	L
Relationship 3	M
Transient effects: heating up	N
Relationship 4	O
Relationship 5	P
Relationship 6	Q
Relationship 7	R
Assumption: one warmth period, one cooling down period in any 24 hours	S
Diagram showing θ_i variations with t (time) (See Figure 30, Chapter 5)	T
Assumption: Rapid response of C-H system	U
Relationship 8	V
Relationship 9	W
Relationship 10	X
Relationship 11	Y
Relationship 12	Z
Relationship 13	AA
Relationship 14	BB
Relationship 15	CC
Relationship 16	DD
Relationship 17	EE
Relationship 18	FF
Relationship 19	GG
Relationship 20	HH
Relationship 21	II
Relationship 22	JJ
Relationship 23	KK

Relationship List

(Numbering refers to relationship numbers above)

Relationship		No
level 0	$H_\ell = \frac{A\kappa}{d}(\theta_i - \theta_0) = K(\theta_i - \theta_0)$ (heat loss)	1
level 0	$H_G - H_\ell = 0$ (steady temperature: heat gained = heat loss)	2
2→3	$\mathcal{L} = £NKC(\theta_i - \theta_o)$ (cost)	3
level 0	$\frac{Td\theta_i}{dt} = H_G - K(\theta_i - \theta_o)$ (heating up)	4
level 0	$T\frac{d\theta_i}{dt} = -K(\theta_i - \theta_o)$ (cooling down)	5
1 & 4→6	$t = \frac{1}{A}\ \ell n\left[\frac{B - A\theta}{B - A\theta_i}\right]$	6
6→7	$\mathcal{L} = H_G tC$	7
1 & 3→8	Heat loss = heat generated (night temperature $\theta_c > \theta_{min}$) $= K(\theta_c - \theta_o)(t_{1b} - t_4)$	8
level 0	Heat generated = $H_G(t_{1b} - t_{1a})$ (temperature allowed to fall to θ_{min})	9
8 & 9→10	Difference in costs = £C(RHS of 8 - RHS of 9)	10
6→11	$t_{1b} - t_{1a} = \frac{1}{A}\ \ell n\left[\frac{B - A\theta_{min}}{B - A\theta_c}\right]$	11
level 0	Heat gained by house = $T(\theta_c - \theta_{min})$	12

5→13 $$t_4 = \frac{1}{A} \ell n \left[\frac{\theta_r - \theta_o}{\theta_c - \theta_o} \right] \qquad 13$$

13→14 $$\theta_i = (\theta_r - \theta_o)\, e^{-At} + \theta_o \qquad 14$$

13→15 $$t_{1a} = \frac{1}{A} \ell n \left[\frac{\theta_r - \theta_o}{\theta_{min} - \theta_o} \right] \qquad 15$$

13→16 $$\theta_c = (\theta_r - \theta_o) e^{-At_4} + \theta_o \qquad 16$$

13→17 $$\theta_{min} = (\theta_r - \theta_o) e^{-At_{1a}} + \theta_o \qquad 17$$

14 & 17→18 $$\theta_i - \theta_{min} = (\theta_r - \theta_o)(e^{-At} - e^{-At_{1a}}) \qquad 18$$

12 & 18→19 Heat gained by house

$$= T(\theta_c - \theta_{min})$$

$$= H_G(t_{1b} - t_{1a}) + (\theta_r - \theta_o)\,[T(e^{-At_{1b}} - e^{-At_{1a}}) + Ke^{-At_{1a}}(t_{1b} - t_{1a})] \qquad 19$$

11 & 17 & 19→20

$$T(\theta_c - (\theta_r - \theta_o) e^{-At_{1a}} - \theta_o) = \text{RHS of } 19 \qquad 20$$

8 & 13 & 20→21

Solve for t_{1a}, t_{1b} 21

20→22 Difference in heating amounts for given θ_c 22

10 & 21 & 22→23

Difference in costs in terms of θ_c 23

A: atomic
I: intermediate
G: global

Specificity level (SL)

Complexity level (CL)	A	I	G
L	(A) (C) (D) (H) (J1) (K) (L2) (M3) (04) (P5) (Q6) (R7) (V8) (W9) (X10) (Y11) (Z12) (AA13) (BB14) (CC15) (DD16) (EE17) (FF18) (GG19) (HH20) (II21) (JJ22) (KK23)		
M		(E) (F) (I) (N) (S) (U)	
H		(G)	

L: low
M: medium
H: high

Figure 34: Concept matrix: Central-heating problem: Author's initial modelling attempts

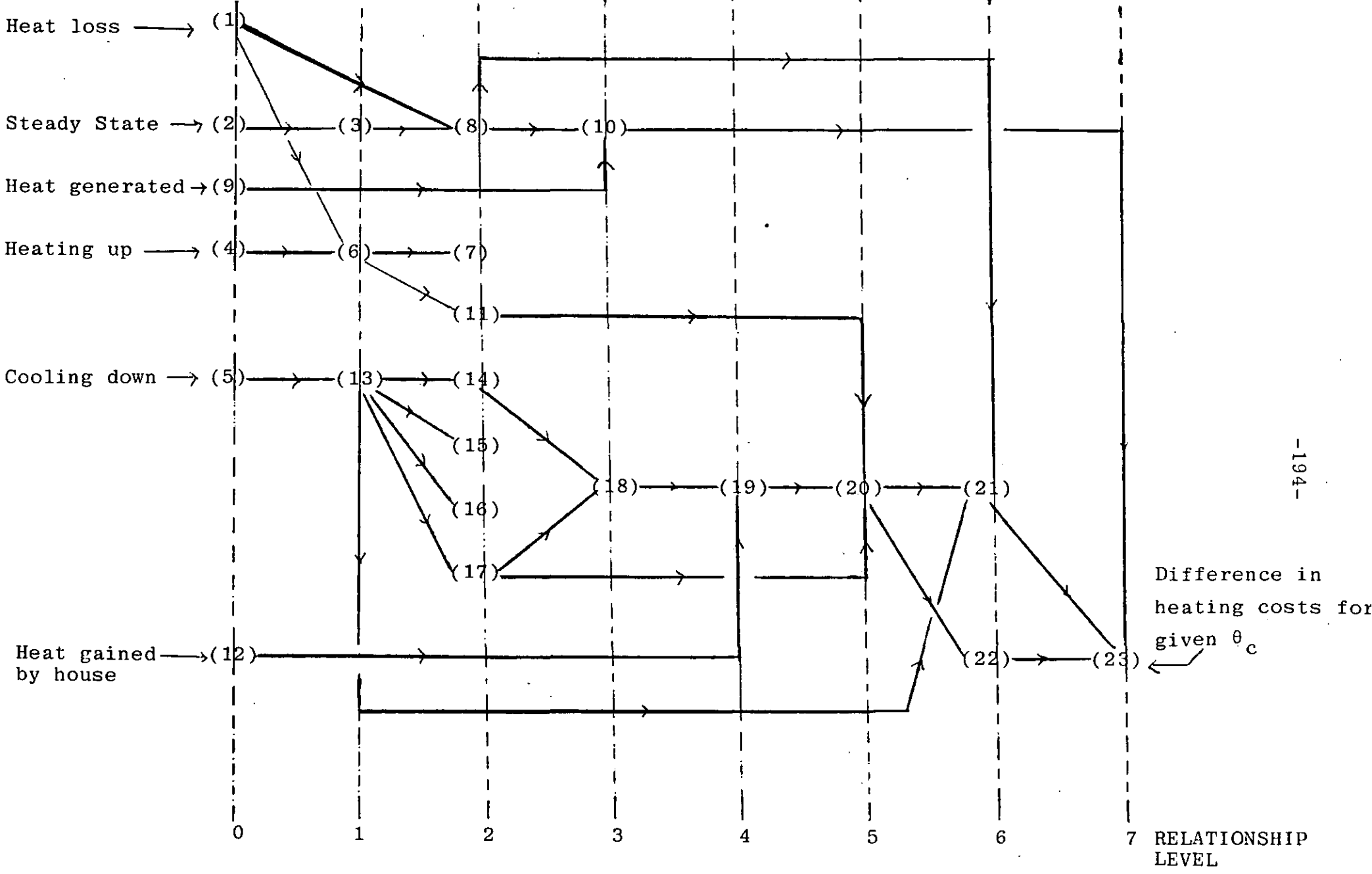


Figure 35: Relationship level graph: Central-heating problem: Author's initial modelling attempts

It should be noted that:

1) Features, including relationships, are listed in the order in which they occurred in the model development.

2) A separate relationship list is provided to indicate for each relationship:

   (i) Its form

   (ii) Other relationships from which it is derived, eg: 8 & 9 → 10 implies that relationships 8 & 9 are used to derive relationship 10.

3) The concept matrix shows each member of the features list. Where letter(s) and a number appear together then the feature is a relationship, eg: (HH20) means that feature HH is relationship 20.

4) The relationship level graph shows all relationships from the list. Note that relationships 1, 2, 9, 4, 5, 12 are each at level 0; these relationships require no mathematical derivation and depend solely on interpretation of the problem statement and associated basic physics. Relationships at level 1 and above are derived mathematically eg: relationship 3 is derived from relationship 2(level 0) and hence relationship 3 is at level 1. Intermediate mathematical detail in deriving a relationship is not shown.

5) An element of subjectivity is inevitably involved in the construction of the concept matrix and the relationship level graph, particularly in the former. However, a number of colleagues have constructed both for this problem and close agreement has been observed in each case.

The points, numbered 1-5 above, are general and refer to the essential characterisitcs of an analysis of any modelling attempt using a concept matrix and relationship level graph, no matter what the original problem. The following are

interpretations of Figures 34 and 35 and thus relate specifically to the author's initial attempts at modelling the central-heating problem. However, several of the interpretations have a wider significance for modelling processes in general and these are examined as they arise.

I Distribution of features in concept matrix

Most of the features, even the ones identified initially, tend to be highly specific to the problem and also tend to be the most easily quantified (at least in principle). Hence the cluster of features in the top left hand corner and the sparsity of features in the other squares of the concept matrix. It is the sparsity in the other squares that is the most noteworthy characteristic, since relationships are defined in general to fit in the top left hand corner.

The M.Sc Math. Ed. group which also attempted this problem (see section 6.2, Chapter 6) had most of their initial features in the bottom right hand corner and middle square of the concept matrix. As reported in Chapter 6, the M.Sc group (inexperienced in modelling, and with only 3 hours at most spent on the problem) made less progress than the author. The latter who has considerable experience both of modelling industrial problems as well as those problems used in teaching in higher education, showed a stronger sense of direction at the outset as to be expected.

II Generation of variables and constants

Variables and constants are largely generated as relationships are formed. An analysis of the features list shows that out of 15 symbols (variables and constants) generated, only 3 (θ_i, θ_o, T) were thought of before relationships were formed. An additional 5 symbols were introduced in level 0 relationships, an additional 3 symbols at level 1, and the final 4 at level 2; the last symbol to be introduced

is θ_{min}, which occurs in relationship 11. Symbols such as A and B which were introduced solely for mathematical convenience are not included.

As to be expected, towards the end of goal seeking (higher relationship levels), symbols of prime importance to the problem are no longer generated.

III Level 0 relationships

The hardest part in getting started with any modelling problem is the formation of the first level 0 relationships. In this case, relationships 1 and 2 provide the starting point. Experience in modelling as well as in the problem class (elementary heat exchange) appear to be important factors which lead to improvement. As the solution progresses, however, additional insights are gained and these 'prompt' the need for further information. Hence the generation of relationships 4, 5, 9 and 12 each at level 0. Mathematics (solution) has helped in the intuitive (level 0) understanding of the posed problem.

IV Formation of relationships at levels 1, 2, ...

Relationships are often generated by working simultaneously at a variety of levels, eg: relationship 19 (level 4) from 18 (level 3) and 12 (level 0). Note that not all relationships generated at a given level are subsequently used, eg: relationships 7, 15, 16 (all at level 2) make no contribution to the 'solution' (relationship 23), and therefore are redundant.

V Sub-problem identification

The relationship level graph is partitioned into two distinct regions as far as relationship 19(level 4); the upper region starts with relationships 1, 2, 9, 4 and the lower region starts with relationships 5 and 12, all at level 0. Not until relationship 20 (level 5) is

reached is a link formed between the two regions. The upper concentrates mainly on heating up and the lower mainly on cooling down of the house. Each region therefore represents the development of a sub-problem where the two sub-problems are combined at relationship 20. The author was totally unaware whilst modelling that these two sub-problems were in fact being tackled; it felt like working on one problem only.

With some problems however, it is not only possible to identify sub-problems at the outset, but it is quite clear that the problem can be broken down into very distinct parts. For example, with the record player problem (see section 5.5, Chapter 5 and an analysis of students' attempts of this in a later section of this chapter) it is clear <u>before any mathematics is attempted</u> that the following are key sub-problems:

(a) Minimisation of sound distortion from a purely geometrical approach

(b) Minimisation of sound distortion from a signal analysis approach

A further break-down occurs by considering straight and off-set pick-up arms separately.

In the case of the central-heating problem it was decided to 'polish up' and produce a simpler solution. The two chief guides used in the production of the 'polished' solution were obtained from IV and V above, namely:

IV Avoid redundancy

V Concentrate on the sub-problems of cooling down and heating up at the outset

The 'solution', namely getting an expression for the difference in heating costs in terms of θ_c (the 'tick-over' warmer night temperature $\theta_r > \theta_c > \theta_{min}$) is provided in section 5.8, Chapter 5.

Only ten relationships are generated in the new solution, where the starting point consists of:

Cooling down, relationship 5 level 0

Heating up, relationship 4 level 0

from the original attempt (see Figure 35). The new relationship list and level graph are shown in Appendix 1A.

7.3 Analysis of case studies

A selection of students' attempts at modelling various problems is now analysed in terms of the concept matrix and relationship level graph. Since the experiments involved, as reported in Chapter 6, are of short to medium duration and also because the students referred to have little or no modelling experience, the relationship level graphs are considerably less developed than the one illustrated in Figure 35 in section 7.2. For the same reasons, fewer features appear in each concept matrix. For comparison purposes, the author's 'polished' modelling approaches are analysed where appropriate and the details are provided in appendices.

7.3.1 Modelling the heating of a baby's milk bottle

The following is an analysis of the M.Sc Math. Ed. group 1 attempts at tackling this problem. Observations and reports of the students (teachers) taking part are to be found in sections 6.2 and 6.3 of Chapter 6.

Baby's milk bottle problem

M.Sc Math. Ed. group 1 attempts

Feature list in order of occurrence

Feature	Order of Occurrence
What temperature must milk reach ($35^{o}C$)	A
Temperature of milk from 'fridge ($10^{o}C$)	B
Better to overheat milk and let it cool - especially as baby may take a while to drink it?	C
Rate of heating: Heat water extremely quickly or bring heat up gradually?	D
Material bottle is made of	E
Copper based pan or otherwise	F
No heat convection in milk or water	G
Relationship 1	H
Relationship 2	I
Relationship 3	J
Heat lost to surroundings is negligible	K
Relationship 4	L
Relationship 5	M
Relationship 6	N
Relationship 7	O

Relationship 8	P
Relationship 9	Q
Relationship 10	R
Relationship 11	S
Bottle will slow down heat transfer from water to milk	T
Heat from system will be lost to surroundings	U
Heat input in fact is not constant as ring starts from room temperature and builds up to constant temperature	V
Air above milk in bottle?	W

Relationship list

(Numbering refers to relationship numbers above)

Relationship		No.
level 0	$m_w \times 4200(100-\theta_f) = m_B \times c_B + (\theta_f-10) + m_m \times c_m (\theta_f - 10)$	1
	(Heat lost by water = Heat gained by milk and bottle)	
level 0	$\frac{dQ}{dt} = - mc \frac{d\theta}{dt}$	2
2→3	$\ell n (\theta - \theta_R) = - \frac{kst}{mc}$	3
level 0	$t \propto m \times c \times (\theta_f - 10)$	4
level 0	Mass = volume × density	5
level 0	Heat capacity = mass × specific heat	6
level 0	Total heat capacity of system = Σ capacities	7
5 & 6 & 7 → 8	Total heat capacity $= [c_S + c_B + 4200(H_w(A_s-a_B)+h_m a_B)]$	8
8→9	Heat gained by system in reaching 35°C from 10°C = 25 × right hand side of 8	9
level 0	Heat supplied = Heat gained = wt	10
9 & 10 → 11	$t = \frac{\text{right hand side of 9}}{w}$ (time to heat system)	11

Specificity level

	A	I	G
L	(A)(B)(H1)(I2)(J3)(L4) (P8)(Q9)(S11)	(M5)(N6)(07)(R10)	
M	(V)	(E)(F)(G)(K)(U)(W)	
H		(C)(D)(T)	

Complexity Level

Figure 36: Concept matrix:
Heating of a baby's milk bottle:
M.Sc Math. Ed. group 1

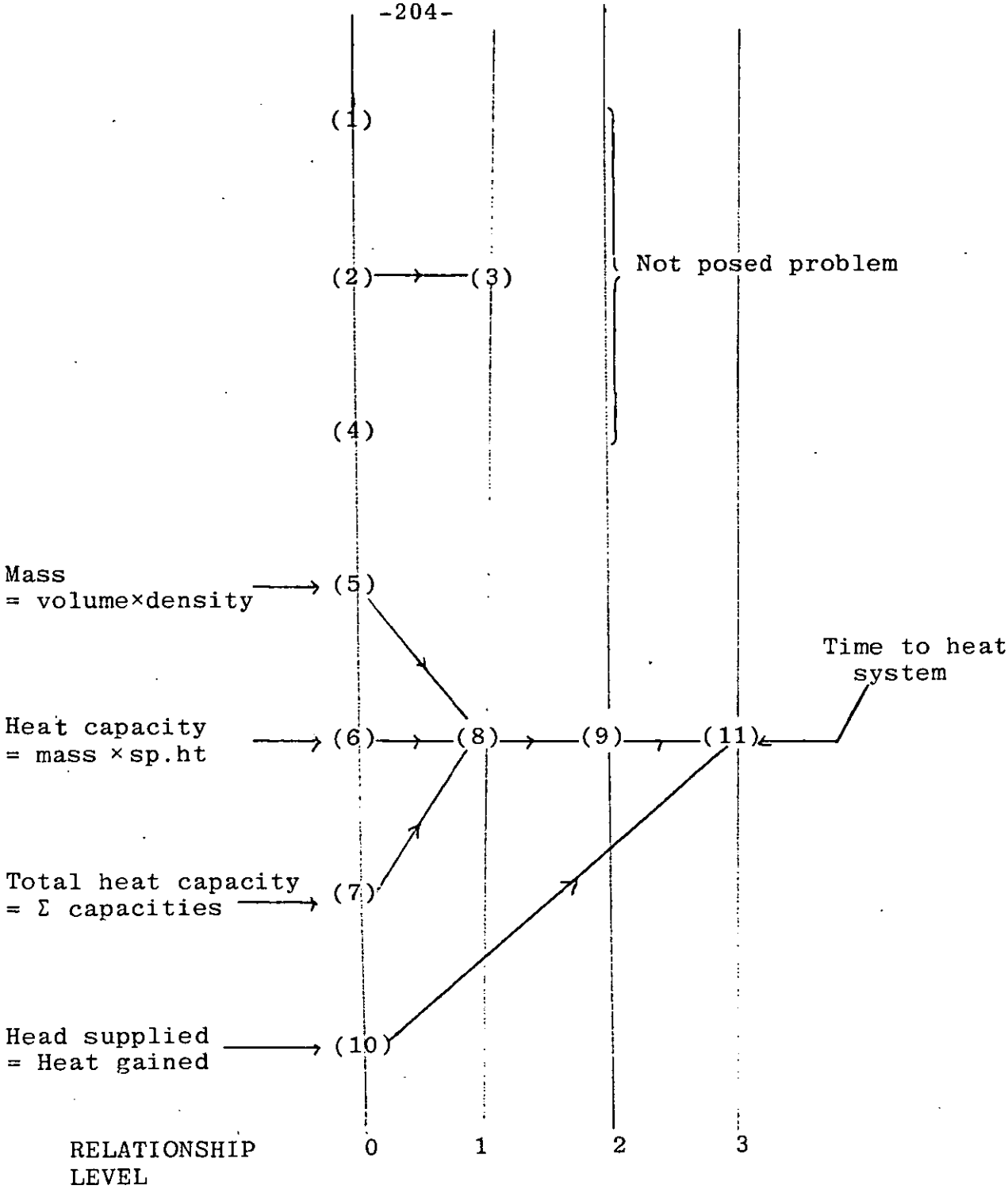


Figure 37: Relationship level graph:
Heating of a baby's milk bottle:
M.Sc Math. Ed. group 1

For comparison purposes in this case an analysis of M.Sc Math Ed. group 4 is shown in Appendix 1B.

In each case the concept matrices, shown in Figure 36 for group 1 and in appendix 1B for group 4, indicate an early concentration on specifics. However, as reported in sections 6.2 and 6.3, Chapter 6, the teachers' initial reactions were to look at general features and even in some instances to want to change the problem into quite a different one. Groups logs omit such features in spite of the author asking for their inclusion. In the next sub-section, 7.3.2 on the record player problem, general discussion with the author took place before groups worked on their own (B.Sc 2 Applied Physics), and a concept matrix showing the features that arose in such a discussion is presented.

Although relationships are generally placed in the (L, A) position (top left hand corner of a concept matrix), it should be noted that for group 1, relationships 5, 6, 7 and 10 have been placed in the (L, I) position; for group 4 only one relationship (No. 1) is in the (L, I) position. The reason for such placements is that the relationships concerned, eg: relationship 5, group 1:

$$\text{Mass} = \text{Volume} \times \text{density}$$

are of general applicability, and thus do not relate solely ('specifically') to the problem - hence they are I (intermediate); however, the relationships involve features or concepts which are easily quantified (eg: mass = 0.2 kg, volume = 200 ml = 0.0002 m^3, density = 1000 kg m^{-3}) and so they are of L (low) complexity level. It is interesting to note that group 1 generated more relationships in the (L, I) position than group 4; the latter may consciously or sub-consciously have done the same and <u>then</u> <u>wrote</u> the relationship down in a form most specific to the problem.

Referring to the relationship level graphs, Figure 37 for group 1 and appendix 1B for group 4, the following points emerge:

1 Ignoring heat loss and treating the milk bottle as a perfect conductor, group 1 reached relationship 11, level 3; a comparable stage of development is reached by group 4 in relationship 4 (level 1) and 5 (level 0). Group 4 continue to model heat losses as far as relationship 12 (level 2). Group 4 has made more progress with the problem than group 1 and has done so with less intermediate mathematical working (as measured by relationship level).

2 Apart from considering a different (not posed) problem (water heated then bottle immersed), group 1 have concentrated on one sub-problem. Group 4 have worked on two sub-problems, with linkage at relationship 7 (level 1); their log does not show awareness of sub-problem identification at the outset.

3 Some relationships are different forms of the same key idea, but in order to distinguish the forms each is given a different relationship number. For example, for group 4, relationships 2 and 3 (both at level 0) express the same idea: 'rate of heat input = mass × specific heat × rate of temperature rise.' Relationship 2 refers to milk only in the saucepan, whereas relationship 3 refers to water and milk in the saucepan - hence the distinction.

7.3.2 Minimisation of sound distortion in a record player

The following analysis refers to the B.Sc 2 Applied Physics groups of students tackling this problem. Reports on the observations of the students taking part are to be found in sections 6.2 and 6.5 of Chapter 6.

As explained in Chapter 6, an hour was spent with the class on 'clarifying' discussion before students were split into groups. The purpose of this preliminary discussion was to ensure that students were quite clear on what was expected of them over a two-week period of modelling the problem. The discussion took the form of the students asking the author questions, often quite general in nature, and direct answers were only provided if technical or design matters were queried. No direct advice was given on the physics of the problem. A features list for this part of the activity is now provided, together with a concept matrix (Figure 38). The broader nature of the features identified is shown and a selection of features is used to show how lecturer-student interactions took place. The list and concept matrix are based on a transcript of an audio recording made.

Minimisation of sound distortion in a Record Player

B.Sc.2 Appd Physics: Class discussion

Feature list in order of occurrence

Feature	Order of occurrence
Are we considering the usual polyvinyl discs or laser discs?	A
Disc in vertical plane?	B
Quality of apparatus?	C
Geometry of recording groove?	D
Information from side of groove?	E
Arm is pivoted, stylus movement on circular arc	F
Position of pivot of arm important?	G
Nature of recorded signal?	H
Maximum signal amplitude?	I
Consider different types of cartridge?	J
Do we need to consider the method of translating the initial mechanical signal?	K
Is flutter (bouncing up and down) involved?	L
Is sound distortion noise or not true representation of sound?	M
Signal on disc is true	N
Transverse force on stylus negligible?	O
Keep stylus rigid with arm	P
Representation of signal - single line or with width (double line)	Q

Specificity level

Complexity Level	A	I	G
L	(I)	(F) (G)	(B)
M		(D) (O) (P)	(A)
H	(N)	(E)(H)(K)(L)(M) (Q)	(C) (J)

Figure 38: Concept matrix: Minimisation of sound distortion in a record player: B.Sc 2 Appd. Physics, preliminary class discussion.

In order to give an indication of the way in which the features arose in the preliminary discussion, the following examples taken from the transcript are provided.

Feature C

Student A "What quality apparatus are you considering? Economic considerations are important in the design."

Lecturer: "Yes, they are important."

Student B "I think that costings only come into the last stages of design."

Lecturer: "I would have thought that the economics of the design would have come in at an early stage. However, I would have thought that my problem statement did not emphasize economic considerations (see section 5.5, Chapter 5). So you may choose to design an expensive apparatus, or you may decide that the problem is pretty tough and not worry about money considerations. You may consider that any analysis or investigation applies to both cheap and expensive equipment."

Feature D

Student C "Are we to determine the geometry of the recording groove?"

Lecturer: "The pick-up arm and stylus are picking up sideways movements, not vertical movements." (Sketch drawn of a typical signal on the blackboard).

Feature G

Student B "Is it a major assumption, the position of the pivot of the arm?"

Lecturer: "Well, I will interpret that as a question to me as a design engineer and say that the position of the pivot is due to marketing considerations. (Brief discussion clarifying position of pivot - not much space left on deck for pivot other than corner.) However, you may decide that there are such overriding advantages putting the pivot somewhere else - it is up to you, to pursue this if you wish with supporting mathematics and physics."

Feature 0

Student D "Is the transverse force of the stylus in the groove negligible?"

Lecturer: "I think our discussion so far should guide you. Imagine the signal as being a little trough dug into the surface of the record, which enables the diamond stylus to sit in and be forced to wobble about as the groove passes beneath it. Once you see that you may realise that the transverse force may not be of much interest to you."

As far as possible the lecturer (author) has avoided imposing any solution paths on the students and has also avoided giving hints away. The main purpose of the experiment was to leave the students to work by themselves in groups.

The concept matrix in Figure 38 shows that most early features identified by the students tend to be more general than those reported in their logs; the latter, as with the M.Sc Math. Ed. groups, tend to concentrate more on specific ideas that can be more readily symbolised and/or quantified. It is interesting to note that in spite of the problem statement and its

associated sketch diagram (section 5.5, Chapter 5) that features A and B should arise; perhaps these G (global) features were considered as an 'opening shot' just to get one's concentration started.

The analysis of group 4 is now considered. For comparison purposes, group 1 is also analysed and details are to be found in Appendix 1C. The author's 'polished' approach is detailed in Appendix 1D. For the sake of brevity, features lists and their subsets (relationship lists) are omitted, but key characteristics are clearly identified.

Specificity level

Complexity Level			
L	(C) ((H1)) (J3)(L5) ((M6)) (N7)(08)(P9) (Q10)(R11)	(A)(B)(E)(G) ((K4))	
M	(I2)	(D) (F)	
H			

(( )) denotes level 0 relationships

Figure 39: Concept matrix: Minimisation of sound distortion in a record player: B.Sc 2 Appd. Physics, Group 4.

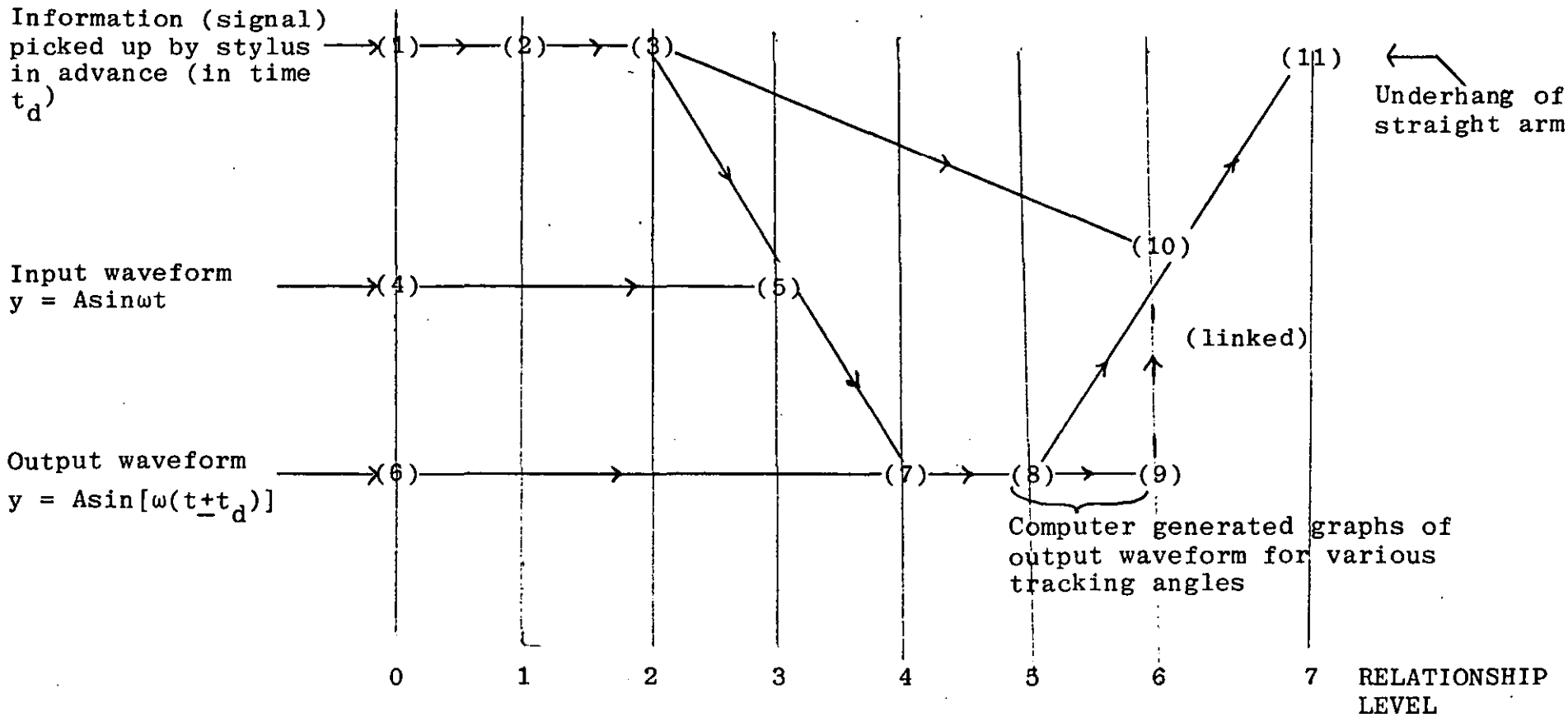


Figure 40: Relationship level graph:
Minimisation of sound distortion in a record player:
B.Sc 2 Appd. Physics, group 4

As is quite common then, the features recorded in the students' group logs tend to concentrate on more specific and more readily symbolised/quantified ideas. Thus, the concept matrices shown in Figure 39 (group 4) and in appendix 1C (group 1) have features (apart from relationships) clustering towards the top left hand corner in each case. However, group 1 has identified some more global and less easily quantified ideas at an early stage, viz:

Feature A	How is signal produced?
Feature C	Type of cartridge used?
Feature D	Balance mechanism of arm?

A little later on, but just before the first relationships (level 0) are generated, there is feature H:

Feature H	Centripetal forces in bringing arm to centre of disc?

The latter could have led to a quite different problem involving sources of sound distortion not covered in the problem statement. The author dissuaded the group (1) from spending much time on these aspects.

Referring to the relationship level graphs, Figure 40 for group 4, appendix 1C for group 1, and appendix 1D for the author's 'polished' approach, the following points can be made:

1 All three graphs show how few initial relationships (level 0 type) are used to tackle the problem, this is so even in the case of the author's more extensive treatment (although 'polishing' does tend to eliminate some relationships - see section 7.2). This is one of the characteristics of a conceptually harder problem where it is important to start with relationships in a form amenable to quite rapid mathematical development.

2 Closely coupled with 1 above is the observation that quite a high relationship level is reached before a definite deduction or 'solution' is obtained. This implies that quite a few mathematical deductive steps (as measured by relationship level) have to be carried out in the formulation-solution process for this type of problem.

3 Apart from the author, where three distinct sub-problems were recognised at the outset (with a single link between them: relationships 3, 16, 23), groups 1 and 4 have concentrated on one sub-problem:

group 1 geometrical approach
group 4 signal analysis

Both groups have also concentrated on the straight arm pick-up.

Group 1, relationships 5-9 are comparable in development to the author's relationships 1-6. Group 1 has however made some (uncorrected) algebraic slips.

Although the details are not provided, Polymodel 3 group has a similar development to B.Sc Appd. Physics, group 1. However, in the case of Polymodel 3, the participants had a much clearer idea of the effects of 'underhang' (improvement) and 'overhang' (worsening) in the straight arm case as the sketch graph in Figure 33, Chapter 6, shows.

4 In the case of group 1, the relationship level graph has two distinct regions: the upper showing scale-diagram and graph development, the lower showing the cosine rule application. The group log shows that the students were attempting to confirm some measurements from their diagrams by mathematical deductive means

(cosine rule and its interpretation). Apart from algebraic slips, the students were able to get close agreement between their measured and calculated values.

The latter, point 4, is an illustration of intermediate validation that can be carried out in model development. It has the virtue of providing more confidence in what is a quite difficult analytical modelling exercise. However, the student log shows that a better overall appreciation of the problem is gained from the deductive part (cosine rule) rather that relying on the scaled diagram measurements (descriptive modelling).

7.3.3 Speed-wobble in motorcycles

The analysis refers to M.Sc Math. Ed. group 4 tackling this problem. Reports on the observations of students (teachers) taking part are to be found in sections 6.2 and 6.4 of Chapter 6.

As pointed out in Chapter 6, the M.Sc groups involved are the same as those who tackled the baby's milk bottle problem. All five groups found the castor problem very difficult, and only group 4 made any progress beyond considering SHM approaches once a hint had been given (on the motion of the point of contact of the castor). It is for this reason that an analysis of group 4 only is provided.

The problem is included in this chapter in order to show how only a few features and a low final relationship level are needed to gain very good insights into a well-posed and well-structured real situation. Getting started with the problem is difficult, but once started only relatively little mathematics is subsequently required to find 'solutions'. The author's 'polished' approach is represented by a relationship level graph, presented in Appendix 1E.

Speed-wobble in motorcycles

M.Sc Math. Ed. group 4 attempts

Feature list in order of occurrence

Feature	Order of Occurrence
Shape of castor	A
Level and material of ground	B
Material of castor	C
Direction of forces	D
Velocity of castor	E
Amount (area) of castor in contact with the ground	F
Weight immediately above point of contact	G
Pushing force returns castor to path's direction	H
Moment of inertia	I
Inertial force causing wobble	J
Friction	K
Relationship 1	L
Relationship 2	M
Relationship 3	N
Relationship 4	O
Relationship 5	P

Relationship list

(Numbering refers to relationship numbers above)

Relationship		No.
level 0	$F \sin \theta . x = -I\ddot{\theta}$ (x is trail, F is friction - direction not clear, I is moment of inertia (axis not specified), θ is angular displacement of castor from forward motion of steering axis)	1
1→2	$\theta = a \cos \sqrt{\frac{Fxt}{I}}$	2
level 0	Hint: Diagrams showing motion of point of contact of castor with ground	3
3→4	$F.L = I\ddot{\theta}$ (L is trail)	4
4→5	$\int F.L\,dt = \ldots$	5

Specificity level

Complexity Level	A	I	G
L	(E)(G)(I)(K)(L1)(M2)(N3)(04)(P5)	(H)(J)	
M		(A)(B)(D)(F)	
H		(C)	

Figure 41: Concept matrix: Speed-wobble in motorcycles: M.Sc Math. Ed. group 4

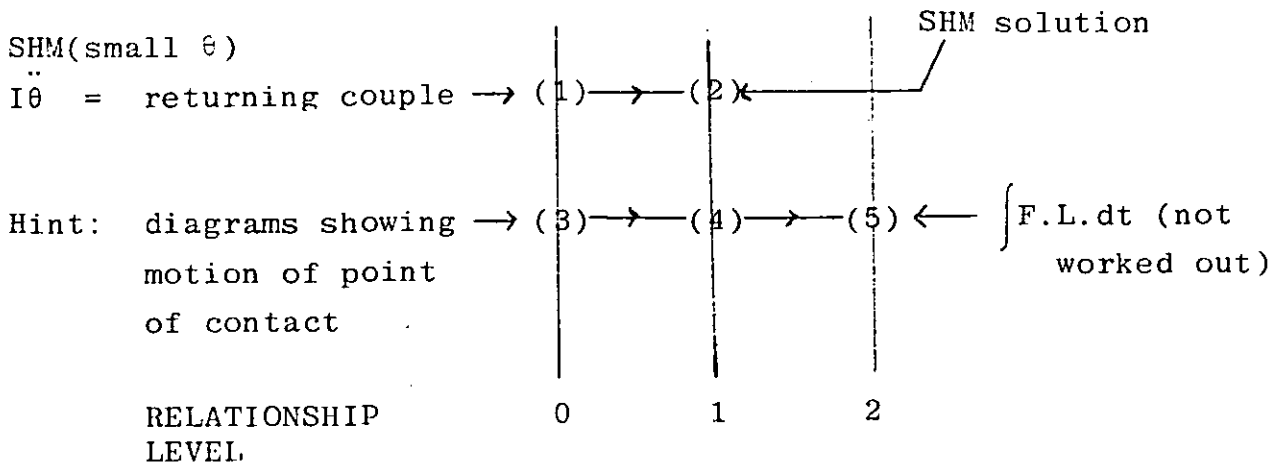


Figure 42: Relationship level graph: Speed-wobble in motorcycles: M.Sc Math. Ed. group 4

The concept matrix in Figure 41 shows that initial features are less specific and less easily quantified, apart from feature E. Features G, I, K even though they are most specific and are easily symbolised/quantified, are presented in the group 4 log as verbal statements (often only one word long) and are not identified as variables or constants in symbolic form. Not until relationships are generated are variables introduced; this is a slightly extreme, but nevertheless quite common observation on variable/constant generation. In other words, variable/constant generation cannot be divorced from relationship generation; this is in contradistinction to the approach of Treilibs (1979) (see section 3.3, Chapter 3).

Referring to the relationship level graphs in Figure 42 for M.Sc Math. Ed. group 4, and in appendix 1E for the author's 'polished' approach, the following points emerge:

1 Both graphs show only two relationships (level 0 type) that are used in tackling the problem. This characteristic, namely that very few starting relationships are used in the modelling, is shared by the record player approaches as typical of a class of conceptually harder problems. Note that relationship 3 in Figure 42 has been included to show the significant effect of the lecturer's hint; it could be strictly argued that this relationship should have been included at level 0 in appendix 1E, but it would have added little to the latter in view of the more extensive structure portrayed.

2 Unlike the record player problem, a low final relationship level produces significant results. Apart from the numerical solution which would be required for predicting a range of numerical values for θ (angular displacement of plane of castor wheel) and for $\dot{\theta}$ (angular velocity), for validation purposes, very little deductive mathematics is used.

3 M.Sc Math. Ed. group 4's relationships 3 (hint), 4, and 5 are comparable in development with the author's relationships 1, 3, and 4. Writing $F = \mu R$ and carrying out the integration, group 4's relationship 5 would have led to $I\dot{\theta} = \mu RL.t$ + constant; a further integration would have led to a result for $\ddot{\theta}$. Apart from the sign of F (group 4 now have the correct direction), these results are comparable with the author's relationships 3 and 4. It should be noted that the author's relationships 5, 6, and 7 are modifications to 1, 3 and 4 for the purposes of generating a numerical solution: they do not therefore contribute to the formulation of the problem.

4 The author's 'polished' solution shows two sub-problems: exponential decay, and oscillations. The author did not recognise the two sub-problems at the outset, having initially concentrated on oscillations only - the possibility of the oscillations terminating and subsequent motion decaying exponentially was arrived at after having carried out some mathematics. The exponential decay implies that the point of contact cannot slide in a direction perpendicular to the plane of the castor; a modeller with better physical intuition might have realised this at the outset.

Because of the tight structure and fewer possibilities for modelling approaches, this problem is better placed before students with more experience of this type of study. As pointed out in Chapter 6, a graded approach of setting lead-up problems which encouraged students to model the motion of the point of contact of the castor with the ground would have made the approach easier.

7.3.4 Evacuation of a school

The analysis, with only very slight modifications, is based on the work of Wilson (1983). Reports on the interactive experiment with sixth formers are to be found in sections 6.2 and 6.6. of Chapter 6.

Wilson was asked to construct a concept matrix and relationship level graph of his students' responses in order to test:

(a) His ability to carry out such constructions
(b) The usefulness of such constructions for analysing an organisational problem (although still analytical and deterministic).

It certainly seems that Wilson had no difficulty with either construction, although he mentions the inevitable element of subjectivity involved. He even entered features (other than relationships) on an extended relationship level graph, although this is a much more subjective exercise and is not pursued here.

The author also constructed both the concept matrix and relationship level graph based on features identified in the transcript of an audio recording made by Wilson. Both representations are in very close agreement with those of Wilson.

For the sake of brevity, only Wilson's relationship list and graph (the latter with very minor corrections) are provided in this section, although some key characteristics of the concept matrix are also identified.

Evacuation of a school (Wilson, 1983)

Lower sixth form mathematics students

Relationship list (See section 6.6, Chapter 6)

Relationship		No.
level 0	$T_1 = \frac{4}{5}N$ (Time to evacuate one class)	1
level 0	$T_2 = \frac{d_1}{s_1}$ (Time for any student to walk distance d_1 at speed s_1)	2
1 & 2→3	$T = T_1 + T_2 = \frac{4}{5}N + \frac{d}{s}$ (addition of times)	3
level 0	$T_3 = \frac{d_2}{s_2}$ (Relating stairs to distance and speed)	4
3 & 4→5	$T = \frac{4}{5}N + \frac{d_1}{s_1} + \frac{d_2}{s_2}$	5
1 & 2→6	Delay time (for second class) $= \frac{4}{5}N - \frac{d_1}{s}$	6
6→7	Delay time (for second class in position x) $= \frac{4}{5}N - (x - 1)\frac{d_1}{s_1}$	7

5 & 7→8 $\quad T = \frac{2}{5}N + (x - 1)D$ 8

(Time for last person to exit second class in position x, double-file)

8→9 $\quad T = \frac{2}{5}N + (x - 1)D + \frac{D_\ell}{s}$ 9

($\frac{D_\ell}{s}$ = time for last pair to walk to single exit)

6→10 Delay time (for second class, double-file) 10

$= \frac{2}{5}N - \frac{d_1}{s}$

10→11 Delay time for nth class 11

$$= \sum_{i=1}^{n-1} \left(\frac{2}{5}N_i - \frac{d_i}{s}\right)$$

9 & 11→12 $\quad T = \frac{D_\ell}{s} + \sum_{i=1}^{n} \frac{2}{5} N_i - \sum_{i=1}^{n-1} \frac{d_i}{s}$ 12

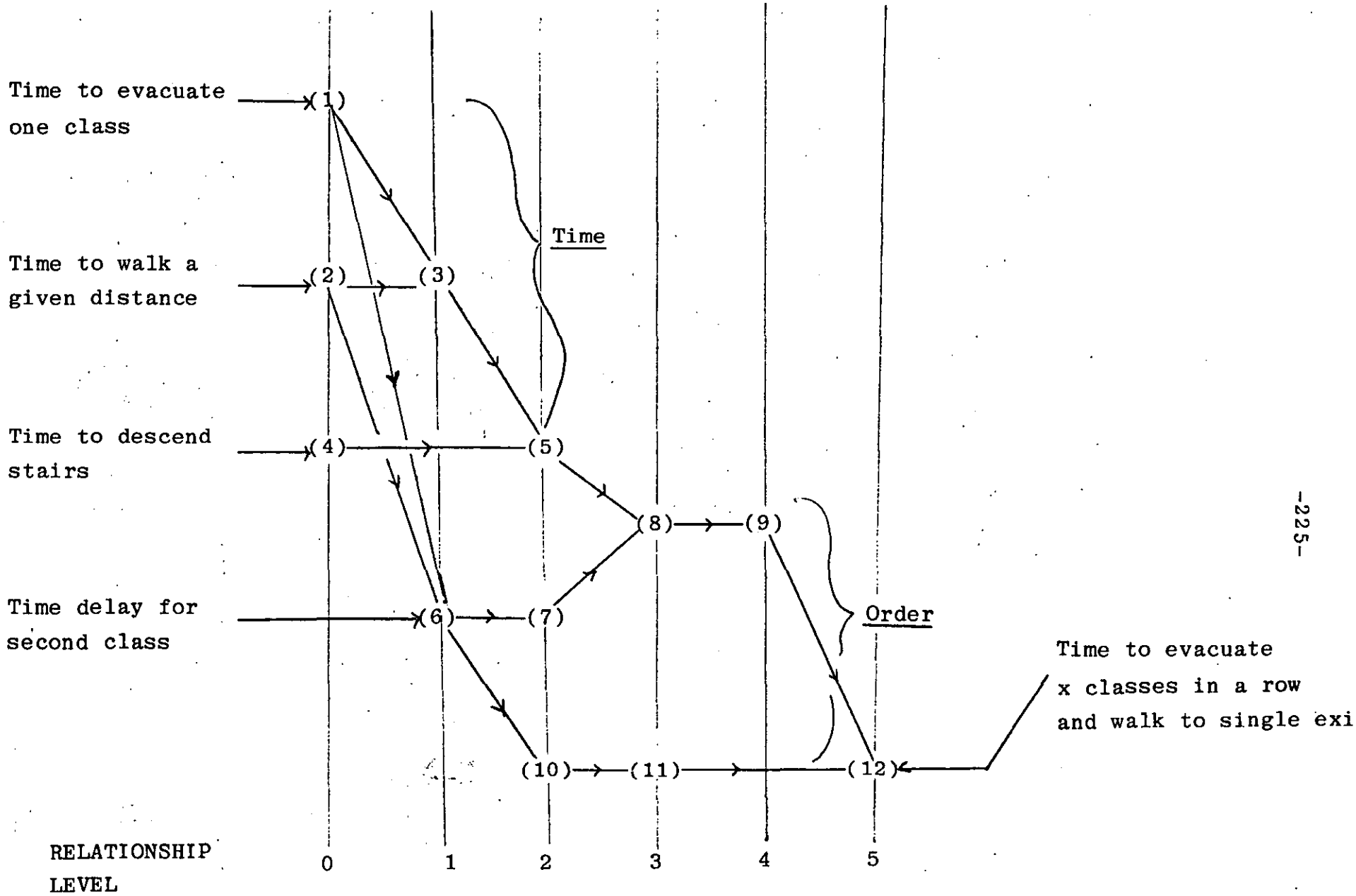


Figure 43: Relationship level graph: Evacuation of a school: Lower sixth form students (Wilson, 1983)

The concept matrix, Wilson (1983), shows that most features are concentrated in the (L,A) and(M,A) positions; that is, most are in the most easily equantified/symbolised and highly specific position: (L,A), or are in the less easily quanitfied/ symbolised but still highly specific position: (M,A). The author would have put some of the latter into the (L,I)position. Because Wilson entered the features from his transcript, rather than asking students to keep a log and then abstracting details from this, more less well-defined and global features will have been recorded. Even with the sixth formers, variables/constants were not identified until relationships were generated: for example, T and N in relationship 1.

Referring to the relationship level graph shown in Figure 43, the following key points emerge:

1 In view of the importance of relationship 6, referring to the ordering concept of a time-delay to avoid 'bottle-necks', it might be better placed at level 0. It tends not to depend or be directly derived from relationships 1 and 2.

2 Two sub-problems have been developed:

relationships 1-5: Time (to evacuate and travel corridor)

relationships 6-12: Order (avoidance of bottle-necks or orderly flow)

There are two linkage points at:

relationship 6 (although this is weak, see 1 above)

relationship 8

Neither Wilson nor the students had recognised these aspects (sub-problems) at the outset. The nature of the linkage also illustrates that sub-problems are not necessarily hierarchically developed, ie: one following another.

3 As the relationship list shows, smaller steps have been taken by the sixth formers in the development of the model than would be expected by undergraduates. Consequently, rather more relationships have been generated to reach the stage of time to evacuate. However, the relationship level graph has been useful in portraying the formulation-solution strategy adopted by the students. Some frequently occurring characteristics in modelling, eg: simultaneity of working, are illustrated.

7.4 Summary and Conclusions

Two theoretical constructs, namely a concept matrix and a relationship level graph, have been devised and used in the analysis of formulation and solution processes in a range of problems. The analyses in the preceding sections have illustrated the complex nature of the processes involved and have shown that formulation and solution are intimately interwoven. The relationship level graph, in particular, has shown that much of the modelling process is non-linear in nature and that several activities are often carried out by working at a variety of stages simultaneously. The relationship level graph has provided the most powerful tool in the analysis by illustrating the dynamic nature of modelling through relationship generation. The concept matrix has been a useful aid in classifying features in model development. The emphasis throughout has been on students who are inexperienced in modelling and who have in general had only a short time in which to tackle the problems involved.

The following are the main points that have emerged from the previous sections:

1 Distribution of features

Relevance of features at any stage is not considered since this is only known when a 'solution' is obtained. In other words, from the modeller's point of view, relevance is determined only in an *a posteriori* sense rather than *a priori*.

There is no discernible order in which features are recognised although there is a general movement from the bottom right hand corner of the concept matrix in early stages to the top left hand corner in the later stages (onset of solution). This general movement is to be expected, since a mathematical solution will generally require more specific (A) as well as more easily quantified/symbolised concepts (L) to operate on.

2 Basic relationships are often generated as solution proceeds

Level 0 relationships (those basic relationships which are not derived mathematically)are needed before any mathematics can be carried out. However, in order to reach any significant solution stage, further level 0 relationships are often required. The mathematical solution itself helps with further understanding and hence formulation of the problem by prompting such relationships.

3 Relationships can occur in various forms

Relationships can occur in various forms throughout their generation:

(L,I)type — General: that is they are applicable to a wide range of situations and not just to the problem in hand.

(L,A)type — Specific: that is those relationships which are written in a form directly related to the problem. Minor variations of the same form occur as the solution develops.

4 Relationship level as goal seeking

As with features generally, relationships often occur in no discernible order. However, a measure of the general progress made in finding a solution is provided by relationship level.

Those students who have a strong sense of direction and make good progress, reach a certain solution stage at a lower relationship level. However, this can only be judged by comparing a relationship level graph with another where the sub-problems identified are roughly the same. For example, a judgement can be made by comparing groups of the same class and/or with a lecturer's approach.

5 Most variables and constants are generated with relationships

Very few variables and constants are identified, at least in symbolic form, before the first relationships are formed. As mathematical deductions are made in the generation of relationships, so variables and constants are more naturally introduced.

6 Sub-problem identification

It is difficult to find a general rule regarding the recognition of sub-problems. Sometimes sub-problems are identified at the outset, ie: before any relationships are generated; this may be referred to as *a priori* recognition. On other occasions, sub-problems are only recognised by partitions formed in the relationship level graph; this may be referred to as *a posteriori* recognition. In the latter case, sub-problems are connected by numerous linkages and are certainly not developed hierarchically or 'end-on'.

'Polished' solutions, for example what one normally expects students to produce in a written report for assessment, may be produced by avoiding redundancy

(relationships not used in original 'crude' approach) and by presenting sub-problems as a priori (in either case).

There are certain disadvantages or defects with the analysis of formulation-solution processes that has been carried out. The creative leap that is required in the formation of the first level 0 relationships to get the solution started has not been investigated. Clearly this is a very difficult matter and all that can be said for the present is that students improve, as with modelling expertise generally, with more practice; it also seems very important for students to gain practice by modelling a particular class of problems where common features arise. The strength or importance of relationships, apart from the basic or level 0 type, has also not been investigated. Deeper insights into the direction or main thrust of formulation and solution would no doubt accrue if such strengths could be defined.

In spite of the defects of the analysis, however, there are some important implications for teaching, learning and assessment in mathematical modelling. The work of this chapter supports the choice of learning heuristics that is presented in section 6.7, Chapter 6, in particular:

Establish a clear statement of objectives

See 1 and 6 of this section. Encourage students to keep a log of all rough work done and to include initial 'vague' thinking; from this initial work, it is easier to get some reasonable objectives on how far to go, ie: what type of solution or solutions are being sought. Do not insist on initial partitioning of problem, ie: identificatio of sub-problems; the partitioning might well evolve naturally at a later stage (of the formulation-solution process).

Don't write a vast list of features

With experience, students appreciate the virtue of this heuristic (see section 6.7, Chapter 6). 1, 2 and 5 of this section show that features are identified as the solution is developed.

Simplify

Start with the simplest ideas to get level 0 relationships (crudest assumptions made).

Get started with mathematics as soon as possible

Don't try and discover all the basic (starting) relationships at the outset. Proceeding with a solution will prompt the need for additional information (level 0 relationships). See 1, 2 and 5 of this section.

Carry out some mathematics on initial relationships

See 1, 2, and 5 of this section.

With regard to assessment, the work of this chapter, together with Chapter 6, would indicate that more time would have to be granted to students in order to carry out more modelling of the problem (furtherance of solution and some validation). Additional time would also have to be provided for the writing of reports. Only one class reported on in this chapter (and Chapter 6), namely B.Sc 2 Applied Physics on the record player problem, was assessed; additional time of one week was provided for the students in which to write up their reports (including their log). This assessment together with the assessment of extended modelling course-works of the M.Sc Math. Ed. groups is discussed in the next chapter. The work of Chapters 6 and 7 which illustrate some key processes in modelling provide a guide to expectation of student performance working under various conditions; a fairer judgement of students' efforts at modelling should now be possible.

CHAPTER 8

ASSESSMENT METHODS

8.1 Introduction

This chapter deals briefly with the implications of the last chapter on Formulation-Solution Processes for assessment as well as some additional considerations based on the author's experience of assessing courses in mathematical modelling at the Polytechnic of the South Bank.

As pointed out in section 3.4, Chapter 3, the three main forms of assessment are:

Homework/Course-work (small/medium assignment)

Project/Dissertation (major assignment)

Written examination (formal, fixed time)

The terms used to define an assessment form are somewhat flexible, and so a brief explanation of each regarding this investigation is now provided. Homework is meant to indicate an extension of class work on a modelling problem, eg, carrying out some initial mathematics on expressions (relations) so far identified. In view of the small range of modelling activities carried out for homework it is considered inappropriate to award marks or other grading for such work. Instead, informal comments and guidance (if necessary) are all that is given by the lecturer. Course-work on the other hand is intended to provide an opportunity for students to carry out

a fairly extensive range of modelling activities; depending on the nature of the problem and the course of which modelling is a part, students may be expected to spend in time anything from about 12 hours (eg, BSc 2 Appd. Physics on record player problem, see Chapters 6 and 7) to 40 hours (eg, O.U. students on MST 204, see Chapter 3) or even 52 hours (average time for MSc Math. Ed., see later) in carrying out an investigation and writing up a report. The awarding of marks or other grading for course-work is considered to be most appropriate by many, if not by all. Project is sometimes used as an alternative term to course-work, but in this discussion it is used only to refer to a major assignment such as a dissertation (eg, for the MSc Math. Ed.). Assessment of a dissertation is not considered in this chapter. Written examination, as pointed out in Chapter 3, is considered to be the most inappropriate form of assessment of mathematical modelling activities. It may consist entirely of unseen questions, or it may consist of seen questions (handed out a few days or more before the examination is due to start), or a combination. A later section of this chapter discusses written examinations in modelling with examples of questions set for the MSc Math. Ed.

The two main forms of assessment discussed in this chapter are:

Written examinations (section 8.2)

Course-work (section 8.3)

in the senses defined earlier.

Associated with any assessment form are the issues of formal and informal grading (the latter is sometimes referred to as impression marking). As discussed in section 3.4, Chapter 3, there are arguments for and against each method of grading. These issues are taken up again in the subsequent sections of this chapter, but suffice it to say at this juncture that although there are strong arguments in favour of informal grading (even for externally assessed assignments), a formal

marking scheme which awards marks for each of well-defined attributes or sections of a student's modelling attempt may be commended for the lecturer inexperienced in the teaching and assessment of mathematical modelling.

Some key considerations which guide assessment, no matter in which form or whether a formal marking scheme is used, are indicated by the findings of Chapter 7 on formulation-solution processes.

As pointed out in Chapter 7, it is the relationship level graph (RLG) rather than the concept matrix (CM) that has provided the deeper insights into modelling processes. Consequently, the results of analysing formulation-solution processes using RLG are the most relevant in providing guidance for assessment. The RLG has shown that formulation and solution are intimately interwoven (carrying out some mathematics prompts the need for further understanding of the problem - generation of further level 0 relationships). So, formulation and solution may best be marked together. Analysis, using RLG, of students attempts at modelling has shown that although 'interpretation' and 'validation' are often an integral part of 'formulation-solution', they can be more naturally separated out for marking. The RLG has also shown, through demonstrating relationship generation and the possible evolution of sub-problems, that model development and improvement take place naturally; consequently, it is unreasonable to insist on students in all cases to make a separate development of models in a hierarchical sense. Both the CM and RLG show that simplifying assumptions, relationships, variables and constants are generated naturally with the development of a model(s), and so it is artificial to ask for a list of such items in the initial part of a report - such items could only be listed with hindsight and out of their natural context. The latter point is not encouraging lack of clarity, on the contrary, students should be encouraged to identify most clearly any assumptions and variables they create as they develop their model(s

The above points may be summarised as follows:

1 Formulation and solution are intimately interwoven, even in 'polished' model developments, and so are best treated as a single entity

2 Interpretation and validation can be more easily separated out for marking. A warning must be issued even here, though, since these latter activities are a vital part of the modelling process and are themselves often integrated with formulation-solution activities

3 Improvement of the model can take place in natural development and so it is unreasonable to insist on separate treatment

4 Sub-problems are often only identified with hindsight, consequently it is unreasonable to ask for separate treatment of each

5 Simplifying assumptions, relationships, variables and constants are generated naturally with model development. Consequently it is artificial to ask for a list of such items at the outset

Additional considerations bearing in mind points 1 - 5 above which are taken into account in assessment are the following:

Credit to be given for:

A Interpretation of problem including clear statements of initial objectives

B Generation of relationships consistent with initial objectives

C Technical competence in mathematics in generating additional relationships

D Rational simplifications making clear any assumptions made

E Recognition of a solution - ability to interpret and validate. Checking for logical errors

F Conclusions and general discussion - awareness of strengths and weaknesses of model development, suggestions for further work

G Overall presentation - ability to communicate clearly in written form; clear diagrams and sketches

In the subsequent sections the fundamental points made earlier will be embodied in discussions on assessment of examination papers and of course-work assignments. Additional considerations specific to a group of students as well as the form of assessment will also be identified.

8.2 Written examinations

This section refers to written examinations in mathematical modelling and, in particular, illustrates with examples of questions set in the MSc Math. Ed. final year (second year) assessment.

The MSc Math. Ed., the only course of its kind in the public sector of higher education, started running in 1977. The course is intended mainly for secondary school teachers and college of further education lecturers who have a degree or equivalent qualification in mathematics. The structure of the mathematical modelling component of the course is briefly outlined in section 4.3, Chapter 4; more extensive reporting

on the running of the course may be found in Oke (1980, 1984). Reports on a selection of modelling activities with a year 1 class are provided in Chapters 6 and 7.

The examination paper, which is taken at the end of year 2, is of three hours duration. The paper consists of two sections:

Three questions are to be attempted (1 hour per question), with one question only selected from Section A.

Section A (Seen one week before examination)

Three questions, each stating a practical problem, to be modelled from scratch. Only initial approaches are expected, but they must include some mathematics and interpretation. One question is based on a problem in the social and organisational area, one on physics/engineering area, and one on life sciences/biology.

Example (Physics/engineering area)

Modern office blocks, particularly of the high rise type, have large glazed areas on the outside to permit entry of as much natural light as possible. By concentrating on the forces involved on an individual glass unit or pane, try to identify some key design features. Is there an optimum pane size, and if so, does double glazing affect this? In your development, consider simple models and make clear any assumptions you feel are necessary.

(June 1983 paper)

Section B (Unseen)

Approximately 5 or 6 questions, each based on general modelling and/or pedagogic issues. Essay type answers expected.

Example

Make out a case for teaching mathematical modelling, indicating clearly the level and background of the students involved. Refer to relevant articles as far as possible.

(June 1983 paper)

Further examples of questions set may be found in Appendix 2A where the complete June 1982 and June 1983 examination papers appear.

In order to provide an indication of the extent of the initial modelling development that is expected in response to a Section A type question, the following outlines a possible approach to the office block glazing problem above:

Office block glazing (Section A, June 1983)

Outline notes on possible approach:

Consider single-glazing. Size of glass-pane is limited by risk of glass breakage; pane needs to be as large as possible to allow maximum amount of light entry - too many panes over a large area will involve loss of light entry due to area of supporting frames. Consequently, there appears to be an optimum size for a given pane.

Key methods by which pane is assumed to break:

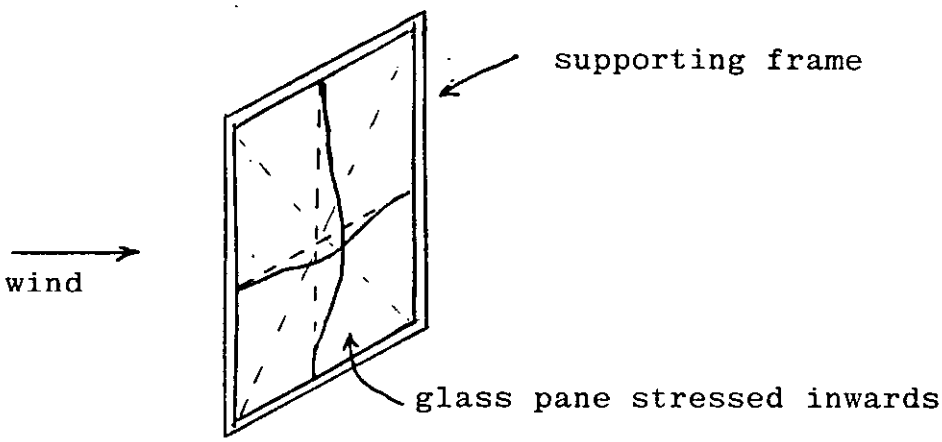
(a) Wind causing flexure

(b) By crushing under own weight

(c) Thermal cracking - pane not allowed to expand (or contract) in frame

With a well-designed frame, it can be assumed that (c) will not occur. Before (b) takes place, whole side of high-rise office-block would consist of single pane of glass! Wind forces causing flexure, as in (a), seem to be the single most important cause of breakage (ignoring accidents).

Flexure due to wind forces:



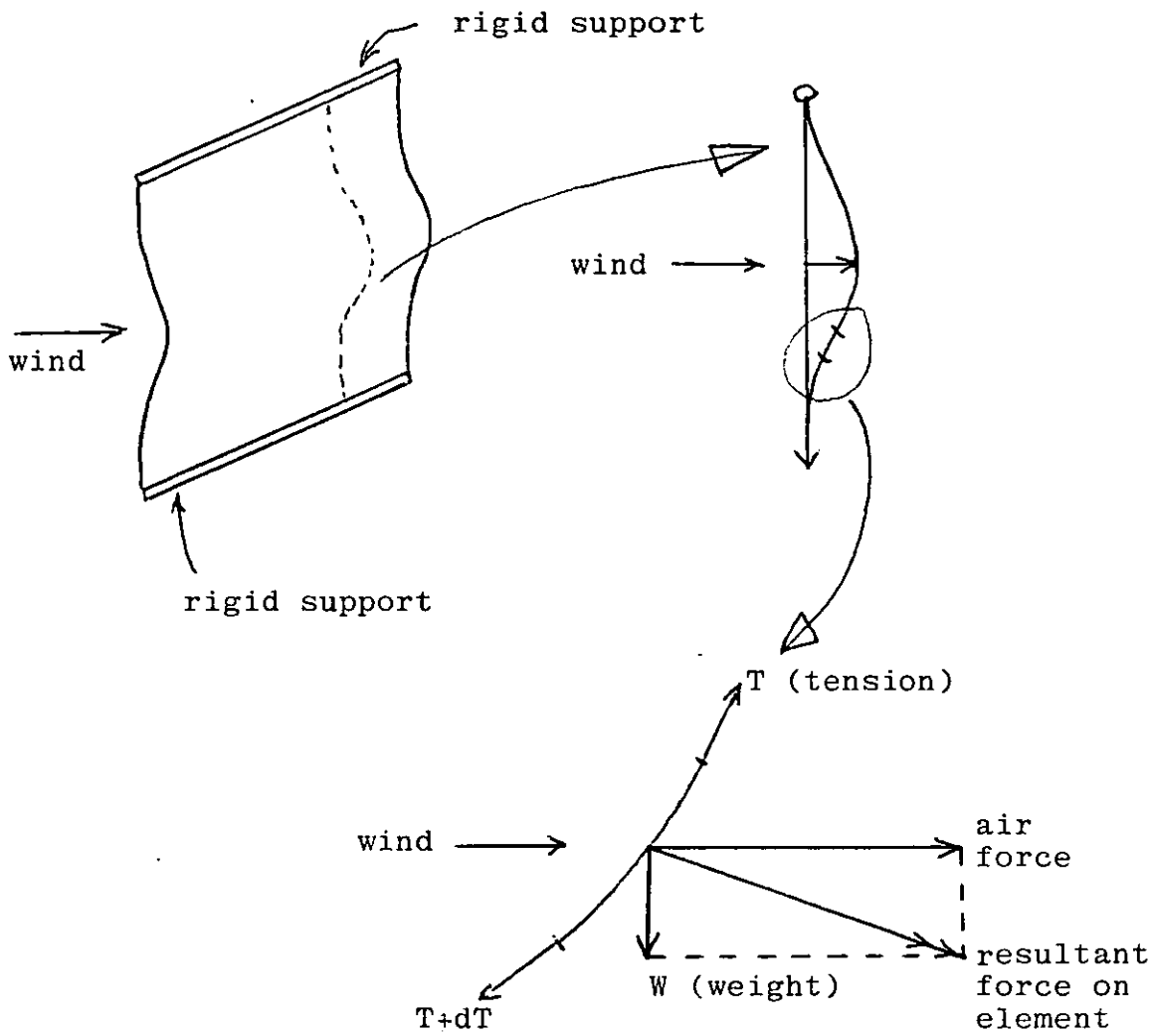
Assuming frame is rigid on all four sides of pane, then problem reduces to 2-D stress type (assuming small displacements).

If wind speed is v, then $\frac{d}{dt}(mv) = \dot{m}v = (\rho Av)v = \rho Av^2$ can be assumed from Newton's second law to be force (normal on pane of area A). ($\dot{m} = \rho Av$ is flow-rate of wind, ρ is density of air). This approach would provide simplified boundary conditions. By solving the biharmonic stress equation, maximum stresses can be found (near centre of pane). The design would involve knowledge of maximum possible wind speed v (over the year, in a given location), so that maximum stress is much less (50% less?) than breaking (yield?) stress of glass involved. Hence size of pane. For double glazing, air is trapped between 2 panes of glass and would be partly compressed - this might strengthen structure and hence permit a larger unit for given wind speed; stressing of inner pane would also have to be taken into account.

So far, the mathematics that would be involved would be fairly complicated and beyond expectation in the time allowed (one week to prepare modelling approach, and one hour in the examination in which to write out the development). So, it is wise to consider an even cruder approach in order to get some upper-bound for stress at the middle of the pane.

Crude model

Consider a single-pane of glass, rigidly supported along upper and lower edges only, then problem reduces to one in 1D:



Maximum stress (at mid-point) would be greater than for 2-D model and hence would be an upper-bound. Solution follows from elementary beam theory, using resultant of air force and weight for external loading.

NB If an approach along the lines of the above development were followed, then some attempt at solving the beam problem identified above would be expected.

Full credit would be given for a comparable development.

Section A (one question) and Section B (two questions) are allocated equal marks by informal (impression) marking. It was decided by the marking team at South Bank (the author and two colleagues) that formal marking was inappropriate in view of the possibly wide variation in approaches that could be adopted in tackling any one question. For example, the outline approach provided above (the author's) represents mainly initial formulation, with reasons, of a crude model; little mathematics is used or intended (elementary beam theory and solution of a differential equation is the most expected). Consequently most credit, for a comparable development, would have to be given to initial arguments of the type used above in creating a specific problem to be solved. On the other hand, a student may decide (this actually happened in one case in 1983) that only the briefest (half-page) discussion would suffice, and then proceed with a solution with some numerical values (from a text-book) being inserted. Credit would, in the latter case, concentrate on solution and interpretation. As a measure of the standards set for the course, the approach which has been outlined together with some solution and interpretation of the elemental beam would attract full marks (33); without the latter solution, a mark of two-thirds of the total would be awarded (22). Section B questions are marked as essays, where content, presentation, relevance, and clarity in communication are given credit.

Clearly, it is not reasonable to expect an extensive modelling development for a Section A question. In fact, all that is insisted upon are points A, B, D and G with some attention paid to the remaining from the credit list provided in section 8.1.

Examination papers for the years 1979 - 1980 had the same structure except that Section A was unseen. The poor standards achieved in Section A pursuaded the teaching team to adopt a 'seen' approach from 1981 onwards, which resulted in considerable improvements in student performance. However, in view of the realistic expectation of few modelling activities being carried out in the time available and under the stress

conditions of a formal fixed-time examination, it has now been decided to discontinue with this mode of assessment from 1985 onwards. The reason for the inclusion of a written examination paper in the first place was an attempt to balance the assessment modes in what was a completely new experience (running an MSc Math. Ed.) both for the South Bank Polytechnic and for the CNAA.

Mathematical Modelling is also assessed by course-work on the MSc and this mode will be the main mode of assessment in 1985 and subsequent years. The next section discusses course-work assignments, with illustrations of the assignments involved with the MSc and BSc Applied Physics courses at South Bank.

8.3 Course-work assignments

8.3.1 MSc Math. Ed.

In the case of MSc Math. Ed., one course-work assignment is set towards the end of year 1. Originally, two assignments were set, but largely due to a policy of reducing the overall number of assessments on the course in all subjects, a concession had to be made in mathematical modelling.

This assignment consists of each student (teacher) finding their own problem, in any area they wish, and developing a mathematical model relating to this problem. Teachers are expected to define the learning aims appropriate to a level of student with which they are familiar, and to provide self-assessment questions for their students - these questions may test understanding of the developed model as well as test ability to extend or model a similar situation. Originally this course-work was assessed according to the following formal marking scheme:

		%
1	Statement of problem. To include how the problem was identified in the first place	10
2	Learning aims (broad and specific teaching aims, including level of students for whom material might be wholly or partially appropriate)	10
3	Construction of model	50
4	Analysis of model (including validation)	
5	Discussion (general and conclusions)	20
6	Self-assessment question(s) (for intended students)	10
	Total	100

Assessments (1), (3), (4) and (5) would be appropriate to any modelling exercise, whereas (2) and (6) are specifically relevant to the teachers on the MSc course. Note that whether formal marking is used or informal (impression) marking is used, the above serves as a useful check-list. Note also, that in view of the comments made in section 8.1, a further break-down of modelling activities is avoided although points A - G do provide an additional overall guide. As the teaching team gained experience in marking course-works, impression marking has taken over. This approach is further supported since teachers have considerable choice in how they present their work, and because of the completely free choice they have in the problem (which they find) to model.

The whole matter of assessment, regarding both examination papers and course-work assignments, has been discussed at length on the 'Advisory Committee for Mathematical Education' (South Bank), chaired by Professor A C Bajpai. The committee agreed that mathematical modelling would be more appropriately assessed by course-work rather than by formal fixed-time examination. The external examiners of the M.Sc. course have agreed that whilst a formal marking scheme for course-work can be of value, the most important criterion for judging a particular piece of work is based on knowledge of standards that have been developed as a result of running the course over several years. These 'standards' are established by 'impression' marking whereby the internal examiner, in final concurrence with the external examiner(s), arrives at a final mark (grade) by appraising the overall quality of a piece of course-work using points A - G as guidance.

A list of titles giving an indication of the wide range of problems that have been considered by teachers is provided in Appendix 2B for the years 1980 and 1983. Course-works have been found on average over the years to take 52 hours to complete; this is considered to be quite extensive, and the teachers carry out the work in their own time during the latter

part of the summer term. Staff are available for consultation throughout most of the period, but no help is provided with details.

Teachers are asked to find their own problem and to develop a modelling approach comparable in extent to some samples provided in the earlier part of the course. In other words, although a thoroughly competent development is expected, any attempts at elaborate mathematics and/or attempts at introducing an abundance of detail into an analysis is discouraged. Credit is given for a development that is consistent with the learning aims that must be identified at the beginning of each report. On the whole, teachers produce work within the reasonable perspectives outlined here, however there are one or two exceptions where quite voluminous and over-ambitious reports have been presented; in the latter cases, excessive enthusiasm had led to attempts to study a problem in a manner which is much more appropriate to a team of professional modellers with much more time available. In the other extreme, some reports contain a large amount of descriptive material with little mathematical content and consequently the benefits of modelling are barely achieved.

In order to give an indication of standards reached by teachers in their modelling course-work, the author's comments on three reports selected from the 1983 group (titles in Appendix 2B) are provided in Appendix 2C. The three reports and the reasons for their selection are: (Pass mark 50%)

1 The Shower Problem

Assessment: Highest mark awarded (for 1983)
Grade A (75%)

To illustrate the strengths and weaknesses of a well-developed modelling approach which is also very well presented

2 Heating and Heat Loss for a Domestic Immersion Heater

Assessment: Grade B- (62%)

To illustrate an over-ambitious piece of work with masses of detail and presented in a complicated and unclear manner

3 Recreational Carrying Capacity

Assessment: Grade E (35%). Lowest mark awarded. Fail

To illustrate a report with a large amount of descriptive material with virtually no mathematics involved

It is very important for students in their development of mathematical modelling skills to receive comments on their assessed work in order that they may improve on their weaknesses. A balance between encouragement and criticism is required, especially with part-time students where there is inevitably less contact between lecturing staff and students (teachers) than is the case with full-time students. The comments in Appendix 2C illustrate the author's attempts at achieving such a balance. Significant or major criticism is intended to be positive, and so suggested alternative approaches are indicated in the comments. For example, in connection with report 2 mentioned earlier, an alternative layout is suggested in order to make the presentation clear and easier to follow. In the case of report 3, some suggestions are made on how to focus on specific aspects of the problem chosen and on how a modelling development could take place based on these aspects.

8.3.2 BSc Applied Physics

In contrast to the extensive course-work that is expected of the MSc Math. Ed. teachers, taking an average of 52 hours and where a problem has first to be found, course-work on mathematical

modelling takes approximately 12 - 15 hours in the BSc Appd. Physics. A problem, or set of problems, is presented to the physicists in the form of a problem statement (see Chapter 5).

Mathematical modelling was first introduced on the BSc Appd. Physics degree four years ago. At present it is taken only in the second year of the course, but it is planned to include modelling in the first year as well from 1985 onwards. The subject forms a compulsory part of the curriculum and it is assessed; marks contribute towards the final part I of the degree.

The course-work assignment consists of a practical problem that is presented to the class which is then split into groups; the groups then work for two weeks (3 hours per week) as part of their normal course where contact may be made with a lecturer. At the end of the two-week period, students have an additional week in which to write up group reports in their own time. The mode of working in class time is illustrated in Chapters 6 and 7 where the record player problem is considered.

In order to illustrate the assessment of this type of assignment, the groups referred to above who worked on the record player problem will now be considered. A formal marking scheme was adhered to on this occasion as follows:

Group report to be in following format

1 Problem statement (see section 5.5, Chapter 5)

2 Report on class discussion (see sections 6.2 and 6.5, Chapter 6; section 7.3.2, Chapter 7)

3 Log consisting of minute by minute group development of model(s). This must be an honest and accurate record of what actually happened

4 Report consisting of model(s) with interpretation of results based on 3 above

5 Conclusions

6 References if any

Marks awarded as follows:	%
Overall presentation	20
Log (Section 3 above)	30
Main report (Section 4 above)	40
Conclusions (Section 5 above)	10
Total	100

The decision to assess each group, rather than individuals, seemed to be a natural one since groups worked together as teams. The disadvantage of assessing in this manner, however, is that the less able or less hard working get the same credit as the stronger members of their group. Little discord was observed on the latter point, although each group did tend to produce a leader. Most reports show evidence of a genuinely co-operative effort, at least to the extent of sharing the writing of sections amongst group members.

It was decided to assess according to a formal marking scheme by triple-blind marking; one marker was the author, another was a moderately experienced lecturer in modelling (and its assessment), and the third marker was relatively inexperienced in modelling. The final mark awarded was an average of the three markers. As pointed out in Chapter 6, three members of staff observed the groups working in class time and made observation notes; these three staff are the same ones referred to above who independently marked each report. The marks produced are shown in Table 10. Also shown in Table 10 is the maximum relative discrepancy (MRD) between markers, where

MRD = Numerical value of maximum difference between markers ÷ Average mark

(For example, marks for presentation for group 1 are respectively 13, 15, 14. Hence MRD = 2/14 = 0.14 (approx))

	GROUP 1	GROUP 2	GROUP 3	GROUP 4
PRESENTATION (max. 20)	13 15 (0.14) 14	11 13 (0.17) 12	11 16 (0.35) 16	16 12 (0.28) 15
LOG (max. 30)	20 18 (0.11) 19	16 18 (0.17) 19	19 20 (0.05) 20	23 21 (0.09) 21
REPORT (max. 40)	27 22 (0.20) 27	16 18 (0.12) 18	24 24 (0.08) 26	35 35 (0.15) 30
CONCLUSIONS (max. 10)	5 8 (0.45) 7	4 5 (0.40) 6	6 3 (0.60) 6	3 4 (0.50) 5
TOTAL	65 63 (0.06) 67	47 54 (0.15) 55	60 63 (0.13) 68	77 72 (0.08) 71
AVERAGE TOTAL	65%	52%	64%	74%

First number in each box: author's mark
Second " " " " moderately experienced marker
Third " " " " relatively inexperienced marker
Number in brackets in each box: maximum relative discrepancy between markers

Table 10

BSc 2 Appd. Physics course-work group marks: Minimisation of sound distortion in a record player

The table shows no consistent difference between the total marks given in any group across the groups, in fact there is surprisingly close agreement. However, there are more significant (although still not consistent) differences in the marks given to each section as shown by the higher MRD values. The most striking differences occur for marks awarded to the conclusions section; these differences (highest MRD is for group 3) will not contribute much to the total marks, however, since this section can at most contribute 10 out of 100 in weighting. No doubt the overall close agreement between the markers can be explained by the fact that all three were closely involved with the observation of the groups.

Note that more pronounced differences in marking might have been predicted in view of there being no break-down in marks for the main report section, where the model(s) development takes place. That such close agreement amongst the markers (highest MRD is 0.20 for group 1) has been achieved is another instance of support for informal (impression) marking.

8.4 Summary and conclusions

This chapter covers general points for guidance in the assessment of mathematical modelling assignments. The two main forms of assignment considered are written examinations and course-work. Illustrations of the points have been made by referring to the assessment methods used in the MSc Math. Ed. and BSc Appd. Physics courses offered at the South Bank Polytechnic.

The overall implications of Chapter 7 for assessment as well as the presentation of a credit guidance list are covered in section 8.1.

A subset of modelling activities is all that can be expected in a formal written examination and consequently this form of assessment is not recommended. The limited scope for assessing modelling in this manner is illustrated in the case of the MSc Math. Ed. in section 8.2.

By contrast, the less stressful mode of course-work, where much more time is made available, is considered to be a most appropriate form for assessment. Examples of marking schemes used in assessing modelling assignments in the MSc Math. Ed. and BSc Appd. Physics courses are provided in section 8.3. Irrespective of the marking schemes considered, all points in section 8.1 are expected to be covered for full credit to be given. A case for informal (impression) marking is made, where the assessor has an eye for attributes in the credit list appearing in some form or other in a course-work report. Formal marking schemes may best be used by inexperienced lecturers, although even then a large element of judgement is needed in attributing marks to any section. Close agreement is often achieved between several markers, even where a vaguely defined section is part of the marking scheme; this is illustrated in section 8.3.2 in the marking of the record player problem. Such close agreement may well be due to lecturers (markers) being closely involved in observing students modelling a particular problem or may be due to lecturers working closely together as a team over several years (as in the case with the MSc Math. Ed.).

CHAPTER 9

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

9.1 Summary of the Research Investigations

The growth of interest in the teaching of mathematical modelling since the late 1960's was identified and reviewed in Chapter 2. For the purposes of the subsequent investigations, a working definition of mathematical model was proposed (see section 2.2):

> A simplified and solvable mathematical representation of an aspect of a practical problem.

The reasons for this choice of definition are explained. The definition is broad enough to cover both deterministic and stochastic models, although it emphasises analytical rather than descriptive or empirical models.

Some of the most recent and significant research which is related to this thesis is reported on in Chapter 3. Teaching styles, learning modes, and assessment methods have been identified as well as the research need for a fuller investigation of the formulation-solution interface. Regarding the latter, although the flow-chart and similar representations of modelling processes provide a valuable overall guide, the suggestions of Clements (1982) based on the systems work of Checkland (1975) most closely relate to the non-linear and holistic approach adopted in this thesis.

The work mentioned above provides the back-drop to the main aims and scope of the research project which are delineated in Chapter 4. The principal aim of the project has been to investigate formulation-solution processes and the extent to which these processes lead to better guidance and understanding of teaching, learning, and assessment in mathematical modelling.

The following activities have been carried out in support of this aim:

The development of case studies of the mathematical modelling approaches that may be used in the solution of practical problems

The design of teaching and learning experiments carried out mainly with undergraduates and teachers on an M.Sc course in mathematical education.

The theoretical development of formulation-solution processes by means of:

A concept matrix (CM)

A relationship level graph (RLG)

The analysis of a selection of student's modelling attempts using CM and RLG

The implications of the theoretical development of formulation-solution processes for assessment

The development of case studies

In Chapter 5, nine practical problems with outlines of possible modelling approaches have been presented. Each approach is deterministic and analytical, and most of the problems require some background knowledge of physics (at approximately GCE 'A' level).

The case studies were based on the following design features:

Motivation
Level of difficulty
Scope
Content
Duration of modelling exercise

Details of these design features may be found in section 5.1. The appropriateness of the design features is tested in Chapter 6 on teaching and learning experiments with further analysis on students' attempts in Chapter 7.

The design of teaching and learning experiments

Reports on the observation of seventeen experiments based on the nine case studies presented in Chapter 5 are given in Chapter 6. The experiments mainly involved undergraduates with mathematics, physics/engineering backgrounds, with teachers on the M.Sc in Mathematical Education, and occasionally with secondary school students. All the students involved had little or no modelling experience. All the experiments were based on short to medium duration activities, that is students spending time ranging from one hour to ten hours on a given problem. Long duration project-type work is usually given to students who have some experience of modelling and the assessment of such projects is covered in Chapter 8.

The main findings on students' difficulties which are exemplified in sections 6.2-6.6, are as follows:

- Tendency to want to work on problem other than posed

- Variables and constants: which to choose as dependent, independent, parameters (particular difficulty for school students)

- Relationship and variables: level of detail (too much detail leads to confusion, tool little or excessive 'lumping' leads to general mathe-

- matical solutions which are difficult to interpret)

- Tendency to: keep listing features, draw many diagrams/graphs, carry out large amounts of computation rather than use analytical techniques (even elementary ones). School students particularly prefer arithmetic to algebraic or other methods (this has also been found by Treilibs (1979))

- Lack of confidence in making simplifications. Even when simplifications are made, difficulties are experienced in interpreting mathematical solutions arising from them

- Tendency to drift and lose sight of objectives. Fixations formed (unwilling to try other more fruitful paths).

The experiments reported in sections 6.3-6.6 cover the two basic teaching/learning styles: interactive and group, or a combination. It has been observed that the interactive approach is suitable for modelling activities that are being tackled for the first time, especially in the case of school students, but that group work enables students to gain confidence and ability once the first one or two interactive sessions have been experienced. It has also been illustrated that lecturer intervention is needed at certain key points in order to prevent 'frustration', 'fixation', and other difficulties from taking over. Research generally has shown (see section 3.4, Chapter 3) that <u>work done in groups is useful in the early stages of feature identification, but that the solution stage is best carried out on an individual basis</u>. This has been confirmed by the experiments conducted in Chapter 6, and furthermore that much of the formulation of a problem is still being carried out at the solution stage. There is no research recommending group size, where group working is carried out, but that judgement of the author and others is that 4 <u>seems to be optimum</u>.

A set of learning heuristics has been devised in an attempt to provide some 'rules-of-thumb' for the student inexperienced in modelling. The heuristics, which are described with student opinion in section 6.7, may briefly be listed as:

1 Establish a clear statement of objectives
2 Don't write a vast list of features
3 Simplify
4 Get started with maths as soon as possible
5 Carry out some mathematics on initial relationships
6 Got a solution yet?
7 Know when to stop
8 Interpret your solution
9 Validate your solution
10 If stuck
11 Have frequent rests

The most popular (useful) heuristics were deemed to be 1, 3, and 4, whilst the least useful was 2. The description of 2 has now been modified to a form almost identical to that used at the Open University, Berry and O'Shea (1982).

The theoretical development of formulation - solution processes

In order to gain a fuller understanding of formulation and solution together with the complex linkages between them, two theoretical ideas were developed:

Concept matrix (CM)

Relationship level graph (RLG)

The ideas are introduced in section 4.5, Chapter 4 and are developed fully in Chapter 7.

Section 7.2 defines and illustrates in detail the ideas in the construction of both CM and RLG. Sections 7.3.1-7.3.4 analyse, in terms of CM and RLG, the results of students' attempts at modelling from a selection of experiments reported in Chapter 6.

The analysis has shown that formulation-solution processes are not merely highly oscillatory, see section 3.3 of Chapter 3, but that they are largely non-linear in nature and that several activities are often carried out by working at a variety of stages simultaneously. The most powerful tool in providing insights into modelling processes is the RLG rather than the CM; the latter is mainly an aid in classifying features in a model development. The main findings of the analyses, which can be found in more detail in section 7.4, may be briefly summarised as follows:

1 Distribution of features

Although there is no discernible order in which features of a problem are recognised, there is a general movement whilst modelling from the (global)/(difficult to quantify) to the (highly specific)/(easily quantified) concepts.

2 Basic relationships are often generated as solution proceeds

Apart from the initial relationships which are needed to get started (which are based on the first understanding of a problem and are not derived mathematically), the mathematical solution as it progresses often prompts the need for more initial relationships.

3 Relationships can occur in various forms

Relationships occur in two basic forms throughout their generation:

General: Applicable to a wide range of problems and not just to the problem in hand.

Specific: Related directly to a specific problem.

4 Relationship level as goal seeking

As with features generally, relationships often occur in no discernible order but a measure of the general progress made in finding a solution is provided by relationship level.

5 Most variables and constants are generated with relationships

Very few variables and constants are identified at the outset, instead they appear naturally as the solution progresses.

6 Sub-problem identification

Partioning a problem into sub-problems may be possible initially, but such break-down into distinct parts is often only possible after having seen a pattern of linkages in a RLG.

The work of this chapter in its support for the choice of the learning heuristics discussed earlier, is also detailed in section 7.4.

The two chief weaknesses of the analysis have been identified as its inability to:

Explain how the initial relationship to get the solution started are obtained (creative leap)

Describe the strength or importance of relationships (apart from initial ones).

However, in spite of these weaknesses, the CM and RLG (particularly the latter) have shown considerable insights into the modelling process and are capable of being used in the analysis of a variety of different students' attempts at modelling (from school to HE).

An investigation of assessment methods

Implications of the theoretical analysis on formulation-solution processes for assessment of mathematical modelling are examined in section 8.1, Chapter 8. These implications may be briefly summarised as follows:

1 Treat formulation and solution as a single entity

2 Interpretation and validation can be more easily separated out for marking

3 Model improvement evolves naturally, difficult to mark sections (models) separately

4 Don't insist on separate treatment of sub-problems

5 Don't insist on lists of assumptions, relationships, and variables at outset.

These implications follow naturally and logically from the main findings of the analysis of formulation-solution processes listed earlier (and found in detail in section 7.4, Chapter 7).

Consistent with this list of implications is a credit list which may be used as an overall guide in assessment. The credit list, details of which may be found in section 8.1, is further based on the experience of the author and colleagues at South Bank Polytechnic. Briefly summarised the list is:

Credit to be given for:

A Initial interpretation of problem
B Generation of relationships consistent with initial objectives
C Technical competence in mathematics
D Rational simplifications based on assumptions
E Recognition of a solution
F Conclusions - awareness of strengths/weaknesses of model development
G Overall presentation - clear communication.

Examples of assessment of examination papers and of course-works are provided in sections 8.3.1-8.3.2. Along with others, examination papers are considered to be an inappropriate form for assessing mathematical modelling since only a sub-set of activities can be expected even with 'seen' questions.

A discussion on informal (impression) and formal marking shows that informal marking, bearing in mind the implications and credit lists earlier, is preferred. For the inexperienced lecturer, formal marking may have a place. In the experience of the author it is beneficial to students to discuss how credit will be given, in general terms (c.f. lists), and to provide detailed comments on a course-work assignment after grading. The latter points form an integral part of the teaching of modelling.

9.2 Suggestions for Further Research

The case study problems in Chapter 5, upon which subsequent work covered in Chapters 6-8 has been based, require some acquaintance with physics (except 'Evacuation of a school' and 'Motorway versus A-Road travel costs'). In particular, the physics involved has either been mechanics (including elementary fluids) or simple heat transfer; the record player problem additionally benefits from a familiarity with waves. These case studies have been devised for students who have approximately GCE 'A' level background knowledge in physics. Consequently, although the applications areas are diverse, the case studies have two main themes: mechanics and heat. It has been observed by the author, that if two or more modelling problems with the same theme are presented to students, not necessarily successively, then students improve in modelling by benefitting from the implicit analogies. Consequently the following suggestions for further research:

1 PROBLEMS WITH COMMON THEME	To gauge the improvement in modelling skills by presenting practical problems based on a common theme (eg: heat-transfer). How many problems per theme and how many themes should be tackled for a given curriculum?

Closely related to the above suggestion is the question of grading modelling activities in order of difficulty. This was first raised in sections 3.2 and 3.4, Chapter 3. Level of difficulty of a given problem has been extensively illustrated in Chapter 6 by reporting on student performance; further analysis of the difficulties experienced by students is illustrated in Chapter 7. One of the most difficult problems was found to be 'speed-wobble in motorcycles' (sections 6.2 and 6.4, Chapter 6 and section 7.3.3, Chapter 7). In the 'conclusions' part of section 6.4, it was suggested that the speed-wobble problem might better have been presented as the culmination of a set of graded problems starting with a rolling wheel which did not wobble. Having identified the need for graded problems, there is the danger that a carefully constructed sequence of problems would remove most of the initial formulation of the final problem. However, this does seem to warrant further research in this direction.

2 GRADED MODELLING PROBLEMS	To develop problems graded in difficulty in a given application domain (theme). To investigate any resulting improvement in modelling skills, eg: ability to develop one's own graded development for a given problem.

Heuristics were first discussed in section 3.2, Chapter 3, in the context of more highly structured problem-solving processes. These heuristics are largely posed in general form, see for example the discussions on Polya (1957), Kilpatrick (1969), and Gagné (1966) (section 3.2). In the case of mathematical modelling

processes (less well structured), although the term 'heuristics' is not used, one learns of the noteworthy work of Morris (1967), Bajpai et al (1974, 1975), and Burkhardt (1979, 1981) (all discussed in section 2.3, Chapter 2) each of whom in effect consider heuristics or 'guidance for students' in detailed form. The heuristics presented in Chapter 6, based on the work mentioned above in addition to the new work of this thesis, are also in detailed form. Examples of general heuristics are: Devise a plan, Carry out the plan, Look back, Polya (1957); examples of more detailed heuristics are: Simplify, Get started with mathematics as soon as possible, Carry out some mathematics on initial relationships (see section 6.7, Chapter 6). Neither the general nor the more detailed heuristics are 'task-specific' (eg: use Newton's second law of motion when considering momentum changes of air impinging on a windmill blade). The reason why task-specific heuristics are not devised, especially in modelling, is because they would constitute very strong hints for a given task (problem) and thereby largely destroy students' opportunities to learn for themselves. However, given the difficulty of modelling, especially for inexperienced students, a case may be made for further development of learning heuristics.

3 LEARNING HEURISTICS	To investigate the possibilities of further development of learning heuristics for mathematical modelling. In particular, to investigate the level of detail needed in each heuristic for a range of problems.

As pointed out in section 7.4, Chapter 7 and in section 9.1 of this chapter, two important aspects of the analysis on formulation-solution process, namely initial relationship formation (before any mathematics is carried out), and strength or importance of each relationship, need further attention. The first of these aspects may properly fall into a field of psychological research on problem solving although no known

work relates very closely to this (see Chapter 3). An investigation into the second aspect, on the strength or importance of each relationship (apart from level 0 types which are fundamental), may lead to more efficient and easier model development.

4 (BEFORE MATHS) RELATIONSHIP CREATION	To investigate those factors affecting the creation of initial relationships before any mathematics is carried out
5 STRENGTH OF RELATIONSHIPS	To investigate the strength or importance of each relationship and the implications for the linkages between each.

The evolution of assessment methods in mathematical modelling is inevitably even more in its infancy than the development of teaching and learning styles. A better understanding of modelling processes is an aid to developing assessment criteria, as is exemplified in Chapter 8, but the issue of how much weight should be attached to each criterion is still open. How important each criterion is considered to be depends, inter alia, on the problem being modelled and the flexibility allowed in students' presentations. The implications of the research on formulation-solution processes (Chapter 7) for assessment, which are provided in Chapter 8 and in abbreviated form in section 9.1. of this chapter, would strongly indicate however that there should be a change in emphasis in student presentation. For example, referring to Berry and O'Shea (1982) and Table 2 in section 3.4, Chapter 3, students are asked to state variables and assumptions in the first stage of their modelling development. The research in this thesis has shown that such items occur with the development of a model and so are most naturally presented as they occur.

The extent to which attributes occur in any given modelling development depends, as pointed out earlier, on the problem being tackled and on the particular presentation being asked for. Illustrations of a broad or general way in which presentation of a model development may be made are provided in sections 8.3.1-8.3.2 in Chapter 8, where a natural flexibility is incorporated. These illustrations are not totally dissimilar to other assessment specifications, even Berry and O'Shea's, and it would seem that further work here would be beneficial. Given a broad and hence flexible framework in which students may present their work then leaves the lecturer to use his/her judgement in making an overall assessment using the credit list (sections 8.1 and 9.1).

6 ASSESSMENT METHODS	To investigate further: possible course-work presentation frameworks and the ways these affect student opportunities to gain credit according to the list of attributes provided in section 8.1, Chapter 8. In particular, whether formal or informal (impression) marking is used, the feasibility of attaching a weight to each attribute to be tested.

9.3 Wider Implications

The research has centred mainly upon analytical and deterministic modelling of problems requiring a background knowledge, or at least an intuitive understanding, of physics. Two problems, 'evacuation of a school' and 'motorway versus A road travel costs', which are organisational in nature have also successfully been analysed in terms of a concept matrix and relationship level graph. There appears to be no reason why analytical and deterministic modelling of any problem, no matter in which application area, cannot be analysed by the same means. The findings of the analysis of Chapter 7 should still apply to a very broad range of problems whether in the physical sciences and technology, the life sciences, or in the

social and organisational sciences. The question that remains is whether or not the analysis applies to stochastic modelling, or even more broadly to empirical modelling. Some of the difficulties experienced in the latter are certainly in common with the types of problems considered in this thesis, eg: postulating the first relationships. One is thus encouraged to believe that much of the research carried out in this thesis is relevant also to a much wider class of modelling activities, although this remains to be tested.

The connection between problem-solving and mathematical modelling has been examined in Chapter 3. It would appear that much of the work reported on in Chapters 6 and 7 would also be applicable to the more highly structured and well-posed problems involved. For example, that carrying out some mathematics itself prompts the need for 'further information' (level 0 relationships) at an intermediate juncture would certainly seem to be true for problem-solving. Once again, this conjecture needs to be tested. The implications for solving mathematical problems in general would be considerable if the analysis could be shown to relate to problem-solving processes in a wide range of activities that arise in various topics, eg: algebra, discrete mathematics, and analysis.

At the higher and less detailed level of methodologies, the analysis of formulation-solution processes would seemto offer some scope for guidance in systems design in a wide range of human activity systems. The philosophy of approach has a bearing on Checkland's (1975) work which, in turn, has implications for the design of computer systems which operate in a social and organisational environment. In the field of expert systems (part of the fifth generation computing development), in particular 'Intelligent Knowledge Based Systems' (IKBS) research, the Alvey Report (1982) identifies the need to develop understanding of human concept formation, reasoning and use of heuristics. The work of Chapters 6 and 7 would appear to have some bearing here, and possibly could make a contribution to the design of an expert system as an aid in the teaching of mathematical modelling.

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APPENDICES

1 Concept Matrices and Relationship Level Graphs

1A Central-heating problem (author's 'polished' modelling approach)

1B Baby's milk bottle problem (M.Sc Math. Ed. group 4)

1C Minimisation of sound distortion in a record player (B.Sc 2 Appd. Physics, group 1)

1D Minimisation of sound distortion in a record player (author's 'polished' modelling approach)

1E Speed-wobble in motorcycles (author's 'polished' approach)

2 Assessment used in M.Sc Math. Ed

2A M. Sc Math. Ed., Year 2 Final examination papers in Mathematical Modelling. June 1982 and June 1983

2B M.Sc Math. Ed., Mathematical Modelling Course-work titles. 1980 and 1983.

2C M.Sc Math. Ed., Assessment comments on Mathematical Modelling Course-works. 1983.

APPENDIX 1A

Central-heating problem

(author's 'polished' modelling approach)

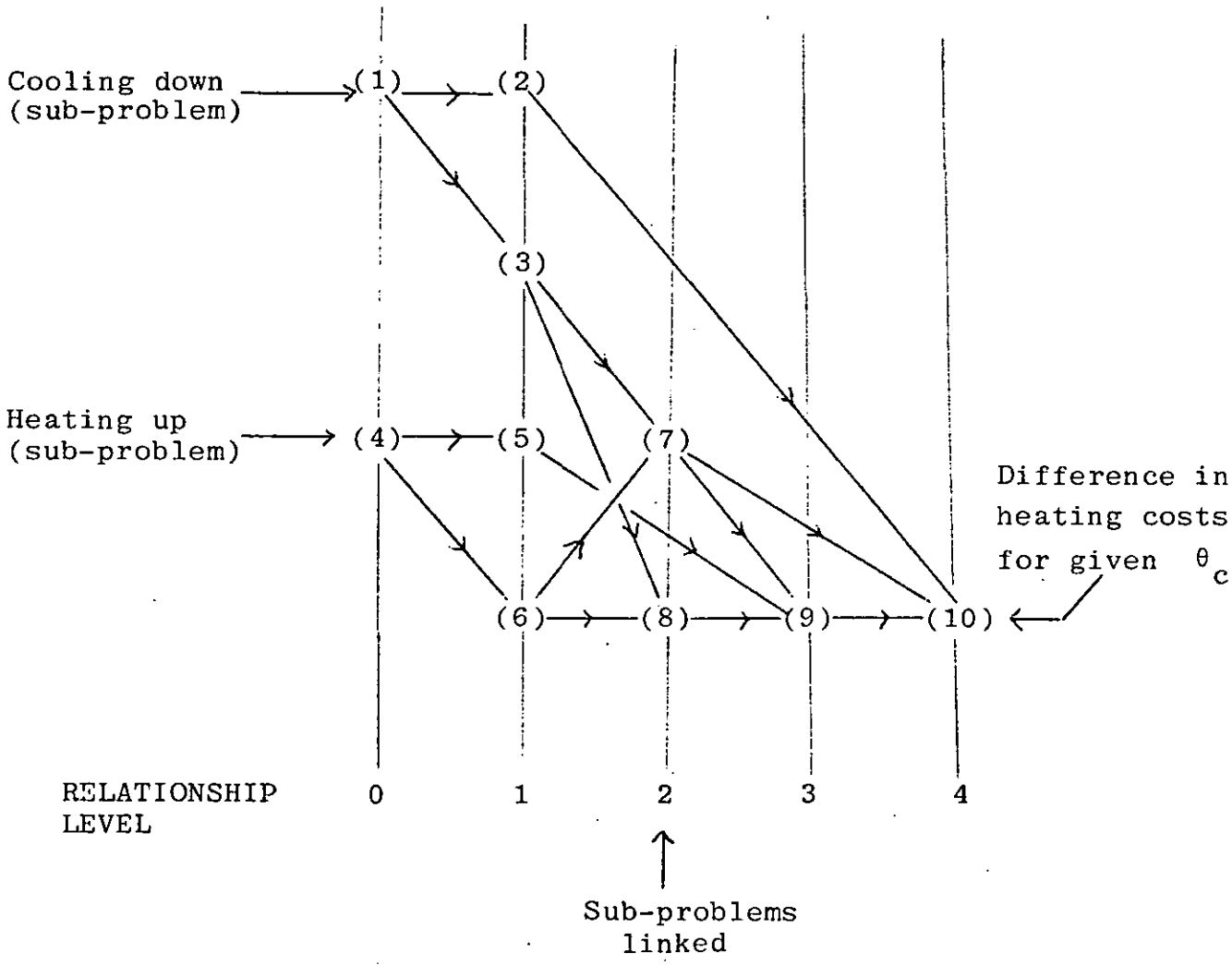
Relationship list in order of occurrence

Relationship		No.
level 0	$T\frac{d\theta_i}{dt} = -K(\theta_i - \theta_o)$	1 *(5, level 0)
1→2	$t_4 = \frac{1}{A} \ell n \left[\frac{\theta_r - \theta_o}{\theta_c - \theta_o}\right]$	2 *(13, level 1)
1→3	$t_{1a} = \frac{1}{A} \ell n \left[\frac{\theta_r - \theta_o}{\theta_{min} - \theta_o}\right]$	3 *(15, level 2)
level 0	$T\frac{d\theta_i}{dt} = H_G - K(\theta_i - \theta_o)$	4 *(4, level 0)
4→5	$t_{1b} - t_{1a} = \frac{1}{A} \ell n \left[\frac{B - A\theta_{min}}{B - A\theta_c}\right]$	5 *(11, level 2)
4→6	$t_2 - t_{1a} = \frac{1}{A} \ell n \left[\frac{B - A\theta_{min}}{B - A\theta_r}\right]$	6
3 & 6→7	Solve 3 & 6 iteratively for t_{1a}	7

3 & 6→8	Solve 3 & 6 iteratively for θ_{min}	8
5 & 7 & 8→9	Solve for t_{1b}	9
2 & 7 & 9→10	Difference in costs in terms of θ_c	10 *(23, level 7)

*Figures in brackets refer to original modelling approach (See Figure 35, Chapter 5).

Relationships level graph



APPENDIX 1B

Baby's milk bottle problem (M.Sc Math. Ed. group 4)

Feature list in order of occurrence

Feature	Order of occurrence
Range of temperature required	A
Temperature of bottle from 'fridge	B
Shape and material of saucepan	C
Material of bottle	D
Specific heats of milk (s_m) and of water (s_w)	E
Consider milk only in saucepan	F
Relationship 1	G
Relationship 2	H
So for a fixed m_m (mass of milk) there is a fixed time (for heating)	I
Saucepan has also to be heated but remains constant throughout problem	J
Relationship 3	K
Relationship 4	L
Heat provided by stove = heat needed for heating water plus heat lost to outside	M
Relationship 5	N
Relationship 6	O
Relationship 7	P
What areas to include in heat loss calculation?	Q
Relationship 8	R
Relationship 9	S
Relationship 10	T
Relationship 11	U
Relationship 12	V

Relationship list

(numbering refers to relationship numbers above)

Relationship		No.
level 0	$\frac{\text{heat}}{\text{time}} = \text{mass} \times \text{sp.ht} \times \frac{\text{temp}}{\text{time}}$	1
level 0	$Q = m_m s_m \frac{d\theta}{dt}$ (milk only in saucepan) (rate of heat input = mass × sp. ht. × rate of temperature rise)	2
level 0	$Q = (m_w s_w + m_m s_m) \frac{d\theta}{dt}$ (water surrounding bottle with milk)	3
3→4	$\frac{d\theta}{dt} = \frac{Q}{m_w s_w + m_m s_m}$ as $m_w \to 0$	4

Heat loss considerations:

level 0	mass of liquid $= [\pi R^2 H + \pi r^2 (h-H)]\rho$ $= \lambda H + \mu$	5
level 0	Newton's law of cooling: rate of heat loss $= k(\theta - \theta_a)$ depends on area exposed to air temp. of air	6

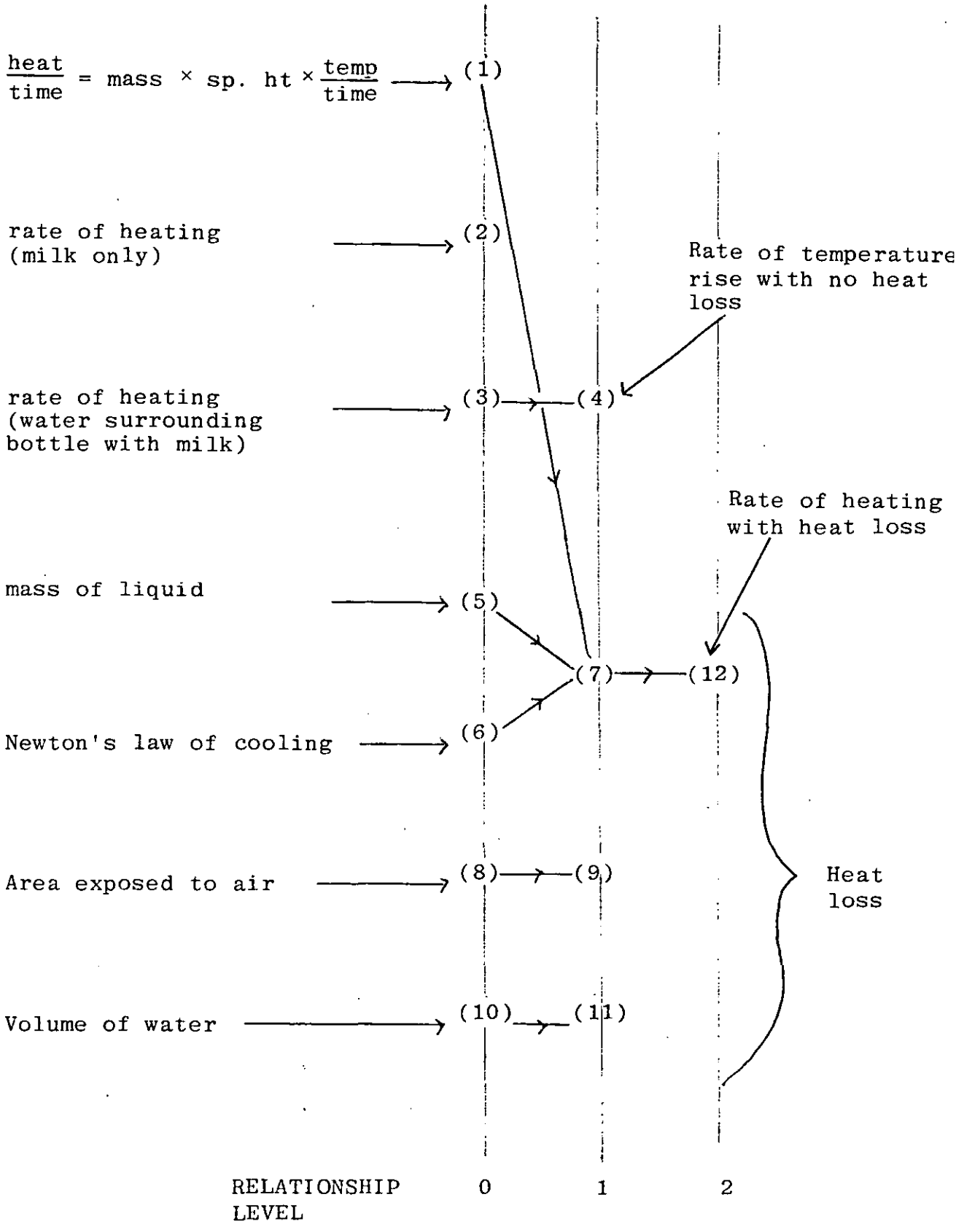
1 & 5 & 6→7	$Q = s(\lambda H + \mu)\frac{d\theta}{dt} + k(\theta-\theta_a)$	7
level 0	Area exposed to air (A) $= 2\pi RH + \pi R^2 + 2\pi r(h-H)$	8
8→9	$\frac{dA}{dH} = 2\pi(R-r)$	9
level 0	Volume of water $= (\pi R^2 - \pi r^2)H$	10
10→11	$\frac{dV}{dH} = \pi(R^2 - r^2)$	11
7→12	$Q' - k\theta = Le^{-kt}$ $(Q' = Q + k\theta_a)$	12

Concept matrix

Specificity level

Complexity Level	A	I	G
L	(B)(E) (H2)(K3) (L4)(N5)(06)(P7) (R8)(S9)(T10)(U11) (V12)	(A)(G1)(I)(Q)	
M	(M)	(C)(D)(F)(J)	
H			

Relationship level graph



APPENDIX 1C

Minimisation of sound distortion in a record player

(B.Sc 2 Appd. Physics, group 1)

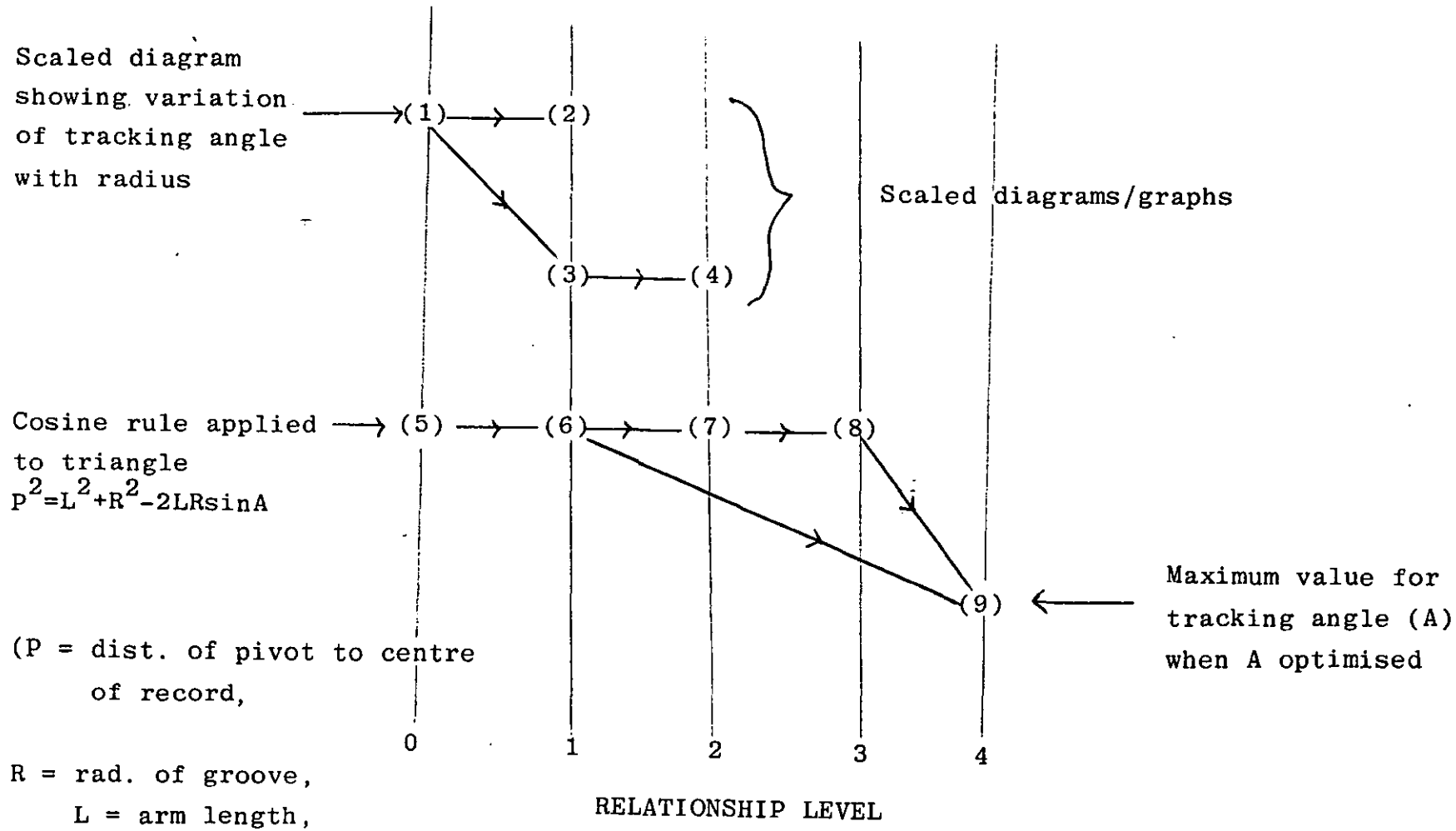
Specificity level

Complexity level	A	I	G
L	(F)(G)(I)((J1))(K2)(L3)(M4)((N5))(O6)(P7)(Q8)(R9)	(E)	
M		(B)	(D)
H		(A)(H)	(C)

(( )) denotes level O relationships

Concept matrix

Relationship level graph



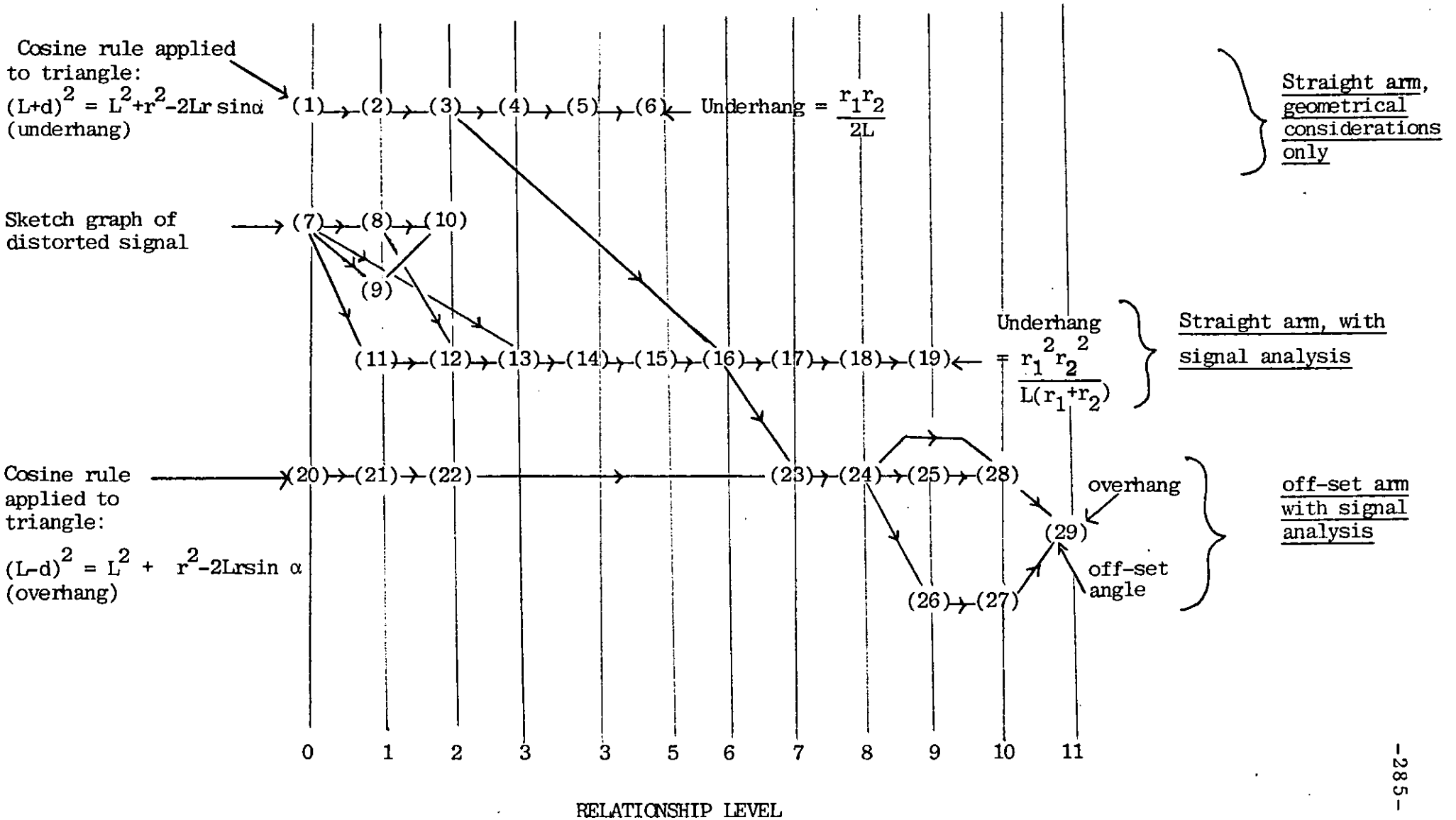
(P = dist. of pivot to centre of record,

R = rad. of groove,

L = arm length,

A = tracking angle).

Minimisation of sound distortion in a record player (Author's 'polished' modelling approach)



APPENDIX 1D

APPENDIX 1E

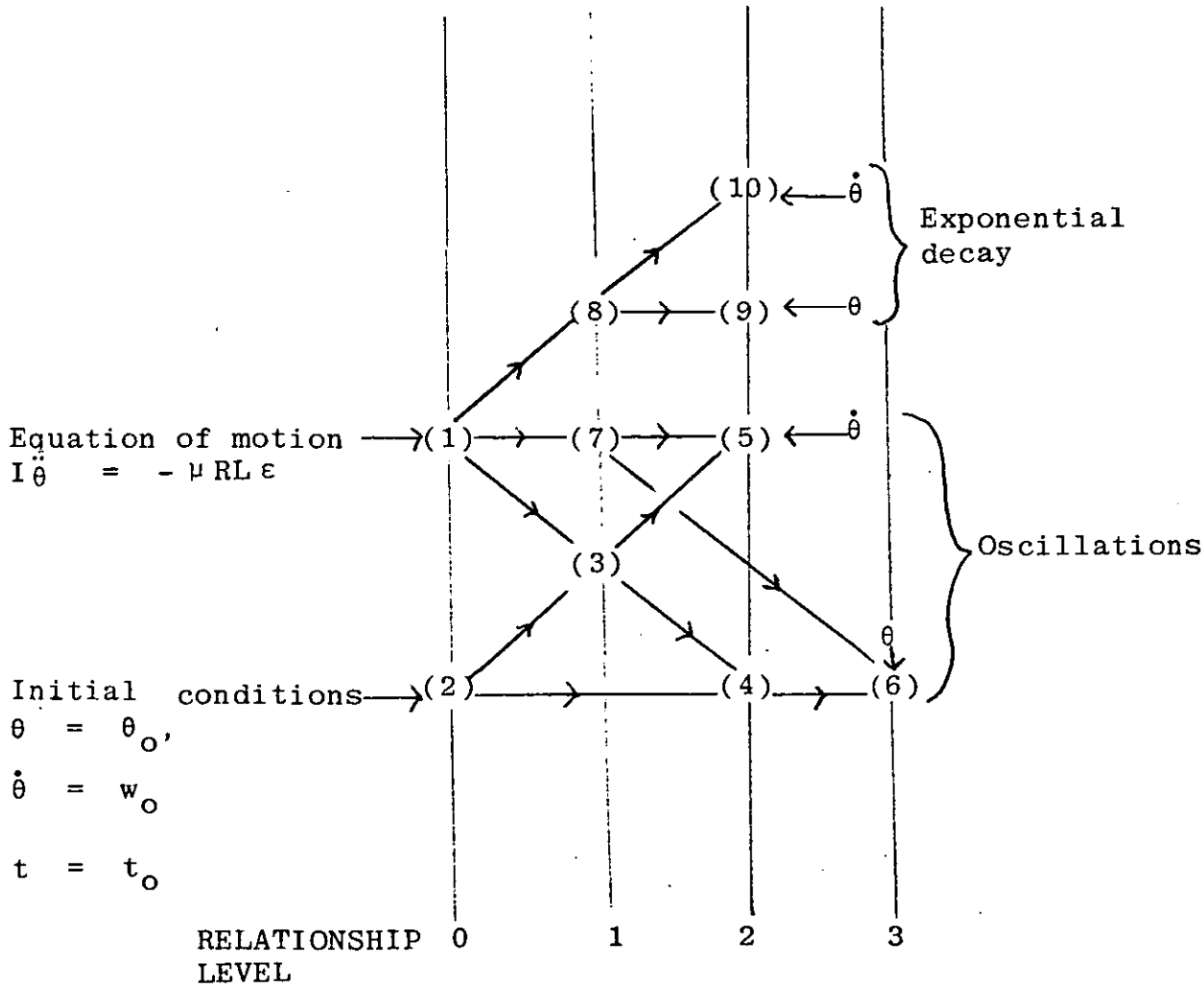
Speed-wobble in motorcycles (author's 'polished' approach)

Relationship list in order of occurrence

Relationship		No.
level 0	$I\ddot{\theta} = -\mu RL\varepsilon$ where $\varepsilon = \text{sign}\,(L\dot{\theta} + v\sin\theta)$ (see section 5.4. Chapter 5)	1
level 0	initial conditions: $\theta = \theta_o,\ \dot{\theta} = w_o,\ t = t_o$	2
1 & 2→3	$\dot{\theta} = -\frac{\mu RL\varepsilon}{I}(t - t_o) + w_o$	3
2 & 3→4	$\theta = -\frac{\mu RL\varepsilon}{2I}(t-t_o)^2 + w_o(t-t_o) + \theta_o$	4
3 & 7→5	$\dot{\theta} = -\mu RL\varepsilon_r(t-t_r)^2 + w_r$	5
4 & 7→6	$\theta = -\frac{\mu RL}{2I}\varepsilon_r(t-t_r)^2 + w_r(t-t_r) + \theta_r$	6
1→7	$\varepsilon_{r+1} = -\varepsilon_r = -\text{sign}\,(Lw_r + v\theta_r)$	7
1→8	$L\dot{\theta} + v\theta = 0$ (no side-slip condition leading to exponential decay)	8

$$\underline{8 \to 9} \qquad \theta = -K \exp\left[-\frac{v}{L}(t-t_s)\right] \qquad 9$$

$$\underline{8 \to 10} \qquad \dot{\theta} = K \frac{v}{L} \exp\left[-\frac{v}{L}(t-t_s)\right] \qquad 10$$



APPENDIX 2A

MSc Math. Ed.

Year 2, Final examination papers in Mathematical Modelling

June 1982 and June 1983

Polytechnic of the South Bank
Faculty of Science & Engineering
Department of Mathematical Sciences and Computing

MSc IN MATHEMATICAL EDUCATION 1981/82

Year II Examination for Part-Time students Wednesday 16 June 1982
1.30pm to 4.30 pm

MATHEMATICAL MODELLING (Part seen paper)

Attempt ONE question from Section A (seen)
and TWO questions from Section B (unseen).

The use of electronic pocket calculators is
allowed in the examination.

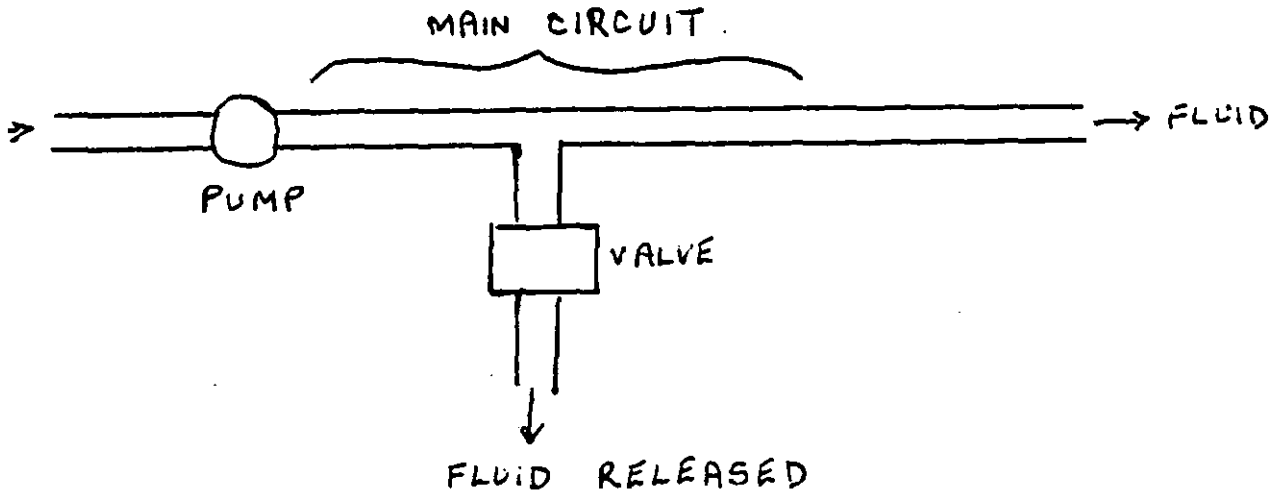
SECTION A (seen questions)

A local authority is concerned to control a dangerous road T-junction, an "accident black spot", by means of traffic lights. The authority is seeking advice on the mode of operation and the phasing of the traffic lights. Consider and discuss the process of model development in order to assist the authority in its decisions. Concentrate on the consultation, data collection, assumptions and validation aspects of the model rather than its computer implementation.

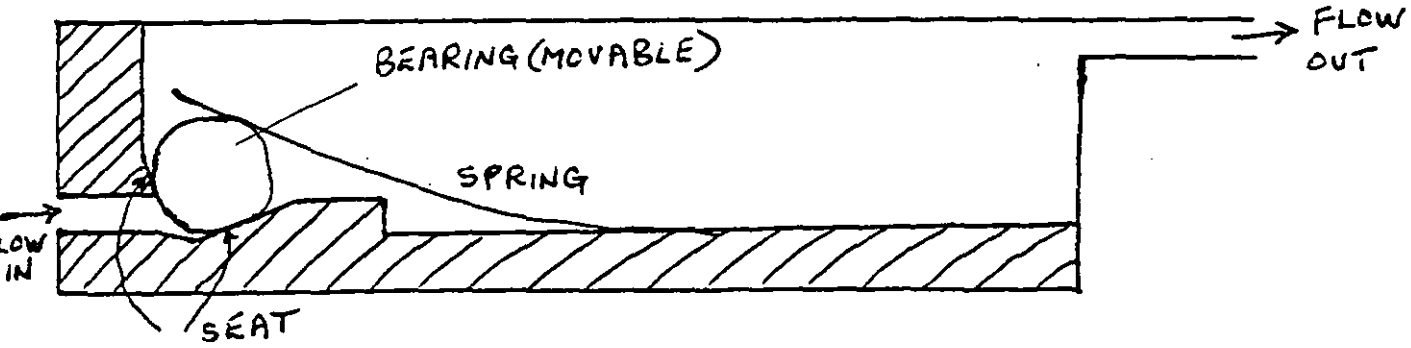
Sugar diabetes is a metabolic disorder. The body cannot burn off excess sugar and so too much sugar builds up in the blood. To diagnose it a patient fasts overnight, reports to hospital the following morning and the concentration of sugar in his blood is measured. He then drinks some glucose very quickly and the concentration is measured every half hour for three or four hours. Model the level G of glucose in the blood and use it to estimate a "period" that indicates diabetes.

continued ...

he sketch below shows part of a main pipe circuit, through which fluid is pumped. he pump should ideally produce a constant flow rate at a constant pressure, but n practice the pressure (and hence the flow rate) varies to some extent. The urpose of the valve is to prevent pressure build-up above the required value; ince the pipe diameters in the main circuit and to the valve are equal, the ressures in all parts of the system may be assumed equal at any instant of time.



A possible valve design is shown below:



Try to identify the features involved and attempt to construct a simple mathematical model which could serve as a design aid for the valve.

SECTION B (Unseen questions)

Mathematical Modelling may be described as an activity covering the following four stages:

Formulation
Solution
Interpretation
Validation.

Discuss the nature of each of these stages, showing how (if at all) each stage is linked with the others.

Are there any additional stages that you would like to include for a fuller description of the modelling process ?

Reference should be made to Case Studies of Mathematical Modelling, Eds, James and McDonald, 1981, as well as other relevant texts and papers.

Discuss critically the article "Mathematical Modelling: A Unifying Theme for Applications of Mathematics" by D N Burghes, Bull.IMA, 17, 8/9 1981.
To what extent does this article:

(a) Illuminate the modelling process, and
(b) Provide material that could be used (possibly modified) for teaching modelling?

Compare and contrast "Mathematics by Modelling" and "Modelling by Mathematics" in pedagogical terms.

continued.....

The treatment of Linear Programming in most textbooks begins by the statement of a "Standard" problem and moves rapidly to a detailed description of its solution by the Simplex method. Discuss, with examples, how some genuine modelling activity can be introduced into the treatment of L.P.

Discuss the Malthusian model of population growth and show with reference to more recent models why it is unsatisfactory as a model of national population growth.

Discuss what Richard Levins means by robust and non-robust theorems. Would you consider M.May's discussion on the relationships among various types of population models to be relevant?

Polytechnic of the South Bank
Faculty of Science & Engineering
Department of Mathematical Sciences and Computing

MSc DEGREE IN MATHEMATICAL EDUCATION 1982-83

Year II Final Examination for Part-Time Students Monday 13 June 1983
1.30 - 4.30 pm.

MATHEMATICAL MODELLING (PART SEEN PAPER)

Attempt ONE question from Section A (seen)
and TWO questions from Section B (unseen)

The use of electronic pocket calculators is
allowed in the examination.

SECTION A (seen)

An Urban District Council has many winding country roads which are only wide enough for a single vehicle. They are investigating a policy of providing "passing places" whereby if two vehicles approach each other one can wait in the passing place until the other passes. Passing places are difficult and costly to arrange since they require negotiation with and payment to landowners.

Develop, as fully as you can, a mathematical model to assist the council with formulating a policy.

Modern office blocks, particularly of the high-rise type, have large glazed areas on the outside to permit entry of as much natural light as possible. By concentrating on the forces involved on an individual glass unit or pane, try to identify some key design features. Is there an optimum pane size, and if so, does double glazing affect this? In your development, consider simple models and make clear any assumptions you feel are necessary.

contd...

contd

In the human bloodstream potassium ions (K^+) are constantly moving into and out of the red blood cells; ie the surfaces of the red blood cells are permeable to K^+ ions. Ions move from the plasma into the red cells at a certain rate, while other ions within the cells move out into the plasma at a certain rate. It is required to determine these two rates (ie of the permeability of the cells' surfaces to K^+ ions in both directions). A technique to achieve this works as follows:
A fixed quantity S of radioactive K^{42+} ions is introduced into the blood. Initially, all these ions are in the plasma. The amount P(t) remaining in the plasma at various subsequent times is determined by taking blood plasma samples and measuring the radioactivity caused by the presence of the K^{42+} isotope ions.

Establish a mathematical model that will enable the required permeabilities to be determined from the raw data collected. Be completely explicit about what assumptions you make. The following data might be of help to you:

t(min)	0	500	1000	1500	2000	2500	3000	3500	4000	4500	5000
P(t)(mg)	5.00	2.96	2.01	1.49	1.14	1.01	0.97	0.92	0.87	0.85	0.85

contd...

SECTION B

Discuss critically the activities involved in the formulation stage of mathematical modelling. To what extent is it possible to categorize identified features? Illustrate points made by choosing relevant examples.

Fig I

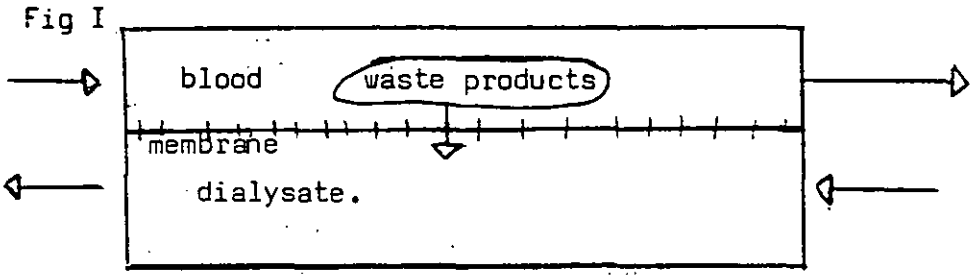


Fig I shows a schematic diagram of a dialyser. Make a mathematical model of the action of this machine and obtain an expression for or relating to the clearance. State clearly any assumptions you make.

Answer parts (a) and (b) below paying particular attention to pedagogical aspects.

(a) Define a situation for which both an analytical and a simulation model can be devised outlining briefly the stages that are involved in the modelling process in each case.

(b) Use your example in (a), among others, to compare and contrast analytical and simulation modelling.

To what extent has the development of modelling in the Life Sciences been influenced by the original discipline (if not a Life Science) of its major contributors. You may if you wish refer specifically to population dynamics.

Make out a case for teaching mathematical modelling, indicating clearly the level and background of the students involved. Refer to relevant articles as far as possible.

APPENDIX 2B

MSc Math. Ed.

Mathematical Modelling Course-work Titles

1980

Two Secretaries, Five Solicitors (office organisation)

Rent or Buy Television

Traffic Flow at Roundabouts

Parking a Car

Petrol Purchase - Company Car (Private versus Business usage)

Lottery Tickets

Travel Time to School

Rocket-Satellite System

Size and Position of Advertisement Signs

Vehicle Braking

Costs of Journeys to Work

Discounts on Sales of Goods

Squash Service

Costing in Book Publication

Hire or Buy a Car

Petrol Filling Station Trade - Monte-Carlo methods

Water Wheel and Impulse Turbine

1983

Maps (Size and scale to cover a given country)

The Tennis Service

Some Aspects of Fielding a Cricket Ball

Street Lighting

Heating and Heat Loss for a Domestic Immersion Heater

Clothes Budget

Investigations into the Problems of Screen Display for a Computerised Flight Simulator

The Shower Problem (flow of hot and cold water)

Recreational Carrying Capacity

The Dividing Society (Investment in Friendly Society)

APPENDIX 2C

MSc Math. Ed. 1983

Assessment Comments on Mathematical Modelling Course-Work

1 The Shower Problem (Grade A: 75%)

Presentation - diagrams, etc, exemplary

Pp 1-34 Identical in both Teacher's and Pupil's texts, so I will refer to page Nos. in Teacher's text throughout. (By the way, self-assessment question starts on p.31 and not 39 in Pupil Text)

Pp 7-8 I take it that you assume your 2nd yr GCE A/L students know little or no physics. By the 'method of mixtures' your 3.1.1 would be:

(Temperature of water from shower) : $T = \dfrac{\dot{m}_h T_h + \dot{m}_c T_c}{\dot{m}_h + \dot{m}_c}$; $\dot{m}_h$, $\dot{m}_c$, mass flow-rates of hot and cold water respectively

From your diagram on p 8, we could have for velocity v:

$$\tfrac{1}{2}v^2 = gd \qquad \text{(energy conservation)}$$

$$\therefore \quad \dot{m}_h = \rho \times \text{area} \times v_h = \rho A_h \sqrt{2gd_h} \quad \text{for hot water.}$$

Similarly for cold.

So $T = (A_h\sqrt{2gd_h} + \ldots)/(\ldots)$ and thus T depends on $\sqrt{d}$ and not d. (Assumes uniform flow).

Interesting to see that you start considering flow-rates for the first time on p 15.

p 11 My only experience of temperature surge is when my wife uses hot water in the kitchen and I freeze in the shower! This seems more understandable somehow than your problem.

p 16 Flow-rates at shower head - is there a 'bottle-neck' somewhere?

p 17, Ex 11 & p 27 } But your measurements show total flow-rate is less than sum of separate flow-rates. So what's wrong?

p 17 Ex 12 and solution on p 27. I don't understand.

3.19 Expression for flow-rate of cold water into shower head according to 3.18?

p 18 It would be useful to remind the reader where you get these numerical quantities from.

p 31 Self-assessment question. Good, but taken after the exercises (no doubt your intention).

General Comments

A splendidly presented piece of work which highlights several features of the modelling activity. Nice build-up of ideas and exercises interspersed in development (with solutions). I agree that the flows in the shower head are not satisfactorily explained physically (I also found your development a little confusing in places). Generally, a first class effort.

2 Heating and Heat Loss for a Domestic Immersion Heater (Grade B - : 62%)

I dare not think how long this project took you to complete (100 hours?). Very ambitious piece of work, to put it mildly.

p 3 "... aimed at students in the last year of the sixth form..."

"... particularly appropriate for engineering, ..., students in further education...".

I suppose that, given great enthusiasm, a very bright student on an honours degree might approximate what you have done in something like one term's work.

Pp 5-11 Models 1, 2 — Good. There is plenty of material here for the sixth former/undergraduate. You then get rapidly very complicated.

Pp 11-48 — It would have been helpful to have seen a table of your model(s) development, including main conclusions. As it stands, your report is jolly hard work to read - masses of detail, masses of mathematics. Morris: "enrich gradually".

A few simple sentences, rather than discursive descriptions, would have helped the reader. For example, p 26 could be written:

> The calculations (Model 7) show that for a temperature fall from 65°C to 60°C, cooling time is very large for thick insulation:
>
> | Insulation thickness | Cooling time |
> |---|---|
> | 7.5 cm | 24 hours |
> | 2.5 cm | 8¾ hours |
>
> This assumes no 'draw-off', ie, water not used.
>
> Validation proved difficult for the following reasons:
>
> 1 Difficulty in isolating heater electrically from other apparatus in use
>
> 2 Difficulty in measuring water temperature without disturbing domestic use ('draw-off')
>
> 3 Variability of temperature of surroundings
>
> 4 Thermostat temperature setting inaccurate

Pp 33-44 — Best placed in an appendix (mathematical and arithmetical detail not central to model development)

Pp 51-55 Some very good points made, particularly under the heading of 'General Observations'. However, even Morris (often quoted in the report) would not attempt so much as you have.

General Comments

You have obviously worked extremely hard on this course-work. Enthusiasm shown is astonishing.

However, I think that you have confused modelling for the professional with modelling for the teacher. Even the professional would attempt less than you have; he would concentrate on a narrower range of objectives. The professional team leader would draw together the work of several individuals.

In future, aim for something more modest (like the modelling activities so far introduced on the MSc course). Also, try and present your work with a few sentences, keep detail to a minimum, and use tabular and diagrammatic representation.

You must have learnt an enormous amount from this exercise - you have yourself pointed out how difficult it is to validate a model. Average values are usually all one has to play with - don't despair, they still provide valuable insights.

Finally, I would like to commend you most highly for taking such an earnest interest in this assignment.

3 Recreational Carrying Capacity (Grade E: 35%)

p 2 General aims. "... for both students of mathematics and town and country planning." Level of students in both cases? Undergraduate?

p 6 Diagrammatic map. Of what site? Scale? Imagined typical layout of park?

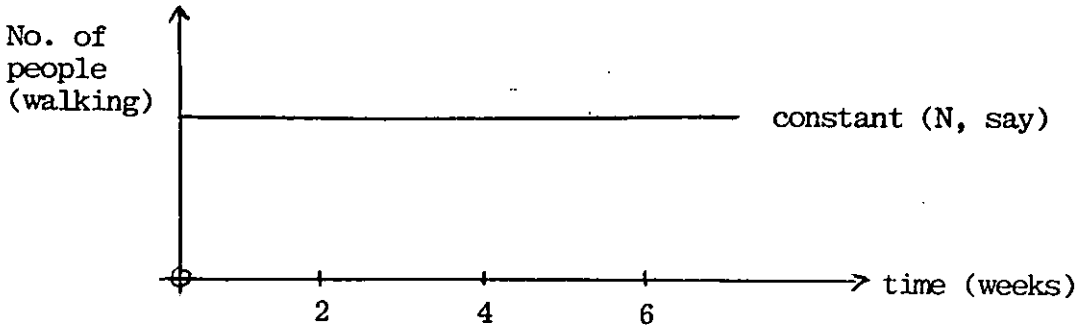
p 7 Weighting numbers according to percentages not clear, eg, square C : 11% of picnicers = 38 - 48? Should be 11 surely?

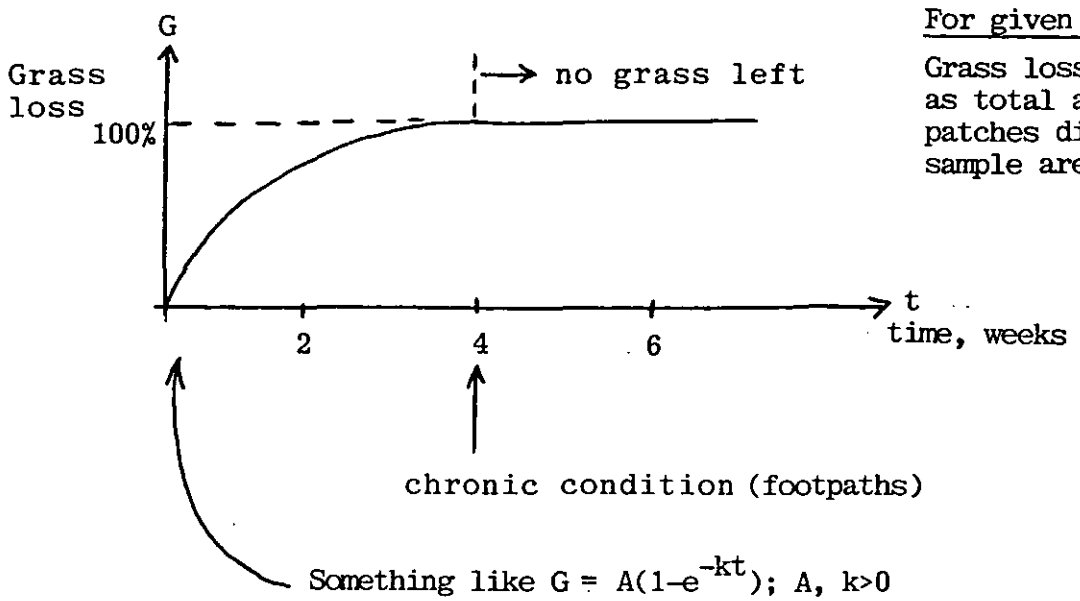
p 12 Last para. "There are therefore 55 spaces in the car park, ..." Where does the '55' come from?

p 14 (and elsewhere) You refer to the term 'model'. Mathematical model? Mathematical models have mathematical solutions of some kind (eg, solution of equations - analytically or graphically). There is no mathematics yet - you still seem to be formulating the problem at the initial stages (identifying features, but no mathematical relationships yet).

A modelling exercise at this juncture could be to try and predict the rate of a particular type of degradation with visitor numbers, eg, wearing down of grass to earth? You would need some data though (ask a friendly gardener?). How long for grass to re-grow? Factors involved: type of grass, wet/dry season, slope (people slip), ...

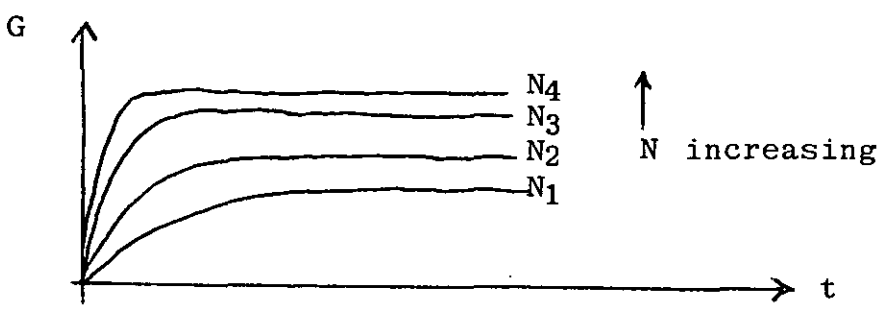
For given grass, perhaps something like the following is reasonable:





For given N

Grass loss measured as total area of earth patches divided by sample area.



Choose a simple more specific problem to model

p 26 Self-assessment question for your students. Terribly broad. How to model?

General Comments

You have, in effect, produced a report on the difficulties of controlling and maintaining large recreational areas. You have highlighted the difficulties of management, covering a wide range of features. As you say in your report, a large simulation program (as carried out in the USA for example) is one attempt at handling such a large and complex problem. You

are obviously very interested in this field of investigation, but unfortunately you have not been able to identify specific problems which you might have modelled mathematically in the time you had available. My suggestion of grass loss is the sort of 'simple' model you might have tried (other 'simple' examples have been presented on the MSc).

In future, try something very specific. Given more time (in another context) you could obtain data for validation.