Pathwise anticipating random periodic solutions of SDEs and SPDEs with linear multiplicative noise

Pathwise Anticipating Random Periodic Solutions of SDEs and SPDEs with Linear Multiplicative Noise

by

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Abstract

In this thesis, we study the existence of pathwise random periodic solutions to both the semilinear stochastic differential equations with linear multiplicative noise and the semilinear stochastic partial differential equations with linear multiplicative noise in a Hilbert space. We identify them as the solutions of coupled forward-backward infinite horizon stochastic integral equations in general cases, and then perform the argument of the relative compactness of Wiener-Sobolev spaces in $C([0,T], L^2(\Omega, \mathbb{R}^d))$ or $C([0,T], L^2(\Omega \times \mathcal{O}))$ and Schauder's fixed point theorem to show the existence of a solution of the coupled stochastic forward-backward infinite horizon integral equations.

Keywords: random periodic solution, random dynamical system, semilinear stochastic partial differential equation, linear multiplicative noises, coupling method, relative compactness, Malliavin derivative, coupled forward-backward infinite horizon stochastic integral equations.

To my mother and father with deep respect and love.

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Chapter 1

Introduction

Periodicity is widely exhibited in a large number of natural phenomena like oscillations, waves, or even lying behind many complicated ensembles such as biological, and economic systems. Only until 19th century, after Henri Poincaré's work in 1880s (Poincaré, 1881, 1882, 1885, 1886), periodic solutions gradually aroused attention and discussion and have occupied an important position in the theory of the deterministic dynamical system ever since then. The existence and construction of periodic solutions is a challenging problem in the study of dynamical systems though they are relative simple trajectories themselves. For example, periodic solutions of partial differential equations of parabolic type have been studied by a number of authors, Hess (Hess, 1991), Vejvoda (Vejvoda, 1982), Fife (Fife, 1964), Lieberman (Lieberman, 1999, 2001), to name but a few. Even today, this topic is still one of the most interesting nonlinear problems in dynamical systems, and as the development of stochastic analysis, periodic behaviours are often found to be subject to random perturbation or under the influence of noises. However, understanding the complexities of stochastic systems are far from clear even for stationary solutions. The concept of stationary solutions is the stochastic counter part of fixed points to deterministic dynamical systems. A fixed point is the simplest equilibrium and large time limiting set of a deterministic dynamical system. A periodic solution is a more complicated limiting set. From periodic solutions, more complicated solutions can be built in. Since the theory of the existence of the solution of the stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) become better understood (Prato and Zabczyk, 1992)

(Prévôt and Röckner, 2007), we need to study more detailed question about the behaviour of solutions of SDEs and SPDEs. Mathematicians have been very much interested in the study of the existence of stationary solutions of SDEs and SPDEs, and invariant manifolds near stationary solutions. For results about SPDEs, see Sinai (Sinai, 1991, 1996), Mattingly (Mattingly, 1999), E, Khanin, Mazel and Sinai (E et al., 2000), Caraballo, Kloeden and Schmalfuss (Caraballo et al., 2004), Liu and Zhao (Liu and Zhao, 2009), Zhang and Zhao (Zhang and Zhao, 2007, 2010), Duan, Lu and Schmalfuss (Duan et al., 2003, 2004), Mohammed, Zhang and Zhao (Mohammed et al., 2008), and Lian and Lu (Lian and Lu, 2010), though there are still many problems that need to be understood. In literature, there were only a few works on the periodicity of stochastic systems. For linear stochastic differential equations with periodic coefficients in the sense of distribution, see Chojnowska-Michalik (Chojnowska-Michalik, 1988, 1990), and for one-dimensional random mappings, see Klünger (Klünger, 2001). Now I would like to mention the work by Zhao and Zheng (Zhao and Zheng, 2009), in which the definition of pathwise random periodic solutions for C^1-cocycles was firstly introduced and studied. However, for random dynamical systems, it is very difficult, if not impossible, to define a useful Poincaré map and to find its fixed point as the trajectory does not return to the same set with certainty. In 2011, Feng, Zhao and Zhou (Feng et al., 2011) gave the definition of pathwise random periodic solutions for semiflows as follows:

Consider a semi-flow u: $\Delta \times H \times \Omega \to H$, where H is a separable Banach space.

Definition 1.0.1 (Random Periodic Solutions for Semiflows)**.** *If there exist an $\mathcal{F}$-measurable map $\varphi : \mathbb{R} \times \Omega \to H$ and a costant τ such that*

$$\begin{cases} u(t,s,\varphi(s,\omega),\omega) = \varphi(t,\omega), & \forall s \le t \\ \varphi(s+\tau,\omega) = \varphi(s,\theta_\tau\omega), & \forall s \in \mathbb{R} \end{cases} \tag{1.1}$$

for any $\omega \in \Omega$, where $\Delta := \{(t,s)|t \ge s, t, s \in \mathbb{R}\}$, then φ is called ***a random periodic solution of period τ of the semi-flow u***

They showed the existence of random periodic solutions to the semilinear τ-periodic SDEs with additive noise, i.e.,

$$\begin{cases} du(t) = Au(t)\,dt + F(t,u(t))\,dt + B_0(t)dW(t), & t \ge s \\ u(s) = x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where F, A and B_0 satisfy certain conditions, with a new analytical method for coupled infinite horizon forward-backward integral equations instead of the traditional geometric method of establishing the Poincaré mapping.

Not long after, Feng and Zhao (Feng and Zhao, 2012) studied the existence of pathwise random periodic solutions to semilinear τ-periodic SPDEs with additive noise,

$$\begin{cases} du(t,x) = \mathcal{L}u(t,x)\,dt + F(t,u(t,x))\,dt + \sum_{k=1}^{\infty}\sigma_k(t)\phi_k(x)dW^k(t), \quad t \geq s, \\ u(s) = \psi \in L^2(D), \\ u(t)|_{\partial D} = 0, \end{cases} \tag{1.3}$$

where $\mathcal{L}$ is the second order differential operator with Dirichlet boundary condition on D,

$$\mathcal{L}u = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u. \tag{1.4}$$

In this thesis we consider the pathwise random periodic solutions to semiliar τ-periodic SDEs and SPDEs with linear multiplicative noise.

The problem is much more difficult than one would initially expect. Firstly as one will see, under the hyperbolicity assumption of A, the random periodic solution depends on the whole path of the Brownian motion, so it is not a process adapted to the filtration generated by the Brownian motion. In the case that the eigenvalues of A are all negative or positive, however, the random periodic solution only depends on either the future path or the past path of the Brownian motion. To overcome this difficulty, we use the random evolution operator generated by the linear part of the stochastic differential equations and stochastic variations of constant formula method.

The structure of the thesis is as follows: first, in Chapter 2, besides the fundamental knowledge on random dynamic systems, the Multiplicative Ergodic Theorems and Malliavin calculus, I briefly summarise the standard relative compactness criteria and some extension, say, relative compactness criteria in Wiener-Sobolev spaces, which includes results in $L^2([0,T]\times\Omega\times\mathcal{O})$ (Bally and Saussereau, 2004) and $C([0,T],L^2(\Omega\times\mathcal{O}))$ (Feng et al., 2011). Also a modified version of Schauder's fixed point theorem is presented. They are the essential tools throughout the entire work.

In Chapter 3, we discuss the existence of pathwise random periodic solutions to the following semilinear τ-periodic SDEs with linear multiplicative noises in finite-dimensional Euclidean space $\mathbb{R}^d$,

$$\begin{cases} du(t) = Au(t)\,dt + F(t,u(t))\,dt + \sum_{k=1}^{M} B_k u(t) \circ dW_t^k, & t \geq s \\ u(s) = \xi \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

where $\{B_k,\ 1 \leq k \leq M\}$ is in $\mathcal{L}(\mathbb{R}^d)$, and $W(t)$, $t \in \mathbb{R}$, is an M-dimensional Brownian motion under the filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in\mathbb{R}}, \mathbb{P})$. Assume A is hyperbolic, i.e., all its eigenvalues have nonzero real part. Then $\mathbb{R}^d$ can be decomposed into

$$\mathbb{R}^d = E^- \oplus E^+ \tag{1.6}$$

where

$$E^- = span\{v : v \text{ is an eigenvector of an eigenvalue } \lambda \text{ of } A \text{ with } Re(\lambda) < 0\}$$

and

$$E^+ = span\{v : v \text{ is an eigenvector of an eigenvalue } \lambda \text{ of } A \text{ with } Re(\lambda) > 0\}.$$

Let P^- be the projection of $\mathbb{R}^d$ to E^- along E^+ and P^+ be the projection of $\mathbb{R}^d$ to E^+ along E^-. Following the idea of Feng, Zhao and Zhou (Feng et al., 2011) and Feng and Zhao (Feng and Zhao, 2012), the random periodic solution of SDE (1.5) should be a solution of the infinite horizon stochastic integral equation

$$\begin{aligned} Y(t) &= \int_{-\infty}^{t} T_{t-s}P^-F(s,Y(s))ds - \int_{t}^{+\infty} T_{t-s}P^+F(s,Y(s))ds \\ &\quad + \int_{-\infty}^{t} T_{t-s}P^-BY(s) \circ dW(s) - \int_{t}^{+\infty} T_{t-s}P^+BY(s) \circ dW(s), \end{aligned} \tag{1.7}$$

where $T_t := e^{At}$ is a hyperbolic linear flow induced by A. In H.Amann (Amann, 1990), the deterministic version of the infinite horizon integral equations were shown to be periodic solutions to affine ordinary differential equations. Note the first difficulty to solve (1.7) is that Y depends on the past and future of the Brownian motion, therefore the stochastic integral is not integral with adapted integrand. We have to deal with anticipating stochastic integrals. However, essential difficulty to study (1.7) directly arises when we study the Malliavin derivative of Y, from the anticipating stochastic integral.

Note that (1.7) can be rewritten as

$$Y(t,\omega) = T_{t-s}Y(s,\omega) + \int_s^t T_{t-\hat{s}}F(\hat{s},Y(\hat{s}))d\hat{s} + \int_s^t T_{t-\hat{s}}BY(\hat{s}) \circ dW(\hat{s}) \tag{1.8}$$

for any $t \geq s$. This is in connection with the stochastic differential equations with anticipating initial condition studied by Nualart (Nualart, 2000). Note here the anticipating initial condition is not given but to be found in our problem, while it was assumed known in Nualart's work.

To overcome the difficulty to deal with (1.7) directly, we use the linear random evolution operator $\Phi : \mathbb{R} \times \Omega \to \mathcal{L}(\mathbb{R}^d)$ defined by

$$\begin{cases} d\Phi(t) = A\Phi(t)\,dt + \sum_{k=1}^{M} B_k\Phi(t) \circ dW^k(t) \\ \Phi(0) = I \in \mathcal{L}(\mathbb{R}^d). \end{cases} \tag{1.9}$$

Assume $\theta : \mathbb{R} \times \Omega \to \Omega$ is defined as $\theta_s W_t = W_{t+s} - W_s$. Then the solution of equation (1.5) is given by

$$u(t, s, \xi, \omega) = \Phi(t - s, \theta_s\omega)\xi + \int_s^t \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)F(\hat{s}, u(\hat{s}, s, \xi, \omega))d\hat{s}. \tag{1.10}$$

In the further development, the celebrated Osledets Theorem (c.f. Arnold (1998)) plays a key role. Similar to (1.6), $\mathbb{R}^d$ can also be decomposed as

$$\mathbb{R}^d = E_d \oplus E_{d-1} \oplus \cdots \oplus E_{m+1} \oplus \cdots E_1, \qquad a.s.$$

and

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \log |\Phi(t, \omega)x|, \text{ for } x \in E_i \setminus \{0\}. \tag{1.11}$$

We can order $\{\mu_i\}_{i=1,\cdots,d}$ such that $\mu_d \leq \mu_{d-1} \leq \cdots \leq \mu_m \leq \cdots \leq \mu_1$, and assume there exists $m \in \{0, 1, 2, \cdots, d\}$ such that $\mu_{m+1} < 0 < \mu_m$. Here we use the convention $\mu_{d+1} = -\infty$ if $m = d$, and $\mu_0 = \infty$ when $m = 0$. With this we have

$$\mathbb{R}^d = E^- \oplus E^+, \tag{1.12}$$

where $E^- = E_d \oplus \cdots \oplus E_{m+1}$, and $E^+ = E_m \oplus \cdots \oplus E_1$. Let $P^\pm : \mathbb{R}^d \to E^\pm$ the projection along $E^\mp$. Then

$$\begin{cases} \|\Phi(t, \theta_s\omega)P^+\| \leq C(\theta_s\omega)e^{\frac{1}{2}\mu_m t}, & \text{when } t \leq 0, \\ \|\Phi(t, \theta_s\omega)P^-\| \leq C(\theta_s\omega)e^{\frac{1}{2}\mu_{m+1} t}, & \text{when } t \geq 0. \end{cases} \tag{1.13}$$

Here $C(\cdot)$ is a random variable tempered from above.

The idea is to consider the following stochastic integral equations of infinite horizon

$$
\begin{aligned}
Y(t,\omega) &= \int_{-\infty}^{t} \Phi(t-s,\theta_s\omega)P^{-}(\theta_s\omega)F(s,Y(s,\omega))ds \\
&\quad - \int_{t}^{+\infty} \Phi(t-s,\theta_s\omega)P^{+}(\theta_s\omega)F(s,Y(s,\omega))ds. \qquad (1.14)
\end{aligned}
$$

Similar to (1.8), one can see that if we can solve equation (1.14), then Y satisfies the following:

$$
Y(t,\omega) = \Phi(t-s,\theta_s\omega)Y(s,\omega) + \int_{s}^{t} \Phi(t-\hat{s},\theta_{\hat{s}}\omega)F(\hat{s},Y(\hat{s},\omega))d\hat{s} \qquad (1.15)
$$

for any $t \geq s$. Now we can compare the equation (1.15) and (1.10) and get

$$
Y(t,\omega) = u(t,s,Y(s,\omega),\omega).
$$

Here $u(t,s,Y(s,\omega),\omega)$ is the solution of equation (1.10) with initial condition $Y(s,\omega)$. But equation (1.15) and equation (1.5) are equivalent, at least for deterministic initial condition ξ. Then by Nualart's substitution theorem, we can see that $Y(t,\omega) = u(t,s,Y(s,\omega),\omega)$ is the solution of (1.5) with anticipating initial condition $Y(s,\omega)$.

The main challenge is to solve the infinite horizon integral equation (1.14). Though the equation seems to be represented for each ω, but it is not enough to solve this equation for each ω. Typically this equation does not guarantee uniqueness. We cannot glue the solutions for each individual ω's together to make the solution measurable for ω due to the lack of the uniqueness.

We will develop a stochastic functional analysis method to solve this equation in the lifted space

$$
C_{\tau}^{\Lambda}(\mathbb{R}, L^2(\Omega,\mathbb{R}^d)) := \{f \in C^{\Lambda}(\mathbb{R}, L^2(\Omega,\mathbb{R}^d)) : \text{ for any } \ t\in\mathbb{R},\ f(t+\tau,\omega) = f(t,\theta_\tau\omega)\},
$$

where the norm of the metric space $C^{\Lambda}(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$ is given as follows,

$$
\|f\|_{\Lambda} := \sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}\|f(t,\cdot)\|_{L^2(\Omega,\mathbb{R}^d)}.
$$

Due to the infinity of the time horizon, the Banach fixed point theorem becomes powerless to this problem. We will use Schauder's fixed point theorem to find a fixed point of an appropriate map $\mathcal{M} : C_{\tau}^{\Lambda}(\mathbb{R}, L^2(\Omega,\mathbb{R}^d)) \to C_{\tau}^{\Lambda}(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$. This is a powerful fixed point theorem which allows us to find a fixed point in a quite general situation. Note when we view a map $\hat{\mathcal{M}} : f(0,\omega) \to f(\tau,\omega)$, it is impossible to find a real fixed point in the classical sense. But we lift the solution to a space of a random field $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$, and then we can study the fixed point of a map $\mathcal{M}$ in the classical sense which gives a random periodic solution. The second difficulty that immediately arises is that the long time behaviours of $\Phi(t,\theta_s\omega)P^+$ and $\Phi(t,\theta_s\omega)P^-$ in the desired space $L^2(\Omega,\mathbb{R}^d)$ and in probability 1 given by the ergodic theorem are very different. Here we naturally use the classical truncation technique to make truncated random functions $\|\Phi^N(t,\theta_s\omega)P^+\|$ and $\|\Phi^N(t,\theta_s\omega)P^-\|$ bounded. The main step to use a truncation technique is to make $N \to \infty$. Unfortunately we cannot prove the sequence of the solution Y^N of the truncated equation is a Cauchy sequence. We suspect this may not be true. We observe in Chapter 3 that we can define an increasing sequence of subsets $\Omega_1^* \subset \Omega_2^* \subset \cdots$ with limit $\bigcup_{k=1}^\infty \Omega_k^*$ such that $\mathbb{P}(\bigcup_{k=1}^\infty \Omega_k^*) = 1$ and every Ω_k^* is invariant w.r.t. $\theta_{n\tau}$, for any $n \in \mathbb{N}$. Moreover on Ω_k^*, the solution of the truncated equation is the solution of the original equation. That means in fact the truncated technique leads to local solutions in the probability space. With this observation we can measurably glue local solutions together to a global solution which is defined for almost all $\omega \in \Omega$. The global solution is measurable w.r.t. $\mathcal{F}$, and solves the original equation for almost all $\omega \in \Omega$.

Note it is not clear whether or not the local solution actually converges to the global solution.

In Chapter 4, we deal with pathwise random periodic solutions of τ-periodic semi-linear stochastic partial differential equations with linear multiplicative noises in an infinite-dimensional Hilbert space H, i.e.,

$$\begin{cases} du(t,x) = \mathcal{L}u(t,x)\,dt + F(t,u(t,x))\,dt + Bu(t,x)\circ dW(t), & t \geq s \\ u(s) = \psi \in H, \\ u(t)|_{\partial\mathcal{O}} = 0. \end{cases} \tag{1.16}$$

We have the same anticipating difficulty as for the case of SDEs. To overcome the difficulty, we use the linear stochastic evolution operator $\Phi(t,\theta_s\omega)$ introduced in Mohammed et al. (2008) and consider the following random integral directly due to the restriction of the substitution theorems available,

$$\begin{cases} u(t,s,\psi,\omega)(x) = \Phi(t-s,\theta_s\omega)\psi(x) + \int_s^t \Phi(t-\hat{s},\theta_{\hat{s}}\omega)F(\tilde{s},u(\hat{s},s,\psi,\omega))(x)\,d\hat{s}, & t \geq s \\ u(s) = \psi \in H, \\ u(t)|_{\partial\mathcal{O}} = 0. \end{cases}$$

Φ is given in terms of Wiener-Itô chaos expansion of the semigroup generated from the second order differential operator and the noise. With the help of the Wiener-Itô chaos expansion, we obtain a number of useful properties of Φ, especially a more explicit form of Φ, (4.14). From this, we obtain the explicit dichotomous decomposition of $L^2(\mathcal{O})$ and Lyapunov exponents in the multiplicative ergodic theorem. The explicit form of the tempered random variable plays a key role in the analysis to use the Schauder's fixed point theorem, Wiener-Sobolev embedding theorem and localization to find a fixed point in the space $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O})) = \{f \in C(\mathbb{R}, L^2(\Omega\times\mathcal{O})) : f(t+\tau,\omega) = f(t,\theta_\tau\omega)\}$.

Chapter 2

Preliminaries

2.1 Random Dynamical Systems

In this section, basic concepts and classical results in random dynamical systems are recalled for later use, and most contents are based on Arnold (1998). From now on, $\mathbb{T}$ stands for the following (semi)groups:

- groups:
  - $\mathbb{T} = \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$: *two-sided discrete time*,
  - $\mathbb{T} = \mathbb{R}$: *two-sided continuous time*,
- semigroups:
  - $\mathbb{T} = \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ (sometimes $\mathbb{T} = \mathbb{Z}^- := -\mathbb{Z}^+$ or $\mathbb{T} = \mathbb{N} := \{1, 2, \ldots\}$): *one-sided discrete time*,
  - $\mathbb{T} = \mathbb{R}^+$ (sometimes $\mathbb{T} = \mathbb{R}^- := -\mathbb{R}^+$): *one-sided continuous time.*

$\mathbb{T}$ is always endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{T})$.

Definition 2.1.1 (Homomorphism, Isomorphism, Endomorphism)**.** *Let θ be a measurable mapping of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ to $(\Omega_2, \mathcal{F}_2)$. The measure $\theta\mathbb{P}_1$ on $\mathcal{F}_2$ defined by $\theta\mathbb{P}_1(A) := \mathbb{P}_1(\theta^{-1}A)$, $A \in \mathcal{F}_2$, is the image of $\mathbb{P}_1$ with respect to θ.*

A measurable mapping θ of $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ to $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ with $\theta\mathbb{P}_1 = \mathbb{P}_2$ is called ***homomorphism*** *of the corresponding probability spaces. It's called* ***isomorphism*** *if, in addition, it's measurably invertible. A homomorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ to itself, i.e. $\theta\mathbb{P} = \mathbb{P}$, is called an* ***endomorphism****, and $\mathbb{P}$ is said to be invariant with respect to θ.*

Definition 2.1.2 (Measurable Dynamical System)**.** *A family $(\theta(t))_{t\in\mathbb{T}}$ of mappings of $(\Omega, \mathcal{F})$ into itself is called a* ***measurable dynamical system*** *with $\mathbb{T}$ if it meets the following conditions:*

1. *$(\omega, t) \mapsto \theta(t)\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$, $\mathcal{F}$-measurable,*
2. *If $0 \in \mathbb{T}$, $\theta(0) = id_\Omega$=identity on Ω,*
3. *(Semi)flow property: $\theta(s + t) = \theta(s) \circ \theta(t)$ for all $s, t \in \mathbb{T}$.*

Remark 2.1.3.

1. *If $\mathbb{T}$ is a group, 0 will be included. Then we can apply the flow property by putting $t = -s$ with condition 2, which yields $id_\Omega = \theta(0) = \theta(s - s) = \theta(s) \circ \theta(-s)$ for all $s \in \mathbb{T}$. On the other side, $id_\Omega = \theta(-s) \circ \theta(s)$ for all $s \in \mathbb{T}$ due to the symmetry of flow property with respect to s and t. Therefore $\theta(s)$ are measurably invertible with $\theta(s)^{-1} = \theta(-s)$.*
2. *A measurable DS $(\theta(t))_{t\in\mathbb{T}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which each $\theta(t)$ is an endomorphism is called a* **measure preserving** *or* **metric DS** *and denoted by $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$, or $\theta(\cdot)$ or θ for short.*

Definition 2.1.4 (Random Dynamical System)**.** *A* ***measurable random dynamical system*** *on the measurable space $(X, \mathcal{B})$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ with time $\mathbb{T}$ is a mapping*

$$\varphi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

with the following properties:

1. *Measurability: φ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}$, $\mathcal{B}$-measurable.*
2. *Cocycle property: The mappings $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$: $X \to X$ form a cocycle over $\theta(\cdot)$, i.e. they satisfy*

$$\varphi(0, \omega) = id_X \quad for\ all \quad \omega \in \Omega \quad if \quad 0 \in \mathbb{T},$$

$$\varphi(t+s,\omega)=\varphi(t,\theta(s)\omega)\circ\varphi(s,\omega) \quad \textit{for all} \quad s,t\in\mathbb{T}, \quad \omega\in\Omega.$$

Definition 2.1.5 (Continuous RDS)**.** *A **continuous or topological RDS** on the topological space X over the metric DS $(\Omega,\mathcal{F},\mathbb{P},(\theta(t))_{t\in\mathbb{T}})$ is a measurable RDS which also satisfies the following property: For each $\omega\in\Omega$ the mapping*

$$\varphi(\cdot,\omega,\cdot):\mathbb{T}\times X\to X, \quad (t,x)\mapsto\varphi(t,\omega,x),$$

is continuous.

Definition 2.1.6 (Linear RDS)**.** *A continuous RDS on a finite-dimensional vector space is called a **linear RDS**, if $\varphi(t,\omega)\in\mathcal{L}(X)$ for each $t\in\mathbb{T}$, $\omega\in\Omega$, where $\mathcal{L}(X)$ is the space of linear operators of X.*

Theorem 2.1.7. *Suppose $\mathbb{T}$ is a group.*

1. *Let φ be a measurable RDS on a measurable space $(X,\mathcal{B})$ over θ. Then for all $(t,\omega)\in\mathbb{T}\times\Omega$, $\varphi(t,\omega)$ is a bimeasurable bijection of $(X,\mathcal{B})$ and*

$$\varphi(t,w)^{-1}=\varphi(-t,\theta(t)\omega) \quad \textit{for all} \quad (t,\omega)\in\mathbb{T}\times\Omega, \tag{3.3}$$

   or, equivalently,

$$\varphi(-t,w)=\varphi(t,\theta(t)^{-1}\omega)^{-1} \quad \textit{for all} \quad (t,\omega)\in\mathbb{T}\times\Omega. \tag{3.4}$$

   Moreover, the mapping $(t,\omega,x)\mapsto\varphi(t,\omega)^{-1}x$ is measurable.

2. *Let φ be a continuous RDS on a topological space X. Then for all $(t,\omega)\in\mathbb{T}\times\Omega$ we have $\varphi(t,\omega)\in$ Homeo(X). If*

   (a) $\mathbb{T}=\mathbb{Z}$, or

   (b) $\mathbb{T}=\mathbb{R}$ and X is a topological manifold, or

   (c) $\mathbb{T}=\mathbb{R}$ and X is a compact Hausdorff space

   then $(t,x)\mapsto\varphi(t,\omega)^{-1}x$ is continuous for all $\omega\in\Omega$.

Now it is natural to discuss the filtration for two-sided continuous time. We recall some work of Crauel (1991, 1993).

Definition 2.1.8 (Past, Future of an RDS). *Let φ be a measurable RDS on a standard space $(X, \mathcal{B})$ with two-sided time. We call a sub-σ-algebra $\mathcal{F}^- \subset \mathcal{F}$ a **past** of φ if it satisfies for all $t \geq 0$,*

1. *$\varphi(-t, \cdot)$ is $\mathcal{F}^-$-measurable,*
2. *$\theta(-t)^{-1}\mathcal{F}^- \subset \mathcal{F}^-$.*

*The past $\mathcal{F}^-$ is called **exhaustive** if $\mathcal{F}^-_\infty := \sigma\{\theta(t)^{-1}\mathcal{F}^- : t \geq 0\} = \mathcal{F}$.*

*Analogously, $\mathcal{F}^+ \subset \mathcal{F}$ is called **future** of φ if it satisfies for all $t \geq 0$*

1. *$\varphi(t, \cdot)$ is $\mathcal{F}^+$-measurable,*
2. *$\theta(t)^{-1}\mathcal{F}^+ \subset \mathcal{F}^+$.*

*The future $\mathcal{F}^+$ is called **exhaustive** if $\mathcal{F}^+_{-\infty} := \sigma\{\theta(-t)^{-1}\mathcal{F}^+ : t \geq 0\} = \mathcal{F}$.*

Remark 2.1.9. *The smallest possible (but in general not exhaustive) choice for $\mathcal{F}^\pm$ is of course the past*

$$\mathcal{F}^- = \sigma\{\varphi(-t, \cdot)x : t \geq 0, x \in X\}$$

and the future

$$\mathcal{F}^+ = \sigma\{\varphi(t, \cdot)x : t \geq 0, x \in X\}$$

generated by φ.

Definition 2.1.10 (Two-Parameter Filtration). *Assume $\mathcal{F}^t_s$, $s, t \in \mathbb{R}$, $s \leq t$, is a two-parameter family of sub-σ-algebra of $\mathcal{F}$ with the following properties:*

1. *$\mathcal{F}^t_s \subset \mathcal{F}^v_u$ for $u \leq s \leq t \leq v$,*
2. *$\mathcal{F}^{t+}_s := \bigcap_{v>t} \mathcal{F}^v_s = \mathcal{F}^t_s$, $\mathcal{F}^t_{s-} := \bigcap_{u<s} \mathcal{F}^t_u = \mathcal{F}^t_s$ for $s \leq t$,*
3. *$\mathcal{F}^t_s$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ for every $s \leq t$.*

*Then $\mathcal{F}^t_s$, $s \leq t$, is called a **(two-parameter) filtration** on $(\Omega, \mathcal{F}, \mathbb{P})$. And we define*

$$\mathcal{F}^t_{-\infty} := \bigvee_{s \leq t} \mathcal{F}^t_s, \quad \mathcal{F}^\infty_s := \bigvee_{t \geq s} \mathcal{F}^t_s.$$

Next we will introduce a concept, which can be considered as one of the characteristic features of RDS.

Definition 2.1.11 (Tempered Random Variables)**.** *1. A random variable $C : \Omega \to (0, \infty)$ is called* ***tempered*** *with respect to DS θ if for the associated stationary stochastic process $t \to C(\theta(t)\cdot)$ the invariant set for which*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log C(\theta(t)\omega) = 0$$

($t \to -\infty$ applies only to two-sided time) has full $\mathbb{P}$-measure.

2. *$C : \Omega \to [0, \infty)$ is called* ***tempered from above*** *if*

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^{+} C(\theta(t)\omega) = 0 \quad \mathbb{P} - a.s.,$$

while $C : \Omega \to (0, \infty]$ is called ***tempered from below*** *if $\frac{1}{C}$ is tempered from above, equivalently, if, with $\log^{-} x := \max(0, -\log x)$,*

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^{-} C(\theta(t)\omega) = 0 \quad \mathbb{P} - a.s.$$

3. *A random variable $V : \Omega \to \mathbb{R}^d$ is called* ***tempered*** *(from above or below) with respect to DS θ if the stationary stochastic process $t \to |V(\theta(t)\cdot)|$ is tempered (from above or below), where $|\cdot|$ is the standard Euclidean norm.*

Proposition 2.1.12 (Dichotomy for Linear Growth of Stationary Process)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$ be a metric DS and let $C : \Omega \to \mathbb{R}$ be measurable. Then*

$$\limsup_{t\to\infty} \frac{1}{t} C(\theta(t)\omega) = \limsup_{t\to-\infty} \frac{1}{|t|} C(\theta(t)\omega) \in \{0, \infty\} \quad \mathbb{P} - a.s.$$

and

$$\liminf_{t\to\infty} \frac{1}{t} C(\theta(t)\omega) = \liminf_{t\to-\infty} \frac{1}{|t|} C(\theta(t)\omega) \in \{-\infty, 0\} \quad \mathbb{P} - a.s.$$

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- *For discrete time*

$$C^{+} \in L^1 \Rightarrow \limsup_{n\to\pm\infty} \frac{1}{|n|} C(\theta^n\omega) = 0 \quad \mathbb{P} - a.s.$$

and

$$C^{-} \in L^1 \Rightarrow \limsup_{n\to\pm\infty} \frac{1}{|n|} C(\theta^n\omega) = 0 \quad \mathbb{P} - a.s.$$

- *For continuous time*

$$\sup_{0\le t\le 1} C^{+}(\theta(t)\cdot) \in L^1 \Rightarrow \limsup_{t\to\pm\infty} \frac{1}{|t|} C(\theta(t)\omega) = 0 \quad \mathbb{P}-a.s.$$

and

$$\sup_{0\le t\le 1} C^{-}(\theta(t)\cdot) \in L^1 \Rightarrow \liminf_{t\to\pm\infty} \frac{1}{|t|} C(\theta(t)\omega) = 0 \quad \mathbb{P}-a.s,$$

where the $\mathbb{P}-a.s.$ statements hold on an invariant set of full $\mathbb{P}-$measure.

Moreover, if θ is ergodic the above limsup's and liminf's are constant on an invariant set of full $\mathbb{P}-$measure.

Remark 2.1.13. *(Caraballo et al., 2010) We note in the case of ergodicity, a random variable is either tempered from above or alternatively there exists a $\{\theta_t\}_{t\in\mathbb{T}}$-invariant set $\hat{\Omega}$ of full measure such that*

$$\limsup_{t\to\pm\infty} \frac{\log^{+} C(\theta(t)\omega)}{|t|} = \infty, \quad \omega \in \hat{\Omega}.$$

Thus a random variable is tempered from above if and only if there exists a positive constant Λ and a positive variable $C_\Lambda(\omega)$ such that

$$C(\theta(t)\omega) \le C_\Lambda(\omega) e^{\Lambda|t|}, \quad \forall\, t \in \mathbb{T} \;\; \mathbb{P}-a.s. \tag{2.1}$$

2.2 The Multiplicative Ergodic Theorem

This section is mainly devoted to the presentation of multiplicative ergodic theorems (c.f. Arnold (1998)), including both finite and infinite dimensional cases.

Definition 2.2.1 (Lyapunov Index of a Function)**.** *Let $\mathbb{T} = \mathbb{R}^+$ or $\mathbb{Z}^+$ or $\mathbb{N}$, and $f : \mathbb{T} \to \mathbb{R}^d$, and name*

$$\lambda(f) =: \limsup_{t\to\infty} \frac{1}{t} \log|f(t)| \in \mathbb{R} \cup \{-\infty, \infty\}$$

the ***Lyapunov index of*** *f. Then*

1. *$\lambda(c) = 0$ if $c \neq 0$ is constant,*

2. *$\lambda(\alpha f) = \lambda(f)$ for all $\alpha \in \mathbb{R}\setminus\{0\}$,*

3. $\lambda(f+g) \le \max\{\lambda(f), \lambda(g)\}$ *with equality if* $\lambda(f) = \lambda(g)$.

4. *If* $\mathbb{T} = \mathbb{R}^+$ *and* f *is locally integrable, then*

$$\lambda\left(\int_0^t f(s)ds\right) \le \lambda(f) \quad if \quad \lambda(f) \ge 0,$$

$$\lambda\left(\int_t^\infty f(s)ds\right) \le \lambda(f) \quad if \quad \lambda(f) < 0,$$

If f *is measurable and locally bounded and* g *is locally integrable, then*

$$\lambda\left(\int_0^t \langle f(s), g(s)\rangle ds\right) \le \lambda(f) + \lambda\left(\int_0^t |g(s)|ds\right)$$

if $\lambda(f) \ge 0$.

Similarly for $\mathbb{T} = \mathbb{Z}^+$ *or* $\mathbb{N}$ *with integrals replaced by sums.*

5.

$$\lambda(\langle f, g\rangle) \le \lambda(f) + \lambda(g),$$

$$\lambda(|f|^\alpha) = \alpha\lambda(f) \qquad for \quad \alpha \in \mathbb{R}.$$

Definition 2.2.2 (Lyapunov Exponents)**.** *The* ***forward Lyapunov exponent*** *of the solution* $\Phi(t)x$ *of a non-autonomous linear differential equation* $\dot{x}_t = A(t)x_t$ *starting at time* $t = 0$ *at the state* $x \in \mathbb{R}^d$ *is defined to be the Lyapunov index of* $\Phi(t)x$,

$$\lambda^+(x) = \lambda(x) := \limsup_{t\to\infty} \frac{1}{t}\log|\Phi(t)x|,$$

and for two-sided time, the ***backward Lyapunov exponent*** *of* $\Phi(t)x$ *is defined as the Lyapunov index of* $\Phi(-t)x$,

$$\lambda^-(x) =: \limsup_{t\to\infty} \frac{1}{t}\log|\Phi(-t)x| = \limsup_{t\to-\infty} \frac{1}{|t|}\log|\Phi(t)x|.$$

2.2.1 MET for Euclidean spaces

Theorem 2.2.3 (MET for One-Sided Time)**.** *Let* Φ *be a linear cocycle with one-sided time over the metric DS* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{T}})$. *Then the following statements hold:*

(A) **Non-invertible case** $\mathbb{T} = \mathbb{N}$**:** *If the generator* $A = \Phi(1,\cdot) : \Omega \to \mathbb{R}^{d\times d}$ *satisfies*

$$\log^+ ||A(\cdot)|| \in L^1(\Omega, \mathcal{F}, \mathbb{P}),$$

where $\|\cdot\|$ *is the norm on* $\mathcal{L}(\mathbb{R}^d)$*, then there exists a forward invariant set* $\hat{\Omega} \in \mathcal{F}$ *of full measure such that for each* $\omega \in \hat{\Omega}$

1. *The limit* $\lim_{n\to\infty}(\Phi(n,\omega)^*\Phi(n,\omega))^{1/2n} =: \Psi(\omega) \geq 0$ *exists.*
2. *Let* $e^{\lambda_{p(\omega)}(\omega)} < \cdots < e^{\lambda_1(\omega)}$ *be different eigenvalues of* $\Psi(\omega)$ *(possibly* $\lambda_{p(\omega)}(\omega) = -\infty$*) and let* $U_{p(\omega)}(\omega), \cdots, U_{\lambda_1(\omega)}$ *be the corresponding eigenspaces with multiplicities* $d_i(\omega) := \dim U_i(\omega)$*. Then*

$$p(\theta\omega) = p(\omega),$$

$$\lambda_i(\theta\omega) = \lambda_i(\omega) \qquad \forall\, i \in \{1, \cdots, p(\omega)\},$$

$$d_i(\theta\omega) = d_i(\omega) \qquad \forall\, i \in \{1, \cdots, p(\omega)\}.$$

3. *Put* $V_{p(\omega)+1}(\omega) := \{0\}$ *and for* $i = 1, \cdots, p(\omega)$

$$V_i(\omega) := U_{p(\omega)}(\omega) \oplus \cdots \oplus U_i(\omega),$$

so that

$$V_{p(\omega)}(\omega) \subset \cdots \subset V_i(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d$$

defines a filtration of $\mathbb{R}^d$*. Then for each* $x \in \mathbb{R}^d \setminus \{0\}$ *the Lyapunov exponent*

$$\lambda(\omega, x) := \lim_{n\to\infty} \frac{\log|\Phi(n,\omega)x|}{n}$$

exists as a limit and

$$\lambda(\omega, x) = \lambda_i(\omega) \Longleftrightarrow x \in V_i(\omega) \setminus V_{i+1}(\omega),$$

equivalently

$$V_i(\omega) = \{x \in \mathbb{R}^d : \lambda(\omega, x) \leq \lambda_i(\omega)\}.$$

4. *For all* $x \in \mathbb{R}^d \setminus \{0\}$

$$\lambda(\theta\omega, A(\omega)x) = \lambda(\omega, x),$$

whence

$$A(\omega)V_i(\omega) \subset V_i(\theta\omega) \quad \forall\, i \in \{1, \cdots, p(\omega)\}.$$

5. *If* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n\in\mathbb{N}})$ *is ergodic, then the function* $p(\cdot)$ *is constant on* $\hat{\Omega}$*, and the functions* $\lambda_i(\cdot)$ *and* $d_i(\cdot)$ *are constant on* $\{\omega \in \hat{\Omega} : p(\omega) \geq i\}$*,* $i = 1, \cdots, d$*.*

(B) **Invertible case $\mathbb{T} = \mathbb{N}$:** *If $A : \Omega \to Gl(d, \mathbb{R})$ and*

$$\log^+ ||A|| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$$

and

$$\log^+ ||A^{-1}|| \in L^1(\Omega, \mathcal{F}, \mathbb{P}),$$

then the set $\hat{\Omega}$ of full measure on which (A) holds (and on which in the ergodic case $p(\cdot)$, $\lambda_i(\cdot)$,$d_i(\cdot)$ are constant) can be chosen to be invariant. Further, $\lambda_{p(\omega)}(\omega) > -\infty$ on $\hat{\Omega}$, and

$$A(\omega)V_i(\omega) = V_i(\theta\omega) \quad \forall\ i \in \{1, \cdots, p(\omega)\}.$$

(C) **Invertible case $\mathbb{T} = \mathbb{R}^+$:** *Let $\Phi(t, \omega) \in Gl(d, \mathbb{R})$. Assume $\alpha^+ \in L^1$ and $\alpha^- \in L^1$, where*

$$\alpha(\omega)^{\pm} := \sup_{0 \le t \le 1} \log^+ ||\Phi(t, \omega)^{\pm 1}||.$$

Then all statements of part (B) hold with n, θ and $A(\omega)$ replaced with t, $\theta(t)$ and $\Phi(t, \omega)$, and the set $\hat{\Omega} \in \mathcal{F}$ of full measure is now invariant with respect to $(\theta(t))_{t \in \mathbb{R}^+}$.

(D) **Measurability:** *The function $\omega \mapsto p(\omega) \in \{1, \cdots, d\}$ (measurably extended from $\hat{\Omega}$ to Ω) is measurable. The functions $\omega \mapsto \lambda_i(\omega) \in \mathbb{R} \cup \{-\infty\}$, $\omega \mapsto d_i(\omega) \in \{1, \cdots, d\}$, $\omega \mapsto U_i(\omega) \in \bigcup_{k=1}^d G_k(d)$ and $\omega \mapsto V_i(\omega) \in \bigcup_{k=1}^d G_k(d)$, $G_k(d)$ the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^d$(measurably extended to $\{\omega : p(\omega) \ge i\} \in \mathcal{F}$) are measurable.*

Theorem 2.2.4 (MET for Two-Sided Time)**.** *Let Φ be a linear cocycle with two-sided time over the metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t \in \mathbb{T}})$. Then for the continuous time $\mathbb{T} = \mathbb{R}$, with the assumption that $\alpha^+ \in L^1$ and $\alpha^- \in L^1$, where*

$$\alpha(\omega)^{\pm} := \sup_{0 \le t \le 1} \log^+ ||\Phi(t, \omega)^{\pm 1}||,$$

there exists an invariant set $\hat{\Omega}$ of full measure on which the statements of the MET for $\mathbb{T} = \mathbb{R}^+$ hold. Moreover, for each $\omega \in \hat{\Omega}$ there exists an Oseledets splitting

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)}(\omega)$$

of $\mathbb{R}^d$ into random subspaces $E_i(\omega)$ with dimension $\dim E_i(\omega) = d_i(\omega)$ *with the following properties: For $i \in \{1, \cdots, p(\omega)\}$,*

1. *If* $P_i(\omega) : \mathbb{R}^d \to E_i(\omega)$ *denotes the projection onto* $E_i(\omega)$ *along* $F_i(\omega) := \oplus_{j\neq i} E_j(\omega)$, *then*

$$\Phi(t,\omega)P_i(\omega) = P_i(\theta_t\omega)\Phi(t,\omega),$$

or equivalently

$$\Phi(t,\omega)E_i(\omega) = E_i(\theta_t\omega),$$

where θ_t *is short for* $\theta(t)$.

2. *We have*

$$\lim_{t\to\pm\infty} \frac{\log|\Phi(t,\omega)x|}{t} = \lambda_i(\omega) \Longleftrightarrow x \in E_i(\omega) \setminus \{0\},$$

3. *The above convergence is uniform with respect to* $x \in E_i(\omega) \cap S^{d-1}$ *for each fixed* ω, *where*

$$S^{d-1} := \{x \in \mathbb{R}^d : |x|^2 = 1\} \subset \mathbb{R}^d.$$

2.2.2 MET for infinite dimensional spaces

Theorem 2.2.5 (Ruelle's One-Sided MET)**.** *(Ruelle, 1982) Let* $\Phi : \mathbb{R} \times \Omega \to L(H)$ *be strongly measurable, such that* (Φ, θ) *is an* $L(H)$*-valued cocycle, with each* $\Phi(t,\omega)$ *compact. Suppose that*

$$\mathbb{E} \sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)\|_{L(H)} + \mathbb{E} \sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\cdot)\|_{L(H)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ *such that* $\theta_t\Omega_0 \subseteq \Omega_0$ *for all* $t \in \mathbb{R}^+$, *and for each* $\omega \in \Omega_0$, *the limit*

$$\Lambda(\omega) := \lim_{t\to\infty} (\Phi(t,\omega)^*\Phi(t,\omega))^{1/2t}$$

exists in the uniform operator norm. Each linear operator $\Lambda(\omega)$ *is compact, non-negative and self-adjoint with a discrete spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the λ_i*'s are distinct and non-random. Each eigenvalue* $e^{\lambda_1} > 0$ *has a fixed finite non-random multiplicity* m_i, *and a corresponding eigenspace* $E_i(\omega)$, *with* $m_i = dim E_i(\omega)$. *Set* $i = \infty$ *when* $\lambda_i = -\infty$. *Define*

$$V_1(\omega) := H, \quad V_i(\omega) := [\oplus_{j=1}^{i-1} E_j(\omega)]^\perp, \; i > 1, \; V_\infty := \ker \Lambda(\omega).$$

Then

$$V_\infty \subset \cdots \subset \cdots \subset V_{i+1}(\omega) \subset V_i(\omega) \cdots \subset V_2(\omega) \subset V_1(\omega) = H,$$

$$\lim_{t\to\infty} \frac{1}{t} \log |\Phi(t,\omega)x| = \begin{cases} \lambda_i, \ if \ x \in V_i(\omega) \setminus V_{i+1}(\omega), \\ -\infty, \ if \ x \in V_\infty(\omega), \end{cases}$$

and

$$\Phi(t,\omega)(V_i(\omega)) \subseteq V_i(\theta_t \omega)$$

for all $t \geq 0$, $i \geq 1$.

2.3 Malliavin Calculus and Malliavin Derivatives

This section mainly focuses on the principle concepts and conclusions arising from the study of the Malliavin calculus and Malliavin derivatives (c.f. Nualart (2000), Nualart (2009) and Oksendal (1997)).

Denote $\mathbf{T} = [0,T]$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbf{T}, \mathcal{B})$ a Lebesgue measurable space. Consider a one-dimensional Wiener process $W(t) = W(t,\omega)$, $t \in \mathbf{T}$, $\omega \in \Omega$, defined on on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $W(0,\omega) = 0$ $\mathbb{P}$-a.s. And let $\mathcal{F}_t$ be a σ-algebra generated by $W(s,\cdot)$, $0 \leq s \leq t$. From now on, W_t is short for $W(t)$ or $W(t,\omega)$.

Denote

$$I_m(f) = \int_{\mathbf{T}^m} f(t_1, \ldots, t_m) dW_{t_1} \ldots dW_{t_m},$$

where $f(t_1, \ldots, t_m)$ is a symmetric function in $L^2(\mathbf{T}^m)$. We just summarize one result as follows:

$$\mathbb{E}[I_m(f) I_n(g)] = \begin{cases} 0, & if \ n \neq m, \\ m! < f, g >_{L^2(\mathbf{T}^m)}, & if \ n = m. \end{cases} \tag{2.2}$$

Theorem 2.3.1 (Wiener-Itô Chaos Decomposition). *(c.f. Oksendal (1997)) Let F be an $\mathcal{F}_T$-measurable random variable in $L^2(\Omega)$, then there exists a unique sequence $\{f_n\}_{n=0}^\infty$ of deterministic symmetric function $f_m \in L^2(\mathbf{T}^m)$ such that*

$$F = \sum_{m=0}^\infty I_m(f_m), \tag{2.3}$$

where $I_0(f_0) = \mathbb{E}(F)$. Moreover, we have the isometry

$$\|F\|^2_{L^2(\Omega)} = \sum_{m=0}^\infty m! \|f_m\|^2_{L^2(\mathbf{T}^m)}. \tag{2.4}$$

Next I would like to introduce the Skorohod stochastic integral of the random process $u(t,\omega)$.

Definition 2.3.2. *Suppose that $u(t,\omega)$ is a stochastic process such that $u(t,\cdot)$ is $\mathcal{F}_T$-measurable and square-integrable for all $t \in \mathbf{T}$, and with Wiener-Itô chaos expansion*

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)).$$

Then the ***Skorohod integral*** *is defined as*

$$\delta(u) := \int_{\mathbf{T}} u(t,\omega)\delta W_t := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{2.5}$$

where $\tilde{f}_n$ is the symmetrization of $f_n(t_1,\cdots,t_n,t)$ as a function of $n+1$ variables $t_1,\cdots,t_n,t$. We say u is ***Skorohod-integrable*** *and write $u \in Dom(\delta)$ if the series in (2.5) converges in $L^2(\Omega)$.*

Remark 2.3.3.

1. *Note that in Definition 2.3.2, $u(t,\omega)$ is not necessary to be adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathbf{T}}$. Thus we are able to define the integral w.r.t an anticipating integrand.*

2. *The Itô integral is a particular case of the Skorohod integral, with adapted integrand (c.f. Nualart (2000)).*

Another anticipating integral is of Stratonovich type and defined as follows:

Definition 2.3.4. *A measurable process $u(t,\omega)$ such that $\int_{\mathbf{T}} |u_t|dt < \infty$ a.s. is* ***Stratonovich integrable*** *if the familiy S^π*

$$S^\pi := \int_{\mathbf{T}} u_t W_t^\pi dt,$$

where

$$W_t^\pi = \sum_{i=0}^{n-1} \frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i} \chi_{(t_i,t_{i+1}]}(t),$$

converges in probability as $|\pi| \to 0$ and in this case the limit will be denoted by $\int_{\mathbf{T}} u_t \circ dW_t$.

Let $\mathcal{S}$ denote the class of smooth and cylindrical random variables of the form

$$F = f(W(h_1),\cdots,W(h_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$, i.e., f and all its partial derivatives have polynomial growth order, and $h_1, \cdots, h_n \in L^2(\mathbf{T})$, and $n \geq 1$.

The derivative of F is the $L^2(\mathbf{T})$-valued random variable given by

$$\mathcal{D}_r F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n)) h_i(r).$$

Then denote by $\mathcal{D}^{1,2}$ the domain of $\mathcal{D}$ in $L^2(\Omega)$, i.e. $\mathcal{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$||F||_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}||\mathcal{D}.F||_{L^2(\mathbf{T})}^2.$$

Also we are able to define $\mathcal{D}^{1,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$||F||_{1,p} = \left(\mathbb{E}|F|^p + \mathbb{E}||\mathcal{D}.F||_{L^p(\mathbf{T})}^p\right)^{\frac{1}{p}}.$$

Theorem 2.3.5. *(c.f. Oksendal (1997)) Let $F = \sum_{m=0}^{\infty} I_m(f_m) \in L^2(\Omega)$, then $F \in \mathcal{D}^{1,2}$ if and only if*

$$\sum_{m=0}^{\infty} m m! \|f_m\|_{L^2(\mathbb{R})}^2 < \infty,$$

and if this is the case we have

$$\mathcal{D}_r F = \sum_{m=0}^{\infty} m I_{m-1}(f_m(\cdot, r)).$$

Now suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical probability space associated with a M-dimensional Brownian motion $\{W_t^j, t \in \mathbf{T}, 1 \leq j \leq M\}$, then the derivative $\mathcal{D}F$ of a random variable $F \in \mathcal{D}^{1,2}$ will be a M-dimensional process denoted by $\{\mathcal{D}_r^j F, r \in \mathbb{T}, 1 \leq j \leq M\}$. For example

$$\mathcal{D}_r^j W_t^k = \delta_{k,j} \chi_{[0,t]}(r).$$

Consider the M-dimensional stochastic differential equation:

$$X_t = x_0 + \sum_{j=1}^{d} \int_0^t A_j(X_s) dW_s^j + \int_0^t B(X_s) ds, \tag{2.6}$$

where $A_j, B : \mathbb{R}^M \to \mathbb{R}^M$, $1 \leq j \leq d$ are measurable functions.

Proposition 2.3.6 (Differentiability of the solution)**.** *(c.f. Nualart (2009)) Suppose that the coefficients A_i, B are continuously differentiable and satisfy*

$$\max\{|A_j(x) - A_j(y)|, |B(x) - B(y)|\} \leq K|x - y|, \quad \forall x, y \in \mathbb{R}^M.$$

Then for all $t \in \mathbf{T}$ and for all $i = 1, \cdots, M$ the random variable X_t^i belongs to the space $\mathcal{D}^{1,\infty} := \cap_{p \geq 2} \mathcal{D}^{1,p}$ and the derivative $\mathcal{D}_r^j X_t^i$ satisfies the following linear stochastic differential equation for $r \leq t$

$$\mathcal{D}_r^j X_t = A_j(X_r) + \sum_{k=1}^{M} \sum_{l=1}^{d} \int_r^t \partial_k A_l(x_s) \mathcal{D}_r^j X_s^k dW_s^l + \sum_{k=1}^{M} \int_r^t \partial_k B(X_s) \mathcal{D}_r^j X_s^k ds.$$

Finally all the Malliavin calculus results involved above can be naturally extended to infinite time interval, which are mainly based on the following extended Wiener-Itô chaos decomposition.

Theorem 2.3.7 (Wiener-Itô Chaos Decomposition)**.** *(c.f. Oksendal (1997)) Let F be an $\mathcal{F}$-measurable random variable in $L^2(\Omega)$, then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of deterministic symmetric function $f_m \in L^2(\mathbb{R}^m)$ such that*

$$F = \sum_{m=0}^{\infty} I_m(f_m), \tag{2.7}$$

where $I_0(f_0) = \mathbb{E}(F)$ and

$$I_m(f_m) = \int_{\mathbb{R}^m} f_m(t_1, \ldots, t_m) dW_{t_1} \ldots dW_{t_m},$$

and $f_m(t_1, \ldots, t_m)$ is a symmetric element in $L^2(\mathbb{R}^m)$.
Moreover, we have the isometry

$$\|F\|_{L^2(\Omega)}^2 = \sum_{m=0}^{\infty} m! \|f_m\|_{L^2(\mathbb{R}^m)}^2, \tag{2.8}$$

and if $F \in \mathcal{D}^{1,2}$ then we have

$$\mathcal{D}_r F = \sum_{m=0}^{\infty} m I_{m-1}(f_m(\cdot, r)).$$

Lemma 2.3.8. *Suppose that $F(\cdot) \in \mathcal{D}^{1,2}$, then for all $h \in \mathbb{R}$, $F(\theta_h \cdot) \in \mathcal{D}^{1,2}$, and*

$$\|F(\theta_h \cdot)\|_{1,2} = \|F(\cdot)\|_{1,2},$$

where $\theta_h : \Omega \to \Omega$ for all $h \in \mathbb{R}$ is a measure preserving measurable DS on $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. First of all, by the measure-preserving property, it is easy to get

$$\mathbb{E}|F(\theta_h \cdot)|^2 = \mathbb{E}|F(\cdot)|^2.$$

Since $F(\cdot) \in \mathcal{D}_{1,2}$, then according to Theorem 2.3.7, it can be written into Wiener-Itô chaos decomposition as follows,

$$F(\omega) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \cdots, t_k) dW_{t_1} \cdots dW_{t_k},$$

and

$$\begin{aligned} F(\theta_h \omega) &= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \cdots, t_k) d(\theta_h W_{t_1}) \cdots d(\theta_h W_{t_k}) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1 - h, \cdots, t_k - h) dW_{t_1} \cdots dW_{t_k}. \end{aligned}$$

Thus again by Theorem 2.3.7, we have

$$\mathcal{D}_r F(\omega) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1, \cdots, t_{k-1}, r) dW_{t_1} \cdots dW_{t_{k-1}},$$

and

$$\mathcal{D}_r F(\theta_h \omega) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_k(t_1 - h, \cdots, t_{k-1} - h, r - h) dW_{t_1} \cdots dW_{t_{k-1}}.$$

Therefore

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}} \|\mathcal{D}_r F(\theta_h \cdot)\|^2 dr &= \sum_{k=0}^{\infty} (k-1)! \int_{\mathbb{R}} \|f_k(t_1 - h, \cdots, t_{k-1} - h, r - h)\|^2_{L^2(\mathbb{R}^{k-1})} dr \\ &= \sum_{k=0}^{\infty} (k-1)! \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |f_k(t_1 - h, \cdots, t_{k-1} - h, r - h)|^2 dt_1 \cdots dt_{k-1} dr \\ &= \sum_{k=0}^{\infty} (k-1)! \int_{\mathbb{R}^k} |f_k(t_1, \cdots, t_{k-1}, r)|^2 dt_1 \cdots dt_{k-1} dr \\ &= \mathbb{E} \int_{\mathbb{R}} \|\mathcal{D}_r F(\cdot)\|^2 dr. \end{aligned}$$

□

And it is not hard to extend Lemma 2.3.8 to the following result.

Lemma 2.3.9. *Suppose that F is an $\mathcal{F}$-measurable random variable in $L^2(\Omega \times \mathcal{O})$ with*

$$F = \sum_{m=0}^{\infty} \int_{\mathbb{R}^m} f_m(t_1, \ldots, t_m, x) dW_{t_1} \ldots dW_{t_m},$$

where $f_m(t_1, \ldots, t_m, x)$ is a symmetric element in $L^2(\mathbb{R}^m)$ for x fixed, and moreover let $F(\cdot, x) \in \mathcal{D}^{1,2}$, then for all $h \in \mathbb{R}$, $F(\theta_h \cdot, x) \in \mathcal{D}^{1,2}$, and

$$\int_{\mathcal{O}} \|F(\theta_h \cdot, x)\|^2_{1,2} dx = \int_{\mathcal{O}} \|F(\cdot, x)\|^2_{1,2} dx.$$

Proof. Similar to Lemma 2.3.8. □

2.4 Relative Compactness Criteria in Wiener-Sobolev Spaces

Relative Compactness criteria always play an important role when we are dealing with a fixed point problem, and among the criteria the Arzelà-Ascoli lemma is the simplest and mostly widely used one. In this section, a review of some more advanced relative compactness criteria is given, which are be based on the Arzelà-Ascoli lemma as well.

Definition 2.4.1 (Relatively Compact Subspace(or Subset))**.** *Let V be a metric space. A subset $S \subset V$ is called a **relative compact subset** of V if the closure of S is compact, or in another word, if for any sequence in S, there exist a subsequence that converges.*

Theorem 2.4.2 (Arzelà-Ascoli Lemma)**.** *Consider a sequence $\{f_n\}_{n\in\mathbb{N}} \in C([a,b],\mathbb{R})$. If this sequence satisfies the following conditions:*

1. *There is an $M \in \mathbb{R}$ such that*

$$\sup_n \sup_{x\in[a,b]} |f_n(x)| \leq M,$$

2. *For any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sup_n |f_n(x) - f_n(y)| < \epsilon \text{ whenever } |x-y| < \delta.$$

Then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly.

Here follows another classical relative compactness criteria in $L^2([a,b])$ space.

Theorem 2.4.3 (Riesz's Relative Compactness Criteria)**.** *Consider a sequence $\{f_n\}_{n\in\mathbb{N}} \in L^2([a,b])$. Then $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact if for any f_n:*

1. *There is an $M \in \mathbb{R}$ such that*

$$\sup_n \|f_n\|^2_{L^2([a,b])} \leq M,$$

2. *For any $h \in \mathbb{R}$,*

$$\sup_n \|\tau_h f_n - f_n\|_{L^2([a,b])} \to 0 \text{ as } h \to 0,$$

   where $\tau_h f_n := f_n(t+h)$ (set $f_n(t) = 0$, if $t \notin [a,b]$).

Finally I will present two more general versions, one is the generalization of Arzelà-Ascoli Theorem (Kelley, 1991) and another one considers the case when the domain of L^2 is not necessarily bounded (c.f. R. A. Adams (2003)).

Theorem 2.4.4 (Generalized Arzelà-Ascoli Lemma)**.** *Let X be a compact Hausdorff space and Y a metric space. Then a subset S is relatively compact in $C(X, Y)$ if and only if it is equicontinuous and pointwise relatively compact, by which we mean for each $x \in X$, the set $S_x = \{f(x),\ f \in S\}$ is relatively compact in Y.*

Then consider a function defined a.e. on $\mathbf{D} \subset \mathbb{R}^m$, let $\tilde{f}$ denote the zero extension of f outside $\mathbf{D}$:

$$\tilde{f} = \begin{cases} f(x), & \text{if } x \in \mathbf{D}, \\ 0, & \text{if } x \in \mathbb{R}^m \setminus \mathbf{D}. \end{cases}$$

Theorem 2.4.5 (Generalized Riesz's Relative Compactness Criteria)**.** *Let $1 \leq p < \infty$. A bounded subset $S \subset L^p(\mathbf{D})$ is relatively compact in $L^p(\mathbf{D})$ if and only if for any $\epsilon > 0$, there exists a number $\delta > 0$ and a subset $\mathbf{G} \Subset \mathbf{D}$ such that for every $f \in S$ and $h \in \mathbb{R}^m$ with $|h| < \delta$ both of the following inequalities hold:*

$$\int_{\mathbf{D}} |\tilde{f}(x+h) - \tilde{f}(x)|^p dx < \epsilon^p, \tag{2.9}$$

$$\int_{\mathbf{D} \setminus \overline{\mathbf{G}}} |f(x)|^p dx < \epsilon^p. \tag{2.10}$$

2.4.1 Relative Compactness Criteria in $L^2(\mathbf{T} \times \Omega \times \mathcal{O})$

The compactness criterion as a pure random variable version without including time and space variables was firstly investigated by Da Prato, Malliavin and Nualart (Da Prato et al., 1992) and Peszat (Peszat, 1993). In 2004, V. Bally and B. Saussereau transformed the relative compactness criteria in Wiener-Sobolev spaces to classic Hilbert spaces $L^2(\mathbf{T}^m)$ via both Wiener chaos expansion and spectrum decomposition in Hilbert space (Bally and Saussereau, 2004). The Wiener-Sobolev compact embedding provides a powerful method to study the convergence of a sequence of random fields. This is a new direction of Malliavin calculus. The traditional application of Malliavin calculus was in regularity of densities and was studied intensively in the literature.

In this part, I mainly summarized Bally-Saussereau's work. Denote by $\mathcal{O}$ a bounded domain in $\mathbb{R}^d$.

Theorem 2.4.6. *Consider a sequence $(v_n)_{n\in\mathbb{N}}$ of $L^2(\mathbf{T} \times \Omega; H^1(\mathcal{O}))$, where the Hilbert space $H^1(\mathcal{O})$ denotes the completion of $\{v \in C^1(\mathcal{O}), \|v\|_{1,2,\mathcal{O}} < \infty\}$ with respect to the following norm:*

$$\|v\|_{1,2,\mathcal{O}} := \left(\sum_{0\leq|\alpha|\leq 1} \|D^\alpha v\|^2_{L^2(\mathcal{O})} \right)^{\frac{1}{2}},$$

and suppose that

1. $\sup_{n\in\mathbb{N}} \mathbb{E} \int_{\mathbf{T}} \|v_n(t,\cdot,\omega)\|^2_{H^1(\mathcal{O})} dt = C_1 < \infty.$
2. *For all $\varphi \in C_c^\infty(\mathcal{O})$ and $t \in \mathbf{T}$, $v_n^\varphi(t,\cdot)$ belongs to $\mathcal{D}^{1,2}$, where*
$$v_n^\varphi(t,\omega) = \int_{\mathcal{O}} \varphi(x) v_n(t,x,\omega) dx,$$
and
$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_{\mathbf{T}} \|v_n^\varphi(t,\cdot)\|^2_{1,2} dt = C_2 < \infty.$$
3. *For all $\varphi \in C_c^\infty(\mathcal{O})$, $m \in \mathbb{N}$, the sequence $(f^m_{n,\varphi})_{n\in\mathbb{N}}$ is relatively compact in $L^2(\mathbf{T} \times \mathbf{T}^m)$, where the kernels of the Wiener-Itô chaos expansion decomposition of v_n^φ, $f^m_{n,\varphi}$, for any $n \in \mathbb{N}$, are given as*
$$f^m_{n,\varphi}(t, t_1, \cdots, t_m) = \int_{\mathcal{O}} \varphi(x) f^m_n(t, x, t_1, \cdots, t_m) dx,$$
where
$$v_n(t,x,\omega) = \sum_{m=0}^{\infty} I_m(f_n^m(t,x,\cdot))(\omega).$$

Then $\{v_n,\ n \in \mathbb{N}\}$ is relatively compact in $L^2(\mathbf{T} \times \Omega \times \mathcal{O})$.

Hypothesis 3 of the criteria above can be developed in the following version of Theorem 2.4.6, which mainly stemmed from the classical criteria Theorem 2.4.3 and Mallavin derivatives.

Theorem 2.4.7. *Consider a sequence $(v_n)_{n\in\mathbb{N}}$ of $L^2(\mathbf{T} \times \Omega;\ H^1(\mathcal{O}))$ and suppose that*

1. $\sup_{n\in\mathbb{N}} \mathbb{E}\int_{\mathbf{T}} \|v_n(t,\cdot,\omega)\|^2_{H^1(\mathcal{O})}dt = C_1 < \infty$.

2. *For all* $\varphi \in C_c^\infty(\mathcal{O})$ *and* $t \in \mathbf{T}$, $v_n^\varphi(t,\cdot) \in \mathcal{D}^{1,2}$ *and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_{\mathbf{T}} \|v_n^\varphi(t,\cdot)\|^2_{1,2}dt = C_2 < \infty.$$

3. *For all* $\varphi \in C_c^\infty(\mathcal{O})$, *the sequence* $(\mathbb{E}v_n^\varphi)_{n\in\mathbb{N}}$ *satisfies*

   (3i) For any $0 < \alpha < \beta < T$, *and* $h \in \mathbb{R}$ *such that* $|h| < \min\{\alpha, T-\beta\}$, *it holds*

$$\sup_{n\in\mathbb{N}} \int_\alpha^\beta |\mathbb{E}v_n^\varphi(t+h) - \mathbb{E}v_n^\varphi(t)|^2 dt \le C|h|.$$

   (3ii) For any $\epsilon > 0$, *there exists* $0 < \alpha < \beta < T$ *such that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbf{T}\setminus(\alpha,\beta)} |\mathbb{E}v_n^\varphi(t)|^2 dt \le \epsilon.$$

4. *For all* $\varphi \in C_c^\infty(\mathcal{O})$ *the following conditions are satisfied:*

   (4i) For any $0 < \alpha < \beta < T$,$0 < \hat{\alpha} < \hat{\beta} < T$ *and* $h, \hat{h} \in \mathbb{R}$ *such that* $|h| \vee |\hat{h}| < \min\{\hat{\alpha}, \alpha, T-\hat{\beta}, T-\beta\}$, *it holds*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_\alpha^\beta \int_{\hat{\alpha}}^{\hat{\beta}} |\mathcal{D}_{\theta+h}v_n^\varphi(t+\hat{h}) - \mathcal{D}_\theta v_n^\varphi(t)|^2 d\theta dt \le C(|h| + |\hat{h}|).$$

   (4ii) For any $\epsilon > 0$, *there exists* $0 < \alpha < \beta < T$ *and* $0 < \hat{\alpha} < \hat{\beta} < T$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\int_{\mathbf{T}^2\setminus(\alpha,\beta)\times(\hat{\alpha},\hat{\beta})} |\mathcal{D}_\theta v_n^\varphi(t)|^2 d\theta dt < \epsilon.$$

Then $\{v_n,\ n \in \mathbb{N}\}$ *is relatively compact in* $L^2(\mathbf{T} \times \Omega \times \mathcal{O})$.

2.4.2 Relative Compactness Criteria in $C(\mathbf{T}, L^2(\Omega \times \mathcal{O}))$

It's easy to check the following refined versions of relative compactness of Wiener-Sobolev space in Bally and Saussereau (2004) also hold. Feng, Zhao and Zhou (Feng et al., 2011) used the compactness of a sequence of stochastic processes in $C(\mathbf{T}, L^2(\Omega))$ to study periodic solution of stochastic differential equations.

Theorem 2.4.8. (Relative Compactness in $C(\mathbf{T}, L^2(\Omega))$). *Consider a sequence* $(v_n)_{n\in\mathbb{N}}$ *of* $C(\mathbf{T}, L^2(\Omega))$. *Suppose that:*

1. $v_n(t,\cdot) \in \mathcal{D}^{1,2}$ *and* $\sup_{n\in\mathbb{N}} \sup_{t\in\mathbf{T}} ||v_n(t,\cdot)||^2_{1,2} < \infty$.

2. *There exists a constant* $C > 0$ *such that for any* $t, s \in \mathbf{T}$,

$$\sup_n \mathbb{E}|v_n(t) - v_n(s)|^2 < C|t-s|.$$

3. *(3i) There exists a constant* $C > 0$ *such that for any* $h \in \mathbb{R}$, *and any* $t \in \mathbf{T}$,

$$\sup_n \int_{\mathbb{R}} \mathbb{E}|\mathcal{D}_{r+h}v_n(t) - \mathcal{D}_r v_n(t)|^2 dr < C|h|.$$

   (3ii) For any $\epsilon > 0$, *there exists* $-\infty < \alpha < \beta < +\infty$ *such that*

$$\sup_n \sup_{t\in\mathbf{T}} \int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}|\mathcal{D}_r v_n(t)|^2 dr < \epsilon.$$

Then $\{v_n,\ n \in \mathbb{N}\}$ *is relatively compact in* $C(\mathbf{T}, L^2(\Omega))$.

Proof. According to the Generalized Arzelà−Ascoli Lemma 2.4.4, it is sufficient to check with the uniform equicontinuity and pointwise relative compactness.

1. Obviously hypothesis (2) contributes to the uniform equicontinuity in $L^2(\Omega)$-norm.

2. Next define for each $t \in \mathbf{T}$,

$$U_t := \{v_n(t,\cdot),\ n \in \mathbb{N}\}.$$

   I claim that U_t is relatively compact in $L^2(\Omega)$. To achieve this, develop v_n in Wiener-Itô chaos expansions by Theorem 2.3.7,

$$v_n(t,\omega) = \sum_{m=0}^{\infty} I_m(f_n^m(\cdot,t))(\omega), \tag{2.11}$$

   where $f_n^m(\cdot,t)$ are symmetric elements of $L^2(\mathbb{R}^m)$ for each $m \geq 0$ and each $t \in \mathbf{T}$. By Theorem 2.4.6, the relative compactness of $\{v_n\}_{n\in\mathbb{N}}$ is reduced to the relative compactness of $\{f_n^m\}_{n\in\mathbb{N}}$ for each finite $m \in \mathbb{N}$.
   When $m = 0$, $f_n^0(t) = \mathbb{E}v_n(t)$, and for any $t_1, t_2 \in \mathbf{T}$, hypotheses (1) and (2) implies the uniform boundedness of f_n^0,

$$\sup_n \sup_{t\in\mathbf{T}} |f_n^0(t)| \leq \sup_n \sup_{t\in\mathbf{T}} \sqrt{\mathbb{E}|v_n(t)|^2} \leq \sup_{n\in N} \sup_{t\in\mathbf{T}} ||v_n(\cdot,t)||_{1,2} < \infty.$$

Besides, applying with Jensen's inequality gives the uniform equicontinuity of f_n^0,

$$\begin{aligned}\sup_n |f_n^0(t_1) - f_n^0(t_2)| &= \sup_n |\mathbb{E}(v_n(t_1) - v_n(t_2))| \leq \sup_n \mathbb{E}|v_n(t_1) - v_n(t_2)| \\ &\leq \sup_n \sqrt{\mathbb{E}|v_n(t_1) - v_n(t_2)|^2} \\ &\leq \sqrt{C|t_1 - t_2|}.\end{aligned}$$

So $\{f_n^0\}_{n=1}^{\infty}$ is relatively compact in $C(\mathbf{T})$ according to the classical Arzelà-Ascoli lemma 2.4.2.

Using a similar argument as in the proof of Theorem 2.4.7 for each $m \geq 1$ with the general relative compactness criterion 2.4.5, we claim that $\{f_n^m(\cdot, t)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(\mathbb{R}^m)$ for each fixed t.

To see this, let $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$. It holds

$$\begin{aligned}
&\|\tau_h f_n^m - f_n^m\|_{L^2(\mathbb{R}^m)}^2 \\
= \ & \int_{\mathbb{R}^m} |f_n^m(t, t_1 + h_1, \cdots, t_m + h_m) - f_n^m(t, t_1, \cdots, t_m)|^2 dt_1 \cdots dt_m \\
\leq \ & C \sum_{i=1}^m \int_{\mathbb{R}^m} |f_n^m(t, t_1, \cdots, t_{i-1}, t_i + h_i, t_{i+1} + h_{i+1}, \cdots, t_m + h_m) \\
&\qquad - f_n^m(t, t_1, \cdots, t_{i-1}, t_i, t_{i+1} + h_{i+1}, \cdots, t_m + h_m)|^2 dt_1 \cdots dt_m \\
= \ & C \sum_{i=1}^m \int_{\mathbb{R}} \|f_n^m(t, \ldots, t_i + h_i, \ldots) - f_n^m(t, \ldots, t_i, \ldots)\|_{L^2(\mathbb{R}^{m-1})}^2 dt_i \\
= \ & \frac{C}{(m-1)!} \sum_{i=1}^m \mathbb{E}|I_{m-1}(f_n^m(t, \cdots, t_i + h_i, \cdots) - f_n^m(t, \cdots, t_i, \cdots))|^2 dt_i \\
\leq \ & \frac{C}{mm!} \sum_{i=1}^m \int_{\mathbb{R}} \mathbb{E}\Big| \sum_{m\geq 1} m I_{m-1}\big(f_n^m(t, \cdots, t_i + h_i, \cdots) - f_n^m(t, \cdots, t_i, \cdots)\big)\Big|^2 dt_i \\
\leq \ & \frac{C}{m!} \int_{\mathbb{R}} \mathbb{E}\Big|D_{r+h_1} v_n(t) - D_r v_n(t)\Big|^2 dr \\
\leq \ & C|h|,
\end{aligned}$$

here C is a constant only depending on m.

Moreover, for any $\epsilon > 0$, there exists $[\alpha, \beta] \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus[\alpha,\beta]} \mathbb{E}|D_r v_n(t)|^2 dr < \epsilon.$$

Let $G \subset \mathbb{R}^m$ be such that $[\alpha, \beta] \subset G_1$, where G_1 is the interval generated by taking the intersection of G and $span\{(1, 0, \cdots, 0)\}$. Then we have

$$\begin{aligned}
&\int_{R^m \backslash \bar{G}} |f_n^m(t, t_1, \cdots, t_m)|^2 dt_1 \cdots dt_m \\
\leq\ & C \int_{\mathbb{R} \backslash \bar{G}_1} \|f_n^m(t, r, t_2, \cdots, t_m)\|^2_{L^2(\mathbb{R}^{m-1})} dr \\
\leq\ & \frac{C}{m!} \int_{\mathbb{R} \backslash \bar{G}_1} \mathbb{E}\Big| \sum_{m \geq 1} m I_{m-1}(f_n^m(t, r, \cdot))\Big|^2 dr \\
\leq\ & \frac{C}{m!} \int_{\mathbb{R} \backslash \bar{G}_1} \mathbb{E}|D_r v_n(t)|^2 dr \\
\leq\ & C\epsilon.
\end{aligned}$$

By now it has been showed that $\{f_n^m(t, \cdot), n \in \mathbb{N}\}$ is relatively compact in $L^2(\mathbb{R}^m)$ for each finite m and fixed $t \in \mathbf{T}$, which is equivalent to $\{v_n,\ n \in \mathbb{N}\}$ being pointwise relatively compact in $L^2(\Omega)$.

Besides, by (2.11) and hypothesis (2) we can show that $\{f_n^m(\cdot, t)\}_{n \in \mathbb{N}}$ is equicontinous in $L^2(\mathbb{R}^m)$ for each m:

$$\begin{aligned}
\sup_n \|f_n^m(\cdot, t) - f_n^m(\cdot, s)\|_{L^2(\mathbb{R}^m)} &\leq \sup_n \sum_m m! \|f_n^m(\cdot, t) - f_n^m(\cdot, s)\|_{L^2(\mathbb{R}^m)} \\
&= \sup_n \mathbb{E}|v_n(t, \cdot) - v_n(s, \cdot)|^2 \\
&\leq C|t - s|.
\end{aligned}$$

Now applying with generalized Arzela$-$Ascoli Lemma 2.4.4, we conclude that $\{v_n\}_{n=1}^{\infty}$ is relatively compact in $C(\mathbf{T}, L^2(\Omega))$. □

Theorem 2.4.9. (Relative Compactness in $C(\mathbf{T}, L^2(\Omega \times \mathcal{O}))$). *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$. Consider a sequence $(v_n)_{n \in N}$ of $C(\mathbf{T}, L^2(\Omega \times \mathcal{O}))$. Suppose that:*

1. $\sup_{n \in \mathbb{N}} \sup_{t \in \mathbf{T}} E||v_n(t, \cdot)||^2_{H^1(\mathcal{O})} < \infty$.
2. $\sup_{n \in \mathbb{N}} \sup_{t \in \mathbf{T}} \int_{\mathcal{O}} ||v_n(t, x, \cdot)||^2_{1,2} dx < \infty$.
3. *There exists a constant $C > 0$ such that for any $t, s \in \mathbf{T}$,*
$$\sup_n \int_{\mathcal{O}} E|v_n(t, x) - v_n(s, x)|^2 dx < C|t - s|.$$
4. *(4i) There exists a constant C such that for any $h \in \mathbb{R}$, and any $t \in \mathbf{T}$,*
$$\sup_n \int_{\mathcal{O}} \int_{\mathbb{R}} E|\mathcal{D}_{\theta+h} v_n(t, x) - \mathcal{D}_\theta v_n(t, x)|^2 d\theta dx < C|h|.$$

(4ii) For any $\epsilon > 0$, there exist $-\infty < \alpha < \beta < +\infty$ such that

$$\sup_n \sup_{t \in \mathbf{T}} \int_{\mathcal{O}} \int_{\mathbb{R} \setminus (\alpha, \beta)} E|\mathcal{D}_\theta v_n(t,x)|^2 d\theta dx < \epsilon.$$

Then $\{v_n,\ n \in \mathbb{N}\}$ is relatively compact in $C(\mathbf{T}, L^2(\Omega \times \mathcal{O}))$.

The proof is analogous to the proof of Theorem 2.4.8.

2.5 Schauder's Fixed Point Arguments

Theorem 2.5.1 (Schauder's Fixed Point Theorem). *Let S be a closed, convex subset of a normed linear space H. Then every compact, continuous map $\mathcal{M} : S \to S$ has at least one fixed point.*

In Feng et al. (2011), Feng, Zhao and Zhou generalizes the Schauder's fixed point theorem in the sense that the condition the subset S of Banach space H has to be closed is no longer necessary, but imposing that $\mathcal{M}$ be continuous from H to H as the fixed point may not be in S.

Theorem 2.5.2 (Another Version of Schauder's Fixed Point Theorem). *Let H be a Banach space, S be a convex subset of H. Assume a map $\mathcal{M} : H \to H$ is continuous and $\mathcal{M}(S) \subset S$ is relatively compact in H. Then $\mathcal{M}$ has a fixed point in H.*

Chapter 3

Random Periodic Solutions to Nondissipative SDEs with Multiplicative Noise

3.1 Problem Formulation

Above all, note that an SDE is said to be τ-periodic if its coefficients are all periodic in time variable with period τ, where $\tau > 0$.

Consider the τ-periodic semilinear SDE of Stratonovich type with multiplicative linear noises, i.e.,

$$\begin{cases} du(t) = Au(t)\,dt + F(t, u(t))\,dt + \sum\limits_{k=1}^{M} B_k u(t) \circ dW_t^k, & t \geq s \\ u(s) = \xi \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where A, $\{B_k,\ 1 \leq k \leq M\}$ are in $\mathcal{L}(\mathbb{R}^d)$, $W_t := (W_t^1, W_t^2, \cdots, W_t^M)$, $t \in \mathbb{R}$, is an M-dimensional Brownian motion under the filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in\mathbb{R}}, \mathbb{P})$. Here $\Omega = C_0(\mathbb{R}, \mathbb{R}^M) := \{\omega \in C(\mathbb{R}, \mathbb{R}^M) : \omega(0) = 0\}$, $W_t(\omega) := \omega(t)$, and $\mathcal{F}^t := \vee_{s\leq t}\mathcal{F}_s^t$ with $\mathcal{F}_s^t := \sigma(W_u - W_v, s \leq v \leq u \leq t)$. Besides, we define a shift so it leaves Ω invariant by

$$\theta_t : \Omega \to \Omega, \quad \theta_s\omega(t) = \omega(t+s) - \omega(s),\ s, t \in \mathbb{R},$$

and thus the shift θ is measure preserving.

In addition, we assume that

Condition (C). The matrices A, A^*, B_k, and B_k^* are mutually commutative.

To begin our initial problem, we can start at a deterministic ξ. But we will note later that our infinite horizon integral equation (3.13) is reduced to equation (3.1) with anticipating ξ which depends on the whole history of the Brownian motion. Therefore the first challenge in multiplicative case is that the integrals of noise parts are no longer of Ito type, i.e. the relevant integrands are not adapted.

Now define a operator $\Phi : \mathbb{R} \times \Omega \to \mathcal{L}(\mathbb{R}^d)$ by

$$\begin{cases} d\Phi(t) = A\Phi(t)\,dt + \sum\limits_{k=1}^{M} B_k\Phi(t) \circ dW_t^k \\ \Phi(0) = I \in \mathcal{L}(\mathbb{R}^d). \end{cases} \tag{3.2}$$

Due to the commutative property of A and B_k, Φ can be written in the explicit form as

$$\Phi(t,\omega) = \exp\left\{At + \textstyle\sum_{k=1}^{M} B_k W_t^k\right\}.$$

Note that since Φ is a linear perfect cocycle (Mohammed et al., 2008), it is not hard to check that Φ satisfies the conditions of Theorem 2.2.4, the multiplicative ergodic theorem in Euclidean space.

Lemma 3.1.1. *Suppose that $\frac{A+A^*}{2}$ has only nonzero eigenvalues with the order $\mu_p < \mu_{p-1} \cdots < \mu_{m+1} < 0 < \mu_m < \cdots < \mu_1$, $p \leq d$, and the corresponding eigenspaces $U_p, \cdots, U_1$ with multiplicity $d_i :=$ dim U_i and $\sum_{i=1}^{p} d_i = d$. Then*

1. *There exists a non-random splitting such that*

$$\mathbb{R}^d = E_p \oplus E_{p-1} \oplus \cdots \oplus E_{m+1} \cdots \oplus E_1 \quad \mathbb{P} - a.s.,$$

and

$$\mu_i = \lim_{t\to\pm\infty} \frac{1}{t} \log|\Phi(t,\omega)x|, \;\Leftrightarrow\; \quad x \in E_i \setminus \{0\},$$

for any $i \in \{1, 2, \cdots, p\}$, where μ_i is the Lyapunov exponent of Φ, with the corresponding multiplicity d_i, and $E_i = U_i$.

Moreover, $\mathbb{R}^d$ can be decomposed as

$$\mathbb{R}^d = E^- \oplus E^+,$$

where $E^- = E_p \oplus E_{p-1} \oplus \cdots \oplus E_{m+1}$ *is generated by the eigenvectors with negative eigenvalues, while* $E^+ = E_m \oplus E_{m-1} \oplus \cdots \oplus E_1$ *is generated by the eigenvectors with positive eigenvalues.*

2. *If* $P^\pm : \mathbb{R}^d \to E^\pm$ *denotes the projection onto* $E^\pm$ *along* $E^\mp$*, then*

$$\Phi(t, \theta_s\omega)P^\pm = P^\pm \Phi(t, \theta_s\omega) \quad \mathbb{P} - a.s.,$$

with the bounds

$$\begin{cases} \|\Phi(t, \theta_s\omega)P^+\| \leq C(\theta_s\omega)e^{\frac{1}{2}\mu_m t}, & when \;\; t \leq 0, \\ \|\Phi(t, \theta_s\omega)P^-\| \leq C(\theta_s\omega)e^{\frac{1}{2}\mu_{m+1} t}, & when \;\; t \geq 0, \end{cases}$$

for any $t, s \in \mathbb{R}$*, where* $C(\omega)$ *is a tempered random variable from above. Thus there exists an invariant set* $\tilde{\Omega}$ *of full measure in which we have the following bounds instead*

$$\begin{cases} \|\Phi(t, \theta_s\omega)P^+\| \leq C_\Lambda(\omega)e^{\frac{1}{2}\mu_m t}e^{\Lambda|s|}, & when \;\; t \leq 0, \\ \|\Phi(t, \theta_s\omega)P^-\| \leq C_\Lambda(\omega)e^{\frac{1}{2}\mu_{m+1} t}e^{\Lambda|s|}, & when \;\; t \geq 0, \end{cases} \tag{3.3}$$

where Λ *is an arbitrary positive number and* $C_\Lambda(\omega)$ *a positive random variable depending on* Λ.

Proof. 1. It is easy to show that Φ satisfies the condition $\sup_{0\leq t\leq 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\| \in L^1(\Omega)$. In fact we have by the Burkholder-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E} \sup_{0\leq t\leq 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\| &\leq \mathbb{E} \sup_{0\leq t\leq 1} \log^+ \exp\{\| \pm At + \textstyle\sum_{k=1}^M B_k W_{\pm t}^k\|\} \\ &\leq \mathbb{E} \sup_{0\leq t\leq 1} (\|A\|t + \textstyle\sum_{k=1}^M |B_k||W_{\pm t}^k|) \\ &\leq \|A\| + \textstyle\sum_{k=1}^M \|B_k\| \mathbb{E} \sup_{0\leq t\leq 1} |W_{\pm t}^k| \\ &\leq \|A\| + C_1 \textstyle\sum_{k=1}^M \|B_k\| < \infty, \end{aligned}$$

where C_1 is a positive constant. Then the MET theorem holds that ensures the existence of the random Oseledets splitting

$$\mathbb{R}^d = E_p(\omega) \oplus E_{p-1}(\omega) \oplus \cdots \oplus E_{m+1}(\omega) \oplus \cdots \oplus E_1(\omega)$$

and the corresponding random projections $P^\pm(\omega)$.

But if we consider the forward filtration and $\lim_{t\to\infty}(\Phi(t,\omega)^*\Phi(t,\omega))^{1/2t} := \Psi(\omega)$, the mutually commutative property of A, A^*, B_k, and B_k^* leads to

$$\begin{aligned}\Psi(\omega) &= \lim_{t\to\infty} \exp\left\{\frac{1}{2}(A+A^*) + \sum_{k=1}^{M}\frac{(B_k+B_k^*)W_t^k}{2t}\right\}\\ &= \exp\left\{\frac{A+A^*}{2}\right\}\lim_{t\to\infty}\exp\left\{\sum_{k=1}^{M}\frac{(B_k+B_k^*)W_t^k}{2t}\right\}\\ &= \exp\left\{\frac{A+A^*}{2}\right\}.\end{aligned}$$

Let $\mu_p < \mu_{p-1}\cdots < \mu_{m+1} < 0 < \mu_m < \cdots < \mu_1$ be the different eigenvalues of $\frac{A+A^*}{2}$, and let $U_p, \cdots, U_1$ the corresponding eigenspaces, with multiplicity $d_i :=$ dim U_i. Obviously, $e^{\mu_p} < e^{\mu_{p-1}}\cdots < e^{\mu_{m+1}} < 1 < e^{\mu_m} < \cdots < e^{\mu_1}$ are the different eigenvalues of $\exp\{\frac{A+A^*}{2}\}$, and $U_p, \cdots, U_1$ are still the corresponding orthogonal eigenspaces, with multiplicity $d_i :=$ dim U_i.

Then we can define $V_{p+1} := \{0\}$, and for $1 \le i \le p$, $i \in \mathbb{N}$,

$$V_i := U_p \oplus U_{p-1} \oplus \cdots \oplus U_i. \tag{3.4}$$

Therefore

$$V_p \subset V_{p-1} \subset \cdots \subset V_i \subset \cdots \subset V_1 = \mathbb{R}^d \tag{3.5}$$

defines a forward filtration.

Now consider the backward filtration and $\lim_{t\to\infty}(\Phi(-t,\omega)^*\Phi(-t,\omega))^{1/2t} := \tilde{\Psi}(\omega)$,

$$\begin{aligned}\tilde{\Psi}(\omega) &= \lim_{t\to\infty} \exp\left\{-\frac{1}{2}(A+A^*) - \sum_{k=1}^{M}\frac{(B_k+B_k^*)W_{-t}^k}{2t}\right\}\\ &= \exp\left\{-\frac{A+A^*}{2}\right\}\lim_{t\to\infty}\exp\left\{\sum_{k=1}^{M}\frac{-(B_k+B_k^*)W_{-t}^k}{2t}\right\}\\ &= \exp\left\{-\frac{A+A^*}{2}\right\}.\end{aligned}$$

Similarly $\tilde{\mu}_p < \tilde{\mu}_{p-1}\cdots < \tilde{\mu}_{p+1-m} < 0 < \tilde{\mu}_{p-m} < \cdots < \tilde{\mu}_1$ are the different eigenvalues w.r.t $-\frac{A+A^*}{2}$. Let $\tilde{U}_p, \cdots, \tilde{U}_1$ be the corresponding eigenspaces, with multiplicity $\tilde{d}_i :=$ dim $\tilde{U}_i$. Obviously, $\tilde{\mu}_i = -\mu_{p+1-i}$, and therefore $\tilde{U}_i = U_{p+1-i}$. Then we can define $\tilde{V}_{p+1} := \{0\}$, and for $1 \le i \le p$, $i \in \mathbb{N}$,

$$\tilde{V}_i := \tilde{U}_p \oplus \tilde{U}_{p-1} \oplus \cdots \oplus \tilde{U}_i = U_1 \oplus U_2 \oplus \cdots \oplus U_{p+1-i}. \tag{3.6}$$

Therefore

$$\tilde{V}_p \subset \tilde{V}_{p-1} \subset \cdots \subset \tilde{V}_i \subset \cdots \subset \tilde{V}_1 = \mathbb{R}^d \tag{3.7}$$

defines the backward filtration. Then we can construct the space E_i as the intersection of certain spaces from the forward filtration (3.5) and the backward filtration (3.7),

$$E_i := V_i \cap \tilde{V}_{p+1-i} = U_i. \tag{3.8}$$

Thus the Lyapunov exponents of Φ depend on $\frac{1}{2}(A + A^*)$ only. This implies that the Oseledets spaces are non-random and so are the corresponding projections $P^{\pm}$.

2. We have when $t \le 0$,

$$\begin{aligned}
\|\Phi(t,\omega)P^+\| &= \|P^{+*}\Phi(t,\omega)^*\Phi(t,\omega)P^+\|^{1/2} \\
&= \left\|P^{+*}\exp\left\{(A+A^*)t + \textstyle\sum_{k=1}^{M}(B_k + B_k^*)W_t^k\right\}P^+\right\|^{1/2} \\
&\le \left\|P^{+*}\exp\left\{(A+A^*)t\right\}P^+\right\|^{1/2}\left\|\exp\left\{\textstyle\sum_{k=1}^{M}(B_k + B_k^*)W_t^k\right\}\right\|^{1/2} \\
&\le \left\|\exp\left\{\frac{1}{2}(A+A^*)t\right\}P^+\right\|\exp\left\{\frac{1}{2}\textstyle\sum_{k=1}^{M}\|B_k + B_k^*\||W_t^k|\right\} \\
&\le e^{\mu_m t + \sum_{k=1}^{M}\|B\||W_t^k|} \\
&\le e^{\frac{1}{2}\mu t + \sum_{k=1}^{M}\|B\||W_t^k|}e^{\frac{1}{2}\mu_m t},
\end{aligned}$$

where $\|B\| := \frac{1}{2}\max_{k\in\{1,2,\cdots,M\}}\|B_k + B_k^*\|$, $\mu := \min\{-\mu_{m+1}, \mu_m\} > 0$. Define a new random variable

$$C(\omega) := \sup_{t\in\mathbb{R}} C(t,\omega) := \sup_{t\in\mathbb{R}} e^{-\frac{1}{2}\mu|t| + \sum_{k=1}^{M}\|B\||W_t^k|} \ge 1. \tag{3.9}$$

Now it suffices to check that $C(\omega)$ is tempered from above. This can be done similarly as in Duan et al. (2003). Actually we have

$$\lim_{s\to\pm\infty}\frac{1}{|s|}\log C(\theta_s\omega) = \lim_{s\to\pm\infty}\frac{1}{|s|}\log\sup_{t\in\mathbb{R}} C(t,\theta_s\omega),$$

and via the inequality of $|W(t+s)| \le C_{\delta,\omega} + |s|^\delta + |t|^\delta \quad \mathbb{P}-a.s.$ for some $\frac{1}{2} < \delta < 1$ from the iterated logarithm law of Brownian motion, the estimation above could be carried on as follows,

$$\begin{aligned}
&\lim_{s\to\pm\infty}\frac{1}{|s|}\log^+\sup_{t\in\mathbb{R}} C(t,\theta_s\omega) \\
=\ &\lim_{s\to\pm\infty}\frac{1}{|s|}\log\sup_{t\in\mathbb{R}} C(t,\theta_s\omega) \\
=\ &\lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\log e^{-\frac{1}{2}\mu|t| + \sum_{k=1}^{M}\|B\||\theta_s W_t^k|} \\
\le\ &\lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t| + \sum_{k=1}^{M}\|B\||W_{t+s}^k - W_s^k|\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+\sum_{k=1}^{M}\|B\||W^k_{t+s}|\right)+\lim_{s\to\pm\infty}\sum_{k=1}^{M}\|B\|\frac{|W^k_s|}{|s|}\\
&\leq \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+M\|B\|(C_{\delta,\omega}+|s|^\delta+|t|^\delta)\right)+0\\
&\leq \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta\right)\\
&\quad+\sup_{t\in\mathbb{R}}\lim_{s\to\pm\infty}M\|B\|\frac{|s|^\delta}{|s|}+\sup_{t\in\mathbb{R}}\lim_{s\to\pm\infty}\frac{M\|B\||C_{\delta,\omega}|}{|s|}\\
&= \lim_{s\to\pm\infty}\frac{1}{|s|}\sup_{t\in\mathbb{R}}\left(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta\right)\\
&= 0,\quad \mathbb{P}-a.s.,
\end{aligned}$$

where the last inequality holds due to the fact that $\sup_{t\in\mathbb{R}}(-\frac{1}{2}\mu|t|+M\|B\||t|^\delta)<\infty$. This together with the fact that

$$\lim_{s\to\pm\infty}\frac{1}{|s|}\log\sup_{t\in\mathbb{R}}C(t,\theta_s\omega)\geq 0,$$

which leads to

$$\lim_{s\to\pm\infty}\frac{1}{|s|}\log C(\theta_s\omega)=0\quad a.s.$$

Similar argument applies to $\Phi(t,\theta_s\omega)P^-$.

Finally by Remark 2.1.13, we could easily conclude the boundedness of $\Phi(t,\theta_s\omega)P^-$ and $\Phi(t,\theta_s\omega)P^+$.

□

For any $N\in\mathbb{N}$, we now set the truncation of $\Phi(t,\theta_{\hat{s}}\omega)P^{\pm}$ by N. As when $t\geq 0$,

$$\Phi^N(t,\theta_{\hat{s}}\omega)P^- := \Phi(t,\theta_{\hat{s}}\omega)P^-\min\left\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\},\tag{3.10}$$

and when $t\leq 0$,

$$\Phi^N(t,\theta_{\hat{s}}\omega)P^+ := \Phi(t,\theta_{\hat{s}}\omega)P^+\min\left\{1,\frac{Ne^{\frac{1}{2}\mu_m t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|}\right\}.\tag{3.11}$$

Note the solution of (3.1) via (3.2) is written (Mohammed et al., 2008) as

$$u(t,s,x,\omega)=\Phi(t-s,\theta_s\omega)x+\int_s^t\Phi(t-\hat{s},\theta_{\hat{s}}\omega)F(\hat{s},u(\hat{s},s,x,\omega))\,d\hat{s},\quad t\geq s.\tag{3.12}$$

We will look for a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y:\mathbb{R}\times\Omega\to\mathbb{R}^d$ which satisfies the following coupled forward-backward infinite horizon stochastic integral equation,

$$\begin{aligned} Y(t,\omega) &= \int_{-\infty}^{t}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y(\hat{s},\omega))d\hat{s} \\ &\quad -\int_{t}^{+\infty}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{+}F(\hat{s},Y(\hat{s},\omega))d\hat{s}, \end{aligned} \tag{3.13}$$

for all $\omega\in\Omega$, $t\in\mathbb{R}$, and a sequence $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable maps, $\{Y^N\}_{N\geq 1}$, each of which satisfying

$$\begin{aligned} Y^N(t,\omega) &= \int_{-\infty}^{t}\Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\ &\quad -\int_{t}^{+\infty}\Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^{+}F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}, \end{aligned} \tag{3.14}$$

for all $\omega\in\Omega$, $t\in\mathbb{R}$. In the following, we will develop tools to solve equation (3.13) via equation (3.14).

3.2 Main Results

The following substitution theorem for anticipating stochastic differential equations in Nualart (2000) will play an important role in the development of the connection between the infinite horizon integral equation and random perodic solutions.

Theorem 3.2.1. *Consider the following stochastic differential equation*

$$\begin{aligned} X_t &= X_0+\sum_{i=1}^{M}\int_0^t A_i(X_{\hat{s}})\circ dW_{\hat{s}}^i+\int_0^t A_0(X_{\hat{s}})d\hat{s} \\ &= X_0+\sum_{i=1}^{M}\int_0^t A_i(X_{\hat{s}})dW_{\hat{s}}^i+\int_0^t B(X_{\hat{s}})d\hat{s}, \end{aligned} \tag{3.15}$$

where $A_i\in\mathcal{C}(\mathbb{R}^d)$, $0\leq i\leq M$, and $B\in\mathcal{C}(\mathbb{R}^d)$. Assume that $A_i, 1\leq i\leq M$, are of class C^3, and $A_i, 1\leq i\leq M$, and B have bounded partial derivatives of first order. Then for any random vector X_0, the process $X=\{\varphi_t(X_0), t\in[0,1]\}$ satisfies the anticipating stochastic equation (3.15), where $\{\varphi_t(x), t\in[0,1]\}$ denotes the stochastic flow associated with the coefficient of equation (3.15), that is, the solution to the following stochastic differential *equation with initial value $x \in \mathbb{R}^d$:*

$$\begin{aligned}\varphi_t(x) &= x + \sum_{i=1}^{M} \int_0^t A_i(\varphi_{\hat{s}}(x)) \circ dW_{\hat{s}}^i + \int_0^t A_0(\varphi_{\hat{s}}(x))d\hat{s} \\ &= x + \sum_{i=1}^{M} \int_0^t A_i(\varphi_{\hat{s}}(x))dW_{\hat{s}}^i + \int_0^t B(\varphi_{\hat{s}}(x))d\hat{s}. \end{aligned} \tag{3.16}$$

With the help of the substitution result above, we are able to identify the random periodic solution of (3.1) with the solution of (3.13) similar to the additive noise case in Feng et al. (2011). However, we have to deal with the anticipating case, which is a lot more difficult than the additive noise case.

Theorem 3.2.2. *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous map, globally bounded wtih the Jacobian $\nabla F(t, \cdot)$ globally bounded, and assume $F(t, u) = F(t + \tau, u)$ for some fixed $\tau > 0$.*

If (3.1) has a unique solution $u(t, s, x, \omega)$ and the coupled forward-backward infinite horizon stochastic integral equation (3.13) has a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable solution $Y : \mathbb{R} \times \Omega \to \mathbb{R}^d$ such that $Y(t + \tau, \omega) = Y(t, \theta_\tau \omega)$ for any $t \in \mathbb{R}$, $\mathbb{P}$-a.s., then Y is a random periodic solution of equation (3.1) i.e.,

$$\begin{cases} u(t+s, t, Y(t, \omega), \omega) = Y(t+s, \omega), & \mathbb{P} - a.s., \\ Y(t+\tau, \omega) = Y(t, \theta_\tau \omega), & \mathbb{P} - a.s., \end{cases} \tag{3.17}$$

for any $t \in \mathbb{R}$ and $s \geq 0$.

Conversely, if the stochastic differential equation (3.1) has a random periodic solution $Y : \mathbb{R} \times \Omega \to \mathbb{R}^d$ of period τ, the modulus of which is tempered from above for each t, then Y is a solution of the coupled forward-backward infinite horizon stochastic integral equation (3.13).

Proof. If equation (3.13) has a solution $Y(\cdot, \omega)$, then for any $t \geq s$, by the cocycle property of Φ we have

$$\begin{aligned} Y(t, \omega) &= \int_{-\infty}^{s} \Phi(t-s, \theta_s \omega)\Phi(s-\hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &\quad - \int_s^{+\infty} \Phi(t-s, \theta_s \omega)\Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &\quad + \int_s^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} + \int_s^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &= \Phi(t-s, \theta_s \omega)Y(s, \omega) + \int_s^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega)F(\hat{s}, Y(\hat{s}, \omega))d\hat{s}. \end{aligned}$$

This is to say that $Y(t,\omega)$ satisfies (3.12) with initial value $Y(s,\omega)$. Now suppose that there are two solutions $u(t,s,\varphi_1,\omega)$ and $u(t,s,\varphi_2,\omega)$ satisfying equation (3.12), with initial values φ_1 and φ_2, which are $\mathcal{F}$-measurable. Then via the cocycle property of Φ we have that

$$\begin{aligned}
& |u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|^2 \\
\leq\; & 2\|\Phi(t-s,\theta_s\omega)\|^2|\varphi_1-\varphi_2|^2 \\
& +2(T-s)\int_s^t \|\Phi(t-\hat{s},\theta_{\hat{s}}\omega)\|^2\|\nabla F\|^2|u(\hat{s},s,\varphi_1,\omega)-u(\hat{s},s,\varphi_2,\omega)|^2\,d\hat{s}.
\end{aligned}$$

And for any $t>s$,

$$\begin{aligned}
\|\Phi(t-s,\theta_s\omega)\| &= \left\|\exp\{\frac{1}{2}(A+A^*)(t-s)\}\right\|\left\|\exp\{\frac{1}{2}\sum_{k=1}^{M}(B_k+B_k^*)(W_t^k-W_s^k)\}\right\| \\
&\leq e^{\mu_1(t-s)}\prod_{k=1}^{M}\exp\left\{\frac{1}{2}\|B_k+B_k^*\|\|W_t^k-W_s^k\|\right\} \\
&\leq e^{\mu_1(t-s)}\prod_{k=1}^{M}\exp\left\{\|B\|(2C_{\delta,\omega}^k+|t|^\delta+|s|^\delta)\right\} \\
&= e^{\mu_1(t-s)}\exp\left\{2M\|B\|(|t|^\delta+|s|^\delta)\right\}\exp\left\{2\|B\|\sum_{k=1}^{M}C_{\delta,\omega}^k\right\} \\
&\leq e^{\mu_1(t-s)}\exp\left\{2M\|B\|\hat{T}\right\}\exp\left\{2\|B\|\sum_{k=1}^{M}C_{\delta,\omega}^k\right\},
\end{aligned}$$

where $\hat{T}:=\max\{|T|,|s|\}$, and the third line holds due to the fact that there exists Ω_1 of full measure and a constant $\frac{1}{2}<\delta<1$ such that

$$\|W_t^k-W_s^k\|\leq 2C_{\delta,\omega}^k+|t|^\delta+|s|^\delta.$$

Then for any $s\leq t\leq T$,

$$\begin{aligned}
& |u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|^2 \\
\leq\; & 2H_\omega(T-s)|\varphi_1-\varphi_2|^2 \\
& +2(T-s)\|\nabla F\|^2H_\omega(T-s)\int_s^t|u(\hat{s},s,\varphi_1,\omega)-u(\hat{s},s,\varphi_2,\omega)|^2\,d\hat{s},
\end{aligned}$$

where

$$H_\omega(T-s)=e^{2\mu_1(t-s)}\exp\left\{4M\|B\|\hat{T}\right\}\exp\left\{4\|B\|\sum_{k=1}^{M}C_{\delta,\omega}^k\right\}.$$

Thus applying the Gronwall inequality gives

$$|u(t,s,\varphi_1,\omega)-u(t,s,\varphi_2,\omega)|^2\leq 2H_\omega(T-s)|\varphi_1-\varphi_2|^2e^{2\|\nabla F\|_\infty^2H_\omega(T-s)(T-s)^2}\quad \mathbb{P}-\text{a.s.}$$

This $u(t,s,\varphi_1,\omega) \to u(t,s,\varphi_2,\omega)$ when $\varphi_1 \to \varphi_2$, for any $\omega \in \Omega_1$ and $t \in [s,T]$. And from $\mathbb{P}(\Omega_1) = 1$,

$$\mathbb{P}(u(t,s,\varphi_1,\omega) = u(t,s,\varphi_2,\omega), \text{ when } \varphi_1 \to \varphi_2, \forall\, t \in \mathbb{Q} \cap [s,T]) = 1,$$

and from the continuity of $u(t,s,\cdot,\omega)$, we have shown it is a version of $u(t,s,\varphi_2,\omega)$ if $\varphi_1 = \varphi_2$.

This implies the uniqueness of solution to the SDE (3.12) within a finite time interval $[s,T]$. Then by Theorem 3.2.1 and the uniqueness of the solution of the initial value problem (3.12),

$$u(t,s,x,\omega)\Big|_{x=Y(s,\omega)} = u(t,s,Y(s,\omega),\omega) = Y(t,\omega).$$

Conversely, assume equation (3.1) has a random periodic solution which is tempered from above. First note for any non-negative integer l, we have by Theorem 3.2.1,

$$\begin{aligned}
Y(t,\omega) &= u(t \pm l\tau, t, Y(t,\theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega) \\
&= \Phi(\pm l\tau, \theta_{t\mp l\tau}\omega) Y(t,\theta_{\mp l\tau}\omega) \\
&\quad + \int_t^{t\pm l\tau} \Phi(t \pm l\tau - \hat{s}, \theta_{\hat{s}\mp l\tau}\omega) F(\hat{s}, u(\hat{s}, t, Y(t,\theta_{\mp l\tau}\omega), \theta_{\mp l\tau}\omega)) d\hat{s}.
\end{aligned}$$

In particular,

$$\begin{aligned}
P^- Y(t,\omega) &= P^- u(t + l\tau, t, Y(t,\theta_{-l\tau}\omega), \theta_{-l\tau}\omega) \\
&= P^- \Phi(l\tau, \theta_{t-l\tau}\omega) Y(t,\theta_{-l\tau}\omega) \\
&\quad + \int_t^{t+l\tau} P^- \Phi(t + l\tau - \hat{s}, \theta_{\hat{s}-l\tau}\omega) F(\hat{s}, u(\hat{s}, t, Y(t,\theta_{-l\tau}\omega), \theta_{-l\tau}\omega)) d\hat{s} \\
&= \Phi(l\tau, \theta_{t-l\tau}\omega) P^- Y(t,\theta_{-l\tau}\omega) \\
&\quad + \int_t^{t+l\tau} \Phi(t + l\tau - \hat{s}, \theta_{\hat{s}-l\tau}\omega) P^- F(\hat{s}, Y(\hat{s} - l\tau, \omega)) d\hat{s} \\
&= \Phi(l\tau, \theta_{t-l\tau}\omega) P^- Y(t,\theta_{-l\tau}\omega) + \int_{t-l\tau}^{t} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega) P^- F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} \\
&\to \int_{-\infty}^{t} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega) P^- F(\hat{s}, Y(\hat{s},\omega)) d\hat{s} \quad \mathbb{P}-\text{a.s.},
\end{aligned}$$

as $l \to +\infty$. Here P^- commutes with Φ due to the **Condition (C)**. The convergence deserves some justification.

Actually the pointwise convergence $\Phi(l\tau, \theta_{t-l\tau}\omega) P^- Y(t,\theta_{-l\tau}\omega) \overset{l\to+\infty}{\longrightarrow} 0$ follows from the estimation of Φ together with the tempered property of $Y(t,\omega)$ and $C(\omega)$. First it is easy to get

$$\|\Phi(l\tau, \theta_{t-l\tau}\omega)P^{-}\| < C(\theta_{t-l\tau}\omega)e^{\frac{1}{2}l\tau\mu_{m+1}} \quad \mathbb{P}-\text{a.s.},$$

and since $|Y(t, \theta_{-l\tau}\omega)|$ is tempered from above for each $t \in \mathbb{R}$, i.e.,

$$\lim_{l\to\infty} \frac{1}{l\tau} \log^{+} |Y(t, \theta_{-l\tau}\omega)| = 0,$$

which is equal to say for any $\varepsilon_1 > 0$, there exists a T_1, s.t. for all $l\tau > T_1$, we have

$$|Y(t, \theta_{-l\tau}\omega)| < \hat{C}_1(t,\omega)e^{\varepsilon_1 l\tau} \quad \mathbb{P}-\text{a.s.},$$

where $\hat{C}_1(t,\omega)$ is finite $\mathbb{P}-$ a.s. which also depends on t. Similarly consider this random variable $C(\omega)$ which is tempered from above, for any $\varepsilon_2 > 0$, there exists T_2 such that for any $l\tau > T_2$,

$$C(\theta_{t-l\tau}\omega) < \hat{C}_2(t,\omega)e^{\varepsilon_2 l\tau} \quad \mathbb{P}-\text{a.s.},$$

where $\hat{C}_2$ is almost surely finite random variable. Then totally we have

$$|\Phi(l\tau, \theta_{t-l\tau}\omega)P^{-}Y(t, \theta_{-l\tau}\omega)| < \hat{C}_1\hat{C}_2 e^{(\frac{1}{2}\mu_{m+1}+\varepsilon_1+\varepsilon_2)l\tau} \quad \mathbb{P}-\text{a.s.}. \tag{3.18}$$

Letting $\varepsilon_1 + \varepsilon_2 \le \frac{1}{2}\min\{\mu_m, |\mu_{m+1}|\}$, then the right hand side of (3.18) tends to 0 as $l \to +\infty$.

For the last convergence part, we have that by considering the tempered random variable

$$\begin{aligned}
&\lim_{l\to+\infty}\Big|\int_{t-l\tau}^{t}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y(\hat{s},\omega))d\hat{s} - \int_{-\infty}^{t}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y(\hat{s},\omega))d\hat{s}\Big| \\
= &\lim_{l\to+\infty}\Big|\int_{-\infty}^{t-l\tau}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y(\hat{s},\omega))d\hat{s}\Big| \\
\le &\hat{C}_2(t,\omega)\|F\|_\infty \lim_{l\to+\infty}\int_{-\infty}^{t-l\tau} e^{\varepsilon_2|\hat{s}|+\frac{\mu_{m+1}}{2}(t-\hat{s})}d\hat{s} \\
\le &\hat{C}_2(t,\omega)\|F\|_\infty \lim_{l\to+\infty}\int_{-\infty}^{t-l\tau} e^{(\varepsilon_2+\frac{1}{2}\mu_{m+1})(t-l\tau-\hat{s})}d\hat{s}e^{-\varepsilon_2 t}e^{(\varepsilon_2+\frac{1}{2}\mu_{m+1})l\tau} \\
\le &\hat{C}_2(t,\omega)\|F\|_\infty \frac{1}{\varepsilon_2+\frac{1}{2}\mu_{m+1}}e^{-\varepsilon_2 t}\lim_{l\to+\infty}e^{(\varepsilon_2+\frac{1}{2}\mu_{m+1})l\tau} = 0 \;\; \mathbb{P}-\text{a.s.},
\end{aligned}$$

where

$$||F||_\infty := \sup_{t\in\mathbb{R}, u\in\mathbb{R}^d} |F(t,u)|.$$

Thus

$$P^{-}Y(t,\omega) = \int_{-\infty}^{t}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^{-}F(\hat{s},Y(\hat{s},\omega))d\hat{s} \quad \mathbb{P}-\text{a.s.}.$$

Analogously,

$$\begin{aligned}
P^+Y(t,\omega) &= P^+u(t-l\tau,t,Y(t,\theta_{l\tau}\omega),\theta_{l\tau}\omega)\\
&= P^+\Phi(-l\tau,\theta_{t+l\tau}\omega)Y(t,\theta_{l\tau}\omega)\\
&\quad+\int_t^{t-l\tau}P^+\Phi(t-l\tau-\hat{s},\theta_{\hat{s}+l\tau}\omega)F(\hat{s},Y(\hat{s},\theta_{l\tau}\omega))d\hat{s}\\
&= \Phi(-l\tau,\theta_{t+l\tau}\omega)P^+Y(t,\theta_{l\tau}\omega)\\
&\quad+\int_t^{t-l\tau}\Phi(t-l\tau-\hat{s},\theta_{\hat{s}+l\tau}\omega)P^+F(\hat{s},Y(\hat{s}+l\tau,\omega))d\hat{s}\\
&= \Phi(-l\tau,\theta_{t+l\tau}\omega)P^+Y(t,\theta_{l\tau}\omega)-\int_t^{t+l\tau}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^+F(\hat{s},Y(\hat{s},\omega))d\hat{s}\\
&\to -\int_t^{+\infty}\Phi(t-\hat{s},\theta_{\hat{s}}\omega)P^+F(\hat{s},Y(\hat{s},\omega))d\hat{s}\quad \mathbb{P}-\text{a.s.}
\end{aligned}$$

as $l\to+\infty$. Therefore we have proved the converse part as $Y=P^+Y+P^-Y$. □

Before proving the existence theorem, some elementary but essential estimations are given in the following lemma.

Lemma 3.2.3. *For any $t\geq 0$, and $\hat{s}\in\mathbb{R}$, we have*

$$\mathbb{E}\|P^- - \Phi(t,\theta_{\hat{s}}\cdot)P^-\|^2=\mathbb{E}\|P^- - \Phi(t,\cdot)P^-\|^2\leq C(|t|+1)e^{2\|A\||t|+2M\|B\|^2|t|}|t|,$$

where C is a constant that may depend on M, A, B_k, μ_{m+1}, μ_m, F, and τ, and for any $t\leq 0$, and $\hat{s}\in\mathbb{R}$, we have

$$\mathbb{E}\|P^+ - \Phi(t,\theta_{\hat{s}}\cdot)P^+\|^2=\mathbb{E}\|P^+ - \Phi(t,\cdot)P^+\|^2\leq C(|t|+1)e^{2\|A\||t|+2M\|B\|^2|t|}|t|.$$

Moreover, we have for all $t\in\mathbb{R}$,

$$\mathbb{E}\|\Phi(t,\theta_{\hat{s}}\cdot)P^\pm\|^2=\mathbb{E}\|\Phi(t,\cdot)P^\pm\|^2\leq Ce^{2\|A\||t|+2M\|B\|^2|t|}.$$

Proof. We consider P^- case only. The estimation for P^+ case can be derived analogously. From equation (3.2) and the definition of P^-, it is natural to express the projection ΦP^- : $\mathbb{R}\times\Omega\to\mathcal{L}(\mathbb{R}^d,E^-)$ as follows,

$$\begin{cases} d\Phi(t,\omega)P^- = A\Phi(t)P^-\,dt+\sum\limits_{k=1}^{M}B_k\Phi(t)P^-\circ dW_t^k,\\ \Phi(0,\omega)P^- = P^-\in\mathcal{L}(\mathbb{R}^d,E^-). \end{cases}\tag{3.19}$$

$\Phi_{t,\hat{s}}P^{\pm}$ is short for $\Phi(t,\theta_{\hat{s}}\omega)P^{\pm}$, and $\Phi_t P^{\pm}$ is short for $\Phi(t,\omega)P^{\pm}$ from now on. Then for any $t,\hat{s}\in\mathbb{R}$, by the ergodic property of θ and Hölder's inequality we have that

$$
\begin{aligned}
& \mathbb{E}\|P^- - \Phi(t,\theta_{\hat{s}}\cdot)P^-\|^2 \\
= \ & \mathbb{E}\Big\| \int_0^t \Big(A+\frac{1}{2}\sum_{k=1}^M B_k^2\Big)\Phi_{\hat{h},\hat{s}}P^- d\hat{h} + \sum_{k=1}^M \int_0^t B_k \Phi_{\hat{h},\hat{s}}P^- dW^k_{\hat{h}+\hat{s}}\Big\|^2 \\
\leq \ & (M+1)\Big\|A+\frac{1}{2}\sum_{k=1}^M B_k^2\Big\|^2 \mathbb{E}\Big(\int_0^t \|\Phi_{\hat{h},\hat{s}}P^-\|d\hat{h}\Big)^2 + (M+1)\sum_{k=1}^M \mathbb{E}\int_{\hat{s}}^{t+\hat{s}} \|B_k\|^2\|\Phi_{\hat{h}-\hat{s},\hat{s}}P^-\|^2 d\hat{h} \\
\leq \ & (M+1)\Big\|A+\frac{1}{2}\sum_{k=1}^M B_k^2\Big\|^2 |t| \int_0^t \mathbb{E}\|\Phi_{\hat{h}}P^-\|^2 d\hat{h} + (M+1)\sum_{k=1}^M \|B_k\|^2 \int_{\hat{s}}^{t+\hat{s}} \mathbb{E}\|\Phi_{\hat{h}-\hat{s}}P^-\|^2 d\hat{h} \\
\leq \ & 2^M(M+1)\Big(2\|A\|^2|t| + \Big(\textstyle\sum_{k=1}^M \|B_k\|^2\Big)^2|t| + \sum_{k=1}^M \|B_k\|^2\Big)e^{2\|A\||t|+2M\|B\|^2|t|}|t|,
\end{aligned}
$$

where we used

$$
\begin{aligned}
\int_0^t \mathbb{E}\|\Phi_{\hat{h}}P^-\|^2 d\hat{h} &\leq \int_0^t \mathbb{E}\|e^{A\hat{h}+\sum_{k=1}^M B^k W^k_{\hat{h}}}\|^2 d\hat{h} \leq \int_0^t e^{2\|A\|\hat{h}} \prod_{k=1}^M \mathbb{E}e^{2\|B\||W^k_{\hat{h}}|} d\hat{h} \\
&= 2^M \int_0^t e^{2\|A\|\hat{h}} e^{2M\|B\|^2\hat{h}} d\hat{h} \leq 2^M e^{2\|A\||t|+2M\|B\|^2|t|}|t|. \qquad (3.20)
\end{aligned}
$$

Finally the last inequality can be easily drawn from (3.20). □

Theorem 3.2.4. *Let $F:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ be a continuous map, globally bounded and the Jacobian $\nabla F(t,\cdot)$ be globally bounded, and $F(t,u)=F(t+\tau,u)$ for some fixed $\tau>0$. Assume* **Condition (C)**, *the commutativity assumption.*

Then there exists at least one $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y^N:\mathbb{R}\times\Omega\to\mathbb{R}^d$ satisfying equation (3.14) and $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$ for any $t\in\mathbb{R}$ and $\omega\in\Omega$.
Moreover, there exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable map $Y:\mathbb{R}\times\Omega\to\mathbb{R}^d$ satisfying equation (3.13) and $Y(t+\tau,\omega)=Y(t,\theta_\tau\omega)$ for any $t\in\mathbb{R}$ and $\omega\in\Omega$.

The proof of this theorem is quite long, so we break into many parts. Firstly define a Banach space $C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$

$C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d)) := \{f\in C^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d)):$ for any $t\in\mathbb{R},\ f(t+\tau,\omega)=f(t,\theta_\tau\omega)\}$,

where the norm of the metric space $C^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ is given as follows,

$$\|f\|_\Lambda := \sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}\|f(t,\cdot)\|_{L^2(\Omega,\mathbb{R}^d)},$$

which is indeed a weighted norm with $0<\Lambda<\frac{1}{4}\mu=\frac{1}{4}\min\{-\mu_{m+1},\mu_m\}$. Define a map $\mathcal{M}^N$: for any $Y^N\in C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$,

$$\begin{aligned}\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^t \Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^-F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\ &\quad -\int_t^{+\infty}\Phi^N(t-\hat{s},\theta_{\hat{s}}\omega)P^+F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}.\end{aligned}$$

Lemma 3.2.5. *Under the conditions of Theorem 3.2.4, the map*

$$\mathcal{M}^N: C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))\to C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$$

is continuous.

Proof. ***Step 1***: We show that $\mathcal{M}^N$ maps $C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ into itself.

(A) First verify that for any $Y^N\in C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$,

$$\sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2<\infty.$$

Actually by (3.10) and (3.11) we have that

$$\begin{aligned}&e^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2\\ \le\ & 2e^{-2\Lambda|t|}\mathbb{E}\Big|\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N)d\hat{s}\Big|^2+2e^{-\Lambda|t|}\mathbb{E}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N)d\hat{s}\Big|^2\\ \le\ & 2e^{-2\Lambda|t|}\|F\|_\infty^2\Big\{\mathbb{E}\Big(\int_{-\infty}^t\|\Phi^N_{t-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2+\mathbb{E}\Big(\int_t^{+\infty}\|\Phi^N_{t-\hat{s},\hat{s}}P^+\|d\hat{s}\Big)^2\Big\}\\ \le\ & 2N^2\|F\|_\infty^2e^{-2\Lambda|t|}\Big\{\Big(\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2+\Big(\int_t^{+\infty}e^{\frac{1}{2}\mu_m(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2\Big\}.\end{aligned}$$

Now notice that $e^{\Lambda|\hat{s}|}\le e^{-\Lambda\hat{s}}+e^{\Lambda\hat{s}}$, $e^{-2\Lambda|t|}\le e^{-2\Lambda t}$, and $e^{-2\Lambda|t|}\le e^{2\Lambda t}$ for all $\hat{s},t\in\mathbb{R}$. It follows from above that

$$\begin{aligned}&e^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N)(t,\cdot)|^2\\ \le\ & 2N^2\|F\|_\infty^2e^{-2\Lambda|t|}\Big\{\Big(\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-\Lambda\hat{s}}d\hat{s}\Big)^2+\Big(\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda\hat{s}}d\hat{s}\Big)^2\\ &\qquad+\Big(\int_t^{+\infty}e^{\frac{1}{2}\mu_m(t-\hat{s})}e^{\Lambda\hat{s}}d\hat{s}\Big)^2+\Big(\int_t^{+\infty}e^{\frac{1}{2}\mu_m(t-\hat{s})}e^{-\Lambda\hat{s}}d\hat{s}\Big)^2\Big\}\\ \le\ & 2N^2\|F\|_\infty^2\Big\{\Big(\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\Big)^2+\Big(\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\Big)^2\\ &\qquad+\Big(\int_t^{+\infty}e^{(\frac{1}{2}\mu_m+\Lambda)(t-\hat{s})}d\hat{s}\Big)^2+\Big(\int_t^{+\infty}e^{(\frac{1}{2}\mu_m-\Lambda)(t-\hat{s})}d\hat{s}\Big)^2\Big\}\\ \le\ & 8N^2\|F\|_\infty^2\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2}+\frac{1}{(\mu_{m+1}-2\Lambda)^2}+\frac{1}{(\mu_m+2\Lambda)^2}+\frac{1}{(\mu_m-2\Lambda)^2}\Big\}.\end{aligned}$$

(B) Next we show that $\mathcal{M}^N(Y^N)(\cdot,\omega)$ is continuous from $\mathbb{R}$ to $L^2(\Omega,\mathbb{R}^d)$ for any given $Y^N \in C^\Lambda_\tau(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$. First note for any t_1, $t_2 \in \mathbb{R}$ with $t_1 \le t_2$,

$$\begin{aligned}
& \mathbb{E}|\mathcal{M}^N(Y^N)(t_1) - \mathcal{M}^N(Y^N)(t_2)|^2 \\
\le\ & 4\mathbb{E}\Big[\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^- - \Phi^N_{t_2-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N)d\hat{s}\Big|^2 + \Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^-F(\hat{s},Y^N)d\hat{s}\Big|^2 \\
& +\Big|\int_{t_2}^{+\infty}(\Phi^N_{t_1-\hat{s},\hat{s}}P^+ - \Phi^N_{t_2-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N)d\hat{s}\Big|^2 + \Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^+F(\hat{s},Y^N)d\hat{s}\Big|^2\Big] \\
=:\ & \sum_{i=1}^4 T_i.
\end{aligned}$$

It's easy to check that

$$\begin{aligned}
T_2 &:= 4\mathbb{E}\Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \\
&\le 4\|F\|^2_\infty\mathbb{E}\Big(\int_{t_1}^{t_2}\|\Phi^N_{t_2-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2 \\
&\le 4N^2\|F\|^2_\infty\Big(\int_{t_1}^{t_2}e^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 \\
&\le 4N^2\|F\|^2_\infty \max\{e^{2\Lambda|t_2|}, e^{2\Lambda|t_1|}\}|t_2-t_1|^2,
\end{aligned}$$

and similarly

$$T_4 := 4\mathbb{E}\Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \le 4N^2\|F\|^2_\infty \max\{e^{2\Lambda|t_2|}, e^{2\Lambda|t_1|}\}|t_2-t_1|^2.$$

As for T_1, we have the following inequalities through the estimations in Lemma 3.2.3,

$$\begin{aligned}
T_1 &:= 4\mathbb{E}\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^- - \Phi^N_{t_2-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \\
&\le 8\mathbb{E}\Big|\int_{-\infty}^{t_1}(\Phi_{t_1-\hat{s},\hat{s}}P^- - \Phi_{t_2-\hat{s},\hat{s}}P^-)\min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\}F(\hat{s},Y^N)d\hat{s}\Big|^2 \\
&\quad +8\mathbb{E}\Big|\int_{-\infty}^{t_1}\Phi_{t_2-\hat{s},\hat{s}}P^-\Bigg(\min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\} \\
&\qquad\qquad - \min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\Big\}\Bigg)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\Big|^2.
\end{aligned}$$

By using inequality $|\min\{1,a\} - \min\{1,b\}| \le |a-b|$ whenever $a, b \ge 0$, T_1 can be further developed as follows,

$$\begin{aligned}
\le\ & 8\|F\|^2_\infty\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2\Bigg(\int_{-\infty}^{t_1}\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|\min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\Big\}d\hat{s}\Bigg)^2 \\
& +8\|F\|^2_\infty\mathbb{E}\Bigg(\int_{-\infty}^{t_1}\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|\Bigg|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|} - \frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\Bigg|d\hat{s}\Bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \; 8N^2\|F\|_\infty^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2\Big(\int_{-\infty}^{t_1} e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 \\
&\quad +16\|F\|_\infty^2\mathbb{E}\left(\int_{-\infty}^{t_1}\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|\left|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}-\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\right|d\hat{s}\right)^2 \\
&\quad +16\|F\|_\infty^2\mathbb{E}\left(\int_{-\infty}^{t_1}\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|\left|\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}-\frac{Ne^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}\right|d\hat{s}\right)^2 \\
&\leq \; 8N^2\|F\|_\infty^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2\Big(\int_{-\infty}^{t_1} e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 \\
&\quad +16N^2\|F\|_\infty^2\mathbb{E}\left(\int_{-\infty}^{t_1} e^{\Lambda|\hat{s}|}\Big(e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}-e^{\frac{1}{2}\mu_{m+1}(t_2-\hat{s})}\Big)\frac{\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}d\hat{s}\right)^2 \\
&\quad +16N^2\|F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^{t_1} e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}\left|\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|-\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\right|d\hat{s}\Big)^2 \\
&\leq \; 8N^2\|F\|_\infty^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2\Big(\int_{-\infty}^{t_1} e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 \\
&\quad +16N^2\|F\|_\infty^2\mathbb{E}\left(\int_{-\infty}^{t_1} e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}\Big(1-e^{\frac{1}{2}\mu_{m+1}(t_2-t_1)}\Big)\right. \\
&\qquad\qquad\qquad \left.\cdot\frac{\|\Phi_{t_2-t_1,t_1}P^-\|\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}d\hat{s}\right)^2 \\
&\quad +16N^2\|F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^{t_1} e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^- - \Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}d\hat{s}\Big)^2 \\
&\leq \; 96N^2\|F\|_\infty^2\Big\{\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2+\Big(e^{\frac{1}{2}\mu_{m+1}(t_2-t_1)}-1\Big)^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^-\|^2\Big\} \\
&\quad \cdot\Big\{\Big(\int_{-\infty}^{t_1} e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{\Lambda\hat{s}}d\hat{s}\Big)^2+\Big(\int_{-\infty}^{t_1} e^{\frac{1}{2}\mu_{m+1}(t_1-\hat{s})}e^{-\Lambda\hat{s}}d\hat{s}\Big)^2\Big\} \\
&\leq \; 384N^2\|F\|_\infty^2 e^{2\Lambda|t_1|}\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2}+\frac{1}{(\mu_{m+1}-2\Lambda)^2}\Big\} \\
&\quad \cdot(\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^- - P^-\|^2+\mu_{m+1}^2(t_2-t_1)^2\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^-\|^2) \\
&\leq \; CN^2\|F\|_\infty^2 e^{2\Lambda|t_1|}e^{2\|A\||t_2-t_1|+2M\|B\|^2|t_2-t_1|}\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2}+\frac{1}{(\mu_{m+1}-2\Lambda)^2}\Big\} \\
&\quad \cdot\{(1+\mu_{m+1}^2)|t_2-t_1|^2+|t_2-t_1|\},
\end{aligned}
$$

where we have used the cocycle property of Φ,

$$
\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^- - \Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\leq\|\Phi_{t_2-t_1,t_1}P^- - P^-\|,
$$

and the last inequality follows from Lemma 3.2.3. Similarly,

$$
T_3 \;:=\; 4\mathbb{E}\Big|\int_{t_2}^{+\infty}(\Phi^N_{t_1-\hat{s},\hat{s}}P^+ - \Phi^N_{t_2-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2
$$

$$\leq CN^2\|F\|_\infty^2 e^{2\Lambda|t_2|} e^{2\|A\||t_2-t_1|+2M\|B\|^2|t_2-t_1|}\Big\{\frac{1}{(\mu_m+2\Lambda)^2}+\frac{1}{(\mu_m-2\Lambda)^2}\Big\}$$
$$\cdot\{(1+\mu_m^2)|t_2-t_1|^2+|t_2-t_1|\}.$$

(C) We show that $\mathcal{M}^N(Y^N)(t,\theta_{\pm\tau}\omega)=\mathcal{M}^N(Y^N)(t\pm\tau,\omega)$: similar as in Feng, Zhao and Zhou (Feng et al., 2011),

$$\begin{aligned}\mathcal{M}^N(Y^N)(t,\theta_\tau\omega) &= \int_{-\infty}^t \Phi^N_{t-\hat{s},\hat{s}+\tau}P^-F(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}\\ &\quad -\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}+\tau}P^+F(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}\\ &= \int_{-\infty}^t \Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^-F(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\ &\quad -\int_t^{+\infty}\Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^+F(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\ &= \int_{-\infty}^{t+\tau}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^-F(\hat{h},Y^N(\hat{h},\omega))d\hat{h}\\ &\quad -\int_{t+\tau}^{+\infty}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^+F(\hat{h},Y^N(\hat{h},\omega))d\hat{h}\\ &= \mathcal{M}^N(Y^N)(t+\tau,\omega),\end{aligned}$$

since $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$.

Thus we completed the Step 1 and proved that $\mathcal{M}^N$ maps $C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ into itself.

Step 2: We now check the continuity of the map $\mathcal{M}^N : C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d))\to C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$. For this, consider $Y_1^N,Y_2^N\in C^\Lambda_\tau(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$, and $t\in[j\tau,(j+1)\tau)$, $j\in\mathbb{Z}$, then

$$\begin{aligned}&e^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y_1^N)(t,\cdot)-\mathcal{M}(Y_2^N)(t,\cdot)|^2\\ \leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\Big|\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_1^N(\hat{s},\cdot))d\hat{s}-\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_2^N(\hat{s},\cdot))d\hat{s}\Big|^2\\ &+2e^{-2\Lambda|t|}\mathbb{E}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y_1^N(\hat{s},\cdot))d\hat{s}-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y_2^N(\hat{s},\cdot))d\hat{s}\Big|^2,\\ :=\ & \hat{T}_1+\hat{T}_2,\end{aligned}$$

where we have

$$\begin{aligned}\hat{T}_1 &= 2e^{-2\Lambda|t|}\mathbb{E}\Big|\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_1^N(\hat{s},\cdot))d\hat{s}-\int_{-\infty}^t\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y_2^N(\hat{s},\cdot))d\hat{s}\Big|^2\\ &\leq 2\|\nabla F\|_\infty^2e^{-2\Lambda|t|}\mathbb{E}\Big(\int_{-\infty}^t\|\Phi^N_{t-\hat{s},\hat{s}}P^-\||Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\\ &\leq 4N^2\|\nabla F\|_\infty^2e^{-2\Lambda|t|}\mathbb{E}\Big(\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\end{aligned}$$

$$
\begin{aligned}
&\leq 4N^2\|\nabla F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\\
&\quad +4N^2\|\nabla F\|_\infty^2\mathbb{E}\Big(\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|d\hat{s}\Big)^2\\
&\leq 4N^2\|\nabla F\|_\infty^2\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\,\mathbb{E}\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\quad +4N^2\|\nabla F\|_\infty^2\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\,\mathbb{E}\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\leq \frac{8}{|\mu_{m+1}-2\Lambda|}N^2\|\nabla F\|_\infty^2\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\quad +\frac{8}{|\mu_{m+1}+2\Lambda|}N^2\|\nabla F\|_\infty^2\int_{-\infty}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}.
\end{aligned}
$$

$\hat{T}_1$ can be evaluated by the periodic property of Y and the measure-preserving property,

$$
\begin{aligned}
\hat{T}_1 &\leq \frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}\Big\{\int_{j\tau}^t e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\qquad +\sum_{i=-\infty}^{j-1}\int_{i\tau}^{(i+1)\tau} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\Big\}\\
&\quad +\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}\Big\{\int_{j\tau}^t e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\qquad +\sum_{i=-\infty}^{j-1}\int_{i\tau}^{(i+1)\tau} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\Big\}\\
&\leq \frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)j\tau}\mathbb{E}|Y_1^N(\hat{s},\theta_{j\tau}\cdot)-Y_2^N(\hat{s},\theta_{j\tau}\cdot)|^2d\hat{s}\\
&\qquad +\int_0^\tau\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\theta_{i\tau}\cdot)-Y_2^N(\hat{s},\theta_{i\tau}\cdot)|^2d\hat{s}\Big\}\\
&\quad +\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)j\tau}\mathbb{E}|Y_1^N(\hat{s},\theta_{j\tau}\cdot)-Y_2^N(\hat{s},\theta_{j\tau}\cdot)|^2d\hat{s}\\
&\qquad +\int_0^\tau\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\theta_{i\tau}\cdot)-Y_2^N(\hat{s},\theta_{i\tau}\cdot)|^2d\hat{s}\Big\}\\
&= \frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)j\tau}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\qquad +\int_0^\tau\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\Big\}\\
&\quad +\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)j\tau}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\\
&\qquad +\int_0^\tau\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2d\hat{s}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}e^{2\Lambda\tau}\sup_{\hat{s}\in[0,\tau)} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2 \\
& \cdot\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)j\tau}d\hat{s}+\int_0^{\tau}\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\Big\} \\
& +\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}e^{2\Lambda\tau}\sup_{\hat{s}\in[0,\tau)} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2 \\
& \cdot\Big\{\int_0^{t-j\tau} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)j\tau}d\hat{s}+\int_0^{\tau}\sum_{i=-\infty}^{j-1} e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\Big\} \\
=\ & \frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}e^{2\Lambda\tau}\sup_{\hat{s}\in[0,\tau)} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s} \\
& +\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}e^{2\Lambda\tau}\sup_{\hat{s}\in[0,\tau)} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s} \\
\leq\ & 16N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big\{\frac{1}{(\mu_{m+1}-2\Lambda)^2}+\frac{1}{(\mu_{m+1}+2\Lambda)^2}\Big\}\sup_{\hat{s}\in\mathbb{R}} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2,
\end{aligned}
$$

and similarly

$$
\hat{T}_2 \quad\leq\quad 16N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big\{\frac{1}{(\mu_{m}-2\Lambda)^2}+\frac{1}{(\mu_{m}+2\Lambda)^2}\Big\}\sup_{\hat{s}\in\mathbb{R}} e^{-2\Lambda|\hat{s}|}\mathbb{E}|Y_1^N(\hat{s},\cdot)-Y_2^N(\hat{s},\cdot)|^2.
$$

Then the claim that $\mathcal{M}^N : C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d)) \to C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$ is a continuous map is verified. □

Remark 3.2.6. *One can see that it is crucial to use the truncation of the tempered random variable $C(\omega)$ in the Step 2 of the proof. Without the truncation technique, it would be difficult to seperate $\|\Phi_{t-\hat{s},\hat{s}}P^{\pm}\|^2$ and $|Y_1^N(\hat{s},\omega)-Y_2^N(\hat{s},\omega)|^2$ in the estimate of $\hat{T}_1$ and $\hat{T}_2$. For this, classical Hölder's inequality technique loses its power here.*

Next introduce a subset of $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$ as follows,

$$
\begin{aligned}
C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2}) \quad := \quad & \Big\{ f\in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d)) :\ f|_{[0,\tau)}\in C([0,\tau),\mathcal{D}^{1,2}), \\
& \text{and } \forall t\in[0,\tau),\ l\in\{1,\cdots,M\}, \\
& e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l f(t,\cdot)|^2 dr \leq \rho^N(t), \\
& \text{and } \forall\delta\in\mathbb{R},\ \sup_{t\in[0,\tau)} e^{-2\Lambda|t|}\frac{1}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_{r+\delta}^l f(t,\cdot)-\mathcal{D}_r^l f(t,\cdot)|^2 dr<\infty\Big\}.
\end{aligned}
$$

Here

$$
\rho^N(t) := K_1^N\int_0^\tau e^{-\frac{1}{2}\mu|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}+K_2^N, \tag{3.21}
$$

where

$$K_1^N := 12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big(\frac{\sum_{i=-1}^{\infty} e^{-\frac{1}{2}\mu_m i\tau}}{|\mu_m - 4\Lambda|} + \frac{\sum_{i=-1}^{\infty} e^{-\frac{1}{2}\mu_m i\tau}}{|\mu_m + 4\Lambda|} + \frac{\sum_{i=-1}^{\infty} e^{\frac{1}{2}\mu_{m+1} i\tau}}{|\mu_{m+1} - 4\Lambda|} + \frac{\sum_{i=-1}^{\infty} e^{\frac{1}{2}\mu_{m+1} i\tau}}{|\mu_{m+1} + 4\Lambda|}\Big),$$

$$K_2^N := 48\|B\|^2N^2\|F\|_\infty^2(1+d^3)^2\Big(\frac{1}{|\mu_m + 2\Lambda|^3} + \frac{1}{|\mu_m - 2\Lambda|^3} + \frac{1}{|\mu_{m+1} + 2\Lambda|^3} + \frac{1}{|\mu_{m+1} - 2\Lambda|^3}\Big).$$

Lemma 3.2.7. *Under the conditions of Theorem 3.2.4, we have*

$$\mathcal{M}^N(C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})) \subset C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2}).$$

Moreover, $\mathcal{M}^N(C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})|_{[0,\tau)}$ *is relatively compact in* $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$.

Proof. ***Step 1***: In this step we are going to prove that $\mathcal{M}^N$ maps $C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})$ into itself.

1. First we have $\mathcal{M}^N(C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))) \subset C_\tau^\Lambda(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$: the argument here is the same as in ***Step 1*** in the proof of Lemma 3.2.5.

2. Next to illustrate that for any $t \in [0,\tau)$, $l \in \{1,\cdots,M\}$ and any $Y^N \in C_{\tau,\rho}^{\Lambda,N}(\mathbb{R},\mathcal{D}^{1,2})$,

$$e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}} |\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \le \rho^N(t).$$

First we can calculate the Malliavin derivatives of Φ^N by the chain rule: when $t \ge 0$, from Proposition 1.2.3 and Proposition 1.2.4 in Nualart (2000) (or directly obtained from the proof of Proposition 2.1.10 in Nualart (2000)), we know that $\varphi(F) := \min\{1,F\} \in \mathcal{D}^{1,2}$ if $F \in \mathcal{D}^{1,2}$, and for fixed t and s we have that

$$\mathcal{D}_r^l \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\} = \chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\| > Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}\}}(\omega)\mathcal{D}_r^l \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}, \tag{3.22}$$

Then we have, for $l = \{1,2,\cdots,M\}$,

$$\begin{aligned}
&\mathcal{D}_r^l\Phi^N(t,\theta_{\hat{s}}\omega)P^- \\
= \;& \mathcal{D}_r^l\left(\Phi(t,\theta_{\hat{s}}\omega)P^- \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\}\right) \\
= \;& \mathcal{D}_r^l\big(\Phi(t,\theta_{\hat{s}}\omega)P^-\big) \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\} + \Phi(t,\theta_{\hat{s}}\omega)P^-\mathcal{D}_r^l \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\} \\
= \;& \mathcal{D}_r^l\left(\exp\{At + \textstyle\sum_{k=1}^M B_k\theta_{\hat{s}}(W_t)\}P^-\right) \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\} \\
& -\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\| > Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}\}}(\omega)\Phi(t,\theta_{\hat{s}}\omega)P^- \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^2}\mathcal{D}_r^l\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|
\end{aligned}$$

$$
\begin{aligned}
&= \chi_{\{\hat{s}\le r\le t+\hat{s}\}}(r)\mathcal{D}_r^l\left(\exp\{At+\textstyle\sum_{k=1}^M B_k\theta_{\hat{s}}(W_t)\}P^-\right)\min\left\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\}\\
&\quad -\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|>Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}\}}(\omega)\Phi(t,\theta_{\hat{s}}\omega)P^-\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^3}\\
&\quad \cdot\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\mathcal{D}_r^l(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\\
&= \chi_{\{\hat{s}\le r\le t+\hat{s}\}}(r)\Bigg\{B_l\Phi(t,\theta_{\hat{s}}\omega)P^-\min\left\{1,\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|}\right\}\\
&\qquad -\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|>Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}\}}(\omega)\frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^3}\\
&\qquad \cdot\Big(\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}\sum_{k=1}^d(B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{kj}\Big)\Phi(t,\theta_{\hat{s}}\omega)P^-\Bigg\},
\end{aligned}
\tag{3.23}
$$

Due to the equivalence of norms for matrix operators, here we define $\|\Phi(t,\theta_{\hat{s}}\omega)P^-\| := \sqrt{\sum_{i,j=1}^d|(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}|^2}$, $(J)_{ij}$ stands for the ijth element of the matrix J, and $\mathcal{D}_r^l(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{ij}=\sum_{k=1}^d(B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^-)_{kj}$.

Analogously, when $t\le 0$,

$$
\begin{aligned}
&\mathcal{D}_r^l(\Phi^N(t,\theta_{\hat{s}}\omega)P^+)\\
&= \chi_{\{t+\hat{s}\le r\le\hat{s}\}}(r)\Bigg\{-B_l\Phi(t,\theta_{\hat{s}}\omega)P^+\min\left\{1,\frac{Ne^{\frac{1}{2}\mu_m t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|}\right\}\\
&\qquad +\chi_{\{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|>Ne^{\frac{1}{2}\mu_m t}e^{\Lambda|\hat{s}|}\}}(\omega)\frac{Ne^{\frac{1}{2}\mu_m t}e^{\Lambda|\hat{s}|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^+\|^3}\\
&\qquad \cdot\Big(\sum_{i,j=1}^d(\Phi(t,\theta_{\hat{s}}\omega)P^+)_{ij}\sum_{k=1}^d(B_l)_{ik}(\Phi(t,\theta_{\hat{s}}\omega)P^+)_{kj}\Big)\Phi(t,\theta_{\hat{s}}\omega)P^+\Bigg\}.
\end{aligned}
\tag{3.24}
$$

By the chain rule, (3.23) and (3.24), it is easy to write down the Malliavin derivative of $\mathcal{M}^N(Y^N)(t,\omega)$ as follows:

$$
\begin{aligned}
\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^r\chi_{\{r\le t\}}(r)\mathcal{D}_r^l(\Phi_{t-\hat{s},\hat{s}}^N P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad -\int_r^{+\infty}\chi_{\{r\ge t\}}(r)\mathcal{D}_r^l(\Phi_{t-\hat{s},\hat{s}}^N P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad +\int_{-\infty}^t\Phi_{t-\hat{s},\hat{s}}^N P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^l Y^N(\hat{s},\omega)d\hat{s}\\
&\quad -\int_t^{+\infty}\Phi_{t-\hat{s},\hat{s}}^N P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^l Y^N(\hat{s},\omega)d\hat{s},
\end{aligned}
\tag{3.25}
$$

from which we get for any $t \in \mathbb{R}$ the following L^2-estimation,

$$\begin{aligned}
& e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}} |\mathcal{D}_r^l \mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
= \quad & e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^{t} |\mathcal{D}_r^l \mathcal{M}^N(Y^N)(t,\cdot)|^2 dr + e^{-2\Lambda|t|}\mathbb{E}\int_{t}^{+\infty} |\mathcal{D}_r^l \mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
\leq \quad & 3e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^{t} \Big| \int_{-\infty}^{r} \mathcal{D}_r^l(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s}, Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
& +3e^{-2\Lambda|t|}\mathbb{E}\int_{t}^{+\infty} \Big| \int_{r}^{+\infty} \mathcal{D}_r^l(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s}, Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
& +3e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}} \Big| \int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s}, Y^N(\hat{s},\cdot))\mathcal{D}_r^l Y^N(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
& +3e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}} \Big| \int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s}, Y^N(\hat{s},\cdot))\mathcal{D}_r^l Y^N(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
=: \quad & \sum_{i=1}^{4} L_i.
\end{aligned}$$

Together with expressions (3.23) and (3.24), we could estimate each term,

$$\begin{aligned}
L_1 \quad := \quad & 3e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^{t} \Big| \int_{-\infty}^{r} \mathcal{D}_r^l(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s}, Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
\leq \quad & 6\|F\|_\infty^2 e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^{t}\Big(\int_{-\infty}^{r} \|\mathcal{D}_r^l\Phi_{t-\hat{s},\hat{s}}P^-\| \min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|}\Big\} d\hat{s}\Big)^2 dr \\
& +6\|F\|_\infty^2 e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^{t}\Big(\int_{-\infty}^{r} \|\Phi_{t-\hat{s},\hat{s}}P^-\|\mathcal{D}_r^l \min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|}\Big\} d\hat{s}\Big)^2 dr \\
\leq \quad & 6\|B_l\|^2N^2\|F\|_\infty^2(1+d^3)^2 e^{-2\Lambda|t|}\int_{-\infty}^{t}\Big(\int_{-\infty}^{r} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 dr \\
\leq \quad & 12\|B_l\|^2N^2\|F\|_\infty^2(1+d^3)^2 \int_{-\infty}^{t} e^{(\mu_{m+1}+2\Lambda)(t-r)}\Big(\int_{-\infty}^{r} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(r-\hat{s})}d\hat{s}\Big)^2 dr \\
& +12\|B_l\|^2N^2\|F\|_\infty^2(1+d^3)^2 \int_{-\infty}^{t} e^{(\mu_{m+1}-2\Lambda)(t-r)}\Big(\int_{-\infty}^{r} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(r-\hat{s})}d\hat{s}\Big)^2 dr \\
\leq \quad & 48\|B\|^2N^2\|F\|_\infty^2(1+d^3)^2\Big\{\frac{1}{|\mu_{m+1}+2\Lambda|^3} + \frac{1}{|\mu_{m+1}-2\Lambda|^3}\Big\},
\end{aligned}$$

where we have used

$$\begin{aligned}
& \Big|\mathcal{D}_r^l \min\Big\{1, \frac{Ne^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|}\Big\}\Big| \\
= \quad & \Big|\chi_{\{\|\Phi_{t-\hat{s},\hat{s}}P^-\|>Ne^{\frac{1}{2}\mu_{m+1}t}e^{\Lambda|\hat{s}|}\}}(\omega)\frac{Ne^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|^3} \\
& \cdot\Big(\sum_{i,j=1}^{d}(\Phi_{t-\hat{s},\hat{s}}P^-)_{ij}\sum_{k=1}^{d}(B_l)_{ik}(\Phi_{t-\hat{s},\hat{s}}P^-)_{kj}\Big)\chi_{\{s\leq r\leq t\}}(r)\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \|B_l\| \frac{Ne^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|^3} \sum_{i,j=1}^{d}\sum_{k=1}^{d} |(\Phi_{t-\hat{s},\hat{s}}P^-)_{kj}||(\Phi_{t-\hat{s},\hat{s}}P^-)_{ij}| \\
&\leq d^3\|B_l\| \frac{Ne^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t-\hat{s},\hat{s}}P^-\|}.
\end{aligned} \tag{3.26}$$

Similarly,

$$\begin{aligned}
L_2 \;&:=\; 3e^{-2\Lambda|t|}\mathbb{E}\int_t^{+\infty}\Big|\int_r^{+\infty} \mathcal{D}_r^l(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\leq\; 48\|B\|^2N^2\|F\|^2_\infty(1+d^3)^2\Big(\frac{1}{|\mu_m+2\Lambda|^3}+\frac{1}{|\mu_m-2\Lambda|^3}\Big).
\end{aligned}$$

As for terms L_3 and L_4, using Lemma 2.3.8,

$$\begin{aligned}
L_3 \;:=\;& 3e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big|\int_{-\infty}^t \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))\mathcal{D}_r^lY^N(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
\leq\;& 3N^2\|\nabla F\|^2_\infty e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^t e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|d\hat{s}\Big)^2 dr \\
\leq\;& 6N^2\|\nabla F\|^2_\infty e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^t e^{\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|d\hat{s}\Big)^2 dr \\
&+6N^2\|\nabla F\|^2_\infty e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^t e^{-\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|d\hat{s}\Big)^2 dr \\
\leq\;& 6N^2\|\nabla F\|^2_\infty e^{-2\Lambda|t|}\int_{-\infty}^t e^{2\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\,\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2d\hat{s}dr \\
&+6N^2\|\nabla F\|^2_\infty e^{-2\Lambda|t|}\mathbb{E}\int_{-\infty}^t e^{-2\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\,\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2d\hat{s}dr \\
\leq\;& 6N^2\|\nabla F\|^2_\infty\frac{2}{|\mu_{m+1}-4\Lambda|}\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2drd\hat{s} \\
&+6N^2\|\nabla F\|^2_\infty\frac{2}{|\mu_{m+1}+4\Lambda|}\int_{-\infty}^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2drd\hat{s} \\
\leq\;& \Big(\frac{12N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}-4\Lambda|}+\frac{12N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}+4\Lambda|}\Big) \\
&\cdot\Big\{\sum_{i=0}^{\infty}\int_{-i\tau}^{t-i\tau} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2drd\hat{s} \\
&\quad+\sum_{i=0}^{\infty}\int_{t-\tau-i\tau}^{-i\tau} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\cdot)|^2drd\hat{s}\Big\} \\
\leq\;& 12N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_{m+1}-4\Lambda|}+\frac{1}{|\mu_{m+1}+4\Lambda|}\Big)e^{2\Lambda\tau} \\
&\cdot\Big\{\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_0^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\theta_{-i\tau}\cdot)|^2drd\hat{s} \\
&\quad+\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_t^{\tau} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s}+\tau)}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^lY^N(\hat{s},\theta_{-(i+1)\tau}\cdot)|^2drd\hat{s}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \; 12N^2\|\nabla F\|_\infty^2\Big(\frac{1}{|\mu_{m+1}-4\Lambda|}+\frac{1}{|\mu_{m+1}+4\Lambda|}\Big)e^{2\Lambda\tau}\\
&\quad\cdot\Big\{\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_0^t e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-2\Lambda\hat{s}}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(\hat{s},\cdot)|^2drd\hat{s}\\
&\qquad+\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_t^\tau e^{\frac{1}{2}\mu_{m+1}(t-\hat{s}+\tau)}e^{-2\Lambda\hat{s}}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(\hat{s},\cdot)|^2drd\hat{s}\Big\}\\
&\leq \; 12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big(\frac{1}{|\mu_{m+1}-4\Lambda|}+\frac{1}{|\mu_{m+1}+4\Lambda|}\Big)\Big(\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\Big)\\
&\quad\cdot\Big\{\int_0^t e^{\frac{1}{2}\mu_{m+1}|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}+e^{-\frac{1}{2}\mu_{m+1}\tau}\int_t^\tau e^{\frac{1}{2}\mu_{m+1}|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}\Big\}\\
&\leq \; \Big(\frac{12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_{m+1}-4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_{m+1}+4\Lambda|}\Big)\Big(\sum_{i=-1}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\Big)\int_0^\tau e^{\frac{1}{2}\mu|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_4 \;:=&\; 3e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))\mathcal{D}_r^l Y^N(\hat{s},\cdot)d\hat{s}\Big|^2dr\\
\leq&\; 12N^2\|\nabla F\|_\infty^2\Big(\frac{1}{|\mu_m-4\Lambda|}+\frac{1}{|\mu_m+4\Lambda|}\Big)e^{2\Lambda\tau}\\
&\cdot\Big\{\sum_{i=0}^{\infty}e^{-\frac{1}{2}\mu_m i\tau}\int_0^t e^{\frac{1}{2}\mu_m(t-\hat{s}-\tau)}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(\hat{s},\theta_{(i-1)\tau}\cdot)|^2drd\hat{s}\\
&\quad+\sum_{i=0}^{\infty}e^{-\frac{1}{2}\mu_m i\tau}\int_t^\tau e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l Y^N(\hat{s},\theta_{i\tau}\cdot)|^2drd\hat{s}\Big\}\\
\leq&\; 12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}\Big(\frac{1}{|\mu_m-4\Lambda|}+\frac{1}{|\mu_m+4\Lambda|}\Big)\Big(\sum_{i=0}^{\infty}e^{-\frac{1}{2}\mu_m i\tau}\Big)\\
&\cdot\Big\{e^{-\frac{1}{2}\mu_m\tau}\int_0^t e^{\frac{1}{2}\mu_m|t-\hat{s}|}\rho^N(s)d\hat{s}+\int_t^\tau e^{\frac{1}{2}\mu_m|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}\Big\}\\
=&\; \Big(\frac{12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_m-4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2 e^{2\Lambda\tau}}{|\mu_m+4\Lambda|}\Big)\Big(\sum_{i=-1}^{\infty}e^{-\frac{1}{2}\mu_m i\tau}\Big)\int_0^\tau e^{\frac{1}{2}\mu|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}
\end{aligned}
$$

By now it has been shown that

$$
e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2dr\leq\sum_{i=1}^4 L_i\leq K_1^N\int_0^t e^{-\mu|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}+K_2^N=\rho^N(t).
$$

Moreover, the solution $\rho^N(t)$ to equation (3.21) is continous in t, so that for $Y^N\in C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2})$, there exists an integer N_a such that for any $t\in[0,\tau)$,

$$
e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}|\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2dr\leq\rho^N(t)\leq N_a.
$$

3. It remains to show that for any $l\in\{1,\cdots,M\}$ and $\delta\in\mathbb{R}$,

$$
\sup_{t\in[0,\tau)}\frac{e^{-2\Lambda|t|}}{|\delta|}\int_{\mathbb{R}}\mathbb{E}|\mathcal{D}_{r+\delta}^l\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}_r^l\mathcal{M}^N(Y^N)(t,\cdot)|^2dr<\infty.
$$

In fact the left hand side of the above can be separated into three integrals,

$$
\begin{aligned}
& \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|} \int_{\mathbb{R}} \mathbb{E}|\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
= & \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|} \mathbb{E}\int_{-\infty}^{t-\delta} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
& + \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|} \mathbb{E}\int_{t-\delta}^{t} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
& + \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|} \mathbb{E}\int_{t}^{+\infty} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
:= & \hat{K}_1 + \hat{K}_2 + \hat{K}_3.
\end{aligned} \tag{3.27}
$$

To consider $\hat{K}_1$ in (3.27), note when $r \le t-\delta$, by (3.25) we have

$$
\begin{aligned}
\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\omega) = & \int_{-\infty}^{r} \mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\
& + \int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_r Y^N(\hat{s},\omega)d\hat{s} \\
& - \int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_r Y^N(\hat{s},\omega)d\hat{s},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\omega) = & \int_{-\infty}^{r+\delta} \mathcal{D}^l_{r+\delta}(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\
& + \int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta} Y^N(\hat{s},\omega)d\hat{s} \\
& - \int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta} Y^N(\hat{s},\omega)d\hat{s}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\hat{K}_1 = & \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|} \mathbb{E}\int_{-\infty}^{t-\delta} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
\le & \sup_{t\in[0,\tau)} \frac{3e^{-2\Lambda|t|}}{|\delta|} \mathbb{E}\int_{-\infty}^{t-\delta} \Big\{ \Big| \int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta} - \mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2 \\
& + \Big| \int_{-\infty}^{r+\delta} \mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s} - \int_{-\infty}^{r} \mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \\
& + \Big| \int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta} - \mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2 \Big\} dr \\
:= & \sup_{t\in[0,\tau)} \sum_{i=1}^{3} Q_i.
\end{aligned}
$$

First note that Q_1 is bounded via measure preserving result in Lemma 2.3.8,

$$\begin{aligned}
Q_1 \quad &:= \quad \frac{3e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{-\infty}^{t-\delta}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2 dr \\
&\leq \quad \frac{6N^2\|\nabla F\|^2_\infty}{|\delta|}e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^{t}e^{\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|d\hat{s}\Big)^2 dr \\
&\quad +\frac{6N^2\|\nabla F\|^2_\infty}{|\delta|}e^{-2\Lambda|t|}\mathbb{E}\int_{\mathbb{R}}\Big(\int_{-\infty}^{t}e^{-\Lambda\hat{s}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|d\hat{s}\Big)^2 dr \\
&\leq \quad \frac{6N^2\|\nabla F\|^2_\infty}{|\delta|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}-2\Lambda)(t-\hat{s})}d\hat{s} \\
&\quad \cdot\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 d\hat{s}dr \\
&\quad +\frac{6N^2\|\nabla F\|^2_\infty}{|\delta|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}+2\Lambda)(t-\hat{s})}d\hat{s} \\
&\quad \cdot\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 d\hat{s}dr \\
&\leq \quad \Big(\frac{12N^2\|\nabla F\|^2_\infty}{|\delta||\mu_{m+1}+4\Lambda|}+\frac{12N^2\|\nabla F\|^2_\infty}{|\delta||\mu_{m+1}-4\Lambda|}\Big) \\
&\quad \cdot\mathbb{E}\int_{\mathbb{R}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 d\hat{s}dr \\
&\leq \quad 12N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_{m+1}-4\Lambda|}+\frac{1}{|\mu_{m+1}+4\Lambda|}\Big)e^{2\Lambda\tau} \\
&\quad \cdot\Big\{\sum_{i=1}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_0^{\tau}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)Y^N(\hat{s},\theta_{-i\tau}\cdot)|^2 drd\hat{s} \\
&\qquad +\int_0^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)Y^N(\hat{s},\cdot)|^2 drd\hat{s}\Big\} \\
&\leq \quad 12N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\
&\quad \cdot\Big\{\big(\sum_{i=1}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\big)\int_0^{\tau}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}+\int_0^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\Big\} \\
&\quad \cdot e^{2\Lambda\tau}\sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 dr \\
&\leq \quad e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\
&\quad \cdot\sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 dr \\
&< \quad \infty,
\end{aligned}$$

and analogously,

$$
\begin{aligned}
Q_3 &:= \frac{3e^{-2\Lambda|t|}}{|\delta|}\int_{-\infty}^{t-\delta}\mathbb{E}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2dr\\
&\leq 24e^{2\Lambda\tau}N^2\|\nabla F\|^2_\infty\Big\{\frac{1}{\mu_m(\mu_m+4\Lambda)}+\frac{1}{\mu_m(\mu_m-4\Lambda)}\Big\}\\
&\quad\cdot\sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr\\
&< \infty.
\end{aligned}
$$

Secondly, Q_2 can be estimated using (3.23), (3.24) and (3.26),

$$
\begin{aligned}
Q_2 &:= \frac{3e^{-2\Lambda|t|}}{|\delta|}\int_{-\infty}^{t-\delta}\mathbb{E}\Big|\int_{-\infty}^{r+\delta}\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\\
&\qquad\qquad -\int_{-\infty}^{r}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},f(\hat{s},\cdot))d\hat{s}\Big|^2dr\\
&\leq \frac{3e^{-2\Lambda|t|}\|F\|^2_\infty}{|\delta|}\int_{-\infty}^{t-\delta}\mathbb{E}\Big(\int_r^{r+\delta}\|\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2dr\\
&\leq 3N^2e^{-2\Lambda|t|}\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\frac{1}{|\delta|}\int_{-\infty}^{t-\delta}\Big(\int_r^{r+\delta}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2dr\\
&\leq 6N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\frac{1}{|\delta|}\int_{-\infty}^{t-\delta}\Big(\int_r^{r+\delta}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\Big)^2dr\\
&\quad+6N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\frac{1}{|\delta|}\int_{-\infty}^{t-\delta}\Big(\int_r^{r+\delta}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\Big)^2dr\\
&\leq 6N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\int_{-\infty}^{t-\delta}e^{(\mu_{m+1}-2\Lambda)(t-\delta-r)}\int_r^{r+\delta}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(r+\delta-\hat{s})}d\hat{s}dr\\
&\quad+6N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\int_{-\infty}^{t-\delta}e^{(\mu_{m+1}+2\Lambda)(t-\delta-r)}\int_r^{r+\delta}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(r+\delta-\hat{s})}d\hat{s}dr\\
&\leq 12N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\Big\{\frac{1}{(\mu_{m+1}+2\Lambda)^2}+\frac{1}{(\mu_{m+1}-2\Lambda)^2}\Big\}\\
&< \infty.
\end{aligned}
$$

Thus $\hat{K}_1 < \infty$.

To consider $\hat{K}_2$ in (3.27), note that when $r \leq t \leq r+\delta$, the expressions (3.23) and (3.24) gives us

$$
\begin{aligned}
\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\omega) &= \int_{-\infty}^{r}\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^-)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s}\\
&\quad+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s}\\
&\quad-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s},
\end{aligned}
$$

and

$$\begin{aligned}
\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\omega) &= \int_{r+\delta}^{+\infty} \mathcal{D}^l_{r+\delta}(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\
&\quad + \int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s} \\
&\quad - \int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}.
\end{aligned}$$

Thus

$$\begin{aligned}
\hat{K}_2 &= \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
&\leq \sup_{t\in[0,\tau)} \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{-\infty}^{r} \mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\quad + \sup_{t\in[0,\tau)} \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{r+\delta}^{+\infty} \mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\quad + \sup_{t\in[0,\tau)} \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{-\infty}^{t} \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
&\quad + \sup_{t\in[0,\tau)} \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{t}^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
&:= \sup_{t\in[0,\tau)} \sum_{i=4}^{7} Q_i.
\end{aligned}$$

But

$$\begin{aligned}
Q_4 &:= \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{-\infty}^{r} \mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\leq \frac{4\|F\|^2_\infty}{|\delta|}e^{-2\Lambda|t|}\int_{t-\delta}^{t}\mathbb{E}\Big(\int_{-\infty}^{r}\|\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^-\|d\hat{s}\Big)^2 dr \\
&\leq \frac{4}{|\delta|}\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2e^{-2\Lambda t}\int_{t-\delta}^{t}\Big(\int_{-\infty}^{t} e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2 dr \\
&\leq 32\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\Big(\frac{1}{|\mu_{m+1}-2\Lambda|^2}+\frac{1}{|\mu_{m+1}+2\Lambda|^2}\Big) \\
&< \infty.
\end{aligned}$$

and similarly,

$$\begin{aligned}
Q_5 &:= \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t} \Big|\int_{r+\delta}^{+\infty} \mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\leq 32\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\Big(\frac{1}{|\mu_m-2\Lambda|^2}+\frac{1}{|\mu_m+2\Lambda|^2}\Big) \\
&< \infty.
\end{aligned}$$

Besides, we have by similar calculations as in Q_1 and Q_2,

$$\begin{aligned} Q_6 &:= \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2dr \\ &\leq e^{2\Lambda\tau}\frac{32N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\ &\quad\cdot\sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr \\ &< \infty, \end{aligned}$$

and

$$\begin{aligned} Q_7 &:= \frac{4e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_{t-\delta}^{t}\Big|\int_{t}^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2dr \\ &\leq 32e^{2\Lambda\tau}N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_m(\mu_m+4\Lambda)|}+\frac{1}{|\mu_m(\mu_m-4\Lambda)|}\Big) \\ &\quad\cdot\sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2dr \\ &< \infty. \end{aligned}$$

Now we have shown that $\hat{K}_2<\infty$.

To consider $\hat{K}_3$, note that when $r\geq t$, (3.23) and (3.24) gives us

$$\begin{aligned} \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\omega) &= \int_r^{+\infty}\mathcal{D}^l_r(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\ &\quad+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s} \\ &\quad-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_rY^N(\hat{s},\omega)d\hat{s}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\omega) &= \int_{r+\delta}^{+\infty}\mathcal{D}^l_{r+\delta}(\Phi^N_{t-\hat{s},\hat{s}}P^+)F(\hat{s},Y^N(\hat{s},\omega))d\hat{s} \\ &\quad+\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s} \\ &\quad-\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}^l_{r+\delta}Y^N(\hat{s},\omega)d\hat{s}. \end{aligned}$$

Then

$$\begin{aligned}
\hat{K}_3 &= \sup_{t\in[0,\tau)} \frac{e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty} |\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot) - \mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr \\
&\le \sup_{t\in[0,\tau)} \frac{3e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty} \Big\{\Big|\int_{-\infty}^t \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2 \\
&\quad +\Big|\int_{r+\delta_r}^{+\infty} \mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s} - \int_r^{+\infty}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 \\
&\quad +\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2\Big\}dr \\
&= \sup_{t\in\mathbb{R}}\sum_{i=8}^{10} Q_i,
\end{aligned}$$

and now it is easy to write down the following estimations,

$$\begin{aligned}
Q_8 &:= \frac{3e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty}\Big|\int_{-\infty}^t \Phi^N_{t-\hat{s},\hat{s}}P^-\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(t,\cdot)d\hat{s}\Big|^2 dr \\
&\le e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\
&\quad \cdot \sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 dr \\
&< \infty,
\end{aligned}$$

and

$$\begin{aligned}
Q_{10} &:= \frac{3e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty}\Big|\int_t^{+\infty} \Phi^N_{t-\hat{s},\hat{s}}P^+\nabla F(\hat{s},Y^N(\hat{s},\cdot))(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)d\hat{s}\Big|^2 dr \\
&\le 24e^{2\Lambda\tau}N^2\|\nabla F\|^2_\infty\Big(\frac{1}{|\mu_m(\mu_m+4\Lambda)|}+\frac{1}{|\mu_m(\mu_m-4\Lambda)|}\Big) \\
&\quad \cdot \sup_{\hat{s}\in[0,\tau)}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\mathbb{E}\int_{\mathbb{R}}|(\mathcal{D}^l_{r+\delta}-\mathcal{D}^l_r)(Y^N)(\hat{s},\cdot)|^2 dr \\
&< \infty.
\end{aligned}$$

Similarly to Q_2,

$$\begin{aligned}
Q_9 &:= \frac{3e^{-2\Lambda|t|}}{|\delta|}\mathbb{E}\int_t^{+\infty}\Big|\int_{r+\delta}^{+\infty}\mathcal{D}^l_{r+\delta}\Phi^N_{t-\hat{s}}P^+_{\hat{s}}F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s} \\
&\qquad -\int_r^{+\infty}\mathcal{D}^l_r\Phi^N_{t-\hat{s},\hat{s}}P^+F(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2 dr \\
&\le 24N^2\|F\|^2_\infty\|B_l\|^2(1+2d^3)^2\Big\{\frac{1}{(\mu_m+2\Lambda)^2}+\frac{1}{(\mu_m-2\Lambda)^2}\Big\} \\
&< \infty.
\end{aligned}$$

In summary, we have shown that

$$\sup_{t\in[0,\tau)}\frac{e^{-2\Lambda|t|}}{|\delta|}\int_{\mathbb{R}}\mathbb{E}|\mathcal{D}^l_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot)-\mathcal{D}^l_r\mathcal{M}^N(Y^N)(t,\cdot)|^2 dr < \infty.$$

Thus we could conclude that $\mathcal{M}^N$ maps $C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2})$ into itself.

Step 2: Now we can prove that for each $N\in\mathbb{N}$, $\mathcal{M}^N(C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2}))|_{[0,\tau)}$ is relatively compact in $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$.

In fact applying Theorem 2.4.8, result from ***Step 1*** tells us that for any sequence $\{\mathcal{M}^N(f_n)\}_{n\in\mathbb{N}}\in C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2})|_{[0,\tau)}$, there exists a subsequence, still denoted by $\{\mathcal{M}^N(f_n)\}_{n\in\mathbb{N}}$ and $V^N\in C([0,\tau),L^2(\Omega,\mathbb{R}^d))$ such that

$$\sup_{t\in[0,\tau)}\mathbb{E}|\mathcal{M}^N(f_n)(t,\cdot)-V^N(t,\cdot)|^2\to 0 \tag{3.28}$$

as $n\to\infty$. □

Remark 3.2.8. *Note that in Theorem 2.4.8, the relative compactness criterion allows us to apply only with the bounded time intervals rather than $\mathbb{R}$. But we can push it to the whole real line by the random periodicity.*

Proof of Theorem 3.2.4. ***Step 1***: To prove for any fixed N, $\mathcal{M}^N(C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2}))$ is relatively compact in $C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$.

Due to the relative compactness in $C([0,\tau),L^2(\Omega,\mathbb{R}^d))$, we are able to find a subsequence, denoted by $\{\mathcal{M}^N(Y^N_{n_j})\}_{j\in\mathbb{N}}$, from an arbitrary sequence $\{\mathcal{M}^N(Y^N_n)\}_{n\in\mathbb{N}}$ such that it will converge to the accumulation point V^N, denoted as before, in the norm shown in equation (3.28). Now define for any $t\in[m\tau,m\tau+\tau)$,

$$V^N(t,\omega)=V^N(t-m\tau,\theta_{m\tau}\omega),$$

and note

$$\mathcal{M}^N(Y^N_{n_j})(t,\theta_{m\tau}\omega)=\mathcal{M}^N(Y^N_{n_j})(t+m\tau,\omega).$$

With (3.28), the periodic property of $\mathcal{M}^N(Y^N_{n_j})$, and the probability preserving of θ, we obtain

$$\begin{aligned}
&\sup_{t\in[m\tau,m\tau+\tau)} e^{-2\Lambda|t|}\mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\cdot)-V^N(t,\cdot)|^2\\
\le\ &\sup_{t\in[0,\tau)}\mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t+m\tau,\cdot)-V^N(t+m\tau,\cdot)|^2\\
=\ &\sup_{t\in[0,\tau)}\mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\theta_{m\tau}\cdot)-V^N(t,\theta_{m\tau}\cdot)|^2\\
=\ &\sup_{t\in[0,\tau)}\mathbb{E}|\mathcal{M}^N(Y^N_{n_j})(t,\cdot)-V^N(t,\cdot)|^2\to 0,
\end{aligned}$$

Thus

$$\sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}E|\mathcal{M}^N(Y^N_{n_j})(t,\cdot)-V^N(t,\cdot)|^2\to 0,$$

as $j\to\infty$. Therefore $\mathcal{M}^N(C^{\Lambda,N}_{\tau,\rho}(\mathbb{R},\mathcal{D}^{1,2}))$ is relatively compact in $C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$.

Step 2: According to the generalized Schauder's fixed point theorem, $\mathcal{M}^N$ has a fixed point in $C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$. That is to say there exists a solution $Y^N\in C^{\Lambda}_{\tau}(\mathbb{R},L^2(\Omega,\mathbb{R}^d))$ of equation (3.13) such that for any $t\in\mathbb{R}$, $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$. Moreover, $Y^N(t+\tau,\omega)=Y^N(t,\theta_\tau\omega)$.

Now define a subset of Ω as

$$\Omega_N:=\left\{\omega:\ \sup_{s\in\mathbb{R}}\max\left\{\sup_{t\geq 0}\|\Phi(t,\theta_s\omega)P^-\|e^{-\frac{1}{2}\mu|t|-\Lambda|s|},\ \sup_{t\leq 0}\|\Phi(t,\theta_s\omega)P^+\|e^{-\frac{1}{2}\mu|t|-\Lambda|s|}\right\}\leq N\right\},$$

As the random variable $\max\left\{\sup_{t\geq 0}\|\Phi(t,\theta_s\omega)P^-\|e^{-\frac{1}{2}\mu|t|},\ \sup_{t\leq 0}\|\Phi(t,\theta_s\omega)P^+\|e^{-\frac{1}{2}\mu|t|}\right\}$ is tempered from above, it is easy to see that

$$\mathbb{P}(\Omega_N)\to 1,\qquad \text{as}\quad N\to\infty.$$

Note also that Ω_N is an increasing sequence of sets, thus $\cup_N\Omega_N=\hat{\Omega}$ and $\hat{\Omega}$ has the full measure. In fact

$$\hat{\Omega}:=\left\{\omega:\ \sup_{s\in\mathbb{R}}\max\left\{\sup_{t\geq 0}\|\Phi(t,\theta_s\omega)P^-\|e^{-\frac{1}{2}\mu|t|-\Lambda|s|},\ \sup_{t\leq 0}\|\Phi(t,\theta_s\omega)P^+\|e^{-\frac{1}{2}\mu|t|-\Lambda|s|}\right\}<\infty\right\},$$

therefore it is invariant with respect to θ.

Now define

$$\Omega^*_N=\bigcap_{n=-\infty}^{\infty}\theta^{-1}_{n\tau}\Omega_N,$$

then it is easy to see that Ω^*_N is invariant with respect to $\theta_{n\tau}$ for each n. Besides we have $\Omega^*_N\subset\Omega^*_{N+1}$, which leads to

$$\bigcup_N\Omega^*_N=\bigcup_N\bigcap_{n=-\infty}^{\infty}\theta^{-1}_{n\tau}\Omega_N=\bigcap_{n=-\infty}^{\infty}\theta^{-1}_{n\tau}\left(\bigcup_N\Omega_N\right)=\bigcap_{n=-\infty}^{\infty}\theta^{-1}_{n\tau}\hat{\Omega}=\bigcap_{n=-\infty}^{\infty}\hat{\Omega}=\hat{\Omega},$$

with $\mathbb{P}(\hat{\Omega})=1$.

Now we can define $Y:\hat{\Omega}\times\mathbb{R}\to\mathbb{R}^d$ as a combination of Y_N as follows

$$Y:=Y_1\chi_{\Omega^*_1}+Y_2\chi_{\Omega^*_2\setminus\Omega^*_1}+\cdots+Y_N\chi_{\Omega^*_N\setminus\Omega^*_{N-1}}+\cdots. \tag{3.29}$$

Thus it is easy to see that Y is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable and satisfies the following property

$$\begin{aligned}
& Y(t+\tau,\omega) \\
= \;& Y_1(t+\tau,\omega)\chi_{\Omega_1^*}(\omega) + Y_2(t+\tau,\omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\omega) + \cdots + Y_N(t+\tau,\omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\omega) + \cdots \\
= \;& Y_1(t,\theta_\tau\omega)\chi_{\Omega_1^*}(\omega) + Y_2(t,\theta_\tau\omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\omega) + \cdots + Y_N(t,\theta_\tau\omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\omega) + \cdots \\
= \;& Y_1(t,\theta_\tau\omega)\chi_{\Omega_1^*}(\theta_\tau\omega) + Y_2(t,\theta_\tau\omega)\chi_{\Omega_2^*\backslash\Omega_1^*}(\theta_\tau\omega) + \cdots + Y_N(t,\theta_\tau\omega)\chi_{\Omega_N^*\backslash\Omega_{N-1}^*}(\theta_\tau\omega) + \cdots \\
= \;& Y(t,\theta_\tau\omega).
\end{aligned}$$

Moreover Y is a fixed point of $\mathcal{M}$.

We can easily extend Y to the whole of Ω as $\mathbb{P}(\hat{\Omega}) = 1$, which is indistinguishable with Y defined in (3.29). □

Remark 3.2.9. *It is easy to see from (3.29) that $|Y| < \infty$ $\mathbb{P}-a.s.$ Moreover, we don't know whether or not $Y(t,\omega) \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$, though each $Y^N(t,\omega) \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega,\mathbb{R}^d))$.*

3.3 Examples

3.3.1 One Linear Multiplicative Noise Example

Consider the following 2-dimensional Stratonovich SDE with linear multiplicative noise,

$$\begin{cases} dx_1 = -x_1 dt + \sin(x_2)\sin(t)dt + x_1 \circ dW_t^1 \\ dx_2 = x_2 dt + \sin(x_1)\cos(t)dt + x_2 \circ dW_t^2, \end{cases} \tag{3.30}$$

or in a matrix form

$$d\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} \sin(x_2)\sin(t) \\ \sin(x_1)\cos(t) \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \circ dW_t^1 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \circ dW_t^2.$$

Obviously A, B_1, and B_2 are symmetric, where

$$A := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, B_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

And it is easy to check that the matrices A, A^*, B_1, B_1^*, B_2, and B_2^* are mutually commutative. Then we could write down the forward backward infinite horizon stochastic integral equation according to (3.13):

$$\begin{cases} Y_1(t,\omega) = \int_{-\infty}^{t} e^{-(t-\hat{s})+\theta_{\hat{s}}W^1_{t-\hat{s}}} \sin(Y_2(\hat{s},\omega))\sin(\hat{s})d\hat{s} \\ Y_1(t,\omega) = \int_{-\infty}^{t} e^{(t-\hat{s})+\theta_{\hat{s}}W^2_{t-\hat{s}}} \sin(Y_1(\hat{s},\omega))\cos(\hat{s})d\hat{s}, \end{cases} \tag{3.31}$$

Then by the existence theorem (3.2.4) the equation (3.31) is one pathwise periodic solution to the original problem (3.30).

3.3.2 One Additive Noise Example

If we consider a simple 4-dimensional coupled ODE:

$$\begin{cases} \frac{dy_1}{dt} = -y_2 + y_1(1-y_1^2-y_2^2) + \chi_{\{10\le y_3^2+y_4^2\le 20\}} \\ \frac{dy_2}{dt} = y_1 + y_2(1-y_1^2-y_2^2) + \chi_{\{10\le y_3^2+y_4^2\le 20\}} \\ \frac{dy_3}{dt} = y_4 - y_3(1-y_3^2-y_4^2) + \chi_{\{10\le y_1^2+y_2^2\le 20\}} \\ \frac{dy_4}{dt} = -y_3 - y_4(1-y_3^2-y_4^2) + \chi_{\{10\le y_1^2+y_2^2\le 20\}}, \end{cases} \tag{3.32}$$

it is not hard to figure out the limit cycles of this system, that is, $y_1^2 + y_2^2 = 1$ and $y_3^2 + y_4^2 = 1$. But we are interested in the random periodic solutions to the following random perturbation of (3.32) with a white noise perturbation:

$$\begin{cases} dy_1 = -y_2dt + y_1(1-y_1^2-y_2^2)dt + \chi_{\{10\le y_3^2+y_4^2\le 20\}}dt + dW_t^1 \\ dy_2 = y_1dt + y_2(1-y_1^2-y_2^2)dt + \chi_{\{10\le y_3^2+y_4^2\le 20\}}dt + dW_t^2 \\ dy_3 = y_4dt - y_3(1-y_3^2-y_4^2)dt + \chi_{\{10\le y_1^2+y_2^2\le 20\}}dt + dW_t^3 \\ dy_4 = -y_3dt - y_4(1-y_3^2-y_4^2)dt + \chi_{\{10\le y_1^2+y_2^2\le 20\}}dt + dW_t^4, \end{cases} \tag{3.33}$$

Note the nonlinear parts are not bounded, we could not use the existence results we have obtained so far. However, we are able to find the random periodic solutions to the following coupled SDE,

$$\begin{cases} dy_1 = \big(-2y_1 - y_2 + x_1(3-y_1^2-y_2^2)\chi_{\{y_1^2+y_2^2\le 10^{10}\}} + \chi_{\{10\le y_3^2+y_4^2\le 20\}}\big)dt + dW_t^1 \\ dy_2 = \big(y_1 - 2y_2 + y_2(3-y_1^2-y_2^2)\chi_{\{y_1^2+y_2^2\le 10^{10}\}} + \chi_{\{10\le y_3^2+y_4^2\le 20\}}\big)dt + dW_t^2 \\ dy_3 = \big(2y_3 + y_4 - y_3(3-y_3^2-y_4^2)\chi_{\{y_3^2+y_4^2\le 10^{10}\}} + \chi_{\{10\le y_1^2+y_2^2\le 20\}}\big)dt + dW_t^3 \\ dy_4 = \big(-y_3 + 2y_2 - y_4(3-y_3^2-y_4^2)\chi_{\{y_3^2+y_4^2\le 10^{10}\}} + \chi_{\{10\le y_1^2+y_2^2\le 20\}}\big)dt + dW_t^4. \end{cases} \tag{3.34}$$

Under the boundedness of the nonlinear part of (3.34), we could apply the existence result in (Feng et al., 2011), and show the existence of pathwise periodic solutions of (3.34). Moreover, the relevant ODE, obtained by eliminating all the noise parts in (3.34), has the same limit cycles with (3.32). It implies that the random periodic solutions to (3.33) exists and coincides with the ones to (3.34).

Chapter 4

Random Periodic Solutions to Stochastic Partial Differential Equations with Linear Multiplicative Noise

4.1 Problem Formulation

Consider a τ-periodic semilinear SPDEs with multiplicative linear noise, i.e.,

$$\begin{cases} du(t,x) = \mathcal{L}u(t,x)\,dt + F(t,u(t,x))\,dt + Bu(t,x)\circ dW(t), \qquad t \geq s \\ u(s) = \psi \in L_0^2(\mathcal{O}), \\ u(t)|_{\partial\mathcal{O}} = 0. \end{cases} \tag{4.1}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary, and $\mathcal{L}$ is a second order differential operator with Dirichlet boundary condition on $\mathcal{O}$,

$$\mathcal{L}u = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + c(x)u \tag{4.2}$$

with the uniformly elliptic condition

Condition (L): the coefficients a_{ij}, c are smooth functions on $\bar{\mathcal{O}}$, $a_{ij} = a_{ji}$, and there exists $\gamma > 0$ such that $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \gamma|\xi|^2$ for any $\xi = (\xi_1, \xi_2, \cdots, \xi_d) \in \mathbb{R}^d$.

Denote by $L_0^2(\mathcal{O})$ the standard space of all square integrable measurable functions vanishing on the boundary of $\mathcal{O}$ with norm $||\cdot||_{L^2(\mathcal{O})}$. Under the above conditions, $\mathcal{L}$ is a self-adjoint uniformly elliptic operator so it has real-valued eigenvalues $\mu_1 \geq \mu_2 \geq \cdots$ such that $\mu_k \to -\infty$ when $k \to \infty$. Denote by $\{\phi_k \in L^2(\mathcal{O}),\ k \geq 1\}$ a complete orthonormal system of eigenfunctions of $\mathcal{L}$ with corresponding eigenvalues $\mu_k,\ k \geq 1$. A standard notation $H_0^1(\mathcal{O})$ denotes a standard Sobolev space of the square integrable measurable functions having the first order weak derivative in $L^2(\mathcal{O})$ and vanishing at the boundary $\partial\mathcal{O}$. This is a Hilbert space with inner product $(u, v) = \int_{\mathcal{O}} u(x)v(x)dx + \int_{\mathcal{O}} \langle \nabla u(x), \nabla v(x)\rangle dx$, for any $u, v \in H_0^1(\mathcal{O})$. From the uniformly elliptic condition, it's not difficult to know that $\phi_k \in H_0^1(\mathcal{O})$ and there exists a constant C such that

$$||\nabla\phi_k||_{L^2(\mathcal{O})} \leq C\sqrt{|\mu_k|}. \tag{4.3}$$

Besides, with the heat kernel $K(t, x, y)$ of the second order differential operator $\mathcal{L}$,

$$(T_t\phi)(x) = \int_{\mathcal{O}} K(t, x, y)\phi(y)dy, \tag{4.4}$$

defines a linear operator $T_t : L^2(\mathcal{O}) \to L^2(\mathcal{O})$. And by Mercer's theorem (Chapter 3, Theorem 17, Hochstadt (1973)), we have

$$K(t, x, y) = \sum_{k=1}^{\infty} e^{\mu_k t}\phi_k(x)\phi_k(y). \tag{4.5}$$

Assume W_t, $t \in \mathbb{R}$, is an $L^2(\mathcal{O})$-valued Brownian motion defined on the canonical filtered Wiener space, and may be represented by

$$W_t := \sum_{k=1}^{\infty} \sqrt{\lambda_k} W_t^k \phi_k(x), \quad t \in \mathbb{R}, \tag{4.6}$$

where

$$\sum_{k=1}^{\infty} \lambda_k < \infty. \tag{4.7}$$

and the driving noise W^k are mutually independent one-dimensional two-sided standard Brownian motions on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in\mathbb{R}}, \mathbb{P})$, $\mathcal{F}_s^t := \sigma(W_u^k - W_v^k, s \leq v \leq u \leq t)$ and $\mathcal{F}^t := \vee_{s\leq t}\mathcal{F}_s^t$. If define $\theta : (-\infty, \infty) \times \Omega \to \Omega$ as the flow such that $\theta_t\omega^k(s) = W^k(t + s) - W^k(t)$, then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system. Obviously for finite t, W_t converges in $L^2(\Omega, \mathbb{P}, \mathcal{F}; L^2(\mathcal{O}))$. For large t, it is well known that the strong law of large numbers for Brownian motion still holds in the infinite dimension case.

Lemma 4.1.1 (Law of Large Numbers For Infinite-Dimensional Brownian Motions)**.** *Given a infinite-dimensional Brownian motion as defined in (4.6) and (4.7), then*

$$\lim_{t\to\pm\infty} \frac{\|W_t\|_H}{t} = 0 \quad \mathbb{P}-a.s.,$$

where $\|\cdot\|_H$ stands for norm in $H := L^2(\mathcal{O})$.

Now denote by $L(H)$ the Banach space of all linear and bounded opearators $J_1 : H \to H$, $H := L^2(\mathcal{O})$, with the norm

$$\|J_1\| = \sup_{\|v\|_H=1} \|J_1(v)\|_H,$$

where $\|\cdot\|$ denotes the norm of $L(H)$. And denote by $L_2(H)$ the Hilbert space of all Hilbert-Schmidt operators $J_2 : H \to H$, given the norm

$$\|J_2\|_2 := \left[\sum_{k=1}^{\infty} \|J_2(\phi_k)\|_H^2\right]^{1/2},$$

and $B : H \to L_2(H)$ is a bounded linear operator such that

$$B(u)(v) = \sum_{k=1}^{\infty} \sigma_k \langle u, \phi_k\rangle \langle v, \phi_k\rangle \phi_k, \tag{4.8}$$

where $u, v \in H$ and

$$\sum_{k=1}^{\infty} \sigma_k^2 < \infty. \tag{4.9}$$

It is not hard to show the commutation property between T_t and B.

Proposition 4.1.2. *The operator T_t defined by (4.4) and (4.5) commutes with B defined by (4.8). Besides, it also holds that for any $u, v \in H$,*

$$T_t B(u)(v)(x) = B(u)(T_t v)(x). \tag{4.10}$$

Proof. Actually, for any $u, v \in H$,

$$\begin{aligned} T_t B(u)(v)(x) &= \int_{\mathcal{O}} \sum_{j=1}^{\infty} e^{\mu_j t}\phi_j(y)\phi_j(x) \sum_{k=1}^{\infty} \sigma_k \langle u, \phi_k\rangle \langle v, \phi_k\rangle \phi_k(y) dy \\ &= \sum_{j=1}^{\infty} e^{\mu_j t} \sigma_j \langle u, \phi_j\rangle \langle v, \phi_j\rangle \phi_j(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{\infty} \sigma_j \langle u, e^{\mu_j t}\phi_j\rangle \langle v, \phi_j\rangle \phi_j(x) \\
&= \sum_{j=1}^{\infty} \sigma_j \langle T_t u, \phi_j\rangle \langle v, \phi_j\rangle \phi_j(x) \\
&= B(T_t u)(v)(x),
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathcal{O})$. The commutativity follows from the equation above. Moreover,

$$\begin{aligned}
T_t B(u)(v)(x) &= \sum_{j=1}^{\infty} e^{\mu_j t}\sigma_j \langle u, \phi_j\rangle \langle v, \phi_j\rangle \phi_j(x) \\
&= \sum_{j=1}^{\infty} \sigma_j \langle u, \phi_j\rangle \langle v, e^{\mu_j t}\phi_j\rangle \phi_j(x) \\
&= \sum_{j=1}^{\infty} \sigma_j \langle u, \phi_j\rangle \langle T_t v, \phi_j\rangle \phi_j(x) \\
&= B(u)(T_t v)(x).
\end{aligned}$$

So (4.10) follows. □

Set $\Delta := \{(t,s) \in \mathbb{R}^2, s \le t\}$. Equation (4.1) generates a semi-flow $u : \Delta \times L^2_0(\mathcal{O}) \times \Omega \to L^2_0(\mathcal{O})$ when the solution exists uniquely in the space $L^2_0(\mathcal{O})$. As to the continuous function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, define the Nemytskii operator $F : \mathbb{R} \times L^2(\mathcal{O}) \to L^2(\mathcal{O})$ with the same notation

$$\begin{aligned}
F(t, u(t))(x) &= F(t, u(t,x)), \\
F^i(t, u(t))(x) &= \int_{\mathcal{O}} F(t, u(t))(y)\phi_i(y)dy\phi_i(x), \;\; x \in \mathcal{O}, \;\; u_t \in L^2(\mathcal{O}).
\end{aligned}$$

Now we introduce an operator $\Phi : \mathbb{R} \times \Omega \to L(H)$, defined by

$$\begin{cases} d\Phi(t) = \mathcal{L}\Phi(t)\, dt + B\Phi(t) \circ dW_t \\ \Phi(0) = I \in L(H), \end{cases} \tag{4.11}$$

then the mild solution of (4.1) via (4.11) can be written as (c.f. Mohammed et al. (2008))

$$\begin{cases} u(t,s,\psi,\omega)(x) = \Phi(t-s, \theta_s\omega)\psi(x) + \int_s^t \Phi(t-\hat{s}, \theta_{\hat{s}}\omega) F(\hat{s}, u(\hat{s}, s, \psi, \omega))(x)\, d\hat{s}, \quad t \ge s \\ \psi \in L^2_0(\mathcal{O}), \end{cases} \tag{4.12}$$

Note that since Φ is a linear perfect cocycle and T_t and B commute, it is not hard for us to check that Φ has the following properties:

Lemma 4.1.3. *Suppose the order of the eigenvalues w.r.t $\mathcal{L}$ is $\cdots \leq \mu_{m+1} < 0 < \mu_m \leq \cdots \leq \mu_1$, and $L^2(\mathcal{O})$ has a direct sum decomposition*

$$L^2(\mathcal{O}) = \cdots \oplus E_{m+1} \oplus E_m \oplus \cdots \oplus E_1,$$

where $E_k := span\{\phi_k\}$, $k = 1, 2, \cdots$. Then we have for any $v \in L^2(\mathcal{O})$, when $t \leq 0$ and $k \leq m$,

$$\mu_k = \lim_{t \to -\infty} \frac{1}{t} \log \|\Phi(t, \omega) P^k v\|_H \qquad \mathbb{P} - a.s.,$$

and when $t \geq 0$ and $k \geq m+1$,

$$\mu_k = \lim_{t \to +\infty} \frac{1}{t} \log \|\Phi(t, \omega) P^k v\|_H \qquad \mathbb{P} - a.s.,$$

where P^k is the projection onto E^k along $\oplus_{i \neq k} E_i$. And each operator

$$\Phi(t, \theta_{\hat{s}}\omega) P^k = P^k \Phi(t, \theta_{\hat{s}}\omega)$$

with the following estimate

$$\begin{cases} \|\Phi(t, \theta_{\hat{s}}\omega) P^k\| \leq C_\Lambda(\omega) e^{\frac{1}{2}\mu_k t} e^{\Lambda|\hat{s}|} & \mathbb{P} - a.s., \quad when \quad t \leq 0, \ k \leq m, \\ \|\Phi(t, \theta_{\hat{s}}\omega) P^k\| \leq C_\Lambda(\omega) e^{\frac{1}{2}\mu_k t} e^{\Lambda|\hat{s}|} & \mathbb{P} - a.s., \quad when \quad t \geq 0, \ k \geq m+1, \end{cases} \tag{4.13}$$

where Λ is an arbitrary positive number and $C_\Lambda(\omega)$ a positive random variable depending on Λ. Besides, $L^2(\mathcal{O})$ can be decomposed as

$$L^2(\mathcal{O}) = E^- \oplus E^+,$$

where $E^- = \cdots \oplus E_{m+2} \oplus E_{m+1}$ is generated by the eigenvectors with negative eigenvalues, while $E^+ = E_m \oplus E_{m-1} \oplus \cdots \oplus E_1$ is generated by the eigenvectors with positive eigenvalues. If $P^\pm : \mathbb{R}^d \to E^\pm$ denotes the projection onto $E^\pm$ along $E^\mp$, then

$$\Phi(t, \theta_{\hat{s}}\omega) P^\pm = P^\pm \Phi(t, \theta_{\hat{s}}\omega),$$

with the following estimate

$$\begin{cases} \|\Phi(t, \theta_{\hat{s}}\omega) P^+\| \leq C(\theta_{\hat{s}}\omega) e^{\frac{1}{2}\mu_m t}, \quad when \quad t \leq 0, & \mathbb{P} - a.s., \\ \|\Phi(t, \theta_{\hat{s}}\omega) P^-\| \leq C(\theta_{\hat{s}}\omega) e^{\frac{1}{2}\mu_{m+1} t}, \quad when \quad t \geq 0, & \mathbb{P} - a.s., \end{cases}$$

for any $t, \hat{s} \in \mathbb{R}$, where $C(\omega)$ is a tempered random variable from above.

Proof. According to Theorem 1.2.3 in Mohammed et al. (2008), generally Φ can be written in the following representation:

$$\Phi(t,\omega) = T_t + \sum_{n=1}^{\infty} \Phi^n(t,\omega),$$

where

$$\Phi^1(t,\omega) = \int_0^t T_{t-s_1} B T_{s_1} \circ dW(s_1),$$
$$\Phi^n(t,\omega) = \int_0^t T_{t-s_1} B \Phi^{n-1}(s_1,\omega) \circ dW(s_1), \quad n \geq 2.$$

With the commutative property of B and T from Proposition 4.1.2, the expansion above can be further developed,

$$\Phi^1(t,\omega) = T_t BW(t) = T_t \sum_{k=1}^{\infty} \sigma_k \sqrt{\lambda_k} W_t^k \langle \cdot, \phi_k \rangle \phi_k,$$

$$\Phi^n(t,\omega) = T_t \sum_{k=1}^{\infty} \frac{1}{n!} \sigma_k^n \sqrt{\lambda_k^n} (W_t^k)^n \langle \cdot, \phi_k \rangle \phi_k, \quad n \geq 2,$$

or totally

$$\Phi(t,\omega) = T_t e^{BW(t)} = T_t \sum_{k=1}^{\infty} e^{\sigma_k \sqrt{\lambda_k} W_t^k} \langle \cdot, \phi_k \rangle \phi_k, \tag{4.14}$$

where $e^{BW(t)}$ is a well-defined operator mapping from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$. Now by (4.10) and (4.14) it is easy to see that for any $v \in L^2(\mathcal{O})$,

$$\begin{aligned}
P^k \Phi(t, \theta_{\hat{s}}\omega)(v)(x) &= \int_{\mathcal{O}} K(t,x,y) P^k \Big(\sum_{i=1}^{\infty} e^{\sigma_i \sqrt{\lambda_i} \theta_{\hat{s}} W_t^i} \langle v, \phi_i \rangle \phi_i(y) \Big) dy \\
&= \int_{\mathcal{O}} e^{\mu_k t} \phi_k(y) \phi_k(x) e^{\sigma_k \sqrt{\lambda_k} \theta_{\hat{s}} W_t^k} \langle v, \phi_k \rangle \phi_k(y) dy \\
&= e^{\sigma_k \sqrt{\lambda_k} \theta_{\hat{s}} W_t^k} \int_{\mathcal{O}} e^{\mu_k t} \phi_k(y) \phi_k(x) v(y) dy \\
&= \sum_{i=1}^{\infty} e^{\sigma_i \sqrt{\lambda_i} \theta_{\hat{s}} W_t^i} \Big\langle \int_{\mathcal{O}} e^{\mu_k t} \phi_k(y) \phi_k(\cdot) v(y) dy, \phi_i(\cdot) \Big\rangle \phi_i(x) \\
&= \sum_{i=1}^{\infty} e^{\sigma_i \sqrt{\lambda_i} \theta_{\hat{s}} W_t^i} \Big\langle (T_t P^k v)(\cdot), \phi_i(\cdot) \Big\rangle \phi_i(x) \\
&= e^{B\theta_{\hat{s}} W(t)} (T_t P^k v)(x) \\
&= T_t e^{B\theta_{\hat{s}} W(t)} (P^k v)(x) \\
&= \Phi(t, \theta_{\hat{s}}\omega) P^k(v)(x).
\end{aligned}$$

This implies that

$$P^k \Phi(t, \theta_{\hat{s}}\omega) = \Phi(t, \theta_{\hat{s}}\omega) P^k.$$

Moreover, when $k \geq m+1$ and $t \geq 0$, we have from the definition of B and W_t that,

$$\Phi(t,\omega)P^k(v)(x) = e^{\mu_k t+\sigma_k\sqrt{\lambda_k}W_t^k}\langle\phi_k(\cdot),v(\cdot)\rangle\phi_k(x) = e^{\mu_k t+\sigma_k\sqrt{\lambda_k}W_t^k}v^k(x), \tag{4.15}$$

where $v^k(x) := \langle\phi_k(\cdot),v(\cdot)\rangle\phi_k(x)$. And in this way,

$$\begin{aligned}
\|\Phi(t,\omega)P^k\|^2 &= \sup_{\|v\|_H=1}\int_{\mathcal{O}}|\Phi(t,\omega)P^k(v)(x)|^2dx \\
&= \sup_{\|v\|_H=1} e^{2\mu_k t+2\sigma_k\sqrt{\lambda_k}W_t^k}\|v^k\|_H^2 \\
&= e^{2\mu_k t+2\sigma_k\sqrt{\lambda_k}W_t^k}\|\phi_k\|_H^2 \\
&= e^{2\mu_k t+2\sigma_k\sqrt{\lambda_k}W_t^k} \\
&= \|\Phi(t,\omega)P^k\|_2^2.
\end{aligned}$$

Thus we have by Lemma 4.1.1,

$$\begin{aligned}
&\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\|\Phi(t,\theta_{\hat{s}}\omega)P^k\|^2 e^{-\mu_k t-2\Lambda|\hat{s}|} \\
= &\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{\mu_k t+2\sigma_k\sqrt{\lambda_k}\theta_{\hat{s}}W_t^k-2\Lambda|\hat{s}|\} \\
\leq &\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{\mu_k t+2\sigma(\|W_{t+\hat{s}}\|_H+\|W_{\hat{s}}\|_H)-2\Lambda|\hat{s}|\} \\
\leq &\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{\mu_k t+\Lambda|t+\hat{s}|+2\sigma(\|W_{t+\hat{s}}\|_H+\|W_{\hat{s}}\|_H)-\Lambda|t+\hat{s}|-2\Lambda|\hat{s}|\} \\
\leq &\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{\mu_k t+\Lambda t+\Lambda|\hat{s}|+2\sigma(\|W_{t+\hat{s}}\|_H+\|W_{\hat{s}}\|_H)-\Lambda|t+\hat{s}|-2\Lambda|\hat{s}|\} \\
\leq &\sup_{k\geq m+1}\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{(\mu_k+\Lambda)t+2\sigma(\|W_{t+\hat{s}}\|_H+\|W_{\hat{s}}\|_H)-\Lambda(|t+\hat{s}|+|\hat{s}|)\} \\
\leq &\sup_{\hat{s}\in\mathbb{R}}\sup_{t\geq 0}\exp\{2\sigma(\|W_{t+\hat{s}}\|_H+\|W_{\hat{s}}\|_H)-\Lambda(|t+\hat{s}|+|\hat{s}|)\} \\
\leq &\sup_{\hat{s}\in\mathbb{R}} e^{2\sigma\|W_{\hat{s}}\|_H-\Lambda|\hat{s}|}\sup_{\hat{s}+t\in\mathbb{R}} e^{2\sigma\|W_{t+\hat{s}}\|_H-\Lambda|t+\hat{s}|}<\infty, \quad \mathbb{P}-a.s.,
\end{aligned}$$

where $\sigma := \sup_k \sigma_k$. Similar estimation can be applied to the case when $k \leq m$ with $t \leq 0$. Then the boundedness (4.13) can be drawn from those estimations.

The definition of E^+ and E^- naturally permits the property $\Phi(t,\theta_{\hat{s}}\omega)P^{\pm} = P^{\pm}\Phi(t,\theta_{\hat{s}}\omega)$. Now define a new random variable

$$C(\omega) := \sqrt{C_1^2(\omega)+C_2^2(\omega)},$$

where

$$C_1^2(\omega) := \sup_{t\in[0,\infty)}\frac{\|\Phi(t,\omega)P^-\|^2}{e^{\mu_{m+1}t}}, \tag{4.16}$$

and

$$C_2^2(\omega) := \sup_{t\in(-\infty,0]} \frac{\|\Phi(t,\omega)P^+\|^2}{e^{\mu_m t}}. \tag{4.17}$$

It remains to show that $C(\omega)$ is a tempered random variable.

In fact we know from equation (4.15) that

$$\Phi(t,\omega)P^-(v)(x) = \sum_{k=m+1}^{\infty} \Phi(t,\omega)P^k(v)(x) = \sum_{k=m+1}^{\infty} e^{\mu_k t+\sigma_k\sqrt{\lambda_k}W_t^k} v^k(x),$$

and

$$\Phi(t,\omega)P^+(v)(x) = \sum_{k=1}^{m} e^{\mu_k t+\sigma_k\sqrt{\lambda_k}W_t^k} v^k(x).$$

Then from (4.17) we have that

$$C_2^2(\omega) = \sup_{\|v\|_H=1} \sup_{t\in(-\infty,0]} \sum_{k=1}^{m} e^{(2\mu_k-\mu_m)t+2\sigma_k\sqrt{\lambda_k}W_t^k} \|v^k(x)\|_H^2,$$

and from the argument of Lemma 3.1.1 in Chapter 3, it is easy to show $C_2^2(\omega)$ is tempered from above.

Similarly, from (4.16) we have that

$$C_1^2(\omega) := \sup_{\|v\|_H=1} \sup_{t\in[0,\infty)} \sum_{k=m+1}^{\infty} e^{(2\mu_k-\mu_{m+1})t+2\sigma_k\sqrt{\lambda_k}W_t^k} \|v^k(x)\|_H^2.$$

Regarding the temperedness of $C_1(\omega)$ (this proof mainly follows the argument of Theorem 3.4 in Caraballo et al. (2010)), we need to apply the Kingman's subbadditive ergodic Theorem (c.f. Theorem 3.3.2 in Arnold (1998)) to $\log\|\Phi(t,\omega)P^-\|^2$. Thus for every $\varepsilon > 0$, there is a finite-valued random variable C_ε such that when $n > m$,

$$\log\|\Phi(n-m,\theta_m\omega)P^-\|^2 \le 2(n-m)\mu_{m+1} + n\varepsilon + C_\varepsilon(\omega), \quad \mathbb{P}-a.s. \tag{4.18}$$

Now it is sufficient to verify that

$$\lim_{\hat{s}\to\infty} C_1^2(\theta_{\hat{s}}\omega)e^{-\Lambda\hat{s}} = 0,$$

for some sufficiently large Λ. This follows from

$$\begin{aligned}
\|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^2 &\le \|\Phi(1+t-[t]-1-[\hat{s}]+\hat{s},\theta_{[\hat{s}]+[t]}\omega)P^-\|^2 \\
&\quad \cdot\|\Phi([t]-1,\theta_{[\hat{s}]+1}\omega)P^-\|^2 \cdot \|\Phi(1-\hat{s}+[\hat{s}],\theta_{\hat{s}}\omega)P^-\|^2 \\
&\le e^{D(\theta_{[\hat{s}]+[t]+1}\omega)} e^{D(\theta_{[\hat{s}]+[t]}\omega)} e^{2([t]-1)\mu_{m+1}+([t]+[\hat{s}])\varepsilon+C_\varepsilon(\omega)} e^{D(\theta_{[\hat{s}]}\omega)}
\end{aligned}$$

for $t > 1$, where $D(\omega) := \log^+ \sup_{\hat{s},t\in[0,1]} \|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^2$, and

$$\begin{aligned}
\mathbb{E}D &\leq \log^+ \mathbb{E} \sup_{\hat{s},t\in[0,1]} \|\Phi(t,\theta_{\hat{s}}\omega)P^-\|^2 = \log^+ \mathbb{E} \sup_{\hat{s},t\in[0,1]} \|T_t e^{B\theta_{\hat{s}}W_t} P^-\|^2 \\
&= \log^+ \mathbb{E} \sup_{\hat{s},t\in[0,1]} \sup_{\|v\|_H=1} \sum_{j=m+1}^{\infty} e^{2\mu_j t + 2\sigma_j\sqrt{\lambda_j}\theta_{\hat{s}}W_t^j} \|v^j\|_H^2 \\
&\leq \log^+ \sup_{\|v\|_H=1} \sum_{j=m+1}^{\infty} e^{3\sigma_j^2\lambda_j} \sqrt{\mathbb{E}\Big(\sup_{t\in[0,2]} e^{2\sigma_j\sqrt{\lambda_j}W_t^j - 2\sigma_j^2\lambda_j t}\Big)^2} \cdot \\
&\quad \cdot \sqrt{\mathbb{E}\Big(\sup_{\hat{s}\in[0,1]} e^{-2\sigma_j\sqrt{\lambda_j}W_{\hat{s}}^j - 2\sigma_j^2\lambda_j\hat{s}}\Big)^2} \|v^j\|_H^2 \\
&\leq \log^+ C_2 \sup_{\|v\|_H=1} \sum_{j=m+1}^{\infty} e^{9\sigma_j^2\lambda_j} \|v^j\|_H^2 < \infty,
\end{aligned}$$

where we use the Burkholder-Davis-Gundy inequality for the last line and C_2 is a positive constant. □

Remark 4.1.4. *It is easy to check the assumption of Theorem 1.2.3 in Mohammed et al. (2008). In fact in our case, we only need to verify that*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n} \|B(\phi_n)\|_2^2 < \infty.$$

In fact by the defnition of B we have that

$$\Big|\sum_{n=1}^{\infty} \frac{1}{\mu_n} \|B(\phi_n)\|_2^2\Big| \leq \sum_{n=1}^{\infty} \frac{1}{|\mu_n|} \sum_{k=1}^{\infty} \|B(\phi_n)(\phi_k)\|_H^2 = \sum_{n=1}^{\infty} \frac{\sigma_n^2}{|\mu_n|} < \Big(\frac{1}{|\mu_{m+1}|} + \frac{1}{|\mu_m|}\Big) \sum_{n=1}^{\infty} \sigma_n^2 < \infty.$$

Remark 4.1.5. *From (4.15),*

$$\begin{aligned}
\Phi(t,\omega)P^k(v)(x) &= e^{\mu_k t + \sigma_k\sqrt{\lambda_k}W_t^k} \langle \phi_k(z), v(z)\rangle \phi_k(x) \\
&= \langle \phi_k(z), e^{\mu_k t + \sigma_k\sqrt{\lambda_k}W_t^k} v^k(z)\rangle \phi_k(x) \\
&= \langle \phi_k(z), \sum_{j=1}^{\infty} e^{\mu_j t + \sigma_j\sqrt{\lambda_j}W_t^j} v^j(z)\rangle \phi_k(x) \\
&= \langle \phi_k(y), \Phi(t,\omega)v(y)\rangle \phi_k(x), \qquad (4.19)
\end{aligned}$$

and

$$P^k(v)(x) = \langle v(y), \phi_k(y)\rangle \phi_k(x) = v^k(x). \qquad (4.20)$$

For any $N \in \mathbb{N}$, we set the truncation of $\Phi(t,\theta_{\hat{s}}\omega)P^k$ by N according to the boundedness of Φ (4.13) as following: when $t \geq 0$, $j \geq m+1$,

$$\Phi^N(t,\theta_{\hat{s}}\omega)P^k = \Phi(t,\theta_{\hat{s}}\omega)P^k \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_k t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^k\|}\right\}, \qquad (4.21)$$

and when $t \leq 0$, $j \leq m$,

$$\Phi^N(t, \theta_{\hat{s}}\omega)P^k = \Phi(t, \theta_{\hat{s}}\omega)P^k \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_k t}e^{\Lambda|s|}}{\|\Phi(t, \theta_{\hat{s}}\omega)P^k\|}\right\}. \tag{4.22}$$

We will look for a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{O})$-measurable map $Y : \mathbb{R} \times \Omega \to L_0^2(\mathcal{O})$ which satisfies the following coupled forward-backward infinite horizon stochastic integral equation

$$\begin{aligned} Y(t, \omega) &= \int_{-\infty}^{t} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)P^- F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &\quad - \int_{t}^{+\infty} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)P^+ F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &= \int_{-\infty}^{t} \sum_{k=m+1}^{\infty} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)P^k F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &\quad - \int_{t}^{+\infty} \sum_{k=1}^{m} \Phi(t - \hat{s}, \theta_{\hat{s}}\omega)P^k F(\hat{s}, Y(\hat{s}, \omega))d\hat{s}, \end{aligned} \tag{4.23}$$

for all $\omega \in \Omega$, $t \in \mathbb{R}$. The value of $Y(t, \omega) \in L_0^2(\mathcal{O})$ at x is $Y(t, \omega)(x)$, or written as $Y(t, \omega, x)$. Also consider a sequence $\{Y^N\}_{N \geq 1}$ which satisfies

$$\begin{aligned} Y^N(t, \omega) &= \int_{-\infty}^{t} \sum_{k=m+1}^{\infty} \Phi^N(t - \hat{s}, \theta_{\hat{s}}\omega)P^k F(\hat{s}, Y(\hat{s}, \omega))d\hat{s} \\ &\quad - \int_{t}^{+\infty} \sum_{k=1}^{m} \Phi^N(t - \hat{s}, \theta_{\hat{s}}\omega)P^k F(\hat{s}, Y(\hat{s}, \omega))d\hat{s}, \end{aligned} \tag{4.24}$$

for all $\omega \in \Omega$, $t \in \mathbb{R}$.

4.2 Main Results

Theorem 4.2.1. *Assume $F(t, u) = F(t + \tau, u)$ for some fixed $\tau > 0$. If the τ-periodic semilinear random PDE (4.12) has a unique solution $u(t, s, \omega, x)$ and the coupled forward-backward infinite horizon stochastic integral equation (4.23) has a solution $Y : (-\infty, +\infty) \times \Omega \to L_0^2(\mathcal{O})$ such that $Y(t+\tau, \omega) = Y(t, \theta_\tau \omega)$ for any $t \in \mathbb{R}$ a.s., then Y is a random periodic solution of equation (4.12) i.e.*

$$u(t + \tau, t, Y(t, \omega), \omega) = Y(t + \tau, \omega) = Y(t, \theta_\tau \omega) \quad \text{for any} \quad t \in \mathbb{R} \qquad a.s. \tag{4.25}$$

Conversely, if equation (4.12) has a random periodic solution $Y : (-\infty, +\infty) \times \Omega \to L_0^2(\mathcal{O})$ of period τ which is tempered from above for each t, then Y is a solution of the coupled forward-backward infinite horizon stochastic integral equation (4.1).

Proof. Similarly to the proof of Theorem 3.2.2. □

Theorem 4.2.2. *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a continuous map, globally bounded and the Jacobian $\nabla F(t, \cdot)$ be globally bounded. Assume $F(t, u) = F(t + \tau, u)$ for some fixed $\tau > 0$, and* **Condition (L)** *.*

Then there exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable map $Y : \mathbb{R} \times \Omega \to L^2_0(\mathcal{O})$ satisfying equation (4.23) and $Y(t + \tau, \omega) = Y(t, \theta_\tau \omega)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$.

Firstly given a Banach space $C^\Lambda_\tau(\mathbb{R}, L^2(\Omega \times \mathcal{O}))$

$$\begin{aligned} C^\Lambda_\tau(\mathbb{R}, L^2(\Omega \times \mathcal{O})) \quad := \quad & \big\{ f \in C^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O})) \text{ with the following norm :} \\ & \|f\|^\Lambda_{\mathcal{O}} := \sup_{t \in \mathbb{R}} e^{-2\Lambda|t|} \|f\|_{L^2(\Omega \times \mathcal{O})}, \\ & \text{and for any } t \in \mathbb{R},\ f(t + \tau, \omega, x) = f(t, \theta_\tau \omega, x) \big\}, \end{aligned}$$

where $0 < \Lambda < \frac{1}{4}\mu := \frac{1}{4} \min\{-\mu_{m+1}, \mu_m\}$. Then define for any $Y^N \in C_\tau(\mathbb{R}, L^2(\Omega \times \mathcal{O}))$

$$\begin{aligned} \mathcal{M}^N(Y^N)(t, \omega, x) \quad = \quad & \int_{-\infty}^{t} \sum_{k=m+1}^{\infty} \Phi^N(t - \hat{s}, \theta_{\hat{s}} \omega) P^k F(\hat{s}, Y(\hat{s}, \omega))(x) d\hat{s} \\ & - \int_{t}^{+\infty} \sum_{k=1}^{m} \Phi^N(t - \hat{s}, \theta_{\hat{s}} \omega) P^k F(\hat{s}, Y(\hat{s}, \omega))(x) d\hat{s}. \end{aligned}$$

Lemma 4.2.3. *Under the condition of Theorem 4.2.2, we have the following estimation,*

$$\Big(\frac{1}{|\mu_k - 2\Lambda|} + \frac{1}{|\mu_k + 2\Lambda|} \Big) \mathbb{E} \|T_t P^k - P^k\|^2 \leq C \Big(\frac{|\mu_k|}{|\mu_k - 2\Lambda|} + \frac{|\mu_k|}{|\mu_k + 2\Lambda|} \Big) |t|, \tag{4.26}$$

$$\mathbb{E} \|\Phi_t P^k - T_t P^k\|^2 \leq C \max\big\{1, e^{2\mu_k t + 2\sigma_k^2 \lambda_k |t|}\big\} \sigma_k^2 \lambda_k (|t| + |t|^2), \tag{4.27}$$

when $t \geq 0$, $k \geq m + 1$; or when $t \leq 0$, $k \leq m$, and C is a generic constant.

Proof. Firstly it is easy to get (4.26) since

$$\begin{aligned} & \Big(\frac{1}{|\mu_k - 2\Lambda|} + \frac{1}{|\mu_k + 2\Lambda|} \Big) \|T_t P^k - P^k\|^2 \\ = \quad & \Big(\frac{1}{|\mu_k - 2\Lambda|} + \frac{1}{|\mu_k + 2\Lambda|} \Big) \sup_{\|v\|_H = 1} \int_{\mathcal{O}} |(T_t P^k - P^k) v(x)|^2 dx \\ = \quad & \Big(\frac{1}{|\mu_k - 2\Lambda|} + \frac{1}{|\mu_k + 2\Lambda|} \Big) \sup_{\|v\|_H = 1} \int_{\mathcal{O}} |\langle (e^{\mu_k t} - 1) \phi_k(\cdot), v(\cdot) \rangle \phi_k(x)|^2 dx \\ = \quad & \Big(\frac{1}{|\mu_k - 2\Lambda|} + \frac{1}{|\mu_k + 2\Lambda|} \Big) \sup_{\|v\|_H = 1} (1 - e^{\mu_k t})^2 |v^k(\cdot)|^2 \\ \leq \quad & \Big(\frac{|\mu_k|}{|\mu_k - 2\Lambda|} + \frac{|\mu_k|}{|\mu_k + 2\Lambda|} \Big) |t|. \end{aligned}$$

Then from the following SPDEs of Φ_t,

$$\begin{cases} d\Phi_t P^k = \mathcal{L}\Phi_t P^k dt + B\Phi_t P^k \circ dW_t, \\ \Phi_0 P^k = P^k, \end{cases}$$

which gives us for $t \geq 0$,

$$\Phi_t P^k = T_t P^k + \int_0^t T_{t-h} B\Phi_h P^k \circ dW_h.$$

Thus

$$\begin{aligned}
& \mathbb{E}\|\Phi_t P^k - T_t P^k\|^2 \\
= \ & \mathbb{E}\left\|\int_0^t T_{t-h} P^k B\Phi_h P^k \circ dW_h\right\|^2 \\
= \ & \mathbb{E}\sum_{j=1}^{\infty} \lambda_j \int_{\mathcal{O}} \Big| \int_0^t T_{t-h} P^k B\Phi_h P^k(\phi_j)(x) \circ dW_h^j\Big|^2 dx \\
= \ & \lambda_k \mathbb{E}\int_{\mathcal{O}} \Big| \int_0^t T_{t-h} P^k B\Phi_h P^k(\phi_k)(x) \circ dW_h^k\Big|^2 dx \\
= \ & \sigma_k^2 \lambda_k \mathbb{E}\int_{\mathcal{O}} \Big| \int_0^t T_{t-h} P^k \Phi_h P^k(\phi_k)(x) \circ dW_h^k\Big|^2 dx \\
\leq \ & 2\sigma_k^2 \lambda_k \int_{\mathcal{O}} \|T_t P^k\|^2 \mathbb{E}\Big| \int_0^t T_{-h} P^k \Phi_h P^k(\phi_k)(x) dW_h^k\Big|^2 dx \\
& +2\sigma_k^2 \lambda_k \int_{\mathcal{O}} \mathbb{E}\Big| \int_0^t (T_{t-h} P^k)^2 \Phi_h P^k(\phi_k)(x) dh\Big|^2 dx \\
\leq \ & 2\sigma_k^2 \lambda_k \int_{\mathcal{O}} \|T_t P^k\|^2 \int_0^t \|T_{-h} P^k\|^2 \mathbb{E}|\Phi_h P^k(\phi_k)(x)|^2 dh dx \\
& +2\sigma_k^2 \lambda_k \int_{\mathcal{O}} \mathbb{E}\Big| \int_0^t \|T_{t-h} P^k\|^2 |\Phi_h P^k(\phi_k)(x)| dh\Big|^2 dx \\
\leq \ & C \max\left\{1, e^{2\mu_k t + 2\sigma_k^2 \lambda_k |t|}\right\} 2\sigma_k^2 \lambda_k (|t| + |t|^2),
\end{aligned}$$

where

$$\mathbb{E}|\Phi_h P^k(\phi_k)(x)|^2 \leq \mathbb{E}e^{2\mu_k h + 2\sigma_k \sqrt{\lambda_k} W_h^k} |(\phi_k)(x)|^2 \leq e^{2\mu_k h + 2\sigma_k^2 \lambda_k h} |(\phi_k)(x)|^2.$$

The case for $t < 0$, $1 \leq k \leq m$ can be derived analogously. □

Lemma 4.2.4. *Under the conditions of Theorem 4.2.2,*

$$\mathcal{M}^N : C_\tau^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O})) \to C_\tau^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O})),$$

is a continous map. Moreover, $\mathcal{M}^N$ maps $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O}))$ into $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O})) \cap L_\Lambda^\infty(\mathbb{R}, L^2(\Omega, H^1(\mathcal{O})))$.

Proof. ***Step 1***: To show that $\mathcal{M}^N$ maps $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$ to itself.

(A) Firstly show that for any $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$,

$$\sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y^N)(t,\cdot,x)|^2 dx < \infty.$$

In fact, for any $t\in\mathbb{R}$, from equation (4.19) we have

$$\begin{aligned}
& e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y^N)(t,\cdot,x)|^2 dx \\
\leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx \\
& +2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx \\
=\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\langle\phi_k(y),\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^kF(\hat{s},Y^N)(y)\rangle\phi_k(x)d\hat{s}\Big|^2dx \\
& +2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\langle\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^k\phi_k(y),F(\hat{s},Y^N)(y)\rangle\phi_k(x)d\hat{s}\Big|^2dx \\
=\ & 2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\langle\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^k\phi_k(y),F^k(\hat{s},Y^N)(y)\rangle\phi_k(x)d\hat{s}\Big|^2dx \\
& +2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=1}^{m}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\langle\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^k\phi_k(y),F^k(\hat{s},Y^N)(y)\rangle\phi_k(x)d\hat{s}\Big|^2dx \\
\leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\|\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^k\||F^k(\hat{s},Y^N)|d\hat{s}\Big|^2|\phi_k(x)|^2dx \\
& +2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=1}^{m}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\|\Phi^N(t-\hat{s},\theta_{\hat{s}}\cdot)P^k\||F^k(\hat{s},Y^N)|d\hat{s}\Big|^2|\phi_k(x)|^2dx \\
\leq\ & 2e^{-2\Lambda|t|}\Big\{\mathbb{E}\sum_{k=m+1}^{\infty}\Big(\int_{-\infty}^{t}\|\Phi^N_{t-\hat{s},\hat{s}}P^k\||F^k(\hat{s},Y^N)|d\hat{s}\Big)^2 \\
& \qquad +\mathbb{E}\sum_{k=1}^{m}\Big(\int_{t}^{+\infty}\|\Phi^N_{t-\hat{s},\hat{s}}P^k\||F^k(\hat{s},Y^N)|d\hat{s}\Big)^2\Big\} \\
\leq\ & 2e^{-2\Lambda|t|}N^2\Big\{\mathbb{E}\sum_{k=m+1}^{\infty}\Big(\int_{-\infty}^{t}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}|F^k(\hat{s},Y^N)|d\hat{s}\Big)^2 \\
& \qquad +\mathbb{E}\sum_{k=1}^{m}\Big(\int_{t}^{+\infty}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}|F^k(\hat{s},Y^N)|d\hat{s}\Big)^2\Big\} \\
\leq\ & \frac{8N^2}{|\mu_{m+1}-2\Lambda|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=m+1}^{\infty}|F^k(\hat{s},Y^N)|^2d\hat{s} \\
& +\frac{8N^2}{|\mu_{m+1}+2\Lambda|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=m+1}^{\infty}|F^k(\hat{s},Y^N)|^2d\hat{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8N^2}{|\mu_m-2\Lambda|}\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_m-\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=1}^{m}|F^k(\hat{s},Y^N)|^2 d\hat{s}\\
&+\frac{8N^2}{|\mu_m+2\Lambda|}\int_{-\infty}^{t} e^{(\frac{1}{2}\mu_m+\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=1}^{m}|F^k(\hat{s},Y^N)|^2 d\hat{s}\\
\leq\ & 16N^2\|F\|_\infty^2\Big(\frac{1}{|\mu_{m+1}-2\Lambda|^2}+\frac{1}{|\mu_{m+1}+2\Lambda|^2}+\frac{1}{|\mu_m-2\Lambda|^2}+\frac{1}{|\mu_m+2\Lambda|^2}\Big).
\end{aligned}
$$

where

$$
\|F\|_\infty^2 := \sup_{t\in\mathbb{R},x\in\mathbb{R}}|F(t,x)|^2 = \sup_{t\in\mathbb{R},x\in\mathbb{R}}\sum_{k=1}^{\infty}|F(t,x)|^2.
$$

(B) Now we show that $\mathcal{M}^N(Y^N)(t,\omega,x)$ is continuous w.r.t t. For $t_1\leq t_2$,

$$
\begin{aligned}
&\mathbb{E}\int_{\mathcal{O}}|\mathcal{M}(Y^N)(t_1,\cdot,x)-\mathcal{M}(Y^N)(t_2,\cdot,x)|^2dx\\
\leq\ & 4\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^k-\Phi^N_{t_2-\hat{s},\hat{s}}P^k)F(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&+4\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&+4\sum_{k=1}^{m}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_2}^{+\infty}(\Phi^N_{t_1-\hat{s},\hat{s}}P^k-\Phi^N_{t_2-\hat{s},\hat{s}}P^k)F(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&+4\sum_{k=1}^{m}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
=:\ & \sum_{i=1}^{4}T_i.
\end{aligned}
$$

It's easy to check that

$$
\begin{aligned}
T_2\ :=\ & 4\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
=\ & 4\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_1}^{t_2}\Phi^N_{t_2-\hat{s},\hat{s}}P^kF^k(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
\leq\ & 4\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{t_1}^{t_2}\|\Phi^N_{t_2-\hat{s},\hat{s}}P^k\||F^k(\hat{s},Y^N)(x)|d\hat{s}\Big)^2dx\\
\leq\ & 4N^2\max\{e^{2\Lambda|t_2|},e^{2\Lambda|t_1|}\}\sum_{k=m+1}^{\infty}\int_{t_1}^{t_2}e^{\mu_k(t_2-\hat{s})}d\hat{s}\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{O}}|F^k(\hat{s},Y^N)(x)|^2dxd\hat{s}\\
\leq\ & 4N^2\max\{e^{2\Lambda|t_2|},e^{2\Lambda|t_1|}\}\int_{t_1}^{t_2}e^{\mu_{m+1}(t_2-\hat{s})}d\hat{s}\mathbb{E}\int_{t_1}^{t_2}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}|F^k(\hat{s},Y^N)(x)|^2dxd\hat{s}\\
\leq\ & C_N\|F\|_\infty^2\max\{e^{2\Lambda|t_2|},e^{2\Lambda|t_1|}\}|t_2-t_1|^2,
\end{aligned}
$$

where C_N is a general constant depending on N. Similarly

$$\begin{aligned}
T_4 &:= 4\sum_{k=1}^{m}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_1}^{t_2}\Phi^N_{t_1-\hat{s},\hat{s}}P^kF(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&\le C_N\|F\|^2_\infty\max\{e^{2\Lambda|t_2|},e^{2\Lambda|t_1|}\}|t_2-t_1|^2.
\end{aligned}$$

For terms T_1 and T_3, with Lemma 4.2.3 we have the following estimation,

$$\begin{aligned}
T_1 &:= 4\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t_1}(\Phi^N_{t_1-\hat{s},\hat{s}}P^k-\Phi^N_{t_2-\hat{s},\hat{s}}P^k)F(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&\le 8\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t_1}\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_k(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}\Big\}\\
&\qquad\qquad\cdot(\Phi_{t_1-\hat{s},\hat{s}}P^k-\Phi_{t_2-\hat{s},\hat{s}}P^k)F^k(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx\\
&\quad+8\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t_1}\left(\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_k(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}\Big\}-\min\Big\{1,\frac{Ne^{\frac{1}{2}\mu_k(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|}\Big\}\right)\\
&\qquad\qquad\cdot\Phi_{t_2-\hat{s},\hat{s}}P^kF^k(\hat{s},Y^N(\hat{s},\cdot))(x)d\hat{s}\Big|^2dx\\
&\le 8N^2\sum_{k=m+1}^{\infty}\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^k-P^k\|^2\int_{\mathcal{O}}\Big(\int_{-\infty}^{t_1}e^{\frac{1}{2}\mu_k(t_1-\hat{s})}e^{\Lambda|\hat{s}|}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|d\hat{s}\Big)^2dx\\
&\quad+16\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\left(\int_{-\infty}^{t_1}\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|\left|\frac{Ne^{\frac{1}{2}\mu_k(t_1-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}-\frac{Ne^{\frac{1}{2}\mu_k(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}\right|\right.\\
&\qquad\qquad\left.\cdot|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|d\hat{s}\right)^2dx\\
&\quad+16\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\left(\int_{-\infty}^{t_1}\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|\left|\frac{Ne^{\frac{1}{2}\mu_k(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}-\frac{Ne^{\frac{1}{2}\mu_k(t_2-\hat{s})}e^{\Lambda|\hat{s}|}}{\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|}\right|\right.\\
&\qquad\qquad\left.\cdot|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|d\hat{s}\right)^2dx\\
&\le \sum_{k=m+1}^{\infty}\frac{16N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\\
&\quad\cdot\mathbb{E}\Big(\|\Phi_{t_2-t_1,t_1}P^k-P^k\|^2\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k-\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\Big)\\
&\quad+\sum_{k=m+1}^{\infty}\frac{16N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\\
&\qquad\cdot\mathbb{E}\Big(\|\Phi_{t_2-t_1,t_1}P^k-P^k\|^2\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k+\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&+16N^2\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\left(\int_{-\infty}^{t_1}e^{\Lambda|\hat{s}|}\left(e^{\frac{1}{2}\mu_k(t_1-\hat{s})}-e^{\frac{1}{2}\mu_k(t_2-\hat{s})}\right)\frac{\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}\right.\\
&\qquad\left.\cdot|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|d\hat{s}\right)^2dx\\
&+16N^2\sum_{k=m+1}^{\infty}\mathbb{E}\int_{\mathcal{O}}\left(\int_{-\infty}^{t_1}e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_k(t_1-\hat{s})}\left|\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|-\|\Phi_{t_2-\hat{s},\hat{s}}P^k\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^k\|}\right|\right.\\
&\qquad\left.\cdot\sum_{k=m+1}^{\infty}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|d\hat{s}\right)^2dx\\
\leq\;&\sum_{k=m+1}^{\infty}\frac{32N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\|T_{t_2-t_1}P^k-P^k\|^2\\
&\cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\\
&+\sum_{k=m+1}^{\infty}\frac{32N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\|T_{t_2-t_1}P^k-P^k\|^2\\
&\quad\cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\\
&+\|F\|_{\infty}^2\sum_{k=m+1}^{\infty}\frac{32N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\mathbb{E}\|T_{t_2-t_1}P^k-\Phi_{t_2-t_1,t_1}P^k\|^2\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k-\Lambda)(t-\hat{s})}d\hat{s}\\
&+\|F\|_{\infty}^2\sum_{k=m+1}^{\infty}\frac{32N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\mathbb{E}\|T_{t_2-t_1}P^k-\Phi_{t_2-t_1,t_1}P^k\|^2\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k+\Lambda)(t-\hat{s})}d\hat{s}\\
&+\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\left(1-e^{\frac{1}{2}\mu_k(t_2-t_1)}\right)^2\|T_{t_2-t_1}P^k\|^2\\
&\quad\cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\\
&+\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\left(1-e^{\frac{1}{2}\mu_k(t_2-t_1)}\right)^2\|T_{t_2-t_1}P^k\|^2\\
&\quad\cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2d\hat{s}dx\\
&+\|F\|_{\infty}^2\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\left(1-e^{\frac{1}{2}\mu_k(t_2-t_1)}\right)^2\\
&\quad\cdot\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^k-T_{t_2-t_1}P^k\|^2\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k-\Lambda)(t-\hat{s})}d\hat{s}dx\\
&+\|F\|_{\infty}^2\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\left(1-e^{\frac{1}{2}\mu_k(t_2-t_1)}\right)^2\\
&\quad\cdot\mathbb{E}\|\Phi_{t_2-t_1,t_1}P^k-T_{t_2-t_1}P^k\|^2\int_{\mathcal{O}}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k+\Lambda)(t-\hat{s})}d\hat{s}dx\\
&+\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\|T_{t_2-t_1}P^k-P^k\|^2
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1} e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2 d\hat{s}dx \\
& + \sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\|T_{t_2-t_1}P^k-P^k\|^2 \\
& \cdot\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t_1} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2 d\hat{s}dx \\
& +\|F\|_\infty^2\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k-2\Lambda|}\mathbb{E}\|T_{t_2-t_1}P^k-\Phi_{t_2-t_1,t_1}P^k\|^2\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k-\Lambda)(t-\hat{s})}d\hat{s} \\
& +\|F\|_\infty^2\sum_{k=m+1}^{\infty}\frac{64N^2e^{2\Lambda|t_1|}}{|\mu_k+2\Lambda|}\mathbb{E}\|T_{t_2-t_1}P^k-\Phi_{t_2-t_1,t_1}P^k\|^2\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_k+\Lambda)(t-\hat{s})}d\hat{s} \\
\leq\ & C\Big\{\sup_k\frac{|\mu_k|}{|\mu_k-2\Lambda|}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})} \\
& \qquad\cdot\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2dxd\hat{s} \\
& \qquad+\sup_k\frac{|\mu_k|}{|\mu_k+2\Lambda|}\int_{-\infty}^{t_1}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})} \\
& \qquad\cdot\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}|F^k(\hat{s},Y^N(\hat{s},\cdot))(x)|^2dxd\hat{s}\Big\}N^2e^{2\Lambda|t_1|}|t_2-t_1| \\
& +CN^2\|F\|_\infty^2e^{2\Lambda|t_1|}\max\Big\{\sup_{k\geq m+1}e^{2\mu_k|t_2-t_1|+2\sigma_k^2\lambda_k|t_2-t_1|},1\Big\} \\
& \cdot\sum_{k=m+1}^{\infty}\Big(\frac{\sigma_k^2\Lambda_k}{|\mu_k-2\Lambda|^2}+\frac{\sigma_k^2\Lambda_k}{|\mu_k+2\Lambda|^2}\Big) \\
\leq\ & CN^2\|F\|_\infty^2e^{2\Lambda|t_1|}\Big(\frac{1}{|\mu_{m+1}-2\Lambda|}+\frac{1}{|\mu_{m+1}+2\Lambda|}\Big)\sup_k\frac{|\mu_k|}{|\mu_k-2\Lambda|}|t_2-t_1| \\
& +CN^2\|F\|_\infty^2e^{2\Lambda|t_1|}\max\Big\{\sup_k e^{2\mu_k|t_2-t_1|+2\sigma_k^2\lambda_k|t_2-t_1|},1\Big\}\Big(\sum_{k=1}^{\infty}\sigma_k^2\Lambda_k\Big) \\
& \cdot\Big(\frac{1}{|\mu_{m+1}+2\Lambda|^2}+\frac{1}{|\mu_{m+1}-2\lambda|^2}\Big)(|t_2-t_1|+|t_2-t_1|^2),
\end{aligned}
$$

where by the cocycle property of Φ,

$$
\left|\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|-\|\Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\right|\leq\frac{\|\Phi_{t_1-\hat{s},\hat{s}}P^--\Phi_{t_2-\hat{s},\hat{s}}P^-\|}{\|\Phi_{t_1-\hat{s},\hat{s}}P^-\|}\leq\|\Phi_{t_2-t_1,t_1}P^--P^-\|^2.
$$

where C is a generic constant that may depend on $\mathbb{E}C_\Lambda^2(\omega)$. Similarly,

$$
\begin{aligned}
T_3 \ &:= \ 4\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t_2}^{+\infty}\sum_{k=1}^{m}(\Phi^N_{t_1-\hat{s},\hat{s}}P^k-\Phi^N_{t_2-\hat{s},\hat{s}}P^k)F(\hat{s},Y^N)(x)d\hat{s}\Big|^2dx \\
&\leq \ CN^2\|F\|_\infty^2e^{2\Lambda|t_1|}\Big(\frac{1}{|\mu_{m+1}-2\Lambda|}+\frac{1}{|\mu_{m+1}+2\Lambda|}\Big)\sup_k\frac{|\mu_k|}{|\mu_k-2\Lambda|}|t_2-t_1|
\end{aligned}
$$

$$+CN^2\|F\|_\infty^2 e^{2\Lambda|t_1|}\max\Big\{\sup_k e^{-2\mu_k|t_2-t_1|+2\sigma_k^2\lambda_k|t_2-t_1|},1\Big\}\Big(\sum_{k=1}^{\infty}\sigma_k^2\lambda_k\Big)$$
$$\cdot\Big(\frac{1}{|\mu_m+2\Lambda|^2}+\frac{1}{|\mu_m-2\lambda|^2}\Big)(|t_2-t_1|+|t_2-t_1|^2).$$

(C) We show that $\mathcal{M}^N(Y^N)(t,\theta_{\pm\tau}\omega)=\mathcal{M}^N(Y^N)(t\pm\tau,\omega)$:

$$\begin{aligned}
\mathcal{M}^N(Y^N)(t,\theta_\tau\omega) &= \int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}+\tau}P^kF(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}\\
&\quad-\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}+\tau}P^kF(\hat{s},Y^N(\hat{s},\theta_\tau\omega))d\hat{s}\\
&= \int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^kF(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\
&\quad-\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{(t+\tau)-(\hat{s}+\tau),\hat{s}+\tau}P^kF(\hat{s}+\tau,Y^N(\hat{s}+\tau,\omega))d\hat{s}\\
&= \int_{-\infty}^{t+\tau}\sum_{k=m+1}^{\infty}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^kF(\hat{h},Y^N(\hat{h},\omega))d\hat{h}\\
&\quad-\int_{t+\tau}^{+\infty}\sum_{k=1}^{m}\Phi^N_{(t+\tau)-\hat{h},\hat{h}}P^kF(\hat{h},Y^N(\hat{h},\omega))d\hat{h}\\
&= \mathcal{M}^N(Y^N)(t+\tau,\omega).
\end{aligned}$$

Thus we have shown that $\mathcal{M}^N$ maps from $C^\Lambda_\tau(\mathbb{R},L^2(\Omega\times\mathcal{O}))$ to itself.

Step 2: To show $\mathcal{M}^N$ maps from $C^\Lambda_\tau(\mathbb{R},L^2(\Omega\times\mathcal{O}))$ to $L^\infty_\Lambda(\mathbb{R},L^2(\Omega,H^1(\mathcal{O})))$.

In fact for each t and ω fixed, $\mathcal{M}^N(Y^N)(t,\omega,x)$ can be expressed as,

$$\mathcal{M}^N(Y^N)(t,\omega,x)=\sum_{i=1}^{\infty}\int_{\mathcal{O}}\mathcal{M}^N(Y^N)(t,\omega,y)\phi_i(y)dy\phi_i(x), \tag{4.28}$$

where $\phi_k\in H^1_0(\mathcal{O})$, and we have

$$\begin{aligned}
\nabla_x\mathcal{M}^N(Y^N)(t,\omega,x) &= \sum_{i=1}^{\infty}\int_{\mathcal{O}}\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y^N(\hat{s},\omega))(y)d\hat{s}\phi_i(y)dy\nabla_x\phi_i(x)\\
&\quad+\sum_{i=1}^{\infty}\int_{\mathcal{O}}\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y^N(\hat{s},\omega))(y)d\hat{s}\phi_i(y)dy\nabla_x\phi_i(x).
\end{aligned}$$

Then we get

$$
\begin{aligned}
& e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}|\nabla_x\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dx \\
\leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\sum_{i=m+1}^{\infty}\int_{\mathcal{O}}\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y^N(\hat{s},\cdot))(y)d\hat{s}\phi_i(y)dy\nabla_x\phi_i(x)\Big|^2dx \\
& +2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\sum_{i=1}^{m}\int_{\mathcal{O}}\int_{t}^{\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y^N(\hat{s},\cdot))(y)d\hat{s}\phi_i(y)dy\nabla_x\phi_i(x)\Big|^2dx \\
=:\ & L_1+L_2.
\end{aligned}
$$

And by Cauchy-Schwarz inequality, (4.3) and (4.20) we have

$$
\begin{aligned}
L_1 =\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big|\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y^N(\hat{s},\cdot))(y)d\hat{s}\phi_k(y)dy\nabla_x\phi_k(x)\Big|^2dx \\
\leq\ & 2N^2e^{-2\Lambda|t|}\mathbb{E}\sum_{k,j=m+1}^{\infty}\Big(\int_{\mathcal{O}}|\nabla_x\phi_k(x)|^2dx\int_{\mathcal{O}}|\nabla_x\phi_j(x)|^2dx\Big)^{\frac{1}{2}} \\
& \cdot\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}|\phi_k(y)||F^i(\hat{s},Y^N(\hat{s},\cdot))(y)|dyd\hat{s} \\
& \cdot\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_j(t-\hat{s})}e^{\Lambda|\hat{s}|}|\phi_j(y)||F^j(\hat{s},Y^N(\hat{s},\cdot))(y)|dyd\hat{s} \\
\leq\ & CN^2e^{-2\Lambda|t|}\mathbb{E}\Big[\sum_{k=m+1}^{\infty}\Big(\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}\sqrt{|\mu_k|}|\phi_k(y)||F^k(\hat{s},Y^N(\hat{s},\cdot))(y)|dyd\hat{s}\Big)^2 \\
& \cdot\sum_{j=m+1}^{\infty}\Big(\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_j(t-\hat{s})}e^{\Lambda|\hat{s}|}\sqrt{|\mu_j|}|\phi_j(y)||F^j(\hat{s},Y^N(\hat{s},\cdot))(y)|dyd\hat{s}\Big)^2\Big]^{\frac{1}{2}} \\
\leq\ & CN^2e^{-2\Lambda|t|}\mathbb{E}\Big[\sum_{k=m+1}^{\infty}\Big(\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_k(t-\hat{s})}|\mu_k||\phi_k(y)|^2dyd\hat{s} \\
& \cdot\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{2\Lambda|\hat{s}|}|F^k(\hat{s},Y^N(\hat{s},\cdot))(y)|^2dyd\hat{s}\Big) \\
& \cdot\sum_{j=m+1}^{\infty}\Big(\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_j(t-\hat{s})}|\mu_j||\phi_j(y)|^2dyd\hat{s} \\
& \cdot\int_{-\infty}^{t}\int_{\mathcal{O}}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}e^{2\Lambda|\hat{s}|}|F^j(\hat{s},Y^N(\hat{s},\cdot))(y)|^2dyd\hat{s}\Big)\Big]^{\frac{1}{2}}.
\end{aligned}
$$

Thus we have

$$
L_1\leq CN^2\|F\|^2_\infty\big(\frac{1}{|\mu_{m+1}-2\Lambda|}+\frac{1}{|\mu_{m+1}+2\Lambda|}\big)<\infty.
$$

Similarly,

$$
L_2\leq CN^2\|F\|^2_\infty\big(\frac{1}{|\mu_m-2\Lambda|}+\frac{1}{|\mu_m+2\Lambda|}\big)<\infty.
$$

Step 3: Now check the continuity of the map $\mathcal{M}^N$ in $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega \times \mathcal{O}))$. Consider $Y_1^N, Y_2^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$, and $t \in [j\tau, (j+1)\tau)$, $j \in \mathbb{Z}$, then

$$\begin{aligned}
& e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_1^N)(t,\cdot,x) - \mathcal{M}(Y_2^N)(t,\cdot,x)|^2 dx \\
\leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\Big[\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_1^N)(x)d\hat{s} - \int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_2^N)(x)d\hat{s}\Big|^2 \\
& +\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_1^N)(x)d\hat{s} - \int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_2^N)(x)d\hat{s}\Big|^2\Big]dx \\
=:\ & U_1 + U_2,
\end{aligned}$$

where

$$\begin{aligned}
U_1 =\ & 2e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}}\sum_{k=m+1}^{\infty}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_1^N)(x)d\hat{s} - \int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^kF(\hat{s},Y_2^N)(x)d\hat{s}\Big|^2 dx \\
\leq\ & 2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}\|\Phi^N_{t-\hat{s},\hat{s}}P^k\||F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|d\hat{s}\Big)^2 dx \\
\leq\ & 2N^2e^{-2\Lambda|t|}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}|F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|d\hat{s}\Big)^2 dx \\
\leq\ & 4N^2\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_k-\Lambda)(t-\hat{s})}|F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|d\hat{s}\Big)^2 dx \\
& +4N^2\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_k+\Lambda)(t-\hat{s})}|F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|d\hat{s}\Big)^2 dx \\
\leq\ & \frac{8N^2}{|\mu_{m+1}-2\Lambda|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}|F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|^2dxd\hat{s} \\
& +\frac{8N^2}{|\mu_{m+1}-2\Lambda|}\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}|F^k(\hat{s},Y_1^N)(x) - F^k(\hat{s},Y_2^N)(x)|^2dxd\hat{s} \\
\leq\ & e^{2\Lambda\tau}\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|}\sup_{s\in[0,\tau]}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathcal{O}}|Y_1^N(\hat{s},\cdot,x) - Y_2^N(\hat{s},\cdot,x)|^2dx \\
& \cdot\Big\{\int_0^{t-j\tau}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)j\tau}d\hat{s} + \int_0^{\tau}\sum_{i=-j+1}^{\infty}e^{-(\frac{1}{2}\mu_{m+1}-\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\Big\} \\
& +e^{2\Lambda\tau}\frac{8N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|}\sup_{s\in[0,\tau]}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathcal{O}}|Y_1^N(\hat{s},\cdot,x) - Y_2^N(\hat{s},\cdot,x)|^2dx \\
& \cdot\Big\{\int_0^{t-j\tau}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)j\tau}d\hat{s} + \int_0^{\tau}\sum_{i=-j+1}^{\infty}e^{-(\frac{1}{2}\mu_{m+1}+\Lambda)i\tau}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\Big\} \\
=\ & e^{2\Lambda\tau}\Big(\frac{16N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+2\Lambda|^2} + \frac{16N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-2\Lambda|^2}\Big)\sup_{s\in\mathbb{R}}e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathcal{O}}|Y_1^N(\hat{s},\cdot,x) - Y_2^N(\hat{s},\cdot,x)|^2dx,
\end{aligned}$$

where

$$||\nabla F||_\infty := \sup_{t\in\mathbb{R},x\in\mathbb{R}} |\nabla F(t,x)|^2.$$

Analogously,

$$U_2 \le e^{2\Lambda\tau}\Big(\frac{16N^2\|\nabla F\|_\infty^2}{|\mu_m+2\Lambda|^2}+\frac{16N^2\|\nabla F\|_\infty^2}{|\mu_m-2\Lambda|^2}\Big)\sup_{s\in\mathbb{R}} e^{-2\Lambda|\hat{s}|}\mathbb{E}\int_{\mathcal{O}}|Y_1^N(\hat{s},\cdot,x)-Y_2^N(\hat{s},\cdot,x)|^2dx.$$

Then the claim that $\mathcal{M}^N : C_\tau(\mathbb{R}, L^2(\Omega\times\mathcal{O})) \to C_\tau(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$ is a continuous map has been confirmed. □

Next introduce a subset of $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$ as follows,

$$\begin{aligned} C_{\tau,\rho}^{\Lambda,N}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2})) \;:=\Big\{ &\; f\in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O})):\; f|_{[0,\tau)}\in C([0,\tau), L^2(\mathcal{O},\mathcal{D}^{1,2})),\\ &\text{and } \forall t\in[0,\tau),\; e^{-2\Lambda|t|}\sum_{j=1}^\infty\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^j f(t,\cdot,x)|^2dxdr\le\rho^N(t),\\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} e^{-2\Lambda|t|}\sum_{j=1}^\infty\frac{1}{|\delta|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_{r+\delta}^j f(t,\cdot,x)-\mathcal{D}_r^j f(t,\cdot,x)|^2dxdr<\infty\Big\}. \end{aligned}$$

Here

$$\rho^N(t) := K_1^N\int_0^\tau e^{-\frac{1}{2}\mu|t-\hat{s}|}\rho^N(\hat{s})d\hat{s}+K_2^N,\;\mu:=\min\{\mu_m,-\mu_{m+1}\},\tag{4.29}$$

where

$$\begin{aligned} K_1^N &:= 12N^2\|\nabla F\|_\infty^2 e^{(2\Lambda+\frac{1}{2}\hat{\mu})\tau}\Big(\frac{\sum_{i=-1}^\infty e^{-\frac{1}{2}|\mu_{m+1}|i\tau}}{|\mu_{m+1}+4\Lambda|}+\frac{\sum_{i=-1}^\infty e^{-\frac{1}{2}|\mu_{m+1}|i\tau}}{|\mu_{m+1}-4\Lambda|}\\ &\qquad+\frac{\sum_{i=-1}^\infty e^{-\frac{1}{2}\mu_m i\tau}}{|\mu_m+4\Lambda|}+\frac{\sum_{i=-1}^\infty e^{-\frac{1}{2}\mu_m i\tau}}{|\mu_m-4\Lambda|}\Big),\\ \hat{\mu} &:= \max\{-\mu_{m+1},\mu_m\},\\ K_2^N &:= 96\|F\|_\infty^2\Big(\frac{1}{|\mu_{m+1}+2\Lambda|^3}+\frac{1}{|\mu_{m+1}-2\Lambda|^3}\\ &\qquad+\frac{1}{|\mu_m+2\Lambda|^3}+\frac{1}{|\mu_m-2\Lambda|^3}\Big)\sum_{j=1}^\infty\sigma_j^2\lambda_j. \end{aligned}$$

Lemma 4.2.5. *Under the conditions of Theorem 4.2.2, we have*

$$\mathcal{M}^N : C_{\tau,\rho}^{\Lambda,N}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2}))\subset C_{\tau,\rho}^{\Lambda,N}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2})),$$

and $\mathcal{M}^N(C_{\tau,\rho}^{\Lambda,N}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2})))|_{[0,\tau)}$ *is relatively compact in* $C_\tau([0,\tau), L^2(\Omega\times\mathcal{O}))$.

Proof. ***Step 1***: Now we need to calculate the Malliavin derivatives of Φ^j. Firstly note that for any $v \in L^2(\mathcal{O})$ we have for $j \geq m+1$ and $t \geq 0$,

$$\begin{aligned} \mathcal{D}_r^j \Phi(t, \theta_{\hat{s}}\omega) P^j(v)(x) &= \mathcal{D}_r^j e^{\mu_j t + \sigma_j \sqrt{\lambda_j}(W_{\hat{s}+t}^j - W_{\hat{s}}^j)} \langle \phi_j(\cdot), v(\cdot) \rangle \phi_j(x) \\ &= \sigma_j \sqrt{\lambda_j} \chi_{\{\hat{s} \leq r \leq t+\hat{s}\}}(r) e^{\mu_j t + \sigma_j \sqrt{\lambda_j}(W_{\hat{s}+t}^j - W_{\hat{s}}^j)} \langle \phi_j(\cdot), v(\cdot) \rangle \phi_j(x) \\ &= \sigma_j \sqrt{\lambda_j} \chi_{\{\hat{s} \leq r \leq t+\hat{s}\}}(r) \Phi(t, \theta_{\hat{s}}\omega) P^j(v)(x), \end{aligned} \tag{4.30}$$

and when $k \neq j$,

$$\mathcal{D}_r^j \Phi(t, \theta_{\hat{s}}\omega) P^k(v)(x) = \mathcal{D}_r^j e^{\mu_k t + \sigma_k \sqrt{\lambda_k}(W_{\hat{s}+t}^k - W_{\hat{s}}^k)} \langle \phi_k(\cdot), v(\cdot) \rangle \phi_k(x) = 0. \tag{4.31}$$

Besides, we have

$$\mathcal{D}_r^j \|\Phi(t, \theta_{\hat{s}}\omega) P^j\| = \mathcal{D}_r^j e^{\frac{1}{2}\mu_j t + \sigma_j \sqrt{\lambda_j}(W_{\hat{s}+t}^j - W_{\hat{s}}^j)} = \sigma_j \sqrt{\lambda_j} \chi_{\{\hat{s} \leq r \leq t+\hat{s}\}}(r) \|\Phi(t, \theta_{\hat{s}}\omega) P^j\|, \tag{4.32}$$

and when $k \neq j$,

$$\mathcal{D}_r^j \|\Phi(t, \theta_{\hat{s}}\omega) P^k\| = 0. \tag{4.33}$$

Analogously, when $j \leq m$ and $t \leq 0$,

$$\mathcal{D}_r^j \Phi(t, \theta_{\hat{s}}\omega) P^j(v)(x) = -\sigma_j \sqrt{\lambda_j} \chi_{\{\hat{s}+t \leq r \leq \hat{s}\}}(r) \Phi(t, \theta_{\hat{s}}\omega) P^j(v)(x), \tag{4.34}$$

and

$$\mathcal{D}_r^j \|\Phi(t, \theta_{\hat{s}}\omega) P^j\| = -\sigma_j \sqrt{\lambda_j} \chi_{\{t+\hat{s} \leq r \leq \hat{s}\}}(r) \|\Phi(t, \theta_{\hat{s}}\omega) P^j\|. \tag{4.35}$$

And when $k \neq j$,

$$\mathcal{D}_r^j \Phi(t, \theta_{\hat{s}}\omega) P^k(v)(x) = 0, \tag{4.36}$$

and

$$\mathcal{D}_r^j \|\Phi(t, \theta_{\hat{s}}\omega) P^k\| = 0. \tag{4.37}$$

Then we are able to calculate the Malliavin derivatives of Φ^N by the chain rule with (4.30) to (4.37): for $j \geq m+1$ and $t \geq 0$,

$$\begin{aligned}
& \mathcal{D}_r^j \Phi^N(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
= \;& \mathcal{D}_r^j \left(\min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\} \Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x) \right) \\
= \;& \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\} \mathcal{D}_r^j \big(\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x)\big) \\
& + \mathcal{D}_r^j \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\} \Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
= \;& \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\} \sigma_j\sqrt{\lambda_j}\chi_{\{\hat{s}\leq r\leq t+\hat{s}\}}(r)\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
& - \chi_{\{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}<\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|\}}(\omega)\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x)\frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|^2}\mathcal{D}_r^j\|\Phi(t,\theta_{\hat{s}}\omega)P^j\| \\
= \;& \sigma_j\sqrt{\lambda_j}\chi_{\{\hat{s}\leq r\leq t+\hat{s}\}}(r) \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\} \Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
& - \sigma_j\sqrt{\lambda_j}\chi_{\{\hat{s}\leq r\leq t+\hat{s}\}}(r)\chi_{\{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}<\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|\}}(\omega)\frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x),
\end{aligned} \tag{4.38}$$

and when $k \neq j$, it is obviously

$$\mathcal{D}_r^j \Phi^N(t,\theta_{\hat{s}}\omega)P^k(v)(x) = 0. \tag{4.39}$$

Analogously, for $j \leq m$ with $t \leq 0$, we have

$$\begin{aligned}
& \mathcal{D}_r^j \Phi^N(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
= \;& \sigma_j\sqrt{\lambda_j}\chi_{\{t+\hat{s}\leq r\leq \hat{s}\}}(r)\Bigg(- \min\left\{1, \frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\right\}\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x) \\
& + \chi_{\{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}<\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|\}}(\omega)\frac{Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|s|}}{\|\Phi(t,\theta_{\hat{s}}\omega)P^j\|}\Phi(t,\theta_{\hat{s}}\omega)P^j(v)(x)\Bigg),
\end{aligned} \tag{4.40}$$

and when $k \neq j$, it is obviously

$$\mathcal{D}_r^j \Phi^N(t,\theta_{\hat{s}}\omega)P^k(v)(x) = 0. \tag{4.41}$$

And from (4.38) and (4.40), it is easy to obtain that when $j \geq m+1$, $t \geq 0$ or $j \leq m$, $t \leq 0$

$$\|\mathcal{D}_r^j \Phi^N(t,\theta_{\hat{s}}\omega)P^j\| \leq 2\sigma_j\sqrt{\lambda_j}Ne^{\frac{1}{2}\mu_j t}e^{\Lambda|\hat{s}|}. \tag{4.42}$$

Next, show that for any $t \in [0,\tau)$,

$$\sum_{j=1}^{\infty} e^{-2\Lambda|t|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^j\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr < \rho^N(t).$$

By the chain rules and results from (4.38) to (4.41), it is easy to write down the Malliavin derivative of $\mathcal{M}^N(Y^N)(t,\omega,x)$ with respect to the jth Brownian motion for $j \geq m+1$,

$$\begin{aligned}
\mathcal{D}_r^j \mathcal{M}^N(Y^N)(t,\omega,x) &= \int_{-\infty}^{r} \chi_{\{r\leq t\}}(r)\mathcal{D}_r^j(\Phi_{t-\hat{s},\hat{s}}^N P^j)F(\hat{s},Y^N(\hat{s},\omega))(x)d\hat{s} \\
&\quad + \int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^j Y^N(\hat{s},\omega,x)d\hat{s} \\
&\quad - \int_{t}^{+\infty}\sum_{k=1}^{m}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\omega))\mathcal{D}_r^j Y^N(\hat{s},\omega,x)d\hat{s},
\end{aligned} \tag{4.43}$$

from which we get the following L^2-estimation

$$\begin{aligned}
& e^{-2\Lambda|t|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^j\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr \\
= \;& e^{-2\Lambda|t|}\Big(\int_{-\infty}^{t}+\int_{t}^{\infty}\Big)\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^j\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr \\
\leq \;& 3e^{-2\Lambda|t|}\int_{-\infty}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{r}\mathcal{D}_r^j(\Phi_{t-\hat{s},\hat{s}}^N P^j)F(\hat{s},Y^N(\hat{s},\cdot))(x)d\hat{s}\Big|^2dxdr \\
& +3e^{-2\Lambda|t|}\sum_{k=m+1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\mathcal{D}_r^j Y^N(\hat{s},\cdot,x)d\hat{s}\Big|^2dxdr \\
& +3e^{-2\Lambda|t|}\sum_{k=1}^{m}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\omega,x))\mathcal{D}_r^j Y^N(\hat{s},\cdot,x)d\hat{s}\Big|^2dxdr \\
=: \;& \sum_{i=1,2,3}L_i^j.
\end{aligned}$$

Then by (4.42) we have that

$$\begin{aligned}
L_1^j &:= 3e^{-2\Lambda|t|}\int_{-\infty}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{r}\mathcal{D}_r^j(\Phi_{t-\hat{s},\hat{s}}^N P^j)F(\hat{s},Y^N(\hat{s},\cdot))(x)d\hat{s}\Big|^2dxdr \\
&\leq 3\|F\|_\infty^2 e^{-2\Lambda|t|}\int_{-\infty}^{t}\mathbb{E}\Big(\int_{-\infty}^{r}\|\mathcal{D}_r^j\Phi_{t-\hat{s},\hat{s}}^N P^j\|d\hat{s}\Big)^2dr \\
&\leq 12N^2\|F\|_\infty^2\sigma_j^2\lambda_j e^{-2\Lambda|t|}\int_{-\infty}^{t}\Big(\int_{-\infty}^{r}e^{\frac{1}{2}\mu_j(t-\hat{s})}e^{\Lambda|\hat{s}|}d\hat{s}\Big)^2dr \\
&\leq 96N^2\|F\|_\infty^2\left(\frac{\sigma_j^2\lambda_j}{|\mu_j-2\Lambda|^3}+\frac{\sigma_j^2\lambda_j}{|\mu_j+2\Lambda|^3}\right) \\
&\leq 96N^2\|F\|_\infty^2\left(\frac{\sigma_j^2\lambda_j}{|\mu_{m+1}-2\Lambda|^3}+\frac{\sigma_j^2\lambda_j}{|\mu_{m+1}+2\Lambda|^3}\right),
\end{aligned}$$

and

$$
\begin{aligned}
& L_2^j \\
:= \; & 3e^{-2\Lambda|t|}\int_{\mathbb{R}}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N)\mathcal{D}_r^jY^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr \\
= \; & 3e^{-2\Lambda|t|}\int_{\mathbb{R}}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N)(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}d\hat{s}\Big|^2dxdr \\
\leq \; & 3e^{-2\Lambda|t|}N^2\int_{\mathbb{R}}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{\frac{1}{2}\mu_k(t-\hat{s})}e^{\Lambda|\hat{s}|}|\nabla F(\hat{s},Y^N)||(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}|d\hat{s}\Big)^2dxdr \\
\leq \; & 6N^2\int_{\mathbb{R}}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\hat{s})}|\nabla F^k(\hat{s},Y^N)||(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}|d\hat{s}\Big)^2dxdr \\
& +6N^2\int_{\mathbb{R}}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{(\frac{1}{2}\mu_{m+1}-\Lambda)(t-\hat{s})}|\nabla F(\hat{s},Y^N)||(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}|^2d\hat{s}\Big)^2dxdr \\
\leq \; & \frac{12N^2}{|\mu_{m+1}+4\Lambda|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\sum_{k=m+1}^{\infty}|\nabla F(\hat{s},Y^N)|^2|(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}|^2d\hat{s}dxdr \\
& +\frac{12N^2}{|\mu_{m+1}-4\Lambda|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\sum_{k=m+1}^{\infty}|\nabla F(\hat{s},Y^N)|^2|(\mathcal{D}_r^jY^N(\hat{s},\cdot,x))^{(k)}|^2d\hat{s}dxdr \\
= \; & \Big(\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-4\Lambda|}\Big) \\
& \cdot\Big\{\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_0^te^{-2\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^jY^N(\hat{s},\theta_{-i\tau}\cdot,x)|^2dxdrd\hat{s} \\
& \quad+\sum_{i=0}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\int_t^{\tau}e^{-2\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_{m+1}(t+\tau-\hat{s})}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^jY^N(\hat{s},\theta_{-(i+1)\tau}\cdot,x)|^2dxdrd\hat{s}\Big\} \\
\leq \; & e^{(2\Lambda-\frac{1}{2}\mu_{m+1})\tau}\Big(\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}+4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}-4\Lambda|}\Big)\Big(\sum_{i=-1}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau}\Big) \\
& \cdot\int_0^{\tau}e^{\frac{1}{2}\mu_{m+1}|t-\hat{s}|}e^{-2\Lambda|\hat{s}|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^jY^N(\hat{s},\cdot,x)|^2dxdrd\hat{s},
\end{aligned}
$$

where

$$
(\mathcal{D}_r^jY^N(\hat{s},\omega,x))^{(k)}:=\mathcal{D}_r^j\Big(\langle Y^N(\hat{s},\omega),\phi_k\rangle\phi_k(x)\Big).
$$

And similarly

$$
\begin{aligned}
L_3^j \;:=\; & 3e^{-2\Lambda|t|}\int_{\mathbb{R}}\mathbb{E}\sum_{k=1}^{m}\int_{\mathcal{O}}\Big|\int_t^{+\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N)\mathcal{D}_r^jY^N(\hat{s},\cdot,x)d\hat{s}\Big|^2dxdr \\
\leq \; & e^{(2\Lambda+\frac{1}{2}\mu_m)\tau}\Big(\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_m+4\Lambda|}+\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_m-4\Lambda|}\Big)\Big(\sum_{i=-1}^{\infty}e^{-\frac{1}{2}\mu_mi\tau}\Big) \\
& \cdot\int_0^{\tau}e^{\frac{1}{2}\mu|t-\hat{s}|}e^{-2\Lambda|\hat{s}|}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_r^jY^N(\hat{s},\cdot,x)|^2dxdrd\hat{s}.
\end{aligned}
$$

Similar calculation can be applied to the case when $j \leq m$. Therefore we finally reach to

$$\sum_{j=1}^{\infty} L_1^j = 96\|F\|_\infty^2 \left(\frac{\sum_{j=1}^{\infty} \sigma_j^2 \lambda_j}{|\mu_{m+1} - 2\Lambda|^3} + \frac{\sum_{j=1}^{\infty} \sigma_j^2 \lambda_j}{|\mu_{m+1} - 2\Lambda|^3} + \frac{\sum_{j=1}^{\infty} \sigma_j^2 \lambda_j}{|\mu_m - 2\Lambda|^3} + \frac{\sum_{j=1}^{\infty} \sigma_j^2 \lambda_j}{|\mu_m + 2\Lambda|^3} \right) = K_2^N,$$

and

$$\begin{aligned} \sum_{j=1}^{\infty} (L_2^j + L_3^j) &\leq e^{(2\Lambda + \frac{1}{2}\hat{\mu})\tau} \sum_{k=m}^{m+1} \Big(\frac{12N^2\|\nabla F\|_\infty^2}{|\mu_k + 4\Lambda|} + \frac{12N^2\|\nabla F\|_\infty^2}{|\mu_k - 4\Lambda|} \Big) \Big(\sum_{i=-1}^{\infty} e^{-\frac{1}{2}|\mu_k| i\tau} \Big) \\ &\quad \cdot \int_0^\tau e^{\frac{1}{2}\mu|t-\hat{s}|} e^{-2\Lambda|\hat{s}|} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_r^j Y^N(\hat{s}, \cdot, x)|^2 dx dr d\hat{s} \\ &\leq K_1^N \int_0^\tau e^{\frac{1}{2}\mu|t-\hat{s}|} \rho^N(s) d\hat{s}. \end{aligned}$$

To sum up, we have verified the following estimation:

$$\sum_{j=1}^{\infty} e^{-2\Lambda|t|} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr \leq \rho^N(t).$$

Moreover, the solution $\rho^N(t)$ to equation (4.29) is continous in t, so that for $Y^N \in C_{\tau,\rho}^{\Lambda,N}(\mathbb{R}, L^2(\mathcal{O}, \mathcal{D}^{1,2}))$, there exists an integer N_b such that for any $t \in [0, \tau)$,

$$\sum_{j=1}^{\infty} e^{-2\Lambda|t|} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr < \rho^N(t) \leq N_b.$$

It remains to show that

$$\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_{r+\delta}^j \mathcal{M}^N(Y^N)(t, \cdot, x) - \mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr < \infty.$$

According to the calculations in Lemma 3.2.7, the left hand side of the above can be separated into three integrals,

$$\begin{aligned} &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_{r+\delta}^j \mathcal{M}^N(Y^N)(t, \cdot, x) - \mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr \\ = &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{-\infty}^{t-\delta} \mathbb{E} \int_{\mathcal{O}} \mathbb{E} |\mathcal{D}_{r+\delta}^j \mathcal{M}^N(Y^N)(t, \cdot, x) - \mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr \\ &+ \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{t-\delta}^{t} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_{r+\delta}^j \mathcal{M}^N(Y^N)(t, \cdot, x) - \mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr \\ &+ \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{t}^{+\infty} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}_{r+\delta}^j \mathcal{M}^N(Y^N)(t, \cdot, x) - \mathcal{D}_r^j \mathcal{M}^N(Y^N)(t, \cdot, x)|^2 dx dr. \\ =: &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} (\hat{K}_1(t, \delta) + \hat{K}_2(t, \delta) + \hat{K}_3(t, \delta)). \end{aligned} \tag{4.44}$$

To consider $\hat{K}_1(t,\delta)$, note that when $r \le r+\delta \le t$, by (4.38) and (4.40) we have

$$
\begin{aligned}
& \hat{K}_1(t,\delta) \\
= & \frac{e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}^j_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot,x)-\mathcal{D}^j_r\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr \\
\le & \frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big\{\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2 \\
& +\chi_{\{j\ge m+1\}}(j)\Big|\int_{-\infty}^{r+\delta}\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}-\int_{-\infty}^{r}\mathcal{D}^j_r\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2 \\
& +\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2\Big\}dxdr \\
= & \sum_{i=1}^{3}Q_i(t,\delta).
\end{aligned}
$$

First note that Q_1 is bounded as follows,

$$
\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_1(t,\delta) = & \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N) \\
& \qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2dxdr \\
\le & \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3N^2}{|\delta|}e^{-2\Lambda|t|}\sum_{j=1}^{\infty}\mathbb{E}\sum_{k=m+1}^{\infty}\int_{\mathcal{O}}\int_{\mathbb{R}}\Big(\int_{-\infty}^{t}e^{\Lambda|\hat{s}|}e^{\frac{1}{2}\mu_k(t-\hat{s})}|\nabla F(\hat{s},Y^N)| \\
& \qquad\cdot|((\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x))^{(k)}|d\hat{s}\Big)^2dxdr \\
\le & \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{12N^2}{|\delta|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}\int_{-\infty}^{t}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})} \\
& \cdot\sum_{k=m+1}^{\infty}|\nabla F(\hat{s},Y^N)|^2|((\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot))^{(k)}|^2d\hat{s}dxdr \\
\le & 12N^2\|\nabla F\|^2_{\infty}e^{2\Lambda\tau}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\
& \cdot\Big\{(\sum_{i=1}^{\infty}e^{\frac{1}{2}\mu_{m+1}i\tau})\int_0^{\tau}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}+\int_0^{\tau}e^{\frac{1}{2}\mu_{m+1}(t-\hat{s})}d\hat{s}\Big\} \\
& \cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda\hat{s}}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr \\
\le & e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_{\infty}}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big) \\
& \cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr \\
< & \infty,
\end{aligned}
$$

and analogously,

$$\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_3(t,\delta) &= \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big|\int_t^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\\
&\qquad\qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2 dxdr\\
&\le e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_m|}\Big(\frac{1}{|\mu_m+4\Lambda|}+\frac{1}{|\mu_m-4\Lambda|}\Big)\\
&\quad\cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr\\
&< \infty.
\end{aligned}$$

Besides, we have

$$\begin{aligned}
&\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_2(t,\delta)\\
=\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=m+1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{r+\delta}\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\\
&\qquad\qquad -\int_{-\infty}^{r}\mathcal{D}^j_r\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr\\
=\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big|\int_r^{r+\delta}\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^jF^j(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr\\
\le\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big(\int_r^{r+\delta}\|\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^j\||F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2dxdr\\
\le\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\sigma_j^2\lambda_j\frac{12N^2e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big(\int_r^{r+\delta}e^{\frac12\mu_j(t-\hat{s})}e^{\Lambda|\hat{s}|}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2dxdr\\
\le\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\sigma_j^2\lambda_j\frac{24N^2}{|\delta|}\sum_{j=1}^{\infty}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big\{\Big(\int_r^{r+\delta}e^{(\frac12\mu_j-\Lambda)(t-\hat{s})}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2\\
&+\Big(\int_r^{r+\delta}e^{(\frac12\mu_j+\Lambda)(t-\hat{s})}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2\Big\}dxdr\\
\le\ &\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\sigma_j^2\lambda_j\frac{24N^2}{|\delta|}\int_{-\infty}^{t-\delta}\mathbb{E}\int_{\mathcal{O}}\Big\{\int_r^{r+\delta}e^{(\frac12\mu_{m+1}-\Lambda)(t-\hat{s})}d\hat{s}\\
&\cdot\int_r^{r+\delta}e^{(\frac12\mu_{m+1}-\Lambda)(t-\hat{s})}\sum_{j=1}^{\infty}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|^2d\hat{s}\\
&+\int_r^{r+\delta}e^{(\frac12\mu_{m+1}+\Lambda)(t-\hat{s})}d\hat{s}\int_r^{r+\delta}e^{(\frac12\mu_{m+1}+\Lambda)(t-\hat{s})}\sum_{j=1}^{\infty}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|^2d\hat{s}\Big\}dxdr\\
\le\ &\sigma_j^2\lambda_j24N^2\|F\|^2_\infty vol(\mathcal{O})\int_{-\infty}^{t-\delta}e^{(\frac12\mu_{m+1}-\Lambda)(t-\delta-r)}\int_{-\infty}^{r+\delta}e^{(\frac12\mu_{m+1}-\Lambda)(r+\delta-\hat{s})}d\hat{s}dr
\end{aligned}$$

$$
\begin{aligned}
&+24N^2\sigma_j^2\lambda_j\|F\|_\infty^2 vol(\mathcal{O})\int_{-\infty}^{t-\delta} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(t-\delta-r)}\int_{-\infty}^{r+\delta} e^{(\frac{1}{2}\mu_{m+1}+\Lambda)(r+\delta-\hat{s})}d\hat{s}dr\\
\leq\;& \sigma_j^2\lambda_j 96N^2\|F\|_\infty^2 vol(\mathcal{O})\sum_{j=1}^{\infty}\Big(\frac{1}{|\mu_{m+1}+2\Lambda|^2}+\frac{1}{|\mu_{m+1}-2\Lambda|^2}\Big)\\
<\;& \infty.
\end{aligned}
$$

Thus $\sup\limits_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\hat{K}_1(t,\delta)<\infty$.

To consider $\hat{K}_2(t,\delta)$ in (4.44), note that when $r\leq t\leq r+\delta$, (4.38) and (4.40) gives us

$$
\begin{aligned}
\hat{K}_2(t,\delta) \;=\;& \frac{e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}_{r+\delta}^j\mathcal{M}^N(Y^N)(t,\cdot,x)-\mathcal{D}_r^j\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr\\
\leq\;& \frac{4e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t-\delta}^{t}\Big\{\chi_{\{j\geq m+1\}}(j)\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{r}\mathcal{D}_r^j\Phi_{t-\hat{s},\hat{s}}^N P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dx\\
&+\chi_{\{j\leq m\}}(j)\mathbb{E}\int_{\mathcal{O}}\Big|\int_{r+\delta}^{+\infty}\mathcal{D}_{r+\delta}^j\Phi_{t-\hat{s},\hat{s}}^N P^jF(\hat{s},Y^N(\hat{s},\cdot))d\hat{s}\Big|^2dx\\
&+\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}_{r+\delta}^j-\mathcal{D}_r^j)(Y^N)(\hat{s},\cdot,x)d\hat{s}\Big|^2dx\\
&+\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi_{t-\hat{s},\hat{s}}^N P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}_{r+\delta}^j-\mathcal{D}_r^j)(Y^N)(\hat{s},\cdot,x)d\hat{s}\Big|^2dx\Big\}dr\\
=\;& \sup_{t\in[0,\tau)}\sum_{i=4}^{7}Q_i(t,\delta),
\end{aligned}
$$

where

$$
\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}Q_4(t,\delta) \;:=\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{4e^{-2\Lambda|t|}}{|\delta|}\sum_{j=m+1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{r}\mathcal{D}_r^j\Phi_{t-\hat{s},\hat{s}}^N P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr\\
\leq\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{4}{|\delta|}e^{-2\Lambda|t|}\sum_{j=m+1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big(\int_{-\infty}^{r}\|\mathcal{D}_r^j\Phi_{t-\hat{s},\hat{s}}^N P^j\||F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2dxdr\\
\leq\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{16}{|\delta|}e^{-2\Lambda|t|}\sum_{j=m+1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big(\int_{-\infty}^{t}e^{\frac{1}{2}\mu_j(t-\hat{s})}e^{\Lambda|\hat{s}|}|F^j(\hat{s},Y^N(\hat{s},\cdot,x))|d\hat{s}\Big)^2dxdr\\
\leq\;& 128\sigma_j^2\lambda_j\|F\|_\infty^2 vol(\mathcal{O})\Big(\frac{1}{|\mu_{m+1}-2\Lambda|^2}+\frac{1}{|\mu_{m+1}+2\Lambda|^2}\Big)\\
<\;& \infty.
\end{aligned}
$$

and similarly,

$$\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_5(t,\delta) &:= \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{4e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{m}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{r+\delta}^{+\infty}\mathcal{D}_r^j\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr\\
&\le 128\sigma_j^2\lambda_j\|F\|_\infty^2 vol(\mathcal{O})\Big(\frac{1}{|\mu_m-2\Lambda|^2}+\frac{1}{|\mu_m+2\Lambda|^2}\Big)\\
&< \infty.
\end{aligned}$$

Besides, we have by the similar calculations in Q_1 and Q_2,

$$\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_6(t,\delta) &:= \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\\
&\qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2dxdr\\
&\le e^{2\Lambda\tau}\frac{32N^2\|\nabla F\|_\infty^2}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\\
&\quad\cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr\\
&< \infty,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} Q_7(t,\delta) &:= \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t-\delta}^{t}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{t}^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\\
&\qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2dxdr\\
&\le e^{2\Lambda\tau}\frac{32N^2\|\nabla F\|_\infty^2}{|\mu_{m}|}\Big(\frac{1}{|\mu_{m}+4\Lambda|}+\frac{1}{|\mu_{m}-4\Lambda|}\Big)\\
&\quad\cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr\\
&< \infty.
\end{aligned}$$

Thus we have that $\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\hat{K}_2(t,\delta)<\infty$.

Finally to consider $\hat{K}_3(t,\delta)$, note that when $t\le r$, (4.38) and (4.40) gives us

$$\begin{aligned}
&\hat{K}_3(t,\delta)\\
=\ &\frac{e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t}^{+\infty}\mathbb{E}\int_{\mathcal{O}}|\mathcal{D}^j_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot,x)-\mathcal{D}^j_r\mathcal{M}^N(Y^N)(t,\cdot,x)|^2dxdr\\
\le\ &\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_{t}^{+\infty}\mathbb{E}\int_{\mathcal{O}}\Big\{\Big|\int_{-\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)d\hat{s}\Big|^2
\end{aligned}$$

$$
\begin{aligned}
&+\chi_{\{j\leq m\}}(j)\Big|\int_{r+\delta}^{+\infty}\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}-\int_r^{+\infty}\mathcal{D}^j_r\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2\\
&+\Big|\int_t^{+\infty}\sum_{k=1}^m\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)d\hat{s}\Big|^2\Big\}dxdr\\
=&\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\sum_{i=8}^{10}Q_i(t,\delta),
\end{aligned}
$$

and now it is easy to write down the following estimations,

$$
\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}Q_8(t,\delta) \;:=\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_t^{+\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{\infty}^{t}\sum_{k=m+1}^{\infty}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\\
&\qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2dxdr\\
\leq\;& e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_{m+1}|}\Big(\frac{1}{|\mu_{m+1}+4\Lambda|}+\frac{1}{|\mu_{m+1}-4\Lambda|}\Big)\\
&\cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr\\
<\;&\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}Q_{10}(t,\delta) \;:=\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{\infty}\int_t^{+\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_t^{+\infty}\sum_{k=1}^{m}\Phi^N_{t-\hat{s},\hat{s}}P^k\nabla F(\hat{s},Y^N(\hat{s},\cdot,x))\\
&\qquad\cdot(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(t,\cdot,x)d\hat{s}\Big|^2dxdr\\
\leq\;& e^{2\Lambda\tau}\frac{24N^2\|\nabla F\|^2_\infty}{|\mu_{m}|}\Big(\frac{1}{|\mu_{m}+4\Lambda|}+\frac{1}{|\mu_{m}-4\Lambda|}\Big)\\
&\cdot\sup_{\substack{\hat{s}\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{e^{-2\Lambda|\hat{s}|}}{|\delta|}\sum_{j=1}^{\infty}\int_{\mathbb{R}}\mathbb{E}\int_{\mathcal{O}}|(\mathcal{D}^j_{r+\delta}-\mathcal{D}^j_r)(Y^N)(\hat{s},\cdot,x)|^2dxdr\\
<\;&\infty.
\end{aligned}
$$

Similarly to $Q_2(t,\delta)$,

$$
\begin{aligned}
\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}Q_9(t,\delta) \;:=\;& \sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}}\frac{3e^{-2\Lambda|t|}}{|\delta|}\sum_{j=1}^{m}\int_t^{+\infty}\mathbb{E}\int_{\mathcal{O}}\Big|\int_{r+\delta}^{+\infty}\mathcal{D}^j_{r+\delta}\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\\
&\qquad-\int_r^{+\infty}\mathcal{D}^j_r\Phi^N_{t-\hat{s},\hat{s}}P^jF(\hat{s},Y^N(\hat{s},\cdot,x))d\hat{s}\Big|^2dxdr\\
\leq\;& 96\sigma_j^2\lambda_jN^2\|F\|^2_\infty vol(\mathcal{O})\Big(\frac{1}{|\mu_m+2\Lambda|^2}+\frac{1}{|\mu_m-2\Lambda|^2}\Big)\\
<\;&\infty.
\end{aligned}
$$

In summary, we have shown that

$$\sup_{\substack{t\in[0,\tau)\\ \delta\in\mathbb{R}}} \frac{e^{-2\Lambda|t|}}{|\delta|} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \mathbb{E} \int_{\mathcal{O}} |\mathcal{D}^j_{r+\delta}\mathcal{M}^N(Y^N)(t,\cdot,x) - \mathcal{D}^j_r\mathcal{M}^N(Y^N)(t,\cdot,x)|^2 dxdr < \infty.$$

By now we could come to the conclusion that $\mathcal{M}^N$ maps from $C^{\Lambda,N}_{\tau,\rho}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2}))$ to itself.

Step 2: It suffices to prove that for each $N\in\mathbb{N}$, $\mathcal{M}^N(C^{\Lambda,N}_{\tau,\rho}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2}))|_{[0,\tau)}$ is relatively compact in $C([0,\tau), L^2(\Omega\times\mathcal{O}))$.

In fact applying with the Theorem 2.4.9, result from ***Step 1*** tells us that for any sequence $\mathcal{M}^N(f_n)\in C^{\Lambda,N}_{\tau,\rho}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2}))|_{[0,\tau)}$, there exists a subsequence, still denoted by $\mathcal{M}^N(f_n)$ and $V^N\in C([0,\tau), L^2(\Omega\times\mathcal{O}))$ such that

$$\sup_{t\in[0,\tau)} \mathbb{E}\|\mathcal{M}^N(f_n)(t,\cdot,x) - V^N(t,\cdot,x)\|^2_H \to 0 \tag{4.45}$$

as $n\to\infty$. □

Proof of Theorem 4.2.2. ***Step 1***: $\mathcal{M}^N(S)$ is relatively compact in $C^\Lambda_\tau(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$, where

$$S := C^{\Lambda,N}_{\tau,\rho}(\mathbb{R}, L^2(\Omega,\mathcal{D}^{1,2})) \cap L^\infty_\Lambda(\mathbb{R}, L^2(\Omega, H^1(\mathcal{O}))),$$

and from Lemma 4.2.5, for any sequence $\mathcal{M}^N(Y^N_n)\in C^{\Lambda,N}_{\tau,\rho}(\mathbb{R}, L^2(\mathcal{O},\mathcal{D}^{1,2}))$, there exists a subsequence, still denoted by $\mathcal{M}^N(Y^N_n)$, and $V^N\in C([0,\tau), L^2(\Omega\times\mathcal{O}))$ such that

$$\sup_{t\in[0,\tau)} \mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y^N_n)(t,\cdot,x) - V^N(t,\cdot,x)|^2 dx \to 0 \tag{4.46}$$

as $n\to\infty$.

Set for any $t\in[m\tau, m\tau+\tau)$,

$$V^N(t,\omega,x) = V^N(t-m\tau, \theta_{m\tau}\omega, x).$$

Note that by definition

$$\mathcal{M}^N(Y^N_n)(t,\theta_{m\tau}\omega,x) = \mathcal{M}^N(Y^N_n)(t+m\tau,\omega,x).$$

With (4.46) and the probability preserving of θ, we get as $n \to \infty$,

$$\begin{aligned}
& \sup_{t\in[m\tau,m\tau+\tau)} e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_n^N)(t,\cdot,x) - V^N(t,\cdot,x)|^2 dx \\
= & \sup_{t\in[0,\tau)} e^{-2\Lambda|t-m\tau|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_n^N)(t+m\tau,\cdot,x) - V^N(t+m\tau,\cdot,x)|^2 dx \\
\leq & \sup_{t\in[0,\tau)} \mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_n^N)(t,\theta_{m\tau}\cdot,x) - V^N(t,\theta_{m\tau}\cdot,x)|^2 dx \\
= & \sup_{t\in[0,\tau)} \mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_n^N)(t,\cdot,x) - V^N(t,\cdot,x)|^2 dx \\
\to & \ 0.
\end{aligned}$$

Thus

$$\sup_{t\in\mathbb{R}} e^{-2\Lambda|t|}\mathbb{E}\int_{\mathcal{O}} |\mathcal{M}^N(Y_n^N)(t,\cdot,x) - V^N(t,\cdot,x)|^2 dx \to 0,$$

as $n \to \infty$. Therefore $\mathcal{M}^N(S)$ is relatively compact in $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$.

According to the generalized Schauder's fixed point theorem, $\mathcal{M}^N$ has a fixed point in $C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$. That is to say there exists a solution $Y^N \in C_\tau^\Lambda(\mathbb{R}, L^2(\Omega\times\mathcal{O}))$ of equation (4.24) such that for any $t\in\mathbb{R}$, $Y^N(t+\tau,\omega,x) = Y^N(t,\theta_\tau\omega,x)$.

Step 2: Now define a subset of Ω as

$$\Omega_N := \{\omega,\ C_\Lambda(\omega) < N\},$$

As the random variable $C_\Lambda(\omega)$ is tempered from above, it is easy to get

$$\mathbb{P}(\Omega_N) \to 1,$$

as $N\to\infty$. Note also Ω_N is an increasing sequence of sets, thus $\cup_n\Omega_N = \hat{\Omega}$ and $\hat{\Omega}$ has full measure, and is invariant with respect to θ. Then define

$$\Omega_N^* = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\Omega_N,$$

and Ω_N^* is invariant with respect to $\theta_{n\tau}$ for each n. Besides we have $\Omega_N^* \subset \Omega_{N+1}^*$, which leads to

$$\bigcup_N \Omega_N^* = \bigcup_N \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\Omega_N = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\left(\bigcup_N \Omega_N\right) = \bigcap_{n=-\infty}^{\infty} \theta_{n\tau}^{-1}\hat{\Omega} = \bigcap_{n=-\infty}^{\infty} \tilde{\Omega} = \tilde{\Omega},$$

with $\mathbb{P}(\tilde{\Omega}) = 1$.

Now we can define $Y : \hat{\Omega} \times \mathbb{R} \to L^2_0(\mathcal{O})$, as an combinations of Y_N such that

$$Y := Y_1 \chi_{\Omega_1^*} + Y_2 \chi_{\Omega_2^* \setminus \Omega_1^*} + \cdots + Y_N \chi_{\Omega_N^* \setminus \Omega_{N-1}^*} + \cdots , \tag{4.47}$$

and it is easy to see that Y is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable and thus Y also satisfies the following property

$$\begin{aligned}
& Y(t+\tau, \omega) \\
= \; & Y_1(t+\tau, \omega)\chi_{\Omega_1^*}(\omega) + Y_2(t+\tau, \omega)\chi_{\Omega_2^* \setminus \Omega_1^*}(\omega) + \cdots + Y_N(t+\tau, \omega)\chi_{\Omega_N^* \setminus \Omega_{N-1}^*}(\omega) + \cdots \\
= \; & Y_1(t, \theta_\tau\omega)\chi_{\Omega_1^*}(\omega) + Y_2(t, \theta_\tau\omega)\chi_{\Omega_2^* \setminus \Omega_1^*}(\omega) + \cdots + Y_N(t, \theta_\tau\omega)\chi_{\Omega_N^* \setminus \Omega_{N-1}^*}(\omega) + \cdots \\
= \; & Y_1(t, \theta_\tau\omega)\chi_{\Omega_1^*}(\theta_\tau\omega) + Y_2(t, \theta_\tau\omega)\chi_{\Omega_2^* \setminus \Omega_1^*}(\theta_\tau\omega) + \cdots + Y_N(t, \theta_\tau\omega)\chi_{\Omega_N^* \setminus \Omega_{N-1}^*}(\theta_\tau\omega) + \cdots \\
= \; & Y(t, \theta_\tau\omega).
\end{aligned}$$

Moreover Y is a fixed point of $\mathcal{M}$.

We can easily extend Y to the whole Ω as $\mathbb{P}(\hat{\Omega}) = 1$, which is distinguishable with Y defined in (4.47). □

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