CALCULATION OF DOMAINS OF ATTRACTION OF ORDINARY DIFFERENTIAL EQUATIONS USING ZUBOV'S METHOD

by

Paul White B.A.(Oxon.), M.Sc.(Loughborough)

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Summary

In this thesis the method of Zubov for obtaining domains of attraction for systems of autonomous ordinary differential equations is investigated.

The necessary theorems of continuity, existence and uniqueness along with Zubov and Lyapunov theorems are listed in Chapter 1 as a starting point.

In Chapter 2 a survey of previous work in using Zubov's method or of calculating domains of attraction is attempted. The problems of each particular way of doing this are considered, as is some useful background work on numerical computation.

The theory of solution of Zubov's equation is the subject of Chapter 3. Necessary restrictions on the relationship of $\phi(\underline{x})$ to $\underline{f}(\underline{x})$ are derived, and some results on the use of V as the independent variable are also obtained.

The one-dimensional Zubov equation is an O.D.E. and hence a special case. This special case is the subject of Chapter 4 in which a rough asymptotic analysis is shown to provide estimates of the domain of attraction.

In Chapter 5 the Zubov equation is treated as a P.D.E. requiring solution for values of V. The many problems of obtaining the contour $V = \infty$ or $V = 1$ are investigated and some ideas are given on how to get around them.

Chapter 6 contains the algorithm for solving the Zubov equation on characteristics with initialisation from a point near the boundary. Much flexibility is incorporated into this algorithm to suit particular systems.

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Chapter 1

Introduction

1. Discussion

The concept of stability has been present in mathematics for a number of years. Throughout history the question of whether certain situations are stable has arisen. For example, the "ball-in-a-bowl" situation is known to be stable as the ball always accelerates towards the position of least potential energy, while if the bowl is turned upside down then we have an unstable situation. So it has always been a matter of concern whether an equilibrium is returned to after suffering a small displacement.

These practical experiments have led to the mathematical definitions of stability and to the methods and theorems involved in determination firstly of the stability of an equilibrium and secondly of the magnitude and type of displacement which can be permitted. Towards the end of last century the Russian mathematician A.M.Lyapunov developed the functions which bear his name and the associated theorems to determine stability. Since the last war V.I.Zubov, another Russian mathematician, took the analysis further and tied stability in with the solution of a partial differential equation. This thesis is mainly concerned with solving this equation to obtain stability regions.

2. Notation

$\underline{x}$ is a vector of the form $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$x(t)$ or $\underline{x}(t)$ are functions of a scalar variable t.

$\underline{f}(\underline{x},t)$ is a function of $\underline{x}$ and t.

$\underline{f}(\underline{x})$ is a function of $\underline{x}$ alone.

$\underline{x}(t,\underline{x}_o,t_o)$ is a function which depends on the initial conditions

$\dot{\underline{x}}(t)$ is the time derivative of $\underline{x}(t)$.

$\underline{x}^T$ is the transpose of $\underline{x}$. i.e. $\underline{x}^T = (x_1,\ldots,x_n)$.

$\|\underline{x}\|$	is the Euclidean norm given by $\|\underline{x}\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$
$\underline{0}$	is the origin in n-dimensional space. $\underline{0}^T = (0,0,\ldots,0)$
A	is a matrix of the form $\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{bmatrix}$
A^T	is the transpose of A / $\begin{bmatrix} A_{1,1} & A_{2,1} & \cdots & A_{n,1} \\ A_{1,2} & A_{2,2} & \cdots & A_{n,2} \\ \vdots & \vdots & & \vdots \\ A_{1,n} & A_{2,n} & \cdots & A_{n,n} \end{bmatrix}$
$A(t)$	is a matrix whose elements $A_{i,j}(t)$ are functions of t.
$\lambda_i(A)$	are the eigenvalues of A. $i=1,2,\ldots,n$.
$\|A\|$	is the determinant of A.
I	is the matrix $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$
A^{-1}	is the matrix satisfying $AA^{-1} = A^{-1}A = I$ if $\|A\| \neq 0$.
R^n	is all of a Euclidean n-dimensional space.
D	is an open domain in R^n. i.e. $D \subseteq R^n$.
δD	is the boundary of the closure of D.
$D(\underline{f})$	is a domain depending on the function $\underline{f}(\underline{x})$. (Later defined as the Domain of Attraction)
$D_\tau(\underline{f})$	is a domain depending on $\underline{f}(\underline{x},t)$ where $t \geq \tau$.
$V_m(\underline{x})$	is a function of $\underline{x}$ which is homogeneous of degree m.
$\frac{\partial V(\underline{x},t)}{\partial \underline{x}}$	is the vector $\begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}$
$\nabla V(\underline{x},t)$	is another name for $\frac{\partial V(\underline{x},t)}{\partial \underline{x}}$.

3. Definition of Motion Terms

Definition 1.3.1

The system equations are denoted by

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad 1.3.1$$

where $\underline{x}$ is the vector of system variables and $\underline{f}$ is a vector of functions of $\underline{x}$ and t which need not be continuous or differentiable.

System 1.3.1 is the general equation describing all forms of motion, regardless of how many dimensions there are in vector $\underline{x}$. For example a system of numerous interacting particles of gas in an enclosed space can be written in the form 1.3.1.

Theorem 1.3.1

The ordinary differential equation given by $f(x^{(n)}, x^{(n-1)}, \ldots\ldots, x^{(1)}, x, t) = 0$ where $x^{(n)} = \frac{d^n x}{dt^n}$ 1.3.2 may be expressed in the form 1.3.1.

Proof

Define the system variables as

$$\begin{aligned} x_1 &= x \\ x_2 &= \frac{dx}{dt} \\ &\vdots \\ x_n &= \frac{d^{n-1}x}{dt^{n-1}} \end{aligned}$$

Now solving 1.3.2 in terms of $x^{(n)}$ gives

$$x^{(n)} = g(x^{(n-1)}, \ldots\ldots, x^{(1)}, x, t).$$

The system equations now become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= g(x_n, x_{n-1}, \ldots, x_2, x_1, t) \end{aligned}$$

which is in the form 1.3.1. End of proof.

The system 1.3.1 may or may not have a unique solution or even a solution at all. There are several references

from which theorems on existence and uniqueness may be obtained such as Brauer and Nohel (1), Sanchez (2), Coddington and Levinson (3) and Lefschetz (4).
Two theorems will be quoted here without proof after some definitions.

Definition 1.3.2

$\underline{f}(\underline{x},t)$ satisfies a Lipschitz condition with respect to $\underline{x}$, for $\underline{x} \in D$, some $\Upsilon \geqslant 0$, if there exists a constant L such that

$$\| \underline{f}(\underline{x}_1,t) - \underline{f}(\underline{x}_2,t) \| \leqslant L \| \underline{x}_1 - \underline{x}_2 \|$$

for all $\underline{x}_1, \underline{x}_2 \in D, t \geq \Upsilon$.

Definition 1.3.3

$\underline{f}(\underline{x},t)$ is said to be continuous in $\underline{x}$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\underline{x}_1$, $\underline{x}_2 \in D$ such that $\| \underline{x}_1 - \underline{x}_2 \| < \delta$ then $\| \underline{f}(\underline{x}_1,t) - \underline{f}(\underline{x}_2,t) \| < \varepsilon$.

Definition 1.3.4

An integral curve of 1.3.1 is given by

$$\underline{x} = \underline{x}(t,\underline{x}_o,t_o)$$

where $\underline{x} = \underline{x}_o$ at $t = t_o$ and $\underline{x}(t,\underline{x}_o,t_o)$ satisfies 1.3.1 identically.

Theorem 1.3.2

If $\underline{f}(\underline{x},t)$ is continuous in $\underline{x}$,t and satisfies a Lipschitz condition in D, then passing through any point $\underline{x}_o \in D$, $t_o \geqslant \Upsilon$ there exists an integral curve $\underline{x} = \underline{x}(t,\underline{x}_o,t_o)$ for $t > t_o$ which may be extended to the boundary of D.

Theorem 1.3.3

If $\underline{f}(\underline{x},t)$ is continuous and satisfies a Lipshitz candition for $\underline{x}$ in some domain D and if $\underline{x}_o \in D$, $t_o \geqslant \Upsilon$ and $\underline{x}_1(t,\underline{x}_o,t_o)$, $\underline{x}_2(t,\underline{x}_o,t_o)$ are two exact solutions of 1.3.1 such that $\underline{x}_1(t_o,\underline{x}_o,t_o) = \underline{x}_2(t_o,\underline{x}_o,t_o) = \underline{x}_o$ then $\underline{x}_1 \equiv \underline{x}_2$ as long as $\underline{x}_1$, $\underline{x}_2 \in D$.

1.3.1 is the general equation but frequently system motion is determined only by its present state and not by time. This is known as the autonomous case.

Definition 1.3.5

If the system equations may be written as

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 1.3.3$$

then the system is autonomous.

Theorem 1.3.4

If $\underline{x} = \underline{x}(t, \underline{x}_0, t_0)$ is the unique solution of 1.3.3 then $\underline{x}(t-t_1, \underline{x}_0, t_0-t_1) \equiv \underline{x}(t, \underline{x}_0, t_0)$ providing $\underline{x}(t_0, \underline{x}_0, t_0) = \underline{x}(t_0-t_1, \underline{x}_0, t_0-t_1)$. i.e. $\underline{x}$ is only dependent on $t - t_0$.

As we are concerned with critical points we need to establish what critical points are.

Definition 1.3.6

A critical point $\underline{x}'$ of system 1.3.1 is a solution of the equation

$$\underline{f}(\underline{x}', t) \equiv \underline{0} \quad \text{for all } t \geqslant \Upsilon \quad \text{for some } \Upsilon \geqslant 0 \qquad 1.3.4$$

and similarly a critical point $\underline{x}'$ of system 1.3.3 is a solution of the equation

$$\underline{f}(\underline{x}') = \underline{0}.$$

If $\underline{x}'$ is a critical point then we may define a new set of variables by

$$\underline{X} = \underline{x} - \underline{x}'. \qquad 1.3.5$$

Substituting 1.3.5 into 1.3.1 defines the new system of equations

$$\dot{\underline{X}} = \underline{f}(\underline{X} + \underline{x}', t) \qquad 1.3.6$$

Now by reference to 1.3.5 we can see that $\underline{X} = \underline{0}$ is a critical point of 1.3.6. Thus any critical point can be translated to the origin by a simple transformation. It will be assumed throughout this thesis that the origin is a critical point and that it is the stability of the origin which is in question.

$$\text{i.e.} \quad \underline{f}(\underline{0}, t) \equiv \underline{0} \qquad 1.3.7$$

for all $t \geqslant \Upsilon$ for some $\Upsilon \geqslant 0$.

Definition 1.3.7

The linear part of $\underline{f}(\underline{x}, t)$ is denoted by the vector $A(t)\underline{x}$ where $A(t)$ is a matrix and 1.3.1 becomes

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{g}(\underline{x}, t) \qquad 1.3.8$$

where $\underline{g}(\underline{x}, t)$ can only be expanded in terms of powers of $\underline{x}$ greater than one if at all. In the autonomous case $A(t)$ becomes the constant matrix A.

Note also that as the origin is a critical point that in 1.3.8

$$\underline{g}(\underline{0}, t) \equiv \underline{0}$$

for all $t \geqslant \Upsilon$, some $\Upsilon \geqslant 0$.

4. Stability

Having considered the definition of the system equations and subsequent integral curves of motion, we now wish to define the stability of the origin.

Definition 1.4.1

The origin of 1.3.1 is stable if 1.3.7 holds and if there exists $\tau \geqslant 0$ such that for all $\varepsilon > 0$, $t_o \geqslant \tau$ there exists a $\delta(t_o, \varepsilon)$ such that

$$\|\underline{x}_o\| < \delta \Longrightarrow \|\underline{x}(t, \underline{x}_o, t_o)\| < \varepsilon \qquad 1.4.1$$

for all $t \geqslant t_o$.

Definition 1.4.2

The origin of 1.3.1 is quasi-asymptotically stable if there exists $\tau \geqslant 0$ and $\delta^* > 0$ such that for all $t_o \geqslant \tau$

$$\|\underline{x}_o\| < \delta^* \Longrightarrow \lim_{t \to \infty} \|\underline{x}(t, \underline{x}_o, t_o)\| = 0. \qquad 1.4.2$$

Definition 1.4.3

The origin of 1.3.1 is asymptotically stable if it is stable and quasi-asymptotically stable.

Definition 1.4.4

The origin of 1.3.1 is unstable if for all $t_o \geqslant \tau$ there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is an initial point $\underline{x}_o$ with $\|\underline{x}_o\| < \delta$ and the solution $\underline{x}(t, \underline{x}_o, t_o)$ is such that

$$\|\underline{x}(t, \underline{x}_o, t_o)\| \geqslant \varepsilon \qquad \text{for some } t_o \leq t < \infty.$$

Definition 1.4.5

The origin of 1.3.1 is asymptotically stable in the whole or strictly asymptotically stable if 1.4.1 and 1.4.2 hold for unbounded $\delta(t_o, \varepsilon)$ and δ^*.

The above five definitions form the basis of determination whether the origin is stable. Now if the origin is not strictly asymptotically stable we need to define the region of R^n for which it is asymptotically stable.

Definition 1.4.6

Providing the origin of 1.3.1 is asymptotically stable then for $\tau \geqslant 0$, $D_\tau(\underline{f})$ is the domain of attraction of the origin of 1.3.1 where for all $t_o \geqslant \tau$, $\underline{x}_o \in D_\tau(\underline{f})$ the integral curves $\underline{x}(t, \underline{x}_o, t_o)$ satisfying 1.3.1 identically tend to the origin as $t \to \infty$.

For the autonomous case 1.3.3 the D.O.A. given by $D(\underline{f})$ is a fixed subset of R^n as the integral curves are not time-dependent.

The stability of the origin may also be investigated using 1.3.8. As we are concerned with the stability of the origin under small displacements it is apparent from definition 1.3.7 that the matrix $A(t)$ is important. However the autonomous linear part matrix A is much easier to consider.

Theorem 1.4.1

If $\dot{\underline{x}} = A\underline{x} + \underline{g}(\underline{x})$ 1.4.3

where $\underline{g}(\underline{x})$ may be expanded in powers of $\underline{x}$ greater than one, then the origin is asymptotically stable if all the eigenvalues $\lambda_i(A)$ of A have negative real parts.

Definition 1.4.7

If the eigenvalues $\lambda_i(A)$ of A have negative real parts then A is a stability matrix.

Theorem 1.4.2

If A is a matrix as in 1.4.3 the origin is unstable if there exists some $\lambda_i(A)$ for which the real part is positive.

Definition 1.4.8

If all the eigenvalues $\lambda_i(A)$ of A have negative real parts or there exists a $\lambda_i(A)$ with positive real part the matrix A is said to have significant stability.

If the stability of the matrix A is not significant then further information about the stability of the origin of 1.4.3 can only be obtained by considering the higher terms $\underline{g}(\underline{x})$.

Definition 1.4.9

S_r, δS_r are the sets given by

$$S_r = \{\underline{x} : \|\underline{x}\| \leq r\}$$

and $$\delta S_r = \{\underline{x} : \|\underline{x}\| = r\}$$

The sets S_r, δS_r are used extensively in later chapters.

Definition 1.4.10

$S_r(\underline{x}')$, $\delta S_r(\underline{x}')$ are the sets given by

$$S_r(\underline{x}') = \{\underline{x} : \|\underline{x} - \underline{x}'\| \leq r\}$$
$$\delta S_r(\underline{x}') = \{\underline{x} : \|\underline{x} - \underline{x}'\| = r\}.$$

5. Positive Definite Functions

Before proceeding to the tneorems of Lyapunov upon which Zubov's equation is based, it is necessary to introduce and define the concept of positive definite functions.

Definition 1.5.1

The scalar function $V(\underline{x})$ is positive definite in some region D of R^n containing tne origin if

$$V(\underline{0}) = 0$$
$$V(\underline{x}) > 0 \text{ for all } \underline{x} \in D,\ \underline{x} \neq \underline{0}.$$

Negative definite functions are defined similarly.

Definition 1.5.2

The function $V(\underline{x})$ is positive semi-definite in some region D if

$$V(\underline{0}) = 0$$
$$V(\underline{x}) \geqslant 0 \text{ for all } \underline{x} \in D.$$

Definition 1.5.3

$V(\underline{x})$ is strictly positive definite if $D \equiv R^n$. i.e. $V(\underline{x}) > 0$ for all $\underline{x} \in R^n$, $\underline{x} \neq \underline{0}$, and $V(\underline{0}) = 0$.

Definition 1.5.4

$V(\underline{x})$ is radially unbounded if as

$$\|\underline{x}\| \longrightarrow \infty \qquad \text{then } V(\underline{x}) \longrightarrow \infty.$$

Positive definite functions play an important part in Lyapunov theory. However for time-dependent systems we need to use the function $V(\underline{x},t)$ where there are analogous definitions to those above.

Definition 1.5.5

The scalar function $V(\underline{x},t)$ is positive definite in a region D of R^n containing the origin if there exists $\Upsilon \geqslant 0$ and a positive definite function $W(\underline{x})$ such that

$$V(\underline{x},t) \geqslant W(\underline{x}) \text{ for } \underline{x} \in D,\ t \geqslant \Upsilon.$$
$$V(\underline{0},t) \equiv 0 \quad \text{for } t \geqslant \Upsilon.$$

Definition 1.5.6

$V(\underline{x},t)$ is positive semi-definite in D if there exists $\Upsilon \geqslant 0$ such that

$$V(\underline{x},t) \geqslant 0 \text{ for all } \underline{x} \in D,\ t \geqslant \Upsilon.$$
$$V(\underline{0},t) \equiv 0 \text{ for all } t \geqslant \Upsilon.$$

The definitions of strictly positive definite and radially unbounded follow in the same way.

Definition 1.5.7

$V(\underline{x},t)$ is decrescent if there exists $\Upsilon \geqslant 0$ and a positive definite function $W(\underline{x})$ such that

$$-W(\underline{x}) \leqslant V(\underline{x},t) \leqslant W(\underline{x})$$

for all $\underline{x} \in D, t \geqslant \Upsilon$.

Frequently the positive definite functions considered in this thesis will be quadratic functions. i.e. functions in which all terms are of x_i^2 , $i = 1,..,n$, or $x_i x_j$, $i,j = 1,..n$

Such functions may be expressed in matrix form as

$$V(\underline{x}) = \underline{x}^T B \, \underline{x} \tag{1.5.1}$$

Definition 1.5.8

The matrix B in 1.5.1 is a positive definite matrix if

$$\underline{x}^T B \, \underline{x} > 0 \quad \text{for all } \underline{x} \in R^n, \ \underline{x} \neq \underline{0}.$$

For purposes of knowing whether B is a positive definite matrix, the following theorem is useful.

Theorem 1.5.1

The matrix B is a positive definite matrix if and only if $\quad |B_i| > 0 \qquad i = 1,...,n$

where B_i is an i x i matrix taken from the upper left corner of B.

This is Sylvester's theorem and can be found in Rosenbrock and Storey(6) or Barnett and Storey (5).

The quadratic expression 1.5.1 is a special case of a general series expansion of $V(\underline{x})$. We may denote $V(\underline{x})$ by

$$V(\underline{x}) = \sum V_m(\underline{x}) \tag{1.5.2}$$

where $V_m(\underline{x})$ is a polynomial of homogeneous degree m. Such series expressions as 1.5.2 can be positive definite functions under certain conditions.

If the range of m is known the question of whether an expression like 1.5.2 is positive definite usually depends on $V_s(\underline{x})$ where s is the lowest value of m used. Theorems on such series expansions form part of Chapter 3.

Theorem 1.5.2

$V(\underline{x})$ is decrescent if it may be written as

$$V(\underline{x}) = \sum_{m=1}^{\infty} V_m(\underline{x}).$$

The proof of this theorem may be found in Hahn (7).

Theorem 1.5.3

The total derivative of $V(\underline{x},t)$ with respect to time is given by

$$\dot{V}(\underline{x},t) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \dot{x}_i \frac{\partial V}{\partial x_i} \tag{1.5.3}$$

providing the partial derivatives in 1.5.3 exist. It can be shown that if the partial derivatives in 1.5.3 exist and are continuous and $V(\underline{x},t)$ is positive definite and decrescent then $V(\underline{x},t) \approx V_s(\underline{x},t)$ for some even $s \geq 0$, $\underline{x}$ near the origin.

6. Lyapunov functions

The theorems of Lyapunov and his second method are quite well known. They are quoted here without proof as they are fundamental to Zubov's method. La Salle and Lefschetz (8), Barbashin (9), Hahn (10) contain all the background and proofs.

Theorem 1.6.1

The origin of 1.3.1 is stable if there exists a function $V(\underline{x},t)$ and some $\varepsilon > 0, \Upsilon \geq 0$ such that $V(\underline{x},t)$ is positive definite and $\dot{V}(\underline{x},t)$ is negative semi-definite for all $\underline{x} \in S_\varepsilon$, $t \geq \Upsilon$.

Theorem 1.6.2

The origin of 1.3.1 is asymptotically stable if there exists $V(\underline{x},t)$ and some $\varepsilon > 0$, $\Upsilon \geq 0$ such that $V(\underline{x},t)$ is positive definite and decrescent and $\dot{V}(\underline{x},t)$ is negative definite for all $\underline{x} \in S_\varepsilon, t \geq \Upsilon$.

Theorem 1.6.3

The origin of 1.3.1 is asymptotically stable if there exists $V(\underline{x},t)$ and some $\varepsilon > 0, \Upsilon \geq 0$ such that $V(\underline{x},t)$ is positive definite and decrescent and $\dot{V}(\underline{x},t)$ is negative semi-definite and $\dot{V}(\underline{x},t) \not\equiv 0$ on any non-trivial trajectory of 1.3.1, $\underline{x} \in S_\varepsilon, t \geq \Upsilon$.

Theorem 1.6.4

The origin of 1.3.1 is unstable if there exists $\Upsilon \geq 0$ and a decrescent function $V(\underline{x},t)$ with $\dot{V}(\underline{x},t)$ negative definite and such that $V < 0$ at some $\underline{x} \in S_\varepsilon$ for all $\varepsilon > 0, t \geq \Upsilon$.

These four theorems give local information about the stability of the origin of 1.3.1. We are concerned in this thesis to obtain regions within which stability is assured. For this purpose we require the next two theorems:

Theorem 1.6.5

The origin of 1.3.1 is asymptotically stable in the whole if there exists a $V(\underline{x},t)$ such that

a) $V(\underline{x},t)$ is strictly positive definite

b) $\dot{V}(\underline{x},t)$ is strictly negative definite

c) $V(\underline{x},t)$ is decrescent

d) $V(\underline{x},t)$ is radially unbounded.

Theorem 1.6.6

The origin of 1.3.1 is asymptotically stable in a region D of R^n if there exists a function $V(\underline{x},t)$ in D, $\Upsilon \geq 0$ such that

a) $V(\underline{x},t)$ is positive definite for $\underline{x} \in D, t \geq \Upsilon$.

b) $\dot{V}(\underline{x},t)$ is negative semi-definite for $\underline{x} \in D, t \geq \Upsilon$.

c) $\dot{V}(\underline{x},t) \not\equiv 0$ on any non-trivial trajectory in D.

d) $\nabla V(\underline{x},t) \neq 0$ in D except at $\underline{x} = \underline{0}$.

e) The boundary δD of D is given by $V(\underline{x},t) = p$ for some p.

Now that we have these theorems we may define certain terms for later use.

Definition 1.6.1

A function $V(\underline{x},t)$ which is positive definite in a neighbourhood of the origin and is such that $\dot{V}(\underline{x},t)$ is negative semi-definite is a Lyapunov function.

Definition 1.6.2

Denote as $p^*(\underline{f},V)$ the largest value of p for which theorem 1.6.6 holds.

Definition 1.6.3

Denote as $R^*(\underline{f},V)$ the domain in theorem 1.6.6 bounded by $V(\underline{x},t) = p^*(\underline{f},V)$. i.e. the largest domain obtainable for this $V(\underline{x},t)$ and $\underline{f}(\underline{x},t)$.

Definition 1.6.4

The region $R^*(\underline{f},V)$ is known as the region of asymptotic stability given by this $V(\underline{x},t)$ for system 1.3.1. This region is abbreviated from now on to R.A.S..

The system given by 1.4.3 where the linear part of $\underline{f}(\underline{x})$ is isolated as $A\underline{x}$ may also be considered. Then the stability of the origin of 1.4.3 may be investigated by choice of a Lyapunov function such as

$$V(\underline{x}) = \underline{x}^T B \, \underline{x}. \qquad 1.6.1$$

The stability of the origin of

$$\dot{\underline{x}} = A\underline{x} \qquad 1.6.2$$

may be determined by a Lyapunov function such as 1.6.1 and the stability of the origin of 1.4.3 is also determined providing A has significant stability. This gives us

Theorem 1.6.7

The origin of 1.6.2 is stable if there exists a unique positive definite symmetric matrix B which is the solution of

$$A^T B + BA = -C$$

for any symmetric positive definite matrix C.

The function 1.6.1 is the Lyapunov function.

Finally we need a theorem which is important especially where considering methods of constructing Lyapunov functions in such a way that $R^*(\underline{f},V)$ converges to $D(\underline{f})$.

Theorem 1.6.8

If the origin of 1.3.1 is asymptotically stable then $R^*(\underline{f},V)$ is given by

$$V(\underline{x},t) \leqslant p^*(\underline{f},V)$$

where $p^*(\underline{f},V)$ is the largest value of p such that

$V(\underline{x},t) \geqslant p$ for all $\underline{x}$ such that $\dot{V}(\underline{x},t) = 0$.

Since we know that $R^*(\underline{f},V) \subseteq D(\underline{f})$ then we see that if $D(\underline{f})$ is bounded then so is $R^*(\underline{f},V)$, while if $R^*(\underline{f},V)$ is unbounded for any $V(\underline{x},t)$ then so is $D(\underline{f})$.

We may also define $p'(\underline{f},V)$ as the smallest value of p such that $V(\underline{x},t) \leqslant p$ for $\underline{x}$ such that $\dot{V}(\underline{x},t) = 0$ if such a $p'(\underline{f},V)$ exists.

Theorem 1.6.9

If $p'(\underline{f},V)$ exists then

$$D(\underline{f}) \subseteq R'(\underline{f},V)$$

where $R'(\underline{f},V)$ is given by

$$\underline{x} : V(\underline{x},t) \leqslant p'(\underline{f},V)$$

where $V(\underline{x},t) \leqslant p'(\underline{f},V)$ for $\underline{x}$ such that $\dot{V}(\underline{x},t) = 0$.

7. Theorems of Zubov

Having introduced the basis of Lyapunov theory we now proceed to the theorems and method of Zubov for actually obtaining Lyapunov functions to find domains of attraction. Zubov's approach is to ensure that $\dot{V}(\underline{x})$ is negative definite and then solve the resulting equation for $V(\underline{x})$. In this section no reference to time-dependent systems is considered as the the theory has been developed on such systems where the behaviour is dependent only on the position $\underline{x}$ in R^n.

The equation which is solved may be written simply as

$$\dot{V}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 1.7.1$$

where $\phi(\underline{x})$ is any positive definite function and d = 0 or 1.

From theorem 1.5.3 the total derivative $\dot{V}(\underline{x})$ is replaced by a sum of partial derivatives giving

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V(\underline{x})}{\partial x_i} = -\phi(\underline{x})(1 - dV(\underline{x})). \qquad 1.7.2$$

The equation given by d = 1 is known as Zubov's regular equation and when d = 0 we have Zubov's modified equation.

The main theorem of Zubov can now be quoted (11),(12).

<u>Theorem 1.7.1</u>

A necessary and sufficient condition for the origin of 1.3.3 to be asymptotically stable and $\overset{*}{R}(\underline{f},V)$ to be the D.O.A. is the existence of two functions $V(\underline{x}),\phi(\underline{x})$ which satisfy 1.7.3. identically and the following properties:

a) $V(\underline{x})$ is positive definite in $\overset{*}{R}(\underline{f},V)$.

b) $\phi(\underline{x})$ is positive definite and continuous in R^n.

c) For $d = 1$, $V(\underline{x}) < 1$, $\underline{x} \in \overset{*}{R}(\underline{f},V)$ and $V(\underline{x}) = 1$, $\underline{x} \in \delta\overset{*}{R}(\underline{f},V)$
For $d = 0$, $V(\underline{x}) < \infty$, $\underline{x} \in \overset{*}{R}(\underline{f},V)$ and $V(\underline{x}) = \infty$, $\underline{x} \in \delta\overset{*}{R}(\underline{f},V)$

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V(\underline{x})}{\partial x_i} = -\phi_1(\underline{x})(1 + \|\underline{f}\|^2)^{\frac{1}{2}}(1 - dV(\underline{x})) \qquad 1.7.3$$

$$\text{or} \quad \dot{V}(\underline{x}(t)) = -\phi_1(\underline{x}(t))(1 + \|\underline{f}\|^2)^{\frac{1}{2}}(1 - dV(\underline{x}(t))).$$

It is relatively straightforward to prove this theorem given $V(\underline{x}),\phi_1(\underline{x})$ by checking that theorem 1.6.6 is satisfied. It requires some extra details to confirm that

$$\overset{*}{R}(\underline{f},V) = D(\underline{f}).$$

Proving that $V(\underline{x}),\phi_1(\underline{x})$ exist given an asymptotically stable origin of 1.3.3 is more difficult and Zubov bases it on actual construction of $V(\underline{x}),\phi_1(\underline{x})$. The proof of this important theorem can be found in Zubov's book (12).

More details will be considered about the construction of $V(\underline{x})$ and $\phi_1(\underline{x})$ in sections 2.2 and 3.6 as $V(\underline{x})$ with the properties of theorem 1.7.1 does not necessarily exist for all $\phi_1(\underline{x})$.

<u>Theorem 1.7.2</u>

If $\underline{f}(\underline{x})$ is bounded then theorem 1.7.1 holds with 1.7.3 replaced by

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 1.7.4$$

<u>Proof</u>

Since $\phi_1(\underline{x})(1 + \|\underline{f}\|^2)^{\frac{1}{2}} > 0$
$\underline{x} \neq \underline{0}$

while $\phi_1(\underline{0})(1 + \|\underline{f}\|^2)^{\frac{1}{2}} = 0$

then we may re-write $\phi_1(\underline{x})(1 + \|\underline{f}\|^2)^{\frac{1}{2}}$ as $\phi(\underline{x})$ and $\phi(\underline{x})$ has the same properties as the function $\phi_1(\underline{x})$ used in theorem 1.7.1. While if $\underline{f}(\underline{x})$ is unbounded as $\underline{x} \to \underline{x}'$ for some $\underline{x}' \in R^n$ then $\phi_1(\underline{x})(1 + \|\underline{f}\|^2)^{\frac{1}{2}}$ is discontinuous as $\underline{x} \to \underline{x}'$.

End of proof.

From now on 1.7.4 is the equation which will be referred to as Zubov's equation and several results will be established using the regular equation ($d = 1$) all of which have an analogous result for $d = 0$.

Definition 1.7.1

$G(\lambda)$ is the set

$$G(\lambda) = \{ \underline{x} : V(\underline{x}) < \lambda \}.$$

Theorem 1.7.3

For $\underline{x} \in D(\underline{f})$ we have

for $d = 0$ $\qquad 0 \leqslant V(\underline{x}) < \infty.$

$d = 1$ $\qquad 0 \leqslant V(\underline{x}) < 1.$

Proof

Putting $d = 1$ in 1.7.1 and integrating with respect to time we have

$$V(\underline{x}(t)) = 1 - (1 - V(\underline{x}_0))\exp(\int_0^t \phi(\underline{x}(t'))dt'). \qquad 1.7.5$$

Rearranging 1.7.5 gives

$$V(\underline{x}_0) = 1 - (1 - V(\underline{x}(t))\exp(-\int_0^t \phi(\underline{x}(t'))dt'). \qquad 1.7.6$$

Now let $t \to \infty$ in 1.7.6 giving

$$V(\underline{x}_0) = 1 - \exp(-\int_0^\infty \phi(\underline{x}(t))dt).$$

Now if $\underline{x}_0 \neq \underline{0}$ then $\int_0^\infty \phi(\underline{x}(t))dt > 0.$

Hence $V(\underline{x}_0) < 1$ while if $\underline{x}_0 = \underline{0}$ then $\underline{x}(t) \equiv \underline{0}$ and

$\int_0^\infty \phi(\underline{x}(t))dt = 0$ giving $V(\underline{x}_0) = 0.$

The proof for $d = 0$ is by the transformation

$$W(\underline{x}) = -\log(1 - V(\underline{x})) \qquad 1.7.7$$

which when substituted into 1.7.1 gives

$$\dot{W}(\underline{x}) = -\phi(\underline{x}). \qquad 1.7.8$$

End of proof.

For the remainder of this section the notation of 1.7.7, 1.7.8 is used. i.e. $d = 1$ gives $V(\underline{x})$ as the solution of

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - V(\underline{x})) \qquad 1.7.9$$

and $d = 0$ gives $W(\underline{x})$ as the solution of

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial W}{\partial x_i}(\underline{x}) = -\phi(\underline{x}). \qquad 1.7.10$$

Theorem 1.7.4

For $\lambda \in (0,1)$ for $d = 1$ or $\lambda \in (0,\infty)$ for $d = 0$ then $G(\lambda)$ are bounded domains.

Theorem 1.7.5

For each $\phi(\underline{x})$ then if $V(\underline{x})$ exists then the equations $V(\underline{x}) = p$ form a non-intersecting set of curves in R^n.

Theorem 1.7.6

The trajectories $\underline{x}(t,\underline{x}_o,t_o)$ of

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

which tend to the origin cross the contours $V(\underline{x}) = p$ once and once only.

Theorem 1.7.7

If $\lambda_1 < \lambda_2$ then $G(\lambda_1) \subset G(\lambda_2)$.

Theorem 1.7.8

$D(\underline{f}) = G(1)$ for $d = 1$

and $D(\underline{f}) = G(\infty)$ for $d = 0$.

Theorem 1.7.9

The limiting values of $V(\underline{x})$ as $\underline{x} \to \underline{\xi}$ is given by

$$\lim_{\underline{x} \to \underline{\xi}} V(\underline{x}) = V(\underline{\xi}) \text{ for all } \underline{\xi} \in D(\underline{f}).$$

Theorem 1.7.10

The curve $V(\underline{x}) = 1$ or $W(\underline{x}) = \infty$ is an integral curve of 1.3.3.

Theorem 1.7.11

The limiting values of $V(\underline{x})$ as $\underline{x} \to \underline{\xi}$ is given by

$$\lim_{\underline{x} \to \underline{\xi}} V(\underline{x}) = 1 \quad \text{for } \underline{x} \in D(\underline{f}), \quad \underline{\xi} \in \delta D(\underline{f}).$$

Theorem 1.7.12

For each $\phi(\underline{x})$ then if $V(\underline{x}), W(\underline{x})$ exist satisfying theorem 1.7.1 they are unique.

Theorem 1.7.13

The origin of 1.3.3 is asymptotically stable in the whole if $W(\underline{x}) < \infty$ or $V(\underline{x}) < 1$ for all $\underline{x} \in R^n$.

Theorem 1.7.14

$$W(\underline{x}(t)) = W(\underline{x}_o) - \int_o^t \phi(\underline{x}(t'))dt'$$

and $V(\underline{x}(t)) = 1 - (1 - V(\underline{x}_o)) \exp\left(\int_o^t \phi(\underline{x}(t'))dt'\right)$

for $\underline{x}_o \in D(\underline{f})$.

Theorem 1.7.15

$$W(\underline{x}_o) = \int_o^\infty \phi(\underline{x}(t))dt$$

and $V(\underline{x}_o) = 1 - \exp\left(-\int_o^\infty \phi(\underline{x}(t))dt\right)$

Theorems 1.7.3 to 1.7.15 are all a consequence of theorems 1.7.1 and 1.7.2 and all depend on $V(\underline{x})$ actually existing which is not necessarily true for all $\phi(\underline{x})$ as will be seen in later chapters. However if $f(\underline{x})$ has a linear part and $\phi(\underline{x})$ has a quadratic part we can obtain the quadratic approximation to $V(\underline{x})$.

Theorem 1.7.16

If $A\underline{x}$ is the linear part of $\underline{f}(\underline{x})$ and $\underline{x}^T C\, \underline{x}$ is the quadratic part of $\phi(\underline{x})$ then the quadratic approximation to $V(\underline{x})$ is given by $\underline{x}^T B\, \underline{x}$ where

$$A^T B + BA = -C$$

has a unique solution for B.

This theorem introduces the subject of approximations to the actual $V(\underline{x})$. Zubov proposed a construction procedure based on substitution of 1.5.2 into 1.7.9 or 1.7.10. It is assumed that $\phi(\underline{x})$, $\underline{f}(\underline{x})$ may be expanded as

$$\underline{f}(\underline{x}) = \sum_{m=1}^{\infty} \underline{f}_m(\underline{x}) \qquad 1.7.11$$

$$\phi(\underline{x}) = \sum_{m=2}^{\infty} \phi_m(\underline{x}) \qquad 1.7.12$$

Then we may write the unknown functions $V(\underline{x})$, $W(\underline{x})$ as

$$V(\underline{x}) = \sum_{m=2}^{\infty} V_m(\underline{x})$$
$$W(\underline{x}) = \sum_{m=2}^{\infty} W_m(\underline{x}) \qquad 1.7.13$$

In 1.7.11, 1.7.12, 1.7.13 $\underline{f}_m(\underline{x})$, $\phi_m(\underline{x})$, $V_m(\underline{x})$, $W_m(\underline{x})$ contain only terms whose total homogeneous powers are m. Substituting 1.7.11, 1.7.12, 1.7.13 into 1.7.9, 1.7.10 gives

$$\sum_{i=1}^{n} \left(\left[\sum_{m=1}^{\infty} f_{i,m}(\underline{x}) \right] \left[\sum_{m=2}^{\infty} \frac{\partial V_m(\underline{x})}{\partial x_i} \right] \right) = - \left[\sum_{m=2}^{\infty} \phi_m(\underline{x}) \right] \quad \left(1 - d\sum_{m=2}^{\infty} V_m(\underline{x})\right) \qquad 1.7.14$$

The actual details of 1.7.14 are considered in Chapter 2 but if solved systematically for $V_m(\underline{x})$, m = 2,3......, we obtain a succession of approximations to $V(\underline{x})$ given by

$$V^{(N)}(\underline{x}) = \sum_{m=2}^{N} V_m(\underline{x}). \qquad 1.7.15$$

There are some results associated with the series construction procedure which are a consequence of theorems 1.6.6, 1.6.8, 1.6.9 and definitions 1.6.2, 1.6.3, 1.6.4.

Theorem 1.7.17

The curve $V^{(N)}(\underline{x}) = p^*(\underline{f}, V^{(N)})$ is wholly in $D(\underline{f})$. N =2,3....

Theorem 1.7.18

If $D(\underline{f})$ is bounded then so are all $V^{(N)}(\underline{x}) = p^*(\underline{f}, V^{(N)})$.

Theorem 1.7.19

If any $V^{(N)}(\underline{x}) = p^*(\underline{f}, V^{(N)})$ is unbounded then so is $D(\underline{f})$.

Theorems 1.7.17, 1.7.18, 1.7.19 now suggest the following definitions:

Definition 1.7.2

$R_N(\phi, \underline{f})$ is the R.A.S. indicated by $V^{(N)}(\underline{x})$ for a particular $\phi(\underline{x})$.

i.e. $R_N(\phi, \underline{f}) = (\underline{x} : V^{(N)}(\underline{x}) \leqslant p^*(\underline{f}, V^{(N)}))$

where $\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x}))$

and $V^{(N)}(\underline{x})$ is given by 1.7.14 and 1,7.15.

Definition 1.7.3

$R_c(\phi, \underline{f})$ is the set given by

$\underline{x} \in R_c(\phi, \underline{f}) \Longrightarrow V^{(N)}(\underline{x})$ is a convergent series as $N \to \infty$.

Definition 1.7.4

$R(\phi, \underline{f})$ is the set given by

$$R(\phi, \underline{f}) = \lim_{N \to \infty} R_N(\phi, \underline{f}).$$

8. Methods of Solution of P.D.E.s

This thesis is mainly concerned with the solution of 1.7.2 to try to find the contour $V = \infty$ or $V = 1$. There are standard methods for solving partial differential equations of this type and the background of such methods is given here.

The usual analytic method of solving 1.7.2 is by characteristics and the auxiliary equations. It is shown in Sneddon (13):

Theorem 1.8.1

The auxiliary equations of 1.7.2 are given by

$$\frac{dx_1}{f_1(\underline{x})} = \frac{dx_2}{f_2(\underline{x})} = \cdots\cdots\cdots = \frac{dx_n}{f_n(\underline{x})} = \frac{-dV}{\phi(\underline{x})(1 - dV)} \qquad 1.8.1$$

and

Theorem 1.8.2

The solutions of 1.8.1 are given by

$$u_i(x_1, \ldots, x_n, V) = c_i$$

for $i = 1, .., n$

where c_i are arbitrary.

Sneddon (13) gives three methods of solving 1.8.1. The first method is to try to spot functions $P_i(\underline{x},V)$, $i = 1,\ldots,n$, $R(\underline{x},V)$ such that

$$\sum_{i=1}^{n} f_i(\underline{x})P_i(\underline{x},V) - \phi(\underline{x})(1 - dV)R(\underline{x},V) = 0$$

and
$$P_i(\underline{x},V) = \frac{\partial U(\underline{x},V)}{\partial x_i} \qquad i = 1,\ldots,n$$

$$R(\underline{x},V) = \frac{\partial U(\underline{x},V)}{\partial V}$$

then $u(\underline{x},V) = c$
is a solution of 1.8.1.

The second method involves finding functions $P_i(\underline{x},V)$, $P_i'(\underline{x},V)$, $i = 1,\ldots,n$, $R(\underline{x},V)$, $R'(\underline{x},V)$ such that

$$\frac{\sum_{i=1}^{n} P_i(\underline{x},V)dx_i + R(\underline{x},V)dV}{\sum_{i=1}^{n} P_i(\underline{x},V)f_i(\underline{x}) - R(\underline{x},V)\phi(\underline{x})(1 - dV)} = dW(\underline{x},V)$$

and

$$\frac{\sum_{i=1}^{n} P_i'(\underline{x},V)dx_i + R'(\underline{x},V)dV}{\sum_{i=1}^{n} P_i'(\underline{x},V)f_i(\underline{x}) - R'(\underline{x},V)\phi(\underline{x})(1 - dV)} = dW'(\underline{x},V)$$

where dW; dW' are exact derivatives of $\underline{x}$, V.
For then $W(\underline{x},V) = W'(\underline{x},V) + c$
is the required relationship which yields a solution.

The third method is to eliminate certain variables if this is possible. This is particularly applicable to the 2-dimensional version of 1.8.1 which is given by

$$\frac{dx}{f(x,y)} = \frac{dy}{g(x,y)} = \frac{dV}{-\phi(x,y)(1 - dV)} \qquad 1.8.2$$

The first equation in 1.8.2 gives a relationship between x and y which is an O.D.E. and we denote its solution by

$$y = \psi(c_1,x) \qquad 1.8.3$$

where c_1 is an arbirary constant. Substituting for y from

1.8.3 into the first and third terms of 1.8.2 gives a similar relationship between x and V.

The method of characteristics and the auxiliary equation is applicable if we require to obtain the characteristics. However, we are interested in the boundary of the D.O.A. and we cannot necessarily obtain that from the characteristics. Equation 1.8.1 and 1.8.2 are useful equations to use for a numerical method as shown in Chapter 6.

Numerical solutions of a P.D.E. such as 1.7.2 can be done by various methods. The simplest first order P.D.E. which is considered is given by

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \qquad 1.8.4$$

Various methods exist for solving such equations. First, we define u(mh,nk) as the value of u at x = mh, t = nk and u_m^n as the computed value at this point. Then difference schemes usually attempt to compute u_m^{n+1} given u_m^n, u_{m-1}^n, u_{m+1}^n, u_{m-1}^{n+1} or any neighbouring values required to make up the scheme.

Such methods include:

a) $u_m^{n+1} = (1-\rho a)u_m^n + \rho a u_{m-1}^n$ 1.8.5

where $\rho = k/h$.

b) $u_m^{n+1} = (1-\rho^2 a^2)u_m^n - \frac{\rho a}{2}(1-\rho a)u_{m+1}^n + \frac{\rho a}{2}(1+\rho a)u_{m-1}^n$

1.8.6

known as the Lax-Wendroff formula (14).

c) $u_m^{n+1} = (1+\rho a)u_m^n - \rho a u_{m-1}^{n+1}$. 1.8.7

Equation 1.7.2 has variable coefficients and methods have to be adapted to this situation. We use the notation

$$\Delta_x u_m^n = u_{m+1}^n - u_m^n$$
$$\Delta_t u_m^n = u_m^{n+1} - u_m^n$$
$$\nabla_x u_m^n = u_m^n - u_{m-1}^n$$
$$\nabla_t u_m^n = u_m^n - u_m^{n-1}$$

Then if we solve

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0 \qquad 1.8.8$$

then two difference schemes which maintain their accuracy when applied to 1.8.8 as well as 1.8.4 are

a) $$u_m^{n+1} = (1 - \frac{\rho a_m^{n+\frac{1}{2}}}{2} (\Delta_x + \nabla_x) + \frac{\rho^2}{4}(a_m^{n+\frac{1}{2}} \Delta_x a_m^{n+\frac{1}{2}} \nabla_x + a_m^{n+\frac{1}{2}} \nabla_x a_m^{n+\frac{1}{2}} \Delta_x))u_m^n \qquad 1.8.9$$

the Lax-Wendroff formula,

b) $$(1 + \frac{\rho a_m^{n+\frac{1}{2}}}{4}(\Delta_x + \nabla_x))u_m^{n+1} = (1 - \frac{\rho a_m^{n+\frac{1}{2}}}{4}(\Delta_x + \nabla_x))u_m^n \qquad 1.8.10$$

the Crank-Nicolson formula,
where $a_m^n = a(mh, nk)$.

The origin of finite difference schemes such as 1.8.5, 1.8.6, 1.8.7, 1.8.9, 1.8.10 usually lies in truncating an infinite theoretical series. Taylor series is one such method where we know that

$$u(mh,(n+1)k) = \exp(k\frac{\partial}{\partial t}) \; u(mh,nk) \qquad 1.8.11$$

Substituting from 1.8.8 into 1.8.11 and expressing the exponential in terms of its power series, we may truncate the series and obtain a difference formula.

The situation becomes more complicated when we consider an equation such as

$$\frac{\partial u}{\partial t} = a(x,y,t)\frac{\partial u}{\partial x} + b(x,y,t)\frac{\partial u}{\partial y}. \qquad 1.8.12$$

An implicit method for finding u_{m_1,m_2}^{n+1} is the 18 point A.D.I. method by Mitchell and Gourlay (15) which when a,b are constants becomes

$$(1 - \frac{\rho_2 b}{4}(\Delta_y + \nabla_y))(1 - \frac{\rho_1 a}{4}(\Delta_x + \nabla_x))u_{m_1,m_2}^{n+1} \qquad 1.8.13$$
$$= (1 + \frac{\rho_2 b}{4}(\Delta_y + \nabla_y))(1 + \frac{\rho_1 a}{4}(\Delta_x + \nabla_x))u_{m_1,m_2}^{n}$$

where $\rho_1 = k/h_1$, $\rho_2 = k/h_2$, where $u^n_{m_1,m_2}$ is the computed value of u at $x = m_1h_1$, $y = m_2h_2$, $t = nk$.

When a,b are dependent on x,y,t 1.8.13 becomes

$$(1 - \frac{\rho_2}{4} b^{n+\frac{1}{2}}_{m_1,m_2}(\Delta_y + \nabla_y))(1 - \rho_1 a^{n+\frac{1}{2}}_{m_1,m_2}(\Delta_x + \nabla_x))u^{n+1}_{m_1,m_2}$$
$$= (1 + \rho_2 b^{n+\frac{1}{2}}_{m_1,m_2}(\Delta_y + \nabla_y))(1 + \frac{\rho_1}{4} a^{n+\frac{1}{2}}_{m_1,m_2}(\Delta_x + \nabla_x))u^{n}_{m_1,m_2}.$$

In choosing a finite difference method various consideratio have to be taken into account, such as initial conditions, accuracy, stability. The more points covered by a difference formula, the more need to be specified to start computation going. 1.8.5 requires for example u^o_m to be given so that u^n_m may be computed for $n > 0$. Now if u^o_m , $m = M_1,..,M_2$, are given then by systematic application of 1.8.5 we may compute u^n_m for $n > 0$ and $m \leq M_2$, $m \geq M_1 + n$ and computation stops when $u^{M_2-M_1}_{M_2}$ is obtained.

Accuracy is determined by the local truncation error. This is obtained by substituting the actual values of u(mh,nk) into the difference formula. Denoting the Local Truncation Error by L(mh,nk) we write down for the scheme 1.8.6

$$L(mh,nk) = u(mh,(n+1)k) - (1 - \rho^2a^2)u(mh,nk) + \frac{\rho a}{2}(1 - \rho a)u((m+1)h,nk)$$
$$- \frac{\rho a}{2}(1 + \rho a)u((m - 1)h,nk). \qquad 1.8.14$$

By means of expressing u(mh,(n + 1)k), u((m - 1)h,(n + 1)k) in terms of u(mh,nk) and derivatives of u at x = mh, t = nk by Taylor series, and cancelling out terms using 1.8.4 we obtain that

$$L(mh,nk) = O(k^3) + O(h^3)$$

Thus we see that 1.8.6 is a second order method.

Stability is considered by the method of Von Neumann (16) where the computed values of u are subtracted from the actual values. A formula such as 1.8.6 is taken from the corresponding formula 1.8.14 and using the notation

$$e^n_m = u(mh,nk) - u^n_m$$

we obtain

$$L(mh,nk) = e^{n+1}_m - (1 - \rho^2a^2)e^n_m + \frac{\rho a}{2}(1 - \rho a)e^n_{m+1} - \frac{\rho a}{2}(1 + \rho a)e^n_{m-1}$$

. The error is then assumed to have a Fourier series distribution which we write as

$$e^n_m = \lambda^n e^{i\omega mh}$$

and then if $|\lambda| \leq 1$ for all ω we see that errors tend to die away.

Finite element methods are also applicable but are not considered in this thesis.

It is a simple matter of extending solution of 1.8.8 or 1.8.12 to solving 1.7.2 and this is the subject of Chapter 5.

9. Motivation and Contents

The method of Zubov for obtaining Lyapunov functions has been tested on various systems by various authors. It is a reversal of the Lyapunov theory in that instead of selecting $V(\underline{x})$ and then considering the behaviour of $\dot{V}(\underline{x})$ to determine stability, the Zubov method selects $\dot{V}(\underline{x})$ and then considers the behaviour of $V(\underline{x})$. Also instead of finding $R^*(\underline{f},V)$ as a region of asymptotic stability, the Zubov approach determines $D(\underline{f})$ if $V(\underline{x})$ can be found.

The basic question then is one of determining $V(\underline{x})$ given $\dot{V}(\underline{x})$ and the system equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}). \qquad 1.9.1$$

Determination of $V(\underline{x})$ is from a partial differential equation 1.7.2 which in general is not easy to solve analytically. The method Zubov suggested of using power series to build up partial sums has been implemented by such authors as Margolis and Vogt, Yu and Vongsuriya, De Sarker and Rao. In each case they have found non-uniform convergence to the actual D.O.A..

So can we really say that the Zubov approach is useful for general systems? This thesis attempts to solve 1.7.2 by various methods and tries to find some way or ways by which we can establish the usefulness of Zubov's method when applied to systems of equations.

In Chapter 2 the background material is all collected together. A comparison of previous work in various fields together with some comments is presented there. The main method is the series construction which is dealt with in great detail in two dimensions. It is noticeable that examples covered in the literature deal only with systems with linear parts

$$\dot{\underline{x}} = A\underline{x} + \underline{g}(\underline{x}) \qquad 1.9.2$$

and functions $\phi(\underline{x})$ with quadratic parts

$$\phi(\underline{x}) = \underline{x}^T C\, \underline{x} + \underline{h}(\underline{x}). \qquad 1.9.3$$

It is shown in Chapter 2 that this case is straightforward but that situations other than 1.9.2 and 1.9.3 may or may not be soluble by series construction.

The Lie series method is also considered where the equation 1.9.1 and

$$\dot{V}(\underline{x}) = -\phi(\underline{x})(1 - dV)$$

are integrated simultaneously with respect to time by Taylor series. Possible transformations and vector methods are also looked into and all such methods of solving 1.7.2 have their disadvantages. For background purposes some methods are given on obtaining D.O.A.s and obtaining Lyapunov functions.

Two methods of solving 1.7.2 suggested themselves during this research. One method is to treat 1.7.2 as any other partial differential equation and apply numerical techniques for solving it, while the other method was to solve 1.8.1 numerically. In the course of the research certain types of behaviour of the solutions of 1.7.2 and 1.8.1 were experienced. Such behaviour can be explained by theory and Chapter 3 sets out to establish this theory which is concerned with the relative behaviour of $\underline{f}(\underline{x})$, $\phi(\underline{x})$ and $V(\underline{x})$ to each other.

Such relationships have on the whole been overlooked when solving the Zubov equation on the basis that if 1.9.2 and 1.9.3 hold then 1.7.2 can be solved to give $V(\underline{x})$ and hence the D.O.A. is established. For systems not possessing linear parts more care is needed. Zubov warns that "if for any reason whatever we know the rate of decrease of solutions $\underline{x}(t)$ of 1.9.1 then $\phi(\underline{x})$ can always be chosen".

In Chapter 3 the required relationships between $f(\underline{x})$, $\phi(\underline{x})$, $V(\underline{x})$ are established by reference to their behaviour as $\underline{x} \rightarrow \underline{0}$. The "asymptotic degree" of a function at the origin is defined and we show that if the behaviour of $\underline{f}(\underline{x})$ is known then $\phi(\underline{x})$ can be chosen by reference to this definitiom.

The one-dimensional Zubov equation is different from other cases as it becomes an O.D.E. and may be integrated directly with respect to x. The system trajectories either tend to or away from the origin and finding the D.O.A. is reduced to seeking a value of x which is on the boundary. It is worth a special chapter as methods applicable to O.D.E.s can be used. Chapter 4 covers this case and an asymptotic analysis is carried out at the boundary of the D.O.A. which serves to answer the question of what value of V to attain in order to guarantee a predetermined accuracy for computation of the boundary point.

The two methods mentioned earlier of solving 1.7.2 and 1.8.1 are the subject of Chapters 5 and 6. The method of finite differences in Chapter 5 although relatively easy to carry out does run into some problems which stem directly from the fact that the P.D.E. 1.7.2 has variable coefficients. A combination of various factors all of which are considered analytically make it very difficult to obtain good estimates of $D(\underline{f})$ for general systems. Possible variations of the method near certain problem areas are studied in an attempt to compute D.O.A.s.

In Chapter 6 the equation 1.8.1 forms the basis of numerical computation. It is shown that solving 1.8.1 from points $\underline{x}_0$ near the boundary of the D.O.A. is inherently more stable than beginning at

$$V(\underline{0}) = 0$$

and computing outwards from the origin. This method takes account of the particular properties of $V(\underline{x})$ especially its positive definiteness for asymptotically stable systems. Solving for $V < 0$ has some interesting results from which we can establish an estimate of $D(\underline{f})$.

Chapter 2

Other Methods

1. Introduction

We require at this stage a chapter presenting other work done in this field and related subjects. A number of authors have considered the Zubov equation and attempted its solution, usually by the series construction proposed by Zubov.

Five specific methods of tackling the problem are given a section each, and any other methods are collected in sections 7 and 8. The series method in section 2 is the most well-known method associated with the Zubov equation and most widely used. Most examples on which this method has been tried are found to have linear parts. i.e.

$$\dot{\underline{x}} = A\underline{x} + \underline{g}(\underline{x})$$

where $\dfrac{\|\underline{g}(\underline{x})\|}{\|\underline{x}\|} \longrightarrow 0$ as $\underline{x} \rightarrow \underline{0}$.

The series method can still be solved for systems without linear parts but needs some care.

The Lie series method in section 3 has also been studied, but this reduces to Taylor series and relies on complete differentiability of

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

with respect to time.

Rodden's method in section 6 was something of a milestone in stability theory but being a numerical method it does require care to be taken over accuracy and stability of computation. In section 7 the other ideas stemming from Lyapunov functions are considered, while in section 8 note is made of obtaining D.O.A.s without ever refering to $\phi(\underline{x})$, $V(\underline{x})$.

2. Series Method

The series solution of the Zubov equation was first put forward by Zubov himself (12). Since then a number of authors such as Hewit (17), Rodden (18), Margolis and Vogt (19), De Sarker and Rao (20), Yu and Vongsuriya (21) have attempted to use the methods to obtain approximations to

D.O.A.s of various examples, some well-behaved and some not so well-behaved. They have claimed improvements on other methods of estimation although this is done by selecting the "best" domain from a non-uniform procedure. Ferguson (22) generalises the construction to higher order tensors.

For the system of equations

$$\underline{\dot{x}} = \underline{f}(\underline{x})$$

the method relies upon $\underline{f}(\underline{x})$ possessing a power series expansion which we denote as

$$f_i(\underline{x}) = \sum_{m=1}^{\infty} \sum_{m_1+m_2+\ldots+m_n=m} \cdots\cdots\sum F_i(m_1,\ldots,m_n)x_1^{m_1}\, x_2^{m_2}\ldots\ldots x_n^{m_n} \qquad 2.2.1$$

If we chose $\phi(\underline{x})$ in such a way that it also possesses a power series expansion

$$\phi(\underline{x}) = \sum_{m=2}^{\infty} \sum_{m_1+m_2+\ldots+m_n=m} \cdots\cdots\sum Q(m_1,\ldots,m_n)x_1^{m_1}\, x_2^{m_2}\ldots\ldots x_n^{m_n} \qquad 2.2.2$$

then we may construct $V(\underline{x})$ in the same form

$$V(\underline{x}) = \sum_{m=2}^{\infty} \sum_{m_1+m_2+\ldots m_n=m} \cdots\cdots\sum A(m_1,\ldots,m_n)x_1^{m_1}\, x_2^{m_2}\ldots\ldots x_n^{m_n} \qquad 2.2.3$$

We now substitute 2.2.1, 2.2.2, 2.2.3 into the Zubov equation

$$\sum_{i=1}^{n} \dot{x}_i \frac{\partial V}{\partial x_i} = -\phi(1 - dV) \qquad 2.2.4$$

to obtain the unknown coefficients $A(m_1,\ldots,m_n)$.

In (17) Hewit expands 2.2.4 into individual terms to demonstrate how involved the computation becomes. Other authors do not actually write out the full set of equations for $A(m_1,\ldots,m_n)$ and usually concentrate on 2 or at most 3 dimensions. An attempt will be made to generalise the 2-dimensional series construction and establish a theorem on the lowest degrees of f, g, ϕ.

Let the 2-dimensional system be

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned} \qquad 2.2.5$$

and let us denote f, g, ϕ, V as

$$f(x,y) = \sum_{m=s}^{\infty} \sum_{k=0}^{m} f_{m,k}x^k y^{m-k} \qquad 2.2.6$$

$$g(x,y) = \sum_{m=s}^{\infty} \sum_{k=0}^{m} g_{m,k} x^k y^{m-k} \qquad 2.2.7$$

$$\phi(x,y) = \sum_{m=q}^{\infty} \sum_{k=0}^{m} \phi_{m,k} x^k y^{m-k} \qquad 2.2.8$$

$$V(x,y) = \sum_{m=2}^{\infty} \sum_{k=0}^{m} V_{m,k} x^k y^{m-k}. \qquad 2.2.9$$

In 2.2.6, 2.2.7 s is such that

$$\sum_{k=0}^{s} (f_{s,k}^{\ 2} + g_{s,k}^{\ 2}) > 0. \qquad 2.2.10$$

and in 2.2.8 q is such that

$$\sum_{k=0}^{q} \phi_{q,k}^{\ 2} > 0. \qquad 2.2.11$$

Since V(x,y) is the unknown function it is considered to have powers of x,y of degree $\geqslant 2$ in order that it may be positive definite.

Substituting 2.2.6 to 2.2.9 and 2.2.5 into the 2-dimensional version of 2.2.4 gives

$$\begin{aligned}&\Big(\sum_{m=s}^{\infty}\sum_{k=0}^{m} f_{m,k}x^k y^{m-k}\Big)\Big(\sum_{m=2}^{\infty}\sum_{k=0}^{m} kV_{m,k}x^{k-1}y^{m-k}\Big)\\ &+ \Big(\sum_{m=s}^{\infty}\sum_{k=0}^{m} g_{m,k}x^k y^{m-k}\Big)\Big(\sum_{m=2}^{\infty}\sum_{k=0}^{m} (m-k)V_{m,k}x^{k}y^{m-k-1}\Big) \qquad 2.2.12\\ &+ \Big(\sum_{m=q}^{\infty}\sum_{k=0}^{m} \phi_{m,k}x^k y^{m-k}\Big)\Big(1 - d\sum_{m=2}^{\infty}\sum_{k=0}^{m} V_{m,k}x^{k}y^{m-k}\Big) \equiv 0\end{aligned}$$

providing that V(x,y) satisfies conditions (23) in which the terms $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ exist and are expressible in the above form.

We need to isolate homogeneous powers of x,y in 2.2.12. Hence we re-arrange 2.2.12 to obtain

$$\begin{aligned}&\sum_{m=s+1}^{\infty}\sum_{r=s}^{m-1}\Big(\sum_{i=0}^{r} f_{r,i}x^i y^{r-i}\Big)\Big(\sum_{j=0}^{m-r}(j+1)V_{m-r+1,j+1}x^j y^{m-r-j}\Big)\\ &+\sum_{m=s+1}^{\infty}\sum_{r=s}^{m-1}\Big(\sum_{i=0}^{r} g_{r,i}x^i y^{r-i}\Big)\Big(\sum_{j=0}^{m-r}(m-r-j+1)V_{m-r+1,j}x^j y^{m-r-j}\Big)\\ &+\sum_{m=q}^{\infty}\sum_{k=0}^{m}\phi_{m,k}x^k y^{m-k}\\ &-d\sum_{m=q+2}^{\infty}\sum_{r=q}^{m-2}\Big(\sum_{i=0}^{r}\phi_{r,i}x^i y^{r-i}\Big)\Big(\sum_{j=0}^{m-r}V_{m-r,j}x^j y^{m-r-j}\Big) \equiv 0. \qquad 2.2.13\end{aligned}$$

Now we may state and prove a theorem pertaining to solution of 2.2.13.

Theorem 2.2.1

In the construction procedure 2.2.13, $q \geqslant s + 1$.

Proof

The construction procedure to obtain $V(x,y)$ is carried out by comparing coefficients of like terms in 2.2.13. Now suppose $q \leqslant s$. We may then single out homogeneous terms of degree q from 2.2.13 which gives

$$\sum_{k=0}^{q} \phi_{q,k} x^k y^{q-k} \equiv 0. \qquad 2.2.14$$

Since 2.2.14 is an identity in x and y the only solution is that

$$\phi_{q,k} = 0, \qquad k = 0,\ldots,q,$$

and this contradicts the definition of q by the restriction 2.2.11 which thus contradicts the assumption $q \leqslant s$.

End of proof.

This theorem has implications on the choice of ϕ. It is stated by Margolis and Vogt (19) and elsewhere that for arbitrary choice of ϕ, V can be determined in series form by this method. Margolis and Vogt only consider systems in which $s = 1$ and then by reference to theorem 2.2.1 we require $q \geqslant 2$ and any positive definite ϕ may be used. But clearly we see that if $s \geqslant 2$ then solution by construction 2.2.13 breaks down for any ϕ with a quadratic part. However nearly all problems which arise are concerned with systems that possess linear parts. Let us consider the example

$$\begin{aligned} \dot{x} &= -x^3 - xy^2 + x(x^2 + y^2)^2 \\ \dot{y} &= -x^2 y - y^3 + y(x^2 + y^2)^2 \end{aligned} \qquad 2.2.15$$

If we try as a Lyapunov function

$$V(x,y) = x^2 + y^2$$

and differentiate with respect to system 2.2.15 we obtain

$$\dot{V} = \dot{x}\frac{\partial V}{\partial x} + \dot{y}\frac{\partial V}{\partial y} = -2(x^2 + y^2)^2(1 - x^2 - y^2).$$

Hence $V(x,y) = 1$ is tangential to $\dot{V}(x,y) = 0$ and the D.O.A. of 2.2.15 is given by $x^2 + y^2 < 1$. However if we solve 2.2.12 for this example using

$$\phi(x,y) = ax^2 + bxy + cy^2$$

we obtain

$$(-x^3-xy^2+x(x^2+y^2)^2)\left(\sum_{m=2}^{\infty}\sum_{k=0}^{m} kV_{m,k}x^{k-1}y^{m-k}\right)$$
$$+(-x^2y - y^3+ y(x^2+ y^2)^2)\left(\sum_{m=2}^{\infty}\sum_{k=0}^{m}(m-k)V_{m,k}x^k y^{m-k-1}\right)$$
$$+(ax^2+ bxy + cy^2)\left(1 - \sum_{m=2}^{\infty}\sum_{k=0}^{m} V_{m,k}x^k y^{m-k}\right) = 0. \qquad 2.2.16$$

From 2.2.16 we see that the terms of lowest homogeneous degree are

$$ax^2 + bxy + cy^2 = 0.$$

Hence we require $a = b = c = 0$ and $\phi(x,y)$ must be chosen without a quadratic part. It seems to be accepted that any positive definite $\phi(x,y)$ satisfies the Zubov theorems, but it has been demonstrated that for systems without linear parts that this is not the case. The theory in Chapter 3 investigates further the relationship between the lowest degree terms of f, g, ϕ, V.

Now suppose $q > s + 1$. Taking the terms with homogeneous degree $s + 1$ in 2.2.13 we obtain

$$\left(\sum_{i=0}^{s} f_{s,i}x^i y^{s-i}\right)(V_{2,1}y + 2V_{2,2}x)$$
$$+\left(\sum_{i=0}^{s} g_{s,i}x^i y^{s-i}\right)(2V_{2,0}y + V_{2,1}x) \equiv 0. \qquad 2.2.17$$

Now it is possible that if

$$\sum_{i=0}^{s} g_{s,i}x^i y^{s-i} \equiv 0$$

the solution of 2.2.17 becomes

$$V_{2,1} = V_{2,2} = 0$$

with $V_{2,0}$ undetermined. However it is likely that the $s + 2$ simultaneous equations generated by 2.2.17 in 3 unknowns will have the trivial solution

$$V_{2,0} = V_{2,1} = V_{2,2} = 0$$

as its unique solution.

Hence if $q > s + 1$ then $V(x,y)$ may or may not possess a quadratic part. Hence there is no loss of generality in writing $\phi(x,y)$ as

$$\phi(x,y) = \sum_{m=s+1}^{\infty} \sum_{k=0}^{m} \phi_{m,k} x^k y^{m-k}$$

where the lower terms of $\phi(x,y)$ may be zero. i.e. restriction 2.2.11 no longer holds, but s is still defined according to 2.2.6, 2.2.7 and restriction 2.2.10.

Then in general we may compute the series solution for $V(x,y)$ from 2.2.13 including the possibility that V might not possess a quadratic part.

Re-arranging 2.2.13 to further group like terms together we obtain

$$\sum_{m=s+1}^{\infty} \sum_{r=s}^{m-1} \sum_{i=0}^{r} \sum_{j=0}^{m-r} ((j+1)f_{r,i}V_{m-r+1,j+1}+(m-r-j+1)g_{r,i}V_{m-r+1,j})x^{i+j}y^{m-i-j}$$
$$+ \sum_{m=s+1}^{\infty} \sum_{k=0}^{m} \phi_{m,k} x^k y^{m-k}$$
$$-d\sum_{m=s+3}^{\infty} \sum_{r=s+1}^{m-2} \sum_{i=0}^{r} \sum_{j=0}^{m-r} \phi_{r,i} V_{m-r,j} x^{i+j} y^{m-i-j} \equiv 0 \qquad 2.2.18$$

2.2.18 is an identity in (x,y) and we may now equate terms of the same homogeneous degree and this gives

$$\sum_{r=s}^{m-1} \sum_{i=0}^{r} \sum_{j=0}^{m-r} ((j+1)f_{r,i}V_{m-r+1,j+1}+(m-r-j+1)g_{r,i}V_{m-r+1,j})x^{i+j}y^{m-i-j}$$
$$+ \sum_{k=0}^{m} \phi_{m,k} x^k y^{m-k}$$
$$-\left\{ d\sum_{r=s+1}^{m-2} \sum_{i=0}^{r} \sum_{j=0}^{m-r} \phi_{r,i} V_{m-r,j} x^{i+1} y^{m-i-j}\right\} \equiv 0, \quad m \geqslant s+1, \qquad 2.2.19$$

where the term $\{\cdot\}$ is zero if $d = 0$ or if $m = s+1, s+2$.

The expression 2.2.19 is seen to be a sum of powers of homogeneous degree m with coefficients which are linear in the unknown elements

$$V_{u,j} \qquad \begin{array}{l} u = 2,\ldots,m-s+1 \\ j = 0,\ldots,u \end{array} \qquad m \geqslant s+1.$$

For each m we may compute $V_{m-s+1,j}$, $j = 0,..,m-s+1$, having previously computed

$$V_{u,j} \qquad \begin{array}{l} u = 2,\ldots,m-s \\ j = 0,\ldots,u. \end{array}$$

The terms in 2.2.19 are terms in $x^k y^{m-k}$ $k = 0,\ldots,m.$ hence we see that 2.2.19 represents m+1 linear equations in the m-s+2 unknowns $V_{m-s+1,j}$ $j = 0,\ldots,m-s+1.$
We can represent these equations in matrix form

$$C\underline{v} = \underline{b} \tag{2.2.20}$$

where C is a matrix of m+1 rows and m-s+2 columns, $\underline{v}$ is an m-s+2 vector, $\underline{b}$ is an m+1 vector. It remains to fill in the elements of 2.2.20.

$$\underline{v} = \begin{bmatrix} V_{m-s+1,0} \\ V_{m-s+1,1} \\ \vdots \\ V_{m-s+1,m-s} \\ V_{m-s+1,m-s+1} \end{bmatrix}$$

To obtain the elements of C we need to isolate from 2.2.19 the coefficients of $x^k y^{m-k}$ $k = 0,\dots,m$ which contain $V_{m-s+1,j}$ $j = 0,\dots,m-s+1$. Denote these coefficients by $C(x^k y^{m-k})$.

There are four cases:

a) if $0 \leqslant k \leqslant \min(m-s,s)$

$$C(x^k y^{m-k}) = \sum_{j=0}^{k}((j+1)f_{s,k-j}V_{m-s+1,j+1}+(m-s-j+1)g_{s,k-j}V_{m-s+1,j})$$

b) if $s \leqslant k \leqslant m-s$

$$C(x^k y^{m-k}) = \sum_{j=k-s}^{k}((j+1)f_{s,k-j}V_{m-s+1,j+1}+(m-s-j+1)g_{s,k-j}V_{m-s+1,j})$$

c) if $m-s \leqslant k \leqslant s$ 2.2.21

$$C(x^k y^{m-k}) = \sum_{j=0}^{m-s}((j+1)f_{s,k-j}V_{m-s+1,j+1}+(m-s-j+1)g_{s,k-j}V_{m-s+1,j})$$

d) if $\max(m-s,s) \leqslant k \leqslant m$

$$C(x^k y^{m-k}) = \sum_{j=k-s}^{m-s}((j+1)f_{s,k-j}V_{m-s+1,j+1}+(m-s-j+1)g_{s,k-j}V_{m-s+1,j})$$

From 2.2.21 we can define the element $C_{k+1,j+1}$ as the coefficient of $V_{m-s+1,j}$ in $C(x^k,y^{m-k})$. The matrix C is written as in fig. 1.

$$
\left[\begin{array}{ccccc}
(m-s+1)g_{s,0} & 0\ +f_{s,0} & \ddots & & \\
(m-s+1)g_{s,1} & (m-s)g_{s,0}+f_{s,1} & \ddots & 0\ +(m-s)f_{s,0} & \\
\vdots & \vdots & \ddots & g_{s,0}+(m-s)f_{s,1} & (m-s+1)f_{s,0} \\
(m-s+1)g_{s,s} & (m-s)g_{s,s-1}+f_{s,s} & \ddots & \vdots & \vdots \\
 & (m-s)g_{s,s}+\ 0 & \ddots & g_{s,s-1}+(m-s)f_{s,s} & (m-s+1)f_{s,s-1} \\
 & & \ddots & g_{s,s}+\ 0 & (m-s+1)f_{s,s}
\end{array}\right]
$$

Fig. 1.

Finally we may group all the known quantities of 2.2.19 into the vector $\underline{b}$. b_{k+1} is the known terms in the coefficient of $x^k y^{m-k}$. We do not need to express b_{k+1} in form 2.2.21 as this is not necessary for computation. It is sufficient to write down the known parts of 2.2.19 which are

$$-\left\{\sum_{r=s+1}^{m-1}\sum_{i=0}^{r}\sum_{j=0}^{m-r}((j+1)f_{r,i}V_{m-r+1,j+1}+(m-r-j+1)g_{r,i}V_{m-r+1,j})x^{i+j}y^{m-i-j}\right\}$$

$$-\sum_{k=0}^{m}\phi_{m,k}x^k y^{m-k} + \left[d\sum_{r=s+1}^{m-2}\sum_{i=0}^{r}\sum_{j=0}^{m-r}\phi_{r,i}V_{m-r,j}x^{i+j}y^{m-i-j}\right]$$

where $\{\cdot\} = 0$ if $m = s + 1$ and $[\cdot] = 0$ if $d = 0$ or $m = s+1, s+2$.

2.2.20 has a unique solution if $s = 1$. If $s > 1$ there are more equations than unknowns, but this does not necessarily lead to contradictions.

<u>Theorem 2.2.2</u>

If C is an $m \times n$ matrix of rank r then

$$C\underline{v} = \underline{b} \qquad 2.2.22$$

has a consistent solution $\underline{v}$ if and only if $(C,\underline{b})$ the matrix formed by putting C and $\underline{b}$ side by side also has rank r.

The proof of this theorem is found in Heading (24).

Theorem 2.2.2 requires the rank of C and $(C,\underline{b})$ to be the same. C is an $m+1$ by $m-s+2$ matrix, and $(C,\underline{b})$ is an $m+1$

by m-s+3 matrix.

Now if the system equations are given by

$$\dot{\underline{x}} = A\underline{x} + \underline{g}(\underline{x})$$

where $\dfrac{\|\underline{g}(\underline{x})\|}{\|\underline{x}\|} \to 0$ as $\underline{x} \to \underline{0}$

and if A is a stability matrix then we may refer to Appendix A to see that when m = 2, s = 1 the 3 x 3 matrix C has a non-zero determinant, hence 2.2.22 has a unique solution.

Consider the example

$$\dot{x} = -x^3$$
$$\dot{y} = -y^3$$

which is obviously asymptotically stable in the whole.
If we use

$$\phi(x,y) = x^4 + y^4 \qquad 2.2.23$$

then the equation 2.2.20 becomes for m = 4, s = 3

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} V_{2,0} \\ V_{2,1} \\ V_{2,2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad 2.2.24$$

The solution of 2.2.24 becomes

$$\begin{bmatrix} V_{2,0} \\ V_{2,1} \\ V_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

However if we replace 2.2.23 by

$$\phi(x,y) = x^4 + x^2y^2 + y^4$$

then there is no solution for $V_{2,0}, V_{2,1}, V_{2,2}$.

Now consider the slightly different example

$$\dot{x} = -x^3 - xy^2$$
$$\dot{y} = -x^2y - y^3$$

which is also asymptotically stable in the whole.
Now let

$$\phi(x,y) = \sum_{k=0}^{4} \phi_{4,k}x^k y^{4-k}$$

then 2.2.20 becomes

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} V_{2,0} \\ V_{2,1} \\ V_{2,2} \end{bmatrix} = \begin{bmatrix} -\phi_{4,0} \\ -\phi_{4,1} \\ -\phi_{4,2} \\ -\phi_{4,3} \\ -\phi_{4,4} \end{bmatrix} \qquad 2.2.25$$

For 2.2.25 to have a solution we see that

$$\begin{aligned} \phi_{4,0} + \phi_{4,4} &= \phi_{4,2} \\ \phi_{4,1} &= \phi_{4,3} \end{aligned} \qquad 2.2.26$$

is necessary. Then

$$\begin{bmatrix} V_{2,0} \\ V_{2,1} \\ V_{2,2} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}\phi_{4,0} \\ \tfrac{1}{2}\phi_{4,1} \text{ or } \tfrac{1}{2}\phi_{4,3} \\ \tfrac{1}{2}\phi_{4,4} \end{bmatrix} \qquad 2.2.27$$

From 2.2.26, 2.2.27 we see that if $\phi(x,y)$ is positive definite then $V(x,y)$ is also positive definite.

These examples show that if $s > 1$ then the series construction may still be valid but not for every choice of positive definite ϕ.

It is perhaps a matter of further research whether a particular asymptotically stable system will yield to the construction for all ϕ or just some ϕ or none. i.e. given $\underline{f}(\underline{x})$ what class of $\phi(\underline{x})$ will yield a series construction for $V(\underline{x})$? Also in view of these examples, it is pertinent to ask whether all the series construction can be carried out or whether it may break down at some order. It was only shown that the quadratic part exists in these examples.

It is already known that the series construction where it can be carried out yields Lyapunov functions which indicate R.A.S.s which are inferior and do not necessarily converge to the D.O.A.. Also the method does require $f(x,y)$, $g(x,y)$ to have a reasonably well-behaved series expansion.

3. Lie Series Method

3.1 Introduction

The Lie series method was developed by Burnand and Sarlos (25) and basically involves computation of a series which reduces to no more than the Taylor series of the R.H.S. of

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 2.3.1$$

$$\dot{V} = -\phi(\underline{x}) \qquad 2.3.2$$

The authors develop a theory for the Lie series in their paper and then Kormanik and Li (26) also use the Lie series to generate boundary points and fit a continuous curve to them.

3.2 Method

The method used is based on defining the operator D as

$$D \equiv \underline{\theta}^T(\underline{x},V)\frac{\partial}{\partial \underline{x}} + W(\underline{x},V)\frac{\partial}{\partial V} \qquad 2.3.3$$

If we consider the system to be in n-dimensions we may write 2.3.3 as

$$D \equiv \sum_{i=1}^{n} \theta_i(\underline{x},V)\frac{\partial}{\partial x_i} + W(\underline{x},V)\frac{\partial}{\partial V} \qquad 2.3.4$$

and the system 2.3.1 and 2.3.2 may be written as

$$\dot{x}_i = f_i(\underline{x}) \qquad i = 1,\dots,n \qquad 2.3.5$$

$$\dot{V} = -\phi(\underline{x}) \qquad 2.3.6$$

Now we can consider the application of the operator D. From 2.3.4 we see that

$$Dx_i = \theta_i(\underline{x},V) \qquad i = 1,\dots,n \qquad 2.3.7$$

and $$DV = W(\underline{x},V) \qquad 2.3.8$$

Now the authors define θ_i $i = 1,\dots,n$ and W in such a way that D becomes the time derivative. For if D becomes the time derivative then the result of Taylors theorem gives

$$x_i(t) = e^{tD}x_i(0) \qquad i = 1,\dots,n \qquad 2.3.9$$

$$V(t) = e^{tD}V(0) \qquad 2.3.10$$

It is easily seen that D becomes the time derivative by comparing 2.3.5 with 2.3.7 and 2.3.6 with 2.3.8 giving

$$\theta_i(\underline{x},V) \equiv f_i(\underline{x}) \qquad i = 1,\dots,n \qquad 2.3.11$$

$$W(\underline{x},V) \equiv -\phi(\underline{x}) \qquad 2.3.12$$

With D defined by 2.3.4, 2.3.11, 2.3.12 we now have the system equations in the form

$$Dx_i = f_i(\underline{x}) \qquad i = 1,..,n \qquad 2.3.13$$

$$DV = -\phi(\underline{x}) \qquad 2.3.14$$

The following results hold for the operator D.

Theorem 2.3.1

a) $D(x_i + x_j) = Dx_i + Dx_j \qquad i,j = 1,\dots,n$ 2.3.15

b) $D(cx_i) = cDx_i \qquad i = 1,\dots,n$ 2.3.16

c) $D(x_i x_j) = (x_j)(Dx_i) + (x_i)(Dx_j) \quad i,j = 1,\dots,n$ 2.3.17

where x_i could be replaced by V in any of a),b),c) above.

Proof

The proofs are straightforward and will be given for part c) only. From 2.3.4, 2.3.11, 2.3.12 we obtain

$$D(x_i x_j) = \sum_{m=1}^{n} f_m(\underline{x})\frac{\partial}{\partial x_m}(x_i x_j) - \phi(\underline{x})\frac{\partial}{\partial V}(x_i x_j)$$

Now $\frac{\partial}{\partial x_i}(x_i x_j) = x_j, \quad \frac{\partial}{\partial x_j}(x_i\ x_j) = x_i,$

$$\frac{\partial}{\partial x_m}(x_i x_j) = 0 \text{ for } m \neq i,j, \qquad m = 1,\dots,n$$

Therefore $D(x_i x_j) = x_j f_i(\underline{x}) + x_i f_j(\underline{x})$.   2.3.18

We see from substitution of 2.3.13 into 2.3.18 that we obtain 2.3.17. End of proof.

Having defined D so as to obtain the time derivative, we are now able to use 2.3.9, 2.3.10 to obtain trajectories of $\underline{x}(t)$ in state space and corresponding values of the Lyapunov function V(t) given $\underline{x}(0)$ and V(0). To do this 2.3.9 and 2.3.10 must be expanded in power series form

$$x_i(t) = \sum_{m=0}^{\infty} \frac{(tD)^m}{m!} x_i(0) \qquad i = 1,\dots,n \qquad 2.3.19$$

$$V(t) = \sum_{m=0}^{\infty} \frac{(tD)^m}{m!} V(0) \qquad 2.3.20$$

2.3.19 and 2.3.20 form a power series in t whose coefficients are respectively

$$\frac{D^m x_i(0)}{m!} \qquad i = 1,\dots,n \qquad 2.3.21$$

$$\frac{D^m V(0)}{m!} \qquad 2.3.22$$

The terms 2.3.21 and 2.3.22 are computed recursively by differentiating 2.3.13 and 2.3.14 giving

$$D^m x_i(0) = D^{m-1} f_i(\underline{x}(0)) \qquad i = 1,\ldots,n \qquad 2.3.23$$

$$D^m V(0) = -D^{m-1} \phi(\underline{x}(0)) \qquad 2.3.24$$

The R.H.S.s of 2.3.23 and 2.3.24 are obtained in terms of $D^j x_i(0)$ and $D^j V(0)$ $\begin{matrix} j = 0,\ldots,m-1 \\ i = 1,\ldots,n \end{matrix}$ by using the expressions in theorem 2.3.1.

Thus we see that given $\underline{x}(0)$, $V(0)$ we may compute $\underline{x}(t)$, $V(t)$ by the power series 2.3.19, 2.3.20 whose coefficients we previously compute by 2.3.23, 2.3.24 and the expressions 2.3.15, 2.3.16, 2.3.17.

It can be noted that this method is simply taking the Taylor series expansions of solutions of 2.3.1 and 2.3.2. We clearly cannot use $\underline{x}(0) = \underline{0}$ for since $\underline{f}(\underline{0}) = \underline{0}$ we see that $D^m \underline{x}(0) = \underline{0}$, $m \geq 0$. The authors use the initial conditions

$$V(0) = 0$$

$$\|\underline{x}(0)\| \leq \varepsilon \qquad \text{for small } \varepsilon, \qquad 2.3.25$$

and compute 2.3.19, 2,3.20 to give $\underline{x}(t)$, $V(t)$ for negative time to trace out trajectories towards the boundary of the D.O.A.. When

$$V(t) > p \qquad 2.3.26$$

for some p, a boundary point has been defined. The process is carried out for various initial conditions and a set of boundary points are obtained.

3.3 Curve Fitting

Kormanik and Li (26) use the above method to generate N_1 boundary points $\underline{x}^{(m)}$ $m = 1,\ldots,N_1$. They then use an algorithm of Ho and Kashyap (27) to find a polynomial of degree 2q homogeneous in powers of x_i, close to the points $\underline{x}^{(m)}$ $m = 1,\ldots,N_1$. It is basically an attempt to define a computed domain in terms of a closed set $F(\underline{x}) \leq 0$ where

$$F(\underline{x}) = \sum_{j=0}^{2q} F_j(\underline{x}) \qquad 2.3.27$$

where $F_j(\underline{x})$ is homogeneous in the x_i of degree j, rather than in terms of a set of points.

First of all a new set of points "inside" the first set is picked and defined as $\underline{x}^{(m)}$ $m = N_1+1,\ldots,N_1+N_2$. See fig, 2. Then a matrix A is set up with M columns and N_1+N_2

rows. The elements of A are formed by evaluating the expression 2.3.27 term by term. $F(\underline{x})$ has M terms of the form

$$x_1^{p_1} x_2^{p_2} \ldots\ldots x_n^{p_n} \quad \text{where} \sum^{n} p_i \leqslant 2q \qquad 2.3.28$$

These terms are evaluated in an arbitrary fixed order and the element $A_{m,j}$ is the value of the jth expression 2.3.28 at the point $\underline{x}^{(m)}$. The first N_1 rows are computed by evaluating 2.3.28 for all admissible p_i $i = 1,\ldots,n$ at the point $\underline{x}^{(m)}$ $m = 1,\ldots,N_1$. The remaining rows are computed similarly but the signs are reversed.

The algorithm then goes through an iteration procedure given by

$$\underline{y}(N) = A\underline{w}(N) - \underline{b}(N)$$
$$\underline{w}(N+1) = \underline{w}(N) + \mu A^{\#}(\underline{y}(N) + |\underline{y}(N)|) \qquad 0 < \mu \leqslant 1$$
$$\underline{b}(N+1) = \underline{b}(N) + \mu(\underline{y}(N) + |\underline{y}(N)|)$$
$$\text{for } N \geqslant 0$$

starting from $\underline{b}(0) > \underline{0}$ but arbitrary, and arbitrary $0 < \mu \leqslant 1$

$$\underline{w}(0) = A^{\#}\underline{b}(0).$$

If at some stage $\underline{y}(N) \geqslant \underline{0}$ then the algorithm ceases. The M elements of $\underline{w}(N)$ correspond to the M columns of A and hence to the M expressions 2.3.28. These are the coefficients of the terms given by 2.3.28 in $F(\underline{x})$.

If at some stage $y(N) \leqslant \underline{0}$ with at least one element negative then the two sets of points cannot be separated by a polynomial of degree 2q, and a polynomial of degree 2q + 2 must be tried.

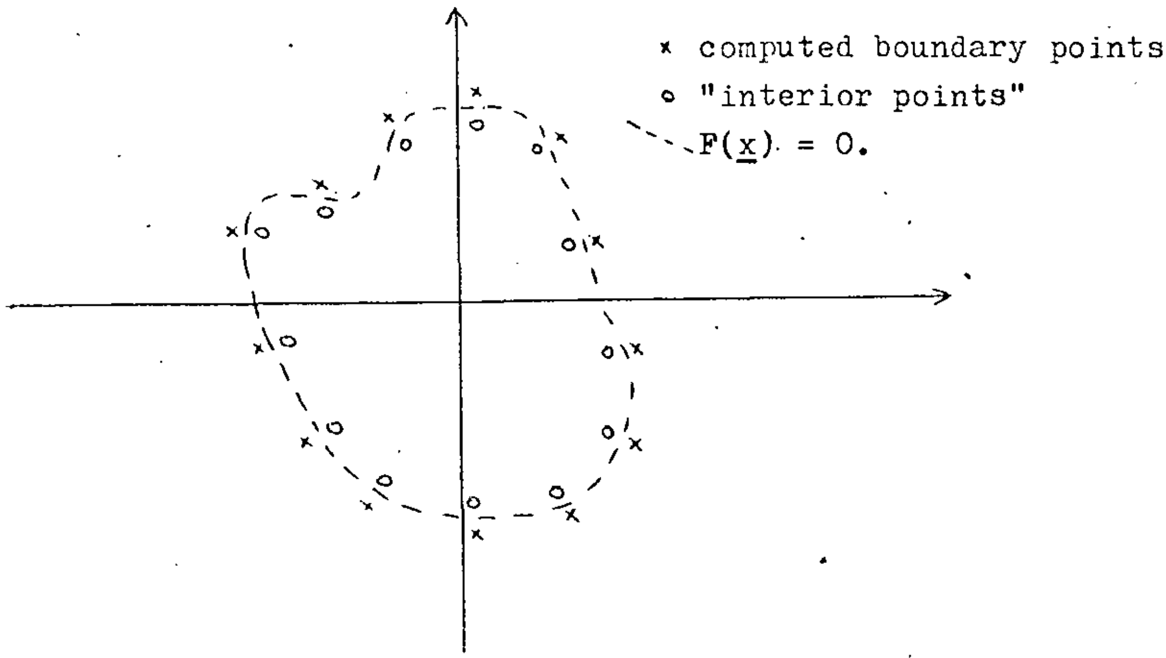


Fig. 2

3.4 Hahn Example

Consider the system

$$\dot{x} = -x + 2x^2y$$
$$\dot{y} = -y \qquad 2.3.29$$
$$\dot{V} = -2(x^2 + y^2)$$

in two dimensions.

Using the auxiliary equation method of section 8 of Chapter 1 and separating the variables we obtain as one solution of 2.3.29a,b

$$x = \frac{ay}{1+ay^2} \quad \text{where} \quad a = \frac{x_o}{y_o(1-x_oy_o)} \qquad 2.3.30$$

From 2.3.29b another solution is given by

$$y = y_oe^{-t} \qquad 2.3.31$$

Substituting 2.3.31 into 2.3.30 gives

$$x = \frac{x_oe^{-t}}{1-x_oy_o(1-e^{-2t})} \qquad 2.3.32$$

finally substituting 2.3.31 and 2.3.32 in 2.3.29c and integrating we obtain

$$V = V_o - \frac{x_o^2(1-e^{-2t})}{1-x_oy_o(1-e^{-2t})} - y_o^2(1-e^{-2t}) \qquad 2.3.33$$

where $x_o = x(0)$, $y_o = y(0)$, $V_o = V(0)$.

2.3.31, 2.3.32, 2.3.33 are the analytic solutions $x(t)$, $y(t)$, $V(t)$ of 2.3.29 given any x_o, y_o, V_o at $t = 0$. It may be noted that if x_o, y_o, V_o satisfy

$$V_o = y_o^2 + \frac{x_o^2}{1-x_oy_o}$$

then
$$V(t) = y(t)^2 + \frac{x(t)^2}{1-x(t)y(t)}.$$

If we take the initial conditions given by 2.3.25 then 2.3.31 and 2.3.32 represent the correct trajectory through (x_o, y_o) while since $\dot{V}$ only depends on x, y and not on V, then $V(t)$ differs from the Lyapunov function

$$V = y^2 + \frac{x^2}{1-xy}$$

by a constant term which is equal to $y_o^2 + \dfrac{x_o^2}{1-x_oy_o}$.

The power series expansion for x(t), y(t), given by 2.3.19 becomes

$$y(t) = y_o \sum_{m=0}^{\infty} \frac{(-t)^m}{m!}$$

$$x(t) = a \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \left(\sum_{i=0}^{\infty} (-b)^i (2i+1)^m \right) \qquad 2.3.34$$

where $a = \dfrac{x_o}{1-x_o y_o}$, $b = \dfrac{x_o y_o}{1-x_o y_o}$.

3.5 Results and conclusions

The series given by 2.3.34 were computed up to m = 39. Fig. 8 shows the results of using various x_o, y_o in the initial conditions. The trajectories given by 2.3.30 are also shown for comparison. The boundary of the D.O.A. is given by

$$xy = 1 \qquad 2.3.35$$

and the Taylor series solution truncated at the terms in t^{39} diverges from the analytical solution some distance from 2.3.35.

An investigation was also carried out into the effects of truncating the series at different powers. The series was truncated at t^{N-1} for N = 30,40,50,70, $x_o = 0{\cdot}01$, $y_o = 0{\cdot}01$ and the trajectories computed are shown in fig. 9 with the analytical curve representing $N = \infty$. Fig. 10 shows the same analysis for N = 30,40,50,60,70 and $x_o = 0{\cdot}01$, $y_o = 0{\cdot}005$.

It is observed from figs. 9,10 that the error in the trajectories as a result of truncation is not consistently one way. From 2.3.34b we see that as $t \to -\infty$

$$x(t) \approx \frac{a(-t)^{N-1}}{(N-1)!} \sum_{i=0}^{\infty} (-b)^i (2i+1)^{N-1}$$

For N = 30,50,70 we have $x(t) \to \infty$
while for N = 40,60 we have $x(t) \to -\infty$.

It seems that the authors go to a great deal of trouble to work out a method that is just a Taylor series expansion. Also the expansion of the R.H.S.s of 2.3.23, 2.3.24 become infeasible for all except certain "well-behaved" functions. Any function containing non-integer powers of x_i i = 1,...,n cannot be computed by this method unless the non-integer terms have a power series expansion themselves.

The results above and in figs. 8,9,10 suggest that even for 70 terms of 2.3.19, 2.3.20 it is difficult to maintain accuracy until a boundary point can be computed and in some cases the computed point will be outside the D.O.A. as shown by figs. 9,10 for N = 30,50,70. Shields(28) notes that computing 2.3.20 as well as 2.3.19 provides a boundary condition 2.3.26 which is arbitrary. We can actually arrange for $V(\underline{x}) = p$ at any $\underline{x} \in D(\underline{f})$, $\underline{x} \neq \underline{0}$, and for any $p > 0$.

Theorem 2.3.2

If $V(\underline{x})$, $\phi(\underline{x})$ satisfy the modified Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})$$

and the conditions of theorem 1.7.1 then by solving the similar Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V_1(\underline{x})}{\partial x_i} = -\frac{p_1 \phi(\underline{x})}{V(\underline{x}_1)} \qquad 2.3.36$$

for $V_1(\underline{x})$ given any $\underline{x}_1 \in D(\underline{f})$, $\underline{x}_1 \neq \underline{0}$ and any $p_1 > 0$ then

$$V_1(\underline{x}_1) = p_1 \qquad 2.3.37.$$

Proof

By theorem 1.7.15

$$V(\underline{x}_1) = \int_0^\infty \phi(\underline{x}(t))dt \qquad 2.3.38$$

where $\underline{x}(0) = \underline{x}_1$.

From 2.3.37 and 2.3.38 we see that

$$V_1(\underline{x}_1) = \frac{p_1}{V(\underline{x}_1)} \int_0^\infty \phi(\underline{x}(t))dt$$

Hence we see that $V_1(\underline{x})$ is the solution of the Zubov equation with $\phi(\underline{x})$ replaced by $\dfrac{p_1\phi(\underline{x})}{V(\underline{x}_1)}$. End of proof.

The curve-fitting method also has its drawbacks. Even supposing the points $\underline{x}^{(m)}$ are conservative approximations to the boundary of the D.O.A., it may be difficult to find the set $\underline{x}^{(m)}$, $m = N_1+1,\dots,N_1+N_2$, to guarantee that they are in the D.O.A..

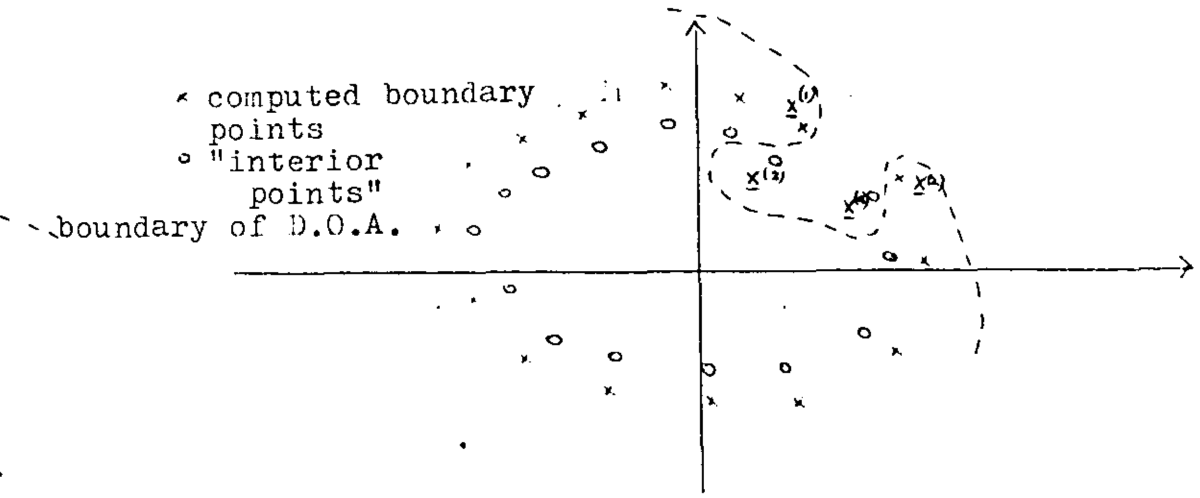


Fig. 3

Consider the situation of fig. 3. The points marked $\underline{x}^{(1)}, \underline{x}^{(2)}$ are computed by Lie series, while the "interior" points $\underline{x}^{(3)}$, $\underline{x}^{(4)}$ are outside the boundary of the D.O.A.. The method must break down in this situation since any $F(\underline{x}) = 0$ obtained by the Ho and Kashyap algorithm contravenes the boundary. It all comes down to the question of what is meant by an "interior" point. The only safe way to obtain $\underline{x}^{(m)}$, $m = N_1+1,\ldots,N_1+N_2$, is to use 2.3.19 and 2.3.20 computing the boundary point from the criterion 2.3.26 but taking a point with $V < p$ as an "interior" point.

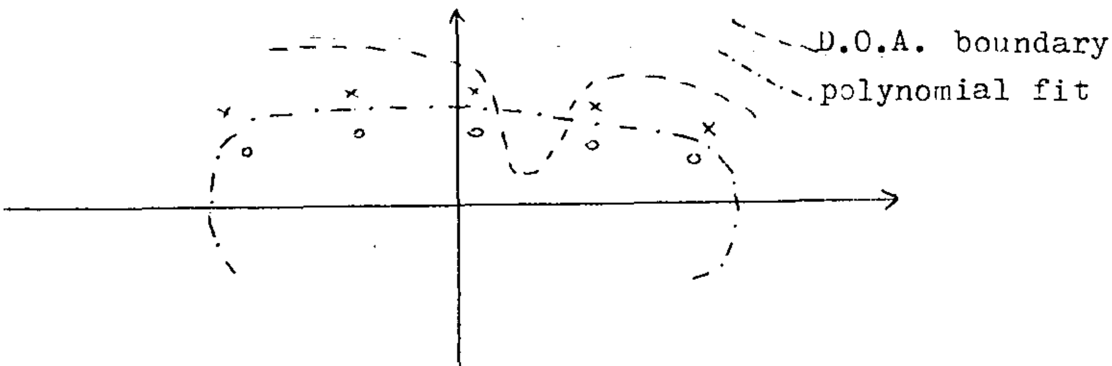


Fig. 4

Now consider the situation of fig. 4. The boundary and "interior" points are assumed correct, but still there is no guarantee of getting a curve to fit which does not contravene the boundary. Taking the suggestion of fitting a polynomial of degree 2q to its natural limit suggests that we might as well connect the boundary points by a piecewise linear path.

4. Transformations

Suppose we are given the system equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{2.4.1}$$

then the question arises as to whether it is possible to transform the variables of 2.4.1 in such a way that the D.O.A. boundary becomes more obvious.

We will consider the 2-dimensional system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{2.4.2}$$

What we are now looking for is a substitution

$$y = h(x,u) \tag{2.4.3}$$

which when substituted into 2.4.2 gives system equations in x,u, which do not affect the asymptotic stability properties of 2.4.2.

Differentiating 2.4.3 with respect to time and using the chain rule gives

$$\dot{y} = \frac{\partial h}{\partial x}(x,u)\dot{x} + \frac{\partial h}{\partial u}(x,u)\dot{u} \tag{2.4.4}$$

Substituting 2.4.2 and 2.4.3 into 2.4.4 we have

$$g(x,h(x,u)) \equiv \frac{\partial h}{\partial x}(x,u)\ f(x,h(x,u)) + \frac{\partial h}{\partial u}(x,u)\dot{u}$$

By this means we have now transformed the system equations 2.4.2 into the new system

$$\begin{aligned} \dot{x} &= f(x,h(x,u)) \\ \dot{u} &= \frac{g(x,h(x,u)) - \frac{\partial h}{\partial x}(x,u)\ f(x,h(x,u))}{\frac{\partial h}{\partial u}(x,u)} \end{aligned} \tag{2.4.5}$$

To consider how such a transformation could facilitate estimating the D.O.A. we consider two examples:

$$\begin{aligned} \dot{x} &= -x + y + x(x^2 + y^2) \\ \dot{y} &= -x - y + y(x^2 + y^2) \end{aligned}$$

If we consider the transformation

$$y = h(x,u) = \pm\sqrt{u^2 - x^2} \tag{2.4.6}$$

then 2.4.5 becomes

$$\begin{aligned} \dot{x} &= -x(1 - u^2) \pm\sqrt{u^2 - x^2} \\ \dot{u} &= -u(1 - u^2) \end{aligned} \tag{2.4.7}$$

2.4.6 and 2.4.7 immediately yield the solution for the D.O.A.

$$-1 < u < 1.$$

If we use the transformation

$$y = \pm\sqrt{u - x^2} \qquad 2.4.8$$

instead the corresponding system equations become

$$\begin{aligned} \dot{x} &= -x(1 - u) \pm\sqrt{u - x^2} \\ \dot{u} &= -2u(1 - u) \end{aligned} \qquad 2.4.9$$

2.4.9 suggests that the D.O.A. is given by $u < 1$ but we observe that the transformation 2.4.8 does not exist in real numbers for $u < 0$. However every point of the (x,y) plane has a unique non-negative value of u and we need only consider $u \geqslant 0$ and we see that the D.O.A. is given by $0 \leqslant u < 1$.

Consider also the Hahn example

$$\begin{aligned} \dot{x} &= -x + 2x^2y \\ \dot{y} &= -y \end{aligned} \qquad 2.4.10$$

The obvious transformation of $u = xy$ yields

$$\begin{aligned} \dot{x} &= -x + 2xu \\ \dot{u} &= -2u(1 - u) \end{aligned}$$

from which we see that the D.O.A. in the (x,u) plane is given by $u < 1$. However it must be recognised that the origin of the (x,u) plane is the y-axis of the (x,y) plane under the transformation $u = xy$ and we have established $xy < 1$ is the D.O.A. of the set $x = 0$. By inspection of 2.4.10 we see that $x = 0$ is an invariant set, and the theory of the stability of such sets is a slight departure from this thesis. It is mentioned as one of the problems incurred by this type of transformation to be included in a section covering possibilities of solving Zubov's equation in this way.

There are two ways of proceeding generally from 2.4.5. The first is to assume that f and g possess only integral powers of x and y. In this case we may attempt to define the general transformation as

$$y = h(x,u) = \sum_m \sum_k a_{m,k} x^k u^{m-k}. \qquad 2.4.11$$

We also assume 2.4.11 is diffentiable term by term in u and x and that f,g may be written in power series form

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{k=0}^{m} f_{m,k}x^k y^{m-k}$$
$$g(x,y) = \sum_{m=1}^{\infty}\sum_{k=0}^{m} g_{m,k}x^k y^{m-k} \qquad 2.4.12$$

Substituting 2.4.11 and 2.4.12 into 2.4.5 gives

$$\dot{u} = \frac{\left[\sum_{m=1}^{\infty}\sum_{k=0}^{m} g_{m,k}x^k\left(\sum_r\sum_i a_{r,i}x^i u^{r-i}\right)^{m-k} - \left\{\sum_{m=1}^{\infty}\sum_{k=0}^{m} f_{m,k}x^k\left(\sum_r\sum_i a_{r,i}x^i u^{r-i}\right)^{m-k}\right\}\left\{\sum_r\sum_i i a_{r,i}x^{i-1}u^{r-i}\right\}\right]}{\sum_r\sum_i (r-i)a_{r,i}x^i u^{r-i-1}} \qquad 2.4.13$$

The range of r,i has not been included as it becomes whatever range is needed for a particular f,g. It is reasonable to require that 2.4.13 is independent of x and hence the ratio of the coefficients of x^m in numerator and denominator should be a function of u alone which is the same for all m. The examples considered and the transformations given can be verified using 2.4.13.

The second way of proceeding from 2.4.5 would be as in 2.4.8 where the transformation used makes u a Lyapunov function of x and y. Since the region $u < 0$ is undefined and the origin is the only point in the (x,y) plane satisfying $u = 0$ we are concerned with knowing regions in which $\dot{u}$ is negative and is going to stay negative for all $t \geq 0$. This is equivalent to finding the region $R^*(\underline{f},u)$ of theorem 1.6.6 and definition 1.6.4.

For transformation 2.4.8 and general system 2.4.2, 2.4.5 becomes

$$\dot{x} = f(x,\pm\sqrt{u-x^2})$$
$$\dot{u} = 2xf(x,\pm\sqrt{u-x^2}) \pm 2\sqrt{u-x^2}\; g(x,\pm\sqrt{u-x^2})$$

Another possibility for a transformation could arise by considering the Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})). \qquad 2.4.14$$

Since we know that the origin of a stable system satisfies $V(\underline{0}) = 0$ and that $V(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$, $\underline{x} \in D(\underline{f})$ and $V(\underline{x}) = \infty$ for $\underline{x} \in \delta D(\underline{f})$, $d = 0$, then we may consider the transformation

$$W(\underline{x}) = \frac{1}{V(\underline{x})} \qquad 2.4.15$$

The transformation 2.4.15 should reveal that

$$W(\underline{0}) = \infty$$

and $$W(\underline{x}) = 0 \text{ for } \underline{x} \in \delta D(\underline{f}).$$

Differentiating 2.4.15 with respect to time gives

$$\dot{W}(\underline{x}) = -W(\underline{x})^2\ \dot{V}(\underline{x}) \qquad 2.4.16$$

Combining 2.4.14 and 2.4.16 we obtain the corresponding P.D.E. for W in terms of $\underline{x}$ as

$$\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial W}{\partial x_i}(\underline{x}) = \phi(\underline{x})W(\underline{x})(W(\underline{x}) - d) \qquad 2.4.17$$

We may apply the series method to 2.4.17 as in section 2. We assume that we have a 2-dimensional system with f,g,ϕ given by 2.2.6, 2.2.7, 2.2.8 and that

$$W(x,y) = \sum_{m=r}^{\infty}\sum_{k=0}^{m} W_{m,k}x^k y^{m-k} \qquad 2.4.18$$

Substituting 2.2.6, 2.2.7, 2.2.8 and 2.4.18 into 2.4.17 gives an equation similar to 2.2.12.

Equating the lowest homogeneous powers of x,y we see that if $d = 0$ or $d = 1$, $r < 0$

$$r = s - q - 1 \qquad 2.4.19$$

while if $d = 1$, $r \geq 0$

$$q = s - 1. \qquad 2.4.20$$

2.4.20 is difficult to achieve if $s = 1$ as $\phi(x,y)$ will not then be positive definite. However since we require $W(0,0) = \infty$ we see that $r < 0$ is necessary. From 2.4.19 we then immediately see that

$$s \leq q$$

which is consistent with Zubov's method.

There is, however, some difficulty with definition of 2.4.18 for negative m. A simple example shows this from Davidson and Cowan (29).

$$\dot{x} = -y$$
$$\dot{y} = x - y + x^2 y$$

Using $\phi(x,y) = x^2 + y^2$, $d = 0$ the lowest powers in 2.4.17 become

$$(-y)\Big(\sum_{k=0}^{-2} kW_{-2,k}x^{k-1}y^{-2-k}\Big) + (x-y)\Big(\sum_{k=0}^{-2} W_{-2,k}(-2-k)x^k y^{-3-k}\Big)$$
$$= (x^2+y^2)\Big(\sum_{i=0}^{-2}\sum_{j=0}^{-2} W_{-2,i}W_{-2,j}x^{i+j}y^{-4-i-j}\Big) \qquad 2.4.21$$

Clearly from 2.4.21 we cannot let the summation for k be confined to $k = 0,-1,-2$. However we can let k increase from 0 to ∞ and then from $-\infty$ to -2 and write down each equation. These become

$$(-y)\Big(\sum_{k=0}^{\infty} kW_{-2,k}x^{k-1}y^{-2-k}\Big) + (x-y)\Big(\sum_{k=0}^{\infty}(-2-k)W_{-2,k}x^k y^{-3-k}\Big)$$
$$=(x^2+ y^2)\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} W_{-2,j}W_{-2,k-j}\Big)x^k y^{-2-k}\Big) \qquad 2.4.22$$

and similarly for negative k. The series generated by 2.4.22 assuming $W_{-2,0}= 1$ gives

$$W_{-2}(x,y) = \frac{1}{y^2}\Big(1 + \frac{x}{y} - \frac{x^2}{2y^2} - \frac{2x^3}{y^3} - \frac{5x^4}{y^4} + \ldots\ldots\Big).$$

It is apparent that the series construction has too many problems to be feasible.

The method of solution using the auxiliary equations is no more helpful. The corresponding equation to 1.8.2 for $W(x,y)$ is

$$\frac{dx}{f(x,y)} = \frac{dy}{g(x,y)} = \frac{dW}{\phi(x,y)W(W-d)} \qquad 2.4.23$$

It can be seen that 2.4.23 is no better than 1.8.2 by comparing the terms in V and W respectively. If $d = 0$ then

$$\int \frac{dW}{W^2} = -\frac{1}{W} = -V = -\int dV$$

while if $d = 1$ then

$$\int\frac{dW}{W(W-1)} = \log(1 - \tfrac{1}{W}) = \log(1 - V) = -\int\frac{dV}{1-V}.$$

Hence we obtain V and $1/_W$ as the same function of x,y.

Having attempted certain transformations which suggested themselves from the nature of the problem, it has become obvious that transformations, if they are any use at all, are only useful by using certain transformations with certain examples. This is equivalent to deciding what the D.O.A. is and fitting the transformation around it.

5. The Vector Method

Infante and Clark (30) in their paper considered that the second method of Lyapunov was essentially geometric in nature and that geometric techniques should therefore be used to generate a Lyapunov function and hence obtain the D.O.A.. They refer to theorem 1.6.3 noting that if $V(\underline{x})$ can be found satisfying this theorem then the origin of

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

is asymptotically stable. The method is developed in 2 dimensions. We denote the system equations as

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y) \qquad 2.5.1$$

Then sufficient conditions for the existence of a time independent integral of 2.5.1

$$h(x,y) = p \qquad 2.5.2$$

are given by

$$\frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y) = 0 \qquad 2.5.3$$

However most systems do not satisfy 2.5.3. A class of systems which do satisfy 2.5.3 are given by

$$\dot{x} = y$$
$$\dot{y} = g(x,y) \qquad 2.5.4$$

where $\frac{\partial g}{\partial y}(x,y) = 0$.

It is system 2.5.4 that we consider to develop the method on, without the assumption that $\frac{\partial g}{\partial y} = 0$.

Suppose now the system 2.5.4 is slightly modified so that we have a system satisfying 2.5.3. The system which achieves this is given by

$$\dot{x} = y - \int_0^x \frac{\partial g}{\partial y}(x',y)dx' \qquad 2.5.5$$

$$\dot{y} = g(x,y).$$

Thus the system 2.5.5 possesses a time independent integral. However, the system 2.5.5 does not necessarily have the same properties as 2.5.4. Thus we have to modify 2.5.5 further to try to preserve the essential properties of 2.5.4. We obtain

$$\dot{x} = y - \int_0^x \frac{\partial g}{\partial y}(x',y)dx' + f_1(x,y) \qquad 2.5.6$$

$$\dot{y} = g(x,y) + g_1(x,y)$$

where $\frac{\partial f_1}{\partial x}(x,y) + \frac{\partial g_1}{\partial y}(x,y) = 0.$ 2.5.7

The final step of the method is to investigate the characteristic vectors of the systems 2.5.4 and 2.5.6. However we do this in three dimensions and so introduce z as a variable the axis of which is perpendicular to the (x,y) plane. As we only have a two-dimensional system we add the third system equation as

$$\dot{z} = 0 \qquad 2.5.8$$

restricting initial conditions to

$$z(0) = 0.$$

Then we may write 2.5.4 and 2.5.6 together with 2.5.8 in vector form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ g(x_1,y_1) \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} y_2 - \int_0^{x_2} \frac{\partial g}{\partial y_2}(x_2',y_2)dx_2' + f_1(x_2,y_2) \\ g(x_2,y_2) + g_1(x_2,y_2) \\ 0 \end{bmatrix}$$

where $(x_1,y_1,z_1)^T$, $(x_2,y_2,z_2)^T$ are respectively the solutions of 2.5.4, 2.5.8 and 2.5.6, 2.5.8. The vector $\underline{W}$ given by

$$\underline{W} = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} \times \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \\ \dot{z}_2 \end{bmatrix} .$$

is a vector which is perpendicular to the (x,y) plane and we denote $\hat{W}$, a signed number, as the magnitude of $\underline{W}$ expressed in the convention that $\hat{W}$ is positive for $\underline{W}$ in the direction of increasing z. (fig. 5)

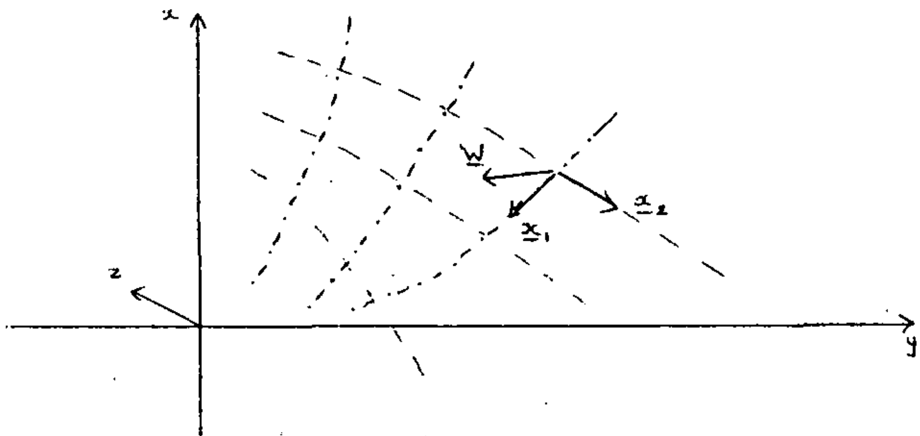


Fig. 5

The basis of the method is thus to choose f_1,g_1 in such a way that the solutions 2.5.2 of 2.5.6 form closed concentric curves round the origin which makes h(x,y) a Lyapunov function.

Now if the origin of 2.5.4 is asymptotically stable then we know by theorem 1.7.6 that trajectories of 2.5.4 cross curves 2.5.2 and hence cross trajectories of 2.5.6 toward the direction of decreasing h(x,y). (Fig. 6). This means that by careful choice of $f_1(x,y)$, $g_1(x,y)$ we may establish $\hat{W}$ to be positive semi-definite in a neighbourhood of the origin. Thus by investigating h(x,y) and $\hat{W}$ we obtain an R.A.S. given by

$$h(x,y) < p'.$$

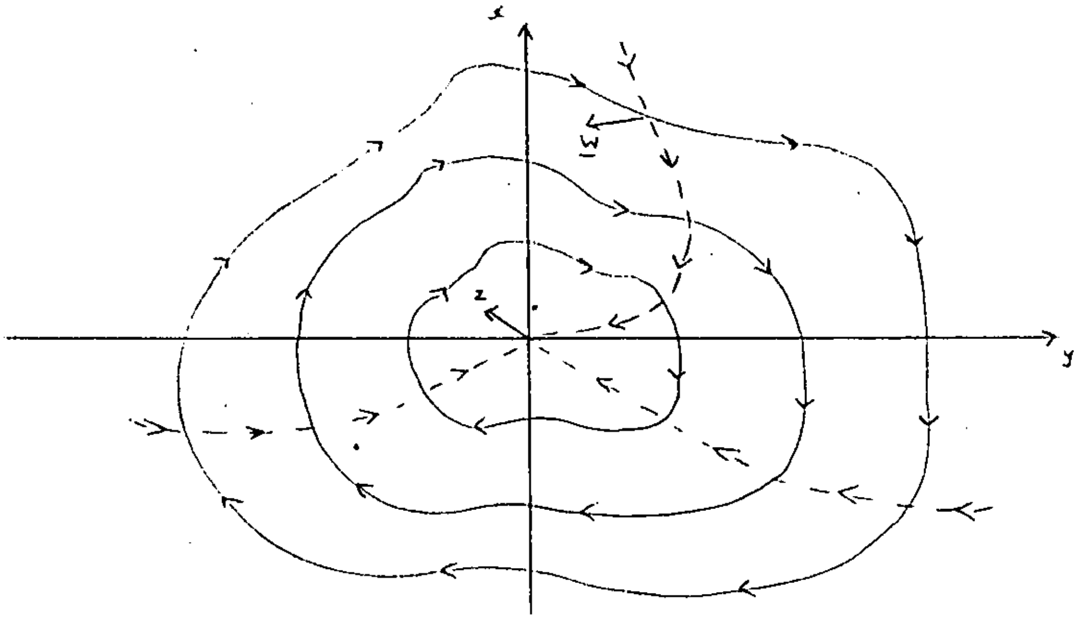


Fig. 6

To illustrate the method we consider the equations

$$\dot{x} = y$$
$$\dot{y} = -x + x^3 - ey$$

The modified system becomes

$$\dot{x} = y + ex + f_1(x,y)$$
$$\dot{y} = -x + x^3 - ey + g_1(x,y) \qquad 2.5.9$$

Then the cross product magnitude $\hat{W}$ becomes

$$\hat{W} = ex^2(1-x^2) + e^2xy + x(1-x^2)f_1(x,y) + eyf_1(x,y) + yg_1(x,y) \qquad 2.5.10$$

We have to find f_1, g_1 satisfying 2.5.7 and such that $\hat{W}$ is positive semi-definite in a region of the origin.

This can be achieved by removing the xy terms

$$f_1(x,y) = 0 \qquad 2.5.11$$
$$g_1(x,y) = -e^2x.$$

Substituting 2.5.11 into 2.5.10 gives

$$\hat{W} = ex^2(1 - x^2)$$

and the integral of 2.5.9 is

$$h(x,y) = x^2((1+e)^2 - \frac{x^2}{2}) + 2exy + y^2.$$

Since the extreme values of x which allow $\hat{W} \geqslant 0$ are $x = \pm 1$ we see that the R.A.S. is given by

$$x^2((1+e)^2 - \frac{x^2}{2}) + 2exy + y^2 < \frac{1 + 2e^2}{2}.$$

Alternatively we may choose

$$f_1(x,y) = 0$$
$$g_1(x,y) = -e^2x - ex^3$$

giving an R.A.S of

$$x^2((1+e^2) - \frac{x}{2}^2(1-e)) + 2exy + y^2 < \frac{1+e+2e^2}{2}.$$

Several points occur about this method:

a) As the authors state they cannot make an algorithm out of the method as the choice of f_1 and g_1 is made by inspection.

b) Since the method relies on inspection it can only be done on systems that can be so inspected, and the D.O.A.s of which can easily be obtained by other methods.

c) Presumably the condition 2.5.3 is sufficient for there to exist closed curves of the form 2.5.2. After all, systems 2.5.1 possess an integral which is the solution of

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} .$$

Margolis and Vogt (19) mention that given a Lyapunov function V(x,y) the system

$$\dot{x} = \frac{\partial V(x,y)}{\partial y}$$
$$\dot{y} = \frac{-\partial V(x,y)}{\partial x} \qquad 2.5.12$$

has as its solution the curves $V(x,y) = p$. System 2.5.12 does satisfy 2.5.3.

d) The systems on which the method is developed are of the form 2.5.4 which is only a subset of 2.5.1.

e) There is no generalisation to higher orders that can easily be seen.

f) There is no guarantee that the system 2.5.6 will have similar properties to 2.5.4 even when f_1, g_1 are chosen.

6. Rodden's Method

Of significant importance in a catalogue of methods of determining regions of asymptotic stability is Rodden's method (18). The method he proposed is a computational method of determining R.A.S.s of given Lyapunov functions which have been chosen or worked out by some method such as those of Zubov (11), Ingwerson (31). Szego (32)(33).

The basis of the method is to use theorem 1.6.8 to find $p^*(\underline{f},V)$ such that

$$V(\underline{x}) \geqslant p^*(\underline{f},V)$$

for all $\underline{x}$ such that

$$\dot{V}(\underline{x}) = 0 \qquad 2.6.1$$

where the system equations are given by

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$
$$f(\underline{0}) = \underline{0}. \qquad 2.6.2$$

The stability of the origin is immediately indicated by checking that theorem 1.6.3 is satisfied

i.e. $V(\underline{0}) = 0$
$V(\underline{x}) > 0 \qquad \underline{x} \neq \underline{0}$
$\dot{V}(\underline{x}) \leqslant 0$

for all $x \in S_\varepsilon$ for some $\varepsilon > 0$.

Having satisfied the conditions required for this choice of $V(\underline{x})$ to indicate that the origin of 2.6.2 is asymptotically stable we may now proceed to determine the region of asymptotic stability.

There are three stages to this method:

a) Find $\underline{x}^{(1)}$ such that $\dot{V}(\underline{x}^{(1)}) = 0$

b) Track along 2.6.1 to find $\underline{x}^{(2)}$ such that

$$V(\underline{x}^{(2)}) \leqslant V(\underline{x})$$

for all $\underline{x}$ satisfying 2.6.1, $\underline{x} \in S_\varepsilon(\underline{x}^{(2)})$ for some $\varepsilon > 0$.

c) Trace out the boundary of the R.A.S. given by

$$V(\underline{x}) = V(\underline{x}^{(2)}). \qquad 2.6.3$$

Stage a) is comparatively simple to carry out compared with stages b), c). The latter require us to keep on a pre-defined curve and to keep adjusting for any errors made in tracking, while the former searches for any point satisfying 2.6.1. Providing we define a reasonably logical method for stage a) we have a lot of freedom for choice of large step-sizes and changes of direction.

The systematic method for stage a) is required to be such that the point $\underline{x}^{(1)}$ will be found no matter where it is. That is, we must not confine our search to one region of R^n when there may not be such a point in this region but there may be elsewhere.

Rodden does this by means of a spiral which he does not actually define but Hewit (17) does. First of all we select a plane in 2 dimensions containing the origin. A 2-dimensional plane in an n-dimensional space requires n-2 linear relationships between the variables $x_1,\ldots,x_n$ given by

$$A\underline{x} = \underline{0} \qquad 2.6.4$$

where A is an n-2 by n matrix, and $\underline{x}$ is the vector of system variables. If we replace $\underline{x}$ by $B\underline{y}$ where B is an n x n matrix satisfying

$$AB = (I,\underline{0},\underline{0}) \qquad 2.6.5$$

and $\underline{y}$ is a new set of variables, then we have the 2-dimensional plane in $\underline{y}$ given by

$$AB\underline{y} = \underline{0}. \qquad 2.6.6$$

$(I,\underline{0},\underline{0})$ is the matrix formed from an n-2 by n-2 unit matrix with 2 extra columns of zeroes.

i.e.

$$(I,\underline{0},\underline{0}) = \begin{bmatrix} 1 & & & 0 & 0 \\ & 1 & O & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ O & & 1 & 0 & 0 \end{bmatrix}$$

From 2.6.6 we see that such a transformation gives the plane formed by setting

$$y_i = 0 \qquad i = 1,\ldots,n-2$$

and allowing y_{n-1}, y_n to vary freely. The matrix equation 2.6.5 represents n sets of linear equations each represented by n-2 equations in the n unknowns $B_{i,j}$ $j = 1,\ldots,n$ for each $i = 1,\ldots,n$.

Rodden illustrates the method in 2 dimensions. Now that it has been shown that by simple linear transformations of the variables that a plane through the origin is equivalent to a plane in which n-2 variables are held to zero and the other 2 allowed to vary freely, we may illustrate stage a) in 2 dimensions.

We define a series of points in the (x,y) plane in 2 dimensions by (x_m, y_m) where

i) $(x_o, y_o) = (r, 0)$

ii) $x_m^2 + y_m^2 = (rm)^2 \qquad m = 0, 1, \ldots$

iii) (x_{m+1}, y_{m+1}) is where the straight line tangential to $x^2 + y^2 = (rm)^2$ passing through (x_m, y_m) intersects the curve $x^2 + y^2 = (r(m+1))^2$. $m = 0, 1 \ldots$

It may be shown (Appendix B) that this definition is equivalent to

i) $(x_o, y_o) = (r, 0)$

ii) $$x_{m+1} = x_m - y_m \frac{\sqrt{2m+1}}{m}$$
$$y_{m+1} = y_m + x_m \frac{\sqrt{2m+1}}{m} \qquad 2.6.7$$

Corresponding to each (x_m, y_m) there is a $V_m, \dot{V}_m$ given by

$$V_m = V(x_m, y_m)$$
$$\dot{V}_m = \dot{V}(x_m, y_m).$$

The series $\dot{V}_m$ is checked so long as $\dot{V}_m < 0$.
When we have found N such that

$$\dot{V}_N < 0$$
$$\dot{V}_{N+1} > 0 \qquad 2.6.8$$

then a process of interval halving along the straight line given by

$$x = x_N + \lambda(x_{N+1} - x_N)$$
$$y = y_N + \lambda(y_{N+1} - y_N)$$

$0 < \lambda < 1$

is carried out until $(x^{(1)}, y^{(1)})$ is found to pre-determined accuracy. This completes stage a).

Rodden does not actually mention how the 2-dimensional plane of the spiral curve is selected when we are solving a system in 3 or more dimensions. Clearly the point $\underline{x}^{(1)}$

in n dimensions depends on which plane is chosen. Rodden chooses the expanding spiral series 2.6.7 with 2 dimensions in mind so that if $\dot{V} > 0$ anywhere in this plane then $\underline{x}^{(1)}$ will be found. The problem in higher dimensions is that $\dot{V}$ is not necessarily strictly negative definite but it can be negative definite when constrained to a particular plane.

Rodden's own third order example shows this problem. Consider the system given by Ingwerson (31)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_1 - 2x_2 - 2x_3 - x_1^3 \end{aligned} \qquad 2.6.9$$

Ingwerson derived the Lyapunov function

$$V(\underline{x}) = 2x_1^2 + 2x_1x_2 + 6x_2^2 + 4x_2x_3 + x_3^2 + x_1^4 + 2x_1^3x_2$$

which has a time-derivative with respect to the system 2.6.9

$$\dot{V}(\underline{x}) = -6(1 - x_1^2)x_2^2 \qquad 2.6.10$$

Now clearly from 2.6.10 $\dot{V}(\underline{x})$ is negative semi-definite for all $\underline{x}$ where

$$\underline{x} \in \{ \underline{x} : x_1 = 0 \}$$

and a value of N defined by 2.6.8 will not be obtained when the spiral search is carried out in the (x_2, x_3) plane for $x_1 = 0$. Rodden says that if the plane 2.6.4 does not intersect $\dot{V} = 0$ then a new plane is chosen. This raises two questions:

1) How far does the spiral go before deciding that a new plane should be chosen?

2) Is there a systematic approach to choosing new planes such that if $\dot{V}$ is not strictly negative definite then $\underline{x}^{(1)}$ will be found?

The obvious answer to question 1) is to confine the spiral search to S_R for some large R. This is equivalent to finding a stability region given by $R^*(\underline{f}, V) \cap S_R$. The answer to question 2) lies in constructing a linear combination of fixed planes. It can be shown that in general (Appendix C) given any $\underline{x} \in R^n$ that $\underline{x}$ satisfies

$$\lambda_1 A_1 \underline{x} + \lambda_2 A_2 \underline{x} + \dots\dots\dots + \lambda_{n-1} A_{n-1} \underline{x} = \underline{0} \qquad 2.6.11$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ where $A_i\underline{x} = \underline{0}$, $i = 1,\ldots,n-1$, are planes in n dimensions as given by 2.6.4. Any systematic method of selecting different planes will in some way be related to variation of λ_i, $i = 1,\ldots,n-1$. Even then the set of $(\lambda_1, \ldots\ldots, \lambda_{n-1})$ such that 2.6.11 intersects 2.6.1 may be missed out due to λ_i, $i = 1,\ldots,n-1$ possibly being varied in steps which are too large.

Given that $\underline{x}^{(1)}$ has been found we now require to proceed along 2.6.1 to find a point where $V(\underline{x})$ is minimised. So we have to find which direction to proceed along the n-1 dimensiona surface 2.6.1 so that $V(\underline{x})$ is decreased at the greatest rate. The gradients of $V(\underline{x})$ and $\dot{V}(\underline{x})$ are respectively the vectors $\nabla V(\underline{x})$, $\nabla\dot{V}(\underline{x})$. From fig. 7 we see that the correct direction to search for $\underline{x}^{(2)}$ is opposite to component $\nabla V(\underline{x})$ on the $\dot{V}(\underline{x}) = 0$ surface. Thus we proceed in the direction of the vector $\underline{H}$ where

$$\underline{H}(\underline{x}) = \left(\frac{\nabla V}{\|\nabla V\|} \cdot \frac{\nabla \dot{V}}{\|\nabla \dot{V}\|}\right)\frac{\nabla \dot{V}}{\|\nabla \dot{V}\|} - \frac{\nabla V}{\|\nabla V\|} \qquad 2.6.12$$

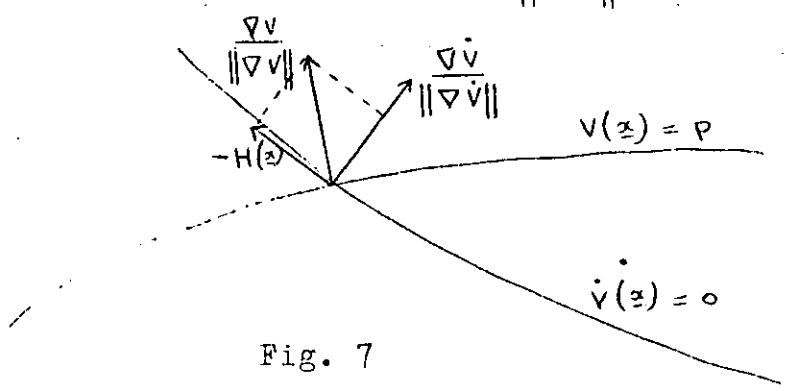


Fig. 7

Now given any point $\underline{x}$ on 2.6.1 we take the next point to be $\underline{x}'$ where

$$\underline{x}' = \underline{x} + \frac{\underline{H}(\underline{x})}{\|\underline{H}(\underline{x})\|}\, ds$$

where ds is a fixed step-size. $\underline{x}'$ will not satisfy 2.6.1 exactly. To return to 2.6.1 we iterate by step search and interval halving along the line given by

$$\underline{x} = \underline{x}' + \lambda\nabla\dot{V}(\underline{x}') \qquad 2.6.13$$

where $-\lambda$ is of the same sign as $\dot{V}(\underline{x}')$. until we find $\underline{x}''$ satisfying 2.6.1 and 2.6.13. The danger is that 2.6.1 and

2.6.13 may not intersect but since

$$\|\underline{x}'' - \underline{x}'\| \approx O(ds^2) \text{ as } ds \rightarrow 0$$

this is unlikely. The procedure to find $\underline{x}''$ from $\underline{x}$ is repeated until we find $\underline{x}$ such that the vector $\underline{H}$ has changed direction. This is done by testing the scalar product of $\underline{H}(\underline{x})$ and $\underline{H}(\underline{x}'')$.

In the unlikely event that the step-size is large enough for the minimum to be missed by the scalar product test, we hope to approach the minimum point by another path.

Once we have $\underline{x}$, $\underline{x}''$ such that

$$\underline{H}(\underline{x}).\underline{H}(\underline{x}'') < 0 \qquad 2.6.14$$

then the step-size ds may be reduced typically to one-third of its previous value until the desired accuracy is achieved. Rodden alters ds in a different way. He takes

$$ds = \frac{\|\underline{H}(\underline{x})\|}{\left\|\frac{d\underline{H}(\underline{x})}{ds}\right\|}$$

the ith component of $\frac{d\underline{H}}{ds}$ is given by

$$\left(\frac{d\underline{H}(\underline{x})}{ds}\right)_i = \left(\frac{\nabla\underline{H}(\underline{x}).\frac{\partial\nabla\underline{H}(\underline{x})}{\partial x_i}}{\|\nabla\underline{H}(\underline{x})\|}\right)\frac{dx_i}{ds}$$

This seems unecessarily complicated when compared to simple ratio reduction of ds each time we find $\underline{x}$, $\underline{x}''$ satisfying 2.6.14.

Rodden has noted the problems which can occur when $\underline{x}^{(2)}$ lies on a saddle point of $V(\underline{x})$. i.e. $\nabla V(\underline{x}) = \underline{0}$. He selects the alternative surface to 2.6.1 given by

$$\dot{V}(\underline{x}) = -\varepsilon \qquad 2.6.15$$

for some small ε. He searches for the points of tangency of 2.6.15 with $V(\underline{x}) = p$. As Hewit notes this again seems unecessary as we are only interested in the direction of $\underline{H}(\underline{x})$ and this always points to the minimum. Stage b) is now complete.

Stage c) has a repeat of the problems of stage a) in 3 or more dimensions. To trace out 2.6.3 in more than 2 dimensions involves using particular planes in which we find curves satisfying 2.6.3 which we can somehow "piece" together to obtain a picture of the complete R.A.S..

Given $\underline{x}^{(2)}$ we may trace 2.6.3 in a plane containing $\underline{x}^{(2)}$. However if we require other plane intersections of R^n we have to track onto them first. A systematic method of building up a picture of 2.6.3 would be carried out in a similar way to that used in stage a) given by 2.6.11.

Since 2.6.3 is built up using planes we consider again the 2-dimensional situation. We suppose that we have a point (x,y) on V(x,y) = p where p is given, and we wish to find the next point (x',y') in a similar way to that used in stage b). The difference in stage c) is that we cannot determine a vector such as $\underline{H}(\underline{x})$ to indicate which direction $\underline{x}'$ lies as we want to trace out the whole of an (n-1)-dimensional surface. This is why we consider 2 dimensions only where we are constrained to look for $\underline{x}'$ in a particular direction.

Rodden uses a second order method to find (x',y') this time.

$$x' = x + \frac{dx}{ds}\,ds + \frac{1}{2}\frac{d^2x}{ds^2}(ds)^2 \qquad 2.6.16$$
$$y' = y + \frac{dy}{ds}\,ds + \frac{1}{2}\frac{d^2y}{ds^2}(ds)^2$$

Using the geometric properties of a fit such as 2.6.16 for small ds we obtain

$$\frac{dx}{ds} = \frac{-\frac{\partial V}{\partial y}}{\|\nabla V\|} \qquad 2.6.17$$
$$\frac{dy}{ds} = \frac{\frac{\partial V}{\partial x}}{\|\nabla V\|}$$

and

$$\frac{d^2x}{ds^2} = \frac{1}{R}\,\frac{\frac{\partial V}{\partial x}}{\|\nabla V\|} \qquad 2.6.18$$
$$\frac{d^2y}{ds^2} = \frac{1}{R}\,\frac{\frac{\partial V}{\partial y}}{\|\nabla V\|}$$

where $R = \dfrac{\|\nabla V\|}{\frac{d^2V}{ds^2}}$ 2.6.19

$$\text{and } \frac{d^2V}{ds^2} = \frac{\partial^2 V}{\partial x^2}\left(\frac{dx}{ds}\right)^2 + \frac{2\partial^2 V}{\partial x \partial y}\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + \frac{\partial^2 V}{\partial y^2}\left(\frac{dy}{ds}\right)^2 \qquad 2.6.20$$

Substituting 2.6.17 into 2.6.20 then 2.6.20 into 2.6.19, 2.6.19 into 2.6.18 and finally 2.6.17, 2.6.18 into 2.6.16 gives (x',y').

(x',y') will not lie on $V(x,y) = p$ and so a search identical to 2.6.13 is carried out along

$$x = x' + \lambda \frac{\partial V(x',y')}{\partial x}$$

$$y = y' + \lambda \frac{\partial V(x',y')}{\partial y}$$

to find (x'',y'') satisfying $V(x'',y'') = p$.

It is interesting that a second order method is used in stage c) and not for a similar process in stage b). The same method 2.6.16 to 2.6.20 could just as well be applied to stage b). If we consider the plane selected by some means for stage c) we see that the vector

$$\begin{bmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{bmatrix} \text{ is perpendicular to } \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix} .$$

Now in stage b) and equation 2.6.12 we see also that $\underline{H}(\underline{x})$ is perpendicular to $\nabla \dot{V}(\underline{x})$. Hence we may define a plane $P\underline{x}$ as the plane containing both $\underline{H}(\underline{x})$ and $\nabla \dot{V}(\underline{x})$ for any $\underline{x}$. Then by identifying $P\underline{x}$ in such a way that 2.6.16 to 2.6.20 can be evaluated we have a new $\underline{x}'$ and then we may use 2.6.13 to find $\underline{x}''$. The only difference in the use of the second order method for each stage is that $P\underline{x}$ changes for each step.

It seems that the extra computation required to compute $\underline{x}'$ using a second order method is not justified anyway. High order accuracy numerical methods are needed only where there is no back-up computation. But in this case we always iterate to get back on to either $\dot{V}(\underline{x}) = 0$ or $V(\underline{x}) = p$ and the accuracy of obtaining $\underline{x}'$ is more readily improved by reducing ds than by computing extra terms.

It seems on the whole that this method is more readily applied to 2 dimensions than higher orders. But in 2 dimension it seems better to evaluate r such that

$$V(r\cos\Theta, r\sin\Theta) = p \qquad 2.6.21$$

for various Θ then compute $\dot{V}(r\cos\Theta, r\sin\Theta)$ adjusting p according to whether 2.6.21 intersects $\dot{V}(r\cos\Theta, r\sin\Theta) = 0$. In this way we are assured of progressing systematically, while in Rodden's method we first try to find a point which may not exist and even if it does then is outside the R.A.S..

7. Other Lyapunov Methods

Other methods of constructing Lyapunov functions than by Zubov's method should be briefly mentioned.

Firstly, Ingwerson's method (31). This method relies on the principle that if

$$\dot{\underline{x}} = \underline{f}(\underline{x}) = A\underline{x} + \underline{g}(\underline{x}) \qquad 2.7.1$$

where $\dfrac{\|\underline{g}(\underline{x})\|}{\|\underline{x}\|} \to 0$ as $\underline{x} \to \underline{0}$

then $A = J(\underline{0})$

where $J(\underline{x})$ is the Jacobean matrix of $\underline{f}(\underline{x})$. So a Lyapunov function using the matrix $J(\underline{x})$ will indicate stability of the origin. We recall by theorem 1.6.7 that the origin of

$$\dot{\underline{x}} = A\underline{x}$$

is asymptotically stable if and only if there exist positive definite matrices B,C such that

$$A^T B + BA = -C. \qquad 2.7.2$$

The method of Ingwerson for obtaining a Lyapunov function for 2.7.1 is to solve the corresponding matrix equation

$$J(\underline{x})^T B(\underline{x}) + B(\underline{x})J(\underline{x}) = -C(\underline{x}) \qquad 2.7.3$$

for a positive definite $B(\underline{x})$ given a positive definite variable matrix $C(\underline{x})$. We may also recall that when 2.7.2 is solved we obtain a Lyapunov function

$$V(\underline{x}) = \underline{x}^T B\, \underline{x} \qquad 2.7.4$$

in which case $$\frac{\partial^2 V(\underline{x})}{\partial \underline{x}^2} = 2B. \qquad 2.7.5$$

We see from 2.7.4, 2.7.5 that if we can find $B(\underline{x})$ from 2.7.3 then we might hope to obtain a Lyapunov function by setting

$$\frac{\partial^2 V(\underline{x})}{\partial \underline{x}^2} = B(\underline{x}) \qquad 2.7.6$$

However the matrix $B(\underline{x})$ may not be the second derivative of a scalar function $V(\underline{x})$ with respect to $\underline{x}$. Clearly if there is such a $V(\underline{x})$ satisfying 2.7.6 then by positive definiteness of $B(\underline{x})$ we see that $V(\underline{x})$ is also a positive definite near the origin.

To ensure that $B(\underline{x})$ satisfies 2.7.6 for some $V(\underline{x})$ we need the condition

$$\frac{\partial B(\underline{x})_{i,j}}{\partial x_k} = \frac{\partial B(\underline{x})_{i,k}}{\partial x_j} \qquad 2.7.7$$

for $i, j, k = 1, \ldots, n$

$i \neq j,\ i \neq k.$

2.7.7 can be guaranteed by setting to zero all variables other than x_i, x_j in the element $B(\underline{x})_{i,j}$ $\quad i, j = 1, \ldots, n.$

Having carried out that operation on $B(\underline{x})$ it is just a matter of integrating 2.7.6 twice to obtain $V(\underline{x})$. The double integration to obtain $V(\underline{x})$ from $B(\underline{x})$ does not actually require 2.7.7 to be satisfied first, but clearly since we have a positive definite matrix $B(\underline{x})$ then it helps considerably if $B(\underline{x})$ is an exact second derivative in obtaining a $V(\underline{x})$ which is itself positive definite.

Schultz and Gibson (47) consider the system

$$\dot{\underline{x}} = A(\underline{x})\, \underline{x} \qquad 2.7.8$$

where $A(\underline{x})$ is in companion form with variable coefficients

$$A(\underline{x}) = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ a_1(\underline{x}) & a_2(\underline{x}) & \cdots & \cdots & & a_n(\underline{x}) \end{bmatrix}.$$

Now they set up a matrix $B(\underline{x})$ with certain restrictions given by

$$B_{n,n}(\underline{x}) = 2.$$

$$B_{i,i}(\underline{x}) > 0 \text{ and functions of } x_i \quad i=1,\ldots,n. \qquad 2.7.9$$

$$B_{i,j}(\underline{x}) \text{ are functions of } x_1,\ldots,x_{n-1} \quad i \neq j.$$

The matrix B is used to assign $\frac{\partial V}{\partial \underline{x}}$ where

$$\frac{\partial V(\underline{x})}{\partial \underline{x}} = B(\underline{x})\,\underline{x} \qquad 2.7.10$$

The restrictions 2.7.9 on $B(\underline{x})$ are to increase the possibility of $V(\underline{x})$ being positive definite near the origin. To fully determine the elements of $B(\underline{x})$ certain other processes need to be carried out.

First we require that $\dot{V}(\underline{x})$ is negative definite. This is done by investigating $\dot{V}(\underline{x})$ using 2.7.10 and 2.7.8 which becomes

$$\dot{V}(\underline{x}) = \underline{x}^T B(\underline{x})^T A(\underline{x})\,\underline{x} < 0. \qquad 2.7.11$$

Secondly we must ensure that 2.7.10 gives an exact gradient of a scalar function $V(\underline{x})$. The required condition is that

$$\text{curl}\,\frac{\partial V(\underline{x})}{\partial \underline{x}} = \underline{0} \qquad 2.7.12$$

With restrictions 2.7.9, 2.7.11 and 2.7.12 on $B(\underline{x})$ we may determine $B(\underline{x})$ and hence by line integration of 2.7.10 we obtain $V(\underline{x})$. This method has been generalised to solve non-companion systems in Schultz (48).

The method of Szego (32), (33) is similar to Zubov's method. He discovered that the origin of

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 2.7.13$$

is asymptotically stable within

$$\alpha^*(\underline{x}) \leq \delta \qquad 2.7.14$$

if there exist $V(\underline{x}), \phi(\underline{x}), \beta(V)$ such that

a) $V(\underline{0}) = 0.$

b) $\phi(\underline{x})$ is positive definite on trajectories of 2.7.13.

c) $\dot{V}(\underline{x}) = \underline{f}(\underline{x})^T \frac{\partial V(\underline{x})}{\partial \underline{x}} = \frac{-\phi(\underline{x})}{\beta(V)}.$

d) $\int_{0}^{V} \beta(t)dt < \infty$.

e) $\alpha^{*}(\underline{x}) = \int_{0}^{V} \beta(t)dt > 0$ for $\underline{x}$ satisfying 2.7.14

It can be seen that $\beta(V) \equiv 1$ converts Szego's method into Zubov's method.

8. Non-Lyapunov Methods

Finally, in this chapter, a mention should be made of other methods of finding regions of asymptotic stability or of integrating the system equations.

Davidson and Cowan (29) use a type of Lyapunov function for their method, but one in which the boundary of the D.O.A. is given by

$$V(\underline{x}) = 0.$$

Their method is most applicable to limit cycle systems where the function $V(\underline{x})$ is defined by

$$V(\underline{x}) = \|\underline{x}(t)\| - \|\underline{x}(t - t')\| \qquad 2.8.1$$

where $\underline{x}(t - t') = c\underline{x}(t)$ for some scalar constant c. This poses the immediate problem of how to decide what t' is. Davidson and Cowan integrate

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 2.8.2$$

by the fourth order Runge-Kutta method using negative time. Now in 2 dimensions it is relatively easy to decide when the origin has been completely encircled. This can be done by using polar co-ordinates and terminating computation when

$$\theta = \theta_0 \pm 2\pi.$$

However in 3 or more dimensions the system trajectories are 1-dimensional curves in n-dimensional space and a lot of computer logic must be required to determine whether or not a complete rotation has been achieved, and even then this only establishes a 1-dimensional section of an (n-1)-dimensional boundary. When the D.O.A. of 2.8.2 is not a limit-cycle Davidson and Cowan still use the function $V(\underline{x})$ given by 2.8.1 but with t' fixed. This leads to problems when considering examples with trajectories which may tend to diverge from the origin initially but are asymptotically stable. The Hahn example is characteristic of this

$$\dot{x} = -x + 2x^2y \qquad 2.8.3$$
$$\dot{y} = -y$$

The solutions of 2.8.3 are given by

$$x(t) = \frac{x_o e^{-t}}{1 - x_o y_o (1 - e^{-2t})}$$
$$y(t) = y_o e^{-t}.$$

If we let the initial point be chosen on the line $y = x$ and so write

$$x_o = y_o = a$$

and then let $u = e^{-2t'}$ where t' is fixed we see that $V(\underline{x}) = 0$ when

$$a^2 = \frac{1 \mp \left(\frac{u}{2-u}\right)^{\frac{1}{2}}}{1-u} \qquad 2.8.4$$

Fig. 11 shows the relationship between a and t' for negative t'. Clearly the more negative the value of t' chosen the greater the value of a and for $t' < -\frac{1}{2}\log 2$ then $V(a,a) < 0$ for all a. The general formula for any (x_o, y_o) corresponding to 2.8.4 is given by

$$r^2 = \frac{1 \mp \left(\frac{u \cos^2\theta}{1 - u \sin^2\theta}\right)^{\frac{1}{2}}}{\sin\theta \cos\theta (1-u)}$$

$x_o = r \cos\theta$, $y_o = r \sin\theta$.

For $u > 1/\sin^2\theta$ we see that $V(r\cos\theta, r\sin\theta) < 0$ for all r and we see that Davidson and Cowan's method is not really applicable in this form.

Texter (34) in his paper gives 3 methods of finding an initial point on the boundary of the D.O.A.. The first of these transforms the Zubov equation to polar co-ordinates and solves it numerically. This is the subject of an extensive study in Chapter 5. His second suggestion is to inspect the system equations which when it can be done is always better than any computation.

The third way Texter gives is that of integrating the system equations from an initial point

$$\underline{x}(0) = c\underline{x}_o$$

for various c to obtain the maximum c for which the trajectories are stable. This is similar to Davidson and Cowan's method except that Texter does not say specifically how he decides whether a trajectory is stable, and indeed neither author says how we alter $\underline{x}_o$ if we do not find a point on the boundary of the D.O.A..

Once a point $c\underline{x}_o$ has been found on the boundary of the D.O.A. the second part of Texter's algorithm is integration of the system trajectories from this initial point. He uses the Euler method of solving the O.D.E. given by (35)

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \qquad 2.8.5$$

which in step form is

$$y_{m+1} = y_m + \frac{\delta x \; g(x_m,y_m)}{f(x_m,y_m)} \qquad 2.8.6$$
$$x_{m+1} = x_m + \delta x, \quad m = 0,1,\ldots$$

Euler's method is only first order accurate and O.D.E.s can be solved by much more accurate methods. Also to integrate the system trajectories for positive t means that if the origin is stable the boundary is unstable and computation of the boundary will diverge from it as errors accumulate.

Finally it can be seen from 2.8.5 and 2.8.6 that the criticism of Davidson and Cowan's method that it is not applicable to higher orders is just as valid here.

The previous two methods have each involved to some extent numerical computation of the system equations either for $\underline{x}$ as a function of t or for relationships between the x_i's $i = 1,\ldots,n$. The methods require fixed step-sizes for either t or one of the x_i's.

However Fox (36) proposes that O.D.E.s such as 2.8.2 should be solved using characteristics. Characteristics were introduced in section 8 of Chapter 1 and as was mentioned there we are not specifically interested in any characteristic curves in this thesis except the one which forms the boundary of the D.O.A..

But Davidson and Cowan's idea of computing the system equations for negative time opens up another possibility. If we choose initial conditions given by

$$\|\underline{x}_o\| = \varepsilon \tag{2.8.7}$$

and then integrate numerically along the characteristics we should obtain a family of curves the envelope of which is the D.O.A.. Kormanik and Li (26) used this idea with Taylor series but here we consider finite difference approximations along the characteristics as suggested by Fox.

Suppose we have the system of equations

$$\dot{x}_i = f_i(\underline{x}) \qquad i = 1,\dots,n$$

then the characteristics are given by

$$\frac{dx_1}{f_1(\underline{x})} = \frac{dx_2}{f_2(\underline{x})} = \dots\dots\dots = \frac{dx_n}{f_n(\underline{x})} \tag{2.8.8}$$

Likewise we may consider the characteristics of a P.D.E. such as the Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i} = -\phi(\underline{x})(1 - dV). \tag{2.8.9}$$

The characteristics of 2.8.9 are given by 1.8.1 which is reproduced here

$$\frac{dx_1}{f_1(\underline{x})} = \frac{dx_2}{f_2(\underline{x})} = \dots\dots\dots = \frac{dx_n}{f_n(\underline{x})} = \frac{-dV}{\phi(\underline{x})(1-dV)} . \tag{2.8.10}$$

If we solve 2.8.8 from initial conditions 2.8.7 we should obtain the desired family of curves. If we solve 2.8.10 instead we should also obtain the variation of V along each characteristic. Theoretically we could integrate 2.8.10 until V reaches a certain prescribed value as advocated by Kormanik and Li, but to illustrate the method we consider 2.8.8 only and in 3 dimensions.

The characteristics of a 3-dimensional system may be written

$$\frac{dx}{f(x,y,z)} = \frac{dy}{g(x,y,z)} = \frac{dz}{w(x,y,z)} \tag{2.8.11}$$

Fox proposes that 2.8.11 is converted into the 2 equations

$$w(x,y,z)dx = f(x,y,z)dz$$
$$g(x,y,z)dx = f(x,y,z)dy \qquad 2.8.12$$

or any other 2 similar equations from 2.8.11. He then uses the Trapezium rule of numerical integration which when applied to 2.8.12 gives

$$\tfrac{1}{2}(w(x_i,y_i,z_i) + w(x,y,z))(x-x_i) = \tfrac{1}{2}(f(x_i,y_i,z_i) + f(x,y,z))(z-z_i)$$
$$\tfrac{1}{2}(g(x_i,y_i,z_i) + g(x,y,z))(x-x_i) = \tfrac{1}{2}(f(x_i,y_i,z_i) + f(x,y,z))(y-y_i) \qquad 2.8.13$$

$i = 0,1,\ldots$

We see that 2.8.13 represents two equations for x,y,z in terms of x_i,y_i,z_i which is the last point calculated on the characteristic. Now we may select any one of x,y,z and let $x = x_{i+1}$ or $y = y_{i+1}$ or $z = z_{i+1}$. By solving 2.8.13 we then obtain y_{i+1},z_{i+1} for a chosen x_{i+1} and likewise for any other way round. This has the advantage over methods which fix say x_{i+1} which have difficulties when

$$\frac{dx}{dy} \approx 0$$

or

$$\frac{dx}{dz} \approx 0.$$

We have no way of knowing beforehand which way a characteristic will go in the (x,y,z) space and we may prefer to choose x_{i+1},y_{i+1},z_{i+1} subject to a condition limiting the line displacement such as

$$(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2 + (z_{i+1}-z_i)^2 \leqslant c^2$$

$i = 0,1,2\ldots.$

Fox's method is simple to apply but if the system equations can be integrated numerically in this way, it is possible to use much more accurate methods than the trapezium rule.

9. Conclusions and Other Possibilities

Various work on solving Zubov's equation or of obtaining Lyapunov functions and corresponding regions of stability, and some other ways of finding D.O.A.s have all been presented here. This is nôt an exhaustive list of work done in these and relative fields but they serve as background material for developing other techniques of solving the Zubov equation.

The Zubov methods examined have had some difficulties particularly of non-convergence and it is hoped to overcome these problems in later chapters.

Other possibilities for solving the Zubov equation may include:

a) Transforming it to an O.D.E. using Green's functions.

b) Using variational techniques to determine contours of $V(\underline{x})$.

c) Using the analytic solution of

$$\dot{\underline{x}} = A\underline{x} + \underline{g}(\underline{x})$$

given by $x(t) = \Phi(t)\underline{x}_0 + \Phi(t)\int_0^t \Phi(-s)\underline{g}(\underline{x}(s))ds$ 2.9.1

where $\Phi(t)\underline{x}_0$ is a solution of

$$\dot{\underline{x}} = A\underline{x} \quad \text{such that} \quad \Phi(0) = I$$

and maybe some iteration for the integral part of 2.9.1.

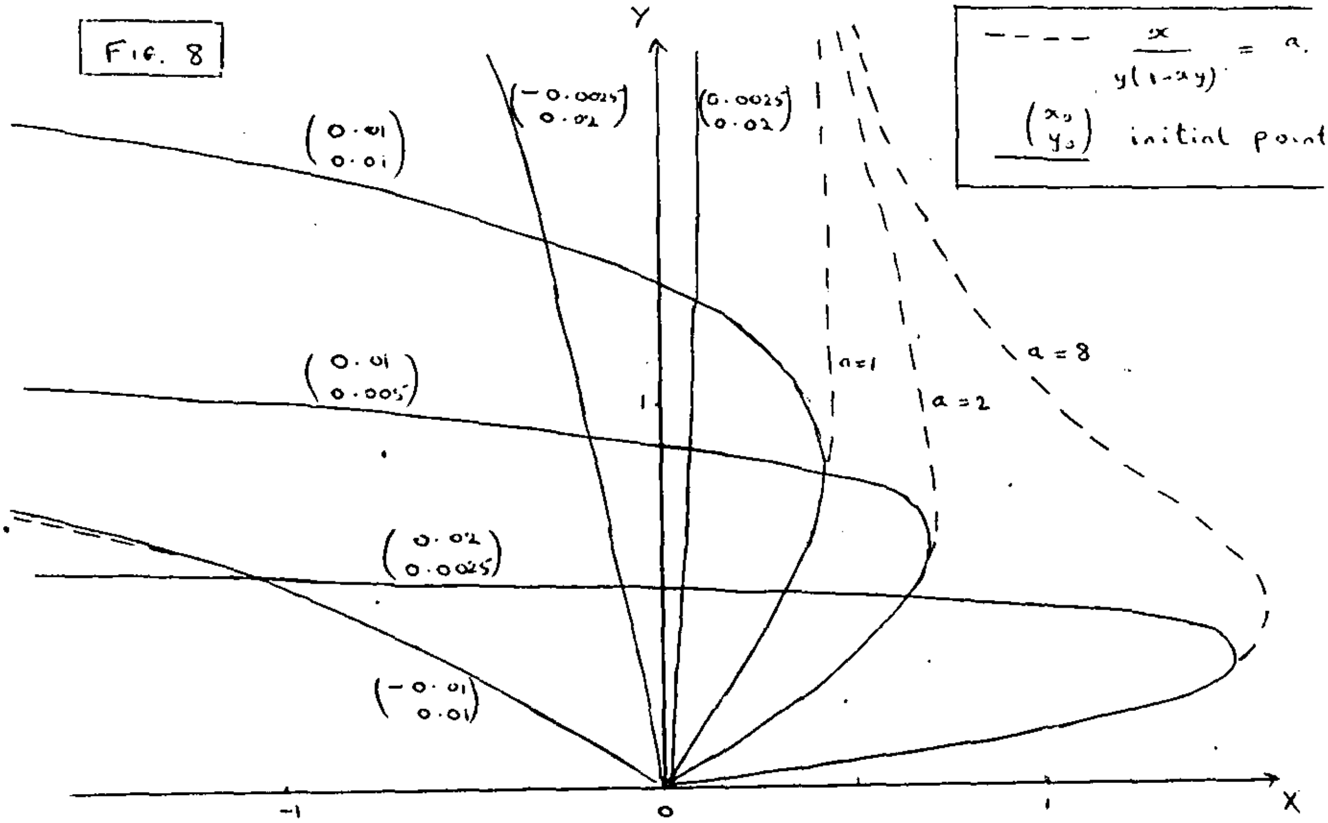


FIG. 8

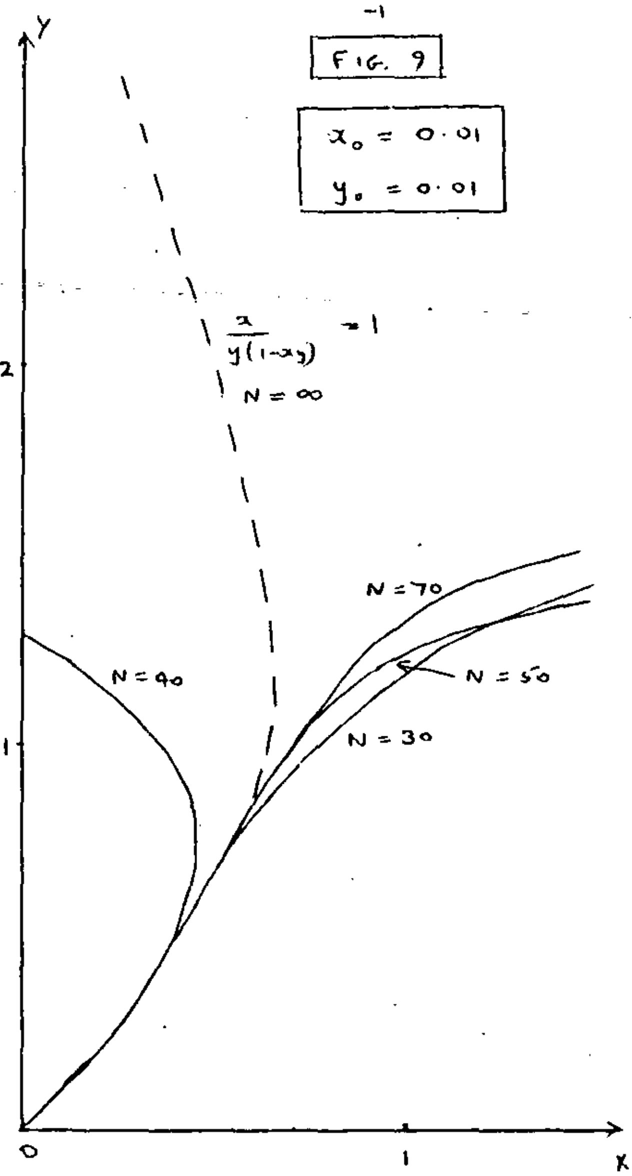


FIG. 9

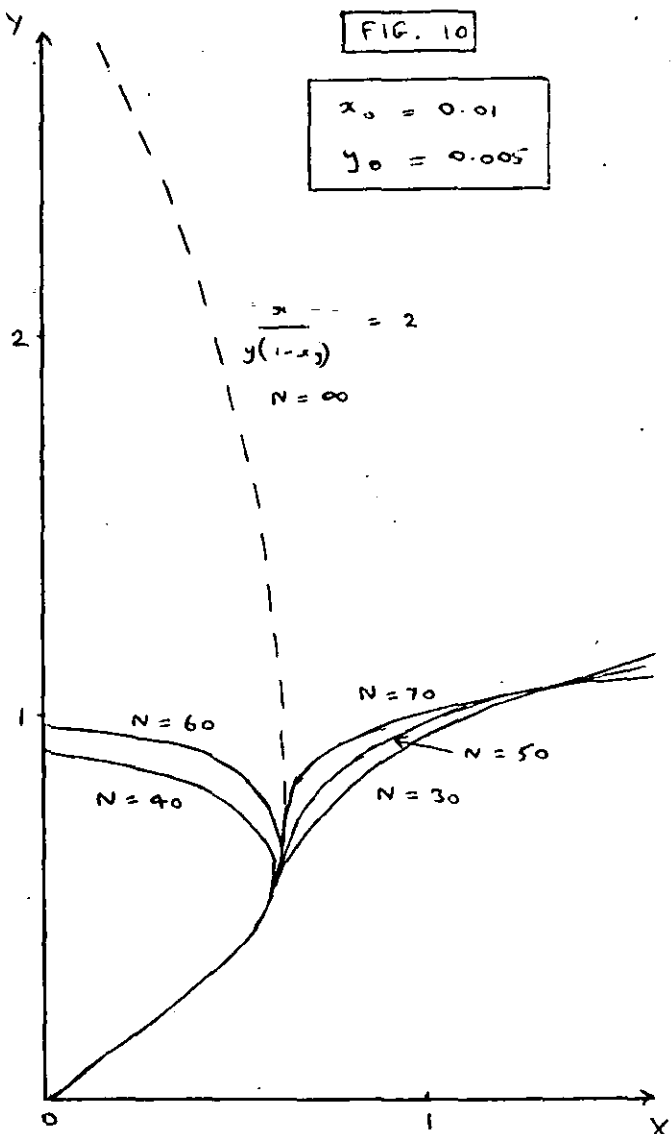
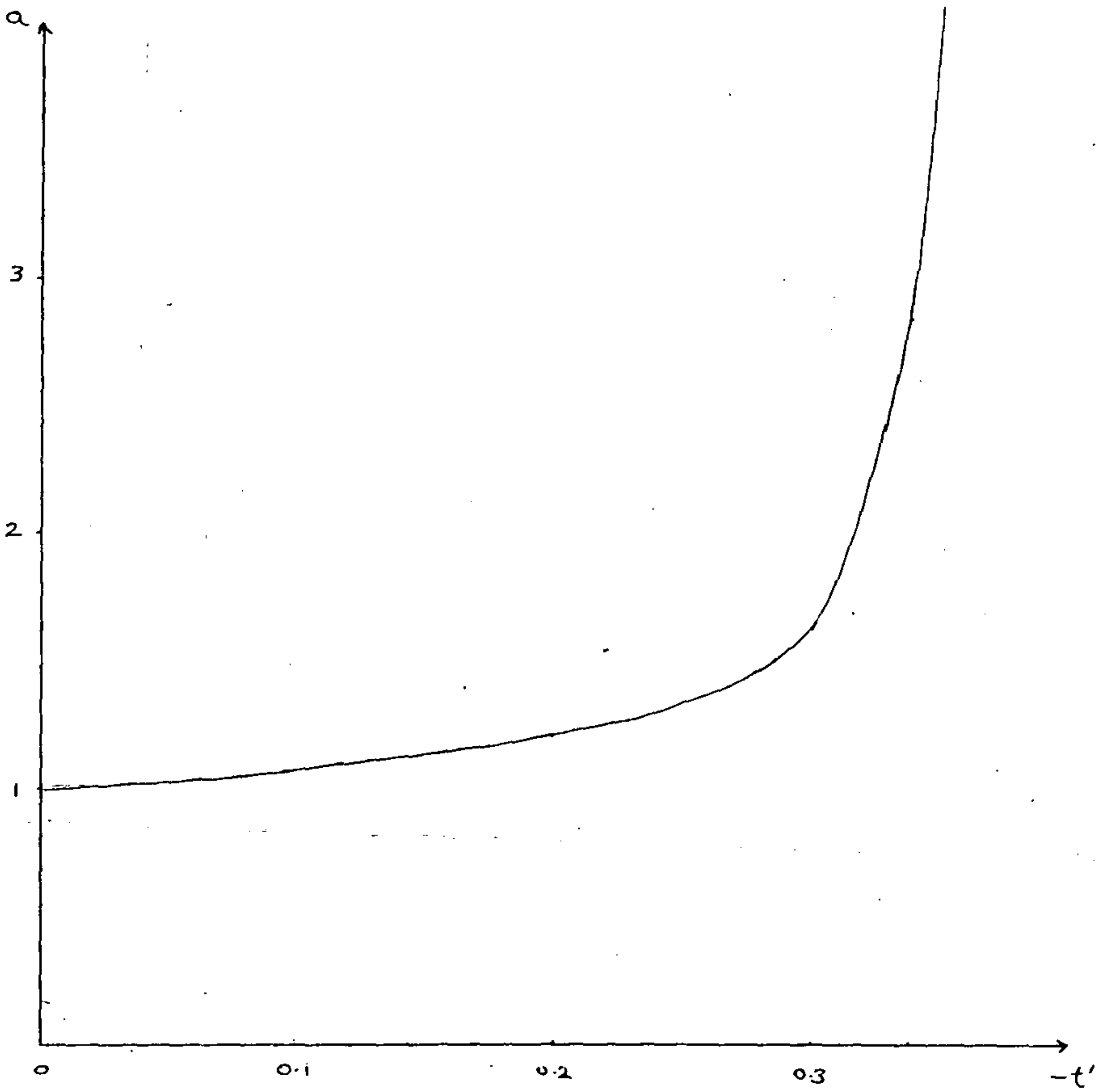


FIG. 10

$$\boxed{\dot{x} = -x + 2x^2 y \qquad \dot{y} = -y}$$



$$a^2 = \frac{\left(\frac{u}{2-u}\right)^{1/2} - 1}{u - 1} \quad \text{where} \quad u = e^{-2t'}$$

FIG. 11

Chapter 3

Theory of Solution of Zubov's Equation.

1. Introduction.

Computation of D.O.A.s by using numerical integration techniques on the Zubov equation will be attempted in the three chapters following this one. The theoretical basis of such computation is analysed here. In this chapter the definitions used for later work are made. Some interesting new results on the Zubov equation are also presented.

It seems to be generally accepted that if the Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 3.1.1$$

can be solved for a positive definite $\phi(\underline{x})$ then the D.O.A. will be obtained by setting $V = 1$ or $V = \infty$. In this chapter it is shown that $\phi(\underline{x})$ cannot always be freely chosen and even that $\underline{f}(\underline{x})$ may be such that the D.O.A. cannot be obtained for any $\phi(\underline{x})$.

In section 2 some definitions are set out followed by further theorems on positive definite functions to supplement those in section 5 of Chapter 1. In section 3 we take a look at the relationship of the Zubov equation in polar co-ordinates to the rectangular co-ordinate version 3.1.1. Section 4 gives some interesting results on systems which are symmetric about the origin. In section 5 the method of solving P.D.E.s by the auxiliary equation or characteristics method is investigated as a concept in order to throw light on the relationship between $\underline{x}$, V, ϕ, t. In section 6 we consider the choice of $\phi(\underline{x})$ and whether some $\phi(\underline{x})$ are not admissible as far as theorem 1.7.1 is concerned. Section 7 covers the situations where the Zubov equation may be soluble but the D.O.A. not indicated by the corresponding Lyapunov function.

2. Degree of Functions

In Chapter 1 some definitions on positive definite functions were given. These definitions mainly concentrated on the general properties of such functions. Theorem 1.5.1 was concerned with a particular class of functions, namely quadratic functions. Following theorem 1.5.3 it was stated that a positive definite decrescent function could be approximated by its lowest order terms near the origin. We now attempt to define and establish some results on this subject.

Definition 3.2.1

The asymptotic degree $P(\underline{f})$ of an n-dimensional continuous function on an n-dimensional vector space is such that

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} \longrightarrow c(\Gamma) \text{ as } \underline{x} \longrightarrow \underline{0}$$

along a general continuous path Γ in R^n where $c(\Gamma)$ exists and is finite for all paths and non-zero for at least one path.

For example suppose we define the function $\underline{f}(\underline{x})$ in 2 dimensions by

$$\underline{f}(\underline{x}) = \begin{bmatrix} x_1 + x_2^2 \\ x_1^2 x_2^2 + x_1^3 \end{bmatrix} \tag{3.2.1}$$

Then $$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} = \frac{((x_1+x_2^2)^2 + (x_1^2x_2^2 + x_1^3)^2)^{\frac{1}{2}}}{(x_1^2 + x_2^2)^{P(\underline{f})/2}} \tag{3.2.2}$$

By inspection of 3.2.2 as $\underline{x} \longrightarrow \underline{0}$ we see that

$$P(\underline{f}) = 1 \tag{3.2.3}$$

which gives a limit $$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|} \rightarrow \cos\theta \tag{3.2.4}$$

for paths Γ which approach the origin along

$$\frac{x_2}{x_1} = \tan\theta.$$

3.2.3 and 3.2.4 satisfy the conditions of definition 3.2.1 for the system 3.2.1.

Definition 3.2.2

The asymptotic degree $P(V)$ of a scalar continuous function $V(\underline{x})$ on an n-dimensional vector space is such that

$$\frac{V(\underline{x})}{\|\underline{x}\|^{P(V)}} \longrightarrow c(\Gamma) \text{ as } \underline{x} \to \underline{0}$$

and $\underline{x}$ varies along a general continuous path Γ in R^n where $c(\Gamma)$ exists and is finite for all paths and non-zero for at least one path.

Now we need to show that these definitions are consistent. The next two theorems establish that.

Theorem 3.2.1

$$P(\underline{f}) = \min(P(f_1), P(f_2), \ldots, P(f_n))$$

where $\underline{f}(\underline{x})^T = (f_1(\underline{x}), f_2(\underline{x}), \ldots, f_n(\underline{x}))$.

Proof

$$\text{Let } P(f_j) = s_j \qquad j = 1, \ldots, n$$
$$P(f) = s$$

and suppose there exists in contradiction to the theorem a j such that $s_j < s$.

We use the definition of the Euclidean norm of $\underline{f}(\underline{x})$ which is

$$\|\underline{f}(\underline{x})\| = \left(\sum_{i=1}^{n} f_i(\underline{x})^2\right)^{\frac{1}{2}} \qquad 3.2.5$$

Dividing 3.2.5 by $\|\underline{x}\|^s$ gives

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^s} = \left(\sum_{i=1}^{n}\left(\frac{f_i(\underline{x})}{\|\underline{x}\|^s}\right)^2\right)^{\frac{1}{2}} \qquad 3.2.6$$

By definition 3.2.2 we know that

$$\frac{f_j(\underline{x})}{\|\underline{x}\|^{s_j}} \longrightarrow c(\Gamma) \quad \text{as } \underline{x} \to \underline{0}$$

where $c(\Gamma)$ exists and is finite for all continuous Γ and non-zero for some Γ. Let Γ' be a path for which $c(\Gamma')$ is non-zero.

$$\text{Hence } \frac{f_j(\underline{x})}{\|\underline{x}\|^s} = \frac{f_j(\underline{x})\,\|\underline{x}\|^{s_j-s}}{\|\underline{x}\|^{s_j}} \longrightarrow c(\Gamma')\|\underline{x}\|^{s_j-s} \qquad 3.2.7$$

as $\underline{x} \to \underline{0}$ along Γ'. Now if $s_j < s$ then we see that the limit as $\underline{x} \to \underline{0}$ of 3.2.7 is infinite and so also is the limit of 3.2.6 infinite upon substituting 3.2.7 into 3.2.6. Hence we have established by definition 3.2.1 that

$$P(\underline{f}) \neq s$$

where $s > s_j$ for some $j = 1,\dots,n$.
Now suppose also in contradiction to the theorem that $s < s_i$, $i = 1,\dots,n$.
This time

$$\frac{f_i(\underline{x})}{\|\underline{x}\|^{s}} = \frac{f_i(\underline{x})}{\|\underline{x}\|^{s_i}} \|\underline{x}\|^{s_i - s} \longrightarrow c(\Gamma)\|\underline{x}\|^{s_i - s} \qquad 3.2.8$$

as $\underline{x} \to \underline{0}$ along any continuous path Γ. Now if $s_i > s$ then we see that the limit as $\underline{x} \to \underline{0}$ of 3.2.8 is zero for all paths Γ in R^n. Substituting 3.2.8 into 3.2.6 and letting $\underline{x} \to \underline{0}$ we see that

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{s}} \longrightarrow 0 \text{ as } \underline{x} \to \underline{0}$$

along any continuous path Γ. Now we have established by definition 3.2.1 that

$$P(\underline{f}) \neq s$$

where $s < s_i$, $i = 1,\dots,n$.

Having eliminated all other possibilities we are left with the result

$$s = \min(s_1, s_2, \dots, s_n).$$

End of proof.

<u>Theorem 3.2.2</u>

$P(\underline{f})$ and $P(V)$ are unique for a particular $\underline{f}(\underline{x})$ or $V(\underline{x})$.

<u>Proof</u>

Suppose in contradiction that there exist s_1, s_2 both satisfying definition 3.2.2 for $V(\underline{x})$.

i.e. $$\frac{V(\underline{x})}{\|\underline{x}\|^{s_1}} \longrightarrow c_1(\Gamma) \text{ as } \underline{x} \to \underline{0}$$

and $$\frac{V(\underline{x})}{\|\underline{x}\|^{s_2}} \longrightarrow c_2(\Gamma) \text{ as } \underline{x} \to \underline{0}$$

and $\underline{x}$ varies along a general continuous path Γ in R^n. $c_1(\Gamma)$, $c_2(\Gamma)$ both exist and are finite and each is non-zero for some Γ. Without loss of generality we may assume that $s_1 > s_2$.

Now $$\frac{V(\underline{x})}{\|\underline{x}\|^{s_2}} = \frac{V(\underline{x})}{\|\underline{x}\|^{s_1}}\|\underline{x}\|^{s_1 - s_2} \longrightarrow c_1(\Gamma)\|\underline{x}\|^{s_1 - s_2} \qquad 3.2.9$$

Now since $c_1(\Gamma)$ exists and is finite, and $s_1 > s_2$ we see that the limit of 3.2.9 as $\underline{x} \to \underline{0}$ is zero for any continuous Γ.

i.e. $$\frac{V(\underline{x})}{\|\underline{x}\|^{s_2}} \longrightarrow 0 \text{ as } \underline{x} \to \underline{0}.$$

Hence by definition 3.2.2 we have shown that

$$P(V)\ P(\underline{f}) \neq s_2.$$

This shows that the assumption that s_1 and s_2 are different is false.

For the vector function $\underline{f}(\underline{x})$ we see that since $P(f_i)$, $i = 1,\ldots,n$, are each unique then by definition 3.2.2 and theorem 3.2.1, so is $P(\underline{f})$. End of proof.

With these definitions and theorems we may consider the properties of a positive definite function near the origin. Future analysis is made easier if we can evaluate $P(\underline{f})$, $P(V)$ by considering only straight line paths. In that case we could replace $\underline{x}$ by $(r,\underline{\theta})$, $\|\underline{x}\|$ by r and $c(\Gamma)$ by $c(\underline{\theta})$ in definitions 3.2.1 and 3.2.2. We thus require to prove that if we considered only straight line paths in definitions 3.2.1 and 3.2.2 $P(\underline{f})$, $P(V)$ would be unaltered.

This result needs proving in two stages, the first of which is on the 2-dimensional version.

<u>Theorem 3.2.3</u>

$$\frac{V(r,\theta)}{r^{P(V)}} \longrightarrow c(\theta) \text{ as } r \to 0 \qquad 3.2.10$$

in two dimensions along any line of constant θ where (r,θ) is the polar co-ordinate location of the point $\underline{x}$, $c(\theta)$ exists and is finite for all θ and is non-zero for at least one θ.

<u>Proof</u>

Suppose first that $c(\theta)$ is infinite or non-existent for some θ. Then since straight line paths are a subset of all possible paths then we see that 3.2.10 contradicts definition 3.2.2.

Now suppose that $c(\theta) = 0$ for all θ. However we know by definition 3.2.2 that there is a path Γ' for which

$$\frac{V(r,\theta)}{r^{P(V)}} \longrightarrow c(\Gamma') \neq 0 \text{ as } r \to 0 \qquad 3.2.11$$

and (r,θ) varies along Γ'. Since Γ' is continuous we may write Γ' as a relationship between the variables in a parametric form

$$r = r(z)$$
$$\theta = \theta(z) \qquad 3.2.12$$

where z varies continuously along Γ' and $r(z),\theta(z)$ are continuous with respect to z, and $r(z) \to 0$ as $z \to z'$ where z' may be finite or infinite. Now if as $r \to 0$ we have from 3.2.12 the result $\theta \to \theta'$ then we may use a result of continuous functions such as in Rudin (49) which implies that

$$c(\Gamma') \to c(\theta') \text{ as } \theta \to \theta'. \qquad 3.2.13$$

Hence comparing 3.2.11 and 3.2.13 we see that $c(\theta') \neq 0$ which is a contradiction.

Finally let us suppose again that $c(\theta) = 0$ for all θ and Γ' is given by 3.2.11, 3.2.12 but θ has no limit as $r \to 0$ in 3.2.12.

However since the path 3.2.12 tends to the origin we see that $\theta(z)$ exists and is continuous for all $r(z)$ as $r(z) \to 0$.

Hence there exists a θ' such that for all $\delta > 0$ there exist z, ε such that $\varepsilon < \delta$ and

$$r(z) = \varepsilon$$
$$\theta(z) = \theta' + 2m\pi \qquad 3.2.14$$

where m is an integer.

3.2.14 indicates not only that $(r(z),\theta(z))$ is on the path given by 3.2.12 but also that by reference to Appendix D $(r(z),\theta')$ is the same point in R^2 as $(r(z), \theta(z))$. So we may define an infinite sequence z_j, r_j such that

$$r_j = r(z_j) \to 0 \text{ as } j \to \infty$$

and (r_j,θ') lies on the path 3.2.12.

Now we have a contradiction since by theorem 4.2 of Rudin (49) we see that

$$\frac{V(r_j,\theta')}{r_j^{P(V)}} \to c(\Gamma') \neq 0 \text{ as } j \to \infty$$

as the sequence (r_j,θ') lies on Γ' while

$$\frac{V(r_j,\theta')}{r_j^{P(V)}} \to c(\theta') = 0 \text{ as } j \to \infty$$

as the sequence (r_j, θ') lies on the straight line $\theta = \theta'$.

Hence there exists θ' such that $c(\theta') \neq 0$.

End of proof.

Theorem 3.2.4

$$\frac{V(r,\underline{\theta})}{r^{P(V)}} \dot{\longrightarrow} c(\underline{\theta}) \text{ as } r \to 0$$

along any line of constant $\underline{\theta}$ where $(r,\underline{\theta})$ is the polar co-ordinate location of the point $\underline{x} \in R^n$, $c(\underline{\theta})$ exists and is finite for all $\underline{\theta}$ and is non-zero for at least one $\underline{\theta}$.

Proof

As with theorem 3.2.3 we see that $c(\underline{\theta})$ cannot be infinite or non-existent.

To prove that $\underline{\theta}'$ exists such that $c(\underline{\theta}')$ is non-zero we use induction. We suppose that $c(\underline{\theta}) = 0$ for all paths constrained such that $\theta_k = \theta_k^{\dagger}$, $k = 1,\ldots,i$, for some values of θ_k' but there exists a path Γ' constrained only such that $\theta_k = \theta_k'$ $k = 1,\ldots,i-1$ on which $c(\Gamma')$ is non-zero. $i = 1,\ldots,n-1$.

As in theorem 3.2.3 we may write the path Γ' in the form

$$\begin{aligned} r &= r(z) \\ \underline{\theta} &= \underline{\theta}(z) \end{aligned} \qquad 3.2.15$$

and establish similarly that there exists a θ_i' such that for all $\delta > 0$ there exist z, ε such that $\varepsilon < \delta$ and

$$\begin{aligned} r(z) &= \varepsilon \\ \theta_k(z) &= \theta_k' \qquad k = 1,\ldots,i-1 \\ \theta_i(z) &= \theta_i' + 2m_i\pi \end{aligned}$$

where m_i is an integer. We again define an infinite sequence z_j, r_j such that

$$r_j = r(z_j) \longrightarrow 0 \text{ as } j \to \infty$$

and $(r_j,\underline{\theta})$ lies on the path 3.2.15 with $\theta_k = \theta_k'$ $k = 1,\ldots,i$.

By theorem 4.2 of Rudin (49) we see that

$$\frac{V(r_j,\underline{\theta})}{r_j^{P(V)}} \longrightarrow c(\Gamma') \neq 0 \text{ as } j \to \infty$$

with $\theta_k = \theta_k'$, $k = 1,\ldots,i$, as the sequence $(r_j,\underline{\theta})$ lies on Γ' while

$$\frac{V(r_j,\underline{\theta})}{r_j^{\,P(V)}} \longrightarrow c(\Gamma) = 0 \text{ as } j \to \infty$$

where $\theta_k = \theta_k'$, $k = 1,\dots,i$ as the sequence $(r_j,\underline{\theta})$ lies in the region constrained such that $\theta_k = \theta_k'$, $k = 1,\dots,i$. This contradiction proves that if there exists a path Γ' constrained such that $\theta_k = \theta_k'$, $k = 1,\dots,i-1$, on which $c(\Gamma') \neq 0$ then there also exists Γ'' constrained on $\theta_k = \theta_k'$, $k = 1,\dots,i$, on which $c(\Gamma'') \neq 0$. Therefore if there exists a path Γ in R^n on which $c(\Gamma) \neq 0$ then there exists a path Γ on $\theta_1 = \theta_1'$ such that $c(\Gamma) \neq 0$ and by successive induction there exists a path Γ' constrained on $\theta_k = \theta_k'$, $k = 1,\dots,n-1$, such that $c(\Gamma') \neq 0$. Since Γ' is the line $\underline{\theta} = \underline{\theta}'$ we see that $c(\underline{\theta}') \neq 0$. End of proof.

By theorem 3.2.4 we may concentrate on straight line paths through the origin in order to evaluate $P(V)$, $P(\underline{f})$. This is useful in the next theorem:

Theorem 3.2.5

If $V(\underline{x})$ is given by a series expansion

$$V(\underline{x}) = \sum_{m=s}^{\infty} V_m(\underline{x}) \qquad 3.2.16$$

where the terms of $V_m(\underline{x})$ are homogeneous of degree m, then $P(V) = S$.

Proof

To prove this theorem we need to change 3.2.16 into polar co-ordinate form. The generalised transformation of $\underline{x}$ into $(r,\underline{\theta})$ is shown in Appendix D where the basic equations are

$$x_1 = r\cos\theta_1$$

$$x_i = r\sin\theta_1 \dots\dots \sin\theta_{i-1}\cos\theta_i$$

$$i = 2,\dots,n-1 \qquad 3.2.17$$

$$x_n = r\sin\theta_1 \dots\dots \sin\theta_{n-2}\sin\theta_{n-1}.$$

Since all terms in $V_m(\underline{x})$ are homogeneous terms of total power m we see that each term in $V_m(\underline{x})$ becomes a function of $\underline{\theta}$ multiplied by r^m.

i.e. $V_m(\underline{x}) \equiv r^m\, \Theta_m(\underline{\theta}) \qquad m = s, s+1, \dots \qquad 3.2.18$

Substituting 3.2.18 into 3.2.16 gives

$$V(r,\underline{\theta}) = \sum_{m=s}^{\infty} r^m\, \Theta_m(\underline{\theta}). \qquad 3.2.19$$

Now dividing 3.2.19 by $\|\underline{x}\|^s$ or equivalently by r^s we obtain the result

$$\frac{V(r,\underline{\theta})}{r^s} = \sum_{m=s}^{\infty} r^{m-s} \Theta_m(\underline{\theta}). \qquad 3.2.20$$

Now we may fix $\underline{\theta}$ and let $r \to 0$ in 3.2.20 giving

$$\frac{V(r,\underline{\theta})}{r^s} \longrightarrow \Theta_s(\underline{\theta}) + O(r) \text{ as } r \longrightarrow 0. \qquad 3.2.21$$

Since $\Theta_s(\underline{\theta}) \not\equiv 0$ (by definition of s) we see that there exists some $\underline{\theta}'$ for which $\Theta_s(\underline{\theta}') \neq 0$. By definition 3.2.2 and theorem 3.2.4 we see from 3.2.21 that

$$P(V) = s.$$

End of proof.

Theorem 3.2.6

$V(\underline{x})$ is positive definite in an S_ε neighbourhood of the origin for some $\varepsilon > 0$ if

$$V(\underline{x}) = \sum_{m=s}^{\infty} V_m(\underline{x}) \qquad 3.2.22$$

where the terms of $V_m(\underline{x})$ are homogeneous of total degree m and where $s \geqslant 2$ and s even and $V_s(\underline{x})$ is positive definite. $V(\underline{x})$ is not positive definite if s odd or $s \leqslant 0$.

Proof

The proof is based on equation 3.2.21. First let s be odd and let $\underline{\theta}'$ be such that $\Theta_s(\underline{\theta}') \neq 0$. Denote by $\underline{x}'$ any point where

$$\underline{x}' \equiv (r,\underline{\theta}') \qquad 3.2.23$$

and let $\underline{\theta}''$, $\underline{x}''$ be such that

$$\underline{x}'' \equiv (r,\underline{\theta}'')$$
$$\underline{x}'' = -\underline{x}'. \qquad 3.2.24$$

Now if $\Theta_s(\underline{\theta}') < 0$ then by 3.2.21 we see that for some $\varepsilon > 0$ and every $r < \varepsilon$, $V(r,\underline{\theta}') < 0$. While if $\Theta_s(\underline{\theta}') > 0$ then similarly for all $r < \varepsilon$ we know that $V(r,\underline{\theta}') > 0$. Now since s is odd we know that

$$V_s(-\underline{x}) = -V_s(\underline{x}) \qquad 3.2.25$$

for all $\underline{x} \in R^n$. Hence by putting m = s in 3.2.18 we see that 3.2.25 implies that

$$\Theta_s(\underline{\theta}'') = -\Theta_s(\underline{\theta}')$$

where $\underline{\theta}'$, $\underline{\theta}''$ are related by 3.2.23, 3.2.24. So if $\Theta_s(\underline{\theta}') > 0$ then $\Theta_s(\underline{\theta}'') < 0$ and we may find some $\varepsilon > 0$ such that for all $r < \varepsilon$ we have $V(r,\underline{\theta}'') < 0$. Hence we have proved for s odd that there are points $\underline{x}$ in every neighbourhood of the origin where $V(\underline{x}) < 0$.

Now if $s \leq 0$ then by 3.2.21 we see that $V(\underline{x})$ becomes infinite at the origin if $r \to 0$ along a line $\underline{\theta} = \underline{\theta}'$ where $\Theta_s(\underline{\theta}') \neq 0$. Hence $V(\underline{0}) \neq 0$ and by definition 1.5.1 this means that $V(\underline{x})$ is not positive definite.

Finally if s even and $s \geq 2$ then we know that

$$V_s(-\underline{x}) = V_s(\underline{x}).$$

Now if there exists $\underline{\theta}'$ such that $\Theta_s(\underline{\theta}') < 0$ then by 3.2.21 and letting $r \to 0$ we see again that there exists $\varepsilon > 0$ such that $V(r,\underline{\theta}') < 0$ for all $r < \varepsilon$. While if $V_s(\underline{x})$ is positive definite then by putting $m = s$ in 3.2.18 we see that $\Theta_s(\underline{\theta}) > 0$ for all $\underline{\theta}$ and then by 3.2.21 we see that for every $\underline{\theta}$ there exists an $\varepsilon(\underline{\theta}) > 0$ such that $r < \varepsilon(\underline{\theta})$ implies $V(r,\underline{\theta}) > 0$. Since by Appendix D we have $0 \leq \theta_i < 2\pi$, $i = 1,\ldots,n-1$ and hence $\underline{\theta}$ has only a finite range. This implies that since $\varepsilon(\underline{\theta}) > 0$ for all $\underline{\theta}$ there exists ε' such that $0 < \varepsilon' < \varepsilon(\underline{\theta})$ for all $\underline{\theta}$. Hence we know that for all $r < \varepsilon'$ and for all $\underline{\theta}$ we have

$$V(r,\underline{\theta}) > 0$$

i.e. there exists a neighbourhood $S_{\varepsilon'}$ of the origin in which $V(\underline{x})$ is positive definite. End of proof.

Theorem 3.2.6 only applies to such $V(\underline{x})$ which admit a continuous series expansion in integral powers of x_i, $i = 1,\ldots,n$ given by 3.2.22. The situation for a general function $V(\underline{x})$ is not the same. For example we may see that the function

$$V(\underline{x}) = \|\underline{x}\| \qquad 3.2.26$$

is positive definite in the whole while

$$P(V) = 1.$$

3.2.26 does not admit a single series expansion in $\underline{x}$ of the form 3.2.22 for all $\underline{x}$ in a neighbourhood of the origin.

The only thing which can be said in the more general case is that $P(V) > 0$ otherwise by definition 3.2.2 we see that $V(\underline{0}) \neq 0$ if $P(V) \leq 0$.

We now prove a theorem which will be useful in the chapter on the one-dimensional Zubov equation, which is the one-dimensional version of theorem 3.2.6.

Theorem 3.2.7

V(x) is a positive definite scalar function of the scalar variable x in a neighbourhood of x = 0 if

$$V(x) = \sum_{m=s}^{\infty} V_m x^m \qquad 3.2.27$$

where $s \geqslant 2$ and s even and $V_s > 0$. V(x) is not positive definite if s odd or $s \leqslant 0$.

Proof

We may write 3.2.27 as

$$V(x) = x^s \sum_{m=s}^{\infty} V_m x^{m-s}$$

Then as $x \rightarrow 0$

$$\frac{V(x)}{x^s} \longrightarrow V_s + O(x) \qquad 3.2.28$$

We therefore see from 3.2.28 that if $s \geqslant 2$ and s even and $V_s > 0$ that V(x) is positive definite by using the same argument as in theorem 3.2.6.

While if $s \leqslant 0$ then $V(0) \neq 0$ and if s odd then from 3.2.28 we know that either $V(x) < 0$ or $V(-x) < 0$ for x in some neighbourhood of x = 0 and hence in either case V(x) is not positive definite. End of proof.

3. Polar Co-ordinate Zubov Equation

The theory in section 2 relied heavily on polar co-ordinates and in Chapter 5 the Zubov equation will be solved numerically in polar co-ordinate form. This section therefore investigates the relationship between rectangular and polar co-ordinates as applied to the Zubov equation.

We are seeking to transform the Zubov equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x}))$$

into the form

$$F(r,\underline{\theta})\frac{\partial V}{\partial r}(r,\underline{\theta}) + \sum_{i=1}^{n-1} G_i(r,\underline{\theta})\frac{\partial V}{\partial \theta_i}(r,\underline{\theta}) = -\Phi(r,\underline{\theta})(1 - dV(r,\underline{\theta})) \qquad 3.3.1$$

by the transformation 3.2.17. Using the result of Appendix E we see that in two dimensions

$$\begin{aligned} \frac{\partial V}{\partial x_1} &= \cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \\ \frac{\partial V}{\partial x_2} &= \sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta} \end{aligned} \qquad 3.3.2$$

From 3.3.2 we can obtain the terms of 3.3.1 and these are

$$F(r,\theta) \equiv \cos\theta\, f_1(r\cos\theta, r\sin\theta) + \sin\theta\, f_2(r\cos\theta, r\sin\theta)$$

$$G(r,\theta) \equiv -\frac{\sin\theta}{r} f_1(r\cos\theta, r\sin\theta) + \frac{\cos\theta}{r} f_2(r\cos\theta, r\sin\theta). \qquad 3.3.3$$

Now we may establish a theorem connecting $P(F)$, $P(G)$ with $P(f_1)$, $P(f_2)$.

Theorem 3.3.1

$$P(F) \geq \min(P(f_1), P(f_2))$$

$$P(G) \geqslant \min(P(f_1), P(f_2)) - 1 \qquad 3.3.4$$

Proof

Let $P(f_1) = s_1$

$P(f_2) = s_2$

$P(F) = s_F \qquad 3.3.5$

$P(G) = s_G$

and suppose that $s_1 \geqslant s_2$. Now from 3.3.3

$$\frac{F(r,\theta)}{r^{s_2}} = \frac{\cos\theta\, f_1(r\cos\theta, r\sin\theta)}{r^{s_2}} + \frac{\sin\theta\, f_2(r\cos\theta, r\sin\theta)}{r^{s_2}} \qquad 3.3.6$$

and

$$\frac{G(r,\theta)}{r^{s_2-1}} = \frac{-\sin\theta\, f_1(r\cos\theta, r\sin\theta)}{r^{s_2}} + \frac{\cos\theta\, f_2(r\cos\theta, r\sin\theta)}{r^{s_2}} \qquad 3.3.7$$

Writing 3.3.6, 3.3.7 in terms of the limit of each function we have

$$\frac{F(r,\theta)}{r^{s_F}} (r^{s_F - s_2}) = \cos\theta\, \frac{f_1(r\cos\theta, r\sin\theta)}{r^{s_1}}(r^{s_1 - s_2}) + \sin\theta\, \frac{f_2(r\cos\theta, r\sin\theta)}{r^{s_2}} \qquad 3.3.8$$

$$\frac{G(r,\theta)}{r^{s_G}} (r^{s_G - s_2 + 1}) = -\sin\theta\, \frac{f_1(r\cos\theta, r\sin\theta)}{r^{s_1}}(r^{s_1 - s_2}) + \cos\theta\, \frac{f_2(r\cos\theta, r\sin\theta)}{r^{s_2}} \qquad 3.3.9$$

Now using theorem 3.2.3 and definition 3.2.2 and letting $r \to 0$ in 3.3.8, 3.3.9 we observe that

$$\frac{F(r,\theta)}{r^{s_F}}\,(r^{s_F - s_2}) \longrightarrow \cos\theta\, c_1(\theta) r^{s_1 - s_2} + \sin\theta\, c_2(\theta) \qquad 3.3.10$$

$$\frac{G(r,\theta)}{r^{s_G}}\,(r^{s_G - s_2 + 1}) \longrightarrow -\sin\theta\, c_1(\theta) r^{s_1 - s_2} + \cos\theta\, c_2(\theta) \qquad 3.3.11$$

as $r \to 0$ with θ constant. Since we assume $s_1 \geqslant s_2$ then the right hand sides of 3.3.10 and 3.3.11 are finite as $r \to 0$ for all θ.

Now let θ' be such that

$$\frac{F(r,\theta')}{r^{s_F}} \longrightarrow c_F(\theta') \neq 0$$

as $r \to 0$ along $\theta = \theta'$.
Substituting $\theta = \theta'$ into 3.3.10 immediately shows that if $s_F < s_2$ is assumed then the left hand side of 3.3.10 is infinite as $r \to 0$ along $\theta = \theta'$ which contradicts the finiteness of the R.H.S. for all θ. Hence if $s_1 \geqslant s_2$ then $s_F \geqslant s_2$. We may prove similarly that if $s_2 \geqslant s_1$ then $s_F \geqslant s_1$.

$$\text{i.e. } s_F \geqslant \min(s_1, s_2) \qquad 3.3.12$$

We may similarly prove using 3.3.11 that

$$s_G \geqslant \min(s_1, s_2) - 1. \qquad 3.3.13$$

Substituting 3.3.5 into 3.3.12, 3.3.13 yields the result 3.3.4. End of proof.

If we now consider the situation for example in 4-dimensional systems then using Appendix E again we have

$$\frac{\partial V}{\partial x_1} = \cos\theta_1 \frac{\partial V}{\partial r} - \frac{\sin\theta_1}{r}\frac{\partial V}{\partial \theta_1} \qquad 3.3.14$$

$$\frac{\partial V}{\partial x_2} = \sin\theta_1 \cos\theta_2 \frac{\partial V}{\partial r} + \frac{\cos\theta_1 \cos\theta_2}{r}\frac{\partial V}{\partial \theta_1} - \frac{\sin\theta_2}{r\sin\theta_1}\frac{\partial V}{\partial \theta_2} \qquad 3.3.15$$

$$\frac{\partial V}{\partial x_3} = \sin\theta_1 \sin\theta_2 \cos\theta_3 \frac{\partial V}{\partial r} + \frac{\cos\theta_1 \sin\theta_2 \cos\theta_3}{r}\frac{\partial V}{\partial \theta_1} + \frac{\cos\theta_2 \cos\theta_3}{r\sin\theta_1}\frac{\partial V}{\partial \theta_2} - \frac{\sin\theta_3}{r\sin\theta_1 \sin\theta_2}\frac{\partial V}{\partial \theta_3}. \qquad 3.3.16$$

$$\frac{\partial V}{\partial x_4} = \sin\theta_1 \sin\theta_2 \sin\theta_3 \frac{\partial V}{\partial r} + \frac{\cos\theta_1 \sin\theta_2 \sin\theta_3}{r}\frac{\partial V}{\partial \theta_1} + \frac{\cos\theta_2 \sin\theta_3}{r\sin\theta_1}\frac{\partial V}{\partial \theta_2} + \frac{\cos\theta_3}{r\sin\theta_1 \sin\theta_2}\frac{\partial V}{\partial \theta_3}. \qquad 3.3.17$$

Substituting 3.3.14 to 3.3.17 into 3.3.1 gives

$$F(r,\underline{\theta})=\cos\theta_1 f_1+\sin\theta_1\cos\theta_2 f_2+\sin\theta_1\sin\theta_2\cos\theta_3 f_3 + \sin\theta_1\sin\theta_2\sin\theta_3 f_4$$

$$G_1(r,\underline{\theta})=(-\sin\theta_1 f_1+\cos\theta_1\cos\theta_2 f_2+\cos\theta_1\sin\theta_2\cos\theta_3 f_3 + \cos\theta_1\sin\theta_2\sin\theta_3 f_4)/r$$

$$G_2(r,\underline{\theta})=(-\sin\theta_2 f_2+\cos\theta_2\cos\theta_3 f_3+\cos\theta_2\sin\theta_3 f_4)/r\sin\theta_1$$

$$G_3(r,\underline{\theta})=(-\sin\theta_3 f_3+\cos\theta_3 f_4)/r\sin\theta_1\sin\theta_2 \qquad 3.3.18$$

Let $P(\underline{f}) = s$, then using theorem 3.2.1 and analysing the limits of 3.3.18 as $r \to 0$ we establisn that

$$P(F) \geq s$$
$$P(G_1) \geq s-1$$
$$P(G_2) \geq s-1$$
$$P(G_3) \geq s-1$$

providing that $\sin\theta_1 \sin\theta_2 \neq 0$. The situation which arises if $\sin\theta_1 \sin\theta_2 = 0$ is more interesting. However the singularities in 3.3.1 caused by this are removed by multiplying 3.3.1 by $\sin\theta_1 \sin\theta_2$. We can do this since we know V has no singularities except on the boundary of the D.O.A.. Now if we let $\sin\theta_1 = 0$ in the revised 3.3.1 we obtain the relationship

$$(-\sin^2\theta_2 f_2+\cos\theta_2\sin\theta_2\cos\theta_3 f_3 + \cos\theta_2\sin\theta_2\sin\theta_3 f_4)\frac{\partial V}{\partial\theta_2}/r$$
$$+(-\sin\theta_3 f_3 + \cos\theta_3 f_4)\frac{\partial V}{\partial\theta_3}/r = 0 \qquad 3.3.19$$

The relationship 3.3.19 holds for any (r,θ_2,θ_3) and is independent of $\Phi(r,\underline{\theta})$. Consideration of 3.2.17 shows why this is the case. If we let $\sin\theta_1 = 0$ in 3.2.17 we see that $\underline{x}$ is constrained to lie on the line

$$x_1 = r$$
$$x_2 = x_3 = x_4 = 0.$$

Hence all points on $\sin\theta_1 = 0$ are equivalent for any given $r > 0$ and since r may be cancelled from 3.3.19 we see that for different values of θ_2,θ_3 3.3.19 expresses the same relationship in an infinite number of ways.

Likewise when $\sin\theta_2 = 0$ we have

$$(-\sin\theta_3 f_3 + \cos\theta_3 f_4)\frac{\partial V}{\partial\theta_3}/r = 0 \qquad 3.3.20$$

As before we see that points with the same r,θ_1 satisfying $\sin\theta_2 = 0$ are equivalent since $\sin\theta_2 = 0$ becomes

$$x_1 = r\cos\theta_1$$
$$x_2 = r\sin\theta_1$$
$$x_3 = x_4 = 0.$$

Hence V is independent of θ_3 as 3.3.20 shows.

We may therefore conclude that since the singularities in 3.3.18 constrain a 3-dimensional subspace of $(r,\theta_1,\theta_2,\theta_3)$ onto a 1 or 2 dimensional subspace of (x_1,x_2,x_3,x_4) these singularities are regular or removable (def. see Burkill (50)). Thus we see that we may evaluate $P(F),P(G_1),P(G_2),P(G_3)$ by considering their behaviour as $r \to 0$, by using 3.3.1 if $\sin\theta_1\sin\theta_2 \neq 0$ and 3.3.19 or 3.3.20 if $\sin\theta_1 = 0$ or $\sin\theta_2 = 0$

We may now summarize the general n-dimensional result in two theorems.

Theorem 3.3.2

$$P(F) \geq P(\underline{f}) \qquad 3.3.21$$

Proof

From Appendix E we have the general result

$$F(r,\underline{\theta}) = \cos\theta_1 f_1(r,\underline{\theta}) + \sum_{i=2}^{n-1}\Big(\prod_{j=1}^{i-1}\sin\theta_j\Big)\cos\theta_i f_i(r,\underline{\theta}) + \Big(\prod_{j=1}^{n-1}\sin\theta_j\Big) f_n(r,\underline{\theta}). \qquad 3.3.22$$

Let $P(\underline{f}) = s_1$ 3.3.23

$P(F) = s_2$

Now suppose in contradiction that $s_1 > s_2$. Dividing 3.3.22 by r^{s_2}

$$\frac{F(r,\underline{\theta})}{r^{s_2}} = \frac{\cos\theta_1 f_1(r,\underline{\theta})}{r^{s_2}} + \sum_{i=2}^{n-1}\Big(\prod_{j=1}^{i-1}\sin\theta_j\Big)\cos\theta_i \frac{f_i(r,\underline{\theta})}{r^{s_2}} + \Big(\prod_{j=1}^{n-1}\sin\theta_j\Big)\frac{f_n(r,\underline{\theta})}{r^{s_2}}. \qquad 3.3.24$$

Now we consider the behaviour of 3.3.24 as $r \to 0$ by investigating each term, re-writing

$$\frac{f_i(r,\underline{\theta})}{r^{s_2}} = \frac{f_i(r,\underline{\theta})}{r^{s_1}}\,(r^{s_1-s_2}) \qquad 3.3.25$$

Now by theorem 3.2.1 we have

$$P(f_i) \geq s_1 \qquad 3.3.26$$

and substituting 3.3.26 into 3.3.25 and letting $r \to 0$ gives

$$\frac{f_i(r,\underline{\theta})}{r^{s_2}} = \frac{f_i(r,\underline{\theta})}{r^{P(f_i)}}\,(r^{P(f_i)-s_2}) \longrightarrow 0$$

since $s_1 > s_2$, $i = 1,2,\ldots,n$, for all $\underline{\theta}$.

Hence $\dfrac{f_i(r,\underline{\theta})}{r^{s_2}} \longrightarrow 0$ as $r \to 0$ for all $\underline{\theta}$. 3.3.27

Substituting 3.3.27 into 3.3.24 shows that

$$\frac{F(r,\underline{\theta})}{r^{s_2}} \longrightarrow 0 \text{ as } r \to 0 \text{ for all } \underline{\theta}. \qquad 3.3.28$$

3.3.28 shows that $P(F) \neq s_2$ contradicting 3.3.23. Hence the assumption $s_1 > s_2$ was incorrect and by definition 3.3.23 of s_1, s_2 we see that 3.3.21 holds. End of proof.

We may similarly establish the corresponding result for $G_i(r,\underline{\theta})$, $i = 1,\ldots,n-1$.

Theorem 3.3.3

$P(G_i) \geq P(\underline{f}) - 1$

The proof of theorem 3.3.3 is exactly as for theorem 3.3.2 providing $\underline{\theta}$ does not satisfy

$$\prod_{i=1}^{n-2} \sin \theta_i = 0 \qquad 3.3.29$$

It has been shown that when 3.3.29 holds, $(r,\underline{\theta})$ is constrained onto a subspace of R^n on which a slightly different version of 3.3.1 holds. The asymptotic behaviour of $G_i(r,\underline{\theta})$ may be considered by letting $r \to 0$ in 3.3.1 or the corresponding version if 3.3.29 holds. The algebra is not carried out here as it is similar to that in theorem 3.3.2.

The result of theorem 3.3.1 will now be illustrated by examples. Consider the example of Davies (46)

$$\dot{x} = 6y - 2y^2$$
$$\dot{y} = -10x - y + 4x^2 + 2xy + 4y^2 \qquad 3.3.30$$

By definition 3.2.1 we see that

$$P(\underline{f}) = 1.$$

The Zubov equation becomes

$$(6y - 2y^2)\frac{\partial V}{\partial x} + (-10x - y + 4x^2 + 2xy + 4y^2)\frac{\partial V}{\partial y} = -\phi(x,y)(1 - dV). \qquad 3.3.31$$

Transforming 3.3.31 into the polar co-ordinate system (r,θ) we obtain

$$F(r,\theta) = -4r \sin\theta \cos\theta - r \sin^2\theta + 4r^2 \sin\theta$$

$$G(r,\theta) = -10\cos^2\theta - \sin\theta\cos\theta - 6\sin^2\theta + 4r\cos\theta + 2r\sin\theta.$$

Thus we see by letting $r \to 0$ that

$$P(F) = 1,\ P(G) = 0$$

and theorem 3.3.1 is satisfied with equality.

However consider the example

$$\begin{aligned} \dot{x} &= y + x^2 \\ \dot{y} &= -x + 3y^3 \end{aligned} \qquad 3.3.32$$

We obtain for this example

$F(r,\theta) = r^2\cos^3\theta + 3r^3\sin^4\theta$

$G(r,\theta) = -1 - r\cos^2\theta\sin\theta + 3r^2\sin^3\theta\cos\theta.$

This time we see

$P(\underline{f}) = 1,\ P(F) = 2,\ P(G) = 0$

which satisfies theorem 3.3.1 with one equality and one inequality. Some results on the situation where theorems 3.3.1, 3.3.2, 3.3.3 are satisfied either with equality or inequality are covered in the next section.

Let us look at what equation 3.3.1 means. Equation 3.3.1 is the Zubov equation corresponding to a system in polar co-ordinates given by

$$\begin{aligned} \dot{r} &= F(r,\underline{\theta}) \\ \dot{\theta}_i &= G_i(r,\underline{\theta}) \quad i = 1,\dots,n-1 \end{aligned} \qquad 3.3.33$$

Now we require a theorem on $P(\underline{f})$.

Theorem 3.3.4

For the origin of $\dot{\underline{x}} = \underline{f}(\underline{x})$ to be a critical point we require $P(\underline{f}) > 0$.

Proof

By definition 3.2.1 we know that

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} \longrightarrow c(\Gamma) \text{ as } \underline{x} \to \underline{0} \qquad 3.3.34$$

where $\underline{x}$ varies along a continuous path Γ in R^n where $c(\Gamma)$ exists and is finite for all Γ and non-zero for some Γ. Let Γ' be a path such that $c(\Gamma') \neq 0$ and letting $\underline{x} \to \underline{0}$, $\Gamma = \Gamma'$ in 3.3.34 we see that if $P(\underline{f}) \leqslant 0$ then $\|\underline{f}(\underline{x})\| \not\to 0$ as $\underline{x} \to \underline{0}$ and by definition 1.3.6 the origin is not a critical point. Hence if the origin is a critical point then $P(\underline{f}) > 0$.

End of proof.

Using theorems 3.3.2, 3.3.3 3.3.4 together we see

$P(F) \geqslant 0,\ P(G_i) \geqslant -1, \quad i = 1,\dots,n-1.$

The origin $\underline{x} = \underline{0}$ of rectangular co-ordinates corresponds to $r = 0$ in polar co-ordinates. This is, in effect, another transformation similar to those discussed in section 2.4. As was stated there, the stability of invariant sets is more complicated than simple analysis of the origin.

It will be noted system 3.3.33 has similarities with both one-dimensional systems and Lyapunov theory. Although $\dot{r}$ is dependent on r and $\underline{\theta}$ we are looking specifically to see if $r \to 0$ given $r = r_0$, $\underline{\theta} = \underline{\theta}_0$ at $t = 0$. Also it may be observed that

$$V(\underline{x}) = \|\underline{x}\| = r$$

is a candidate for a Lyapunov function, and if we can find a region containing the origin in which $\dot{r}$ is negative definite and r is positive definite then stability is assured.

4. Symmetric Systems

Examples 3.3.30 and 3.3.32 showed that theorems 3.3.1, and by implication theorems 3.3.2, 3.3.3 also, may be satisfied with equality or inequality. Now we attempt to define conditions in which the inequalities are strict. Looking back at section 3 we see that the inequalities in theorem 3.3.1 are strict if the R.H.S.s of 3.3.6 or 3.6.7 tend to zero as $r \to 0$ for all θ. It is difficult in general to tell what restrictions on f_1, f_2 may give

$$P(F) > P(f_1, f_2)$$

or

$$P(G) > P(f_1, f_2) - 1.$$

However we can investigate certain situations if f_1, f_2 have a series expansion similar to 3.2.19. By theorems 3.2.1 and 3.2.5 we may write down

$$f_1(r,\theta) = \sum_{m=s}^{\infty} r^m \, Hf_m(\theta)$$
$$f_2(r,\theta) = \sum_{m=s}^{\infty} r^m \, Hg_m(\theta) \qquad 3.4.1$$

where $P(f_1, f_2) = s$

and either $Hf_s(\theta) \not\equiv 0$

or $Hg_s(\theta) \not\equiv 0$.

Now we can establish in two dimensions

Theorem 3.4.1

Given that $P(f_1, f_2) = s$ and f_1, f_2 are expressible by 3.4.1 then $P(F) > P(f_1, f_2)$ if and only if

$\cos\theta\, Hf_s(\theta) + \sin\theta\, Hg_s(\theta) \equiv 0$ for all θ. 3.4.2

Likewise $P(G) > P(f_1, f_2) - 1$ if and only if

$-\sin\theta\, Hf_s(\theta) + \cos\theta\, Hg_s(\theta) \equiv 0$ for all θ. 3.4.3

<u>Proof</u>

Substituting 3.4.1 into 3.3.3 gives

$$F(r,\theta) = \cos\theta \sum_{m=s}^{\infty} r^m\, Hf_m(\theta) + \sin\theta \sum_{m=s}^{\infty} r^m\, Hg_m(\theta)$$

$$G(r,\theta) = -\sin\theta \sum_{m=s}^{\infty} r^{m-1}\, Hf_m(\theta) + \cos\theta \sum_{m=s}^{\infty} r^{m-1}\, Hg_m(\theta). \qquad 3.4.4$$

Dividing the equations 3.4.4 by r^s and r^{s-1} respectively gives

$$\frac{F(r,\theta)}{r^s} = \cos\theta\, Hf_s(\theta) + \sin\theta\, Hg_s(\theta) + O(r)$$

$$\frac{G(r,\theta)}{r^{s-1}} = -\sin\theta\, Hf_s(\theta) + \cos\theta\, Hg_s(\theta) + O(r). \qquad 3.4.5$$

Now if $P(F) > s$ then

$$\frac{F(r,\theta)}{r^s} = \frac{F(r,\theta)}{r^{P(F)}}\left(r^{P(F)-s}\right) \longrightarrow 0 \text{ as } r \to 0 \qquad 3.4.6$$

for all θ.

3.4.6 implies that as $r \to 0$ in 3.4.5 we must have

$$\cos\theta\, Hf_s(\theta) + \sin\theta\, Hg_s(\theta) \equiv 0 \qquad 3.4.7$$

while conversely if 3.4.7 is true then by letting $r \to 0$ in 3.4.5 gives

$$\frac{F(r,\theta)}{r^s} \longrightarrow 0 \text{ as } r \to 0 \text{ for all } \theta.$$

Hence $P(f) \neq s$ and since by theorem 3.3.1 $P(f) \geqslant s$ then we see that $P(f) > s$. This proves the theorem for $P(F)$ and the proof for $P(G)$ is identical using the second equation of 3.4.5. End of proof.

Now let us consider in detail the functions $Hf_s(\theta)$ and $Hg_s(\theta)$. Since 3.4.1 is obtained by collecting together homogeneous terms in (x,y) to the power $m = s, s+1, \ldots,$ we may write down

$$Hf_s(\theta) = \sum_{j=0}^{s} f_{s,j} \cos^j\theta \sin^{s-j}\theta$$

$$Hg_s(\theta) = \sum_{j=0}^{s} g_{s,j} \cos^j\theta \sin^{s-j}\theta \qquad 3.4.8$$

Substituting 3.4.8 into 3.4.2 and 3.4.3 we see that for $P(F) > P(f_1, f_2)$ we require

$$\cos\theta\sum_{j=0}^{s} f_{s,j}\cos^j\theta\,\sin^{s-j}\theta + \sin\theta\sum_{j=0}^{s} g_{s,j}\cos^j\theta\,\sin^{s-j}\theta \equiv 0$$

for all θ. 3.4.9

Likewise $P(G) > P(f_1,f_2) - 1$ if and only if

$$-\sin\theta\sum_{j=0}^{s} f_{s,j}\cos^j\theta\,\sin^{s-j}\theta + \cos\theta\sum_{j=0}^{s} g_{s,j}\cos^j\theta\,\sin^{s-j}\theta \equiv 0$$

for all θ. 3.4.10

3.4.9 and 3.4.10 are identities in θ which require that all coefficients of separate terms are zero. Hence the conditions of theorem 3.4.1 are met respectively if and only if

$$\begin{aligned} g_{s,0} &= 0 \\ g_{s,j} + f_{s,j-1} &= 0 \qquad j = 1,\ldots,s \qquad\qquad 3.4.11 \\ f_{s,s} &= 0 \end{aligned}$$

and

$$\begin{aligned} f_{s,0} &= 0 \\ -f_{s,j} + g_{s,j-1} &= 0 \qquad j = 1,\ldots,s \qquad\qquad 3.4.12 \\ g_{s,s} &= 0 \end{aligned}$$

3.4.11 and 3.4.12 represent the conditions under which the inequalities of theorem 3.3.1 are strict in 2-dimensional systems for any s. There are obvious generalisations into higher order systems although the algebra is complicated. However as in practice most systems have linear parts we may consider the system in n dimensions

$$\dot{\underline{x}} = \underline{f}(\underline{x}) = A\underline{x} + \underline{g}(\underline{x}) \qquad 3.4.13$$

where $\underline{g}(\underline{x})$ is in some sense "small" in comparison to $A\underline{x}$ as $\underline{x}\to\underline{0}$.

For the system 3.4.13 we see that since the linear terms dominate near the origin for non-trivial A we have

$$P(\underline{f}) = 1$$

Hence by theorem 3.3.2 and 3.3.3 we see that

$$\begin{aligned} &P(F) \geqslant 1 \\ &P(G_i) \geqslant 0 \qquad\qquad 3.4.14 \\ &i = 1,\ldots,n-1 \end{aligned}$$

and some results may be established to decide when the inequalities in 3.4.14 are strict. One such will be stated here.

Theorem 3.4.2

A necessary and sufficient condition that $P(F) > 1$ where $\underline{f}(\underline{x})$ is given by 3.4.13 is that

$$A + A^T = \theta. \qquad 3.4.15$$

Proof

Writing out 3.4.13 in full gives

$$f_i(\underline{x}) = \sum_{j=1}^{n} A_{i,j}x_j + g_i(\underline{x}) \qquad 3.4.16$$

$i = 1,\ldots,n.$

Now we may substitute the transformation 3.2.17 into 3.4.16 and then substitute 3.4.16 into 3.3.22. We obtain

$$\begin{aligned}\frac{F(r,\underline{\theta})}{r} = {} & A_{1,1}\cos^2\theta_1 + \cos\theta_1 \sum_{i=2}^{n-1}(A_{1,i}+A_{i,1})\Big(\prod_{k=1}^{i-1}\sin\theta_k\Big)\cos\theta_i \\ & +\cos\theta_1(A_{1,n}+A_{n,1})\Big(\prod_{k=1}^{n-1}\sin\theta_k\Big) \\ & +\sum_{i=2}^{n-1}\sum_{j=2}^{n-1} A_{i,j}\Big(\prod_{k=1}^{i-1}\sin\theta_k\Big)\Big(\prod_{k=1}^{j-1}\sin\theta_k\Big)\cos\theta_i\cos\theta_j \\ & +\sum_{i=2}^{n-1}(A_{i,n}+A_{n,i})\Big(\prod_{k=1}^{i-1}\sin\theta_k\Big)\Big(\prod_{k=1}^{n-1}\sin\theta_k\Big)\cos\theta_i \\ & + A_{n,n}\Big(\prod_{k=1}^{n-1}\sin\theta_k\Big)^2 + O(r).\end{aligned} \qquad 3.4.17$$

If A satisfies 3.4.15 then the linear terms of 3.4.17 are zero and

$$\frac{F(r,\underline{\theta})}{r} \longrightarrow 0 \text{ as } r\to 0 \text{ for all } \underline{\theta}.$$

and here $P(F) > 1$. While conversely if $P(F) > 1$ then

$$\frac{F(r,\underline{\theta})}{r} = \frac{F(r,\underline{\theta})}{r^{P(F)}}\left(r^{P(F)-1}\right) \longrightarrow 0 \text{ as } r\to 0 \text{ for all } \underline{\theta}$$

and the linear terms of 3.4.17 must vanish identically. Hence we require

$$A_{i,j} + A_{j,i} = 0$$
$$i,j = 1,\ldots,n$$

i.e. $A + A^T = 0$. End of proof.

Equation 3.4.15 shows that A is not a stability matrix. This is easily seen by recognising that if

$$V(\underline{x}) = \underline{x}^T\underline{x}$$

is tried as a Lyapunov function of the system

$$\dot{\underline{x}} = A\,\underline{x}$$

then $\dot{V}(\underline{x}) = 0$ results. Stability matrices cannot therefore give $P(F) > P(\underline{f})$

but examples such as the Hahn system show that $P(G) > P(\underline{f}) - 1$ is possible.

These results may seem of academic interest here but the asymptotic degree $P(F), P(G_i)$, $i = 1,\ldots,n-1$, $P(\underline{f})$ etc. become important in later sections when solving the Zubov equation and in Chapter 5 when radial grid methods are considered which are affected by the behaviour of F, G_i, $i = 1,\ldots,n$, Φ near the origin.

5. Concepts of Parameterisation

In section 8 of Chapter 1 mention was made of the auxiliary equation of a partial differential equation and the characteristics. Reference was again made to this in section 8 of Chapter 2 in connection with Fox's method of numerical solution of a P.D.E.. In this section we investigate the connection between t and V created by the function $\phi(\underline{x})$. We may reproduce 2.8.8 and 2.8.10

$$\frac{dx_1}{f_1(\underline{x})} = \frac{dx_2}{f_2(\underline{x})} = \ldots\ldots\ldots = \frac{dx_n}{f_n(\underline{x})} \qquad 3.5.1$$

$$\frac{dx_1}{f_1(\underline{x})} = \frac{dx_2}{f_2(\underline{x})} = \ldots\ldots\ldots = \frac{dx_n}{f_n(\underline{x})} = \frac{-dV}{\phi(\underline{x})(1-dV)} \qquad 3.5.2$$

From 3.5.2 we see that we may construct a new set of ordinary differential equations given by

$$\frac{dx_i}{dV} = \frac{-f_i(\underline{x})}{\phi(\underline{x})(1-dV)}. \qquad 3.5.3$$

The equations 3.5.3 may be compared to the original system equations given by

$$\frac{dx_i}{dt} = f_i(\underline{x}). \qquad 3.5.4$$

The system 3.5.4 yields the solution $x_i(t)$ given $x_i(0), i = 1,\ldots n.$ while from 3.5.3 we may obtain either analytically or numerically the solutions $x_i(V)$ given $x_i(V_o)$, $i = 1,\ldots,n$.

Comparison of 3.5.1 and 3.5.2 shows that the trajectories $x_i(t)$ and $x_i(V)$, $i = 1,\ldots,n$, are the same for both systems in the n-dimensional state space, but they have a different relationship with their respective independent variable.

We can see from this comparison that there is not actually much difference between the system equations 3.5.4 and the Zubov P.D.E.

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial V(\underline{x})}{\partial x_i} = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 3.5.5$$

The presence of the term $\phi(\underline{x})(1 - dV(\underline{x}))$ simply gives the trajectories a parametric representation in terms of a new independent variable. Considering the Zubov equation is always studied in the form 3.5.5 with one dependent variable V and n independent variables x_i, $i = 1,\ldots,n$, this is turning the whole problem upside down!

We can see that any transformation of independent variables is possible by multiplying the R.H.S. of 3.5.4 by a fixed function of $\underline{x}$ and the independent variable. Suppose we consider the simple equation

$$\dot{T} = 1 \qquad 3.5.6$$

where $T(\underline{x})$ is considered as a function of $\underline{x}$ similar to $V(\underline{x})$. Now by using the chain rule of differentiation given in theorem 1.5.3 we obtain a formula similar to 3.5.5

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial T(\underline{x})}{\partial x_i} = 1 \qquad 3.5.7$$

emphasising again the similarity in relationship between $\underline{x}(t)$ and $\underline{x}(V)$ since the solution of 3.5.6 is $T = t$.

The question that remains however is what happens to $\underline{x}(V)$ as V varies. We know for asymptotically stable systems that $\underline{x}(t) \to \underline{0}$ as $t \to \infty$ providing $\underline{x}_0 \in D(\underline{f})$. We also know that if we can find positive definite functions $V(\underline{x})$, $\phi(\underline{x})$ satisfying 3.5.5 then the origin is asymptotically stable by theorem 1.7.1. In this case we know that $\underline{x}(V) \to \underline{0}$ as $V \to 0$ where V decreases from V_0 to zero.

But we do not know that such a Lyapunov function $V(\underline{x})$ necessarily exists given $\phi(\underline{x})$. Indeed, since the transformation of the independent variable t to the independent variable V in 3.5.3 can take on any form we see that seme functions $\phi(\underline{x})$ and d will transform 3.5.4 into a system with similar properties to 3.5.4.

We have already observed that

$\phi(\underline{x}) = -1, \quad d = 0$

yields the P.D.E. 3.5.7 and leaves system 3.5.4 unchanged. So what conditions are required on $\phi(\underline{x})$ and d such that 3.5.3 will be a system such that $\underline{x}(V) \to 0$ as $V \to 0$?

The Zubov theory requires firstly that $\phi(\underline{x})$ be positive definite so 3.5.7 is not admissible. But not just any positive definite function $\phi(\underline{x})$ will do.

Consider the example:

$$\begin{aligned} \dot{x} &= -x(x^2 + y^2) \\ \dot{y} &= -y(x^2 + y^2) \end{aligned} \qquad 3.5.8$$

System 3.5.8 is asymptotically stable for all $(x,y) \in R^2$.

Now if we use

$$\phi(x,y) = x^2 + y^2 \qquad 3.5.9$$
$$d = 0$$

and substitute 3.5.8, 3.5.9 into 3.5.3 we obtain

$$\frac{dx}{dV} = x$$
$$\frac{dy}{dV} = y \qquad 3.5.10$$

3.5.10 represents a system for $x(V), y(V)$ which is stable for all (x,y) as $V \to -\infty$. The solution of 3.5.10 being given by

$$x(V) = x_o e^{V-V_o}$$
$$y(V) = y_o e^{V-V_o}$$

proves that $(x(0), y(0)) \neq (0,0)$ unless $(x_o, y_o) = (0,0)$. The reason for this behaviour is that the origin is still a critical point of the transformed system 3.5.10 and solutions can only tend to critical points as $t \to \pm\infty$ providing the solutions are well defined.

Now suppose 3.5.10 is replaced by the system

$$\frac{dx}{dV} = \begin{cases} -x^{\frac{1}{2}} & x \geq 0 \\ (-x)^{\frac{1}{2}} & x \leq 0 \end{cases}$$
$$\frac{dy}{dV} = \begin{cases} -y^{\frac{1}{2}} & y \geq 0 \\ (-y)^{\frac{1}{2}} & y \leq 0 \end{cases} \qquad 3.5.11$$

The solution of 3.5.11 for $(x_o, y_o) \geq (0,0)$ is

$$-2(x^{\frac{1}{2}} - x_o^{\frac{1}{2}}) = V - V_o$$
$$-2(y^{\frac{1}{2}} - y_o^{\frac{1}{2}}) = V - V_o .$$

Now if $x_o = y_o$ we see that

$$x(V_o + 2x_o^{\frac{1}{2}}) = 0$$
$$y(V_o + 2y_o^{\frac{1}{2}}) = 0$$

and solutions of 3.5.11 reach the origin for finite V. In this case the origin is a critical point of 3.5.11 but the system does not satisfy a Lipschitz condition near the origin and by theorem 1.3.3 the solution at the origin is not necessarily unique. This brings us to the theorem on choice of $\phi(\underline{x})$.

Theorem 3.5.1

For the Zubov equation to be soluble for an asymptotically stable system 3.5.4 to yield a positive definite function $V(\underline{x})$ it is necessary that the system 3.5.3 does not satisfy a Lipschitz condition for all $\underline{x} \in S_\varepsilon$ and all $\varepsilon < \delta$ for some $\delta > 0$.

<u>Proof</u>

Suppose for contradiction that the origin of system 3.5.3 lies in a region for which 3.5.3 does satisfy a Lipschitz condition.

i.e. $$\left\| \frac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} - \frac{\underline{f}(\underline{y})}{\phi(\underline{y})(1-dV)} \right\| \leq L\|\underline{x} - \underline{y}\| \qquad 3.5.12$$

for some fixed L and all $(\underline{x},\underline{y}) \in S_\varepsilon$ for $\varepsilon > 0$. Now by theorems 1.3.2, 1.3.3 we know that there exists a unique solution $\underline{x}(V)$ of system 3.5.3 passing through an initial point

$$\underline{x}(V_0) = \underline{x}_0 \qquad 3.5.13$$

providing $\underline{x}_0 \in S_\varepsilon$ and for as long as $\underline{x}(V) \in S_\varepsilon$. Hence there exists a unique solution $\underline{x}(V)$ such that

$$\underline{x}(0) = \underline{0}. \qquad 3.5.14$$

Now since we assume that the origin is a critical point of 3.5.4 and also that $\phi(\underline{x})$ is positive definite we can say

$$\underline{f}(\underline{0}) = \underline{0}$$

$$\phi(\underline{0}) = 0.$$

There are three possibilities for the behaviour of

$$\frac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} \quad \text{as } \underline{x} \to \underline{0}:$$

i) If $\left\| \dfrac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} \right\| \to \infty$ as $\underline{x} \to \underline{0}$ then it does not satisfy a Lipschitz condition.

ii) If $\dfrac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} \longrightarrow \underline{0}$ as $\underline{x} \to \underline{0}$ then we may put $\underline{y} = \underline{0}$ in 3.5.12

thus giving $$\left\| \frac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} \right\| \leq L\|\underline{x}\|.$$

Now from 3.5.3 we see that a solution satisfying the initial condition 3.5.14 is given by

$$\underline{x}(V) \equiv \underline{0}. \qquad 3.5.15$$

By theorem 1.3.3, 3.5.15 is the unique solution passing through

$$\underline{x}(V_0) = \underline{0} \qquad 3.5.16$$

for any finite V_0. This proves that we cannot obtain a solution of 3.5.3 such that $\underline{x}_0 \neq \underline{0}$ which passes through the origin.

iii) If $\dfrac{\underline{f}(\underline{x})}{\phi(\underline{x})(1-dV)} \longrightarrow c(\Gamma) \neq 0$ as $\underline{x} \to \underline{0}$ and $\underline{x}$ varies along a continuous path Γ in R^n, then we see that in this case a solution $\underline{x}(V)$ of 3.5.3 with initial conditions 3.5.13 passing through the origin satisfies

$$\lim_{V\to 0}\frac{d\underline{x}(V)}{dV} = c(\Gamma) \neq 0$$

and hence there exists a $\delta > 0$ such that for $-\delta < V < 0$ then $\left\| \frac{d\underline{x}(V)}{dV} \right\| > 0$. Let x_i be a component of $\underline{x}$ such that $\left| \frac{dx_i(V)}{dV} \right| > 0$.

Then we see that since

$$x_i(V) = x_i(0) + \int_0^V \frac{dx_i(V')}{dV}dV'$$

then while $-\delta < V < 0$ we have either

$$\begin{array}{ll} & x_i(V) < x_i(0) \\ \text{or} & x_i(V) > x_i(0). \end{array} \qquad 3.5.17$$

Now if we assume that $x_i(0) = 0$ we immediately observe from 3.5.17 that for $-\delta < V < 0$, $\underline{x}(V) \neq \underline{0}$. Hence we have found a point $\underline{x}$ in the neighbourhood of the origin where $V < 0$ and $\dot{V} < 0$, and by theorem 1.6.4 this indicates that the origin of 3.5.4 is unstable which is a contradiction.

Having covered all cases where the Lipschitz condition holds and found that $V(\underline{x})$ is not a Lyapunov function then we have proved that to obtain a Lyapunov function then the Lipschitz condition cannot hold. End of proof.

Zubov himself (12) states that not all $\phi(\underline{x})$ are admissible to be chosen so that the conditions of theorem 1.7.1 may be satisfied. He refers to the "rate of decrease" of solutions of 3.5.4 observing that if for any reason whatever the rate of decrease of $\underline{x}(t)$ is known then $\phi(\underline{x})$ may be chosen and he gives examples covering various rates of decrease of $\underline{x}(t)$ showing how $\phi(\underline{x})$ can be chosen. In this section and the next it is shown that choice of $\phi(\underline{x})$ can be made by considering $\underline{f}(\underline{x})$ rather than $\underline{x}(t)$ which if we knew would render the Zubov equation unnecessary.

Zubov's result on choice of $\phi(\underline{x})$ is:

<u>Theorem 3.5.2</u>

If the origin of 3.5.4 is asymptotically stable and if $\phi(\underline{x})$ satisfies the conditions of theorem 1.7.1 and also

$$\int_0^\infty \phi(\underline{x}(t))dt < \infty$$

given $\underline{x}(0) = \underline{x}_0$,

then for this $\phi(\underline{x})$, a $V(\underline{x})$ can be found satisfying theorem 1.7.1.

It remains to show here that theorem 3.5.1 implies theorem 3.5.2.

Theorem 3.5.3

If the origin of 3.5.4 is asymptotically stable and the system 3.5.3 satisfies a Lipschitz condition then

$$\lim_{t\to\infty}\int_0^t \phi(\underline{x}(t'))dt' = \infty \qquad 3.5.18$$

Proof

It has been shown that if 3.5.3 satisfies a Lipschitz condition then either the origin is unstable, which we discount, or that 3.5.15 is the unique solution of 3.5.3 passing through 3.5.16.

However if we consider the relationship between V and t given by

$$\dot{V} = -\phi$$

we see that V is then given by

$$V(\underline{x}(t)) = V_0 - \int_0^t \phi(\underline{x}(t'))dt' \qquad 3.5.19$$

which since $\phi(\underline{x})$ is positive definite means that $V(\underline{x}(t))$ is a monotonic decreasing function of t along the system trajectories But we know that no finite V exists such that $\underline{x}(V) = \underline{0}$. Hence

$$\lim_{t\to\infty} V(\underline{x}(t)) = -\infty \qquad 3.5.20$$

Now by rearranging 3.5.19 and letting $t\to\infty$ using 3.5.20 we see that 3.5.18 occurs. End of proof.

Before considering the Zubov equation in terms of definition 3.2.1, 3.2.2 we shall state a theorem establishing what the Lipschitz condition means in terms of the definitions.

Theorem 3.5.4

A function $\underline{f}(\underline{x})$ for which $\underline{f}(\underline{0}) = \underline{0}$ satisfies a Lipschitz condition in a region D containing the origin if and only if

$P(\underline{f}) \geqslant 1.$ 3.5.21

Proof

This is similar to the proof of theorem 3.5.1. In that proof the three cases of the behaviour of a function as $\underline{x}\to\underline{0}$ were considered. The case $\underline{f}(\underline{x}) \to \infty$ and $\underline{f}(\underline{x}) \to \underline{c}(\Gamma) \neq \underline{0}$ are discounted as the origin is a critical point of $\underline{f}(\underline{x})$.

The Lipschitz condition can be written as

$$\|\underline{f}(\underline{x}) - \underline{f}(\underline{y})\| \leqslant L\|\underline{x} - \underline{y}\| \qquad 3.5.22$$

for some fixed L and all $(\underline{x},\underline{y})\in D$. Hence substituting $\underline{y} = \underline{0}$ into 3.5.22 gives

$$\|\underline{f}(\underline{x})\| \leqslant L\,\|\underline{x}\|. \qquad 3.5.23$$

Now suppose by contradiction that $P(\underline{f}) < 1$. We divide 3.5.23 by $\|\underline{x}\|^{P(\underline{f})}$ which gives

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} \leqslant L\|\underline{x}\|^{1-P(\underline{f})}$$

Now letting $\underline{x} \to \underline{0}$ along any continuous path Γ in D shows that

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} \longrightarrow 0 \text{ as } \underline{x} \to \underline{0}$$

if $P(\underline{f}) < 1$. This contradicts definition 3.2.1 and so $P(\underline{f}) \geqslant 1$.

Now suppose we are given $P(\underline{f}) \geqslant 1$. This means by definition 3.2.1 that

$$\frac{\|\underline{f}(\underline{x})\|}{\|\underline{x}\|^{P(\underline{f})}} \longrightarrow c(\Gamma) \text{ as } \underline{x} \to \underline{0} \qquad 3.5.24$$

along any continuous path Γ and $c(\Gamma)$ exists and is finite for all Γ and is non-zero for at least one Γ. Let c^* be the supremum of $c(\Gamma)$ over all Γ in D. Then from 3.5.24 we have

$$\|\underline{f}(\underline{x})\| \leq c^*\|\underline{x}\|^{P(\underline{f})} \qquad 3.5.25$$

as $\underline{x} \to \underline{0}$, $\underline{x} \in$ D. Now by 3.5.21 we have

$$\|\underline{x}\|^{P(\underline{f})} \leqslant \|\underline{x}\| \qquad 3.5.26$$

for all $\underline{x}$ in some neighbourhood of the origin. Combining 3.5.25 and 3.5.26 we have arrived at the Lipschitz condition for a neighbourhood of the origin. End of proof.

6. Admissible $\phi(\underline{x})$

Now we have established by an example and a theorem that not all positive definite functions $\phi(\underline{x})$ satisfy theorem 1.7.1 for a given $\underline{f}(\underline{x})$, we may study this problem using definition 3.2.1, 3.2.2. We must assume that we may solve the Zubov equation in the form 3.1.1 or at least attempt to solve it and hence that the partial derivatives $\frac{\partial V(\underline{x})}{\partial x_i}$, $i=1,\ldots,n$, exist.

Theorem 3.6.1

$$P(\underline{f}) + P\left(\frac{\partial V}{\partial \underline{x}}\right) \leqslant \min\left(P(\phi), P(d\phi V)\right) \qquad 3.6.1$$

providing $\frac{\partial V}{\partial \underline{x}}$ exists.

Proof

Let $P(\underline{f}) + P\left(\frac{\partial V}{\partial \underline{x}}\right) = s.$ $\qquad$ 3.6.2

Dividing the Zubov equation by $\|\underline{x}\|^s$ gives

$$\frac{\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^s} = \frac{-\phi(\underline{x})(1 - dV(\underline{x}))}{\|\underline{x}\|^s} \qquad 3.6.3$$

Now we consider the limit of each term on the L.H.S. as $\underline{x}\to\underline{0}$ along a continuous path Γ in R^n. By definition 3.2.2 we may state that

$$\frac{f_i(\underline{x})}{\|\underline{x}\|^{P(f_i)}} \longrightarrow c_i'(\Gamma) \text{ as } \underline{x}\to\underline{0} \qquad 3.6.4$$

and $\underline{x}$ varies along a continuous path Γ in R^n. Also

$$\frac{\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^{P(\frac{\partial V}{\partial x_i})}} \longrightarrow c_i^*(\Gamma) \text{ as } \underline{x}\to\underline{0} \qquad 3.6.5$$

where $c_i'(\Gamma)$ and $c_i^*(\Gamma)$, $i = 1,\ldots,n$, exist and are finite and each is non-zero for some Γ.

By theorem 3.2.1 we have the inequalities

$$P(f_i) \geqslant P(\underline{f})$$

$$P(\frac{\partial V}{\partial x_i}) \geqslant P(\frac{\partial V}{\partial \underline{x}}) \qquad i = 1,\ldots,n. \qquad 3.6.6$$

Now by 3.6.2 we see that

$$\frac{f_i(\underline{x})\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^s} = \frac{f_i(\underline{x})}{\|\underline{x}\|^{P(\underline{f})}}\;\frac{\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^{P(\frac{\partial V}{\partial \underline{x}})}}$$

while from 3.6.6 we see that

$$\frac{f_i(\underline{x})\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^s} = \frac{f_i(\underline{x})}{\|\underline{x}\|^{P(f_i)}}\;\frac{\frac{\partial V(\underline{x})}{\partial x_i}}{\|\underline{x}\|^{P(\frac{\partial V}{\partial x_i})}}\,\|\underline{x}\|^{(P(f_i)-P(\underline{f})+P(\frac{\partial V}{\partial x_i})-P(\frac{\partial V}{\partial \underline{x}}))}$$

Using 3.6.6 and 3.6.4, 3.6.5 upon letting $\underline{x}\to\underline{0}$ we see that

$$\frac{f_i(\underline{x})\frac{\partial V}{\partial x_i}(\underline{x})}{\|\underline{x}\|^s} \text{ is finite as } \underline{x}\to\underline{0}.$$

Refering back to 3.6.3 we see that

$$\frac{-\phi(\underline{x})(1 - dV(\underline{x}))}{\|\underline{x}\|^s} \text{ is finite as } \underline{x}\to\underline{0}. \qquad 3.6.7$$

Hence 3.6.7 indicates that $P(-\phi(1 - dV)) \geqslant s$.

Now if $P(\phi) < s$ then we see that

$$\frac{\phi(\underline{x})}{\|\underline{x}\|^{s}} = \frac{\phi(\underline{x})}{\|\underline{x}\|^{P(\phi)}} \|\underline{x}\|^{P(\phi)-s} \rightarrow \infty \quad \text{as } \underline{x} \rightarrow \underline{0} \qquad 3.6.8$$

along a path Γ' for which

$$\frac{\phi(\underline{x})}{\|\underline{x}\|^{P(\phi)}} \rightarrow c(\Gamma') \neq 0.$$

Substituting 3.6.8 into 3.6.7 leads to a contradiction in limits as $\underline{x} \rightarrow \underline{0}$.

Hence $P(\phi) \geqslant s$ and we may similarly prove that $P(d\phi V) \geqslant s$. This shows that

$$\min\,(P(\phi), P(d\phi V)) \geqslant s \qquad 3.6.9$$

and substituting 3.6.2 into 3.6.9 gives the result 3.6.1. End of proof.

Theorem 3.6.2

$$\min_{i=1,..n} \left(P(f_i) + P(\frac{\partial V}{\partial x_i})\right) \leqslant \min\,(P(\phi), P(d\phi V))$$

Proof

Let $\min_{i=1,..,n} \left(P(f_i) + P(\frac{\partial V}{\partial x_i})\right) = s$

Then using the relationships

$$P(f_i) + P(\frac{\partial V}{\partial x_i}) \geqslant s$$

$i = 1,...,n$

instead of 3.6.6 the proof becomes exactly as for theorem 3.6.1. End of proof.

We note by examples that these inequalities in theorems 3.6.1, 3.6.2 can be strict. Consider the example

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y \end{aligned} \qquad 3.6.10$$

Then if we use $\phi(x,y) = 2x^4 + 4y^4$ 3.6.11

$$d = 0$$

and then substitute 3.6.10, 3.6.11 into 3.1.1 we may solve 3.1.1 to obtain the analytic solution

$$V(x,y) = x^2 + y^4.$$

Hence for this example

$$P(\underline{f}) = 1$$

$$P(\frac{\partial V}{\partial \underline{x}}) = 1$$

$$P(\phi) = 4$$

$$P(d\phi V) = \infty$$

which satisfies theorem 3.6.1 with strict inequality, although

$$P(f_i) + P\left(\frac{\partial V}{\partial x_i}\right) = 4 \qquad i = 1,2,$$

and theorem 3.6.2 is satisfied with equality.

Now consider the example

$$\begin{aligned} \dot{x} &= -2y^3 \\ \dot{y} &= x. \end{aligned} \tag{3.6.12}$$

Then if we solve the equation

$$-2y^3\frac{\partial V}{\partial x} + x\frac{\partial V}{\partial y} = 0 \tag{3.6.13}$$

we obtain the analytic solution

$$V(x,y) = x^2 + y^4. \tag{3.6.14}$$

Hence we have satisfied theorem 3.6.2 with

$$\min_{i=1,2} \left(P(f_i) + P\left(\frac{\partial V}{\partial x_i}\right)\right) = 4$$

$$\min\,(P(\phi),P(d\phi V)) = \infty.$$

However the result 3.6.13, 3.6.14 show that

$$x^2 + y^4 = p\ .$$

are trajectories of the system and hence that 3.6.12 is not an asymptotically stable system. It is later shown that asymptotically stable systems for which the series construction is possible satisfy theorem 3.6.2 with equality.

<u>Theorem 3.6.3</u>

$$\min\,(P(\phi),P(d\phi V)) = P(\phi) \tag{3.6.15}$$

if $V(\underline{0}) = 0$.

<u>Proof</u>

We assume that system 3.5.4 is asymptotically stable and $\phi(\underline{x})$, $V(\underline{x})$ are such that theorem 1.7.1 is satisfied. Now we consider $\phi(\underline{x})V(\underline{x})$ and observe that as $\underline{x}\to\underline{0}$

$$\frac{\phi(\underline{x})V(\underline{x})}{\|\underline{x}\|^{P(V)+P(\phi)}} \longrightarrow c_\phi(\Gamma)\,.\,c_V(\Gamma) \tag{3.6.16}$$

as $\underline{x}$ varies along a continuous path Γ in R^n.

Now we also know that

$$\frac{\phi(\underline{x})V(\underline{x})}{\|\underline{x}\|^{P(\phi V)}} \to c_{\phi V}(\Gamma) \text{ as } \underline{x}\to\underline{0}.$$

Re-arranging 3.6.16 gives

$$\frac{\phi(\underline{x})V(\underline{x})}{\|\underline{x}\|^{P(\phi)+P(V)}} = \frac{\phi(\underline{x})V(\underline{x})}{\|\underline{x}\|^{P(\phi V)}} \quad \|\underline{x}\|^{P(\phi V)-P(\phi)-P(V)} \tag{3.6.17}$$

Now if Γ' be such that $c_{\phi V}(\Gamma') \neq 0$ then letting $\underline{x} \to \underline{0}$ along Γ' in 3.6.17 leads to a contradiction if $P(\phi V) < P(\phi) + P(V)$ as the left hand side becomes infinite contradicting 3.6.16. Hence

$$P(\phi V) \geqslant P(\phi) + P(V).$$

The initial condition $V(\underline{0}) = 0$ implies $P(V) > 0$.
Since $P(V) > 0$ then this yields the result

$$P(\phi V) > P(\phi).$$

Now if $d = 1$ we see that $P(d\phi V) = P(\phi V) > P(\phi)$ while if $d = 0$ we see that $P(d\phi V) = \infty > P(\phi)$ which establishes that

$$P(d\phi V) > P(\phi)$$

and result 3.6.15 is proved. End of proof.

The relationship of $P(V)$ and $P(\frac{\partial V}{\partial \underline{x}})$ is of obvious interest.

Theorem 3.6.4

$$P(V) \leqslant P\left(\frac{\partial V}{\partial \underline{x}}\right) + 1 \qquad 3.6.18$$

providing that the partial derivatives of V exist and are continuous in a neighbourhood of the origin and $V(\underline{0}) = 0$ and $\lim_{\underline{x} \to \underline{0}} \frac{\partial V}{\partial \underline{x}} . \underline{x} \neq 0$ on any path Γ' for which $\dfrac{\left\|\frac{\partial V}{\partial \underline{x}}\right\|}{\|\underline{x}\|^{P(\frac{\partial V}{\partial \underline{x}})}} \to c(\Gamma') \neq 0$.

Proof

The existence and continuity of the partial derivatives $\frac{\partial V}{\partial x_i}$, $i = 1, \ldots, n$, are sufficient to enable us to call on the result of the Mean Value Theorem for partial derivatives (51) which gives

$$V(\underline{x} + \underline{y}) = V(\underline{x}) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\underline{x} + \lambda \underline{y})\, y_i \qquad 3.6.19$$

for some λ where $0 < \lambda < 1$. Now since $V(\underline{0}) = 0$ substituting $\underline{x} = \underline{0}$ into 3.6.19 gives

$$V(\underline{y}) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\lambda \underline{y})\, y_i \qquad 3.6.20$$

Now let us define

$$P(V) = s_1$$
$$P\left(\frac{\partial V}{\partial \underline{x}}\right) = s_2 \qquad 3.6.21$$

and suppose $s_1 > s_2 + 1$.
Expressing 3.6.20 in vector form and dividing by $\|\underline{y}\|^{s_1}$ gives

$$\frac{V(\underline{y})}{\|\underline{y}\|^{s_1}} = \frac{\frac{\partial V}{\partial \underline{x}}(\lambda \underline{y})\cdot \underline{y}}{\|\underline{y}\|^{s_1}} \qquad 3.6.22$$

We may rearrange the R.H.S. of 3.6.22 to give

$$\frac{V(\underline{y})}{\|\underline{y}\|^{s_1}} = \lambda^{s_2}\|\underline{y}\|^{s_2+1-s_1} \frac{\frac{\partial V}{\partial \underline{x}}(\lambda \underline{y})\cdot \underline{y}}{\|\lambda \underline{y}\|^{s_2}\ \|\underline{y}\|} \qquad 3.6.23$$

Now choose $\underline{y}\to\underline{0}$ on a path Γ' for which

$$\frac{\left\|\frac{\partial V}{\partial \underline{x}}(\lambda \underline{y})\right\|}{\|\lambda \underline{y}\|^{s_2}} \longrightarrow c(\Gamma') \neq 0.$$

Then since $0 < \lambda < 1$ we see that letting $\underline{y}\to\underline{0}$ on Γ' in 3.6.23 gives $\dfrac{V(\underline{y})}{\|\underline{y}\|^{s_1}} \longrightarrow \infty$ contradicting the definition 3.6.21.

This contradiction gives $s_1 \leq s_2 + 1$ and hence 3.6.18.
End of proof.

Now that we have some theorems relating $P(\underline{f})$, $p(V)$, $P(\phi)$ we can establish that for Lyapunov functions to exist for an asymptotically stable system which are obtained from the Zubov equation that there is a constraint on $\phi(\underline{x})$.

<u>Theorem 3.6.5</u>

If $\phi(\underline{x})$, $V(\underline{x})$ exist satisfying the conditions of theorem 1.7.1 and if the restrictions of theorem 3.6.4 are satisfied then

$$P(\phi) > P(\underline{f}) - 1. \qquad 3.6.24$$

Alternatively there are no $V(\underline{x})$, $\phi(\underline{x})$ satisfying theorem 1.7.1 where

$$P(\phi) \leq P(\underline{f}) - 1.$$

<u>Proof</u>

Since the Zubov equation requires $V(\underline{x})$ and its partial derivatives at the origin to be continuous so that a solution exists then we may use theorem 3.6.4 which together with the initial conditions gives

$$P\left(\frac{\partial V}{\partial \underline{x}}\right) > -1. \qquad 3.6.25$$

Then putting 3.6.25 into the result 3.6.1 of theorem 3.6.1 we obtain

$$P(\underline{f}) - 1 < \min\,(P(\phi), P(d\phi V)). \qquad 3.6.26$$

Finally using theorem 3.6.3 and substituting 3.6.15 into 3.6.26 leaves us with the relationship 3.6.24.
End of proof.

To see that theorem 3.6.5 is the strongest restriction available we may consider the one-dimensional example

$$\dot{x} = -\mathrm{sign}(x)\ |x|^{3-s} \qquad 3.6.27$$

where $s > 0$. Using $\phi(x) = x^2$, $d = 0$ we may solve the Zubov equation

$$-\mathrm{sign}(x)\ |x|^{3-s}\ \frac{dV(x)}{dx} = -x^2$$

to obtain the analytic solution

$$V(x) = \frac{|x|^s}{s} \qquad 3.6.28$$

For all $s > 0$ 3.6.28 shows that 3.6.27 is asymptotically stable at the origin and we have

$$P(V) = s$$
$$P(\phi) = 2$$
$$P(f) = 3-s$$

The result of theorem 3.6.5 could be arrived at another way. Theorem 3.5.1 requires that system 3.5.3 does not satisfy a Lipschitz condition in a neighbourhood of the origin, and theorem 3.5.4 shows that this requires

$$P\left(\frac{f}{\phi(1-dV)}\right) < 1$$

which implies that

$$P(\underline{f}) < P(\phi) + 1$$

Now since a lot of the Zubov construction is based on constructing a series solution of a P.D.E. and therefore requiring $\underline{f}(\underline{x})$ and $\phi(\underline{x})$ to have a power series in integral powers of x_i, $i = 1,\ldots,n$ we can establish the corresponding relationships for the series construction. We have seen in theorem 2.2.1 that the powers of the lowest degree terms in $\underline{f}(\underline{x})$, $\phi(\underline{x})$ are constrained to satisfy a relationship similar to 3.6.24. We now formally state this relationship in terms of definitions 3.2.1, 3.2.2.

<u>Theorem 3.6.6</u>

If $\underline{f}(\underline{x})$, $\phi(\underline{x})$ have a power series expansion of the form 2.2.1, 2.2.2 with lowest powers s_i, $i=1,\ldots,n$, and q respectively then the Zubov partial differential equation can only yield a positive definite $V(\underline{x})$ in some neighbourhood of the origin for asymptotically stable systems providing

$$P(\phi) \geqslant P(\underline{f}) + 1 \qquad 3.6.29$$

where $P(\phi) = q$

$P(\underline{f}) = \min(s_1,\ldots,s_n)$.

<u>Proof</u>

It has been shown in section 2 of Chapter 2 that since $s = \min(s_1,\ldots,s_n)$ implies the n-dimensional version of 2.2.10, by recourse to theorem 2.2.1 we see that

$$q \geqslant \min(s_1,\ldots,s_n) + 1 \qquad 3.6.30$$

From theorem 3.2.5 we see that

$$P(\phi) = q$$
$$P(f_i) = s_i \quad i = 1,\ldots,n. \qquad 3.6.31$$

By theorem 3.2.1 we see that

$$P(\underline{f}) = \min(P(f_1),P(f_2),\ldots,P(f_n)) \qquad 3.6.32$$

Substituting 3.6.31, 3.6.32 into 3.6.30 leaves us with 3.6.29 . End of proof.

<u>Theorem 3.6.7</u>

If $\underline{f}(\underline{x})$, $\phi(\underline{x})$ have series constructions as 2.2.1, 2.2.2 and the system 3.5.4 is asymptotically stable at the origin then

$$\min_{i=1,\ldots,n} (P(f_i) + P(\frac{\partial V}{\partial x_i})) = P(\phi) \qquad 3.6.33$$

<u>Proof</u>

Using 3.6.32 we may re-write 2.2.1 with the range of m in $f_i(\underline{x})$ from s to ∞ where $P(\underline{f}) = s$. Now suppose that 3.6.33 is not true. Then by theorems 3.6.2, 3.6.3 we see that

$$\min_{i=1,\ldots,n} (P(f_i) + P(\frac{\partial V}{\partial x_i})) < P(\phi) \qquad 3.6.34$$

3.6.34 implies by theorem 3.2.5 that in the series construction 2.2.12 the L.H.S. of 3.1.1 has lowest homogeneous terms of total power $\min((P(f_i) + P(\frac{\partial V}{\partial x_i}))$ while $\phi(\underline{x})(1 - dV(\underline{x}))$ does not. Now we solve another equation

$$\sum_{i=1}^{n} f_i(\underline{x}) \frac{\partial W(\underline{x})}{\partial x_i} = 0 \qquad 3.6.35$$

The same terms of total power $\min(P(f_i) + P(\frac{\partial V}{\partial x_i}))$ can still be the lowest degree terms in the left hand side of 3.6.35. Hence it is possible to find $W(\underline{x})$ such that the dominant terms of $W(\underline{x})$ in a neighbourhood of the origin are the same as those of $V(\underline{x})$. That is, we may write

$$V(\underline{x}) = W(\underline{x}) + u(\underline{x})$$

where $u(\underline{x})$ is "small" in comparison to V,W. If $V(\underline{x})$ is positive definite in a neighbourhood of the origin then $W(\underline{x})$ is also positive definite near the origin. But from 3.6.35 and theorem

1.5.3 we observe that

$$\dot{W}(\underline{x}) = 0$$

and so $W(\underline{x}) = p$ are the system trajectories showing that the origin of 3.5.4 cannot be asymptotically stable. Thus 3.6.34 is a false assumption and we are left with 3.6.33.
End of proof.

7. Admissible $f(\underline{x})$

Consider the one-dimensional example

$$\dot{x} = -x(1 - x)^{\frac{1}{2}} \quad \text{for } x < 1. \qquad 3.7.1$$

3.7.1 is asymptotically stable in the region $(-\infty,1)$. Now let us solve the Zubov equation in the form

$$-x(1 - x)^{\frac{1}{2}} \frac{dV}{dx}(x) = -x^2. \qquad 3.7.2$$

The analytic solution of 3.7.2 is found by integration to be

$$V(x) = \frac{4}{3} - \frac{4}{3}(1 - x)^{1\frac{1}{2}} - 2x(1 - x)^{\frac{1}{2}}. \qquad 3.7.3$$

Thus we see that, although 3.7.3 is a Lyapunov function establishing the asymptotic stability of the origin of 3.7.1, the boundary of the D.O.A. is not given by $V = \infty$. Admittedly $x = 1$ is the boundary of definition of $f(x)$ and hence of $V(x)$, but we can define any system which satisfies 3.7.1 for $x < 1$ and another relationship for $x > 1$ with as many derivatives as required continuous at $x = 1$. Then $V(x)$ will also have similar properties and satisfy 3.7.3 for $x < 1$ and some other solution for $x > 1$.

This result appears to contradict theorems 1.7.1 and 1.7.3 but in fact it does not. The reason for this can be seen by looking at the solution of 3.7.1. The solution of 3.7.1 when integrated is given by

$$x(t) = \frac{4ae^t}{(ae^t+1)^2}$$

where a is arbitrary.

Now we see that $x(t) \equiv 1$ for all t also satisfies 3.7.1. Hence there are two solutions

$$x(t) = 1$$

$$x(t) = \frac{4e^t}{(e^t+1)^2}$$

each satisfying 3.7.1 and the initial condition $x(0) = 1$.

Hence $x = 1$ may also be in the D.O.A. and if $f(\underline{x})$ is defined for $x > 1$ such that $x \to 1+$ in finite time for $x(0) > 1$ then all space may be in the D.O.A.. This phenomenon occurs because the R.H.S. of 3.7.1 does not satisfy a Lipschitz condition in every neighbourhood of $x = 1$. This is also the condition that 3.7.1 has a unique solution. The Zubov theory assumes

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad 3.7.4$$

satisfies conditions that guarantee existence and uniqueness of the solutions $\underline{x}(t)$ given an initial $\underline{x}_0$ for $t = 0$. Clearly if the solutions of 3.7.4 are not unique then the D.O.A. depends on how we define D.O.A.s for such systems.

Comparison of 3.7.2 and 3.7.3 says something about Lyapunov functions for such systems. We can see that as $x \to 1-$, $\frac{dV(x)}{dx} \to \infty$, while $V(x) \to 4/3$. We see that $V(x)$ does not satisfy a Lipschitz condition near $x = 1$ and that this means an infinite gradient $\frac{dV(x)}{dx}$ does not necessarily mean that $V(x)$ is infinite.

Theorem 3.7.1

Given that the one-dimensional Zubov equation

$$f(x)\frac{dV(x)}{dx} = -\phi(x)(1 - dV(x)) \qquad 3.7.5$$

yields a positive definite $V(x)$ in a neighbourhood of the origin for some positive definite $\phi(x)$ then a sufficient condition that $V = \infty$ for $d = 0$ or $V = 1$ for $d = 1$ indicates the boundary of $D(f)$ is that $f(x)$ satisfies a Lipschitz condition in the neighbourhood of x' where

$$f(x') = 0 \qquad 3.7.6$$

and $x' \in \delta D(f)$.

Proof

Suppose that $f(x)$ does satisfy a Lipschitz condition in x in a neighbourhood of $x = x'$. Reference to definition 1.3.2 shows that

$$|f(x) - f(y)| \leq L|x - y| \qquad 3.7.7$$

for all $x, y \in S_\varepsilon(x')$ for some $L, \varepsilon > 0$.

Now substituting $y = x'$ in 3.7.7 and using 3.7.6 we obtain

$$|f(x)| \leq L|x - x'| \qquad 3.7.8$$

for all $x \in S_\varepsilon(x')$ some $\varepsilon > 0$.

By integrating 3.7.5 we find the solution of the Zubov equation given $V = 0$ at $x = 0$ is

$$V(x) = -\int_0^x \frac{\phi(u)}{f(u)}\,du \qquad (d = 0)$$

or $\log(1 - V(x)) = \int_0^x \frac{\phi(u)}{f(u)}\,du \qquad (d = 1).$

Now suppose we fix x and consider $V(y) - V(x)$ where $y \in (x,x')$ if $x' > 0$ or $y \in (x',x)$ if $x' < 0$.

Since this gives

$$V(y) = V(x) - \int_x^y \frac{\phi(u)}{f(u)}\,du \qquad (d = 0) \qquad 3.7.9$$

or $\log(1 - V(y)) = \log(1 - V(x)) + \int_x^y \frac{\phi(u)}{f(u)}\,du \qquad (d = 1)$

We now concentrate on the integrals in 3.7.9 and suppose that $x,y \in S_\varepsilon(x')$ so that 3.7.7 holds. Since $x,y \in (0,x')$ or $x,y \in (x',0)$ then we may assume that $f(u)$ is of constant sign for $u \in (x,y)$ and since $\phi(u)$ is positive definite then $f(u)/\phi(u)$ is of constant sign for $u \in (x,y)$. Hence

$$\left|\int_x^y \frac{\phi(u)}{f(u)}\,du\right| = \int_x^y \frac{\phi(u)}{|f(u)|}\,|du|. \qquad 3.7.10$$

Substituting 3.7.8 into 3.7.10 we obtain

$$\left|\int_x^y \frac{\phi(u)}{f(u)}\,du\right| \geqslant \frac{1}{L}\int \frac{\phi(u)}{|u-x'|}\,|du| \qquad 3.7.11$$

An appropriate transformation of variable in 3.7.11 then shows that as $y \to x'$

$$\int_x^y \frac{\phi(u)}{|f(u)|}\,|du| \longrightarrow \infty$$

and hence the integrals of 3.7.9 also become infinite showing that $V \to \infty$ for $d = 0$ or $V \to 1$ for $d = 1$. End of proof.

The converse of theorem 3.7.1 is an interesting question. Clearly by theorem 1.3.3 we know that if the Lipschitz condition 3.7.7 does not hold the solution of the one-dimensional system

$$\dot{x} = f(x) \qquad 3.7.12$$

is not necessarily unique and so its boundary is not necessarily given by 3.7.6. But one wonders whether a system 3.7.12 which <u>does</u> have a unique solution in a neighbourhood of $S_\varepsilon(x')$ for some $\varepsilon > 0$ but <u>does not</u> satisfy a Lipschitz condition in this neighbourhood can lead to a solution of its corresponding Zubov equation in accordance with theorem 1.7.3.

Corresponding results for 2 or more dimensions are difficult to obtain but it is likely again that if the solution of 3.7.4 is not unique then the boundary of the D.O.A. is not indicated by $V = \infty$ or $V = 1$.

8. Conclusions

Most authors, when solving the Zubov partial differential equation, have concentrated on systems with linear parts and on $\phi(\underline{x})$ with quadratic parts. In such cases there is no problem with obtaining Lyapunov functions which if they are in closed form indicate stability and obtain the D.O.A. in full.

In this chapter we have looked into examples where difficulties arise and have shown that the whole concept of Lyapunov functions and the Zubov equation can be put onto a sound theoretical footing. In later chapters the equation is solved numerically in various ways and the theory of this chapter explains the behaviour of numerical solutions.

It would appear that although theorem 1.7.1 holds for systems with unique solutions there may be care needed in obtaining $\phi(\underline{x})$ and $V(\underline{x})$.

Chapter 4

The One-Dimensional Zubov Equation

1. Introduction

In this chapter we look exclusively at Zubov's equation applied to one-dimensional systems. That is, we consider systems defined by the differential equation

$$\dot{x}(t) = f(x(t)) \qquad 4.1.1$$

The Lyapunov function V(x) is used to test the stability of 4.1.1 and the chain rule expansion of theorem 1.5.3 becomes

$$\dot{V}(x) = \dot{x}(t)\frac{\partial V}{\partial x} \qquad 4.1.2$$

Since V is only dependent on x the partial derivative $\frac{\partial V}{\partial x}$ is the same as the total derivative $\frac{dV}{dx}$. Now if we let

$$\dot{V}(x) = -\phi(x)(1 - dV(x)) \qquad 4.1.3$$

and substitute 4.1.1 and 4.1.3 into 4.1.2 we obtain Zubov's equation in one dimension

$$f(x)\frac{dV}{dx} = -\phi(x)(1 - dV(x)). \qquad 4.1.4$$

Equation 4.1.4 may be solved analytically by series method or separation of variables or by numerical methods in ordinary differential equations.

The one-dimensional Zubov equation is entirely different from the version in 2 or more dimensions in that 4.1.4 can be expressed explicitly for V(x). There is also much more knowledge about numerical computation of O.D.E.s than there is about P.D.E.s, and it is much easier to deal with just one independent variable than with 2 or more. What we are looking for is the $V = \infty$ or $V = 1$ contour, and in the one-dimensional case we need only proceed to vary x until we find the contour, whereas in higher orders we have the extra problem of deciding how to alter the independent variables to obtain a point on the contour $V = \infty$, $V = 1$ and to trace the complete contour.

Therefore techniques are studied and developed in this chapter that cannot be applied to higher orders. The techniques considered for higher orders in later chapters are applicable to the one-dimensional case, but its relative simplicity renders such methods unnecessary.

In sections 2 to 5 an analytical look at 4.1.4 is taken with consideration of conditions for asymptotic stability and the choice of $\phi(x)$ and the method of series solution, together with the question of convergence of series and convergence of R.A.S.s. Sections 7 -10 deal with numerical computation of 4.1.4 bearing in mind problems of computational stability and accuracy. Other work in this field is the subject of section 6 and the chapter is concluded with examples and conclusions.

2. Asymptotic Stability in One Dimension

In order to consider the analytic solution of 4.1.4 we need to establish the behaviour of $f(x)$, $\phi(x)$ as x is small or as x increases. We already know that $\phi(x)$ must be positive definite and continuous so that

$$\begin{aligned} \phi(0) &= 0 \\ \phi(x) &> 0 \quad \text{for } x \neq 0 \end{aligned} \tag{4.2.1}$$

by definition 1.5.1.

Also since the origin is defined to be a critical point of 4.1.1 we know that $f(0) = 0$. To see what happens to $f(x)$ for $x \neq 0$ in a neighbourhood of the origin we need the following theorems in which we assume that $f(x)$ and $x(t)$ are continuous in their respective arguments.

Theorem 4.2.1

Suppose $\dot{x} = f(x)$ 4.2.2

$f(0) = 0$ 4.2.3

then the system 4.2.2, 4.2.3 is asymptotically stable at the origin if and only if there exists $x_1 < 0$, $x_2 > 0$ such that for $x_1 \leq x < 0$ then $f(x) > 0$ and for $0 < x \leq x_2$ then $f(x) < 0$.

Proof

Suppose firstly that x_1, x_2 exist satisfying the second half of theorem 4.2.1. We then prove that this ensures 4.2.2 to be asymptotically stable at the origin.

Consider an initial point x_0 such that $x_1 \leq x_0 < 0$ and let $x(t)$ be the solution of 4.2.2 such that $x(0) = x_0$. Now by integrating $\dot{x}(t)$ with respect to t we see that

$$x(t) = x_0 + \int_0^t \dot{x}(t')dt'. \tag{4.2.4}$$

Substituting 4.2.2 into 4.2.4 gives

$$x(t) = x_o + \int_o^t f(x(t'))dt' \qquad 4.2.5$$

Now we know that $f(x_o) > 0$. Therefore providing x and f are continuous in t and x respectively then there exists $t'(x_o)$ such that for $t < t'$ then $f(x(t)) > 0$.

Hence for $t < t'$ we see from 4.2.5 that

$$x(t) > x_o. \qquad 4.2.6$$

Since 4.2.6 is true for any x_o, $x_1 \leq x_o < 0$ we may say that since

$$x(t + \delta t) = x(t) + \int_t^{t+\delta t} f(x(t'))dt'$$

then $x(t + \delta t) > x(t)$ while $\delta t < t'(x)$. Hence $x(t)$ is an increasing function of t while $x_1 \leq x(t) < 0$.

We have shown that either

a) $x(t) \rightarrow 0$ in finite or infinite time or

b) $x(t) \rightarrow x_3 < 0$.

But b) cannot be true as $f(x_3) > 0$ and there exists $t'(x_3)$ such that

$x(t) > x_3$ for $t < t'(x_3)$ given $x(0) = x_3$.

Hence we know that $x(t)$ either reaches the origin in finite time or infinite time given $x(0) = x_o$ where $x_1 \leq x_o < 0$. We may prove similarly that $x(t)$ tends to the origin in finite or infinite time given $x(0) = x_o$ where $0 < x_o \leq x_2$. This shows that $x \rightarrow 0$ from either side of the origin and that 4.2.2 is asymptotically stable.

In fact the solutions of $x(t)$ of 4.2.2 given $x(0) = x_o$, $x_1 \leq x_o \leq x_2$, can only reach the origin in infinite time providing 4.2.2 satisfies the conditions of theorems 1.3.2 and 1.3.3 for existence and uniqueness of solutions.

For suppose there exists T such that

$$x(T) = 0 \text{ given } x(0) = x_o, \quad x_1 \leq x_o < 0 \text{ or } 0 < x_o \leq x_2. \qquad 4.2.7$$

Now from 4.2.2 and 4.2.3 we see that

$$x(t) \equiv 0 \qquad 4.2.8$$

is a solution of 4.2.2. Comparing 4.2.7 and 4.2.8 we may immediately notice that the system 4.2.2 with time reversed and the initial condition $x(T) = 0$ does not have a unique solution. Hence if the solution is unique then the origin is not reached in finite time by any non-trivial trajectory. This completes the first half of the proof.

Now suppose 4.2.2 is asymptotically stable. Then there exists a D.O.A. (a_1,a_2) where $a_1 < 0$, $a_2 > 0$ and either or both of a_1 or a_2 may be infinite. Let x_1,x_2 be any points such that

$$a_1 < x_1 < 0 \qquad 4.2.9$$
$$0 < x_2 < a_2.$$

We show that x_1,x_2 satisfy the second half of theorem 4.2.1 by contradiction.

Suppose there exists x_o such that $x_1 \leq x_o < 0$ and $f(x_o) \leq 0$. If $f(x_o) = 0$ then by integration with respect to t we see that

$$x(t) = x_o + \int_0^t \dot{x}(t')dt'.$$

From 4.2.2 we see that $\dot{x}(0) = 0$ and hence $x(t) = x_o$ for all $t > 0$. In this case x_o is not within the D.O.A. and contradicts the definition of a_1.

While if $f(x_o) < 0$ then by continuity of x and f there exist b_1,b_2 with $b_1 < x_o < b_2$ where $f(x) < 0$ for $x \in (b_1,b_2)$. Then by the same integration procedure as used previously we see that $x(t)$ is a decreasing function of t while $b_1 < x(t) < b_2$.

But if $f(x_o) > 0$ when $x_o < b_1$ we have $x(t)$ is an increasing function of t while $x(t) < b_1$. This shows that b_1 is the limit of $x(t)$ as $t \to \infty$ again contradicting the definition of a_1.

We have thus proved that there cannot exist x_o, $x_1 \leq x_o < 0$, such that $f(x_o) \leq 0$ if 4.2.2 is asymptotically stable. It can be proved identically that for x_2 in 4.2.9, there does not exist x_o, $0 < x_o \leq x_2$ such that $f(x_o) \geq 0$. This proves that x_1,x_2 given in 4.2.9 satisfy the conditions of theorem 4.2.1 and completes the proof.

Using theorem 4.2.1 and equation 4.2.1 we now know that for $x > 0$ in a neighbourhood of the origin

$$\frac{\phi(x)}{f(x)} < 0 \qquad 4.2.10$$

and for $x < 0$ in a neighbourhood of the origin

$$\frac{\phi(x)}{f(x)} > 0. \qquad 4.2.11$$

Now let us integrate 4.1.4 by the method of separation of variables. This gives

$$\frac{dV}{1-dV} = \frac{-\phi(x)dx}{f(x)}. \qquad 4.2.12$$

In a neighbourhood of the origin $V(x)$ is small. Hence if we consider $dx > 0$, $x > 0$ and using 4.2.10 we obtain $dV > 0$.

Similarly if $dx \nless 0$, $x < 0$ reference to 4.2.11 gives $dV > 0$. Hence $V(x) > 0$ given $V(0) = 0$. Hence $V(x)$ is positive definite in a neighbourhood of the origin.

Integrating 4.2.12 with respect to each element gives

$$V = V_o - \int_{x_0}^{x} \frac{\phi(x)}{f(x)}\,dx \qquad 4.2.13$$

if $d = 0$, or

$$-\log(1-V) = -\log(1-V_o) - \int_{x_0}^{x} \frac{\phi(x)}{f(x)}\,dx \qquad 4.2.14$$

if $d = 1$.

We are now ready to consider the effect on $V(x)$ of different $\phi(x)$.

3. Theory of Different ϕ

To consider the effect on 4.1.4, 4.2.13, 4.2.14 of using different functions $\phi(x)$ we consider the example

$$\dot{x} = -x^3(1 - x^2). \qquad 4.3.1$$

The D.O.A. of 4.3.1 is $(-1,1)$.

Let $\phi(x) = x^2$, $d = 0$ $\qquad$ 4.3.2

and substitute 4.3.1, 4.3.2 into 4.1.4 gives

$$\frac{dV}{dx} = \frac{1}{x(1-x^2)}. \qquad 4.3.3$$

The solution of 4.3.3 becomes

$$V = \log\left(\frac{x}{\sqrt{1-x^2}}\right) + c \qquad 4.3.4$$

where c is an arbitrary constant. The initial conditions we have are given by $V = 0$ at $x = 0$. However as $x \rightarrow 0$ in 4.3.4, $V \rightarrow -\infty$ for any finite c.

To confirm this we can consider 4.1.1 and 4.1.4 as two parametric representations for x in terms of t or V.

These are

from 4.1.1 $\qquad \dfrac{dx}{dt} = f(x)$

from 4.1.4 $\qquad \dfrac{dx}{dV} = \dfrac{-f(x)}{\phi(x)(1-dV)}. \qquad$ 4.3.5

Since $f(0) = 0$ we know within the D.O.A. that

$x \rightarrow 0$ as $t \rightarrow \infty$.

Letting $d = 0$ in 4.3.5 we see that if

$\dfrac{f(x)}{\phi(x)} \rightarrow 0$ as $x \rightarrow 0$

$x(V)$ has the same properties as $x(t)$ and $x \to 0$ as $V \to -\infty$. Example 4.3.1, 4.3.2 and its analytic solution 4.3.4 confirms that this happens.

This is the one-dimensional version of theorem 3.6.6 which insists that for the Zubov equation to be soluble for $V(x)$

$$P(f) \leq P(\phi) - 1$$

where
$$f(x) \approx x^{P(f)}$$
$$\phi(x) \approx x^{P(\phi)} \quad \text{as } x \to 0 \qquad 4.3.6$$

or as definition 3.2.2 where $P(f)$, $P(\phi)$ are integers, and $f(x)$, $\phi(x)$ have a series expansion in a neighbourhood of the origin. To consider the situation where $f(x)$, $\phi(x)$ do not necessarily have power series expansions in integral powers of x, we need theorem 3.6.5 which requires $P(f) < P(\phi) + 1$. 4.3.7

Clearly 4.3.1, 4.3.2 do not satisfy 4.3.7 and no Lyapunov function $V(x)$ exists.

If the Zubov equation can be solved to give a Lyapunov function $V(x)$ satisfying theorem 1.7.1 then we know that $V(x) = 1$ or $V(x) = \infty$ will indicate the boundary of the D.O.A. providing that such a boundary is well-defined, which requires the conditions of theorem 3.7.1.

It has been shown here that not all positive definite continuous functions are admissible to be used as $\phi(x)$.

4. Series Method

The series construction of Zubov's equation has been documented partioularly in 2 dimensions where staightforward integration of 4.2.13 or 4.2.14 is not possible. It is not always possible in one dimension even to directly integrate the Zubov equation and the series construction is mentioned here for that reason.

We assume that $f(x)$, $\phi(x)$, $V(x)$ have power series expansions.

$$f(x) = \sum_{m=s}^{\infty} f_m x^m \qquad 4.4.1$$

$$\phi(x) = \sum_{m=q}^{\infty} \phi_m x^m \qquad 4.4.2$$

$$V(x) = \sum_{m=r}^{\infty} V_m x^m \qquad 4.4.3$$

where $f_s \phi_q V_r \neq 0$. 4.4.4

Now we may substitute 4.4.1, 4.4.2, 4.4.3 into 4.1.4 with the assumption that $V(x)$ is differentiable term by term. This gives

$$\left(\sum_{m=s}^{\infty} f_m x^m\right)\left(\sum_{m=r}^{\infty} mV_m x^{m-1}\right) \equiv -\left(\sum_{m=q}^{\infty} \phi_m x^m\right)\left(1-d\sum_{m=r}^{\infty} V_m x^m\right). \quad 4.4.5$$

Now we wish to equate like terms in the identity 4.4.5. Each side of the identity 4.4.5 has a lowest degree of x. The L.H.S. of 4.4.5 contains terms in x^{s+r-1} and above, while the R.H.S. contains terms in x^q and above if $r \geq 0$, and terms in x^{q+r} and above if $r \leq 0$. However since we require $V(0) = 0$ we see that $r > 0$.

Hence we have that if $s+r-1 < q$ then $rf_sV_r = 0$ which since $r \neq 0$ contradicts the definition of s,r given by 4.4.1 to 4.4.4.

Also if $s+r-1 > q$ then $\phi_q = 0$ which contradicts the definition of q.

Therefore we have established that given the expansions for $f(x)$, $\phi(x)$ in 4.4.1, 4.4.2 that the power series for $V(x)$ is defined as 4.4.3 with

$$r = q - s + 1 \qquad 4.4.6$$

Now it has been shown in theorem 3.2.7 that for $\phi(x)$ to be positive definite we require q integer and even and $\phi_q > 0$.

Now we equate the lowest powers of x in 4.4.5 to obtain

$$(q-s+1)f_sV_{q-s+1} = -\phi_q. \qquad 4.4.7$$

Now if 4.1.1 is asymptotically stable then the solution of Zubov's equation will yield a positive definite $V(x)$. That is, we would obtain r even integer and $V_r > 0$. Also if such r, V_r are obtained we know that 4.1.1 ia asymptotically stable at the origin. This leads to

<u>Theorem 4.4.1</u>

If $\dot{x} = f(x)$, $f(0) = 0$, $\qquad$ 4.4.8

and $\dot{V} = -\phi(x)(1-dV)$

where $\phi(x)$ is positive definite in the whole and continuous and $V(x)$ is positive definite in a neighbourhood of the origin, and if $f(x)$, $\phi(x)$, $V(x)$ have series expansions as in 4.4.1 to 4.4.3 with s odd integer and $f_s < 0$ then 4.4.8 is asymptotically stable.

<u>Proof</u>

If $\phi(x)$ is p.d. then q is an even integer and $\phi_q > 0$. Hence by 4.4.6, we see that if s is an odd integer then r is an even integer. Also since $V(0) = 0$ we know that $r > 0$ and thus by 4.4.6 $q-s+1 > 0$ also. Therefore if s is odd and $f_s < 0$ then by 4.4.7 we see that $V_{q-s+1} > 0$ and so by theorem 3.2.7 and the theorems of Lyapunov we prove that 4.1.1 is asymptotically stable. End of proof.

Now having established r and V_r we want to systematically establish V_m for $m \geqslant r$. Substituting for r from 4.4.6 into 4.4.5 gives

$$\left(\sum_{m=s}^{\infty} f_m x^m\right)\left(\sum_{m=q-s+1}^{\infty} m\dot{V}_m x^{m-1}\right) \equiv -\left(\sum_{m=q}^{\infty} \phi_m x^m\right)\left(1-d\sum_{m=q-s+1}^{\infty} V_m x^m\right). \qquad 4.4.9$$

Expanding the products in 4.4.9 gives

$$\sum_{i=s}^{\infty}\sum_{j=q-s+1}^{\infty} j f_i V_j x^{i+j-1} \equiv -\sum_{m=q}^{\infty}\phi_m x^m + d\sum_{i=q}^{\infty}\sum_{j=q-s+1}^{\infty}\phi_i V_j x^{i+j}. \qquad 4.4.10$$

Collecting coefficients of terms in x^m in 4.4.10 we obtain

$$\sum_{m=q}^{\infty}\left(\sum_{j=q-s+1}^{m-s+1} jV_j f_{m-j+1}\right)x^m \equiv -\sum_{m=q}^{\infty}\phi_m x^m + d\sum_{m=2q-s+1}^{\infty}\left(\sum_{j=q-s+1}^{m-q}\phi_{m-j}V_j\right)x^m. \qquad 4.4.11$$

Since 4.4.11 is an equation which should be identically satisfied by V(x) given by 4.4.3 we may equate coefficients of x^m. This gives

a) for $q \leqslant m \leqslant 2q-s$

$$\sum_{j=q-s+1}^{m-s+1} jV_j f_{m-j+1} = -\phi_m \qquad 4.4.12$$

b) for $2q-s+1 \leqslant m$

$$\sum_{j=q-s+1}^{m-s+1} jV_j f_{m-j+1} = -\phi_m + d\sum_{j=q-s+1}^{m-q}\phi_{m-j}V_j. \qquad 4.4.13$$

4.4.12 and 4.4.13 form the basis for generating V_m, $m = q-s+1, q-s+2, \ldots$.

From 4.4.12 and 4.4.13 we see that given

V_j for $j = q-s+1, \ldots, m-s$

4.4.12 or 4.4.13 may then be used to compute V_{m-s+1}. Hence re-writing 4.4.12, 4.4.13 explicitly in terms of V_{m-s+1} we obtain

a) for $m = q$

$$V_{q-s+1} = \frac{-\phi_q}{(q-s+1)f_s} \qquad 4.4.14$$

b) for $q+1 \leqslant m \leqslant 2q-s$

$$V_{m-s+1} = \frac{-\phi_m}{(m-s+1)f_s} - \frac{1}{(m-s+1)f_s}\sum_{j=q-s+1}^{m-s} jV_j f_{m-j+1} \qquad 4.4.15$$

c) for $2q-s+1 \leqslant m$

$$V_{m-s+1} = \frac{-\phi_m}{(m-s+1)f_s} - \frac{1}{(m-s+1)f_s}\sum_{j=q-s+1}^{m-s} jV_j f_{m-j+1} + \frac{d}{(m-s+1)f_s}\sum_{j=q-s+1}^{m-q}\phi_{m-j}V_j \qquad 4.4.16$$

The final modification to simplify 4.4.14 to 4.4.16 is to express them in terms of V_m explicitly rather than V_{m-s+1}.

Hence the final form is given by

a) for m = q-s+1

$$V_{q-s+1} = \frac{-\phi_q}{(q-s+1)f_s} \qquad 4.4.17$$

b) for $q-s+2 \leq m \leq 2q-2s+1$

$$V_m = \frac{-\phi_{m+s-1}}{mf_s} - \frac{1}{mf_s}\sum_{j=q-s+1}^{m-1} jV_j f_{m-j+s} \qquad 4.4.18$$

c) for $2q-2s+2 \leq m$

$$V_m = \frac{-\phi_{m+s-1}}{mf_s} - \frac{1}{mf_s}\sum_{j=q-s+1}^{m-1} jV_j f_{m-j+s} + \frac{d}{mf_s}\sum_{j=q-s+1}^{m+s-q-1} \phi_{m+s-j-1}V_j \qquad 4.4.19$$

Equations 4.4.17 to 4.4.19 form the recurrence relation from which we compute the series 4.4.3 for V(x).

5. Convergence of Series

Equations 4.4.17 to 4.4.19 can in theory be solved for $m = q-s+1, q-s+2, \ldots, \infty$ and the full series for V(x) obtained. However unless the coefficients V_m, $m \geqslant q-s+1$, can be recognised as being from a series which has a defined infinite sum, we have the problem of convergence of the series.

When computing the value of a series expansion by an algorithm such as 4.4.17 to 4.4.19, somewhere the series has to be truncated. This is usually done when

$$\left| V_m x^m \right| < \varepsilon \qquad 4.5.1$$

for some $\varepsilon > 0$ and for each particular value of x considered. But sometimes V_m can be quite high before 4.5.1 is satisfied, as an example shows.

Consider

$$\dot{x} = -x + x^3 \qquad 4.5.2$$

The D.O.A. of 4.5.2 is given by $x \in (-1,1)$.

Using $\phi(x) = x^2 + x^4 + x^6$ and $d = 0$ the Zubov equation 4.1.4 becomes

$$\frac{dV}{dx} = \frac{x^2+x^4+x^6}{x-x^3} \qquad 4.5.3$$

The solution of 4.5.3 given V(0) = 0 becomes

$$V(x) = -x^2 - \frac{x^4}{4} - \frac{3}{2}\log(1-x^2). \qquad 4.5.4$$

The region of convergence $R_c(\phi, f)$ of 4.5.4 is also given by $x \in (-1,1)$.

The series expansion of $V(x)$ is given as

$$V(x) = \frac{x^2}{2} + \frac{x^4}{2} + \frac{3}{2}\sum_{n=3}^{\infty}\frac{x^{2n}}{n}.$$

Fig. 14 shows plots of $V^{(2N)}(x)$ where

$$V^{(2N)}(x) = \frac{x^2}{2} + \frac{x^4}{2} + \frac{3}{2}\sum_{n=3}^{N}\frac{x^{2n}}{n} \qquad 4.5.5$$

plotted as $V^{(2N)}(x)$ versus N for various x.

However $V(0{\cdot}99999) = 15{\cdot}2$ while $V^{(80)}(0{\cdot}99999) = 5{\cdot}0$ and this is because $V_{80} = 0{\cdot}0375$ and shows that convergence is very slow.

This problem can be avoided with extra computation by acceleration techniques. Details of such are found in Shanks (37)

Now any truncation such as 4.5.5 is a Lyapunov function in its own right and computing x to obtain when $V^{(N)}(x) = 1$ or $V^{(N)}(x) = \infty$ where

$$V^{(N)}(x) = \sum_{m=q-s+1}^{N} V_m x^m \qquad 4.5.6$$

is not valid. By theorem 1.6.8 we require to compute p such that $V^{(N)}(x) = p$ is tangential to $\dot{V}^{(N)}(x) = 0$.

Therefore given $V^{(N)}(x)$ by 4.5.6 having computed the coefficients V_m by 4.4.17 to 4.4.19 we need to obtain $\dot{V}^{(N)}(x)$ and then the boundary of the D.O.A. is given by either

$$\dot{V}^{(N)}(x) = 0$$

$$\text{or} \qquad V^{(N)}(x) = 1 \text{ or } \infty \qquad 4.5.7$$

whichever value of x is closer to the origin.

From 4.4.1 and 4.5.6 using 4.1.2 we may obtain $\dot{V}^{(N)}(x)$ as

$$\dot{V}^{(N)}(x) = \Big(\sum_{m=s}^{\infty} f_m x^m\Big)\Big(\sum_{m=q-s+1}^{N} mV_m x^{m-1}\Big). \qquad 4.5.8$$

Hence $\dot{V}^{(N)}(x) = 0$ if either factor of 4.5.8 is zero.

When $\sum_{m=s}^{\infty} f_m x^m = 0$ then x is a critical point of $\dot{x} = f(x)$ and the origin is the only critical point which is inside the D.O.A.. Hence we are looking for x such that

$$\frac{\partial V^{(N)}}{\partial x}(x) = 0. \qquad 4.5.9$$

As an alternative approach to finding zeroes of 4.5.8 we may expand 4.5.8 exactly as in 4.4.9 to obtain

$$\sum_{m=q}^{\infty}\Big(\sum_{j=q-s+1}^{\min(N,m-s+1)} jV_j f_{m-j+1}\Big)x^m = -\sum_{m=q}^{\infty}\phi_m x^m + d\sum_{m=2q-s+1}^{\infty}\Big(\sum_{j=q-s+1}^{\min(N,m-q)}\phi_{m-j}V_j\Big)x^m$$

in which providing $N \geqslant \min(m-s+1, m-q)$ then terms in x^m disappear by the relationships 4.4.12 and 4.4.13. Given a general $V^{(N)}(x)$ 4.5.9 is the easiest equation to analyse.

Non-uniformity of convergence of x satisfying 4.5.7 may be seen by another example. Let us consider the example

$$\dot{x} = -1 + e^{-x} \qquad 4.5.10$$

which has a D.O.A. given by $x \in (-\infty, \infty)$

Let us denote the value of $x > 0$ satisfying 4.5.7 for any given N as x_N. Hence x_N is defined as

$$x_N = \min x,\ x > 0 \text{ where } x \in \left\{ x : \dot{V}^{(N)}(x) = 0 \text{ or } V^{(N)}(x) = \infty \text{ for } d = 0 \text{ or } V^{(N)}(x) = 1 \text{ for } d = 1 \right\}$$

Using $\phi(x) = (-1 + e^{-x})^2$, $d = 0$ 4.5.11

and substituting 4.5.11, 4.5.10 in 4.1.4 gives

$$\frac{dV}{dx} = 1 - e^{-x} \qquad 4.5.12$$

The analytical solution of 4.5.12 given $V(0) = 0$ becomes

$$V(x) = e^{-x} + x - 1. \qquad 4.5.13$$

Substituting 4.5.10, 4.5.11 and $d = 0$ into 4.4.5 and solving for $V(x)$ by 4.4.17 to 4.4.19 gives the full series form

$$V(x) = \sum_{i=2}^{\infty} \frac{(-x)^i}{i!}.$$

Thus we are looking for x such that

$$\sum_{i=2}^{N} \frac{(-x)^i}{i!} = \infty \qquad 4.5.14$$

or

$$\sum_{i=2}^{N} \frac{(-x)^{i-1}}{(i-1)!} = 0\,. \qquad 4.5.15$$

Clearly there is no finite x satisfying 4.5.14. Now it is shown in Appendix F that if N is even 4.5.15 has one zero $x = 0$, while if N is odd there are two zeroes which are $x = 0$ and the series $\{x_3, x_5, x_7, \dots\}$ where x_N satisfies 4.5.15 which increases uniformly.

By definitions 1.7.2, 1.7.3, 1.7.4 we see that for 4.5.10, 4.5.11

$$R_{2N}(\phi, f) = (-\infty, \infty)$$

$$\text{with } R_{2N-1}(\phi, f) = (-\infty, x_{2N-1}) \qquad 4.5.16$$

and hence $R(\phi, f) = (-\infty, \infty)$

which is the same as $D(f)$. However convergence of $R_N \rightarrow R$ is certainly not uniform.

Let us consider again example 4.5.2, this time using $d = 1$ and

$$\phi_m(x) = 2\sum_{j=1}^{m} x^{2j}. \qquad 4.5.17$$

The region of stability indicated by $R_N(\phi_m, f)$ is quite interestin

The Zubov equation becomes

$$(-x + x^3)\frac{dV}{dx}(x) = -\left(2\sum_{j=1}^{m} x^{2j}\right)(1 - V(x)). \qquad 4.5.18$$

For $m = 1$ the solution of 4.5.18 becomes

$$V(x) = x^2 \qquad 4.5.19$$

therefore $V^{(N)}(x) = x^2$, $N \geqslant 2$.

and $\dot{V}^{(N)}(x) = 2x(-x+x^3)$, $N \geqslant 2$. 4.5.20

Therefore the region of stability indicated by 4.5.19, 4.5.20 is given by $(-1,1)$ for all $N \geqslant 2$ and this is also the D.O.A. of 4.5.2. Hence $R_N(\phi_1,f) = R(\phi_1,f) = D(f)$ for f given by 4,5.2, ϕ_1 given by 4.5.17.

For $m = 2$ the solution of 4.5.18 is

$$V(x) = 1 - e^{x^2}(1 - x^2)^2. \qquad 4.5.21$$

$V = 1$ at $x = \pm 1$ while $\dot{V}^{(N)}(x) = 0$ at $x = \pm 1$ or when $x = x_N$ and

$$\frac{dV^{(N)}}{dx}(x_N) = 0. \qquad 4.5.22$$

The zeroes x_N of 4.5.22 satisfy the relations $x_N > x_{N+1}$, $x_N > 1$, $N \geqslant 2$. The rapid convergence of the series x_N is seen from the fact that $|x_{22} - 1\cdot 0| < 10^{-5}$. Hence the region $R_N(\phi_2,f)$ is bounded by $f = 0$ and just as for $m = 1$, $R_N = R = D$.

But for $m = 3$

$$V(x) = 1 - (1-x^2)^3 e^{(2x^2 + \frac{x^4}{2})} \qquad 4.5.23$$

and the zeroes x_N of $\frac{dV^{(N)}}{dx}(x) = 0$ are such that $|x_N| < 1$ for $N \geqslant 10$. The series x_N is plotted against N in fig. 15. Hence $R_N(\phi_3,f) = (-1,1)$, $N < 10$, but $R_N(\phi_3,f) \subset (-1,1)$, $N \geqslant 10$, while as $N \rightarrow 0$, $R_N(\phi_3,f) \rightarrow (-1,1)$.

In this case we have shown that although $R(\phi_3,f) = D(f)$ using higher order expressions for $\phi(x)$ and more terms in the series expansion of $V(x)$ is inferior to the easier more manipulable lower orders.

This situation of non-uniformity of convergence occurs throughout various attempts to compute the series expansion of $V(x)$ from Zubov's method in one and more dimensions.

The non-uniformity is still a matter of conjecture. Shields and Storey (38) conjecture that if $R_c(\phi,f) \subset D(f)$ then non-uniformity occurs in the sense that there exist m_1, m_2 such that $m_1 > m_2$

while either $R_{m_1}(\phi,f) \subset R_{m_2}(\phi,f)$ 4.5.24

or by some measure of region "size" that R_{m_1} is "less than" R_{m_2}.

Zubov (12) claims that $R_N(\phi,f) \longrightarrow D(f)$ as $N \rightarrow \infty$ if $D(f) \subseteq R_c(\phi,f)$ but even then convergence is not necessarily uniform.

The example 4.5.10 showed this. For this example $D(f)$ is all-space while the region of convergence of 4.5.13 $R_c(\phi,f)$ is also all-space. While from 4.5.16 we see that $R_n \rightarrow D$ but not uniformly.

Example 4.5.2 with ϕ given by 4,5.17 showed that it is possible to achieve the situation where

$$D(f) \subset R_c(\phi,f). \qquad 4.5.25$$

For each function 4.5.19, 4.5.21 or 4.5.23 we see that $R_c(\phi_m,f)$ is all-space, while convergence of $R_N(\phi_3,f)$ to $(-1,1)$ was not uniform in the sense of 4.5.24 but that it is uniform in the sense that for large N the set $R(\phi,f) - R_N(\phi,f)$ diminished uniformly to the null set.

It can be proved that for the modified Zubov equation($d=0$) situation 4.5.25 is never possible while clearly for $d = 1$ the example 4.5.2 shows that 4.5.25 is possible.

Theorem 4.5.1

If $\dot{x} = f(x)$

and $\dot{V}(x) = -\phi(x)$

then $R_c(\phi,f) = (-r,r)$ 4.5.26

where $r = \min(-a,b)$

where $D(f) = (a,b)$ and either of a or b may be infinite.

Proof

The Zubov equation 4.1.4 becomes for $d = 0$

$$\frac{dV}{dx} = -\frac{\phi(x)}{f(x)}. \qquad 4.5.27$$

Now if $D(f) = (a,b)$ then, by assumptions of uniqueness of 4.1.1 such as theorem 3.7.1, a and b are either infinite or critical points of $f(x)$. Therefore we may write the series expansion 4.4.1 down as

$$f(x) = (1 - \frac{x}{a})^{n_1} (1 - \frac{x}{b})^{n_2} g(x) \qquad 4.5.28$$

where $xg(x) < 0$ for $x \in (a,b)$, $x \neq 0$ and $g(x)$ has a power series expansion

$$g(x) = \sum_{m=0}^{\infty} g_m x^m$$

and where n_1, n_2 are positive integers and the case $a = -\infty$ or $b = \infty$ are covered by the expression 4.5.28.

Substituting 4.5.28 into 4.5.27 gives

$$\frac{dV}{dx} = \frac{-\phi(x)}{(1-\frac{x}{a})^{n_1}(1-\frac{x}{b})^{n_2} g(x)}. \qquad 4.5.29$$

Now since by theorem 1.7.1 $\phi(x)$ is taken to be positive definite then by definition 1.5.1 we know that $1-x/a$ and $1-x/b$ are not factors of $\phi(x)$. So we may expand 4.5.29 in partial fractions form to give

$$\frac{-\phi(x)}{(1-\frac{x}{a})^{n_1}(1-\frac{x}{b})^{n_2}.g(x)} = \sum_{m=0}^{\infty} c_m x^m + \frac{g_1(x)}{g(x)} + \sum_{m=1}^{n_1} \frac{d_m}{(1-\frac{x}{a})^m} + \sum_{m=1}^{n_2} \frac{e_m}{(1-\frac{x}{b})^m} \quad 4.5.30$$

Now by multiplication of 4.5.30 by $(1-\frac{x}{a})^{n_1}(1-\frac{x}{b})^{n_2}\, g(x)$ and letting $x = a$ we obtain

$$-\phi(a) = d_{n_1}(1 - \frac{a}{b})^{n_2}\, g(a).$$

Similarly we have

$$-\phi(b) = e_{n_2}(1 - \frac{b}{a})^{n_1}\, g(b).$$

Hence by definition of $g(x)$ in 4.5.28 and of $\phi(x)$ we see that $d_{n_1} \neq 0$, $e_{n_2} \neq 0$. Hence $\frac{dV}{dx}$ contains a term in $\frac{1}{(1-\frac{x}{a})^{n_1}}$ and $\frac{1}{(1-\frac{x}{b})^{n_2}}$ and when 4.5.29 is integrated $V(x)$ contains at least term in $\frac{1}{(1-\frac{x}{a})^{n_1-1}}$ or $\log(1-x/a)$ and a term in $\frac{1}{(1-\frac{x}{b})^{n_2-1}}$ or $\log(1-x/b)$. These terms have a region of convergence given $(a,-a)$ and $(-b,b)$ respectively, proving that $R_c(\phi,f)$ is given by 4.5.26 and that

$$R_c(\phi,f) \subseteq D(f).$$

This ends the proof.

6. Other Algorithms

The one-dimensional form of Zubov's equation is really sufficiently simple a problem to solve that other methods which are significantly different from the series construction or from numerical integration of an O.D.E. are difficult to find. Other methods for solving Zubov's equation in higher dimensions were the subject of Chapter 2 and their application to one dimension will be considered here.

The Lie series method (25), (26) simply involves the computation of $x(t)$, $V(t)$ from the Taylor series expansions

$$x(t) = \sum_{m=0}^{\infty} \frac{x^{(m)}(0)\ t^m}{m!}$$
$$V(t) = \sum_{m=0}^{\infty} \frac{V^{(m)}(0)\ t^m}{m!} \qquad 4.6.1$$

where $x^{(m)}(0)$, $V^{(m)}(0)$ are computed from the recursive relationship

$$x^{(m)}(0) = \frac{d}{dt} x^{(m-1)}(t)\Big|_{t=0} \qquad 4.6.2$$

$$V^{(m)}(0) = \frac{d}{dt} V^{(m-1)}(t)\Big|_{t=0}$$

$m = 1,2,\ldots$

By 4.1.1 and 4.1.3 the R.H.S.s of 4.6.2 may be computed and 4.6.2 becomes

$$x^{(m)}(0) = \frac{d^{m-1}}{dt^{m-1}} f(x(t))\Big|_{t=0} \qquad 4.6.3$$

$$V^{(m)}(0) = \frac{-d^{m-1}}{dt^{m-1}} \phi(x(t))(1 - dV(x(t))\Big|_{t=0}$$

$m = 1,2,\ldots$

Having computed the coefficients in 4.6.1 by 4.6.3 we then let t become negative in 4.6.1 using the relation $x(0) = \pm \delta$ $V(0) = 0$. By this means we obtain $x(t) \rightarrow x'$ where either $x' = \pm\ \infty$ or $f(x') = 0$ and $V(t) \rightarrow \infty$ or 1, providing x' satisfies the conditions of theorem 3.7.1 to guarantee uniqueness of solutions in every neighbourhood of x'. The question has to be asked, whether it is necessary to compute $V(t)$ at all since if x' is finite we could just as easily terminate computation when

$$\left|x(t+\delta t) - x(t)\right| < \varepsilon \quad \text{for some } \varepsilon > 0$$

where δt is the time interval used to compute $x(t)$ from 4.6.1.

The method Troch (39) uses to integrate 4.1.1 and 4.1.3 by analogue computer along system trajectories, and the method of Davidson and Cowan (29) to integrate 4.1.1 and test whether a cycle of the origin is stable or not are not applicable as the system trajectories become trivial in one dimension. Texter's (34) thoughts on polar co-ordinate systems is likewise trivial in one dimension, as no change in co-ordinates significantly affects 4.1.1 and 4.1.3.

Thus we have found that 4.1.3 is solved either by power series for $V(x)$ or by numerical integration or possibly by Taylor series if 4.6.3 can be differentiated. But other methods used for 2 or more dimensions become similar to those three methods when solving the simpler one-dimensional problem.

7. Numerical Methods

In this section we consider solution of 4.1.4 by numerical methods. If we write 4.1.4. as

$$\frac{dV}{dx} = \frac{-\phi(x)(1-dV)}{f(x)} \qquad 4.7.1$$

we obtain an O.D.E. for $\frac{dV}{dx}$ in terms of V and x. For these equations there is a variety of publications on their solution, for example (52), (53), (54), (55). What is proposed in the next three sections is a method which takes into account the properties of 4.7.1.

First of all we require to know something of the behaviour of $\phi(x)/f(x)$ and $V(x)$ as $x \to x'$ where

$$f(x') = 0. \qquad 4.7.2$$

Integrating 4.7.1 with respect to x we obtain

$$\text{for } d = 0 \qquad V(x) = V(x_0) - \int_{x_0}^{x} \frac{\phi(x)}{f(x)}\,dx \qquad 4.7.3$$

$$\text{for } d = 1 \qquad V(x) = 1 - (1-V(x_0))e^{\int_{x_0}^{x} \frac{\phi(x)}{f(x)}dx}.$$

Now it is known that if $\int \frac{\phi(x)}{f(x)}dx$ has a singularity at x' then x' must be a zero of $f(x)$ or a singularity of $\phi(x)$. The converse is not true as seen by the example

$$\phi(x) = |x|^{\frac{1}{2}}$$

$$f(x) = -|x|^{\frac{1}{2}}|1-x|^{\frac{1}{2}}\text{sign}(x)\text{sign}(1-x) \qquad 4.7.4$$

This example serves to illustrate how ϕ/f has a singularity as $x \to 1_-$ but $V(x)$ is finite.

Using $d = 0$ and substituting 4.7.4 into 4.7.3 gives for $0 \leq x \leq 1$

$$V(x) = -2(1-x)^{\frac{1}{2}} + 2 \qquad 4.7.5$$

which is finite as $x \to 1_-$. This phenomenon does not happen in the Zubov theory unless

$$\text{as } x \to x', \qquad f(x) \approx |x-x'|^{P} \quad \text{where } P < 1. \qquad 4.7.6$$

Theorem 3.7.1 confirms that $V(x) \to \infty$ as $x \to x'$ only if $P \geq 1$ in 4.7.6. Generally $P \geq 1$ in 4.7.6 but this is not always the case, However in integrating 4.7.1 we shall need to evaluate 4.7.1 near x' and the computation becomes unstable.

To overcome this we introduce a system of turning 4.7.1 "upside down" and integrating for x in terms of V. Then as

$$x \to x', \quad \frac{f(x)}{\phi(x)} \to 0$$

and as V increases computation of x is stable.

<u>Definition 4.7.1</u>

If x' is given by

$$f(x') = 0$$

then P(f,x') is defined to be such that

$$\frac{f(x)}{|x-x'|^{P(f,x')}} \longrightarrow c(f,x') \quad \text{as } x \to x'$$

and $|x| < |x'|$, c(f,x') finite, non-zero.

Definition 4.7.1 is the one-dimensional equivalent of definitions 3.2.1, 3.2.2 considered at x = x' instead of at the origin.

If $P(f,x') < 1$ then as seen for the example 4.7.4 when solved for V(x) given by 4.7.5 that V(x) does not necessarily become infinite when $x \to x'$. But by theorems 1.3.3, 3.5.4 the solution of 4.1.1 is not necessarily unique and this case is therefore not considered. We assume as up to now that $P(f,x') \geq 1$.

8. Numerical Algorithm

To see what is meant by instability of computation of 4.7.1 we consider integration of 4.7.1 for fixed steps in x. Let x_n be defined by $x_n = nh$, $n = 1,2,\ldots$ with V_n defined as the computed value of $V(x_n)$. If x' is finite where x' satisfies 4.7.2 then for some n we find the situation where $x_n < x' \leq x_{n+1}$. Numerical integration of V_{n+1} leads to a breakdown caused by evaluating $\frac{dV}{dx}$ near a singularity and by integrating 4.7.1 for the same step-size h when V increases much more rapidly than x.

A numerical method for integrating 4.7.1 to give V in terms of x would have to include some means of altering the step-size h when $\frac{dV}{dx}$ became large to ensure that for given x_n we may define h_n and x_{n+1} such that

$$x_{n+1} = x_n + h_n$$

and $x_{n+1} < x'$, $n = 1,2\ldots$

We propose to overcome this problem by re-writing 4.7.1 as

$$\frac{dx}{dV} = \frac{-f(x)}{\phi(x)(1-dV)}. \qquad 4.8.1$$

Then we may allow V to increase to ∞ or 1 without restriction and x cannot go outside the boundary of the D.O.A..

Symbolically we may write 4.7.1 and 4.8.1 as

$$\frac{dV}{dx} = F(x,V)$$

$$\frac{dx}{dV} = F(x,V)^{-1}$$

where F represents any function of 2 variables. It is reasonable to compute 4.7.1 when $\frac{dV}{dx} < 1$ and to compute 4.8.1 for $\frac{dV}{dx} > 1$. For when $\frac{dV}{dx} < 1$ we fix the step-size for x and we have

$$V_{n+1} - V_n < x_{n+1} - x_n. \qquad 4.8.2$$

Likewise if we compute 4.8.1 for $\frac{dV}{dx} > 1$ and fix the step-size in V we have

$$x_{n+1} - x_n < V_{n+1} - V_n. \qquad 4.8.3$$

These bounds on the independent variable given by 4.8.2 and 4.8.3 ensure that the computation is stable.

To actually integrate 4.7.1 and 4.8.1 numerically there are a number of methods to choose from (52), (53), (54), (55). The standard Fourth Order Runge-Kutta method (55) has a lot of advantages in terms of accuracy, function evaluation, initial conditions, and is used here to illustrate the algorithm. We may now define the steps of the algorithm for d = 0 and x positive.

1) Let $x_o = 0$, $V_o = 0$ be the initial conditions.

2) Fix h and define $x_n = nh$.

3) For increasing n compute V_n recursively from 4.7.1 by the fourth order Runge-Kutta method

$$V_{n+1} = V_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = -\phi(x_n)/f(x_n)$

$$k_2 = -\phi(x_n+h/2)/f(x_n+h/2)$$

$$k_3 = -\phi(x_n+h/2)/f(x_n+h/2)$$

$$k_4 = -\phi(x_n+h)/f(x_n+h).$$

4) Test each time step 3) is computed to see if $\frac{dV}{dx} \geq 1$. This may be done approximately by checking to see if $k_4 \geq 1$, and define N as the value where $\frac{dV}{dx} \geq 1$ occurs.

5) Fix p and let $V_{n+1} = V_n + p$ for $n = N, N+1, \ldots$

6) For increasing n compute x_n recursively from 4.8.1 by the fourth order Runge-Kutta method

$$x_{n+1} = x_n + \frac{p}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad n = N, N+1, \ldots$$

where $k_1 = -f(x_n)/\phi(x_n)$

$$k_2 = -f(x_n+\tfrac{pk_1}{2})/\phi(x_n+\tfrac{pk_1}{2})$$

$$k_3 = -f(x_n+\tfrac{pk_2}{2})/\phi(x_n+\tfrac{pk_2}{2})$$

$$k_4 = -f(x_n+pk_3)/\phi(x_n+pk_3)$$

7) If $|x_{n+1} - x_n| < \delta$ for some n and a pre-determined δ then the method is terminated, otherwise 6) is repeated.

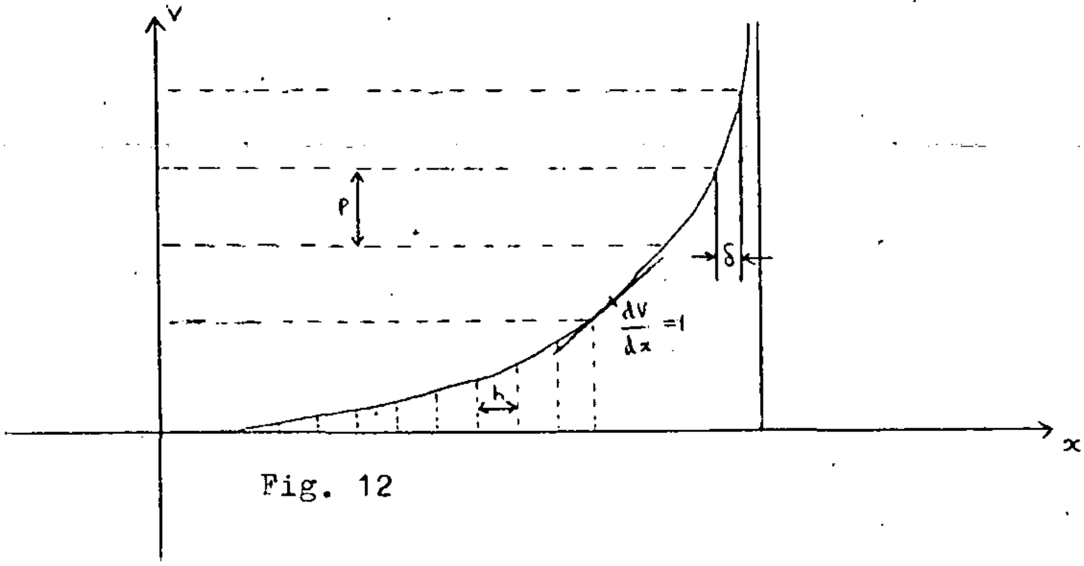


Fig. 12

The algorithm is illustrated in fig. 12 where the curve of V(x) is typical of Lyapunov functions for d = 0. Fig. 12 shows the initial computation where x increases in fixed increments, then when increments of V become too large the changeover occurs to limit the increments of V and compute x which approaches x'.

These two sections have covered how to integrate 4.7.1 numerically for increasing x, but clearly the method for negative x is identical to that for positive x.

When d = 1 the steps of the algorithm are similar. Differences occur in step 4) where the changeover point is given by

$$\frac{1}{(1-V)}\frac{dV}{dx} \geqslant 1$$

and in step 5) where V_n, $n = N, N+1, \ldots$, must be a sequence approaching 1 such as , for example,

$$V_{n+1} = (1-\lambda)V_n + \lambda$$
$$n = N, N+1, \ldots$$

9. Convergence of Numerical Methods

Having provided, in the two previous sections, a numerical algorithm for integrating 4.1.4 we are now interested in the behaviour of x as $V \to \infty$ or 1. Let us assume that d = 0 and that we are interested in the behaviour of x_n as n becomes large in steps 5), 6), 7) of the algorithm.

Fig.13

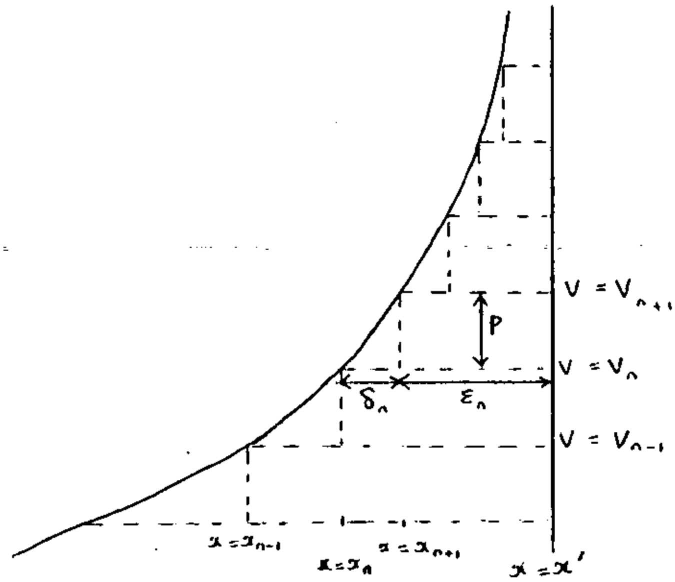


Fig. 13 is a close-up of fig. 12 in the region of x = x'. By analysising fig. 13 we hope to be able to answer the basic question of convergence:

Does there exist an n and a computed V_n such that we ensure that $|x_n - x'| < \varepsilon$ for some given ε ? Also can we establish a sequence n_k such that $|x_{n_k} - x'| < \varepsilon^k$ for some given ε ? 4.9.1

Now we define δ_n, ε_n as shown in fig. 13 as

$$\delta_n = x_{n+1} - x_n \qquad 4.9.2$$
$$\varepsilon_n = x' - x_{n+1}$$

The curve V(x) in fig. 13 is obtained by connecting the data points x_n, $\dot{V}_n$ for n = N,N+1,... . But clearly we may define the functions $\delta(p,V)$ and $\mathcal{E}(p,V)$ in between the data points similarly to 4.9.2 by

$$\delta(p,V) = x(V+p) - x(V) \qquad 4.9.3$$

$$\mathcal{E}(p,V) = x^{i} - x(V+p).$$

Now by the Mean Value Theorem (40) for continuous functions we know that

$$\frac{\delta(p,V)}{p} = \frac{dx}{dV}(V+\lambda p) \qquad 4.9.4$$

where $0 < \lambda < 1$.

We may obtain another expression similar to 4.9.4 by considering Taylor series. The series is given by

$$x(V + p) \equiv e^{pD}x(V)$$

where $D \equiv \frac{d}{dV}$.

Now using the definition of the operator Δ

$$\Delta x(V) \equiv x(V + p) - x(V) \qquad 4.9.5$$

we see that

$$(1 + \Delta)x(V) \equiv e^{pD}x(V). \qquad 4.9.6$$

Thus we have from 4.9.6 a functional relationship

$$pD \equiv \log(1 + \Delta). \qquad 4.9.7$$

Now if we expand 4.9.7 by the power series expansion of the log function, we obtain

$$p\,\frac{dx(V)}{dV} = \left(-\sum_{i=1}^{\infty} \frac{(-\Delta)^i}{i}\right) x(V). \qquad 4.9.8$$

4.9.8 obviously cannot be computed in full but approximations to $\frac{dx}{dV}$ may be obtained by truncation of the series in 4.9.8.

The first two approximations are thus given by

$$\frac{dx(V)}{dV} = \frac{\delta(p,V)}{p} \qquad 4.9.9$$

and $$p\,\frac{dx(V)}{dV} = \frac{3\,\delta(p,V) - \delta(p,V+p)}{2} \qquad 4.9.10$$

where 4.9.5 and 4.9.3 have been used to replace x's by δ's.

For the remainder of the analysis we have to decide which approximate formula to use out of 4.9.10, 4.9.9, some other truncation of 4.9.8 or 4.9.4 with fixed λ. We shall use 4.9.4 which upon integrating with respect to V becomes

$$\int \delta(p,V')dV' = \left[p x(V+\lambda p)\right] + c_1 \qquad 4.9.11$$

where c_1 is the arbitrary constant.

The constant c_1 is eliminated by the condition that

$$x(V) \rightarrow x' \qquad 4.9.12$$

as $V \rightarrow \infty$.

Substituting 4.9.12 into 4.9.11 gives

$$\int_V^\infty \delta(p,V')dV' = \left[px(V'+\lambda p)\right]_V^\infty . \qquad 4.9.13$$

However by equation 4.9.3 we see that 4.9.13 may be simplified to become

$$p\,\varepsilon(p,V+(\lambda-1)p) = \int_V^\infty \delta(p,V')dV'. \qquad 4.9.14$$

Now by reference to 4.9.3 we establish that

$$\delta(p,V) + \varepsilon(p,V) = \varepsilon(p,V-p). \qquad 4.9.15$$

Substituting for ε in terms of δ from 4.9.14 into 4.9.15 gives

$$p\,\delta(p,V) = \int_{V-\lambda p}^{V+(1-\lambda)p} \delta(p,V')dV'. \qquad 4.9.16$$

The solution of the delay-differential equation 4.9.16 is not unique and depends on the initial conditions which must be specified functionally over a range of V. Work on numerical solutions of such equations can be found in references such as (56), (57), (58), (59).

From computed examples it seems that $\delta(p,V)$ usually takes the form

$$\delta(p,V) = a(p)\,e^{b(p)V} \qquad 4.9.17$$

where $b(p)$ is negative so that $\delta(p,V) \rightarrow 0$ as $V \rightarrow \infty$.

Substituting 4.9.17 into 4.9.16 and differentiating 4.9.16 with respect to V gives

$$pa(p)b(p)\,e^{b(p)V} = a(p)(e^{b(p)(V+(1-\lambda)p)} - e^{b(p)(V-\lambda p)}). \qquad 4.9.18$$

4.9.18 is a relationship between a(p), b(p) and p which simplifies to

$$b(p)p = e^{(1-\lambda)b(p)p} - e^{-\lambda b(p)p}. \qquad 4.9.19$$

It is noticeable that 4.9.19 is an equation only of the one function $b(p)p$ and in fact the only solution of 4.9.19 is

$$b(p)p = 0. \qquad 4.9.20$$

A similar analysis of 4.9.10 yields the corresponding equation to 4.9.19

$$2b(p)p(1+e^{b(p)p}) = 3(1-e^{-2b(p)p}) - e^{b(p)p} + e^{-b(p)p} \qquad 4.9.21$$

which also has 4.9.20 as its only solution. 4.9.19 and 4.9.21 each show that the analysis of fig. 13 is only accurate as $p \rightarrow 0$.

To obtain $\varepsilon(p,V)$ we substitute 4.9.17 into 4.9.14 and simplify which gives

$$\varepsilon(p,V) = \frac{-a(p)e^{b(p)(V+(1-\lambda)p)}}{b(p)p}. \qquad 4.9.22$$

There is an analytical explanation for the observed results 4.9.17 and 4.9.22 as shown in the next section. In computation of $\delta(p,V)$, $\varepsilon(p,V)$ we will not know the value of x'. However we may compute the numbers a(p), b(p) by definition 4.9.17 and then substitute into 4.9.22 to give $\varepsilon(p,V)$.

10. Asymptotic Analysis

The results 4.9.17 and 4.9.22 will be obtained in a different way here and will be generalised in this section. For this we need a more general definition of the asymptotic behaviour of a function f(x) than given by definition 3.2.2 or by 4.3.6 when considering various $\phi(x)$.

Such a definition is given by definition 4.7.1 and we shall investigate the solution of 4.8.1 in the neighbourhood $S_\varepsilon(x')$ of x' for small $\varepsilon > 0$ where ε is chosen so that certain assumptions can be made.

Now re-arranging 4.8.1 gives

$$(1 - dV)\frac{dx}{dV} = \frac{-f(x)}{\phi(x)} \qquad 4.10.1$$

and since we know that $\phi(x)$ is positive definite and continuous then $\phi(x') > 0$ for $x' \neq 0$. 4.10.2

Hence using definition 4.7.1 and 4.10.2 we see that as $x \to x'$

$$\frac{f(x)}{\phi(x)|x-x'|^{P(f,x')}} \longrightarrow \frac{c(f,x')}{\phi(x')} \qquad 4.10.3$$

where $0 < |c(f,x')| < \infty$.

Therefore substituting 4.10.1 into 4.10.3 we arrive at the asymptotic relationship between x and V given by

$$\frac{(1-dV)\frac{dx}{dV}}{|x-x'|^{P(f,x')}} \longrightarrow \frac{-c(f,x')}{\phi(x')}$$

where $0 < |c(f,x')| < \infty$.

For the following asymptotic analysis we concentrate on $x' > 0$ and $x \to x'_-$. The procedure is similar for $x' < 0$. We then solve

$$(1-dV)\frac{dx}{dV} = a_1(x' - x)^s \qquad 4.10.4$$

where $s = P(f,x')$, $a_1 = \frac{-c(f,x')}{\phi(x')}$,

instead of 4.10.1. With a_1, s fixed we may readily solve 4.10.4 and integrating by separation of variables method gives for $d = 0$

$$a_1 V = \frac{1}{(s-1)(x'-x)^{s-1}} + c_1 \quad \text{for } s > 1 \qquad 4.10.5$$

$$a_1 V = -\log(x'-x) + c_1 \quad \text{for } s = 1$$

and similarly for $d = 1$ with aV replaced by $-a\log(1-V)$ where c_1 is an arbitrary constant. The initial conditions for finding c_1 given by $V = \infty$ or 1 when $x = x'$ are not helpful. But since we are considering x near x' and V large or near 1, we may take $c_1 = 0$ without loss of generality. Equation 4.10.5a holds for $s < 1$ also but as seen in section 7 by 4.7.6 and by section 7 of Chapter 3, the Lyapunov function and Zubov theory breaks down if $s < 1$. 4.10.5 confirms this and we may summarise in the theorem:

Theorem 4.10.1

If $P(f,x') < 1$ then solution of 4.1.4 yields $V(x)$ where $V(x)$ does not approach ∞ for $d = 0$ or approach 1 for $d = 1$ and the Zubov construction is no longer applicable since the D.O.A. is not indicated by $V = \infty$ or $V = 1$.

The justification for this asymptotic analysis may be found in Murray (41) or Wasow (42). Theorem 4.10.1 is simply theorem 3.7.1 re-written using definition 4.7.1 and theorem 3.5.4.

Having established theorem 4.10.1 for $d = 0$ or 1 we now take $d = 0$ to compare the results 4.10.5 with the analysis of fig. 13. The results of letting $d = 1$ may be similarly derived.

Having obtained 4.10.5 we now see that we have obtained an expression for $\mathcal{E}(p,V)$. Substituting 4.9.3 into 4.10.5 gives

$$a_1(V+p) = \frac{1}{(s-1)\mathcal{E}(p,V)^{s-1}} + c_1 \quad \text{for } s > 1 \qquad 4.10.6$$

or $a_1(V+p) = -\log \mathcal{E}(p,V) + c_1$ for $s = 1$.

4.10.6 may be re-arranged to give $\mathcal{E}(p,V)$ explicitly as

$$\mathcal{E}(p,V) = \left(\frac{1}{(s-1)(a_1(V+p)-c_1)}\right)^{1/s-1} \quad \text{for } s > 1 \qquad 4.10.7$$

or $\mathcal{E}(p,V) = e^{(c_1 - a_1(V+p))}$ for $s = 1$.

The actual value of c_1 depends on the solution of 4.1.4 for $V(x)$. This analysis only holds for

$$\left|1 - \frac{x}{x'}\right| << 1 \qquad 4.10.8$$

and we cannot tell how the solution of 4.10.1 behaves outside the region given by 4.10.8.

The equation 4.10.7b is of the form obtained in 4.9.22 and for most examples the solutions x' of $f(x) = 0$ are obtained from linear factors $(x - x')$ and hence $s = 1$. The expressions 4.9.17, 4.9.22 and 4.10.7b will be verified by examples.

Investigation of 4.10.7b shows that $b(p)$ of 4.9.22 is independent of p as might be expected from the definition of $\varepsilon(p,V)$, $\delta(p,V)$. Further comparison of 4.9.22 and 4.10.7b yields the relationships

$$b(p) \equiv -a_1$$

$$a(p) \equiv a_1 p e^{c_1 - a_1 p}$$

showing that as $p \to 0$ $a(p)$ is asymptotically linear with respect to p as might also be expected from the definitions of $\varepsilon(p,V)$, $\delta(p,V)$.

11. Examples

Example 11.1

$$\dot{x} = -ax - e^{-\frac{1}{x^2}} \qquad 4.11.1$$

This example due to Lehnigk (43) has a varying D.O.A. depending on the value of a. It was shown in (43) that for $0 < a < (2e)^{-\frac{1}{2}}$ that there are three critical points of 4.11.1 of which the origin is one and the other two are negative, while for $a < 0$ 4.11.1 is unstable at the origin and for $a > (2e)^{-\frac{1}{2}}$ there is only one critical point which is the origin. In fig. 16 the magnitude of the negative critical points is plotted against a, showing that at $a = (2e)^{-\frac{1}{2}}$ the roots coincide. Figs. 17, 18 show plots of solutions of 4.11.1 for $x(t)$ for various a which have been obtained by the fourth order Runge-Kutta method.

Using $\phi(x) = x^2$ and $d = 0$ the Zubov equation has been solved and the analysis of sections 9 and 10 investigated. Figs. 19, 20, 21 show the results of plotting $\log \delta(p,V)$, $\log \varepsilon(p,V)$ against V for various a,p. They are seen to be straight lines and the relationships are:

Fig. 19 $a = 0{\cdot}36$, $p = 0{\cdot}25$

$$\left.\begin{aligned} \delta(p,V) &= 0{\cdot}023e^{-0{\cdot}408V} \\ \varepsilon(p,V) &= 0{\cdot}2077e^{-0{\cdot}408V} \end{aligned}\right\} \qquad 4.11.2$$

Fig. 20 $a = 0{\cdot}36$, $p = 0{\cdot}75$

$$\left.\begin{aligned} \delta(p,V) &= 0{\cdot}073e^{-0{\cdot}408V} \\ \varepsilon(p,V) &= 0{\cdot}199e^{-0{\cdot}408V} \end{aligned}\right\} \qquad 4.11.3$$

Fig. 21 $a = 0{\cdot}4, \quad p = 0{\cdot}25$

$$\delta(p,V) = 0{\cdot}0093e^{-0{\cdot}2031V}$$
$$\varepsilon(p,V) = 0{\cdot}1776e^{-0{\cdot}2031V} \qquad 4.11.4$$

Equation 4.10.7b showed that in the expression 4.9.17 and 4.9.22 $b(p)$ is independent of p, and 4.11.2, 4.11.3 confirm that this is true. It is also observed that the approximate relationship $a(p) \approx p$ is verified by 4.11.2, 4.11.3 and the definition of $\delta(p,V)$ given in 4.9.17.

Example 11.2

$$\dot{x} = -x(1 - x) \qquad 4.11.5$$

The D.O.A. of 4.11.5 is given by $(-\infty, 1)$.
Using $\phi(x) = x^2$, $d = 0$ the Zubov equation is solved by the algorithm and figs. 22, 23 show the plots of $\log \delta(p,V)$ and $\log \varepsilon(p,V)$ against V. The relationships are:

Fig. 22 $p = 0{\cdot}5$

$$\delta(p,V) = 0{\cdot}2421\ e^{-1{\cdot}00V}$$
$$\varepsilon(p,V) = 0{\cdot}3665\ e^{-1{\cdot}00V} \qquad 4.11.6$$

Fig. 23 $p = 1{\cdot}0$

$$\delta(p,V) = 0{\cdot}2926\ e^{-0{\cdot}982V}$$
$$\varepsilon(p,V) = 0{\cdot}1719\ e^{-0{\cdot}982V} \qquad 4.11.7$$

For this example 4.7.1 becomes

$$\frac{dV}{dx} = \frac{x}{1-x}$$

the solution of which is

$$V(x) = -x - \log(1-x). \qquad 4.11.8$$

We are interested in the behaviour of 4.11.8 for x near 1. Therefore if we let $x \to 1$ in 4.11.8 we obtain

$$V \approx -1 - \log(1-x)$$

which upon solution for x becomes

$$1 - x(V) \approx e^{-1-V}. \qquad 4.11.9$$

From 4.11.9 we obtain the theoretical results

$$\delta(p,V) \doteq e^{-1-V}(1 - e^{-p})$$
$$\varepsilon(p,V) = e^{-1-V}. \qquad 4.11.10$$

The numerical results 4.11.6, 4.11.7 bear reasonable comparisons with 4.11.10.

12. Conclusions

In this chapter the one-dimensional Zubov equation has been looked at in a way in which higher orders can never be seen. This is because we have no freedom to choose which way we go out from the origin other than whether x is negative or positive. In higher dimensions the boundary of the D.O.A. is not simply obtained by investigating the zeroes of a function f(x) as it can be in one dimension. In this sense the extra function $\phi(x)$ is not really necessary as it transforms the O.D.E. 4.1.1 into another O.D.E. 4.1.4.

In a lot of cases it is indeed simpler to compute solutions of $f(x) = 0$ to obtain the D.O.A.. Various root finding methods exist which alter x in a systematic way to try to obtain x'. Such methods may be either divergent or go past x' without spotting it. There is no possibility of missing the boundary when using methods which solve 4.1.4 as a differential equation letting $V \to \infty$ or 1.

The advantage of the Zubov equation here is that we do not start from an initial point $x = x_0$ at $t = 0$ and try to see if we reach the origin, but instead let $V \to \infty$ or 1 and try to see if we reach $x = x'$. The same situation is achieved here however by solving 4.1.1 and letting $t \to -\infty$. The function $\phi(x)$ is but a transformation which may help to make numerical computation easier if wisely chosen.

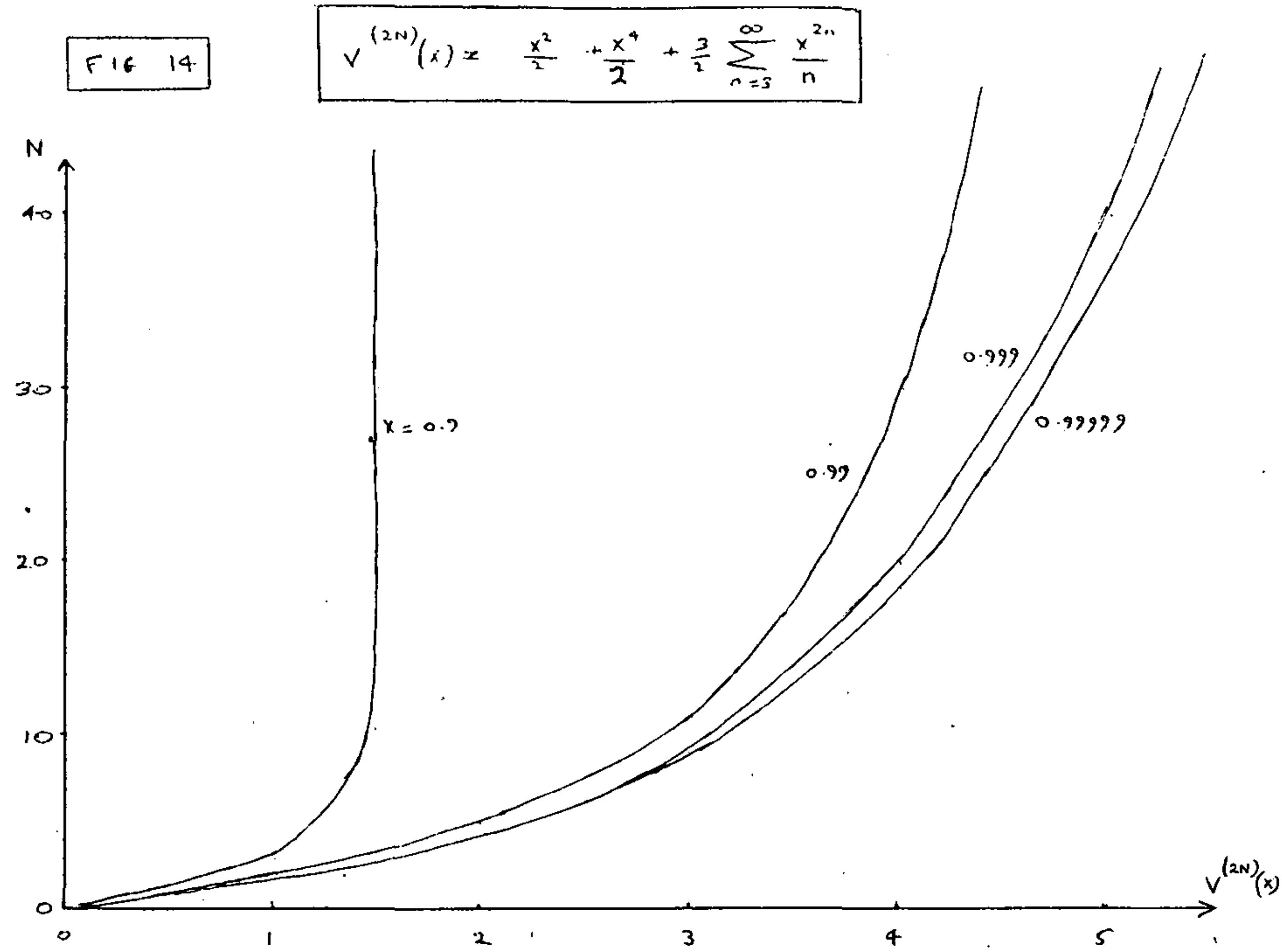
Given that the one-dimensional case is different from the normal Zubov P.D.E. and has less advantages over solving 4.1.1 than its higher order counterpart, we have seen that in return we are able to obtain a greater analysis of what happens to x and V at the boundary of the D.O.A.. The question posed in section 9 by 4.9.1 now has an easy answer. We simply look for functions $\delta(p,V)$ and $\varepsilon(p,V)$ such as 4.9.17, 4.9.22 and for more general s by 4.10.7a. Then once we obtain the relations with actual numbers such as those in 4.11.2, 4.11.3, 4.11.4, 4.11.6, 4.11.7 we can immediately say that we know what value of V to reach to obtain

$$\varepsilon(p,V) < 10^{-n} \quad \text{for some n.} \qquad 4.12.1$$

It must be stressed that x' is unknown and that the relationship for $\delta(p,V)$ is worked out first and then $\varepsilon(p,V)$ is obtained by substituting the results obtained from 4.9.17 into 4.9.22 before solving 4.12.1 for V.

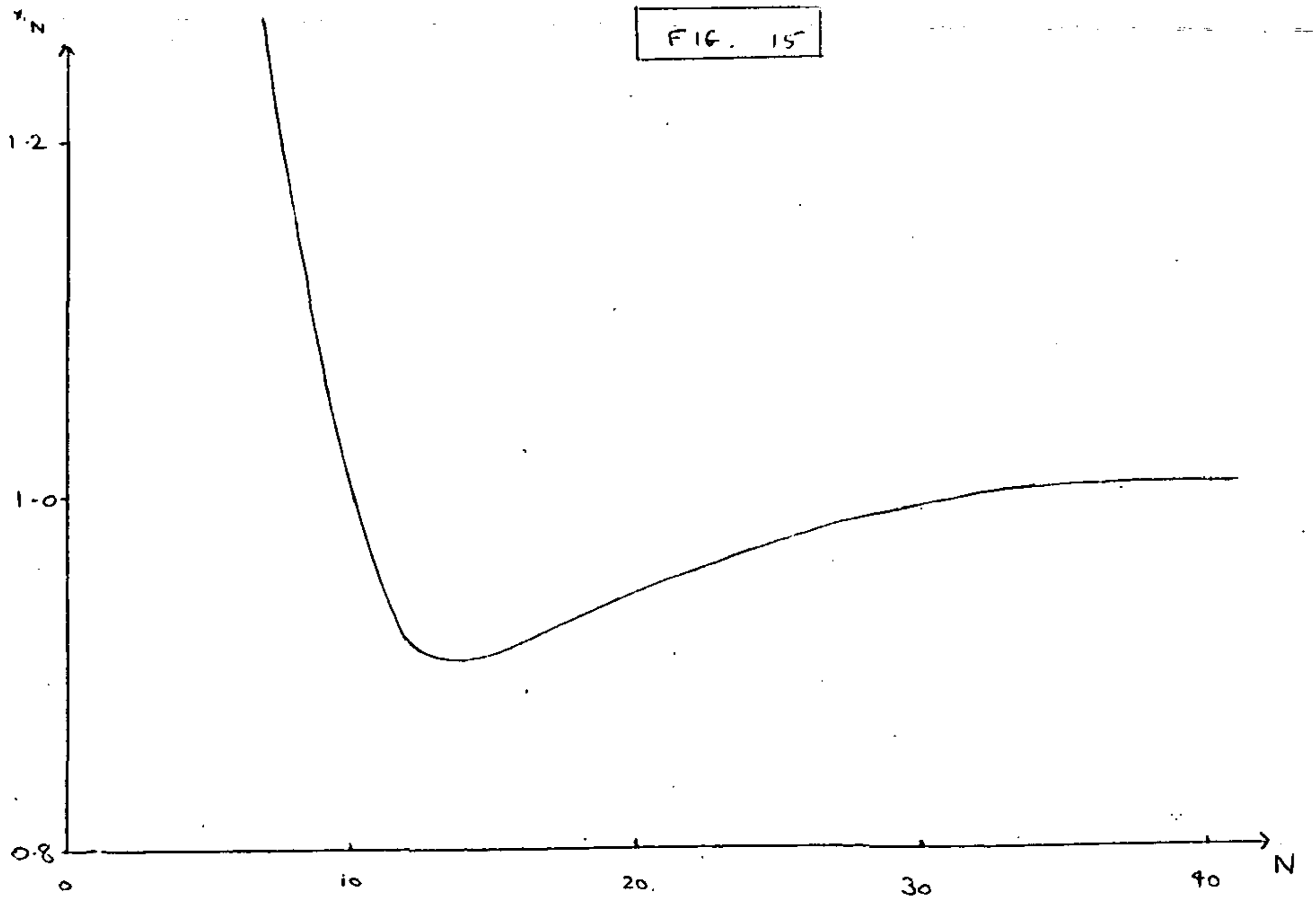
Finally, figs. 19 to 23 show that $\log \varepsilon(p,V)$ flattens out at about $\log \varepsilon(p,V) = -11$ which is a limit of accuracy of the computer used. This does suggest that using a computer will never actually enable us to compute x' as $V \rightarrow \infty$. But x' should be obtained from the definition of $\varepsilon(p,V)$ instead once $\delta(p,V)$ is obtained as a function of V.

FIG 14



$V^{(2N)}(x) = \frac{x^2}{2} + \frac{x^4}{2} + \frac{3}{2}\sum_{n=3}^{\infty} \frac{x^{2n}}{n}$
N
40
30
20
10
0
x = 0.9
0.99
0.999
0.99999
0
1
2
3
4
5
$V^{(2N)}(x)$

FIG. 15



x_N
1.2
1.0
0.8
0
10
20
30
40
N

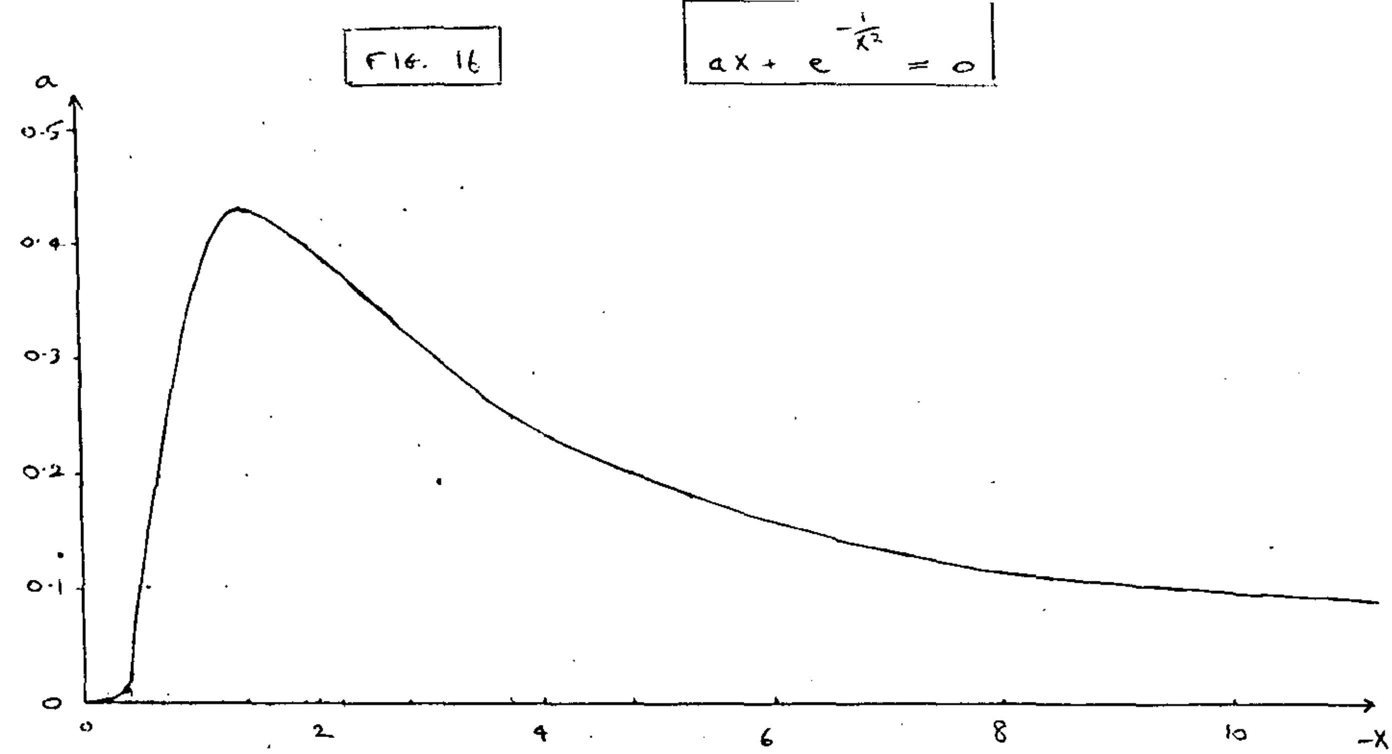


FIG. 16

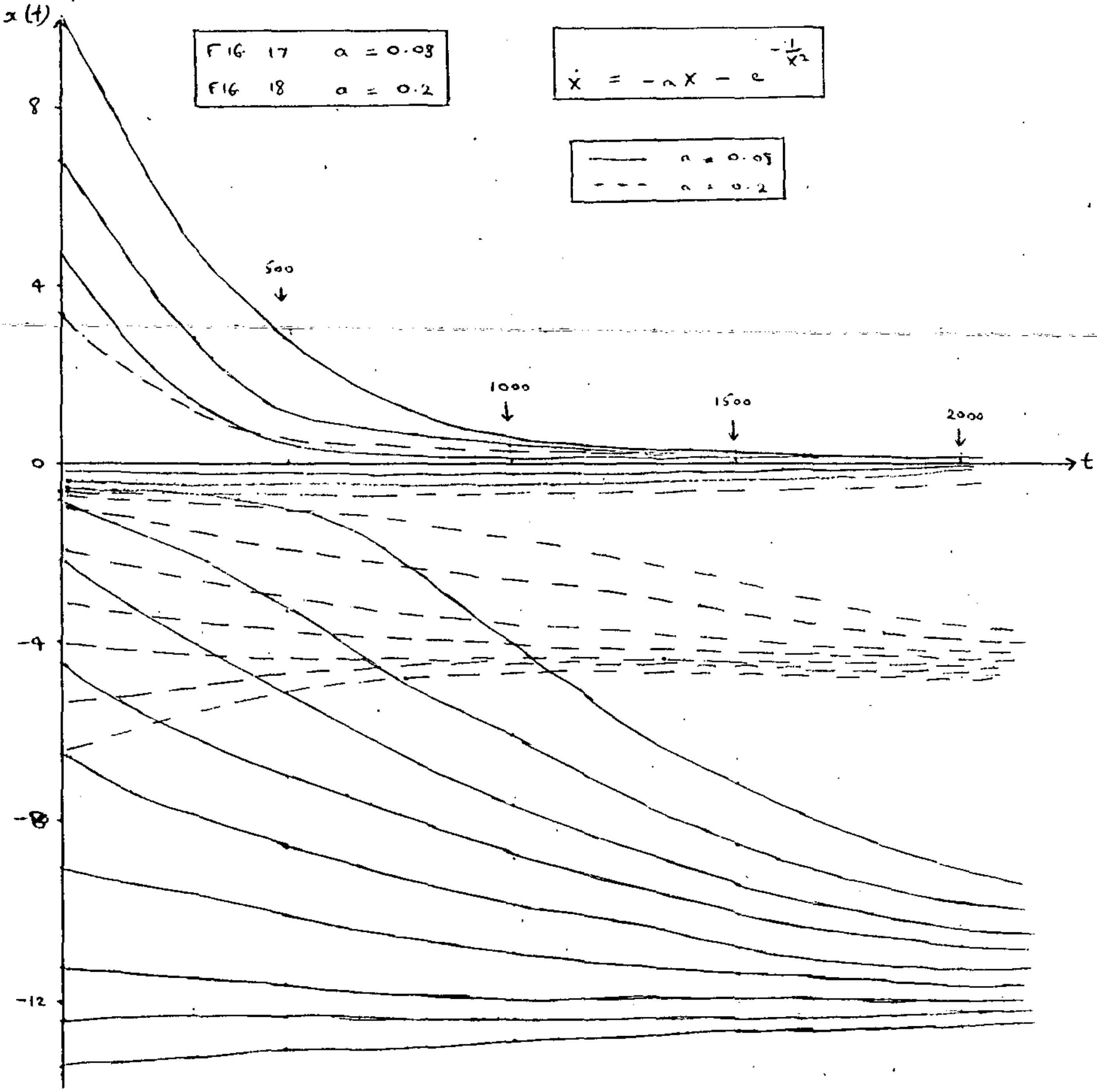


FIG 17 a = 0.08
FIG 18 a = 0.2

FIG 19

$a = 0.36, \; P = 0.25$

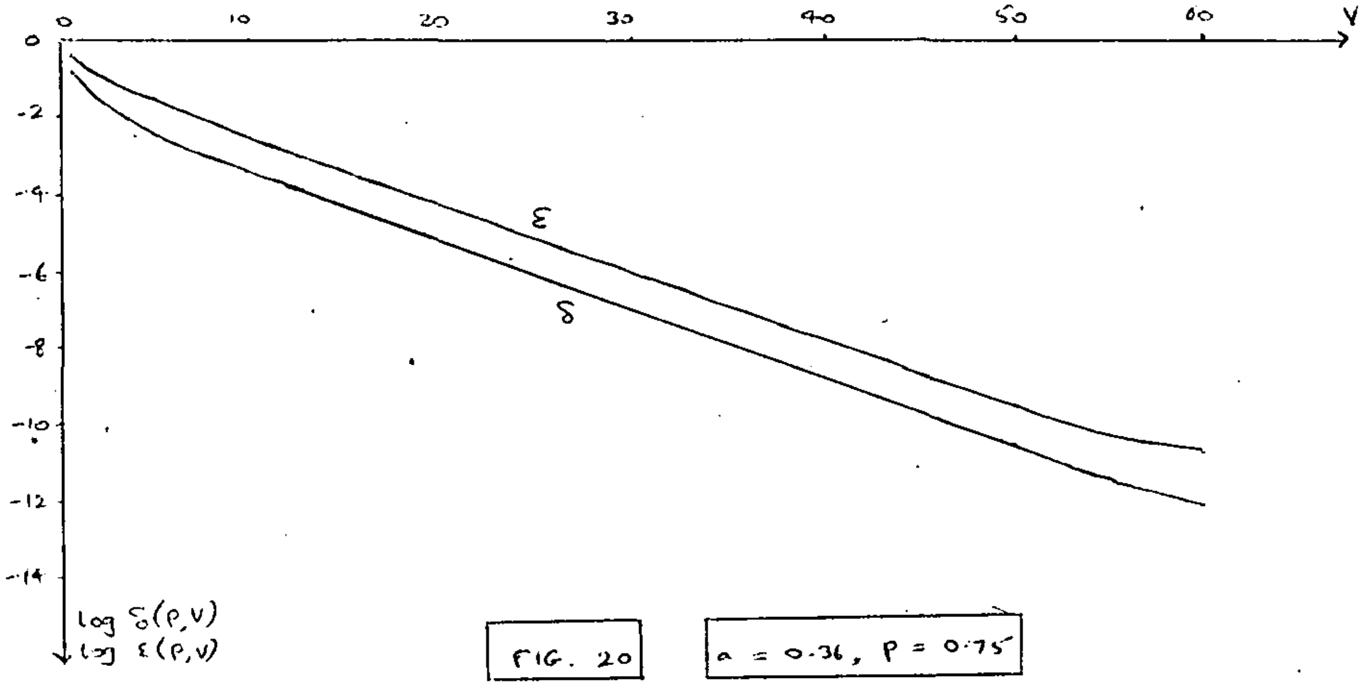


FIG. 20

$a = 0.36, \; P = 0.75$

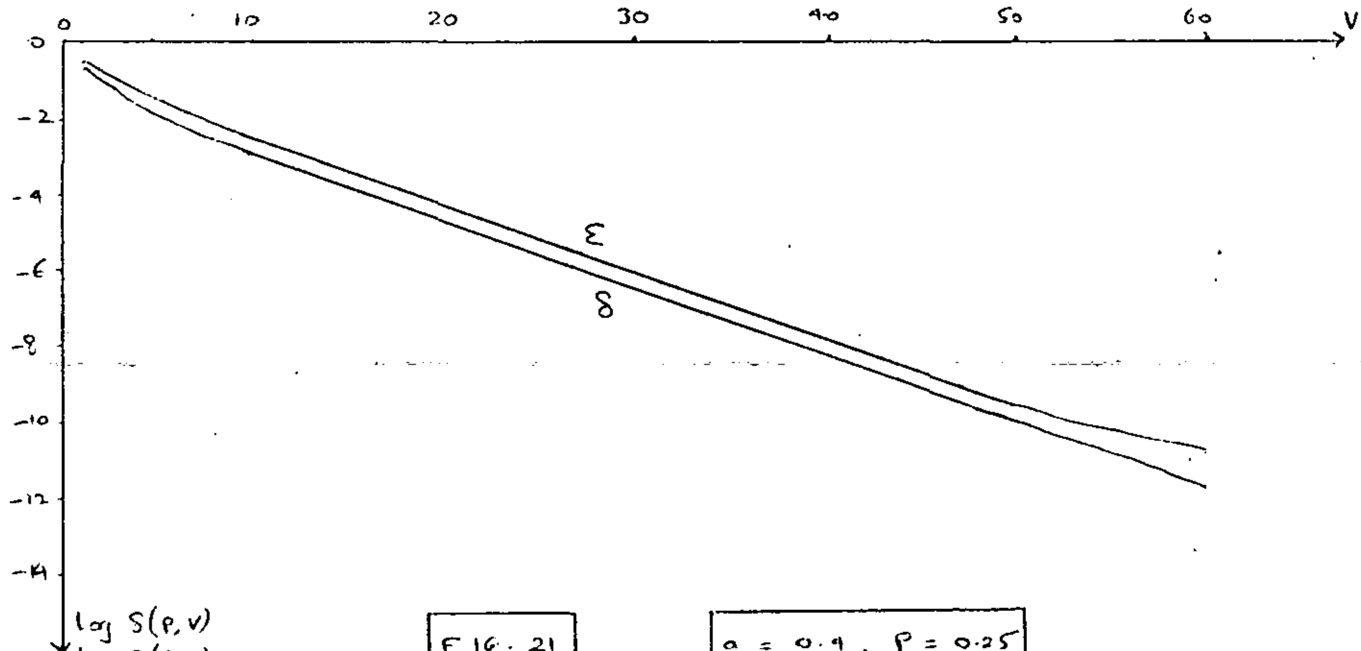
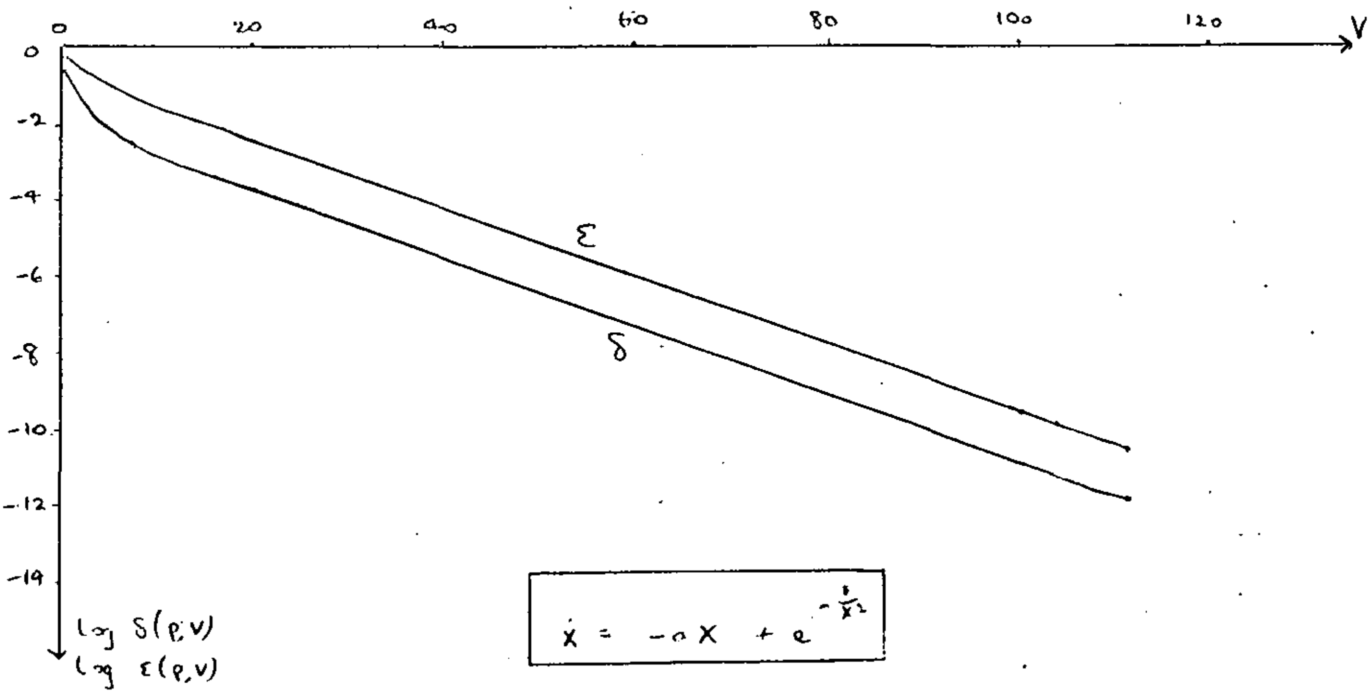


FIG. 21

$a = 0.4, \; P = 0.25$



$$\dot{X} = -aX + e^{-\frac{1}{X^2}}$$

FIG. 22 $p = 0.5$

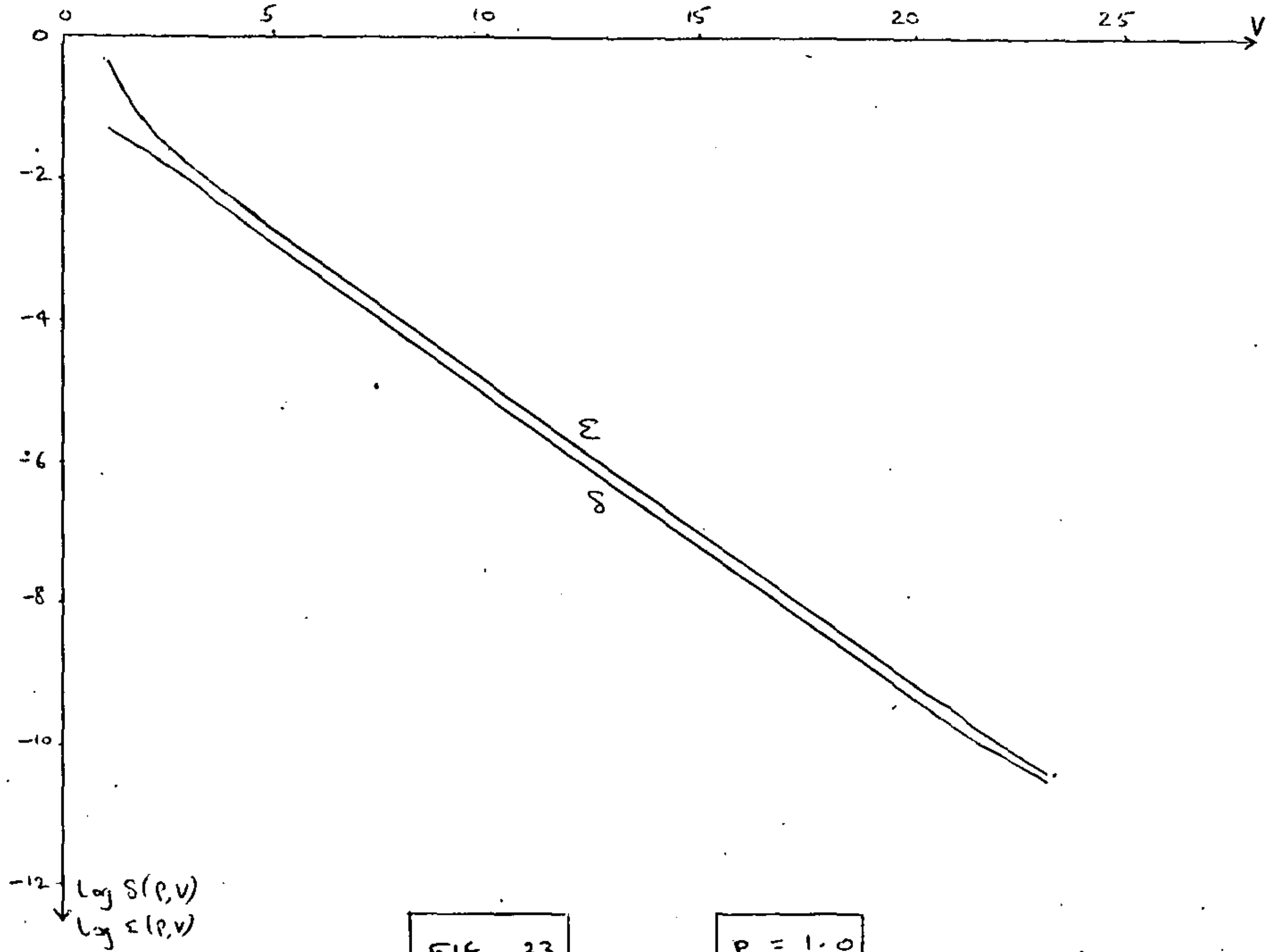
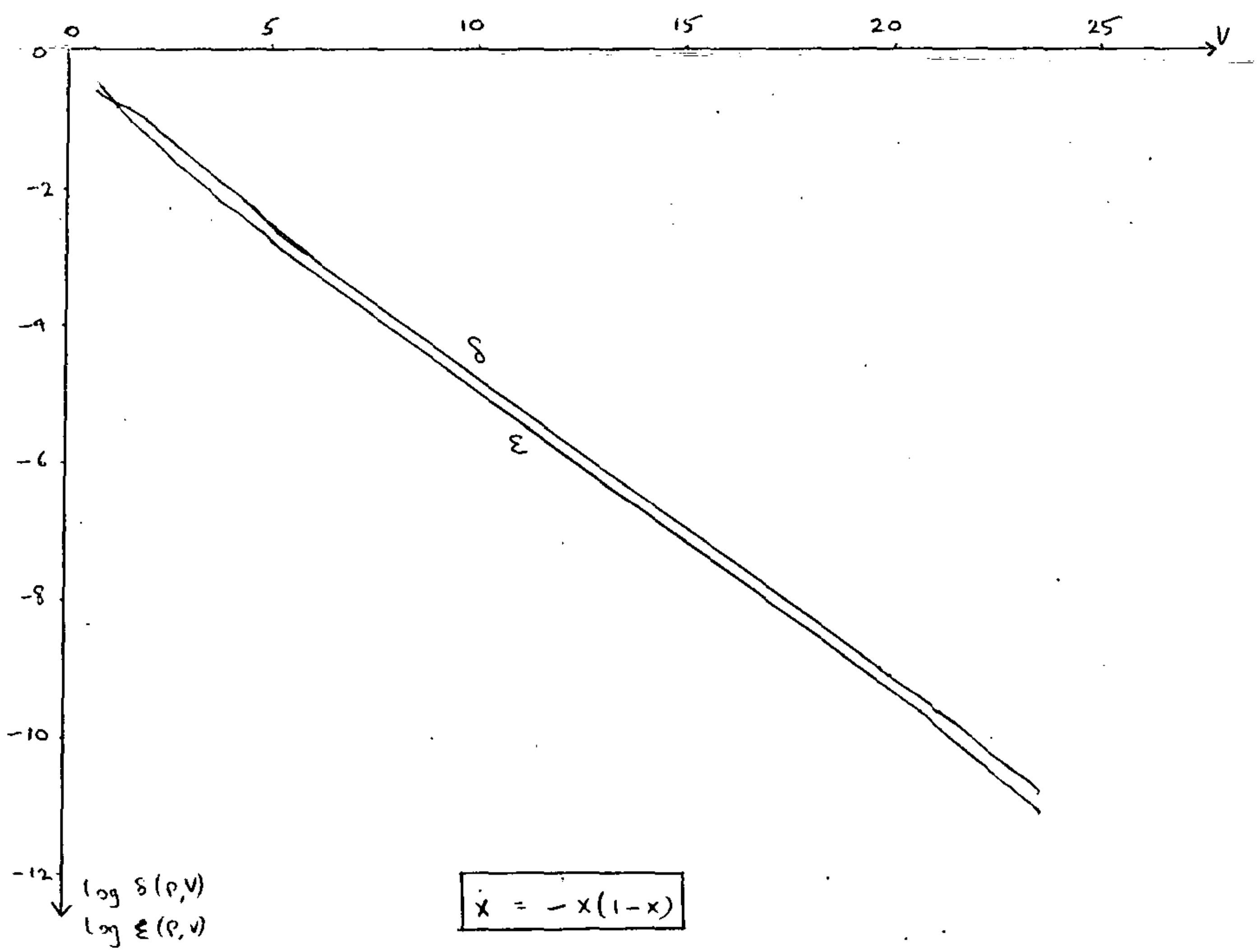


FIG. 23 $p = 1.0$



$\dot{x} = -x(1-x)$

Chapter 5

Finite Difference Methods

1. Introduction

In this chapter a look is made at solving Zubov's partial differential equation by straightforward numerical techniques for computing values of V on a grid system in $x_1,\ldots,x_n$.

Zubov's equation in n dimensions is given by

$$\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 5.1.1$$

where d = 1 for the regular equation and d = 0 for the modified Zubov equation, and f_i are given by the state space equations

$$\dot{x}_i = f_i(\underline{x}) \qquad 5.1.2$$

$i = 1,\ldots,n$

and ϕ is positive definite in $x_1,\ldots,x_n$.

The partial derivatives in 5.1.1 may be replaced by approximate difference formulae involving values of V on a grid, and the step-sizes of each independent variable. Various problems are encountered by this method and section 2 investigates the method. A fundamental problem of such a rectangular grid is that the only initial condition is given by

$$V(0,0,\ldots,0) = 0. \qquad 5.1.3$$

Sneddon shows (13) that a P.D.E. in n dimensions needs initial conditions to be specified on an (n-1)-dimensional subspace. The necessary form of initial conditions is specified as $V = H_1(\underline{x})$ on $H_2(\underline{x}) = 0$. Hence 5.1.3 is only sufficient as initial conditions if n = 1 where 5.1.1 becomes an O.D.E. and Chapter 4 has fully covered this case.

However polar co-ordinate grids overcome the problem of initial conditions, as the theory in section 3 shows. In section 4 various difference formulae for the polar co-ordinate grids are derived, followed in section 5 by a comparison on the basis of errors, stability and convergence. Section 6 looks into reducing the Zubov equation to a set of O.D.E.s along different radial lines. Sections 7 and 8 go into the problems encountered by such methods, and then section 9 sets out to define the optimum method for a general class of systems. Sections 4 to 9 are centred on the 2-dimensional case, but this method is much more easily applicable to higher orders than is the method of Chapter 6.

A number of examples are investigated in section 11 covering all aspects of the chapter, and conclusions drawn in section 12.

2. Rectangular Grid Methods

Given a system 5.1.2 from which we obtain Zubov's P.D.E. 5.1.1 we want to solve 5.1.1 given positive definite ϕ in $x_1,\dots,x_n$ and initial conditions 5.1.3 for V. The required stability condition is that V is positive definite in $x_1,\dots,x_n$ in a neighbourhood of the origin, in which case $V = 1$ (for $d = 1$) or $V = \infty$ (for $d = 0$) are the contours which define the boundary of the D.O.A..

To solve 5.1.1 we can set up a rectangular grid system

$$x_i^{(j_i)} = j_i h_i \qquad 5.2.1$$

$i = 1,\dots,n$

$j_i = -\infty$ to $+\infty$

and denote the analytical (true) value of V at the grid point 5.2.1 as

$$V(j_1h_1,\dots,j_nh_n) \qquad 5.2.2$$

and the computed value of V as

$$V_{j_1,j_2,\dots,j_n}. \qquad 5.2.3$$

The partial derivatives in 5.1.1 may be expressed in terms of the grid values 5.2.3 and the step-sizes h_i, $i = 1,\dots,n$, by, for example,

$$\frac{\partial V}{\partial x_i}(j_1h_1,\dots,j_nh_n) \approx (V_{j_1,j_2,\dots,j_{i-1},j_i+1,j_{i+1},\dots,j_n} - V_{j_1,j_2,\dots,j_{i-1},j_i-1,j_{i+1},\dots,j_n})/2h_i \qquad 5.2.4$$

There are other formulae which are more accurate than 5.2.4, but involve more computation as well.

Let us look at 5.1.1 to 5.1.3, 5.2.1 to 5.2.4 in 2 dimensions, re-writing 5.1.2 as

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\},$$

5.1.1 as

$$f(x,y)\frac{\partial V}{\partial x}(x,y) + g(x,y)\frac{\partial V}{\partial y}(x,y) = -\phi(x,y)(1 - dV(x,y)) \qquad 5.2.5$$

$$d = 0,1.$$

We denote the grid 5.2.1 by

$$x_m = mh$$
$$y_n = nk \qquad 5.2.6$$

$m = -\infty$ to $+\infty$

$n = -\infty$ to $+\infty$.

There are many ways in which $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ can be replaced by values of V on the grid, but to illustrate the method we use the simplest possible which is to write

$$\frac{\partial V}{\partial x}(mh,nk) \approx \frac{V_m^n - V_{m-1}^n}{h} \qquad 5.2.7$$
$$\frac{\partial V}{\partial y}(mh,nk) \approx \frac{V_m^n - V_m^{n-1}}{k}$$

where V_m^n is the computed value of V at $x = mh$, $y = nk$.

The known functions f, g, ϕ may be evaluated at the grid point 5.2.6 and we write

$$f_m^n = f(mh,nk)$$
$$g_m^n = g(mh,nk) \qquad 5.2.8$$
$$\phi_m^n = \phi(mh,nk).$$

Substitution of 5.2.7, 5.2.8 into 5.2.5 gives us a difference formula connecting V_{m-1}^n, V_m^n, V_m^{n-1}

$$\frac{f_m^n}{h}(V_m^n - V_{m-1}^n) + \frac{g_m^n}{k}(V_m^n - V_m^{n-1}) = -\phi_m^n(1 - dV_m^n) \qquad 5.2.9$$

as shown in fig. 24.

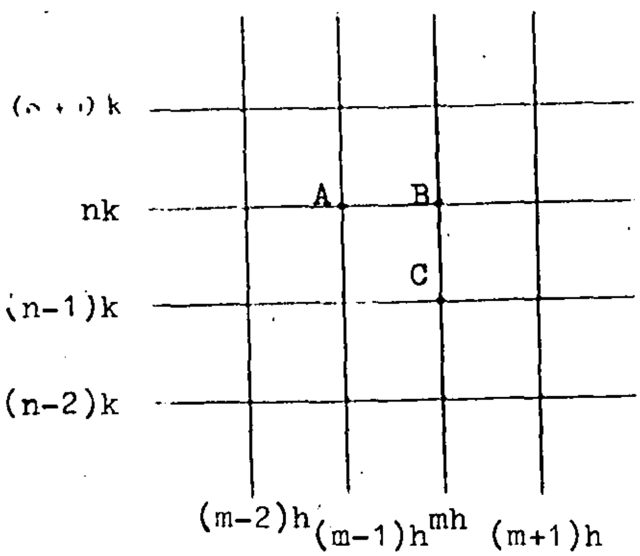


Fig. 24

If any two values of V are known at A, B or C the third point may be computed by the difference formula 5.2.9. However we have only one value which is given as an initial condition which is

$$V_o^o = 0. \qquad 5.2.10$$

In order to use 5.2.9 we require a systematic method where at each point we may calculate V at B knowing its value at A,C or any other way round.

If we are given the values

$$V_M^n \quad n = -\infty \text{ to } +\infty$$

then by using 5.2.9 to compute V_{m-1}^n from V_m^n, V_m^{n-1} we may compute systematically for $m < M$. A different difference formula would be required to compute V_m^n for $m > M$.

Similarly if we are given the values

$$V_m^N \quad m = -\infty \text{ to } +\infty$$

then by using 5.2.9 to compute V_m^{n-1} from V_m^n, V_{m-1}^n we may compute systematically V_m^n for $n < N$. Again a different difference formula would be required to compute V_m^n for $n > N$.

However if we are given

$$V_M^n \quad n = -\infty \text{ to } +\infty \qquad 5.2.11$$

$$\text{and} \quad V_m^N \quad m = -\infty \text{ to } +\infty$$

then we may compute V_m^n for all m,n using 5.2.9 in various ways.

Since 5.2.10 is the given initial condition it seems reasonable to assume initial conditions on V_o^n and V_m^o for all m,n. How we obtain such initial conditions is another matter which will be discussed later.

Re-writing 5.2.9 in terms of V_m^n we obtain

$$V_m^n = \frac{(\rho f_m^n V_{m-1}^n + g_m^n V_m^{n-1} - k\phi_m^n)}{(\rho f_m^n + g_m^n - dk\phi_m^n)} \qquad 5.2.12$$

where $\rho = k/h$.

2.1. Initial Conditions Consideration

Equation 5.2.12 was used to compute values of V on a grid system for the Hahn equation

$$\begin{aligned} \dot{x} &= -x + 2x^2y \\ \dot{y} &= -y \end{aligned} \qquad 5.2.13$$

with $\phi(x,y) = 2x^2 + 2y^2$ 5.2.14

and with initial conditions specified on V_o^n, V_m^o, $m,n \geqslant 0$, computation of V_m^n for $m,n \geqslant 0$ was carried out.

The analytic solution of 5.2.5 given 5.2.13, 5.2.14 is

$$V(x,y) = y^2 + \frac{x^2}{1-xy} \quad 5.2.15$$

Hence initial conditions were set up as

$$\begin{aligned} V_o^n &= n^2k^2 \\ V_m^o &= m^2h^2. \end{aligned} \quad 5.2.16$$

Fig. 34 shows the computed contours of V = 0·4,1,2,4 picked off the grid values, and compares them with the analytical contours given by 5.2.15. h = k = 0·1, d = 0 was used for fig. 34 while h = k = 0·01 for fig. 35 enabled the analytic contours to be reproduced to the accuracy of drawing on paper.

Normally, however, 5.2.15 is not known (we would not need any computation if it was known) and there is no justification for choosing the initial conditions 5.2.16 rather than any other possible conditions. Other initial conditions were looked at to see what difference this made to the results. Also shown on fig. 35 are the results of putting

$$\left.\begin{aligned} V(x,0) &= x^4 \\ V(0,y) &= y^4 \end{aligned}\right\} \quad 5.2.17$$

or

$$\left.\begin{aligned} V_m^o &= m^4n^4 \\ V_o^n &= n^4k^4 \end{aligned}\right\}.$$

Fig. 36 shows the contours of V = 0·4,1,2,4 for the initial conditions

$$\begin{aligned} V(x,0) &= (x^2+x^4)/2 \\ V(0,y) &= (y^2+y^4)/2. \end{aligned} \quad 5.2.18$$

For each of 5.2.17, 5.2.18 we see that the solution of computation although inaccurate on the axes are "correct" again for $x,y \geq 0\cdot1$. It seems that the method is highly stable and reduces errors quickly.

Consider the system

$$\begin{aligned} \dot{x} &= -2x - 2y^4 \\ \dot{y} &= -y. \end{aligned} \quad 5.2.19$$

Using $\phi(x,y) = 4x^2 + 2y^2$ the Zubov equation becomes

$$(-2x - 2y^4)\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} = -4x^2 - 2y^2. \quad 5.2.20$$

The solution of 5.2.20 using the initial condition V(0,0) = 0 becomes

$$V(x,y) = x^2 + y^2 - \frac{2xy^4}{3} + \frac{y^8}{6} \qquad 5.2.21$$

5.2.21 indicates that 5.2.19 is asymptotically stable everywhere. The initial conditions were set up first as

$$\begin{aligned} V_o^n &= n^2k^2, \quad n = 0,1\ldots \\ V_m^o &= m^2h^2, \quad m = 0,1\ldots \end{aligned} \qquad 5.2.22$$

which are correct on the x-axis but not on the y-axis. Fig. 37 shows the results of computation by use of 5.2.12.

The results are inaccurate near the y-axis and errors take longer to die away than the Hahn equation. However the inaccuracy dies out in the region of stability of the method. Fig.38 shows the same results with the initial conditions given by

$$V_o^n = \frac{3n^2k^2 + n^4k^4}{4}$$

$$V_m^o = \frac{3m^2h^2 + m^4h^4}{4}$$

which are now incorrect on both axes. As fig. 38 shows, the errors are quickly eliminated near the x-axis, but take longer near the y-axis.

2.2. Choice of Initial Conditions

We have seen in subsection 2.1 that the difference formula is stable in the sense that errors in the initial conditions are quickly eliminated. This enables us to choose the initial conditions subject only to the condition

$$\begin{aligned} V(x,0) &> 0, \quad x \neq 0 \\ V(0,y) &> 0, \quad y \neq 0 \\ V(0,0) &= 0. \end{aligned}$$

It raises again the questions referred to earlier in this section of how we do in fact choose the initial conditions to be as accurate as possible.

The obvious choice for the initial conditions are lowest degree terms in the series expansion of V(x,y). From 2.2.13 and theorem 3.2.5 we see that if $P(f) = s$, $P(g) = s$, $P(\phi) = q$ then $P(V) = q-s+1$.

The lowest degree terms in V(x,y) are given by

$$V_{q-s+1}(x,y) = \sum_{m=0}^{q-s+1} V_{q-s+1,m}\, x^m y^{q-s+1-m}. \qquad 5.2.23$$

From 5.2.23 we see that a logical choice of initial conditions is given by

$$V(x,0) = V_{q-s+1,q-s+1}x^{q-s+1}$$
$$V(0,y) = V_{q-s+1,0}y^{q-s+1} \qquad 5.2.24$$

For the system 5.2.13, 5.2.14 the initial conditions 5.2.24 become the conditions given by 5.2.16.

2.3 $V = \infty$ Contour

Figs. 34 - 38 were constructed by picking the contours out of a grid of numbers. Such a process is reasonably accurate for the contours constructed, but problems do arise when we try to obtain $V = \infty$. For large V the numbers generated increase rapidly with respect to x or y and any interpolation method becomes inaccurate at some x or y.

If we are looking for the contour V = p we locate it somewhere between grid points with the property

$$\left.\begin{array}{l} V_m^n < p \\ V_m^{n+1} > p \end{array}\right\} \qquad 5.2.25$$

and

$$\left.\begin{array}{l} V_m^n < p \\ V_{m+1}^n > p \end{array}\right\} \qquad 5.2.26$$

Hence as p increases we see that interpolation between the grid points 5.2.25, 5.2.26 to obtain $V_m^{n+c} = p$, $0 \leq c \leq 1$, and $V_{m+c}^n = p$, $0 \leq c \leq 1$, becomes inaccurate and also if $V_m^n < p$ and V_{m+1}^n or V_m^{n+1} are outside the contour $V = \infty$ then the property 5.2.25, 5.2.26 will not necessarily hold. Hence we see that for sufficiently large p it is impossible to pick grid points with the properties 5.2.25, 5.2.26 such that both V_m^n, V_m^{n+1} or both V_m^n, V_{m+1}^n are still inside the D.O.A..

However it can be seen from 5.2.15 that when x or y increases so that $xy > 1$ then $V(x,y)$ becomes negative. So in order to locate the contour $V = \infty$ we look at the grid values to see where V ceases to increase, but suddenly it becomes discontinuous in x or y.

The shaded area in fig. 35 and the $V = \infty$ curve in fig. 36 represent the boundary of continuous results on the grid system for example 5.2.13, 5.2.14 with initial conditions 5.2.17 and 5.2.18 respectively. There is no appreciable difference between them.

Closer investigation of 5.2.12 shows why the discontinuity curve is as on figs. 35,36. From 5.2.12 we expect computation to become unstable if the denominator is small. When the denominator of 5.2.12 is small the equation 5.2.9 becomes a relationship between V_{m-1}^{n} and V_{m}^{n-1} with terms in V_{m}^{n} cancelling out. Hence a small denominator in 5.2.12 should correspond to a small numerator also. However truncation errors in the computation mean that this is not so, and the computation becomes unstable in the neighbourhood of zeroes of the denominator of 5.2.12.

i.e. Instability must occur in the neighbourhood of

$$\rho f_{m}^{n} + g_{m}^{n} - dk\phi_{m}^{n} = 0. \qquad 5.2.27$$

For system 5.2.13, 5.2.14 and d = 0 we obtain

$$\rho(-x + 2x^{2}y) - y = 0$$

which becomes

$$y = \frac{\rho x}{2\rho x^{2}-1}. \qquad 5.2.28$$

The family of curves 5.2.28 for various ρ are shown on fig. 39. The curves of discontinuity on figs. 35, 36 are clearly a combination of the instability of small coefficients along 5.2.28 and the correct discontinuity in V as given by 5.2.15.

Thus $\rho = \frac{1}{2}$ was used by setting h = 0·01, k = 0·005 and using 5.2.12 to compute V_{m}^{n} $m,n \geq 0$ once more. Fig. 40 shows the results with initial conditions given by 5.2.16 and a noticeable improvement is achieved in the region $x \in (1, 2{\cdot}5)$. Not however noticeable enough to recommend reducing ρ lower as this means increasing h and causing errors or reducing k and increasing computation.

In Chapter 4 the difficulty of picking contours of V from a grid was overcome by reversing the Zubov equation to compute x as V increased in discrete steps. This approach was also considered for the 2-dimensional case, as was a general look at how the step-sizes for x and y could be altered as V became large. No detailed method was produced since there are problems of interpolation in each dimension unless the step-size alteration was global.

i.e. If the grid 5.2.6 becomes

$$y_{n} = y_{n-1} + k_{n}$$

$$x_{m} = x_{m-1} + h_{m}$$

where h_{m}, k_{n} and $m,n = 1,2\ldots,\infty$ are varying step-sizes, then 5.2.12 becomes

$$V_m^n = \frac{(\rho_m^n f_m^n V_{m-1}^n + g_m^n V_m^{n-1} - k_n \phi_m^n)}{(\rho_m^n f_m^n + g_m^n - dk_n \phi_m^n)}$$

where $\rho_m^n = \frac{k_n}{h_m}$, $\quad m,n = 1,2,\ldots\infty.$

Any grid involving changing step-sizes such that h_m, k_n vary with y,x involves complicated interpolation and value storage problems as well as the difficulty of working out from V when the step-sizes should be changed.

2.4. Stability

An important aspect in numerical computation is stability. Would we expect errors to be propogated and increase or die away? The classic method of analysising stability is by frequency response. (Mitchell (16)). An initial error of sinusoidal form is assumed and, using 5.2.9 or whichever method is prefered, the magnitude of the propogated error indicates stability if less than one.

We denote the Local Truncation Error (L.T.E.) of the method 5.2.9 by L_m^n where

$$L_m^n = \rho f(mh,nk)(V(mh,nk)-V((m-1)h,nk))+g(mh,nk)(V(mh,nk)-V(mh,(n-1)k)) + k\phi(mh,nk)(1-dV(mh,nk)) \qquad 5.2.29$$

while from 5.2.9 we have

$$\rho f_m^n(V_m^n - V_{m-1}^n) + g_m^n(V_m^n - V_m^{n-1}) + k\phi_m^n(1-dV_m^n) = 0. \qquad 5.2.30$$

If we denote

$$e_m^n = V_m^n - V(mh,nk) \qquad 5.2.31$$

then we may subtract 5.2.30 from 5.2.29 using 5.2.8 and 5.2.31 to obtain

$$L_m^n = \rho f_m^n(e_m^n - e_{m-1}^n) + g_m^n(e_m^n - e_m^{n-1}) - kd\phi_m^n e_m^n. \qquad 5.2.32$$

We may disregard the L.H.S. of 5.2.32 as $L_m^n \to 0$ as $h,k \to 0$. We now assume a sinusoidal input

$$e_o^n = e^{i\omega nk} \qquad 5.2.33$$

for some ω and assume a similar output for e_m^n magnified and denoted as

$$e_m^n = \lambda^m e^{i\omega nk}. \qquad 5.2.34$$

Substituting 5.2.34 into 5.2.32 we obtain

$$\rho f_m^n(\lambda^m e^{i\omega nk} - \lambda^{m-1} e^{i\omega nk}) + g_m^n(\lambda^m e^{i\omega nk} - \lambda^m e^{i\omega(n-1)k}) - kd\phi_m^n \lambda^m e^{i\omega nk} = 0 \qquad 5.2.35$$

Cancelling $\lambda^{m-1}e^{i\omega nk}$ from 5.2.35 we obtain

$$\rho f_m^n(\lambda-1) + g_m^n\lambda(1-e^{-i\omega k}) - kd\phi_m^n\lambda = 0. \qquad 5.2.36$$

Solving 5.2.36 for λ we obtain

$$\lambda = \frac{\rho f_m^n}{(\rho f_m^n+g_m^n-g_m^n e^{-i\omega k}-kd\phi_m^n)} \qquad 5.2.37$$

Now if $|\lambda| \leq 1$ for all ω we see that any sinusoidal error input will be stable, and hence any initial error distribution for which a Fourier Series exists will be stable.

Hence we require from 5.2.37

$$(\rho f_m^n)^2 \leq (\rho f_m^n+g_m^n-kd\phi_m^n-g_m^n\cos\omega k)^2 + (g_m^n\sin\omega k)^2. \qquad 5.2.38$$

Therefore

$$2g_m^n(\rho f_m^n+g_m^n-kd\phi_m^n)(1-\cos\omega k) + kd\phi_m^n(kd\phi_m^n-2\rho f_m^n) \geq 0. \qquad 5.2.39$$

5.2.39 is required to be true for all ω.

Hence either $g_m^n(\rho f_m^n+g_m^n-kd\phi_m^n) \geq 0$

and $kd\phi_m^n(kd\phi_m^n-2\rho f_m^n) \geq 0$ 5.2.40

or $g_m^n(\rho f_m^n+g_m^n-kd\phi_m^n) \leq 0$

and $(2g_m^n-kd\phi_m^n)(2\rho f_m^n+2g_m^n-kd\phi_m^n) \geq 0.$ 5.2.41

5.2.39 or 5.2.40 and 5.2.41 are the required conditions for stability given an initial distribution of the errors on x = 0 by a combination of errors given by 5.2.33. We also need the stability as y increases given initial errors on y = 0. The analysis is exactly the same as above with 5.2.34 replaced by

$$e_m^n = \lambda^n e^{i\omega mh},$$

g_m^n replaced by ρf_m^n and vice versa in 5.2.38 to 5.2.41.

However, 5.2.40 and 5.2.41 are the conditions for stability for increasing x. If 5.2.9 is solved for decreasing x, then putting 5.2.34 into 5.2.32 as before means that we now need $|\lambda| \geq 1$ for all ω. Hence the stability conditions for decreasing x given initial errors on x = 0 are the same as for increasing x but with inequalities reversed. Similarly for decreasing y.

Thus if we are given the initial conditions 5.2.11 with M = N = 0 we may use 5.2.9 to compute V_m^n and then use conditions 5.2.40, 5.2.41 and corresponding conditions for decreasing x, and for y, to see in which regions the initial errors

$$e_o^n = V_o^n - V(0,nk)$$
$$e_m^o = V_m^o - V(mh,0)$$

will die away.

For the Hahn equations we have seen the stability of 5.2.9 given the incorrect initial conditions 5.2.17 and 5.2.18. Let us formally compute the regions in which we expect computation to be stable.

If we substitute d = 0 into 5.2.39 simplification may be made and the stability conditions become dependent only on f, g, ρ :

For increasing x: $g(\rho f+g) \geq 0$ 5.2.42

For decreasing x: $g(\rho f+g) \leq 0$ 5.2.43

For increasing y: $\rho f(\rho f+g) \geq 0$ 5.2.44

For decreasing y: $\rho f(\rho f+g) \leq 0$. 5.2.45

Substituting f, g, ρ from 5.2.13, 5.2.14 into 5.2.42 - 5.2.45 we obtain the regions shown in fig. 25 ($\rho = 1$).

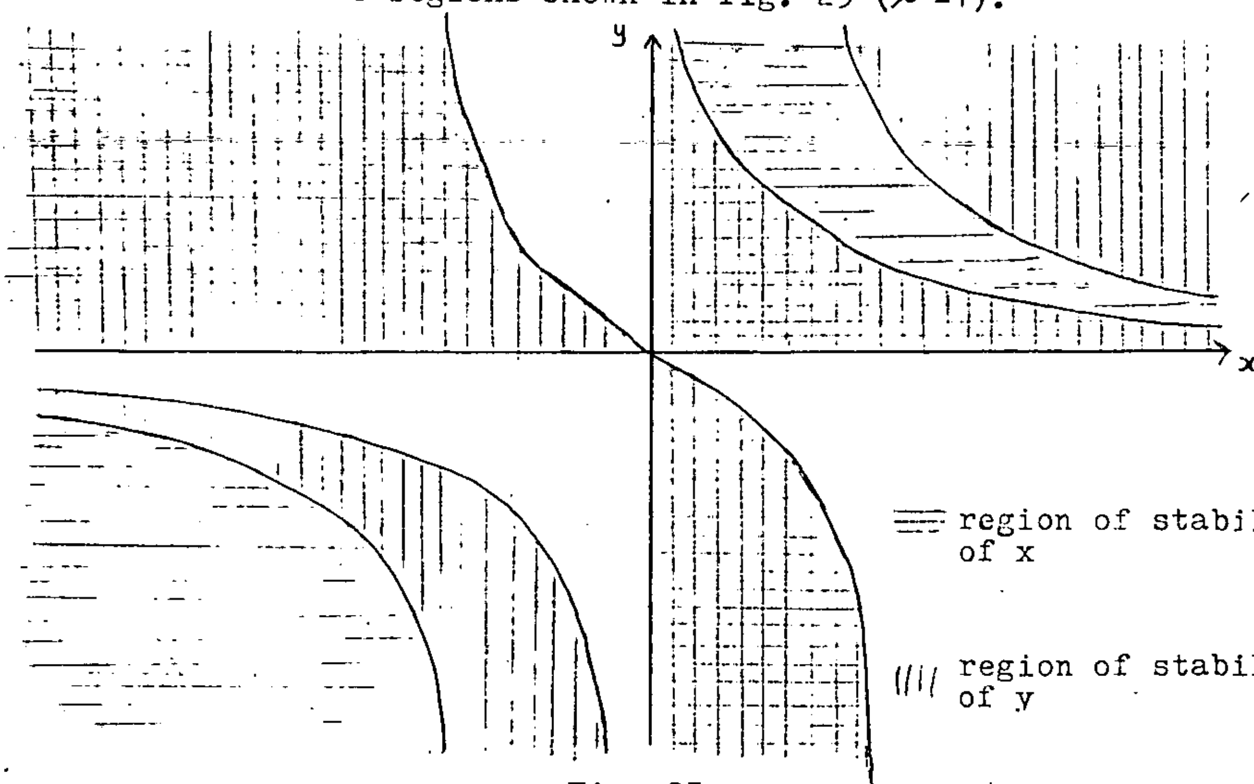


Fig. 25

Fig. 25 illustrates that when 5.2.9 is used to compute V_m^n from V_{m-1}^n and V_m^{n-1} in the positive quadrant that computation is stable up the line given by $xy = \frac{1}{2}$. The lines of discontinuity in figs. 35, 36, 40 are seen to be quite close to $xy = \frac{1}{2}$. Computation of other quadrants using 5.2.9 in different ways cannot guarantee stability in a neighbourhood of the origin.

However it is more likely that to compute results in other quadrants 5.2.9 would still be used to compute V_m^n from V_{m-1}^n, V_m^{n-1} given the definition 5.2.6 of x_m, y_n by simply making h or k negative. Stability in the four quadrants is then determined by considering the conditions for increasing x and increasing y with ρ either positive or negative in alternate quadrants.

Putting f, g, ϕ from 5.2.13, 5.2.14 into 5.2.42 and 5.2.44 with $\rho = +1$ in the first and third quadrants, and $\rho = -1$ in the second and fourth quadrants, we obtain the stability regions of fig. 26. Thus we see that using 5.2.9 in the form 5.2.12 is highly stable and that initial conditions can be chosen arbitrarily without unduly affecting computation of the boundary of the D.O.A., as shown in sub-section 2.1.

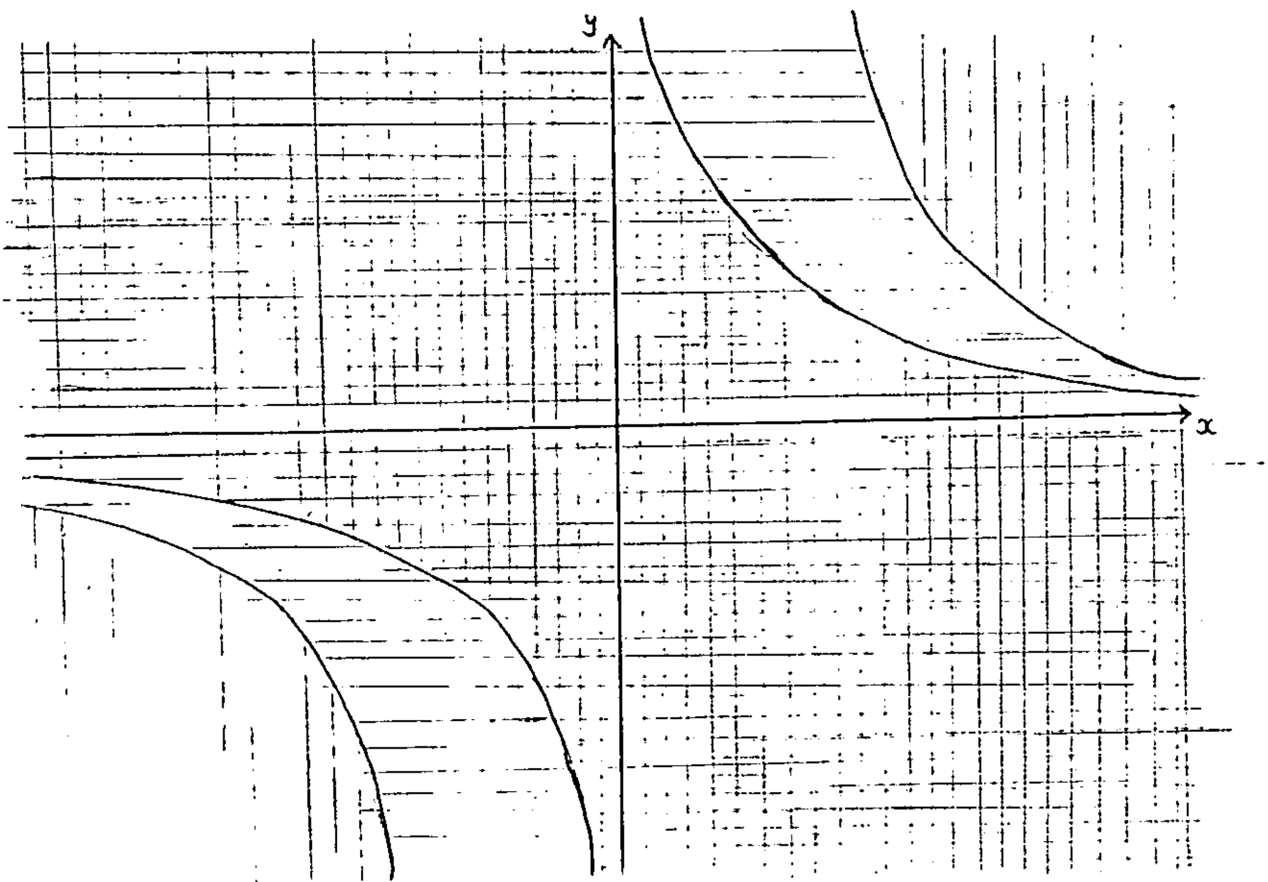


Fig. 26
— region of stability of x

||| region of stability of y

For the system 5.2.19 the stability regions may be evaluated to be given by 5.2.42, 5.2.44 in the positive quadrant. From 5.2.19 we see that $f \leq 0$, $g \leq 0$ in the first quadrant. Hence the stability condition in both x and y becomes

$$\rho f + g = -2\rho x - 2\rho y^4 - y \leq 0. \qquad 5.2.46$$

Hence we need $x \geq -y^4 - y/2\rho$.

5.2.46 shows that this method is stable for system 5.2.19 in the first quadrant.

The explanation of the fact that the errors in the initial conditions took longer to die away near the y-axis can be found by investigation of g, ϕ and the initial conditions. Near the y-axis the following approximations hold:

$$V_m^n - V_m^{n-1} \approx 2nk^2 \qquad 5.2.47$$

(follows from the initial conditions 5.2.22)

$$\phi_m^n \approx 2n^2k^2. \qquad 5.2.48$$

Substituting 5.2.47, 5.2.48 into 5.2.9 yields another approximation

$$V_m^n - V_{m-1}^n \approx 0 \qquad 5.2.49$$

Fig, 37,38 shows that 5.2.49 holds near the y-axis and that the errors die out but not as quickly as the errors caused by initial conditions on the x-axis.

Comparison of 5.2.27 and 5.2.40, 5.2.41 indicates that the boundary of stability of the numerical method and the line of instability caused by the denominator of 5.2.12 being small can often be the same. The next example shows that where instability of the method occurs first, that the results quickly become inaccurate even when the initial conditions are correct.

Now consider the system

$$\dot{x} = -x + y + x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2).$$

Solving the Zubov equation using $d = 1$, $\phi(x,y) = 2x^2 + 2y^2$ we obtain

$$V(x,y) = x^2 + y^2.$$

By reference to 5.2.40, 5.2.41 we see that the stability region in the first quadrant is given by

$$r^2 \leq \frac{c+s}{s}$$

$$\text{and } r \leq \frac{k + \sqrt{k^2 + (\rho c + s)((1+\rho)c + (1-\rho)s)}}{\rho c + s} \qquad 5.2.50$$

$$\text{and } r \leq \frac{k + \sqrt{k^2 + 4\rho^2 c(c-s)}}{2\rho c} \quad \text{or } r \geq \frac{k - \sqrt{k^2 + 4\rho^2 c(c-s)}}{2\rho c}$$

A combination of the equations 5.2.50 is shown on fig. 41 along with the attempts to compute V_m^n using $h = 0{\cdot}01$, $k = 0{\cdot}005$ and equation 5.2.12 with the initial conditions 5.2.16. It can be seen that even when the initial conditions are correct the results will be highly inaccurate if the method used with the system is unstable.

2.5. Conclusion

The method of solution of 5.1.1 in 2 dimensions by a numerical difference formula has certain problems.

The obvious problem is that of initial conditions. However it is seen that it is possible to choose a method such that errors in initial conditions and errors in computation should be swamped. Indeed, the method 5.2.9 gives rise to stability conditions 5.2.39 or 5.2.40, 5.2.41.

Obtaining the boundary of the D.O.A. is more difficult. Figs, 25, 26 indicate that for the Hahn equation the stability region is inside the boundary and the computed results of figs. 35, 36, 40 show this. The reason for this is that when 5.2.27 is satisfied the coefficient of V_m^n is small and a way of avoiding this has to be added to the method.

3. Theory of P.D.E.s

In section 1 it was noted that for a P.D.E. in n dimensions such as 5.1.1 to have a unique solution, initial conditions need to be specified on an (n-1)-dimensional subspace. However if the initial conditions are specified on characteristics of the system, as they are for the Zubov equation, then initial conditions in (n-1) dimensions are not necessary. This will be proved here and then it will be shown that initial condition problems encountered in section 2 are overcome by conversion to polar co-ordinates.

Theorem 5.3.1

A P.D.E.

$$\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1 - dV(\underline{x})) \qquad 5.3.1$$

with initial conditions given by

$$x_i = x_i(t_1,\ldots,t_{\mathfrak{J}}) \qquad i = 1,\ldots,n \qquad 5.3.2$$

$$V = V(t_1,\ldots,t_{\mathfrak{J}})$$

which do not lie on the system characteristics where $t_1,\ldots,t_j$ are freedom parameters, has a unique solution if and only if $j = n-1$.

Proof

It has been noted by theorem 1.8.1 that 5.3.1 may be transformed into a set of O.D.E.s representing the characteristic equations:

$$\frac{dx_i}{f_i(\underline{x})} \quad (i = 1,\ldots,n) = \frac{dV}{-\phi(\underline{x})(1-dV(\underline{x}))} \qquad 5.3.3$$

Also by theorem 1.8.2 the solution of 5.3.3 is given by

$$u_i(\underline{x},V) = c_i \qquad 5.3.4$$

$$i = 1,\ldots,n.$$

Equations 5.3.4 are the characteristics of system 5.3.1. The general solution then becomes

$$W(u_1,\ldots,u_n) = 0. \qquad 5.3.5$$

To determine the function W we need to use the constants c_i in 5.3.5 rather than u_i. When W is found in terms of c_i, reference to 5.3.4 immediately gives W in terms of u_i.

First suppose the initial conditions are as given in 5.3.2 but on a j-dimensional hyperspace.

i.e. Initially
$$x_i = x_i(t_1,\ldots,t_j) \qquad 5.3.6$$
$$V = V(t_1,\ldots,t_j)$$

where j may be any non-negative integer. Since 5.3.4 is true for the initial conditions, we may substitute 5.3.6 in 5.3.4 giving

$$u_i(\underline{x}(t_1,\ldots,t_j),V(t_1,\ldots,t_j)) = c_i \qquad 5.3.7$$

$$i = 1,\ldots,n.$$

Since the initial conditions 5.3.2 do not identically lie on any system characteristic then 5.3.7 represents n equations from which we need to eliminate the j parameters to obtain a relationship between the c_i, $i = 1,\ldots,n$ of the form 5.3.5.

If $j < n$ then eliminating $t_1,\ldots,t_j$ from 5.3.7 gives $n-j$ relations between $c_1,\ldots,c_n$. Hence if $j < n-1$ the solution is not unique.

If $j \geqslant n$ then $t_1,\ldots,t_j$ cannot all be eliminated. Hence we see that $j = n-1$ is necessary. For sufficiency we see that if $j = n-1$ then upon elimination of $t_1,\ldots,t_{n-1}$ from 5.3.7 we are left with one equation connecting all the c_i and using 5.3.4

we see that this becomes the unique solution. End of proof.

Theorem 5.3.2

The initial conditions 5.3.2 may be replaced by the initial conditions

$$x_i = x_i(t_1,\dots,t_j) \quad i = 1,\dots,n$$
$$V = V(t_1,\dots,t_j) \quad j \leq n-1 \qquad 5.3.8$$

where the system 5.3.8 lies identically on n-j-1 separate characteristics of the system 5.3.1.

Proof

If the system 5.3.8 lies identically on n-j-1 characteristics of the system then we may substitute 5.3.6 into 5.3.4. n-j-1 equations of the form 5.3.7 will vanish identically leaving j+1 equations from which we eliminate the j parameters $t_1,\dots,t_j$. The remainder of the proof is as for theorem 5.3.1. End of proof.

It may be, however, that the initial conditions 5.3.8 when substituted into the characteristics 5.3.4 produce the result that 2 or more characteristics 5.3.4 become the same, or dependent. In this case we have identities which when eliminated give other relationships between $t_1,\dots,t_j$ and $c_1,\dots,c_n$. Then we have

Theorem 5.3.3

If the initial conditions 5.3.8 lie identically on n-j+k-1 characteristics of 5.3.1 where k such characteristics are the same as or dependent on some of the n-j-1 other characteristics also satisfied by 5.3.8 the solution of 5.3.1 is again unique.

Alternatively when the initial conditions 5.3.8 are substituted into the characteristics 5.3.4 we find that one or more characteristic may become indeterminate. In this situation less characteristics are satisfied identically, but when singularities are removed we are left with the correct number of identities. This gives

Theorem 5.3.4

If the initial conditions 5.3.8 lie identically on n-j-k-1 characteristics of 5.3.1 and cause k different singularities in other characteristics which when eliminated result in k further characteristics which are identical or dependent then the solution of 5.3.1 is again unique.

These theorems may be illustrated by some examples. Consider first the system

$$\dot{x} = -x + y \mp x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2).$$

The auxiliary equations 5.3.3 become

$$\frac{dx}{-x+y+x(x^2+y^2)} = \frac{dy}{-x-y+y(x^2+y^2)} = \frac{dV}{-2(x^2+y^2)} \qquad 5.3.9$$

By conversion to polar co-ordinates the independent solutions of 5.3.9 may be found and are

$$\frac{1-x^2-y^2}{e^{-V}} = c_1 \qquad 5.3.10$$
$$\frac{(x^2+y^2)e^{-2\tan^{-1}y/x}}{1-x^2-y^2} = c_2. \qquad 5.3.11$$

The initial conditions 5.1.3 may be written parametrically as

$$x = 0,\ y = 0,\ V = 0. \qquad 5.3.12$$

5.3.12 is a set of equations in 0 dimensions while theorem 5.3.1 requires initial conditions in 1 dimension.

Substituting 5.3.12 into 5.3.10, 5.3.11 gives

$$c_1 = 1 \qquad 5.3.13$$
$$c_2 = 0.$$

The result in 5.3.13 satisfies identically the characteristic given by $x = 0$, $y = 0$ i.e. $c_2 = 0$ in 5.3.11. Thus we see that the initial conditions 5.3.12 satisfy identically one characteristic of the system 5.3.9 and theorem 5.3.2 is satisfied by $j = 0$, $n = 2$. The solution is given by

$$1-x-y^2 = e^{-V}.$$

The Hahn system 5.2.13, 5.2.14 is a good system to consider these theorems on. The auxiliary equations 5.3.3 become

$$\frac{dx}{-x+2x^2y} = \frac{dy}{-y} = \frac{dV}{-2(x^2+y^2)} \qquad 5.3.14$$

The solutions of 5.3.14 are the surfaces

$$\frac{x}{y(1-xy)} = c_1 \qquad 5.3.15$$
$$V + \frac{x}{y} - y^2 = c_2 \qquad 5.3.16$$

Substituting 5.3.12 into 5.3.15 and 5.3.16 gives c_1, c_2 indeterminate. If 5.3.15, 5.3.16 are transformed such that x/y is eliminated between them and then 5.3.12 substituted we obtain the relation $c_1 = c_2$ which leads to the analytic solution

$$V = y^2 + \frac{x^2}{1-xy}.$$

We can see that the initial conditions 5.3.12 give no freedom parameters. i.e. $j = 0$ in theorems 5.3.1, 5.3.2, 5.3.3, 5.3.4. Now 5.3.12 lies on both characteristics of 5.3.14 and the conditions of theorem 5.3.1 are not satisfied. Likewise theorem 5.3.2 is not satisfied as 5.3.12 results in 5.3.15 and 5.3.16 being indeterminate. However theorem 5.3.4 is satisfied by $j = 0$, $n = 2$, and $k = 1$. When 5.3.12 is substituted into 5.3.15, 5.3.16 the two characteristics are indeterminate and the singularity caused by consideration of x/y at $(0,0)$ may be eliminated to leave one relationship between c_1 and c_2.

The example illustrates what must in fact happen to all systems where the origin is asymptotically stable. It turns out that since f_i $(i=1,\ldots,n)$ are not functions of V that from 5.3.3 we may obtain n-1 relations of the form 5.3.7 which do not depend on V and one which does depend on V. The n-1 equations are independent of ϕ and V and are the system trajectories given by

$$u_i(x_1,\ldots,x_n) = c_i \qquad i = 1,\ldots,n-1$$

with $u_n(x_1,\ldots,x_n,V) = c_n$.

Since the system trajectories all tend towards the origin we see that

$$u_i(0,\ldots,0) = c_i \qquad i = 1,\ldots,n-1$$

are identically satisfied. There are in fact n-1 singularities or identities which when eliminated similarly to that in 5.3.10, 5.3.11 or 5.3.15, 5.3.16 lead to one equation in c_i, $i=1,\ldots,n$, after substituting $x_i = 0$, $V = 0$, $i=1,\ldots,n$. Hence in the terms of theorem 5.3.4 the Zubov equation 5.3.1 satisifes $j = 0$, $k = n-1$ and a unique solution of 5.3.1 exists.

Hence a unique solution of the Zubov equation 5.3.1 exists for systems where the origin is a critical point by eliminating singularities in the general solution of 5.3.3 at the origin.

3.1. Numerical Solution

However to solve 5.3.1 numerically on a grid system is another matter. The usual method is to compute the solution of V at $x_i^{(j_i)} = j_i h_i$ for some $1 \leq i \leq n$ given values of V at $x_i^{(j_i-1)} = (j_i-1)h_i$, $1 \leq i \leq n$. However to begin computation we need initial conditions of V on $x_i^{(M_i)} = M_i h_i$ for $1 \leq i \leq n$, some M_i. 5.3.17

5.3.17 is an $(n-1)$-dimensional hyperspace in the variables x_i, $i = 1,\ldots,n$, $i \neq j$. This is in accordance with theorem 5.3.1 and by solving 5.3.1 there is no let-out from this problem afforded by singularities except by means of solving numerically along the characteristic lines. This is the method used by Fox (36) which was considered in Chapter 2. This method computes the characteristics $\xi(x,y)$ and the value of V along them. Since all the characteristics converge to the origin we see that the initial conditions 5.1.3 are sufficient for Fox's method.

Since the initial conditions given by 5.1.3 are zero-dimensional and we require $(n-1)$-dimensional conditions then we need to transform the independent variables such that the corresponding initial condition to 5.1.3 in the new variables is $(n-1)$-dimensional.

The polar co-ordinate system given by 3.2.17 is such a system.

$$x_1 = r\cos\theta_1$$

$$x_i = r\sin\theta_1 \ldots\ldots\ldots \sin\theta_{i-1}\cos\theta_i, \qquad i = 2,\ldots,n-1 \qquad 5.3.18$$

$$x_n = r\sin\theta_1 \ldots\ldots\ldots \sin\theta_{n-1}.$$

The initial condition 5.1.3 becomes

$$V(0,\theta_1,\ldots,\theta_{n-1}) = 0 \qquad 5.3.19$$

which is $(n-1)$-dimensional. In the notation of theorem 5.3.1 we obtain the characteristic solutions

$$u_i(r,\theta_1,\ldots,\theta_{n-1},V) = c_i$$
$$i = 1,\ldots,n$$

with initial conditions as

$$r = 0$$
$$\theta_i = t_i \qquad i = 1,\ldots,n-1 \qquad 5.3.20$$
$$V = 0$$

which are in the form 5.3.2.

Now by the chain rule of partial derivatives we know that

$$\frac{\partial V}{\partial r} = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial r}$$

$$\frac{\partial V}{\partial \theta_i} = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial \theta_i} \qquad i = 1,\ldots,n-1. \qquad 5.3.21$$

Now from 5.3.18 we may obtain the partial derivatives

$\frac{\partial x_j}{\partial r}$ $\quad j = 1,\ldots,n$

and $\frac{\partial x_j}{\partial \theta_i}$ $\quad i = 1,\ldots,n-1$, $\quad j = 1,\ldots,n.$

Thus 5.3.21 represents n linear equations of $\frac{\partial V}{\partial r}, \frac{\partial V}{\partial \theta_i}$ $(i=1,\ldots,n-1)$ in terms of $\frac{\partial V}{\partial x_j}$ $(j = 1,\ldots,n)$. These equations may be inverted to obtain $\frac{\partial V}{\partial x_j}$ explicitly and then by substituting in 5.3.1 we obtain a polar co-ordinate P.D.E.

$$F(r,\theta_1,\ldots,\theta_{n-1})\frac{\partial V}{\partial r}(r,\theta_1,\ldots,\theta_{n-1}) + \sum_{i=1}^{n-1} G_i(r,\theta_1,\ldots,\theta_{n-1})\frac{\partial V}{\partial \theta_i}(r,\theta_1,\ldots,\theta_{n-1})$$

$$= -\Phi(r,\theta_1,\ldots,\theta_{n-1})(1-dV(r,\theta_1,\ldots,\theta_{n-1})). \qquad 5.3.22$$

The actual equations 5.3.21 and their inversion are carried out in Appendix E. It is equation 5.3.22 and the initial condition 5.3.19 which form the basis of the numerical method in the rest of this chapter.

4. Radial Methods

Having transformed 5.3.1 into 5.3.22 by means of 5.3.18 we now require to set up a grid system to solve 5.3.22 numerically. From here on we consider the 2-dimensional version of 5.3.1, 5.3.22 which become

$$f(x,y)\frac{\partial V}{\partial x} + g(x,y)\frac{\partial V}{\partial y} = -\phi(x,y)(1-dV(x,y))$$

$$x = r\cos\theta, \quad y = r\sin\theta \qquad 5.4.1$$

$$F(r,\theta)\frac{\partial V}{\partial r} + G(r,\theta)\frac{\partial V}{\partial \theta} = -\Phi(r,\theta)(1-dV(r,\theta)). \qquad 5.4.2$$

Differentiating 5.4.1 with respect to r,θ to obtain the $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta}$ and substituting into 5.4.2 we obtain

$$F(r,\theta) = f(r\cos\theta, r\sin\theta)\cos\theta + g(r\cos\theta, r\sin\theta)\sin\theta$$

$$5.4.3$$

$$G(r,\theta) = (g(r\cos\theta, r\sin\theta)\cos\theta - f(r\cos\theta, r\sin\theta)\sin\theta)/r$$

with $\Phi(r,\theta) = \phi(r\cos\theta, r\sin\theta)$.

Various methods may be used to solve 5.4.2 but three are outlined here. First we require some notation.

The grid is set up in the same manner as the rectangular grid 5.2.1.

$$r_m = mh$$

$$\theta_n = nk$$

where $k = \frac{2\pi}{N}$ $n = 1,\dots,N$ 5.4.4

$m \geqslant 0.$

The analytic solution of 5.4.2 is denoted by

$V(mh,nk)$

and the computed value of V by

V_m^n $m \geqslant 0,\ n = 1,\dots,N$ 5.4.5

4.1 Shields' Method

The method suggested by Shields (28) approximates the partial derivatives by

$$\frac{\partial V}{\partial r} \approx \frac{1}{2h}(V_{m+1}^{n+1}+V_{m+1}^{n}-V_m^{n+1}-V_m^n)$$

$$\frac{\partial V}{\partial \theta} \approx \frac{1}{2k}(V_{m+1}^{n+1}-V_{m+1}^{n}+V_m^{n+1}-V_m^n) \qquad 5.4.6$$

$$V \approx \frac{1}{4}(V_{m+1}^{n+1}+V_{m+1}^{n}+V_m^{n+1}+V_m^n).$$

Substituting 5.4.6 into 5.4.2 and rearranging gives

$$\left(F+\frac{hG}{k}-\frac{hd\Phi}{2}\right)V_{m+1}^{n+1} + \left(F-\frac{hG}{k}-\frac{hd\Phi}{2}\right)V_{m+1}^{n}$$

$$= \left(F-\frac{hG}{k}+\frac{hd\Phi}{2}\right)V_m^{n+1} + \left(F+\frac{hG}{k}+\frac{hd\Phi}{2}\right)V_m^n - 2h\Phi. \qquad 5.4.7$$

For $n = 1,\dots,N$ the equation 5.4.7 represents a set of linear equations for values of V on $r = (m+1)h$ given values on $r = mh$ and known functions F,G,Φ .

A matrix equation may be set up

$$A_m \underline{V}_{m+1} = \underline{B}_m \qquad 5.4.8$$

where the nth element of $\underline{V}_{m+1}$ is V_{m+1}^n. The solution of 5.4.8 at each stage provides values of V on concentric circles.

4.2 Explicit Method

Shields' method is seen to be implicit. This is not a drawback to computation of V_{m+1}^n, $n = 1,\dots,N$, but should computational instability affect the value of V_{m+1}^n for any n then this instability is transmitted to V_{m+1}^n for all $n = 1,\dots,N$ i.e. instability affects the computation of whole circles. To overcome the problem we need an explicit method given by

$$\frac{\partial V}{\partial r} \approx \frac{V^n_{m+1}-V^n_m}{h}$$

$$\frac{\partial V}{\partial \Theta} \approx \frac{(1-a)V^{n+1}_m+2aV^n_m-(1+a)V^{n-1}_m}{2k} \qquad 5.4.9$$

$$V \approx bV^n_{m+1}+(1-b)V^n_m$$

Substituting 5.4.9 into 5. 4.2.gives

$$(F-bhd\Phi)V^n_{m+1} = -\frac{hG}{2k}((1-a)V^{n+1}_m-(1+a)V^{n-1}_m)$$

$$+(F-\frac{ahG}{k}+(1-b)hd\Phi)V^n_m -h\Phi. \qquad 5.4.10$$

5.4.10 is an explicit method in fact not as accurate as 5.4.7. Thus we need to obtain a more accurate method.

4.3 Second Order Method

To develop methods of higher order accuracy we may start from the Taylor series expansion given by

$$V((m+1)h,nk) = \exp(h\frac{\partial}{\partial r})\ V(mh,nk). \qquad 5.4.11$$

The exponential series in 5.4.11 may be truncated and for this method we truncate after the second order term and the method becomes

$$V^n_{m+1} = (1 + h\frac{\partial}{\partial r} + \frac{h^2}{2}\frac{\partial^2}{\partial r^2})\ V^n_m \qquad 5.4.12$$

In order to approximate for the terms in 5.4.12 we need to rearrange 5.4.2 to become

$$\frac{\partial V}{\partial r} = \frac{-G}{F}\frac{\partial V}{\partial \Theta} - \frac{\Phi}{F}(1-dV). \qquad 5.4.13$$

Let us denote the computed value of $\frac{\partial V}{\partial r}$ at $r = mh$, $\Theta = nk$ as $\frac{\partial V^n_m}{\partial r}$ with corresponding notation for $\frac{\partial V}{\partial \Theta}$ and for F, G, Φ, V.

Writing

$$a^n_m = \frac{-G^n_m}{F^n_m}$$

$$b^n_m = \frac{-\Phi^n_m}{F^n_m} \qquad 5.4.14$$

and substituting 5.4.14 into 5.4.13 and setting $x = mh$, $y = nk$ we see that

$$\frac{\partial V_m^n}{\partial r} = -a_m^n \frac{\partial V_m^n}{\partial \theta} - b_m^n(1-dV_m^n). \qquad 5.4.15$$

Now we may differentiate 5.4.15 with respect to r again and so compute $\dfrac{\partial^2 V_m^n}{\partial r^2}$.

$$\frac{\partial^2 V_m^n}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial V_m^n}{\partial r}\right) = \frac{\partial}{\partial r}\left(-a_m^n \frac{\partial V_m^n}{\partial \theta} - b_m^n(1-dV_m^n)\right)$$

$$= -\frac{\partial a_m^n}{\partial r}\frac{\partial V_m^n}{\partial \theta} - \frac{\partial b_m^n}{\partial r}(1-dV_m^n) + db_m^n \frac{\partial V_m^n}{\partial r} - a_m^n \frac{\partial^2 V_m^n}{\partial r \partial \theta} \qquad 5.4.16$$

$$= -\frac{\partial a_m^n}{\partial r}\frac{\partial V_m^n}{\partial \theta} - \frac{\partial b_m^n}{\partial r}(1-dV_m^n) + \left(db_m^n - a_m^n \frac{\partial}{\partial \theta}\right)\left(-a_m^n \frac{\partial V_m^n}{\partial \theta} - b_m^n(1-dV_m^n)\right).$$

Substituting 5.4.15 and 5.4.16 into 5.4.12 gives the next stage of setting up the method:

$$V_{m+1}^n = V_m^n + \left(-hb_m^n + \frac{h^2}{2}\left(-\frac{\partial b_m^n}{\partial r} - db_m^n b_m^n + a_m^n \frac{\partial b_m^n}{\partial \theta}\right)\right)(1-dV_m^n)$$

$$+\left(-ha_m^n + \frac{h^2}{2}\left(-\frac{\partial a_m^n}{\partial r} - 2da_m^n b_m^n + a_m^n \frac{\partial a_m^n}{\partial \theta}\right)\right)\frac{\partial V_m^n}{\partial \theta} \qquad 5.4.17$$

$$+ \frac{h^2 a_m^n a_m^n}{2}\frac{\partial^2 V_m^n}{\partial \theta^2}.$$

The equation 5.4.17 expresses V_{m+1}^n in terms of V, $\dfrac{\partial V}{\partial \theta}$, $\dfrac{\partial^2 V}{\partial \theta^2}$ on the circle $r = mh$, and known terms. The only approximation so far has been truncation of the Taylor series. If we can now approximate all the partial derivatives in 5.4.17 by grid values of the corresponding functions combined in such a way that the accuracy will be maintained then we have a second order method.

The central difference formula for a first derivative has second order accuracy, so we may write

$$\frac{\partial V_m^n}{\partial \theta} \approx \frac{V_m^{n+1} - V_m^{n-1}}{2k} \qquad 5.4.18$$

$$\frac{\partial b_m^n}{\partial r} \approx \frac{b_{m+1}^n - b_{m-1}^n}{2h}$$

and similarly for the others.

The second derivative term in 5.4.17 is written as

$$\frac{\partial^2 V_m^n}{\partial \theta^2} \approx \frac{V_m^{n+1} - 2V_m^n + V_m^{n-1}}{k^2}. \qquad 5.4.19$$

Substituting 5.4.18 and 5.4.19 into 5.4.17 gives the difference scheme. Some simplification can be achieved by writing

$$B_m^n = hb_m^n$$
$$A_m^n = \frac{ha_m^n}{k} \qquad 5.4.20$$

The final version of the difference scheme becomes

$$V_{m+1}^n = V_m^n + \left(-B_m^n + \frac{B_{m-1}^n}{4} - \frac{B_{m+1}^n}{4} - \frac{dB_m^n B_m^n}{2} + \frac{A_m^n B_m^{n+1}}{4} - \frac{A_m^n B_m^{n-1}}{4}\right)(1 - dV_m^n)$$
$$+ \left(-\frac{A_m^n}{2} + \frac{A_{m-1}^n}{8} - \frac{A_{m+1}^n}{8} - \frac{dA_m^n B_m^n}{2} + \frac{A_m^n A_m^{n+1}}{8} - \frac{A_m^n A_m^{n-1}}{8}\right)(V_m^{n+1} - V_m^{n-1})$$
$$+ \frac{A_m^n A_m^n}{2}(V_m^{n+1} - 2V_m^n + V_m^{n-1}). \qquad 5.4.21$$

5. Comparison of Radial Methods

The radial methods given by 5.4.7, 5.4.10, 5.4.21 all serve as numerical routines for integrating 5.4.2 but the question is raised as to whether one method is better than another, and if so, why.

5.1 Initial Conditions

All the methods given in section 4 provide means of computing V on r = (m+1)h given V on r = mh. Thus sufficient initial conditions to start computation are given by $V_0^n = 0$, n = 1,...,N, providing there are no problems caused by evaluating F, G, Φ near the origin. We know that at r = 0, $F = \Phi = 0$ by putting r = 0 into 5.4.3. G is not necessarily small for small r but we must not attempt to evaluate 5.4.7, 5.4.10, 5.4.21 such that the coefficients of the L.H.S. are zero or terms on the R.H.S. become large. To see how putting m = 0 into 5.4.7, 5.4.10, 5.4.21 affects the coefficients of V_{m+1}^n, V_m^n we need to consider the behaviour of F, G, Φ as $r \to 0$.

We make use of definitions 3.2.1, 3.2.2 and associated results on the asymptotic properties of functions near the

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(2) D.C. Crane & P.A. Fox, A comparative study of Computer programs for Integrating Differential Equations, Numerical Mathematics Program Library Project, Bell Telephone Lab., 1969.

(3) C.W. Gear, The Numerical Integration of Ordinary Diff Eqn., Math. Comp., Vol 21 pp. 146-156 1967

(4) Lambert, (J.D).

Maths of Computation

C FOR COMMENT

STATEMENT NUMBER

CONTINUATION

FORTRAN STATEMENT

IDENTIFICATION

$$\ddot{x} + \frac{\delta \dot{x}}{2} + (\omega^2 + \varepsilon \cos 2t)\, x = 0$$

$$|\varepsilon| < \max\left[\left(\sqrt{\delta^2+4} - 2\right)\omega,\ \frac{\delta \omega}{\sqrt{\delta^2+4}}\right]$$

$$x:\quad \ddot{x} + D\,2\dot{x} + (\delta + \varepsilon \cos 2t)\, x = 0$$

$$|\varepsilon| < \omega^2$$

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origin. Now from theorems 3.2.1, 3.6.5 we know that

$$P(\phi) > \min(P(f),P(g))-1 \qquad 5.5.1$$

By reference to 5.4.3 and theorems 3.3.1, 3.4.1 we see that for general f,g

$$P(F) = \min(P(f),P(g))$$
$$P(G) = \min(P(f),P(g))-1 \qquad 5.5.2$$
$$P(\Phi) = P(\phi)$$

except in particular systems where lower degree terms in 5.4.3 cancel out. The Hahn example is one system in which

$$P(f) = 1$$
$$P(g) = 1$$
$$P(F) = 1 \qquad 5.5.3$$
$$P(G) = 2$$

but in general equation 5.5.2 is true.

By reference to theorem 3.4.1 we see that 5.5.2 is true except in certain circumstances where significant terms vanish identically. Generally speaking we must assume that 5.5.2 is true although other possibilities must also be mentioned.

Thus we see from 5.5.1 and 5.5.2 that as $r \to 0$ 5.4.7 asymptotically becomes

$$V_{m+1}^{n+1} - V_{m+1}^{n} \approx -V_{m}^{n+1} + V_{m}^{n} \qquad 5.5.4$$

which is quite stable for computational purposes. Terms in 5.4.7 are evaluated at $r = (m+\frac{1}{2})h$, $\theta = (n+\frac{1}{2})k$ and so putting $m = 0$ into 5.4.7 is no problem. If such a situation as 5.5.3 arises then it is easily seen from 5.4.7 that there is a similar asymptotic relationship to 5.5.4.

Consideration of 5.4.10 using 5.5.1 and 5.5.2, however, shows that as $r \longrightarrow 0$ there will be problems caused by G/F or G/Φ becoming large. The known terms are evaluated at $r = (m+c)h$, $\theta = nk$ and we see that for $m = 0$ c should be as large as possible. Hence in general letting $m = 0$ in 5.4.10 is unstable except in examples such as the Hahn system.

Consideration of 5.4.21 is a difficult task. Using the definitions 5.4.14 and 5.4.20 we see that in general $P(A) = -1$, $P(B) > -1$. Using an assumption that $P(V) > 0$ we may possibly eliminate certain terms of 5.4.21 when $m = 0$, leaving instability caused by such terms as $A_{\theta}^{n}B_{0}^{n+1}$ and clearly we cannot in general use 5.4.21 with m = 0. We need, in fact, $m \geqslant 2$ to use 5.4.21 and this means using 5.4.7 or 5.4.10 to get V_1^n and V_2^n, $n=1,\dots N$, or by setting initial conditions to V_2^n.

5.2 Computation

As has already been noted 5.4.7 is an implicit method. Let us consider the situation at $r = mh$ where the $r = mh$ circle is wholly inside the D.O.A., but $r = (m+1)h$ is not. See fig. 27.

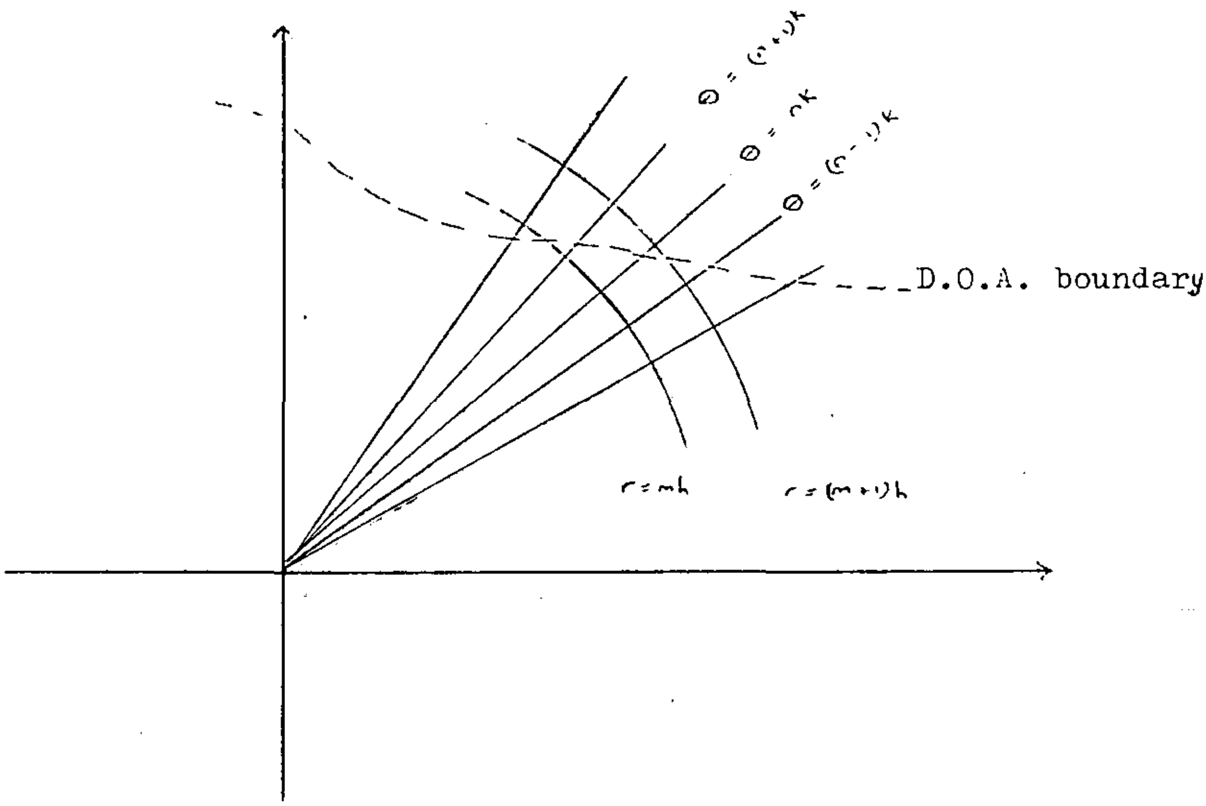


Fig. 27

Assuming no error in computation of V_m^n and the method used is accurate then V_{m+1}^n will in some way, by being unstable, indicate that the boundary has been crossed. Now if $r = (m+1)h$, $\theta = (n+1)k$ is inside the D.O.A., computation of V_{m+1}^{n+1} by 5.4.7 knowing V_{m+1}^n will cause instability in V_{m+1}^{n+1} also. In fact for any V_{m+1}^n that is unstable, continuous use of 5.4.7 will make all V_{m+1}^n, $n = 1,\dots,N$, unstable. Thus the largest computed R.A.S. that can be obtained by 5.4.7 is the largest circle wholly inside the D.O.A..

Methods 5.4.10 and 5.4.21 are explicit and if V_{m+1}^n is unstable for some m,n it does not affect computation of V_{m+1}^n for all $n = 1,\dots,N$. However V_{m+1}^n is used to compute V_{m+2}^{n-1},

V^n_{m+2}, V^{n+1}_{m+2} and instability will spread to these points unless we can find a way of keeping useful information only and not using unstable values of V in further computation. That is the subject of section 8.

5.3 Errors

Each method has associated with it a Local Truncation Error. The order of this error is the factor influencing accuracy of computation. Let us consider the error of method 5.4.10 taken at the point of evaluation of F, G, Φ. i.e. at $r = (m+c)h$, $\theta = nk$.

The Local Truncation Error is obtained by inserting into the numerical method the correct values. Thus we obtain

$$L(r,\theta) \equiv (F-bhd\Phi)^n_{m+c}V((m+1)h,nk)$$

$$-(F-\frac{ahG}{k}+(1-b)hd\Phi)^n_{m+c}V(mh,nk) \qquad 5.5.5$$

$$+\frac{hG}{2k}((1-a)V(mh,(n+1)k)-(1+a)V(mh,(n-1)k))+h\Phi^n_{m+c}.$$

We now obtain $L((m+c)h,nk)$ by substituting for $V((m+1)h,nk)$, $V(mh,nk)$, $V(mh,(n+1)k)$, $V(mh,(n-1)k)$ in 5.5.5 in terms of V, and partial derivatives of V with respect to r,θ at $r = (m+c)h$ $\theta = nk$. To save space the subscripts m+c and n will be dropped. Hence we obtain

$$L \equiv (F-bhd\Phi)(V+(1-c)h\frac{\partial V}{\partial r}+\frac{(1-c)^2h^2}{2}\frac{\partial^2 V}{\partial r^2}+\frac{(1-c)^3h^3}{6}\frac{\partial^3 V}{\partial r^3}+\ldots.)$$

$$+(-F+\frac{ahG}{k}-(1-b)hd\Phi)(V-ch\frac{\partial V}{\partial r}+\frac{c^2h^2}{2}\frac{\partial^2 V}{\partial r^2}-\frac{c^3h^3}{6}\frac{\partial^3 V}{\partial r^3}+\ldots.)$$

$$+(1-a)\frac{hG}{2k}(V-ch\frac{\partial V}{\partial r}+\frac{c^2h^2}{2}\frac{\partial^2 V}{\partial r^2}-\frac{c^3h^3}{6}\frac{\partial^3 V}{\partial r^3}+\ldots..$$

$$+k\frac{\partial V}{\partial \theta}-chk\frac{\partial^2 V}{\partial r\partial\theta}+\frac{c^2h^2k}{2}\frac{\partial^3 V}{\partial r^2\partial\theta}-\frac{c^3h^3k}{6}\frac{\partial^4 V}{\partial r^3\partial\theta}+\ldots..$$

$$+\frac{k^2}{2}\frac{\partial^2 V}{\partial\theta^2}-\frac{chk^2}{2}\frac{\partial^3 V}{\partial r\partial\theta^2}+\frac{c^2h^2k^2}{4}\frac{\partial^4 V}{\partial r^2\partial\theta^2}-\frac{c^3h^3k^2}{12}\frac{\partial^5 V}{\partial r^3\partial\theta^2}+\ldots$$

$$+\frac{k^3}{6}\frac{\partial^3 V}{\partial\theta^3}-\frac{chk^3}{6}\frac{\partial^4 V}{\partial r\partial\theta^3}+\frac{c^2h^2k^3}{12}\frac{\partial^5 V}{\partial r^2\partial\theta^3}-\frac{c^3h^3k^3}{36}\frac{\partial^6 V}{\partial r^3\partial\theta^3}+\ldots.$$

$$\vdots$$

$$)$$

$$-(1+a)\frac{hG}{2k}\Big(V - ch\frac{\partial V}{\partial r} + \frac{c^2h^2}{2}\frac{\partial^2 V}{\partial r^2} - \frac{c^3h^3}{6}\frac{\partial^3 V}{\partial r^3} + \cdots\cdots$$

$$-k\frac{\partial V}{\partial \theta} + chk\frac{\partial^2 V}{\partial r\partial \theta} - \frac{c^2h^2k}{2}\frac{\partial^3 V}{\partial r^2\partial \theta} + \frac{c^3h^3k}{6}\frac{\partial^4 V}{\partial r^3\partial \theta} - \cdots\cdots$$

$$+\frac{k^2}{2}\frac{\partial^2 V}{\partial \theta^2} - \frac{chk^2}{2}\frac{\partial^3 V}{\partial r\partial \theta^2} + \frac{c^2h^2k^2}{4}\frac{\partial^4 V}{\partial r^2\partial \theta^2} - \frac{c^3h^3k^2}{12}\frac{\partial^5 V}{\partial r^3\partial \theta^2} + \cdots\cdots$$

$$-\frac{k^3}{6}\frac{\partial^3 V}{\partial \theta^3} + \frac{chk^3}{6}\frac{\partial^4 V}{\partial r\partial \theta^3} - \frac{c^2h^2k^3}{12}\frac{\partial^5 V}{\partial r^2\partial \theta^3} + \frac{c^3h^3k^3}{36}\frac{\partial^6 V}{\partial r^3\partial \theta^3} - \cdots\cdots$$

$$\vdots$$

$$\Big) \qquad 5.5.6$$

$$+h\Phi.$$

Collecting together terms in 5.5.6 and using 5.4.2 for some cancellation we obtain the Local Truncation Error as

$$L \equiv \Big(\Big(\frac{1-2c}{2}\Big)h^2F\frac{\partial^2 V}{\partial r^2} + (c-b)h^2d\Phi\frac{\partial V}{\partial r} - ch^2G\frac{\partial^2 V}{\partial r\partial \theta} - \frac{ahkG}{2}\frac{\partial^2 V}{\partial \theta^2}\Big)$$

$$+\Big(\Big(\frac{1-3c+3c^2}{6}\Big)h^3F\frac{\partial^3 V}{\partial r^3} + \Big(\frac{-c^2(1-b)-b(1-c)^2}{2}\Big)h^3d\Phi\frac{\partial^2 V}{\partial r^2} + \frac{c^2h^3G}{2}\frac{\partial^3 V}{\partial r^2\partial \theta}$$

$$+\frac{ach^2kG}{2}\frac{\partial^3 V}{\partial r\partial \theta^2} + \frac{hk^2}{6}G\frac{\partial^3 V}{\partial \theta^3}\Big) + O(h^4 + k^4) \qquad 5.5.7$$

Thus we see that the explicit method 5.4.10 is first order. There is no combination of a,b,c that eliminates second order terms in 5.5.7. However if we locally assign

$$a = 0$$

$$b = c = \frac{\frac{1}{2}F\frac{\partial^2 V}{\partial r^2}}{F\frac{\partial^2 V}{\partial r^2} + G\frac{\partial^2 V}{\partial r\partial \theta}} \qquad 5.5.8$$

then we have a second order method. Some means of obtaining the derivatives in 5.5.8 to sufficient accuracy is needed.

The Local Truncation Errors of the other methods are obtained in exactly the same way.

Shields' method:

$$L \equiv \frac{h}{4}\left(\frac{h^2F}{3}\frac{\partial^3 V}{\partial r^3} + h^2G\frac{\partial^3 V}{\partial r^2\partial\theta} - h^2 d\Phi\frac{\partial^2 V}{\partial r^2} + k^2F\frac{\partial^3 V}{\partial r\partial\theta^2} + \frac{k^2G}{3}\frac{\partial^3 V}{\partial\theta^3} - k^2 d\Phi\frac{\partial^2 V}{\partial\theta^2}\right) + O(h^4 + k^4).$$

Second Order method:

$$L \equiv \frac{h^3}{6}\frac{\partial^3 V}{\partial r^3} + k^3A'\frac{\partial^3 V}{\partial\theta^3} + O(k^4 + h^4).$$

Thus we see that both 5.4.7 and 5.4.21 are second order methods and are therefore more accurate than the Explicit method 5.4.10. The Second Order method is the one that is explicit and second order but requires arbitrary initial conditions.

5.4 Convergence

The Courant-Friedrichs-Lewy (44) condition applies to explicit difference methods such as 5.4.10, 5.4.21. If we write down a general scheme for V on $r = (m+1)h$ in terms of V on $r = mh$ and known functions as

$$V^n_{m+1} = W(r,\theta,h,k) + \sum_{j=n-j_1}^{n+j_2} W_{j-n}(r,\theta,h,k)V^j_m \qquad 5.5.9$$

then the C-F-L condition for convergence of the method is given by:

C-F-L condition

If the characteristic curve of 5.4.2 passing through $r = (m+1)h$, $\theta = nk$ intersects $r = mh$ at θ' then convergence takes place if

$$(n-j_1)k \leq \theta' \leq (n+j_2)k$$

where j_1, j_2 are as in 5.5.9.

For the two explicit schemes 5.4.10 and 5.4.21 V^n_{m+1} is computed from V^{n+1}_m, V^n_m, V^{n-1}_m. Hence in this case $j_1 = j_2 = 1$.

The C-F-L condition quite simply requires the characteristic through $r = (m+1)h$, $\theta = nk$ to intersect $r = mh$ within the range of points used in computation of V^n_{m+1}. Thus for 5.4.10, 5.4.21 we require the characteristic to be as in fig. 28 but not as in fig. 29.

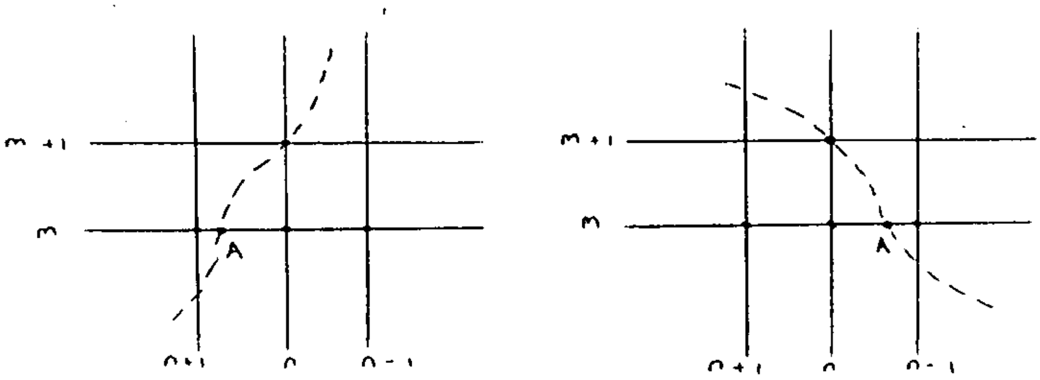


Fig. 28

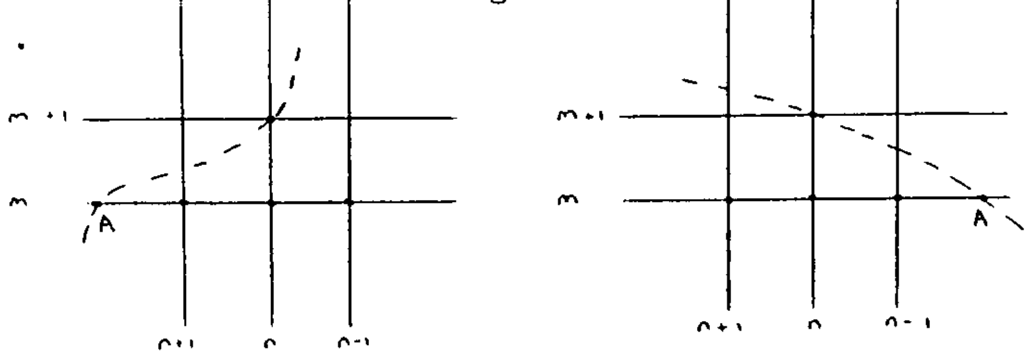


Fig. 29

Consider the point A on fig. 28 which is given by $r = mh$, $\theta = (n+\lambda)k$. Hence the condition for convergence becomes $-1 \leqslant \lambda \leqslant 1$. Now we let $h,k \to 0$ while h/k is constant. The grid "closes up" and we see that

$$\frac{dr}{d\theta} \approx \frac{-h}{\lambda k} \tag{5.5.10}$$

where $\frac{dr}{d\theta}$ is the gradient of the characteristic. But the characteristic of 5.4.2 is given by

$$\frac{dr}{d\theta} = \frac{F}{G}. \tag{5.5.11}$$

Hence letting $h,k \to 0$ with h/k constant and combining 5.5.10 and 5.5.11 we see that

$$\lambda \approx -\frac{hG}{kF}.$$

Thus in order to satisfy the C-F-L condition of convergence we require

$$\frac{h}{k} \leqslant \left|\frac{F}{G}\right|. \tag{5.5.12}$$

Thus we have now imposed an upper bound on h or a lower bound on k to guarantee convergence of 5.4.10, 5.4.21.

5.5 Stability

An important basis of comparison between methods is stability. An accurate method is of little use if the errors which occur are magnified and the results become unstable. But if errors die away then a less accurate method may be more acceptable.

The analysis is by the sinusoidal input method exactly as in sub-section 2.4 and the corresponding versions of 5.2.32 for the methods 5.4.7, 5.4.10, 5.4.21 are

Shields' method:

$$L_m^n = \left(F+\frac{hG}{k}-\frac{hd}{2}\Phi\right)e_{m+1}^{n+1} + \left(F-\frac{hG}{k}-\frac{hd}{2}\Phi\right)e_{m+1}^{n} - \left(F-\frac{hG}{k}+\frac{hd}{2}\Phi\right)e_m^{n+1} - \left(F+\frac{hG}{k}+\frac{hd}{2}\Phi\right)e_m^{n}. \qquad 5.5.13$$

Explicit method:

$$L_m^n = (F-bhd\Phi)e_{m+1}^n - \left(F-\frac{ahG}{k}+(1-b)hd\Phi\right)e_m^n + \frac{hG}{2k}\left((1-a)e_m^{n+1}-(1+a)e_m^{n-1}\right) \qquad 5.5.14$$

Second Order method:

$$L_m^n = e_{m+1}^n - e_m^n + d\left(-B_m^n + \frac{B_{m-1}^n}{4} - \frac{B_{m+1}^n}{4} - \frac{dB_m^nB_m^n}{2} + \frac{A_m^nB_m^{n+1}}{4} - \frac{A_m^nB_m^{n-1}}{4}\right)e_m^n$$
$$-\left(-\frac{A_m^n}{2} + \frac{A_{m-1}^n}{8} - \frac{A_{m+1}^n}{8} - \frac{dA_m^nB_m^n}{2} + \frac{A_m^nA_m^{n+1}}{8} - \frac{A_m^nA_m^{n-1}}{8}\right)\left(e_m^{n+1}-e_m^{n-1}\right)$$
$$-\frac{A_m^nA_m^n}{2}\left(e_m^{n+1}-2e_m^n+e_m^{n-1}\right). \qquad 5.5.15$$

Unlike in the rectangular grid method we only require stability in one dimension. Computation in the Θ-direction is periodic and Von-Neumann stability only gives information on the basis that computation is continued indefinitely. Thus we need to substitute the version of e_m^n in 5.2.34 into 5.5.13, 5.5.14, 5.5.15 to obtain λ in each case:

Shields' method:

$$\lambda_m^n = \frac{\left(F-\frac{hG}{k}+\frac{hd\Phi}{2}\right)e^{i\omega k} + \left(F+\frac{hG}{k}+\frac{hd\Phi}{2}\right)}{\left(F+\frac{hG}{k}-\frac{hd\Phi}{2}\right)e^{i\omega k} + \left(F-\frac{hG}{k}-\frac{hd\Phi}{2}\right)}. \qquad 5.5.16$$

Explicit method:

$$\lambda_m^n = \frac{\left(F-\frac{ahG}{k}+(1-b)hd\Phi\right) - \frac{hG}{2k}\left((1-a)e^{i\omega k} - (1+a)e^{-i\omega k}\right)}{(F-bhd\Phi)} \qquad 5.5.17$$

Second Order method:

$$\lambda_m^n = 1 - d\left(\frac{-B_m^n + B_{m-1}^n}{4} \frac{-B_{m+1}^n}{4} \frac{-dB_m^n B_m^n}{2} \frac{+A_m^n B_m^{n+1}}{4} \frac{-A_m^n B_m^{n-1}}{4}\right)$$

$$+2i \sin \omega k\left(\frac{-A_m^n}{2} \frac{+A_{m-1}^n}{8} \frac{-A_{m+1}^n}{8} \frac{-dA_m^n B_m^n}{2} \frac{+A_m^n A_m^{n+1}}{8} \frac{-A_m^n A_m^{n-1}}{8}\right)$$

$$+ A_m^n A_m^n (\cos \omega k - 1), \qquad 5.5.18$$

From 5.5.16, 5.5.17, 5.5.18 we are able to obtain regions where the magnitude of λ_m^n is less than one for all ω.

The procedure for Shields' method is comparitively simple and in fact

$$\left|\lambda_m^n\right| \leqslant 1 \text{ when either } d = 0 \text{ or } F \leqslant 0. \qquad 5.5.19$$

The Explicit method takes a little more analysing. Taking the modulus of λ from 5.5.17 we obtain for $d = 0$

$$\left(F - \frac{ahG}{k}(1-\cos \omega k)\right)^2 + \left(\frac{hG}{k}\sin \omega k\right)^2 \leqslant F^2. \qquad 5.5.20$$

5.5.20 is a quadratic function of $\cos \omega k$ for $-1 \leqslant \cos \omega k \leqslant 1$. The coefficient of $\cos^2 \omega k$ is $\left(\frac{ahG}{k}\right)^2 - \left(\frac{hG}{k}\right)^2$. Since we assume that in the approximation of $\frac{\partial V}{\partial \Theta}$ given by 5.4.9 that $|a| \leqslant 1$ then we have an inverted parabola for 5.5.20 with one maximum value. However we may note that if $\cos \omega k = 1$ then 5.5.20 is satisfied as an equality. Hence we require the value of $\cos \omega k$ which maximises the L.H.S. of 5.5.20 to be such that $\cos \omega k \geqslant 1$.

Hence we obtain

$$G\left(\frac{hG}{k} - aF\right) \leqslant 0 \qquad 5.5.21$$

as the condition for stability of the explicit method for $d = 0$. If $d = 1$ analysis becomes much harder so we only consider the symmetric version of 5.4.9 given by $a = 0$.

In this case we require

$$(1-2b)hd\Phi + 2F \leqslant \frac{-h\,G^2}{k^2\Phi}. \qquad 5.5.22$$

For the Second Order method we simplify 5.5.18 to become denoted as

$$\lambda = u \cos \omega k + w + iz \sin \omega k.$$

Then if i) $u^2 > z^2$ and $uw \geqslant 0$ then we need $(u+w)^2 \leqslant 1$,

or $u^2 < z^2$ and $uw \geqslant z^2 - u^2$

ii) $u^2 > z^2$ and $uw \leqslant 0$ then we need $(u-w)^2 \leqslant 1$,

or $u^2 < z^2$ and $uw \leqslant u^2 - z^2$

iii) $u^2 < z^2$ and $u^2-z^2 \leqslant uw \leqslant z^2-u^2$ then we need

$$w^2 + z^2 - \frac{u^2w^2}{u^2-z^2} \leqslant 1.$$

From the above analysis we see that unlike the situation of constant coefficient differential equations it is not always easy to analyse stability when the coefficients of the equation are variable. The parameters in 5.4.10 can be chosen with 5.5.21 or 5.5.22 in mind but only for a particular F, G, Φ.

5.6 Discussion

It is clear that there is no one numerical method which will solve all problems it has to tackle in the simplest and quickest way. There are many more methods than those discussed in this section, but the three that have been covered serve to show the comparisons that can be made.

Each method has its strengths and its drawbacks. The Shields' method is second order, stable everywhere if $d = 0$ and has no difficulties with initial conditions. Yet it is implicit and can only obtain circles completely contained in the D.O.A..

The Explicit method is explicit and does not have problems with initial conditions unless c is small, but is only first order unless $a = 0$ and b,c are locally determined, and imposes conditions on h,k for convergence and stability.

The Second Order method is second order and explicit, but cannot be computed near the origin and also puts bounds on h,k for convergence and stability.

The best combination is probably to use Shields' method until the D.O.A. is breached, then one of the other two methods depending on which happens to be less restrictive on choice of h,k, preferably maintaining second order accuracy.

6. Radial Runge-Kutta Method

Runge-Kutta methods are known to possess certain advantages over other difference formulae. They are one-step methods and yet can achieve greater accuracy than any one-step difference formula. Their drawback is heavy calls on function evaluation.

Up to now we still only have a method which computes

V_m^n at $r = mh$, $\theta = nk$, $n = 1,\ldots,N$, $k = 2\pi/N$, and we still have to try to obtain the contour $V = \infty$ or $V = 1$ from the grid values. Therefore it is a logical suggestion that if the variation of V with respect to θ can be replaced by a difference formula, then we may reduce 5.4.2 to a system of O.D.E.s to obtain

$$V_n(r),\ n = 1,\ldots,N, \qquad 5.6.1$$

where $V_n(r)$ is the computed function of V with respect to r on $\theta = nk$.

Hence let us approximate in 5.4.2 for $\frac{\partial V}{\partial \theta}$ by

$$\frac{\partial V}{\partial \theta} \approx \frac{V_{n+1}(r) - V_{n-1}(r)}{2k}. \qquad 5.6.2$$

Also we denote known functions similarly to 5.6.1 and write

$$\begin{aligned} F_n(r) &= F(r,nk) \\ G_n(r) &= G(r,nk) \qquad\qquad 5.6.3 \\ \Phi_n(r) &= \Phi(r,nk) \qquad n = 1,\ldots,N. \end{aligned}$$

Substituting 5.6.1, 5.6.2, 5.6.3 in 5.4.2 we obtain the required set of O.D.E.s

$$F_n(r)V_n'(r) = \frac{-G_n(r)}{2k}(V_{n+1}(r) - V_{n-1}(r)) - \Phi_n(r)(1 - dV_n(r))$$
$$n = 1,\ldots,N. \qquad 5.6.4$$

5.6.4 is a set of N simultaneous differential equations and the nature of the solution depends on the functions F, G, Φ. The analytic solution of 5.6.4 involves a complementary function and a particular integral. The complementary function takes the form

$$\text{C.F.} = \exp\left\{\int \frac{d\,\Phi_n(r)}{F_n(r)}\,dr\right\}$$

and it is difficult to obtain for all but the simplest systems.

Equation 5.6.4 may be written in matrix form

$$\underline{V}'(r) = A(r)\,\underline{V}(r) + \underline{b}(r) \qquad 5.6.5$$

where the elements of A(r) are given by

$$A_{n,n}(r) = \frac{d\,\Phi_n(r)}{F_n(r)}, \qquad n = 1,\ldots,N.$$

$$A_{n,n+1}(r) = \frac{-G_n(r)}{2kF_n(r)} \qquad n = 1,\ldots,N-1.$$

$$A_{n,n-1}(r) = \frac{G_n(r)}{2kF_n(r)} \qquad n = 2,\ldots,N.$$

$$A_{N,1}(r) = \frac{-G_N(r)}{2kF_N(r)}$$

$$A_{1,N}(r) = \frac{G_1(r)}{2kF_1(r)}$$

with all other elements zero and

$$b_n(r) = -\frac{\dot{\Phi}_n(r)}{F_n(r)}, \quad n = 1,\ldots,N.$$

5.6.5 also has a vector solution for $\underline{V}(r)$ in terms of a complementary function, $\exp\left\{\int_0^r A(r)dr\right\} \underline{V}_0$ provided $A(r)$ and $\int A(r)dr$ commute, and a particular solution.

Numerical solution of 5.6.4 may take place by any numerical method which includes the methods in section 4.

To solve 5.6.4 by Runge-Kutta methods we denote 5.6.4 in the form

$$V_n'(r) = H(r, V_{n-1}(r), V_n(r), V_{n+1}(r)) \qquad 5.6.6$$

The Fourth Order Runge-Kutta method applied to 5.6.6 becomes using the notation of 5.4.4, 5.4.5

$$r_{m+1} = r_m + h$$
$$V_{m+1}^n = V_m^n + \frac{h}{6}(k_1^n + 2k_2^n + 2k_3^n + k_4^n)$$

where $k_1^n = H(r_m, V_m^{n-1}, V_m^n, V_m^{n+1})$

$$k_2^n = H\left(r_m + \frac{h}{2}, V_m^{n-1} + \frac{hk_1^{n-1}}{2}, V_m^n + \frac{hk_1^n}{2}, V_m^{n+1} + \frac{hk_1^{n+1}}{2}\right) \qquad 5.6.7$$

$$k_3^n = H\left(r_m + \frac{h}{2}, V_m^{n-1} + \frac{hk_2^{n-1}}{2}, V_m^n + \frac{hk_2^n}{2}, V_m^{n+1} + \frac{hk_2^{n+1}}{2}\right)$$

$$k_4^n = H(r_m + h, V_m^{n-1} + hk_3^{n-1}, V_m^n + hk_3^n, V_m^{n+1} + hk_3^{n+1})$$

5.6.7 may be solved systematically:

a) Given r_m and V_m^n, $n = 1,\ldots,N$, we calculate k_1^n, $n = 1,\ldots,N$.

b) Next compute k_2^n, $n = 1,\ldots,N$ followed by k_3^n, $n = 1,\ldots,N$, and k_4^n, $n = 1,\ldots,N$.

c) From $k_1^n, k_2^n, k_3^n, k_4^n, V_m^n$, $n = 1,\ldots,N$, we compute V_{m+1}^n.

Initial conditions would seem to present a problem upon considering 5.6.4 at $r = 0$. However since we know that (0,0) is a local minimum of $V(x,y)$ then we see that

$$V_n'(0) = 0$$
$$V_n(0) = 0. \qquad 5.6.8$$

Substituting 5.6.8 we see that $H(0,0,0,0) = 0$. So now we may consider the result of putting $m = 0$ into 5.6.7:

$$r_1 = h$$
$$V_1^n = \frac{h}{6}(k_1^n + 2k_2^n + 2k_3^n + k_4^n)$$
$$k_1^n = H(0,0,0,0) = 0$$
$$k_2^n = H(h/2,0,0,0) = -\frac{\Phi_n(h/2)}{F_n(h/2)}.$$

Thus we see that 5.6.8 is sufficient initial conditions to solve 5.6.7 given the definition 5.6.6 and expression 5.6.4.

The major drawback to 5.6.7 is however the problem of instability approaching $V = 1$ or $V = \infty$. In sub-section 5.2 we encountered the problem that computation of V_{m+1}^n depended on V_m^{n-1}, V_m^n, V_m^{n+1} and that if any one of those had encountered instability, then V_{m+1}^n would be unstable whether inside the D.O.A. or not. The situation in 5.6.7 is even worse. Not only does V_{m+1}^n depend on V_m^{n-1}, V_m^n, V_m^{n+1} but on evaluation of H at other points as well. If any of k_i^{n-1}, k_i^n, k_i^{n+1}, $i = 1,2,3,4$ are large and unstable then so must V_{m+1}^n be.

7. Small Coefficients

The comparison of radial methods in section 5 was carried out on the basis of the classical numerical analysis results for stability, convergence and accuracy. There is, however, one problem which occurs when we solve a P.D.E. 5.4.2 with variable coefficients. This is the problem encountered in sub-section 2.3 which occurs when the denominators in the difference formula become small. It was noted in sub-section 2.3 that when the denominators are small that the corresponding numerators should be small also, but due to errors in computation that this does not happen.

The Shields' method 5.4.7 has problems when the coefficients of V_{m+1}^{n+1} or of V_{m+1}^n are zero or small. However providing that not both coefficients are zero there is not too much difficulty.

Let us suppose that

$$\left(F + \frac{hG}{k} - \frac{hd\Phi}{2}\right)_{m+\frac{1}{2}}^{n'+\frac{1}{2}} = 0. \qquad 5.7.1$$

Setting $n = n'$ in 5.4.7 gives a relationship explicitly for V_{m+1}^n in terms of V_m^{n+1} and V_m^n and known functions.

Letting $n = n' - 1$ in 5.4.7 then enables computation of V_{m+1}^{n-1} from V_{m+1}^{n}, V_{m}^{n-1}, V_{m}^{n} and known functions. 5.4.7 is then used to compute successively

$$V_{m+1}^{n'}, V_{m+1}^{n'-1}, \ldots, V_{m+1}^{1}, V_{m+1}^{N}, V_{m+1}^{N-1}, \ldots, V_{m+1}^{n'+1}$$

providing n' as defined by 5.7.1 is unique and

$$\left(F - \frac{hG}{k} - \frac{hd\Phi}{2}\right)_{m+\frac{1}{2}}^{n+\frac{1}{2}} \neq 0 \text{ for any } n = 1,\ldots,N.$$

Likewise if there exists an n' for which

$$\left(F - \frac{hG}{k} - \frac{hd\Phi}{2}\right)_{m+\frac{1}{2}}^{n'+\frac{1}{2}} = 0 \qquad 5.7.2$$

then 5.4.7 may be used to compute successively

$$V_{m+1}^{n'+1}, V_{m+1}^{n'+2}, \ldots, V_{m+1}^{N}, V_{m+1}^{1}, \ldots, V_{m+1}^{n'}.$$

A particular difficulty arises when 5.7.1 and 5.7.2 are satisfied for different values of n'.

Suppose that

$$\left(F + \frac{hG}{k} - \frac{hd\Phi}{2}\right)_{m+\frac{1}{2}}^{n_1'+\frac{1}{2}} = 0 \qquad 5.7.3$$

and

$$\left(F - \frac{hG}{k} - \frac{hd\Phi}{2}\right)_{m+\frac{1}{2}}^{n_2'+\frac{1}{2}} = 0 \qquad 5.7.4$$

where $n_2' < n_1' < N$.

Since 5.7.3 is true we may compute

$$V_{m+1}^{j}, \quad j = n_1', n_1' - 1, \ldots, n_2' + 2, n_2' + 1$$

and since 5.7.4 is true we may compute

$$V_{m+1}^{j}, \quad j = n_2' + 1, n_2' + 2, \ldots, n_1' - 1, n_1'$$

and the two sets of values will not agree exactly due to accumulated errors in computation. It is reasonable in this case to compute both sets and take their average values for V_{m+1}^{j}, $j = n_2' + 1, \ldots, n_1'$.

The Explicit method 5.4.10 is much simpler to analyse. Specifically the small coefficient problem occurs when the L.H.S. of 5.4.10 is small.

i.e. when

$$F - bhd\Phi = 0. \qquad 5.7.5$$

For the modified Zubov equation ($d = 0$) 5.7.5 becomes

$$F(r,\theta) = 0. \qquad 5.7.6$$

To consider the type of effect which 5.7.6 can cause let us consider the example

$$\dot{x} = -6x - 7y + y^2$$
$$\dot{y} = 4x + y + x^2 \qquad 5.7.7$$

Using 5.7.7 in 5.4.3 we see that

$$F(r,\theta) = \sin\theta\cos\theta(\sin\theta+\cos\theta)r^2+(\sin^2\theta-3\sin\theta\cos\theta-6\cos^2\theta)r$$
$$= 0. \qquad 5.7.8$$

Fig. 42 shows the graph of 5.7.8 in the (x,y) plane and fig. 43 shows r as a function of θ. Both roots of 5.7.8 are zero when

$$\sin^2\theta - 3\sin\theta\cos\theta - 6\cos^2\theta = 0 \qquad 5.7.9$$

i.e. $\theta \approx 77^\circ, 126^\circ, 257^\circ, 306^\circ$.

Hence given initial conditions for V_m^n, $n = 1,\dots,N$, then computation of V_{m+1}^n, $n = 1,\dots,N$, should show irregularities at values of n where $nk = \theta$ and θ satisfies 5.7.9.

Computation was attempted with initial conditions $V_{20}^n = (20h)^2$, $n = 1,\dots,100$, $h = 0{\cdot}0125$. Computation of V_{21}^n, $n = 1,\dots,100$, showed irregularities given by $V_{21}^{21} = 0{\cdot}14$, $V_{21}^{35} = 0{\cdot}40$, $V_{21}^{71} = 0{\cdot}09$, $V_{21}^{85} = -0{\cdot}12$ corresponding to $r = 0{\cdot}2625$, $\theta = 75{\cdot}6^\circ$, 126°, $255{\cdot}6^\circ$, 306° in accordance with 5.7.9.

Three methods were attempted to overcome the effects of 5.7.6.

a) To compute $\dfrac{F_{m+c}^{n-1} + F_{m+c}^{n+1}}{2}$ in the L.H.S. of 5.4.10 instead of F_{m+c}^n, $n = 1,\dots,N$, giving the method

$$\left(\frac{F^{-1}+F^{+1}}{2} - bhd\Phi\right)V_{m+1}^n = -\frac{hG}{2k}\left((1-a)V_m^{n+1}-(1+a)V_m^{n-1}\right)$$
$$+\left(F-\frac{ahG}{k}+(1-b)hd\Phi\right)V_m^n - h\Phi. \qquad 5.7.10$$

b) As 5.7.10 except replacing F_{m+c}^n by $\dfrac{F_{m+c}^{n-1}+F_{m+c}^{n+1}}{2}$ on the R.H.S. of 5.7.10 as well to preserve accuracy.

c) Using 5.4.10 as usual providing

$$F_m^n\, F_{m+1}^n > 0 \qquad 5.7.11$$

Where 5.7.11 is violated we know that

$F_{m+c}^n = 0$ for some $0 \leqslant c \leqslant 1$

and we fix $c = 0$ if $|F_m^n| \geqslant |F_{m+1}^n|$

$c = 1$ if $|F_{m+1}^n| \geqslant |F_m^n|$

	V_{21}^{21}	V_{21}^{35}	V_{21}^{71}	V_{21}^{85}
Conventional	0·14	0·40	0·09	−0·12
Mod. a)	0·12	0·20	0·08	−0·29
Mod. b)	0·13	0·23	0·09	−0·38
Mod. c)	0·14	0·40	0·09	−0·12

Fig. 30

The results are summarised in fig. 30 and it is noted that none of the modifications shows any improvement. This is only to be expected as further investigation of fig. 43 shows. As $r \to 0$ the curves $F = 0$ approach the origin along lines of constant θ. Hence the function $\frac{F^{+1} + F^{-1}}{2}$ can also become small which rules out modifications a), b), and also if F^n_{m+c} is small for some $0 \leqslant c \leqslant 1$ then little improvement is obtained by varying c.

It can be observed however that the regular Zubov equation ($d = 1$) does not suffer from this problem. Providing $r > 0$ then b can be varied to avoid 5.7.5 being satisfied.

It has been shown that G/F becomes large as $r \to 0$ in 5.4.10. Here we have found another reason for choosing initial conditions away from the origin.

Investigation of 5.4.21 shows that the Second Order method suffers from this type of instability when the terms on the R.H.S. become large. From the definitions 5.4.14, 5.4.20 we see that zeroes of F are again where we get instability. However we not only require $F^n_m \neq 0$, $n = 1,\ldots,N$, but also we need F^n_{m-1}, F^n_{m+1}, F^{n-1}_m, F^{n+1}_m to be non-zero. This means that instability occurs where the curve $F(r,\theta) = 0$ passes between the points at which $F(r,\theta)$ is being evaluated at in 5.4.21 as drawn in fig. 31.

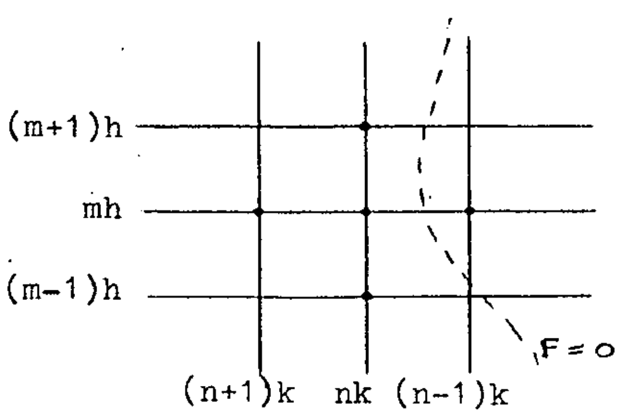


Fig. 31

8. Computation of $V = \infty$ Contour

Up to now methods have been developed in which values of V are computed on a grid system in r and θ. What we are really interested in is to compute the contour $V = \infty$. So we require some means of obtaining this contour which will be more accurate than just picking values from a grid. Three ideas for improving the method are given here.

8.1 Keeping Useful Data

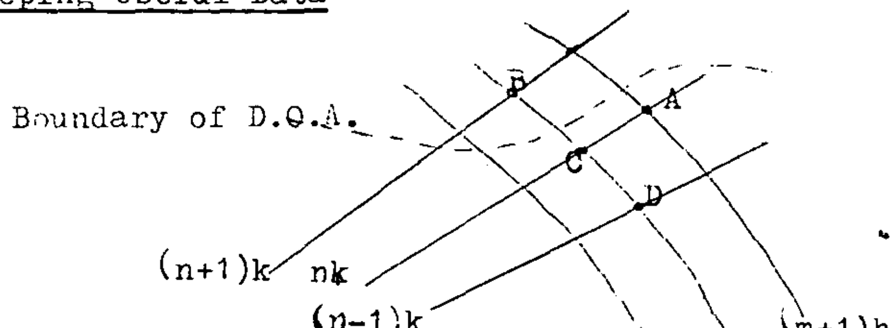


Fig. 32

The Explicit method and the Second Order method both compute the value of V at A from values of V at B,C,D. However, the point B is outside the boundary of the D.O.A. and the computed value will indicate that B is outside the region in which the results are continuous.

But if a discontinuity occured in computation of V_m^{n+1} then by either 5.4.10 or 5.4.21 there must also be a discontinuity in computation of V_{m+1}^n although A is inside the D.O.A..

The Hahn system 5.2.13, 5.2.14 shows up this problem quite well. The boundary of the D.O.A. of 5.2.13 is given by $xy = 1$. Solution of 5.2.13, 5.2.14 is attempted with $N = 100$, $h = 0{\cdot}0125$. The line $\theta = 3{\cdot}6^{\circ}$ corresponds to $n = 1$ and the boundary of the D.O.A. is satisfied by $\theta = 3{\cdot}6^{\circ}$, $r = 3{\cdot}99$ whilst when

$\theta = 7{\cdot}2^{\circ}$, $r = 2{\cdot}84$.

Hence computation of V_m^1 for $m = 226,\ldots,319$ involves using data of points outside the boundary of the D.O.A.. Fig. 44 shows a plot of $V_m^n - V(mh,nk)$ against r along lines of constant θ. The error on $\theta = 3{\cdot}6^{\circ}$ is seen to oscillate and in fact oscillates smoothly for $r \leq 3{\cdot}1$ due to the effect of computing A from B in fig. 32.

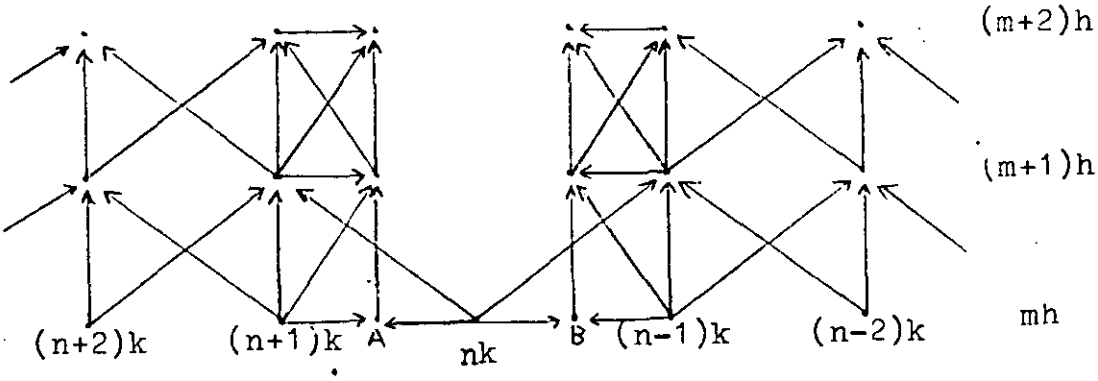


Fig. 33

The method to be used is shown in fig. 33. Computation of V_{m+1}^{n} has been found to be outside the region of continuity. So to compute V_{m+2}^{n-1} and V_{m+2}^{n+1} from correct data we need to find $V_{m+1}^{n-c_1}$ and $V_{m+1}^{n+c_2}$ which are within the region of continuity and $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$.

Step 1). Compute $V_{m}^{n+\frac{1}{2}} = \frac{1}{2}(V_{m}^{n} + V_{m}^{n+1})$

$$V_{m}^{n-\frac{1}{2}} = \frac{1}{2}(V_{m}^{n} + V_{m}^{n-1}) \qquad 5.8.1$$

These are A, B in fig. 33.

Step 2). Compute $V_{m+1}^{n-\frac{1}{2}}$ from $V_{m}^{n-\frac{1}{2}}$, V_{m}^{n-1}, V_{m+1}^{n-1} by Shields' method and $V_{m+1}^{n+\frac{1}{2}}$ from $V_{m}^{n+\frac{1}{2}}$, V_{m}^{n+1}, V_{m+1}^{n+1} similarly with θ step-size set to be negative.

After this V_{m+2}^{n+1} is computed from $V_{m+1}^{n+\frac{1}{2}}$, V_{m+1}^{n+1}, V_{m+1}^{n+2} and similarly for V_{m+2}^{n-1}, and then $V_{m+2}^{n+\frac{1}{2}}$, $V_{m+2}^{n-\frac{1}{2}}$ are computed by step 2) above.

By this method we compute values of V such that if V_{m}^{n} is inside the D.O.A. then so are $V_{m}^{n-c_1}$, $V_{m}^{n+c_2}$ where $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$. Then V_{m+1}^{n} can always be computed from 5.4.10 using step-size k' in θ given by

$$k' = \frac{(c_1 + c_2)k}{2}$$

and $V_{m+1}^{n-c_1}$ can be computed from 5.4.10 if $c_1 = 1$ and 5.4.7 if $c_1 < 1$, similarly for $V_{m+1}^{n+c_2}$.

The only remaining item is to decide when (mh,nk) is outside the continuous region. This is bound to be slightly subjective but from a computational point of view some suggestions are

a) $V_{m+1}^{n} < V_{m}^{n}$

b) $V_{m+1}^{n} < V_{m}^{n}$

or $V_{m+1}^{n} > p$

c) $V^n_{m+1} < 0$

or $V^n_{m+1} > p$

d) $V^n_{m+1} < 0,$

where p is large and positive when d = 0 and p is slightly less than 1·0 when d = 1.

8.2 Tangential Accuracy

The grid system 5.4.4 takes no account of the fact that grid points become more widely spaced out as r increases.

To get round this the grid could be made denser by doubling N at a suitable value of r. It is suggested that if $r > h/k$ then k is halved and the intermediate points are first obtained by

$$V^{n+\frac{1}{2}}_m = \frac{V^{n+1}_m + V^n_m}{2} \qquad 5.8.2$$

The computation of intermediate points by 5.8.1 or 5.8.2 is of second order accuracy and so does not reduce the accuracy of the method, but may introduce extra computer errors caused by more computation.

8.3 Radial Step-size Change

If we find that V^n_m is inside the region of continuity but V^n_{m+1} is not (n=1,...,N) it is reasonable to try to reduce the value of h. The suggested criterion is that when

$$2^s < \left(\frac{\partial V}{\partial r}\right)_{\theta = nk} < 2^{s+1}$$

the step-size h_n for this particular n should be given by

$$h_n = \frac{h}{2^{s+1}} \qquad s = 0, 1, \ldots$$

where h is the original step-size used to begin computation and n = 1,...,N. This method suffers from problems of storage of necessary data to compute V at all grid points.

9. Definition of Optimum Method

Much has been written in previous sections on the three methods given in section 4. Now we can put the relative merits of the methods and their modifications together and suggest the best scheme for computing D.O.A.s from a general 5.4.2.

Sections 5,7,8 have shown up several areas of comparison between the methods and problems that occur with their use. The optimum method is clearly going to be one in which the three methods are possibly all used with switching taking place upon satisfaction of certain criteria.

We have seen that Shields' method is most easily applicable using the initial conditions

$$V_0^n = 0, \quad n = 1,\dots,N. \qquad 5.9.1$$

Also we know that if $d = 0$ Shields' method is stable everywhere and is convergent. Therefore it is best to use Shields' method as much as possible. We could use it until a circle V_m^n, $n = 1,\dots,N$, is computed which is not continuous from the previous circle. However since it can be difficult to decide which circle has breached the region of continuity, this is not recommended. It is better to stop before there is any doubt and we recommend stopping when

$$V = 10 \qquad 5.9.2$$

is breached by a circle $r = mh$.

There is no theoretical basis for 5.9.2 but this has been found to be a good value of V which terminates computation not too close to the overflow line, but near enough to justify not using Shields' method. The best value of **V** to use depends on the scaling of F, G, Φ.

After this we must switch to an explicit method. The Second Order method has all the advantages of accuracy, while for convergence and computation there is nothing to choose. The problem of small coefficients given by $F(r,\theta) = 0$ affects both methods. The only possible disadvantage of the Second Order method may be stability. The Von-Neumann amplification factor must be checked on $r = mh$ and if it becomes large then we must revert to the Explicit method.

If $d = 1$ we may need to terminate Shields' method earlier than by 5.9.2 since it is only stable if $F \leq 0$. Shields' method is most difficult to deal with when coefficients become small, while the Explicit method has a facility for changing b to ensure that $F - bhd\bar{\Phi} \neq 0$.

Thus we can write down the steps of the best algorithm. All the values given are empirical and are suggested on the basis of experience. As already mentioned values of V depend on scalings of F, G, Φ.

a) Use $d = 1$. (Zubov's regular equation).
b) Use Shields' method with the initial conditions 5.9.1.
c) Terminate Shields' method at either

i) $V = 0{\cdot}99$,

or ii) $F + \frac{hG}{k} - \frac{hd\Phi}{2}$ or $F - \frac{hG}{k} - \frac{hd\Phi}{2}$ become "small" in the sense that they become less than $0{\cdot}1$ in magnitude of the other coefficients of 5.4.7,

iii) the stability region $F \leq 0$ is breached.

d) Then use the Second Order method providing either

i) λ_m^n does not become too "large" in the sense of being more than twice that of the Explicit method,

ii) F is not "small" in the sense of c)ii) such that coefficients of 5.4.21 exceed 10.

e) Otherwise use Explicit method with adjustment of b to ensure $F - bhd\Phi$ is not "small". Within the region $0 \leq b \leq 1$, $F - bhd\Phi$ should be as large as possible.
f) The method of keeping useful data in sub-section 8.1 should always be used to obtain the boundary accurately.
g) For convergence of the Second Order or Explicit methods a check must be made to ensure that

$$h \leq k\left|\frac{F}{G}\right|$$

reducing h if necessary.
h) The method of section 8.2 and 8.3 are not recommended as they involve extra computation, but are available if such accuracy of computation of the boundary is desired.

10. Higher Orders

Radial methods of solving Zubov's equation are particularly amenable to extension to higher order systems.

Consider the polar co-ordinate system 5.3.18, and the initial conditions given by 5.3.19 or 5.3.20. Zubov's equation then takes the form 5.3.22

$$F(r,\theta_1,..,\theta_{n-1})\frac{\partial V(r,\theta_1,..,\theta_{n-1})}{\partial r} + \sum_{i=1}^{n-1} G_i(r,\theta_1,..,\theta_{n-1})\frac{\partial V(r,\theta_1,..,\theta_{n-1})}{\partial\theta_i}$$
$$= -\Phi(r,\theta_1,..,\theta_{n-1})(1 - dV(r,\theta_1,..,\theta_{n-1})) \qquad 5.10.1$$

To illustrate how the methods are applicable to 5.10.1 we consider 4 dimensions for ease of notation. For Shields' method the corresponding 4-dimensional finite difference approximations become

$$\frac{\partial V}{\partial r} \approx \frac{1}{8h}\Big(V_{m+1}^{j+1,k+1,l+1} + V_{m+1}^{j+1,k+1,l} + V_{m+1}^{j+1,k,l+1} + V_{m+1}^{j+1,k,l}$$
$$+ V_{m+1}^{j,k+1,l+1} + V_{m+1}^{j,k+1,l} + V_{m+1}^{j,k,l+1} + V_{m+1}^{j,k,l}$$
$$- V_{m}^{j+1,k+1,l+1} - V_{m}^{j+1,k+1,l} - V_{m}^{j+1,k,l+1} - V_{m}^{j+1,k,l}$$
$$- V_{m}^{j,k+1,l+1} - V_{m}^{j,k+1,l} - V_{m}^{j,k,l+1} - V_{m}^{j,k,l}\Big) \qquad 5.10.2$$

and similarly for $\frac{\partial V}{\partial \theta_1}, \frac{\partial V}{\partial \theta_2}, \frac{\partial V}{\partial \theta_3}$, V.

Substitution of 5.10.2 and corresponding terms for $\frac{\partial V}{\partial \theta_1}, \frac{\partial V}{\partial \theta_2}, \frac{\partial V}{\partial \theta_3}$, V into 5.10.1 then gives an implicit formula for 8 points on the sphere $r = (m+1)h$ in terms of known formulae and 8 points on the circle $r = mh$.

There are $N_1N_2N_3$ such equations for fixed m and they may be solved by standard routines for solving linear equations to give $V_{m+1}^{j,k,l}$, $j = 1,\ldots,N_1, k = 1,\ldots,N_2, l = 1,\ldots,N_3$.

The other two methods are much simpler however, since they are explicit. The 4-dimensional version of 5.4.10 is obtained by writing down the 4-dimensional equations for 5.4.9

$$\frac{\partial V}{\partial r} \approx (V_{m+1}^{j,k,l} - V_m^{j,k,l})/h$$

$$\frac{\partial V}{\partial \theta_1} \approx ((1-a_1)V_m^{j+1,k,l} + 2a_1V_m^{j,k,l} - (1+a_1)V_m^{j-1,k,l})/2k_1$$

$$\frac{\partial V}{\partial \theta_2} \approx ((1-a_2)V_m^{j,k+1,l} + 2a_2V_m^{j,k,l} - (1+a_2)V_m^{j,k-1,l})/2k_2$$

$$\frac{\partial V}{\partial \theta_3} \approx ((1-a_3)V_m^{j,k,l+1} + 2a_3V_m^{j,k,l} - (1+a_3)V_m^{j,k,l-1})/2k_3$$

$$V \approx bV_{m+1}^{j,k,l} + (1-b)V_m^{j,k,l}$$

with known terms evaluated at

$$r = (m+c)h,\ \theta_1 = \frac{2\pi j}{N_1},\ \theta_2 = \frac{2\pi k}{N_2},\ \theta_3 = \frac{2\pi l}{N_3}$$

The Second Order method is relatively difficult to evaluate in 4 dimensions. 5.4.12 stays the same except for notation of V, while 5.4.13, 5.4.14, 5.4.15 have obvious generalisations to higher orders. The procedure carried out in 5.4.16 to obtain 5.4.17 is still quite straightforward but involves extra terms. The approximations 5.4.18 are replaced by n approximations of the same form, and the approximation 5.4.19 has to be considered by reference to 5.4.16 and 5.4.17. The last term of 5.4.17 becomes

$$\sum_{i_1=1}^{3}\sum_{i_2=1}^{3} \frac{h^2}{2} a^{j,k,l}_{i_1,m} \cdot a^{j,k,l}_{i_2,m} \frac{\partial^2 V^{j,k,l}_m}{\partial\theta_{i_1}\partial\theta_{i_2}} .$$

Hence 5.4.19 becomes

$$\frac{\partial^2 V^{j,k,l}_m}{\partial\theta_1\partial\theta_2} \approx \frac{V^{j+1,k+1,l}_m + V^{j,k,l}_m - V^{j+1,k,l}_m - V^{j,k+1,l}_m}{4k_1k_2}$$

and similarly for $\dfrac{\partial^2 V^{j,k,l}_m}{\partial\theta_1\partial\theta_3}$, $\dfrac{\partial^2 V^{j,k,l}_m}{\partial\theta_2\partial\theta_3}$

and

$$\frac{\partial^2 V^{j,k,l}_m}{\partial\theta_1^2} \approx \frac{V^{j+1,k,l}_m - 2V^{j,k,l}_m + V^{j-1,k,l}_m}{k_1^2}$$

and similarly for $\dfrac{\partial^2 V^{j,k,l}_m}{\partial\theta_2^2}$, $\dfrac{\partial^2 V^{j,k,l}_m}{\partial\theta_3^2}$.

The other features of this method such as finding the $V = \infty$ contour as described in section 8 are also applicable to higher orders but the algebra is too involved to write down here.

As an example we consider a generalisation of the Hahn system into 3 dimensions given by

$$\begin{aligned} \dot{x} &= -x + 3x^2yz \\ \dot{y} &= -y \\ \dot{z} &= -z \end{aligned} \qquad 5.10.3$$

the D.O.A. of 5.10.3 is given by $xyz < 1$. The series construction of Zubov's equation yields the solution of

$$(-x+3x^2yz)\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} - z\frac{\partial V}{\partial z} = -2(x^2+y^2+z^2) \qquad 5.10.4$$

as being

$$V(x,y,z) = y^2 + z^2 + x^2\sum_{m=0}^{\infty}\left\{\prod_{i=1}^{m}\left(\frac{3i+3}{3i+2}\right)\right\}(xyz)^m$$

Now if we try

$$V(x,y,z) = y^2 + z^2 + x^2 s(u) \qquad 5.10.5$$

where $u = xyz$

in 5.10.4 we obtain the differential equation

$$\frac{ds}{du} + \frac{2(1-3u)s}{3u(1-u)} - \frac{2}{3u(1-u)} = 0 \qquad 5.10.6$$

The solution of 5.10.6 as far as can be obtained is

$$u^{2/3}(1-u)^{4/3}S = \frac{2}{3}\int\left(\frac{1-u}{u}\right)^{1/3} du.$$

Solution of 5.10.4 was attempted numerically in polar co-ordinates using the 3-dimensional version of the Explicit method which is

$$\begin{aligned}(F-bhd\Phi)V_{m+1}^{i,j} = & -\frac{hG_1}{2k_1}((1-a_1)V_m^{i+1,j} - (1+a_1)V_m^{i-1,j}) \\ & -\frac{hG_2}{2k_2}((1-a_2)V_m^{i,j+1} - (1+a_2)V_m^{i,j-1}) \\ & +(F-\frac{a_1hG_1}{k_1} - \frac{a_2hG_2}{k_2}+ (1-b)hd\Phi)V_m^{i,j} - h\Phi\end{aligned}$$

where known terms are evaluated at $r = (m+c)h$, $0 \leq c \leq 1$, $\theta_1 = ik_1$, $\theta_2 = jk_2$, $i = 1,\ldots,N_1$, $j = 1,\ldots,N_2$.

The method of keeping useful data was used, and this remains 2-dimensional even in higher orders. For example, if $V_{m+1}^{i,j}$ is accepted as indicating a point inside the D.O.A. but $V_{m+1}^{i,j+1}$ is not, then we compute such terms as $V_{m+1}^{i,j+\lambda}$ using the Shields' method

$$(F+\frac{hG_2}{\lambda k_2}-\frac{hd\Phi}{2})V_{m+1}^{i,j+\lambda} +(F-\frac{hG_2}{\lambda k_2}-\frac{hd\Phi}{2})V_{m+1}^{i,j}$$

$$=(F-\frac{hG_2}{\lambda k_2}+\frac{hd\Phi}{2})V_m^{i,j+\lambda} +(F+\frac{hG_2}{\lambda k_2}+\frac{hd\Phi}{2})V_m^{i,j} - 2h\Phi$$

$-1 \leq \lambda \leq 1$, and similarly for $V_m^{i+\lambda,j}$.

The results are shown in fig. 45 in the form of a (θ_1,θ_2) plane with $V = 2{\cdot}0$ contours superimposed for different values of r. Although computation was performed for $0 \leq \theta_1, \theta_2 \leq 2\pi$ there is a lot of symmetry as can be seen from the polar co-ordinate transformations

$$\begin{aligned} x &= r\cos\theta_1 \\ y' &= r\sin\theta_1\cos\theta_2 \qquad 5.10.7 \\ z &= r\sin\theta_1\sin\theta_2 .\end{aligned}$$

Comparing 5.10.7 and the analytic solution 5.10.5 we see that

$$V(r,\theta_1,\theta_2) \equiv V(r,\theta_1,\pi+\theta_2)$$
$$V(r,\theta_1,\theta_2) \equiv V(r,2\pi-\theta_1,\theta_2)$$
which is why fig. 45 has been plotted for $0 \leq \theta_1,\ \theta_2 \leq \pi$.
We may further establish the relationships
$$V(r,\theta_1,\theta_2) \equiv V(r,\pi-\theta_1,\pi-\theta_2)$$
$$V(r,\theta_1,\theta_2) \equiv V(r,\theta_1,\pi/2-\theta_2)$$
which can be seen on fig. 45.

It has been shown that radial methods can be extended to higher orders where the principle is still that V on $r = (m+1)h$ is computed from V on $r = mh$.

11. Examples

Example 11.1

$$\dot{x} = -6x - 7y + y^2$$
$$\dot{y} = 4x + y + x^2$$
This system is actually the system
$$\dot{x} = -6x + y + y^2$$
$$\dot{y} = y + x^2$$
transformed to its stable critical point at (2,-4).
Using $\phi(x,y) = 2x^2 + 2y^2$, $d = 0$. Solution of the Zubov equation by series method yields the quadratic part of $V(x,y)$ as
$$V_2(x,y) = (39x^2 + 62xy + 107y^2)/110.$$
The Explicit method was used to solve the Zubov equation
$$(-6x - 7y + y^2)\frac{\partial V}{\partial x} + (4x + y + x^2)\frac{\partial V}{\partial y} = -2x^2 - 2y^2 \qquad 5.11.1$$
using the step-sizes $h = 0{\cdot}0125$, $k = \pi/50$, system parameters $a = 0$, $b = \frac{1}{2}$, $c = \frac{1}{2}$ and initial conditions
$$V_m^n = (mh)^2 \quad \text{for } m = 0 \text{ and } m = 20.$$
It has already been shown in section 7 that computation of 5.11.1 by the Explicit method breaks down on the curves 5.7.8 which are shown graphically on figs. 42, 43.

Solution of 5.11.1 was also attempted using the Second Order method, using the same step-size and initial conditions. With the consideration of where instability will occur shown in fig. 31 we would expect the same points on $r = 21h$ to be affected as in fig. 30, but also neighbouring points. The values obtained are

$V_{21}^{20} = 0{\cdot}33,\quad V_{21}^{21} = -0{\cdot}62,\quad V_{21}^{22} = 0{\cdot}58,\quad V_{21}^{23} = 0{\cdot}01$

$V_{21}^{34} = -0{\cdot}59,\quad V_{21}^{35} = 1{\cdot}37,\quad V_{21}^{36} = -0{\cdot}30$

$V_{21}^{70} = 0{\cdot}13,\quad V_{21}^{71} = -0{\cdot}30,\quad V_{21}^{72} = 0{\cdot}47,\quad V_{21}^{73} = -0{\cdot}06$

$V_{21}^{84} = 0{\cdot}33,\quad V_{21}^{85} = -0{\cdot}67,\quad V_{21}^{86} = 0{\cdot}35$

while $0{\cdot}04 \leq V_{21}^{n} \leq 0{\cdot}09$ for $n = 1,\ldots,100$ but not any of the above.

<u>Example 11.2</u>

$$\dot{x} = -x + y + x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2) \qquad 5.11.2$$

Using $\phi(x,y) = 2(x^2 + y^2)$, $d = 1$ 5.11.3

the analytic solution of the Zubov equation

$$(-x+y+x(x^2+y^2))\frac{\partial V}{\partial x} + (-x-y+y(x^2+y^2))\frac{\partial V}{\partial y} = -2(x^2+y^2)(1-V) \qquad 5.11.4$$

is given by

$$V(x,y) = x^2 + y^2 \qquad 5.11.5$$

which indicates that the boundary of the D.O.A. is given by

$$x^2 + y^2 = 1. \qquad 5.11.6$$

Solution of 5.11.4 was attempted by the Explicit method and the step-sizes

$$h = 0{\cdot}0125,\ 0{\cdot}01,\ 0{\cdot}005$$
$$k = \pi/50 \qquad 5.11.7$$

parameters $a = 0$, $b = \frac{1}{2}$, $c = \frac{1}{2}$ and the initial conditions

$$V_0^n = 0, \quad n = 1,\ldots,100. \qquad 5.11.8$$

The results from the grid points indicated that on $r = mh$, $V_m^n = (mh)^2$, $n = 1,\ldots,100$, $m = 1,2,\ldots$ to at least 4 decimal places. An alternative to 5.11.3 was then used as in Shields (28)

$$\phi(x,y) = 10x^2 + y^2. \qquad 5.11.9$$

Figs. 46,47 show the results of computing the contours using the Explicit method and the method of keeping useful data in sub-section 8.1. The criterion for rejecting data was

$$V_{m+1}^n < V_m^n$$
$$\text{or} \quad V_{m+1}^n > p. \qquad 5.11.10$$

The θ step-size was as 5.11.7 and the initial conditions as 5.11.8. Various h,p were used in the computation. Fig. 46 shows contours given by $p = 0{\cdot}9$, $0{\cdot}95$ and $h = 0{\cdot}0125$. Other values of h were used with no significant difference in the results. 5.11.10 was then replaced by the criterion

$$V^n_{m+1} < V^n_m \qquad 5.11.11$$

which resulted in the three contours shown in fig. 46 for $h = 0{\cdot}0125,\ 0{\cdot}025,\ 0{\cdot}05$. The contours resemble 5.11.6 but by reference to 5.11.5 we see that criterion 5.11.11 is nowhere satisfied by the analytic function 5.11.5. The reason that we obtain a contour which resembles 5.11.6 lies in the old problem of instability caused by small coefficients.

Substitution of F, b, h, d, Φ for this example into 5.7.5 gives

$$-r + r^3 - \frac{hr^2}{2}(9\cos^2\theta + 1) = 0 \qquad 5.11.12$$

when 5.11.9 is used and

$$-r + r^3 - hr^2 = 0 \qquad 5.11.13$$

where 5.11.3 is used. Solving 5.11.12 and 5.11.13 explicitly for r gives respectively

$$r = \frac{h(1+9\cos^2\theta)}{4} \pm \left(1 + \frac{h^2(1+9\cos^2\theta)^2}{16}\right)^{\frac{1}{2}} \qquad 5.11.14$$

$$\text{and } r = \frac{h}{2} \pm \left(1 + \frac{h^2}{4}\right)^{\frac{1}{2}}.$$

Substituting $h = 0{\cdot}0125$ into 5·11·14 and letting $0 \leq \theta \leq 2\pi$ we see that

$$1{\cdot}0031 \leq r \leq 1{\cdot}0317. \qquad 5.11.15$$

Hence in the band given by 5.11.15 we expect computation to break down and criterion 5.11.11 satisfied.

In fig. 47 we see results of using criterion 5.11.10 with $h = 0{\cdot}0125$, $p = 0{\cdot}9,\ 0{\cdot}95,\ 0{\cdot}98,\ 0{\cdot}99,\ 0{\cdot}995,\ 1{\cdot}0$. Fig. 48 shows the same contours of r varying with θ. The areas where the Von-Neumann amplification factor given by 5.5.17 is less than one in magnitude were computed and shown in figs. 47,48. The areas are given by 5.5.22 and for this example becomes

$$(-1 + r^2)r^3(9\cos^2\theta + 1) \leq \frac{-125}{8\pi^2}$$

which is all space except for the enclosed areas marked on figs. 47,48. It can be seen that the contour $V = 1{\cdot}0$ on fig. 48 is more accurate for those θ where the line $\theta = nk$ passes through the area of computational stability.

The Courant-Friedrichs-Lewy condition for convergence of computation given by 5.5.12 becomes for this example

$$r - r^3 \geqslant \frac{h}{k}. \qquad 5.11.16$$

For $h = 0{\cdot}0125$, $k = \pi/50$, 5.11.16 gives $0{\cdot}2079 \leqslant r \leqslant 0{\cdot}8797$. An attempt was made to compare the contours $V = 1{\cdot}0$ obtained using criterion 5.11.10 with $p = 1$ and letting $h,k \to 0$ such that h/k is constant.

Fig. 49 shows the result for $h/k = 5/8\pi$. The results show that the contours do not converge to 5.11.6 as $h,k \to 0$. Convergence might be expected if the initial conditions were given on $r > 0{\cdot}2079$ as $V_m^n = r^2H(\theta)$ where $mh > 0{\cdot}2079$ and the quadratic part of $V(x,y)$ is given by $V_2(r\cos\theta, r\sin\theta) = r^2H(\theta)$ Fig. 50 is a plot of r against θ for the contour $V = 1$ using criterion 5.11.10 and the initial condition

$V_m^n = (0{\cdot}25)^2(3.875\cos^2\theta + 2.25\sin\theta\cos\theta + 1.625\sin^2\theta)$

where $mh = 0{\cdot}25$, $\theta = nk$, with h/k kept constant as before. As seen from 5.11.13 convergence does not occur for $r > 0{\cdot}8797$.

Fig. 51 shows a plot using 5.11.8 as the initial conditions fixing $k = \pi/50$ and letting $h \to 0$, which has the effect of enlarging the region of r for which 5.11.16 holds. Convergence does seem to take place in this case.

Fig. 52 is produced by a simple variation of the optimum method using Shields' method to obtain whole circles until a circle touched the contour $V = 0{\cdot}98$ after which the Explicit method was used.

<u>Example 11.3</u>

$$\begin{aligned}\dot{x} &= 6y - 2y^2 \\ \dot{y} &= -10x - y + 4x^2 + 2xy + 4y^2\end{aligned} \qquad 5.11.17$$

System 5.11.17 has a D.O.A. given by

$$(x - \tfrac{1}{2})^2 + y^2 < 1.$$

Solution of the Zubov equation was attempted using

$$\phi(x,y) = 2(x^2 + y^2).$$

The problem of instability when the L.H.S. of 5.4.10 is small occurs when 5.7.6 is satisfied.

For 5.11.17 we have

$f(r\cos\theta, r\sin\theta) = 6rs - 2r^2s^2$,

$g(r\cos\theta, r\sin\theta) = -10rc - rs + 4r^2c^2 + 2r^2sc + 4r^2s^2$,

where $s = \sin\theta$, $c = \cos\theta$.

Then from 5.4.3 we have

$$F(r,\theta) = -4rsc - rs^2 + 4r^2s = 0. \qquad 5.11.18$$

The solution of 5.11.18 is $r = 0$, $r = \cos\theta + \sin\theta/4$, $\sin\theta = 0$. Hence we would expect the instability in computation given the initial conditions $V_0^n = 0$, $n = 1,\ldots,100$, and step-sizes $h = 0{\cdot}0125$, $k = \pi/50$ to occur when

$$\theta = n\pi$$
$$\text{or} \quad \theta = n\pi - \tan^{-1}4. \qquad 5.11.19$$

Figs. 53,54 show the contours produced by the Explicit method for $V = 5{\cdot}0$, $10{\cdot}0$ respectively and various h. It appears that criterion 5.11.10 is satisfied for small r in the region of θ given by 5.11.19.

This example was also tried using $d = 1$ to consider variation of b to avoid 5.7.5. The system used was to consider the product $F^n_{m+c}(F^n_{m+c} - h\Phi^n_{m+c})$. Similarly to 5.7.11, a constant b was used if this product was positive but otherwise either $b = 0$ or $b = 1$ according to which resulted in a larger magnitude of $F^n_{m+c} - bh\Phi^n_{m+c}$. However since $P(F) = 1$, $P(\Phi) = 2$ and $h = 0{\cdot}0125$ the presence of the extra term $bhd\Phi$ in 5.7.5 made little difference near the origin.

Example 11.4

$$\dot{x} = y$$
$$\dot{y} = -x + \mu(x^2-1)y$$

a Van der Pol system.

Using $\phi(x,y) = x^2 + y^2$, $d = 0$ the solution of the Zubov equation

$$y\frac{\partial V}{\partial x} + (-x+\mu(x^2-1)y)\frac{\partial V}{\partial y} = -(x^2 + y^2)$$

was attempted in various ways. The Explicit method was used but computational instability was discovered in a neighbourhood of the origin. The initial conditions used were $V_0^n = 0$, $n = 1,\ldots,100$, with step-sizes $h = 0{\cdot}0125$, $k = \pi/50$ and computation parameters $a = 0$, $b = c = \frac{1}{2}$ and criterion for discontinuity as 5.11.11. The resulting discontinuity line passed through the origin and was asymptotic to the x-axis. This is explained by looking at the coefficient of V^n_{m+1} in 5.4.10 which is zero when

$$\mu(r^2\cos^2\theta - 1)r\sin^2\theta = 0. \qquad 5.11.20$$

An attempt was also made by using Shields' method until a circle crossed the contour $V = 5{\cdot}0$ then reverting to the Explicit method. Fig. 55 shows the discontinuity line by criterion 5.11.11 and it also touches the circle where the changeover took place in the vicinity of $x = 1$ which is also a solution of 5.11.20. Fig.56 shows the contour $V = 2,3,5$ and the discontinuity line when solving throughout by Shields' method. The discontinuity line was determined by applying criterion 5.11.11 to the grid print-out.

Finally the regular Zubov equation ($d = 1$) was solved similarly by changing methods at $V = 0{\cdot}98$. Fig. 57 shows the discontinuity line. Variation of b to avoid the problem of small coefficients was attempted and fig. 57 shows an improvement in the vicinity of $x = 1$, $y = 1{\cdot}3$.

Example 11.5

$$\dot{x} = -x + y$$
$$\dot{y} = -x - y \qquad 5.11.21$$

Using $\phi(x,y) = x^2 + y^2$, $d = 0$ 5.11.22

the Zubov equation is satisfied by

$$V(x,y) = \frac{x^2 + y^2}{2}.$$

Thus we see that 5.11.21 is asymptotically stable in the whole. The radial Zubov equation 5.4.2 becomes for 5.11.21, 5.11.22

$$-r\frac{\partial V}{\partial r} - \frac{\partial V}{\partial \theta} = -r^2. \qquad 5.11.23$$

5.11.23 may be solved by the Explicit method using $a = 0$, $b = \frac{1}{2}$, $c = \frac{1}{2}$ and the general finite difference formula is given by

$$-rV^n_{m+1} = \frac{h}{2k}(V^{n+1}_m - V^{n-1}_m) - rV^n_m - hr^2 \qquad 5.11.24$$

where $r = (m+\frac{1}{2})h$. 5.11.24 is independent of θ and so we may say that

$$V^1_m = V^2_m = \ldots\ldots = V^N_m. \qquad 5.11.25$$

Substituting $r = (m+\frac{1}{2})h$ and 5.11.25 into 5.11.24 gives

$$-(m+\tfrac{1}{2})hV^n_{m+1} = -(m+\tfrac{1}{2})hV^n_m - (m+\tfrac{1}{2})^2h^3 \qquad 5.11.26$$

5.11.26 simplifies to $V^n_{m+1} = V^n_m + (m+\frac{1}{2})h^2$. 5.11.27

The difference equation 5.11.27 has a solution

$$V^n_m = \frac{m^2h^2}{2} \quad n = 1,\ldots,N. \qquad 5.11.28$$

It is interesting to compare the Von-Neumann stability regions for the Shields' and Explicit methods. We know from 5.5.19 that the Shields' method is stable everywhere if $d = 0$ and the results do indeed yield 5.11.28. The Explicit method with $a = 0$, $d = 0$ yields $|\lambda^n_m| \geqslant 1$ everywhere. The results of computation start diverging from the solution 5.11.28 at about $r = 0{\cdot}55$.

The reason for this divergence is that while 5.11.24 is a formula which is independent of θ, the computer program does not explicitly work out 5.11.24. For a given r,θ the program works out $F(r,\theta)$, $G(r,\theta)$ by the formulae 5.4.3. Substituting from 5.11.21 into 5.4.3 we obtain

$$F(r,\theta) \equiv (-rc + rs)c + (-rc - rs)s$$

$$G(r,\theta) \equiv ((-rc - rs)c - (-rc + rs)s)/r. \qquad 5.11.29$$

The computed values of $F(r,\theta)$, $G(r,\theta)$ are approximately $-r$ and -1 respectively but round off errors occur in computing 5.11.29. It is these errors which become magnified by an unstable method of computation.

Example 11.6

$$\dot{x} = -x + y + x^2$$

$$\dot{y} = -y + xy \qquad 5.11.30$$

If we try

$$V(x,y) = x^2 + y^2$$

as a Lyapunov function for 5.11.30 we obtain

$$\dot{V}(x,y) = -2x^2 + 2xy - 2y^2 + 2x^3 + 2xy^2.$$

Computation of the Zubov equation

$$(-x+y+x^2)\frac{\partial V}{\partial x} + (-y+xy)\frac{\partial V}{\partial y} = -2(x^2-xy+y^2) \qquad 5.11.31$$

was thus attempted using the Explicit method together with the method of keeping useful data. The parameters used were the usual ones and fig. 58 shows computed contours $V = 2,5,10$ and the curve $F = 0$ and the boundary of the D.O.A.. The problem here is instability. Substituting $a = 0$ into 5.5.21 shows that the method is nowhere stable and significantly the results become unstable in the quadrant nearest the boundary. i.e. where V increases most rapidly.

Fig. 59 shows the discontinuity line when 5.11.31 is solved using Shields' method changing over at $V = 2{\cdot}0$. Fig. 60 shows the results of solving the regular equation

$$(-x+y+x^2)\frac{\partial V}{\partial x} + (-y+xy)\frac{\partial V}{\partial y} = -2(x^2-xy+y^2)(1-V)$$

changing methods at $V = 0{\cdot}98$. The extra computational stability achieved over the modified equation 5.11.31 by considering when V reaches a finite limit can be observed.

Example 11.7

$$\dot{x} = -x + 2x^2y$$

$$\dot{y} = -y \qquad 5.11.32$$

This example due to Hahn (10) has been much used to illustrate certain points in this and other chapters. It has the advantage of possessing a well-known D.O.A. boundary given by $xy = 1$ which indicates that the D.O.A. is unbounded. The Shields' method of solution of the Zubov equation was accurate until about $r = 1 \cdot 3$.

The Explicit method was used to solve the Zubov equation

$$(-x+2x^2y)\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} = -2(x^2+y^2)(1-dV) \qquad 5.11.33$$

incorporating the method of keeping useful data to obtain conservative estimates of contours. The criterion for terminatin computation was

$$V^n_{m+1} < 0$$
$$V^n_{m+1} > p. \qquad 5.11.34$$

The step-sizes were $h = 0 \cdot 0125$, $k = \pi/50$ and system parameters given by $a = 0$, $b = \frac{1}{2}$, $c = \frac{1}{2}$ and the initial conditions

$$V^n_0 = 0, \quad n = 1,\ldots,100. \qquad 5.11.35$$

5.11.32 satisfies the conditions of theorem 3.4.1 and from 5.5.3 we see that as $r \to 0$, G/F tends to 0 also. So there is no problem with the initial conditions 5.11.35.

Fig. 61 shows the curves obtained for $p = 2$, $p = 3$. Apart from grid scatter (the grid points are a conservative estimate) the curves are accurate. Fig. 62 shows the curves obtained for $p = 5$, 10 and fig. 63 likewise for $p = 10, 25$. The curves obtained become ragged when criterion 5.11.34 is satisfied by the first part of 5.11.34. That is, computation has become inaccurate and unstable. As seen in section 5 by equation 5.5.21 when $d = 0$, $a = 0$ the Explicit method is nowhere stable.

Fig. 64 shows the attempt to solve 5.11.33 given $d = 1$, $p = 0 \cdot 95$. This is again accurate to the resolution of the grid points, but when $p = 0 \cdot 99$ as shown on fig. 65 the ragged effect can again be seen.

Figs. 66,67 are more promising. They show the attempt to compute 5.11.33 using $d = 0$, $p = 10, 25$ and the criterion

$$V^n_{m+1} < V^n_m$$
$$V^n_{m+1} > p$$

comparing them with the analytical curves given by

$$y^2 + \frac{x^2}{1-xy} = p.$$

The stability analysis for $d = 1$ is rather interesting. The

Von-Neumann amplification factor is less than unity when 5.5.22 is satisfied. In this case $b = \frac{1}{2}$ and the areas of computational stability are given by

$$\frac{-1+2r^2 \sin\theta \cos^3\theta}{2r \sin^4\theta \cos^4\theta} \leqslant \frac{-h}{2k^2}$$

i.e. $$2r^2 \sin\theta \cos^3\theta + \frac{hr\sin^4\theta \cos^4\theta}{k^2} - 1 \leq 0 \qquad 5.11.36$$

For 5.11.36 certain deductions may be made

a) If $\cos\theta = 0$ or $\sin\theta = 0$ then 5.11.36 is true.

b) If $\sin\theta \cos\theta > 0$ then 5.11.36 becomes

$$r \leq \frac{-h \sin^3\theta \cos\theta}{4k^2} \pm \left(\frac{h^2 \sin^7\theta \cos^5\theta/k^4 + 8}{16 \sin\theta \cos^3\theta}\right)^{\frac{1}{2}}.$$

When $\sin\theta = \cos\theta = 1/\sqrt{2}$, $h = 0{\cdot}0125$, $k = \pi/50$ we have $r \leq 1{\cdot}2301$.

c) If $\sin\theta \cos\theta < 0$ and if $h \geqslant 21{\cdot}6997k^2$ then there are certain values of $\cos\theta$, $\sin\theta$ such that 5.11.36 is not satisfied for r in the neighbourhood of

$$r = \frac{-h\sin^3\theta \cos\theta}{4k^2}.$$

With 5.11.32 an attempt was made to see if there was any difference in computation by varying ϕ rather than p in 5.11.33, 5.11.34, 5.11.33 was solved with ϕ replaced by

$$\phi(u,x,y) = u(x^2 + y^2) \qquad 5.11.37$$

and computation of $V = p$ with $u = 2$ corresponds exactly to computation of $V = 2$ with $u = 4/p$. This can be proved by defining $V(u,x,y)$ as the solution of the Zubov equation using 5.11.37

$$V(u,x,y) = -\int_0^t \phi(u,x,y)dt' + V(u,x_0,y_0). \qquad 5.11.38$$

Substituting 5.11.37 into 5.11.38 gives

$$V(u,x,y) = V(u,x_0,y_0) - u\int_0^t (x(t')^2 + y(t')^2)dt'. \qquad 5.11.39$$

$x(t)$, $y(t)$ are independent of u. Hence putting $u = 2$ in 5.11.39 gives

$$V(2,x,y) = V(2,x_0,y_0) - 2\int_0^t (x(t')^2 + y(t')^2)dt'. \qquad 5.11.40$$

Eliminating the integral from 5.11.39, 5.11.40 gives

$$V(u,x,y) = V(u,x_0,y_0) - u/2(V(2,x_0,y_0) - V(2,x,y)). \qquad 5.11.41$$

5.11.41 is true for all x_0, y_0. Hence letting $x_0 = 0$, $y_0 = 0$ and observing $V(u,0,0) = 0$ we obtain

$$V(u,x,y) = \frac{u}{2} V(2,x,y). \qquad 5.11.42$$

Hence we see that $V(2,x,y) = p$, $V(u,x,y) = 2$, are equivalent if $u = 4/p$.

By reference to 5.4.10 we see that each term in computation is linear in V or Φ and that V_m^n has the same properties of $V(u,x,y)$.

Figs, 68, 69, 70 show computation of the Zubov equation with $d = 0$, $p = 2$ with $u = 10,5,3,2,4/3,4/5,2/5,4/25$, and are exactly the same as obtained by $u = 2$, $p = 2/5,4/5,4/3,2,3,5,10,2$

5.11.33 was also computed by the Second Order method. Figs. 71,72,73 show a comparison between the Explicit method and the Second Order method by plotting $V(mh,nk) - V_m^n$ against θ for $m = 96$, 104, 120 respectively. The improvement in accuracy is clearly seen. Fig. 74 shows the contours obtained by the Second Order method.

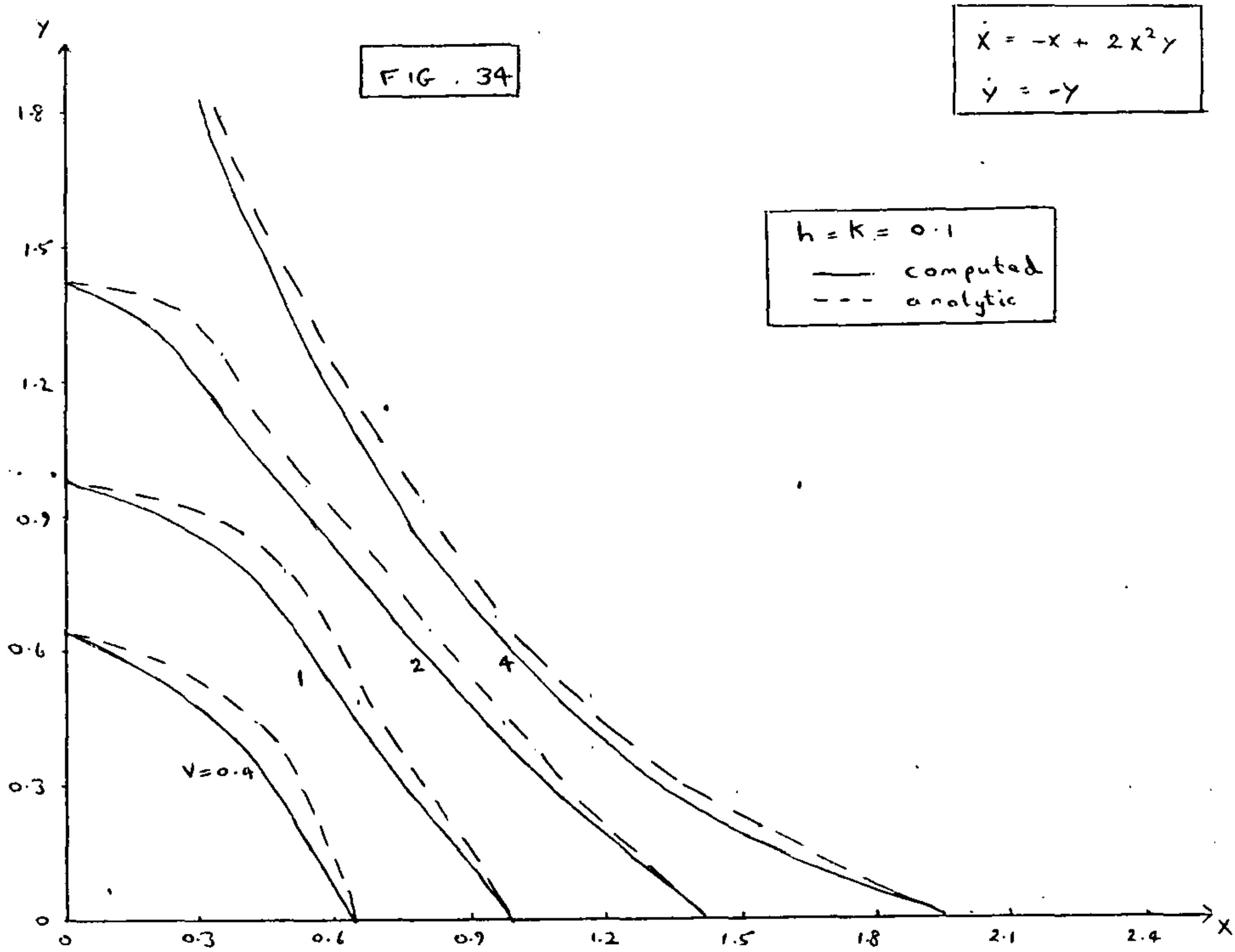
The radial Runge-Kutta methods considered in section 6 were also tried. The boundary of continuous results (fig.75) is inferior to that obtained by finite differences and bears out the points made in section 6 about the stability of computation.

12. Conclusions

In this chapter an investigation has been made into solving Zubov's equation by treating 5.1.1 as a P.D.E. and approximating the partial derivatives. Various problems have been encountered along the way and each in turn has been overcome, culminating in the definition of the optimum way of combining the three methods considered. The biggest problem by a long way is that caused by computing values of V near curves on which certain crucial coefficients are zero, or certain terms become infinite. The regular Zubov method ($d = 1$ in 5.1.1) must be used to avoid this problem.

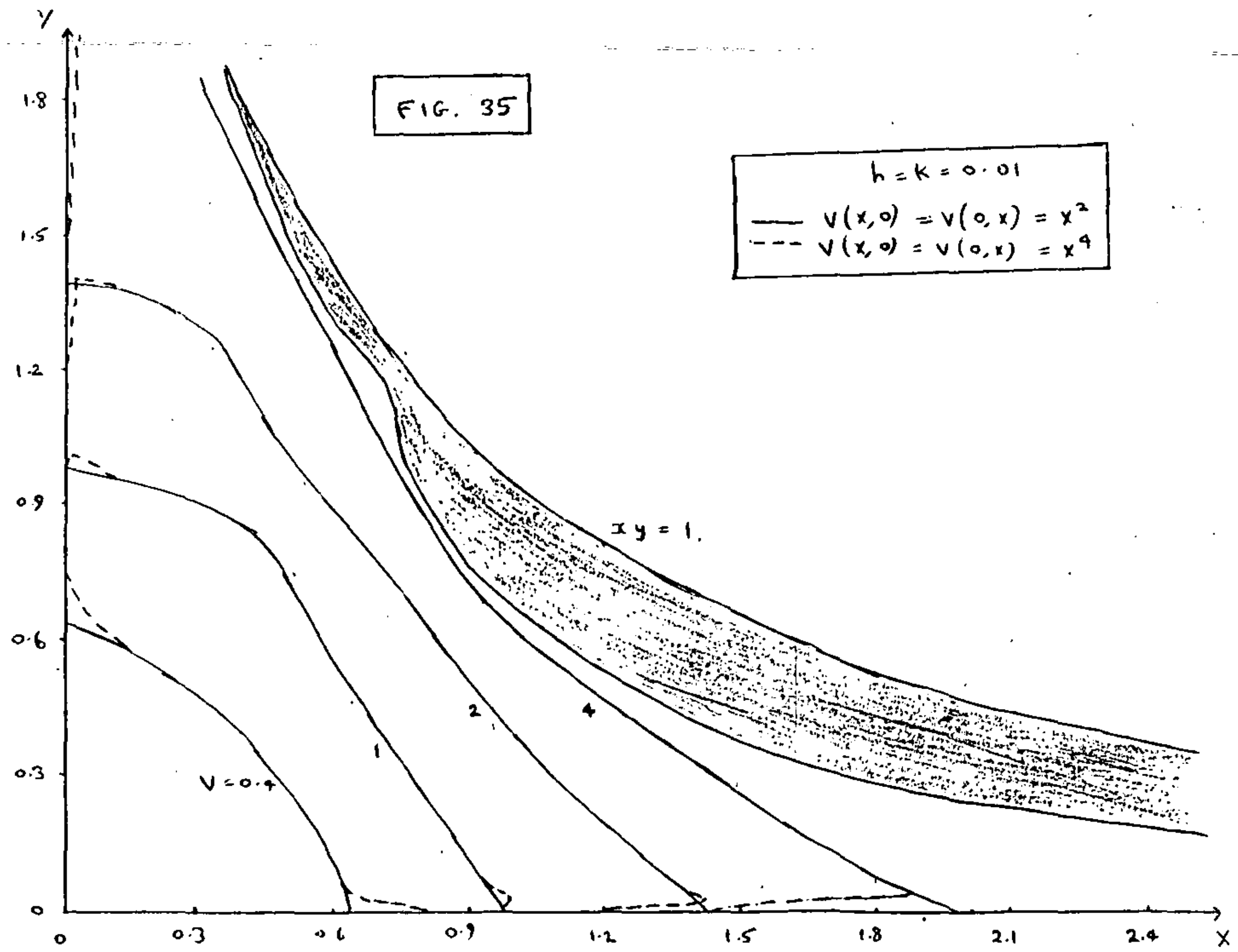
Instability, inaccuracy and lack of convergence all need to be taken care of in this type of method, and we have found that computation can easily fall short of the true boundary of the D.O.A. for any one of these reasons. This method is much simpler than that of Chapter 6 although it is not as good at finding the boundary of the D.O.A..

FIG. 34

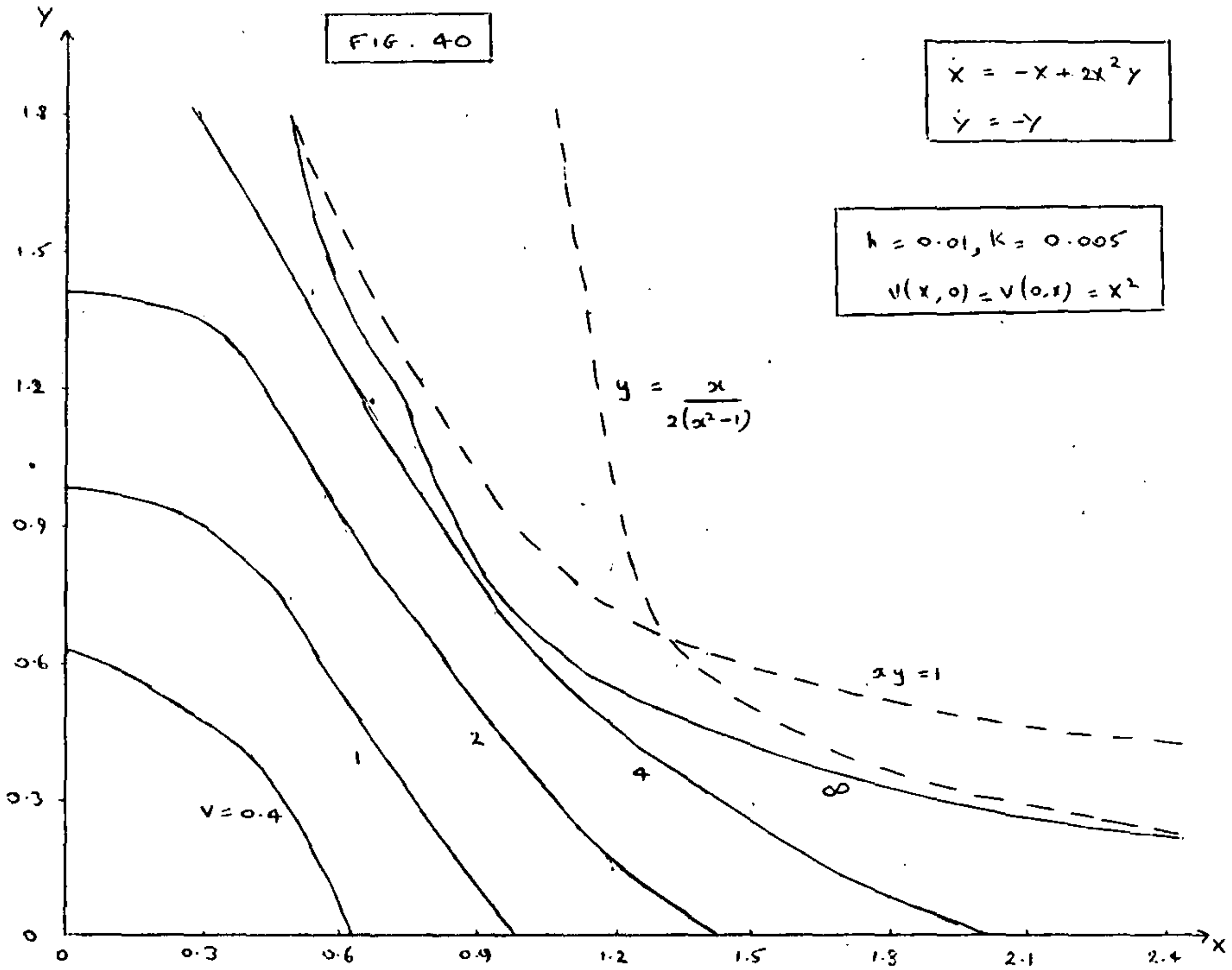


$\dot{x} = -x + 2x^2 y$
$\dot{y} = -y$
h = k = 0.1
computed
analytic
V = 0.4
1
2
4
Y
X
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4

FIG. 35

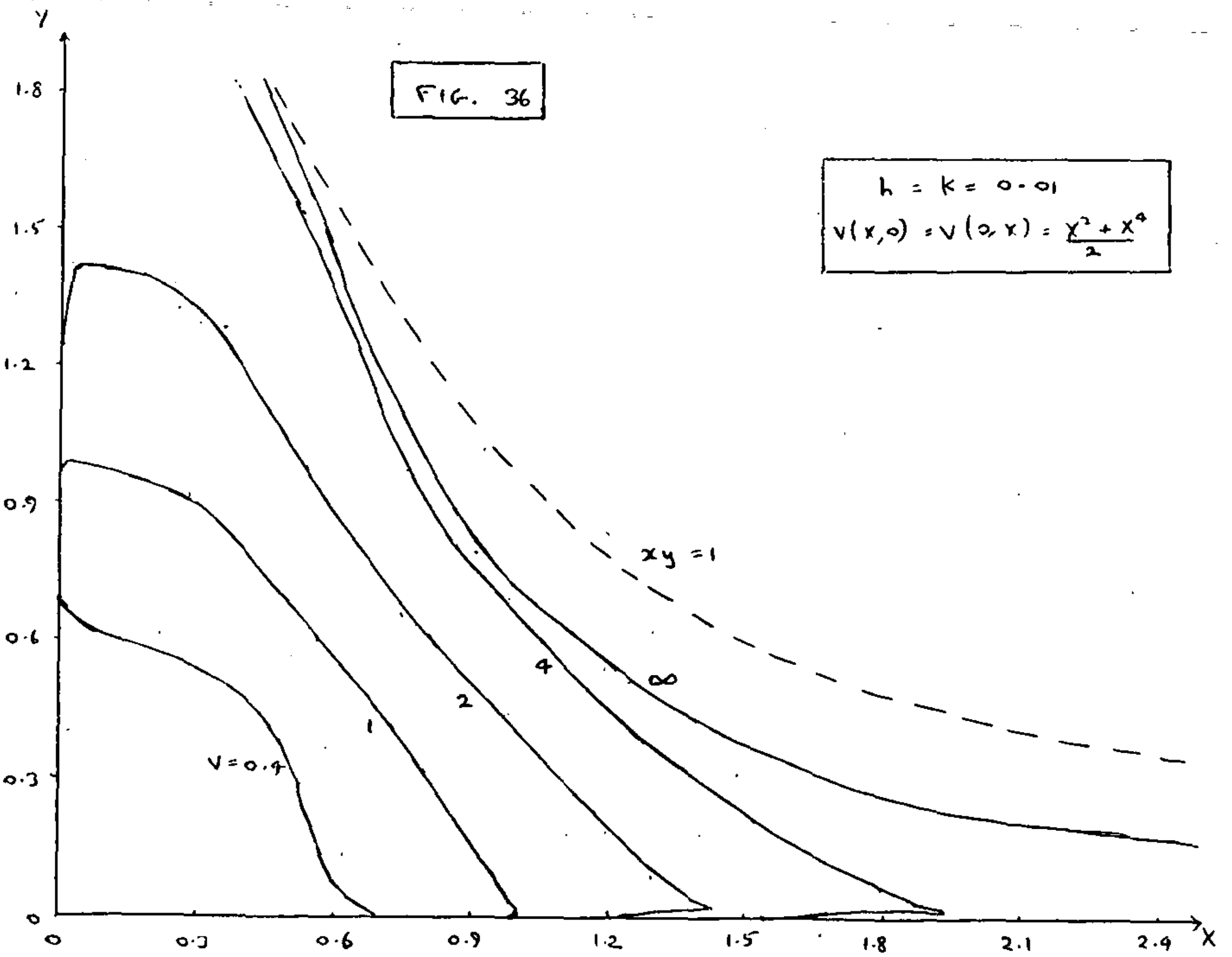


h = k = 0.01
$V(x,0) = V(0,x) = x^2$
$V(x,0) = V(0,x) = x^4$
xy = 1.
V = 0.4
1
2
4
Y
X
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4



$\dot{x} = -x + 2x^2 y$
$\dot{y} = -y$
$h = 0.01, k = 0.005$
$v(x,0) = v(0,x) = x^2$
$y = \frac{x}{2(x^2-1)}$
$xy = 1$
$V = 0.4$
1
2
4
∞
Y
X
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4

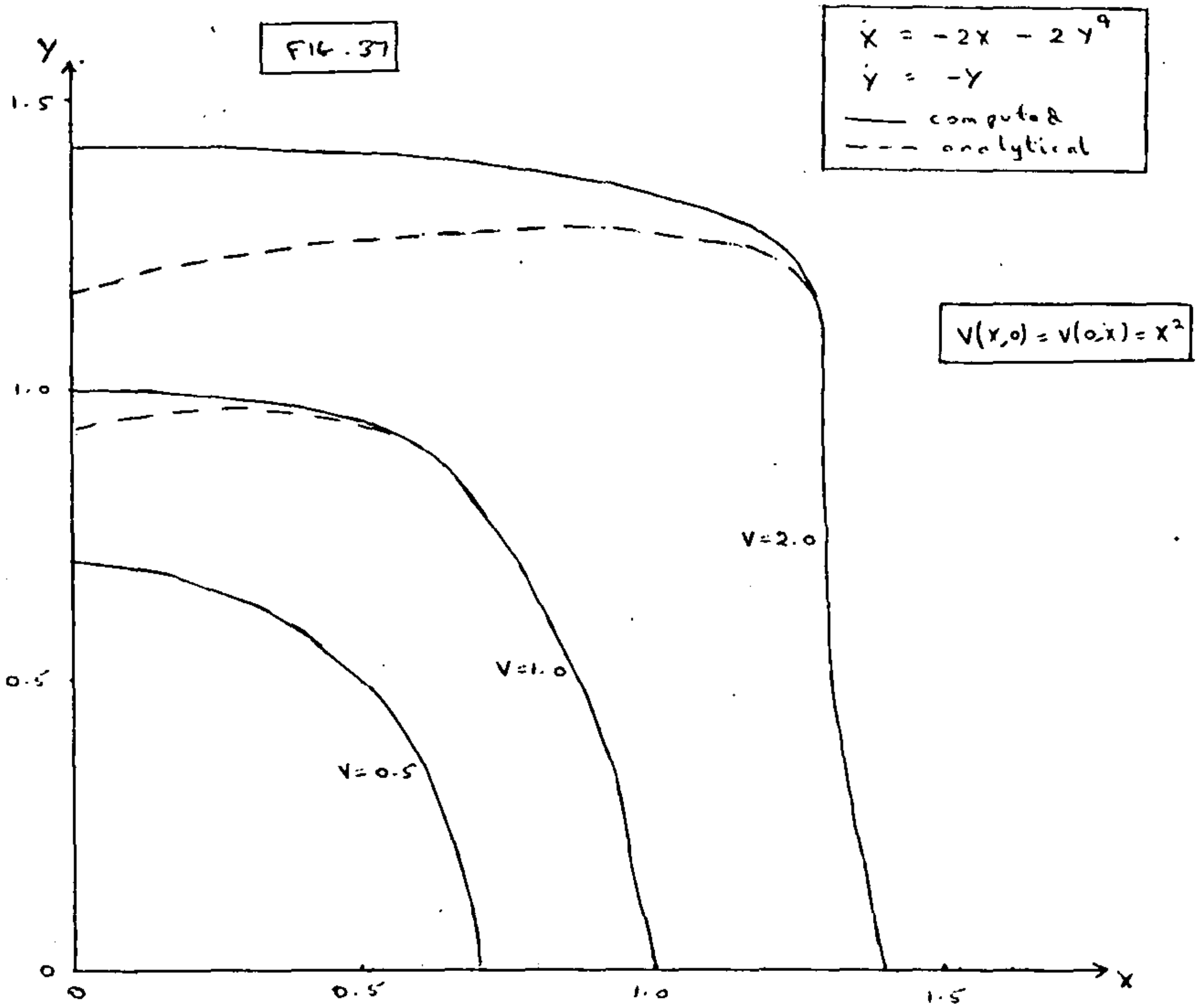
FIG. 40



$h = k = 0.01$
$v(x,0) = v(0,x) = \frac{x^2 + x^4}{2}$
$xy = 1$
$V = 0.4$
1
2
4
∞
Y
X
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4

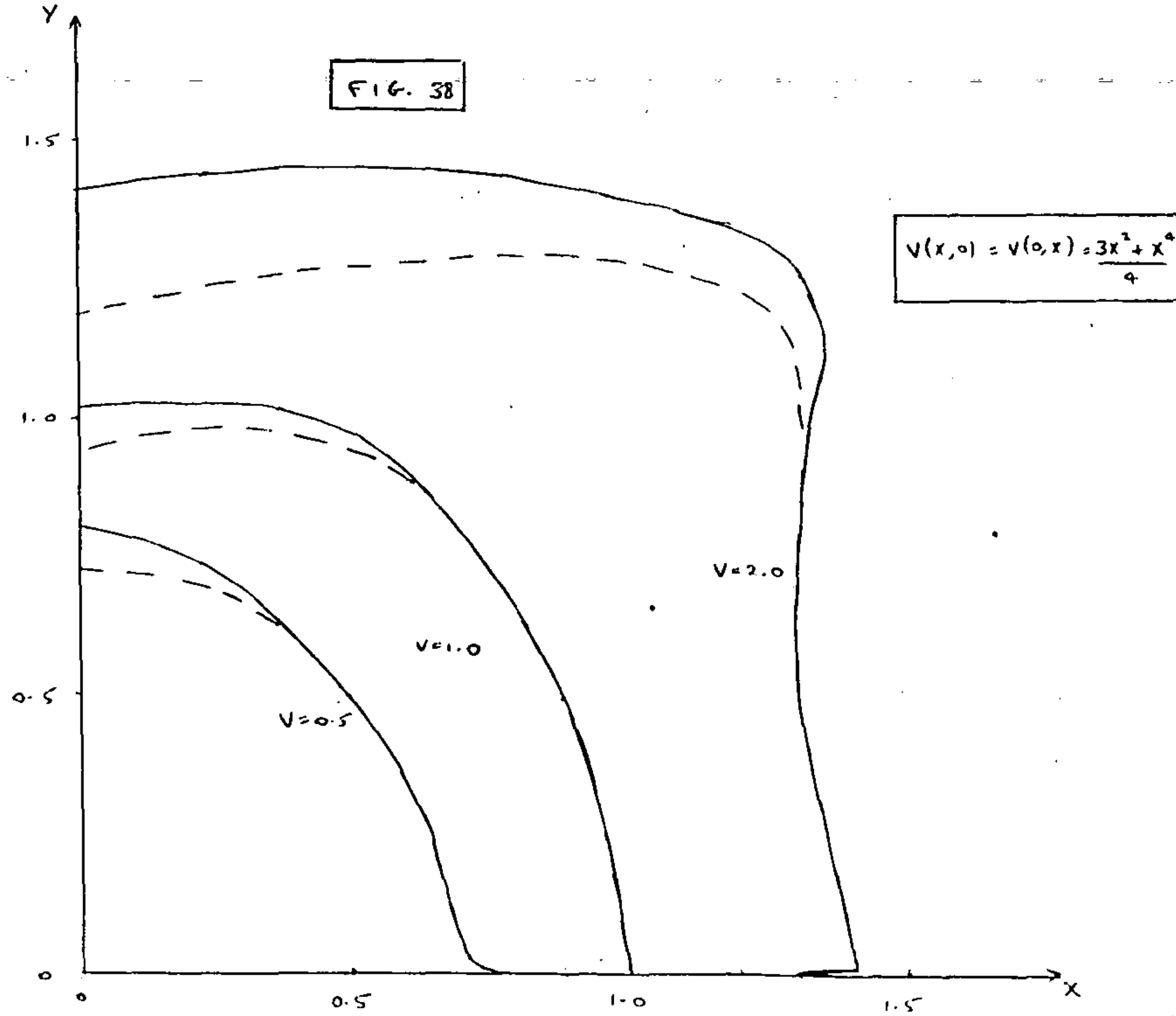
FIG. 36

FIG. 37



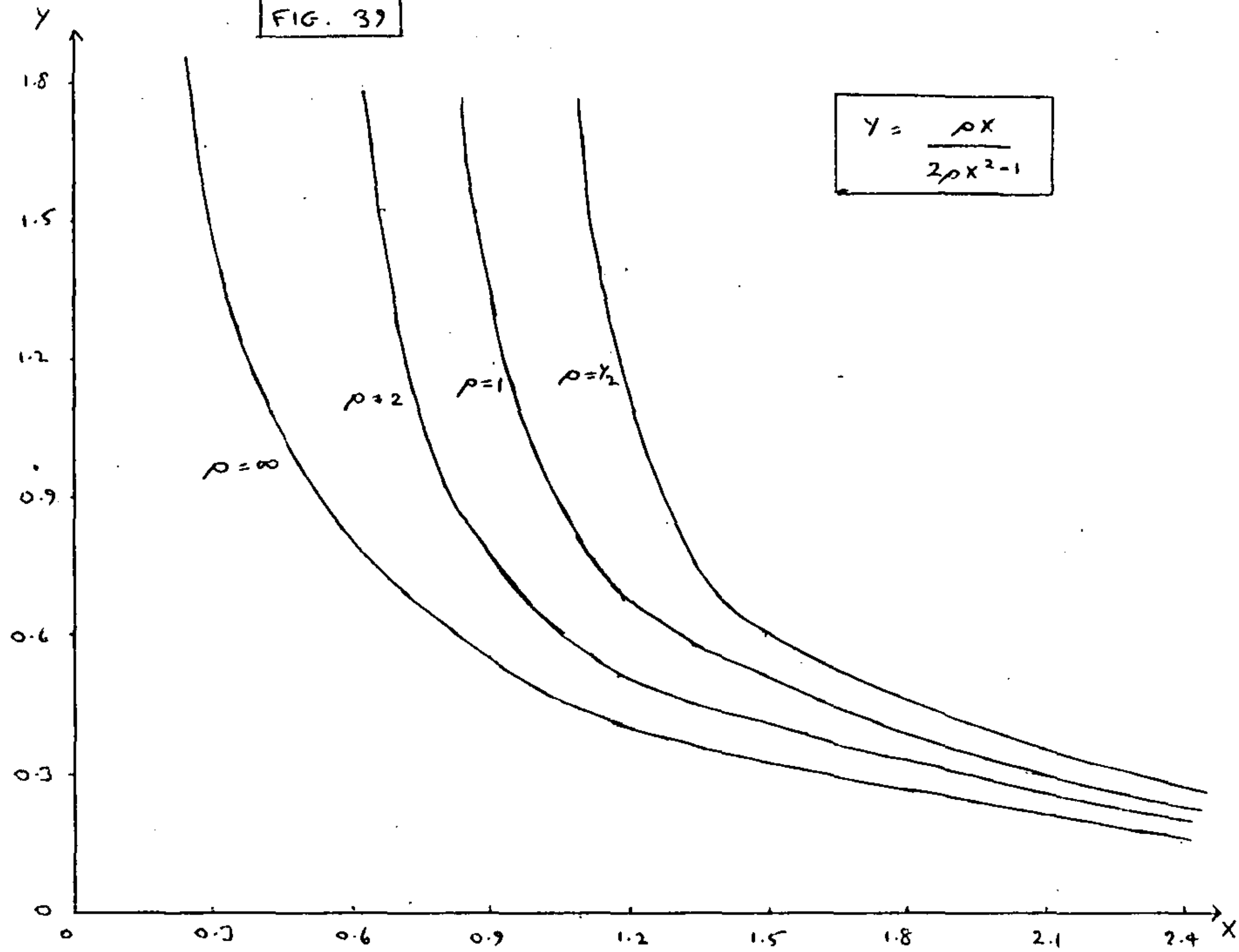
$\dot{X} = -2X - 2Y^9$
$\dot{Y} = -Y$
computed
analytical
$V(X,0) = V(0,X) = X^2$
V=2.0
V=1.0
V=0.5
Y
X
0
0.5
1.0
1.5

FIG. 38



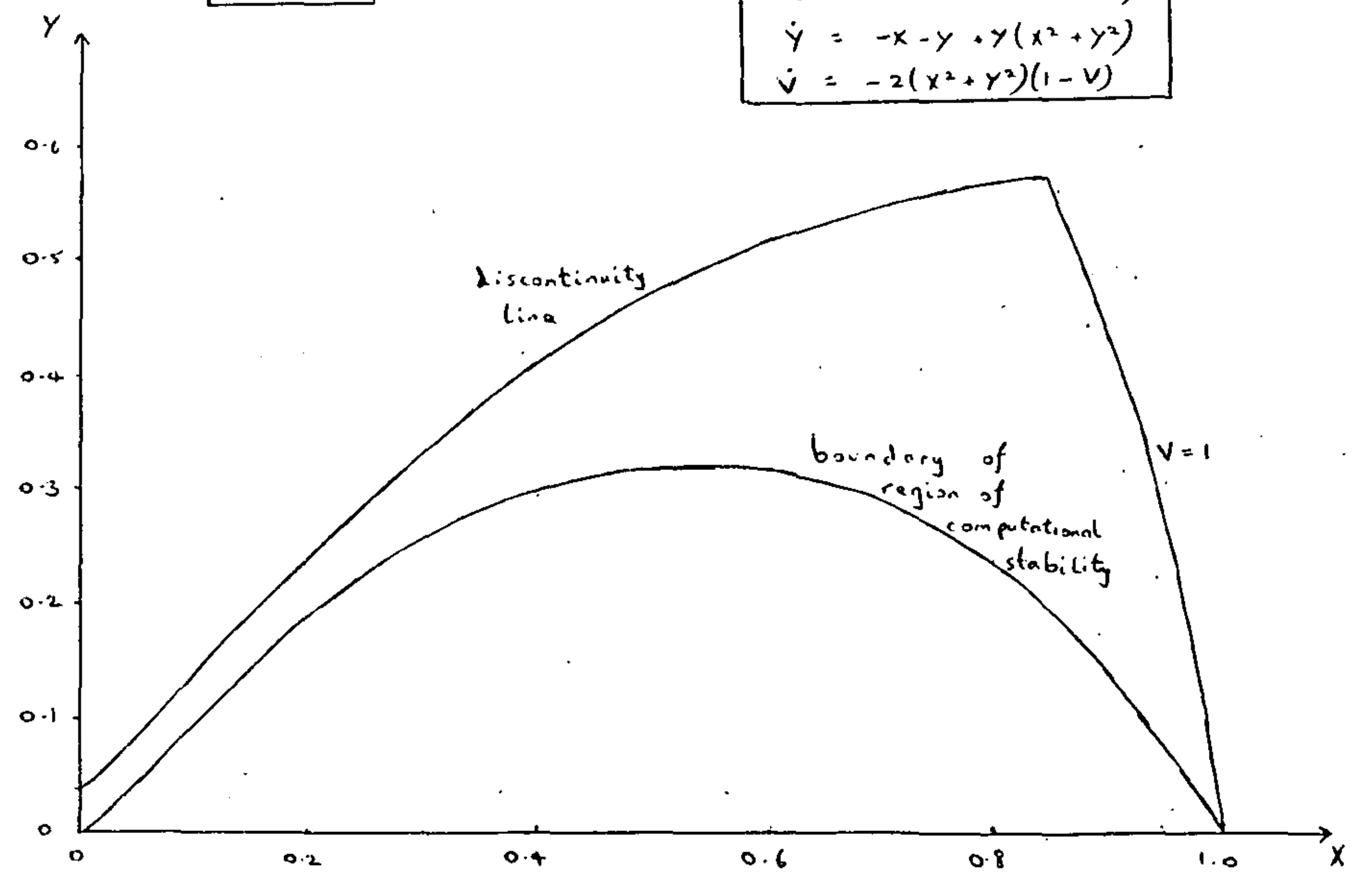
$V(X,0) = V(0,X) = \frac{3X^2 + X^4}{4}$
V=2.0
V=1.0
V=0.5
Y
X
0
0.5
1.0
1.5

FIG. 39



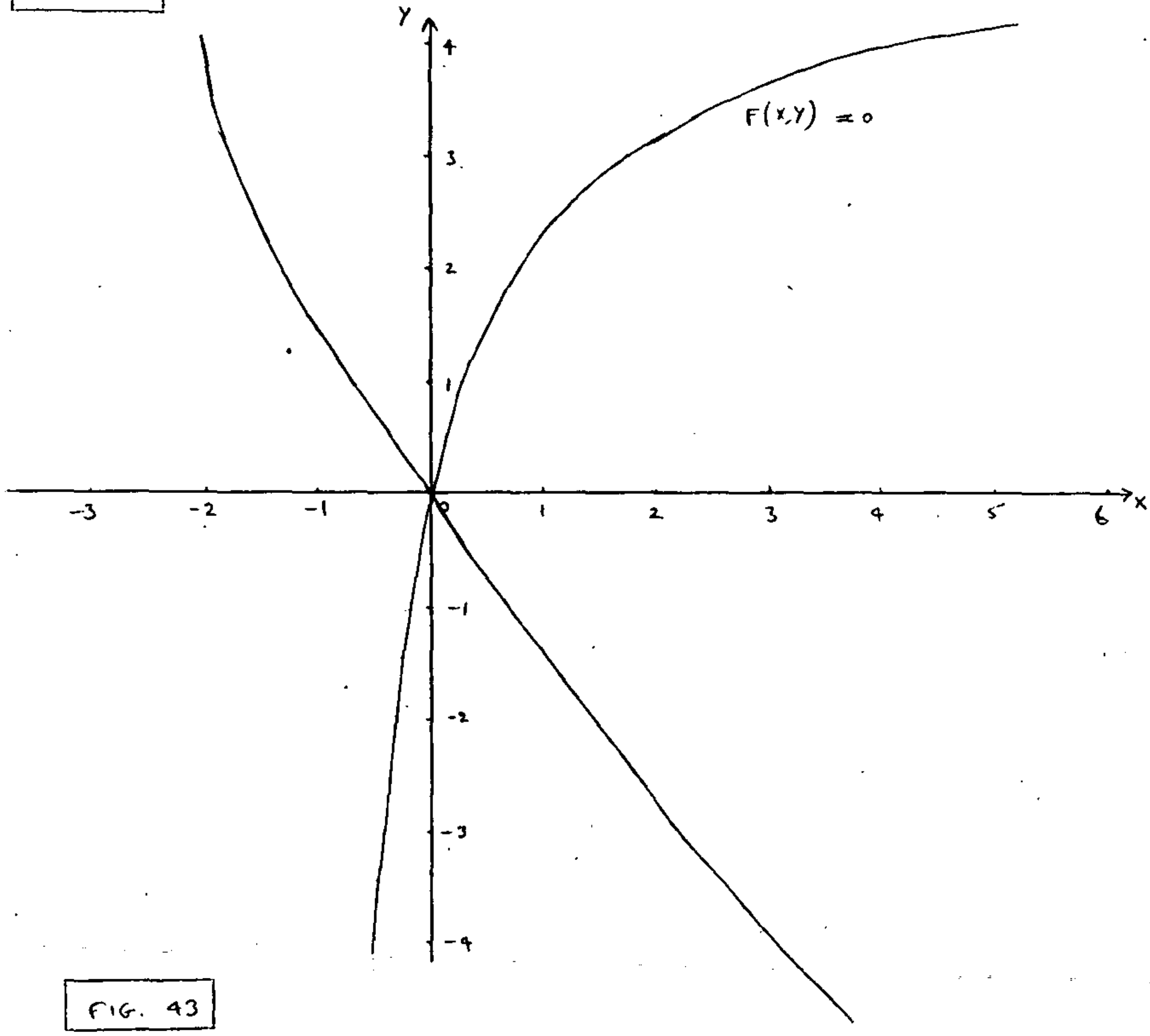
Y
X
1.8
1.5
1.2
0.9
0.6
0.3
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4
$Y = \frac{\rho X}{2\rho X^2 - 1}$
$\rho = \infty$
$\rho = 2$
$\rho = 1$
$\rho = 1/2$

FIG. 41



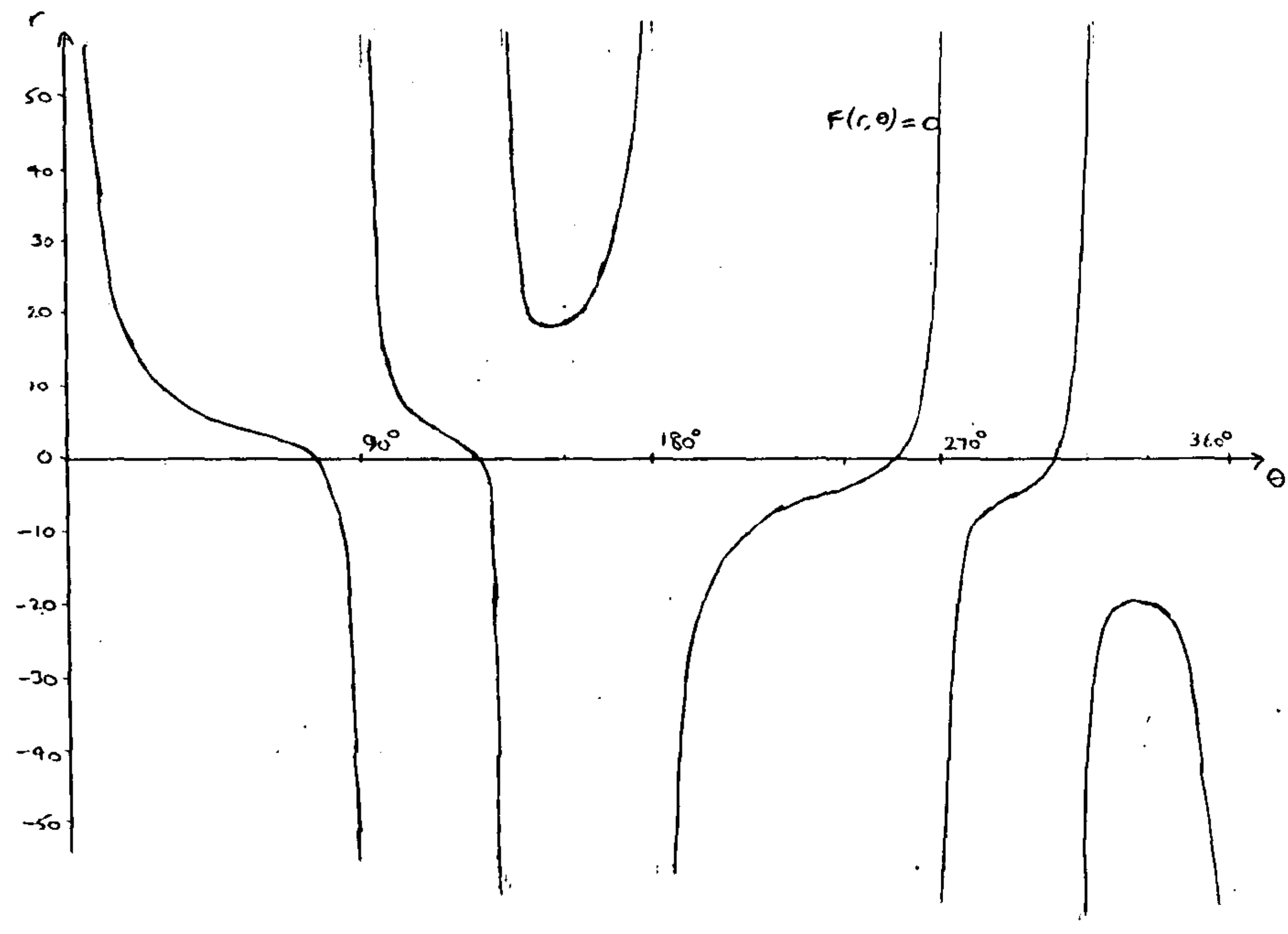
$\dot{X} = -X + Y + X(X^2 + Y^2)$
$\dot{Y} = -X - Y + Y(X^2 + Y^2)$
$\dot{V} = -2(X^2 + Y^2)(1 - V)$
Y
X
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2
0.4
0.6
0.8
1.0
discontinuity line
boundary of region of computational stability
V = 1

FIG. 42



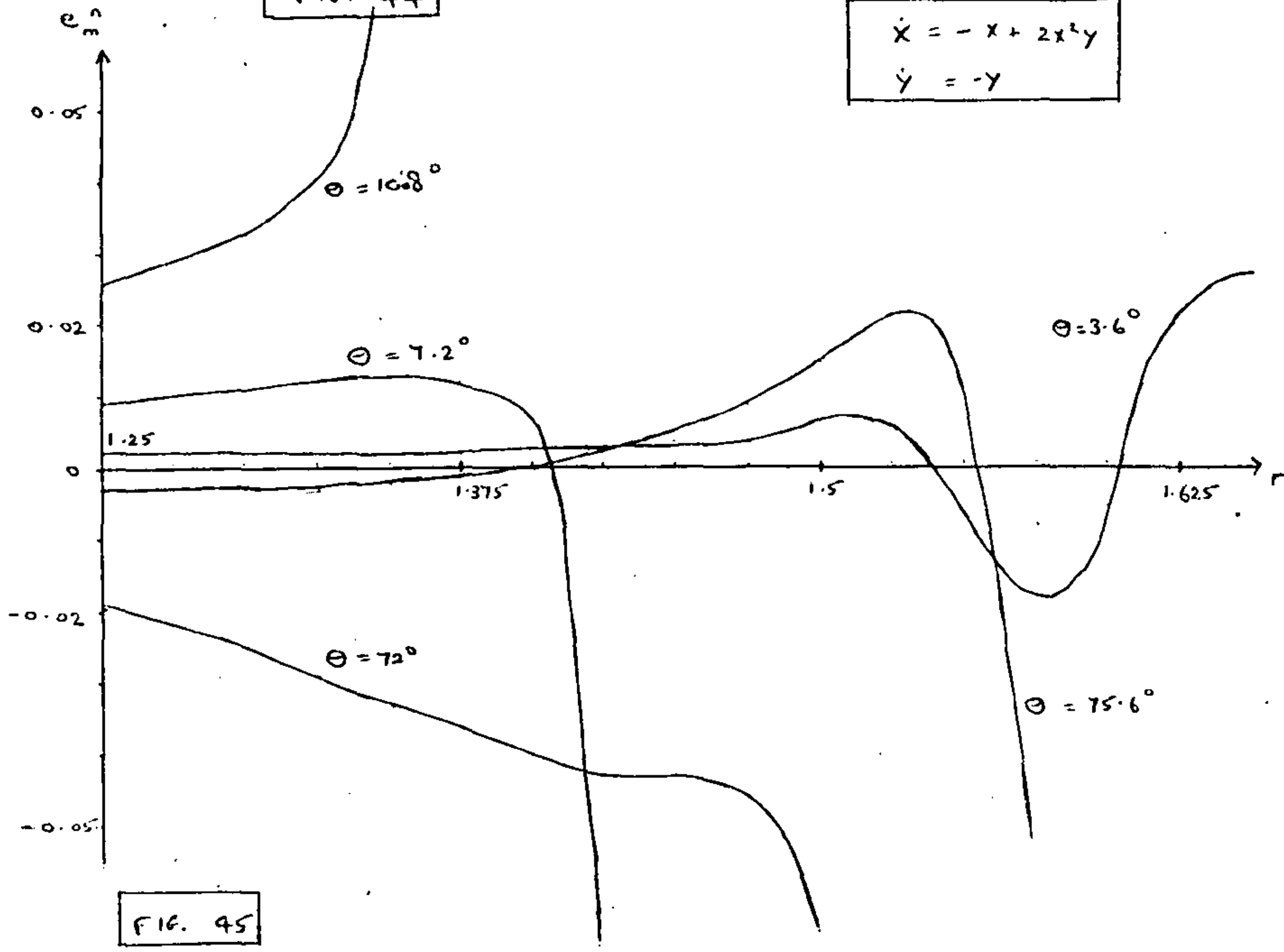
Y
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
1
2
3
4
5
6
X
F(X,Y) = 0

FIG. 43



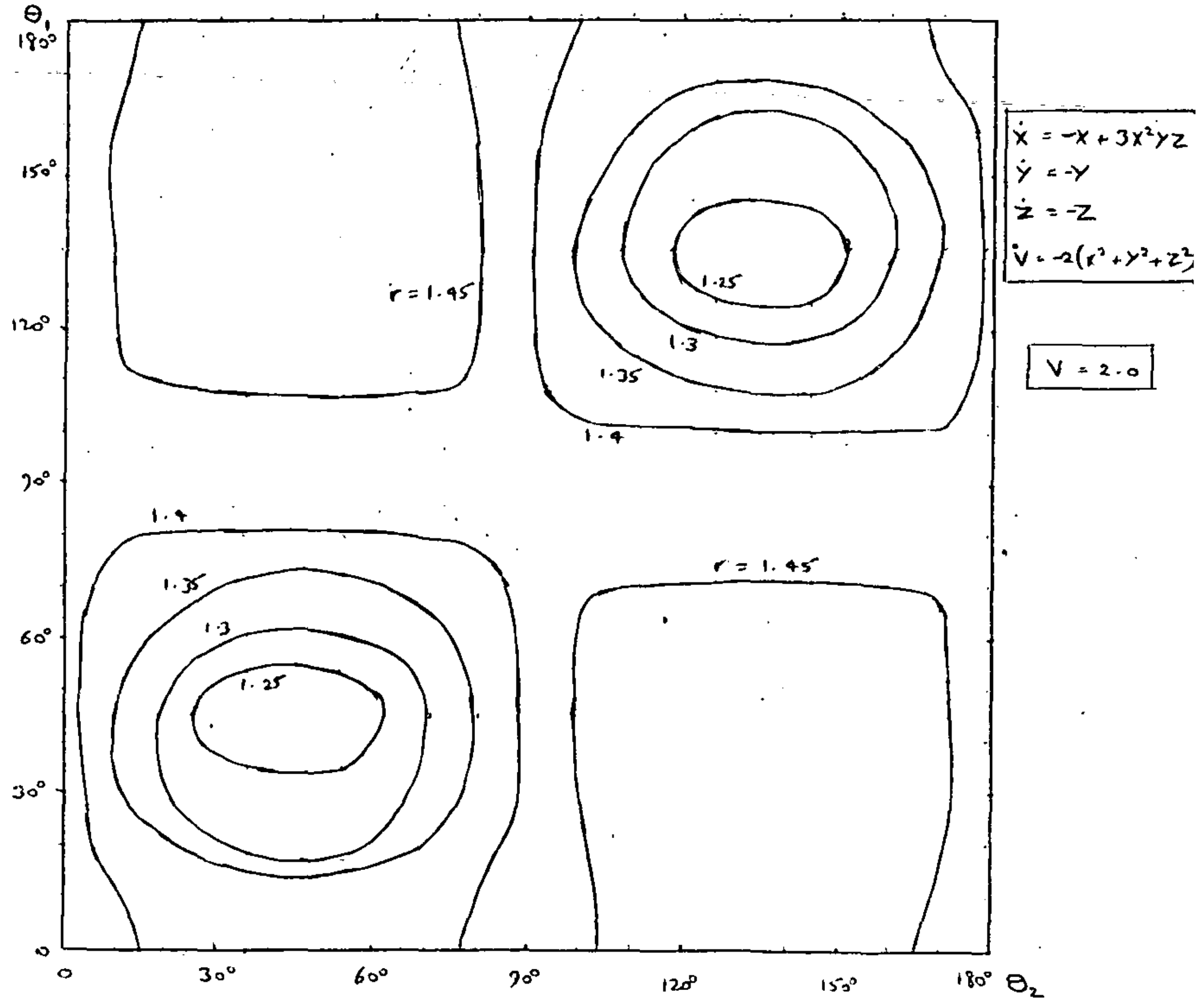
r
50
40
30
20
10
0
-10
-20
-30
-40
-50
90°
180°
270°
360°
θ
F(r,θ)=0

FIG. 44



$\dot{x} = -x + 2x^2y$
$\dot{y} = -y$
e_m^n
0·05
0·02
0
-0·02
-0·05
1·375
1·5
1·625
r
$\Theta = 108°$
$\Theta = 7.2°$
$\Theta = 3.6°$
1.25
$\Theta = 72°$
$\Theta = 75.6°$

FIG. 45



$\dot{x} = -x + 3x^2yz$
$\dot{y} = -y$
$\dot{z} = -z$
$\dot{V} = -2(x^2+y^2+z^2)$
V = 2.0
Θ_1
180°
150°
120°
90°
60°
30°
0
30°
60°
90°
120°
150°
180°
Θ_2
r = 1.45
1.25
1.3
1.35
1.4
1.4
1.35
1.3
1.25
r = 1.45

FIG. 47

$$\dot{x} = -x + y + x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2)$$
$$\dot{v} = -2(x^2 + y^2)(1 - v)$$

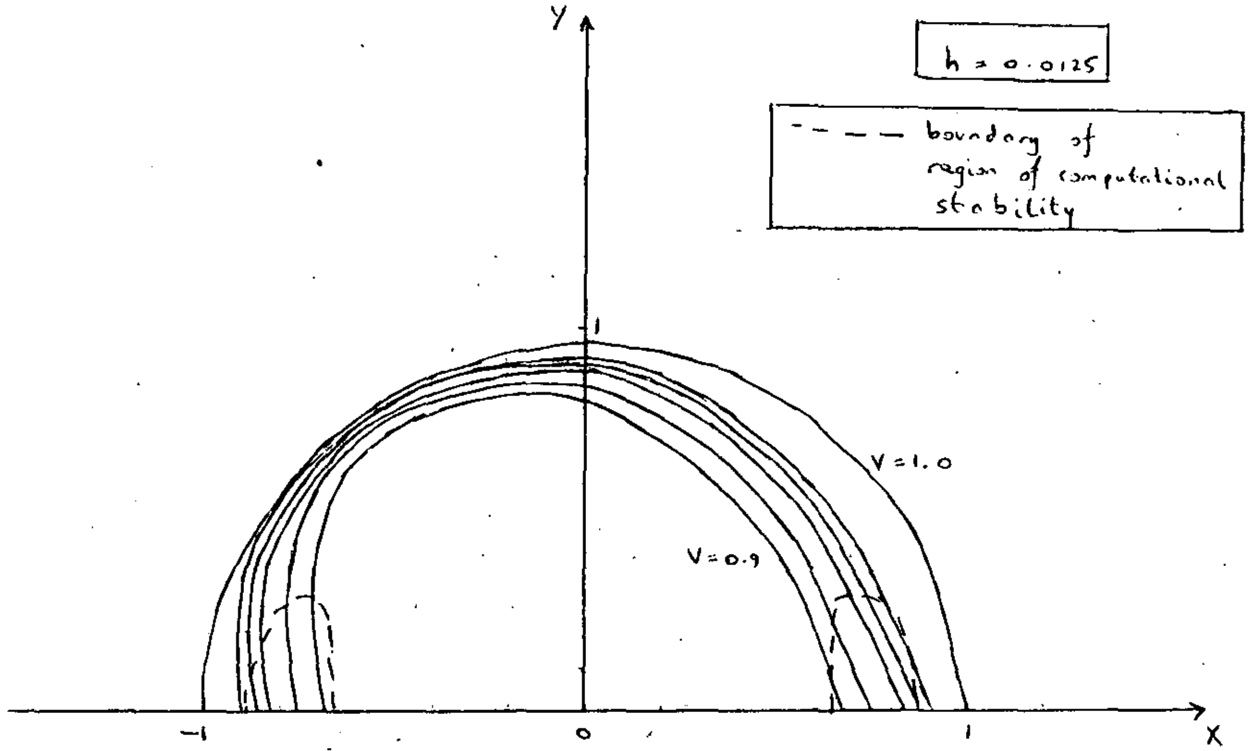


FIG. 46

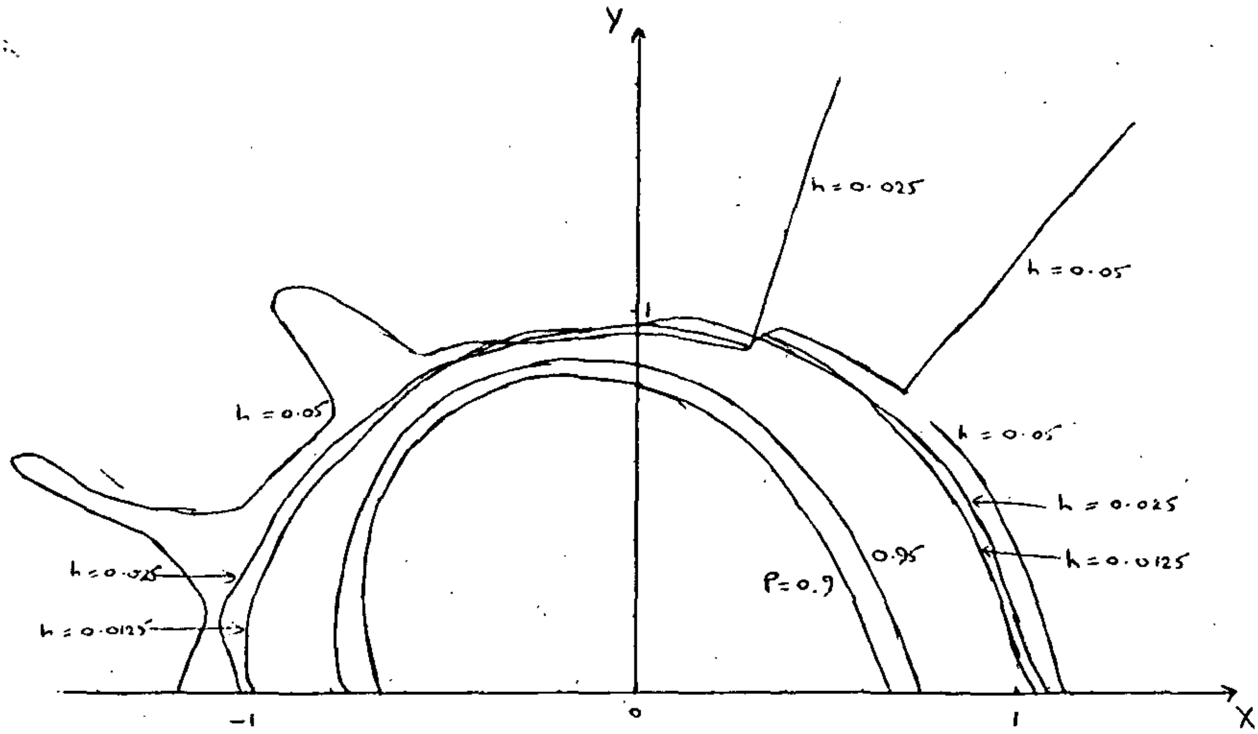
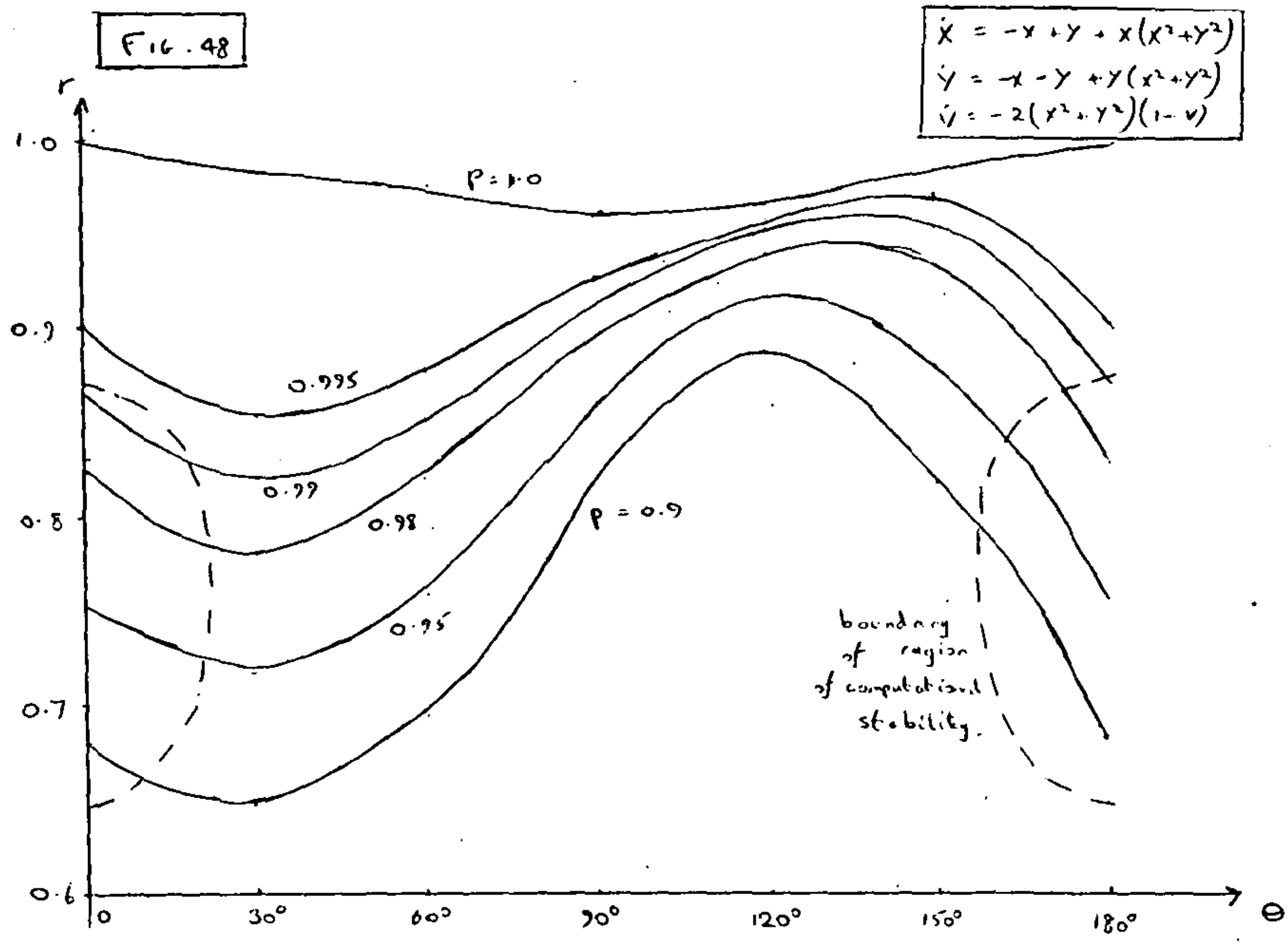
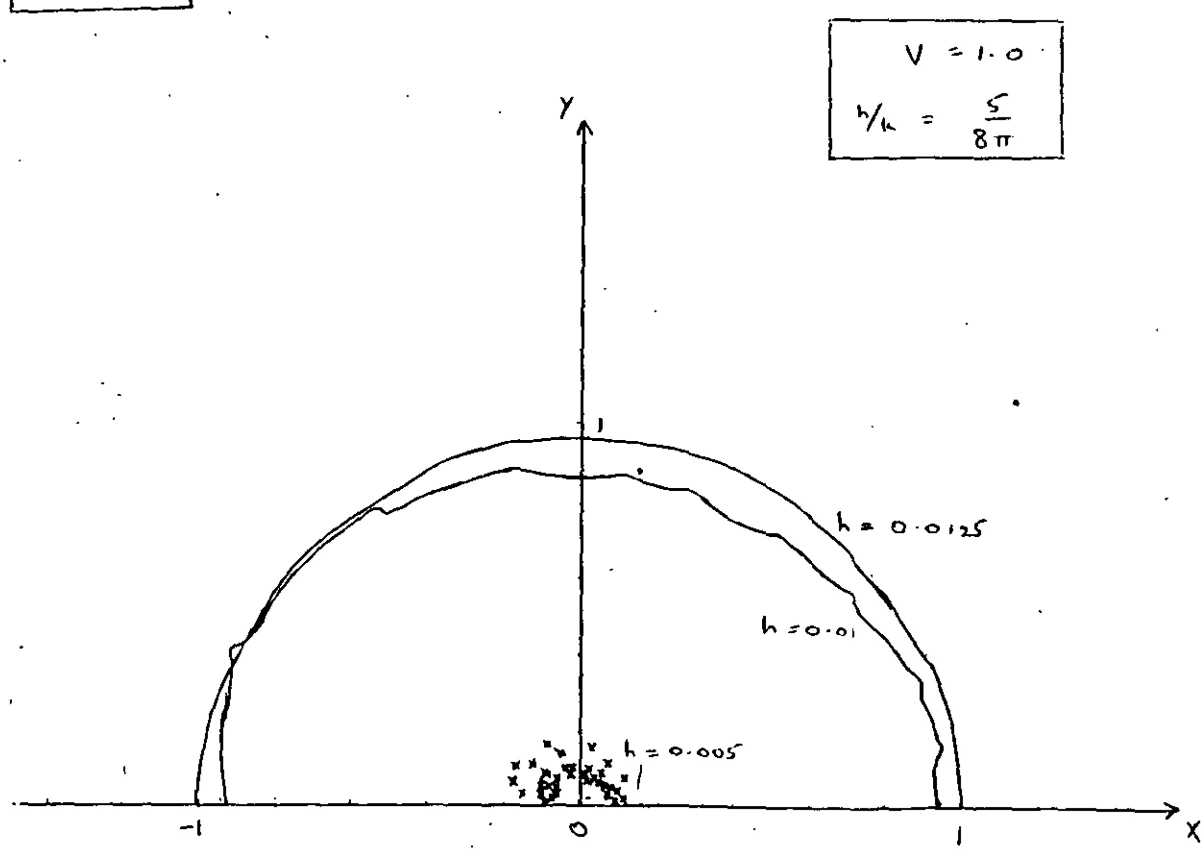


FIG. 48



$\dot{x} = -x + y + x(x^2+y^2)$
$\dot{y} = -x - y + y(x^2+y^2)$
$\dot{v} = -2(x^2+y^2)(1-v)$
r
1.0
0.9
0.8
0.7
0.6
p = 1.0
0.995
0.99
0.98
0.95
p = 0.9
boundary of region of computational stability.
0
30°
60°
90°
120°
150°
180°
θ

FIG. 49



V = 1.0
h/k = $\frac{5}{8\pi}$
Y
1
h = 0.0125
h = 0.01
h = 0.005
-1
0
1
X

$$\dot{x} = -x + y + x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2)$$
$$\dot{v} = -2(x^2 + y^2)(1 - v)$$
$$v = 1.0$$

FIG. 50

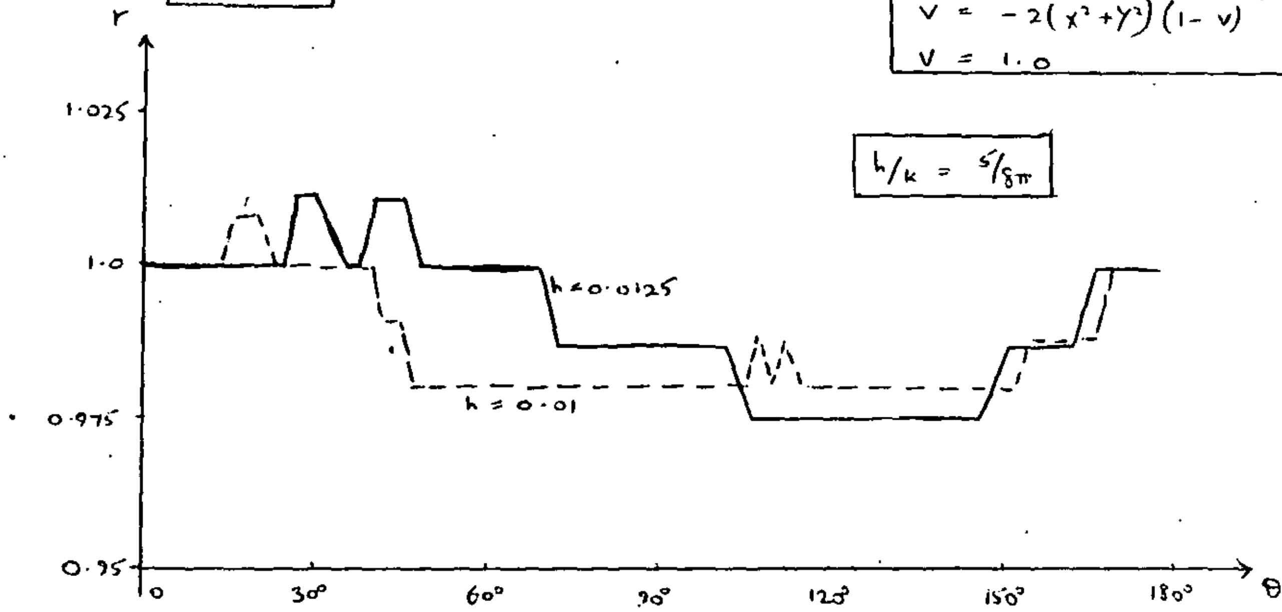


FIG. 51

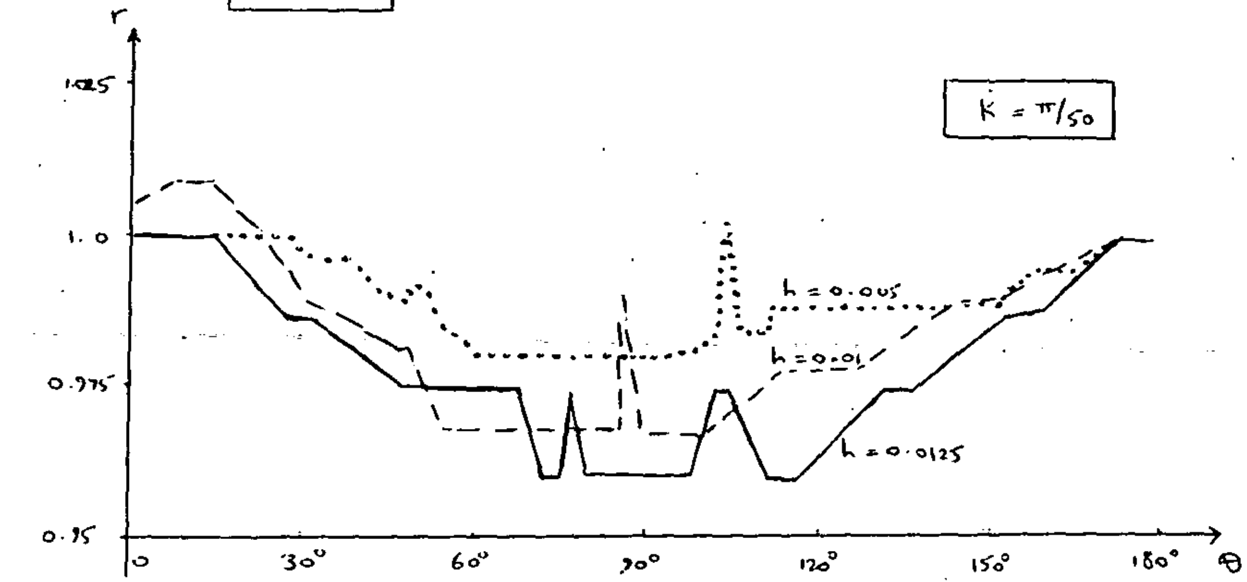
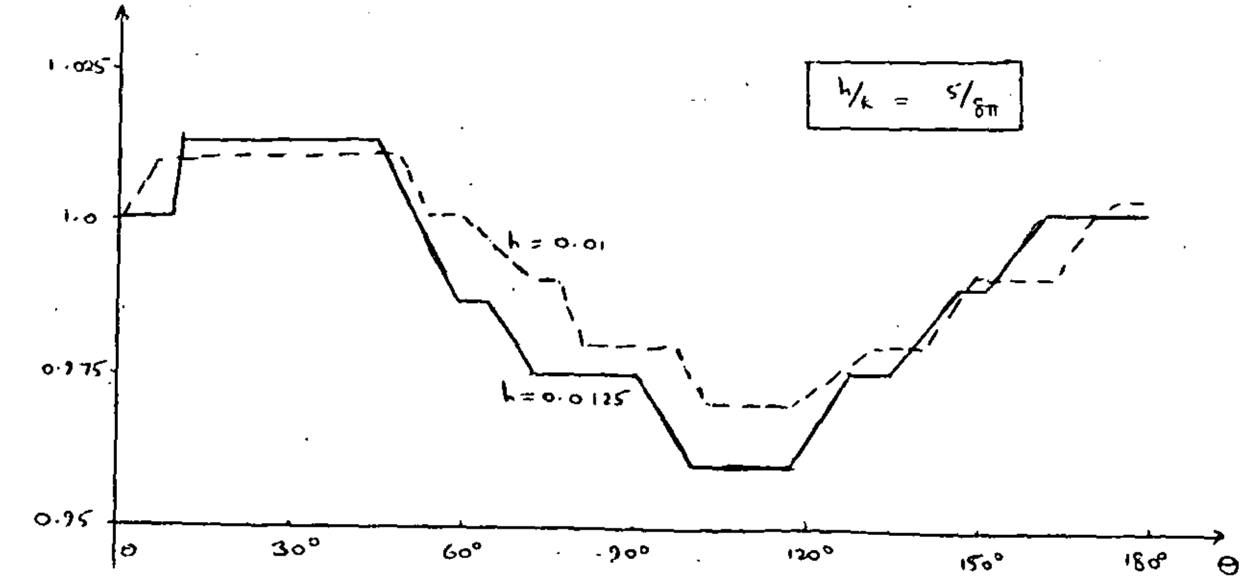


FIG. 52



$$\dot{x} = 6y - 2y^2$$
$$\dot{y} = -10x - y + 4x^2 + 2xy + 4y^2$$
$$\dot{V} = -2(x^2 + y^2)$$

FIG. 53

$V = 5\cdot0$

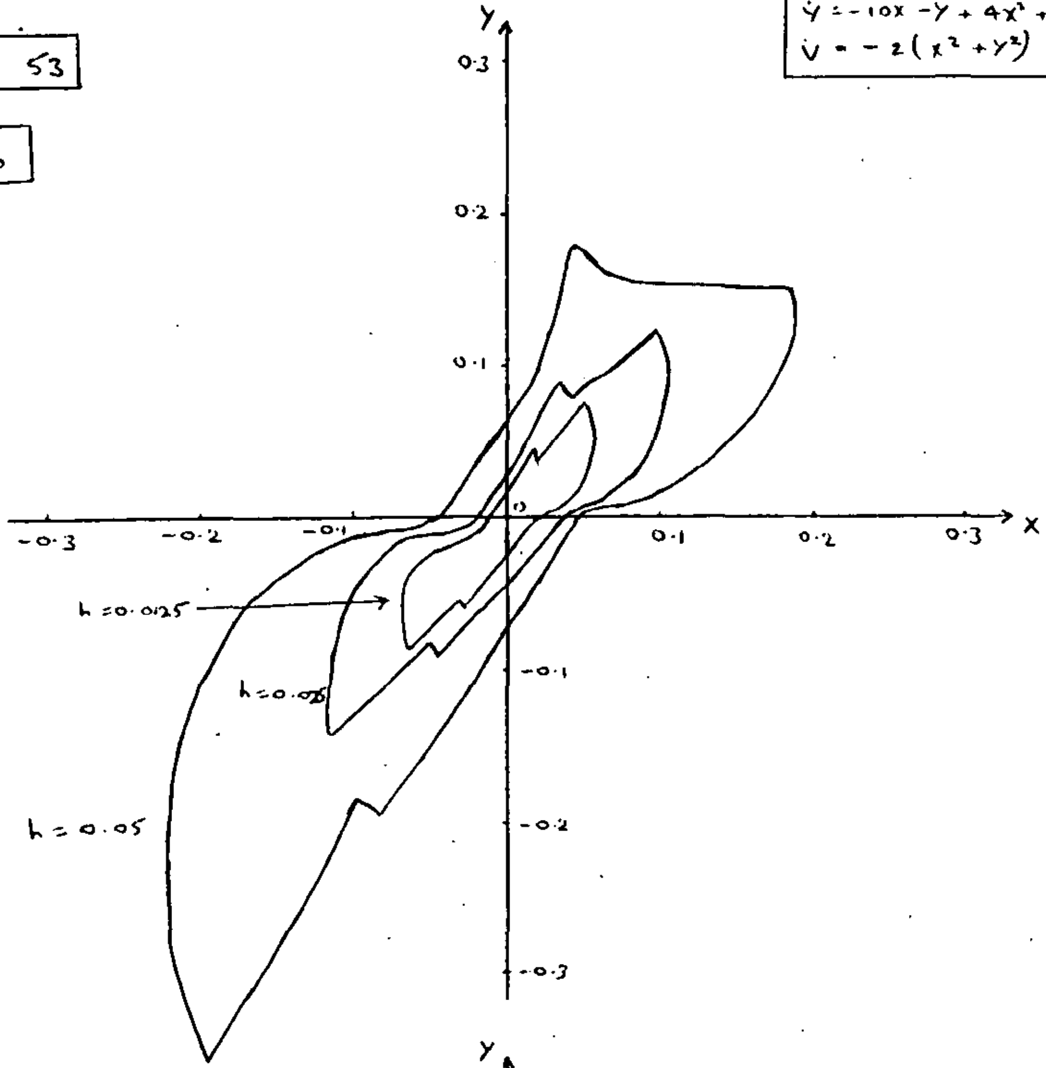
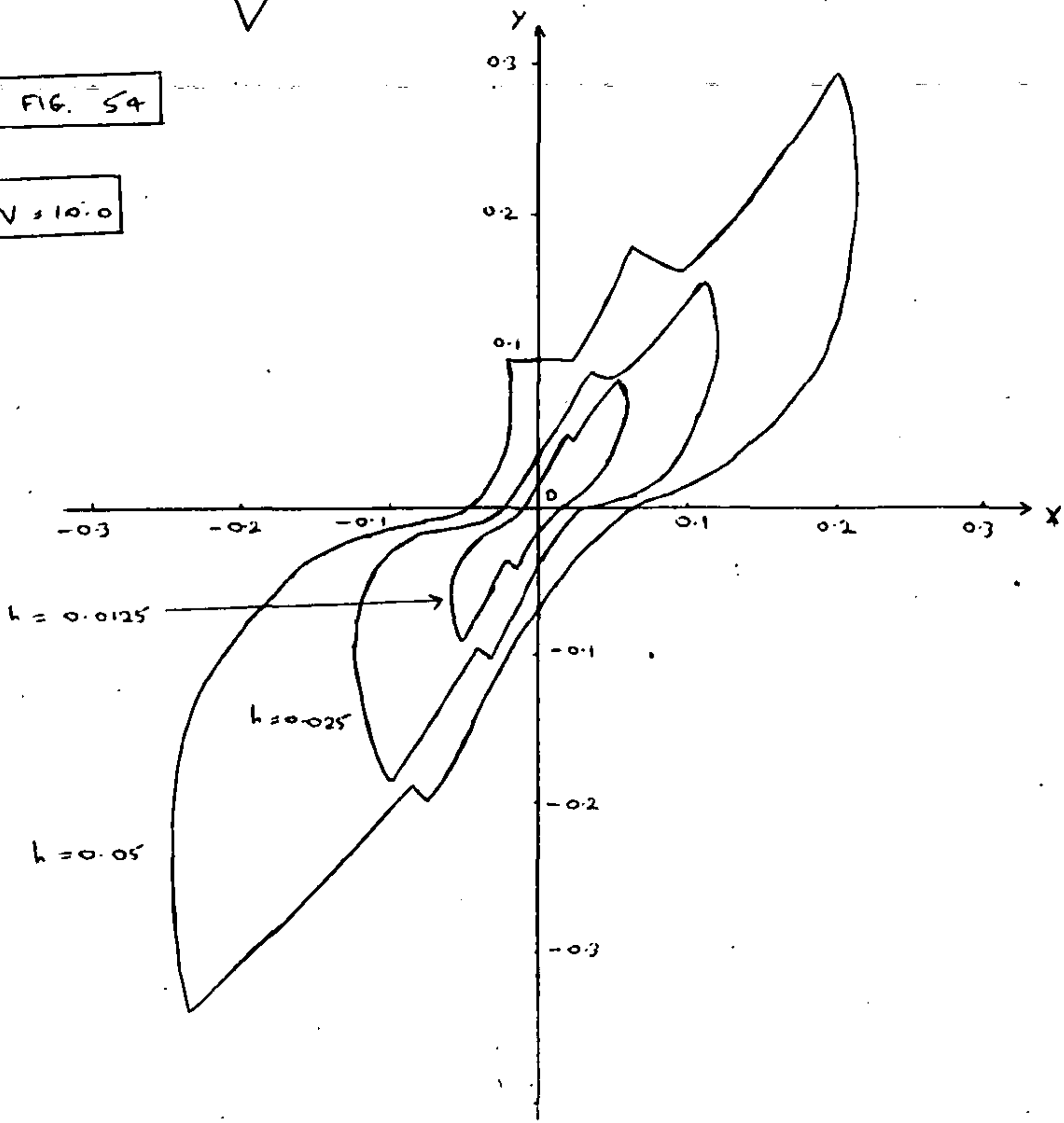


FIG. 54

$V = 10\cdot0$



$$\dot{x} = y$$
$$\dot{y} = -x + \mu(x^2 - 1)y$$
$$\dot{V} = -(x^2 + y^2)(1 - dV)$$

FIG. 55

$d = 0$
SHICLOS + EXPLICIT

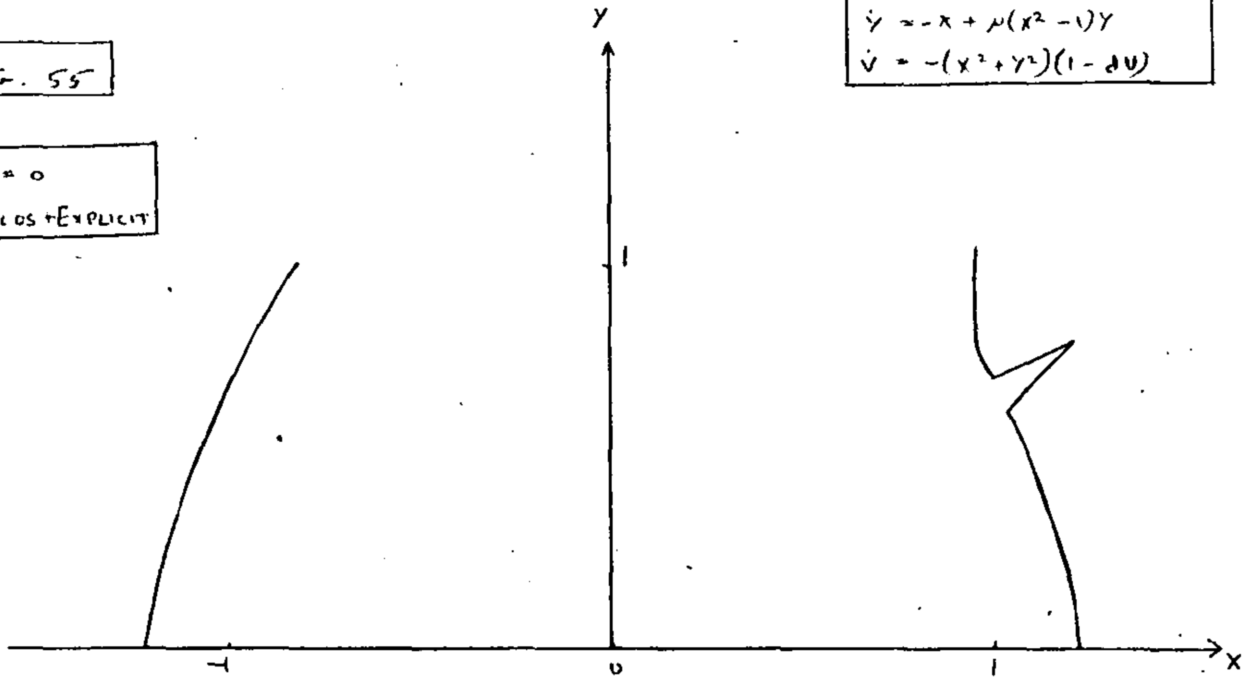


FIG. 56

$d = 0$
SHICLOS

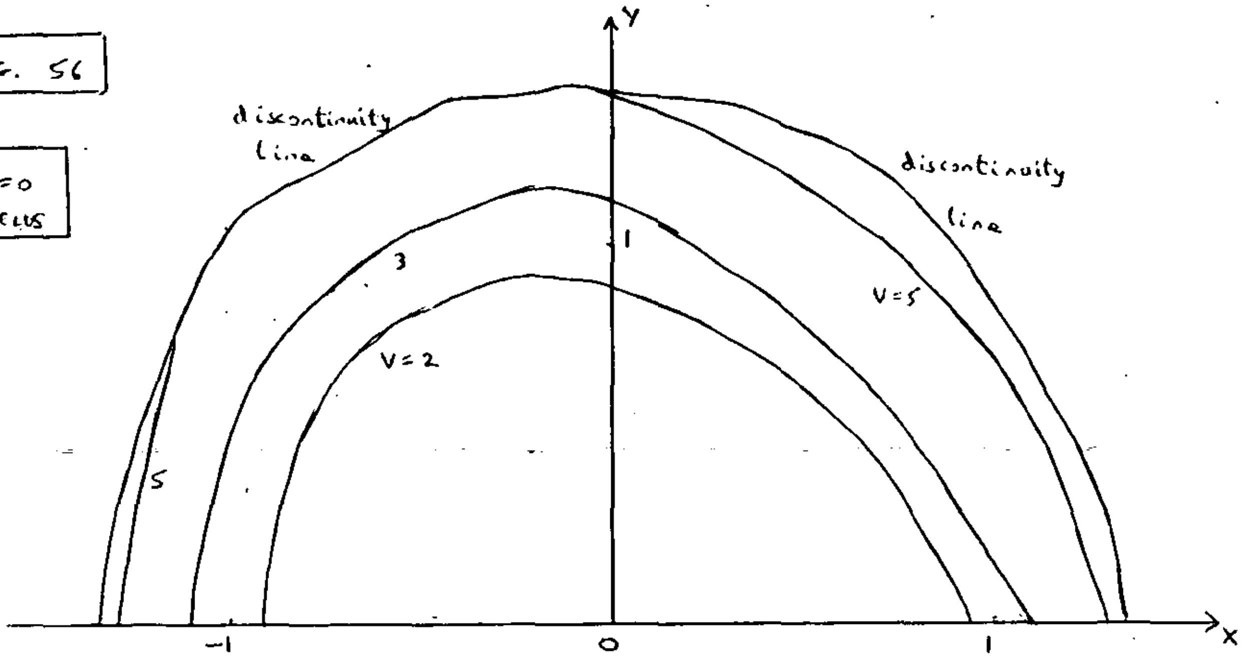
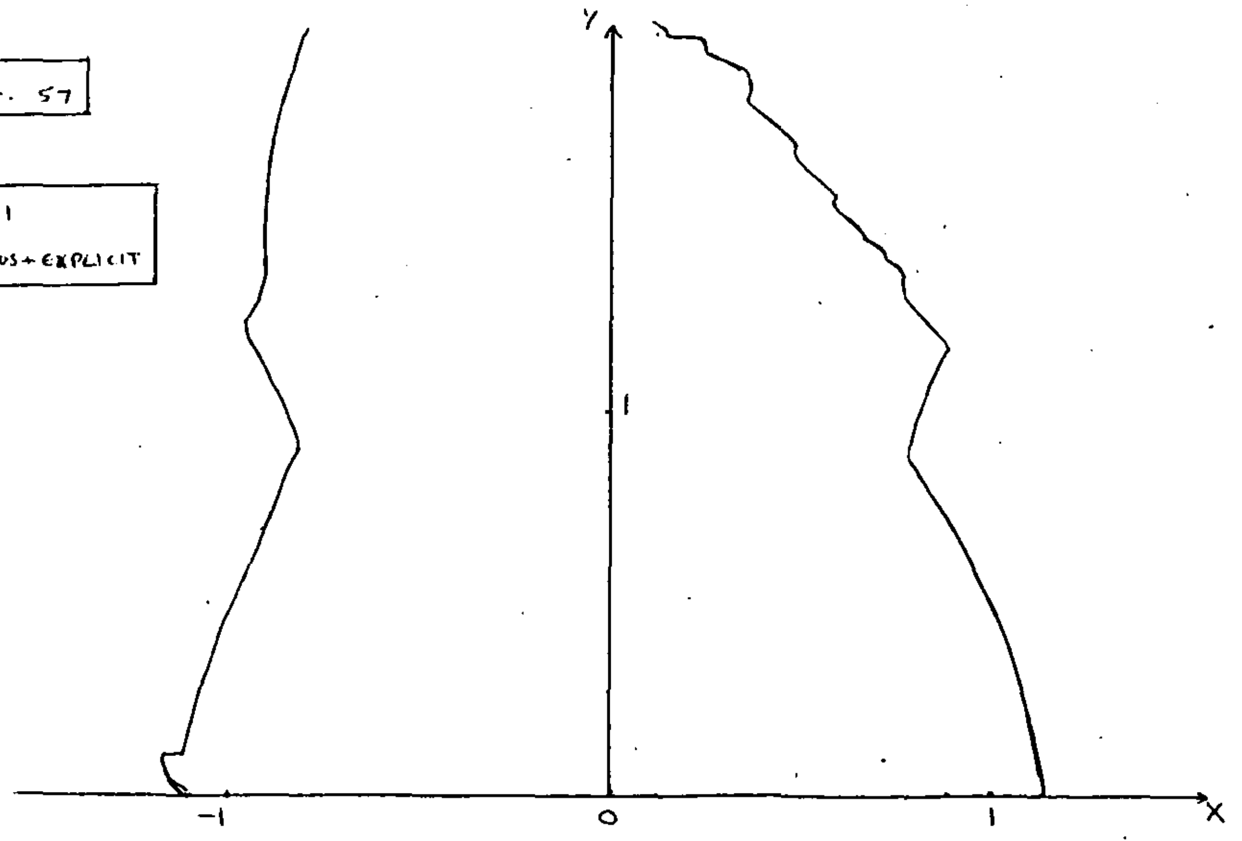


FIG. 57

$d = 1$
SHICLOS + EXPLICIT



$$\dot{x} = -x + y + x^2$$
$$\dot{y} = -y + xy$$
$$\dot{V} = -2(x^2 - xy + y^2)(1.$$

FIG. 58

$d = 0$
EXPLICIT

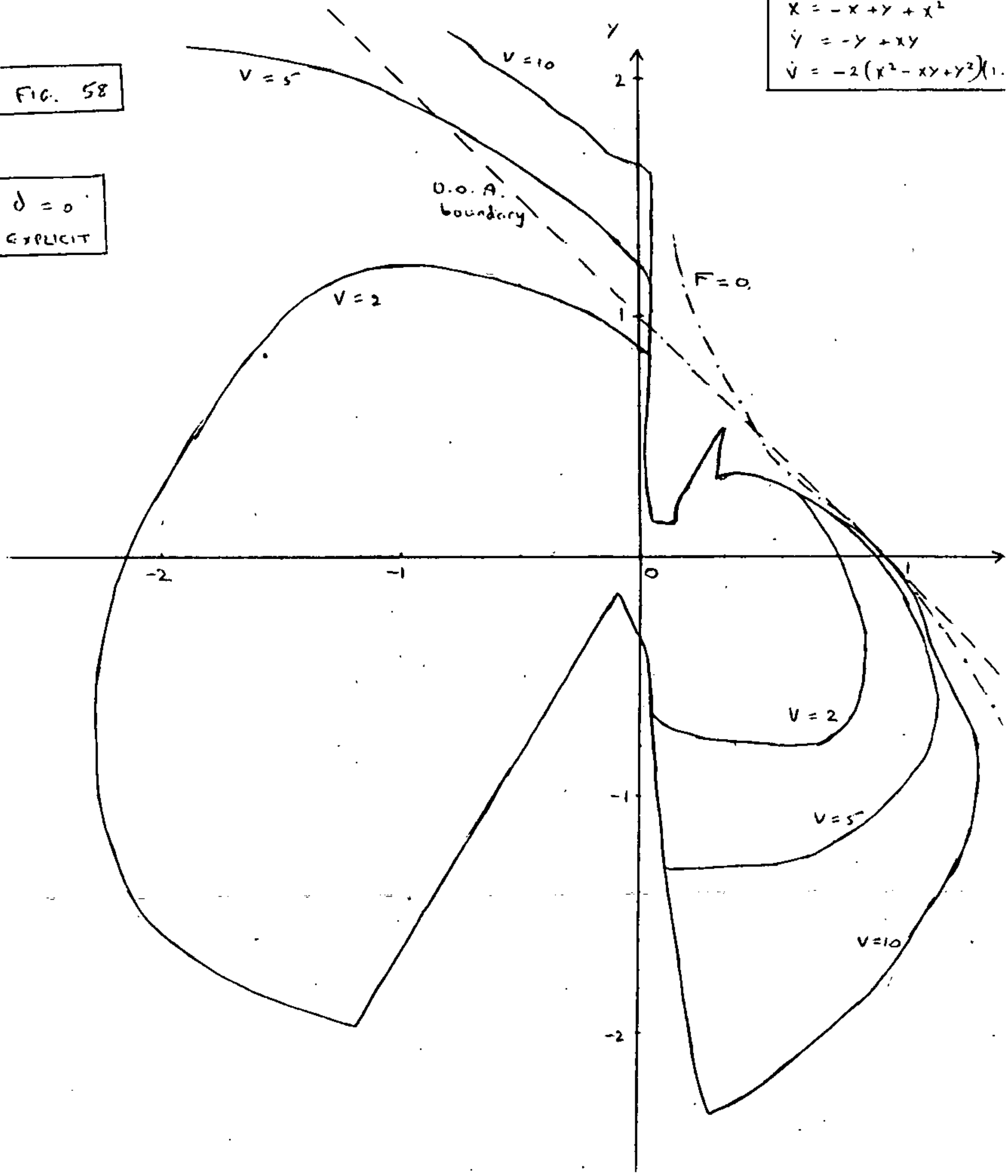


FIG. 59

$d = 0$
SHIELDS + EXPLICIT

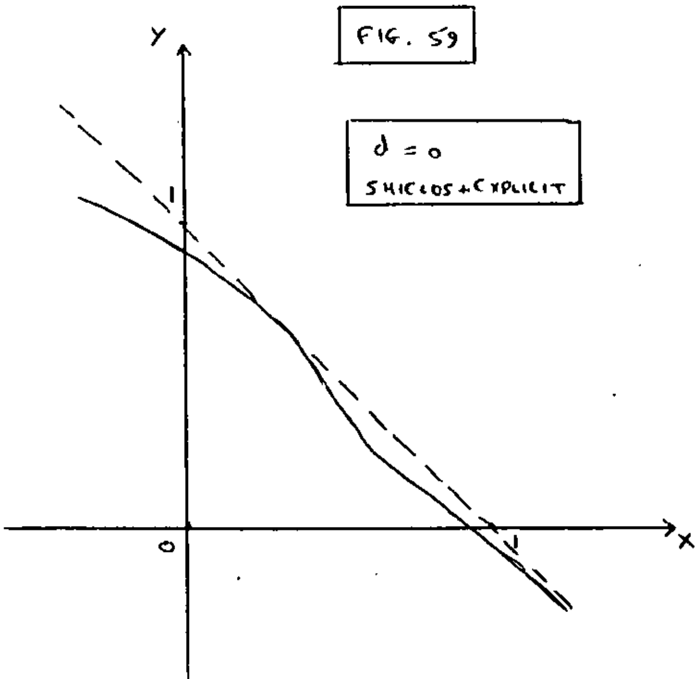


FIG. 60

$d = 1$
SHIELDS + EXPLICIT

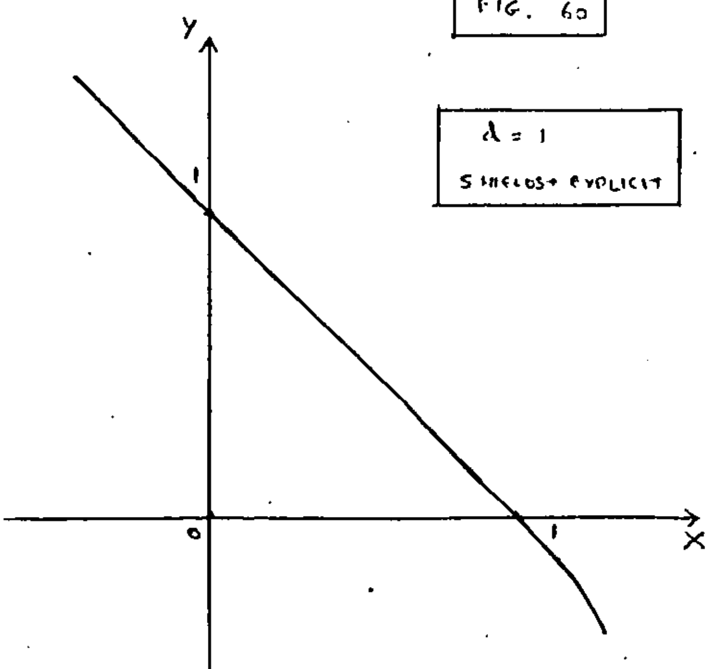


FIG. 61

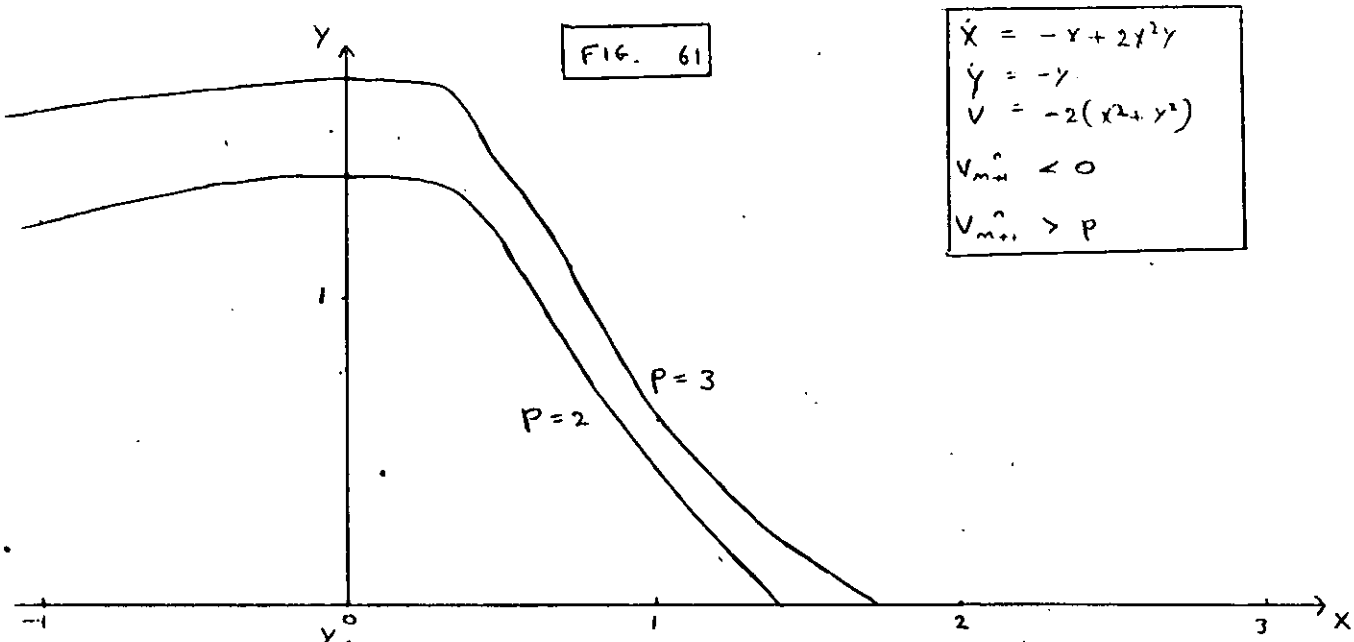


FIG. 62

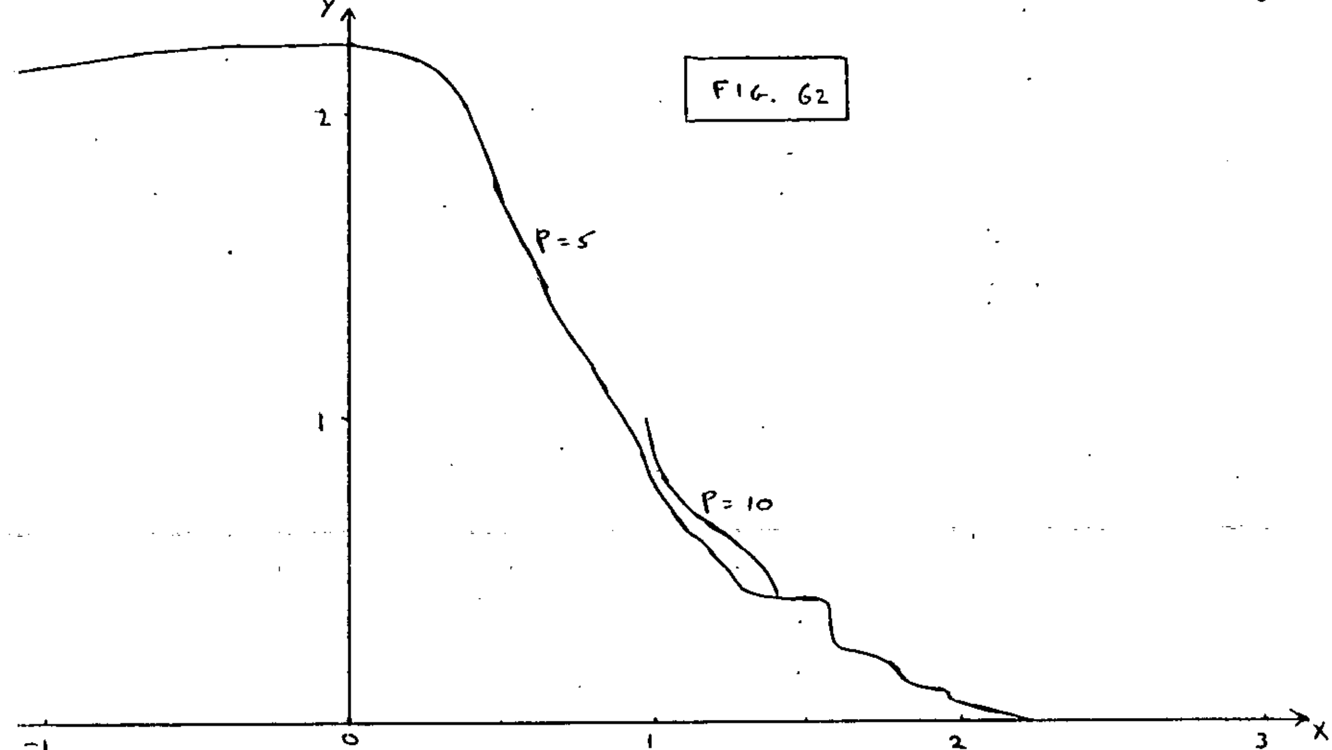
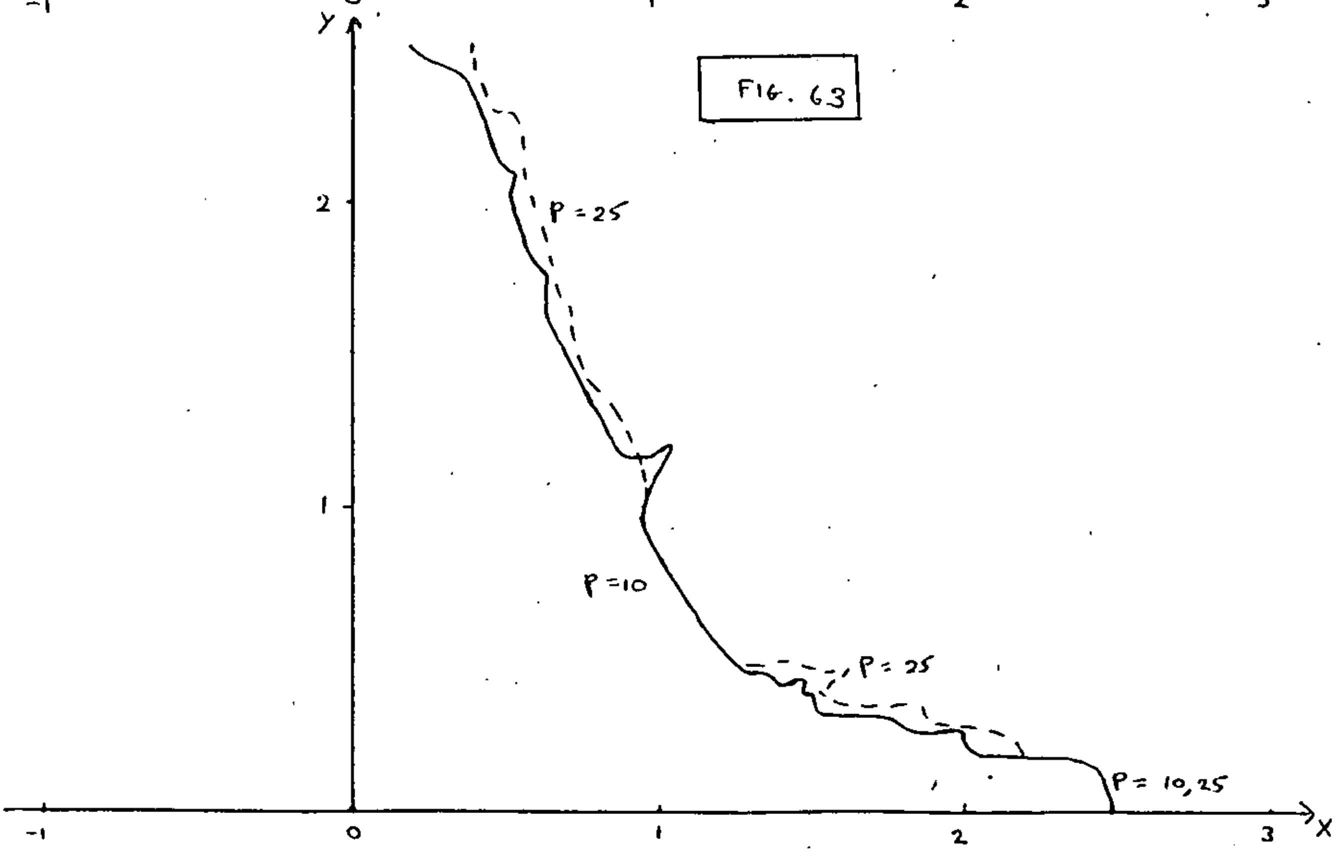


FIG. 63



$$\dot{x} = -x + 2x^2y$$
$$\dot{y} = -y$$
$$\dot{V} = -2(x^2+y^2)(1-V)$$
$$\hat{V}_{m+1} < 0$$
$$\hat{V}_{m+1} > P$$

FIG. 64

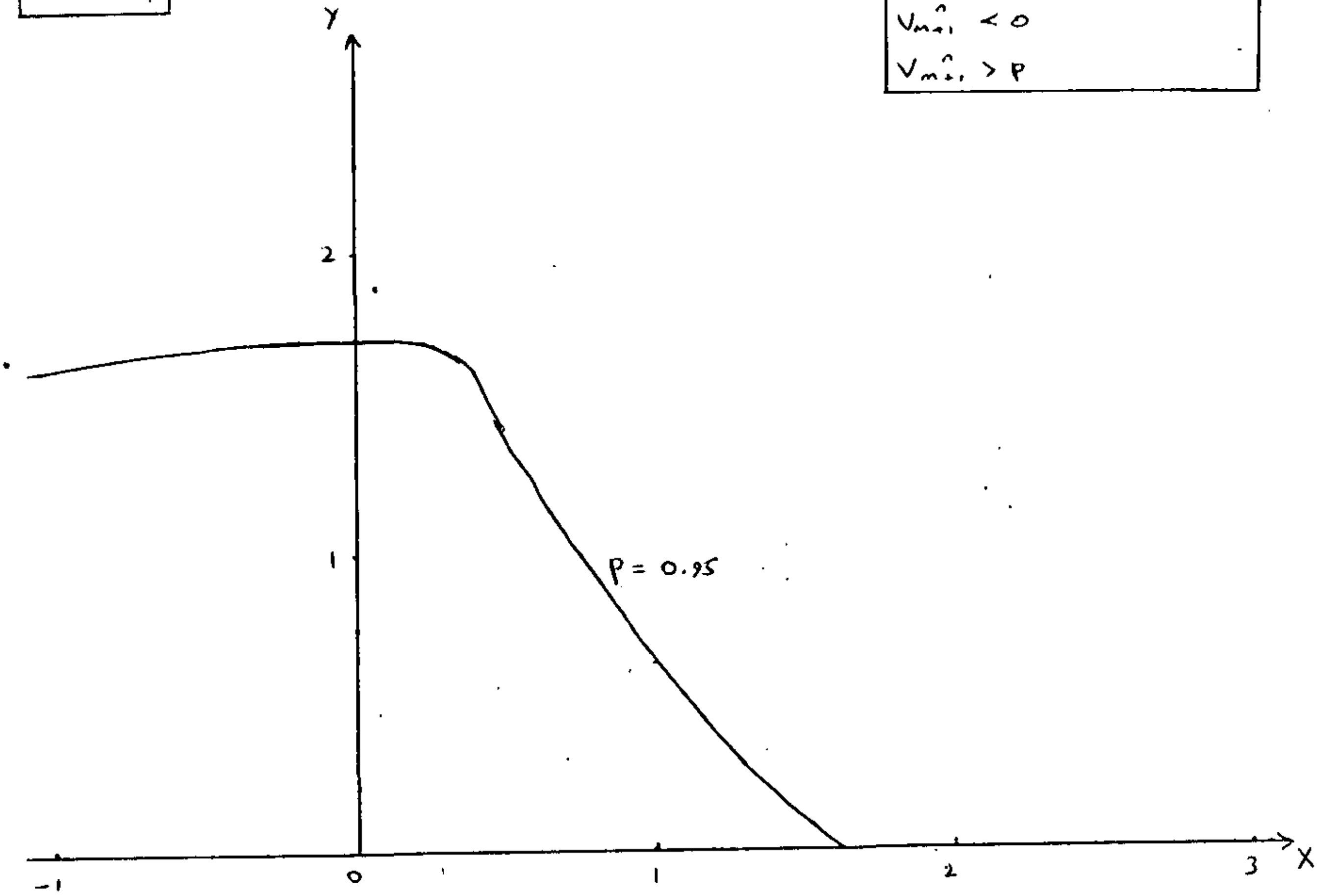


FIG. 65

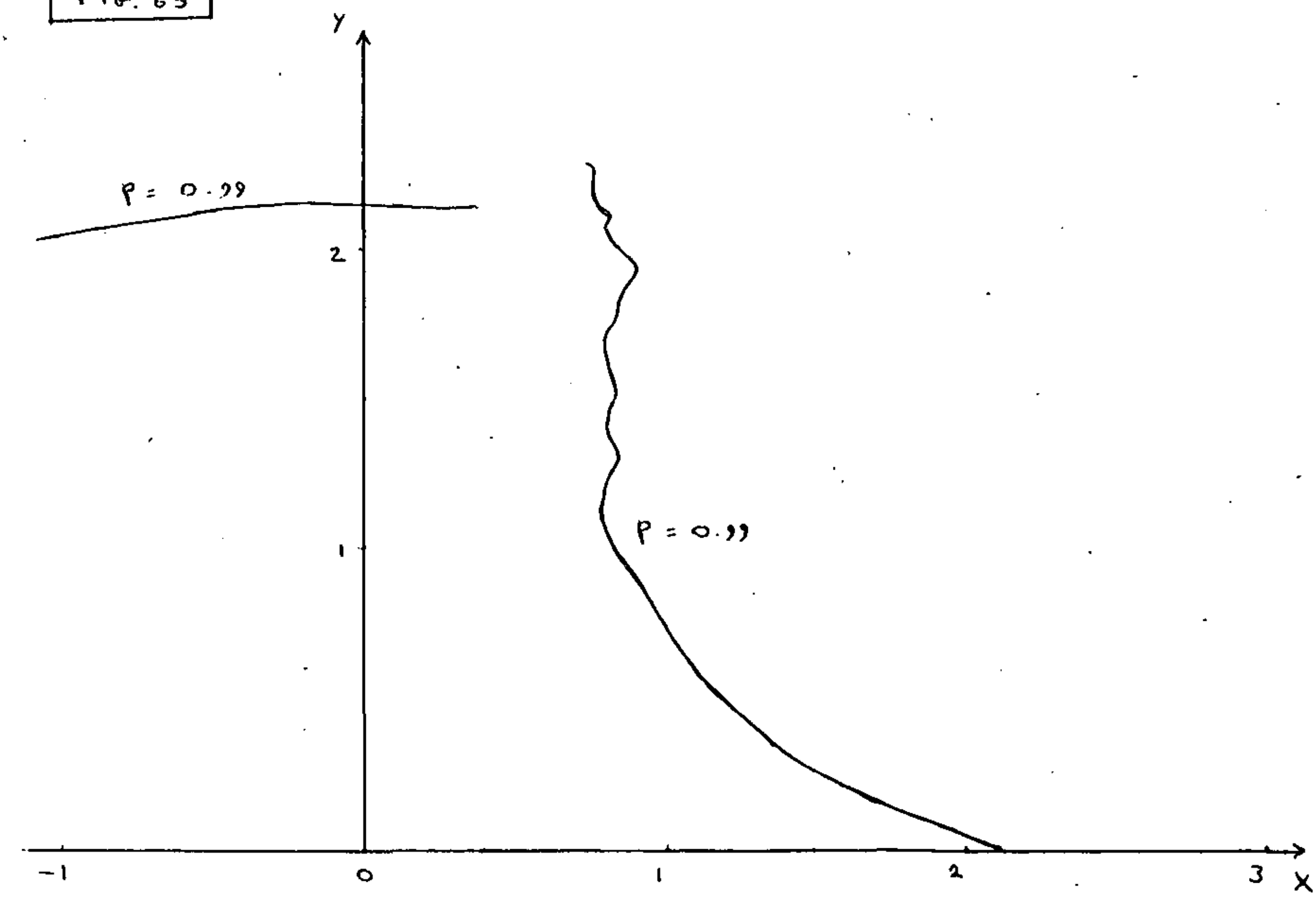


FIG. 66

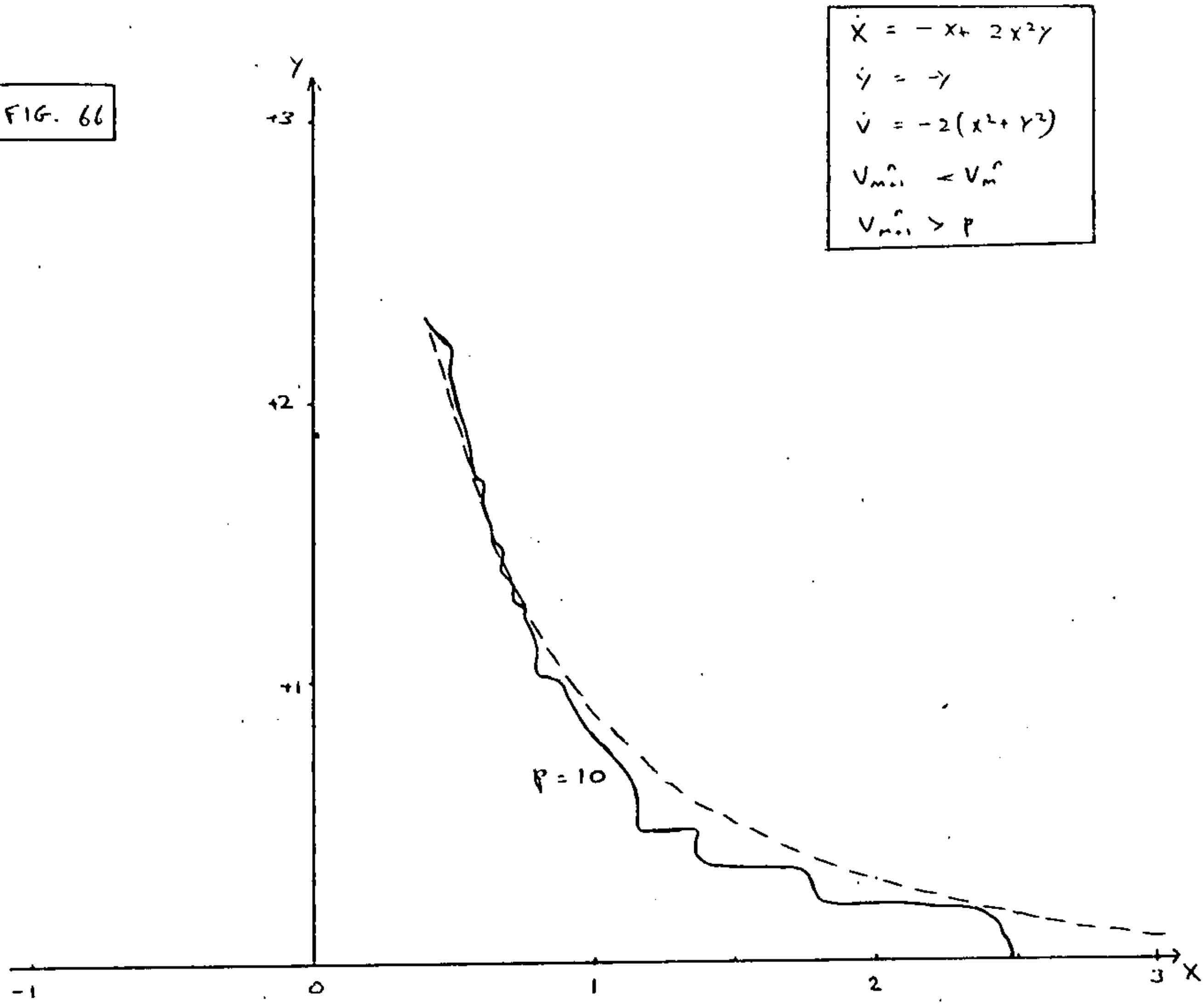

$\dot{X} = -X + 2X^2Y$
$\dot{Y} = -Y$
$\dot{V} = -2(X^2+Y^2)$
$V_{min} < V_m$
$V_{min} > p$
Y
+3
+2
+1
p = 10
-1
0
1
2
3
X

FIG. 67

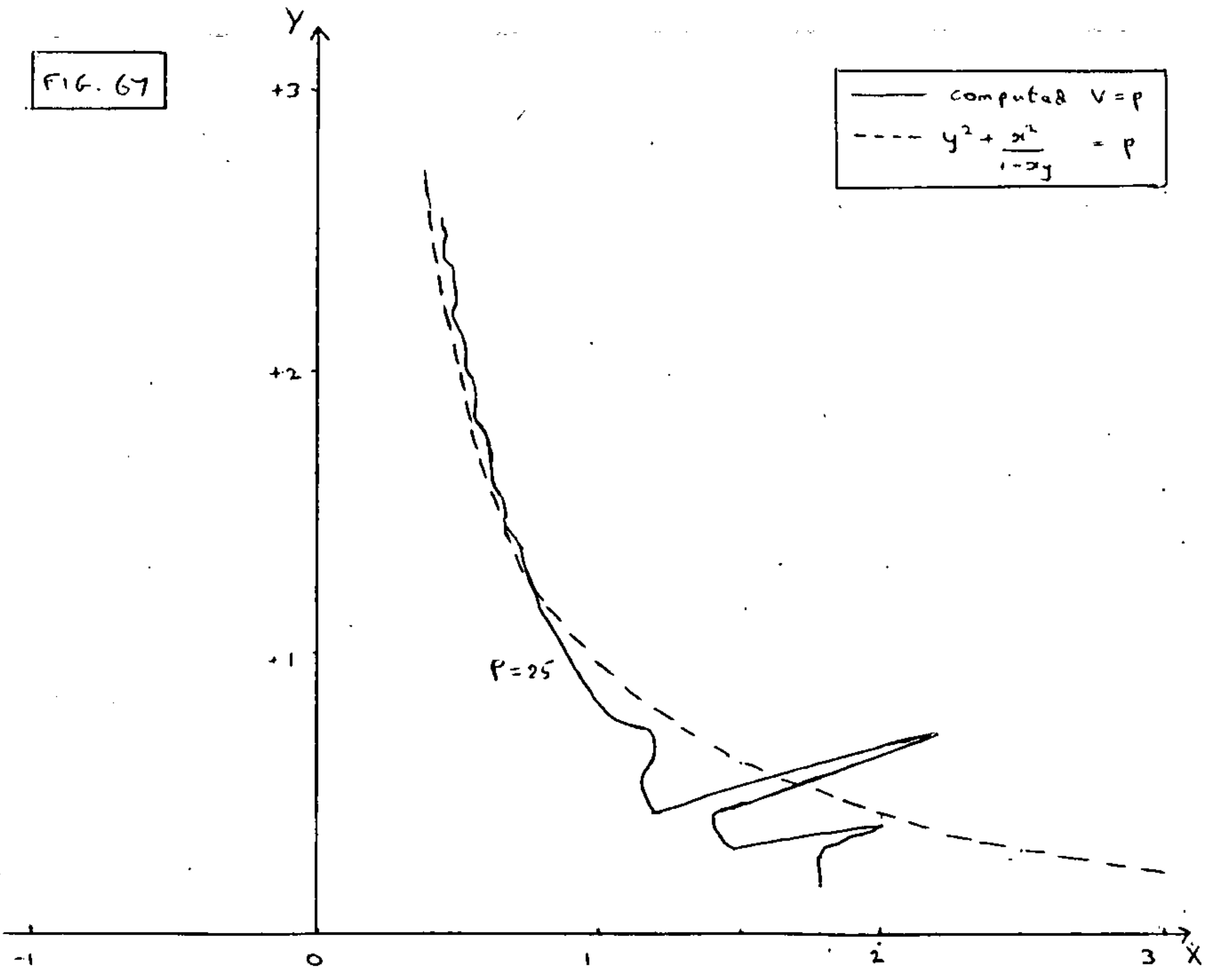

computed V = p
$y^2 + \frac{x^2}{1-xy} = p$
Y
+3
+2
+1
p = 25
-1
0
1
2
3
X

FIG. 68

$$\dot x = -x + 2x^2 y$$
$$\dot y = -y$$
$$\dot V = -u(x^2+y^2)$$
$$V = 2.0$$

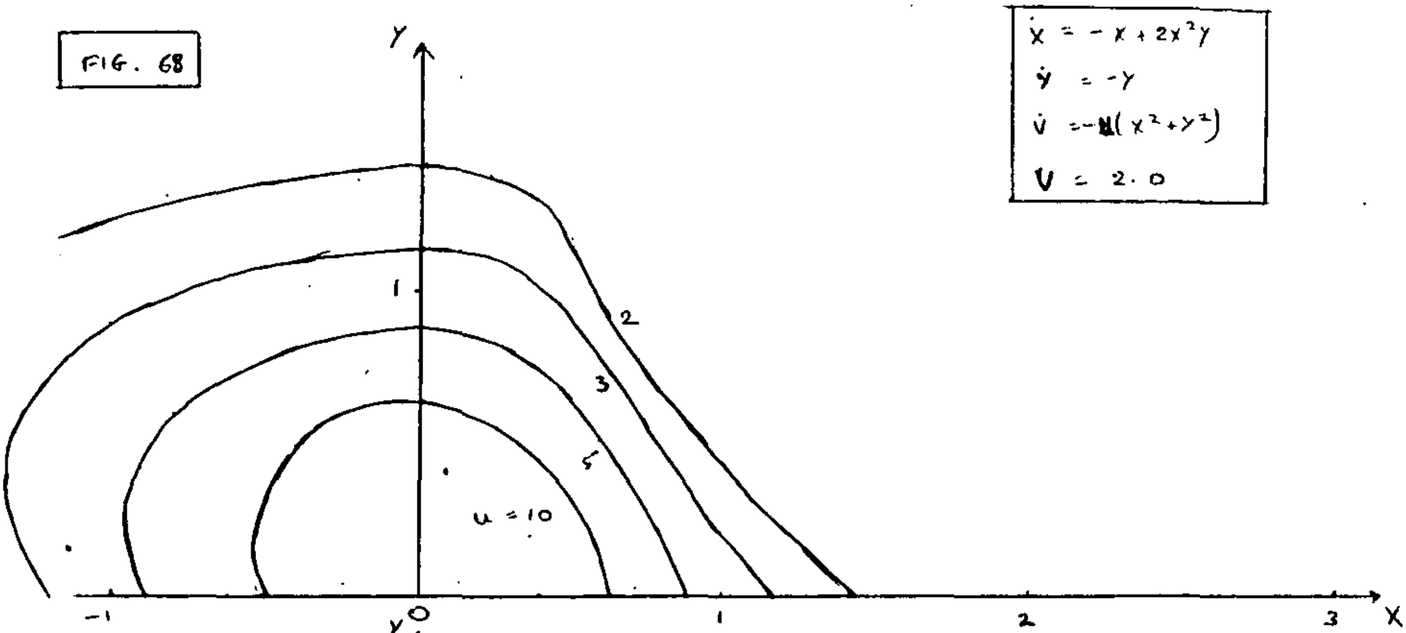


FIG. 69

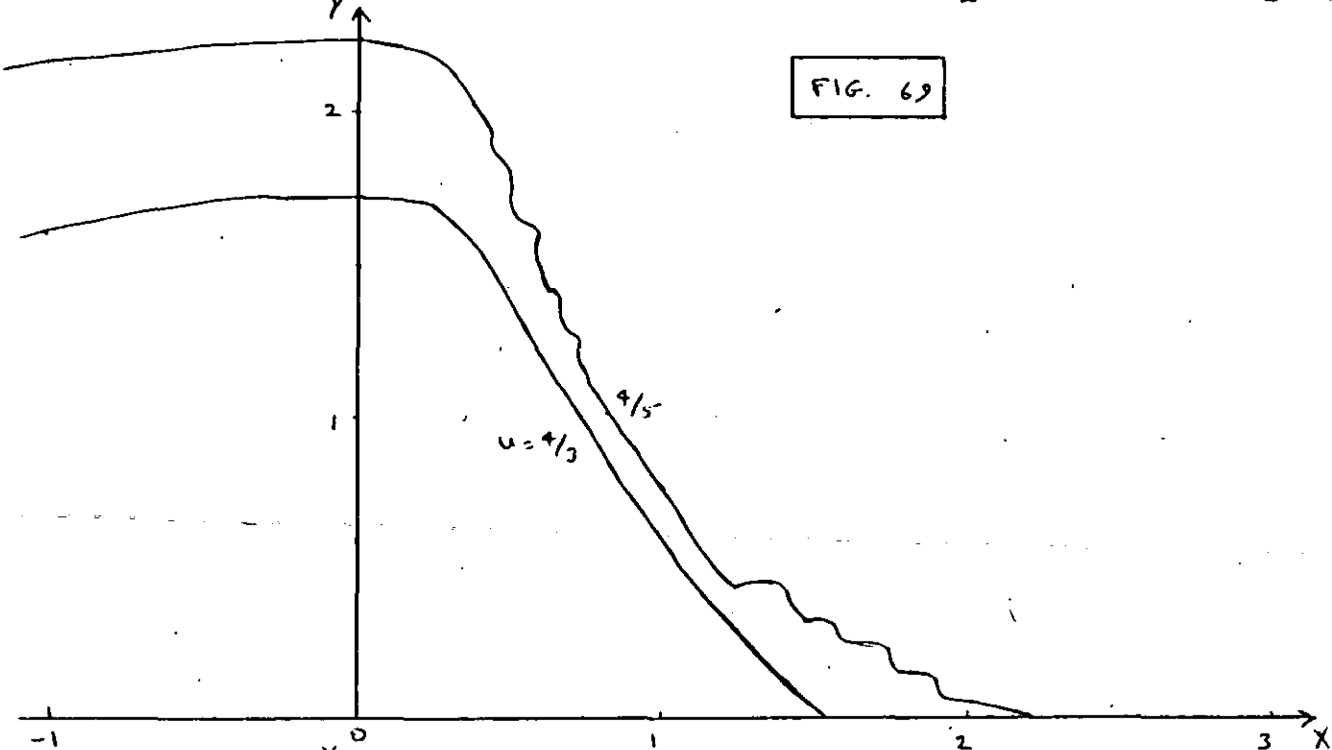
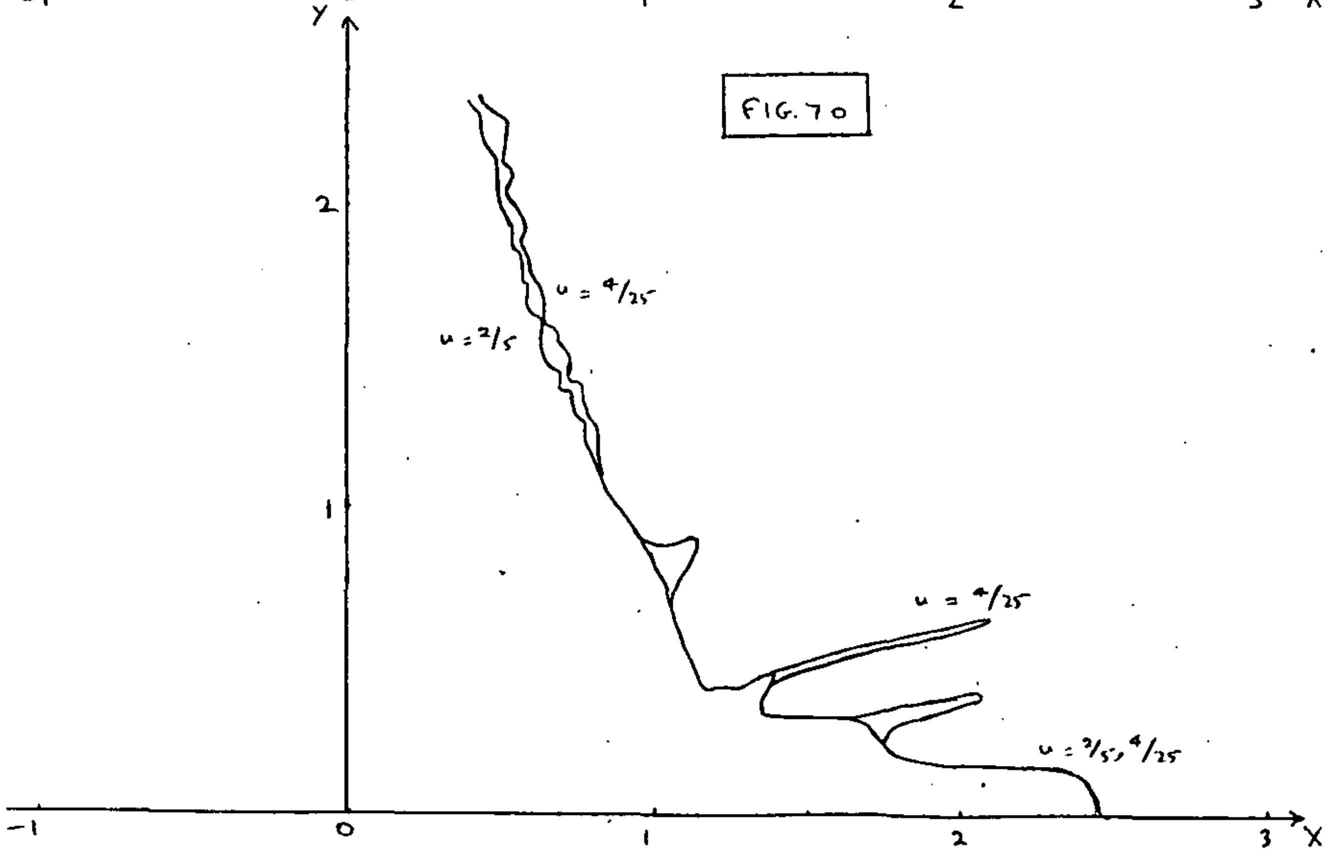


FIG. 70



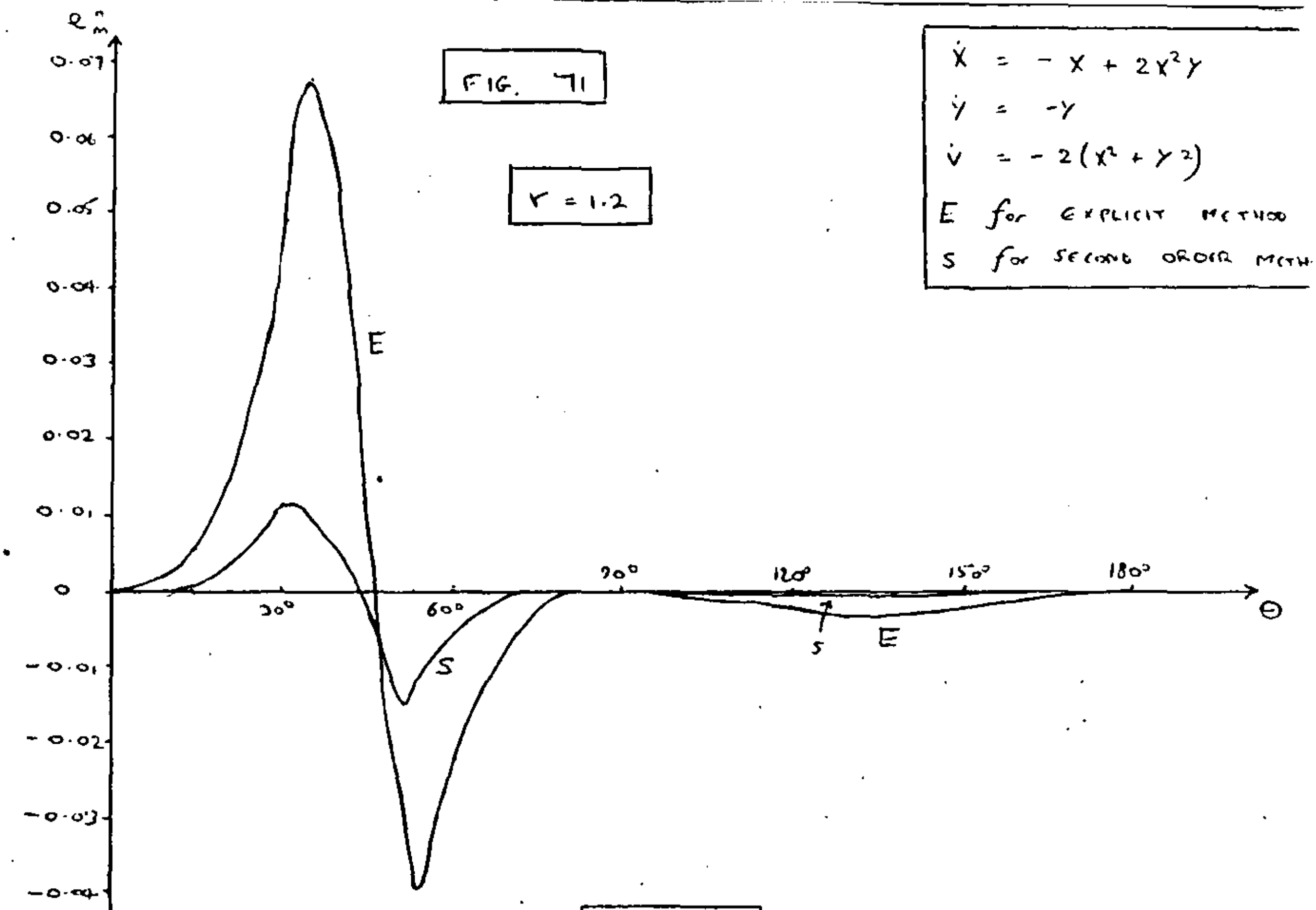


FIG. 71

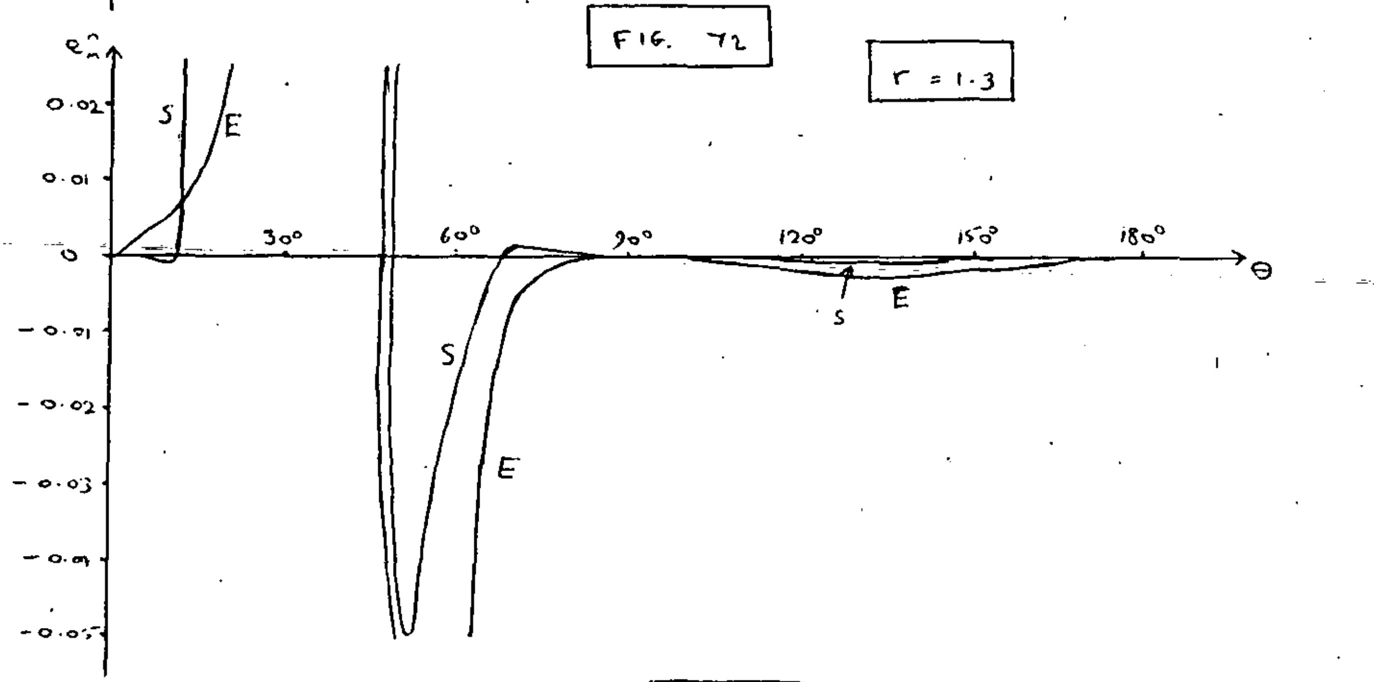


FIG. 72

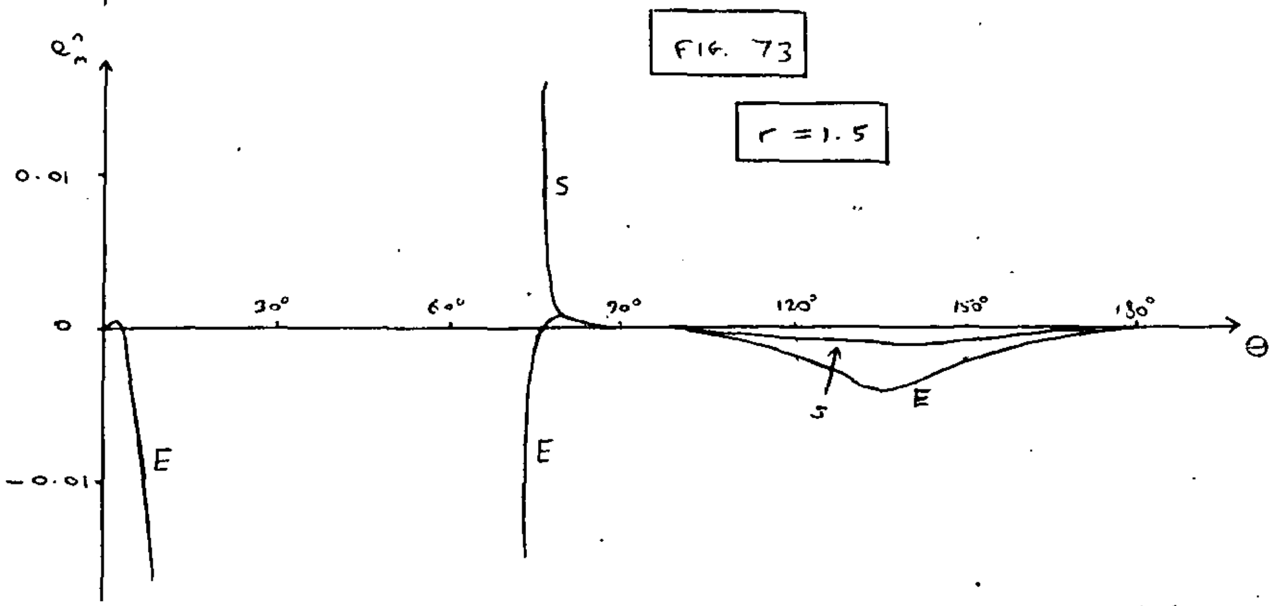


FIG. 73

$$\dot{x} = -x + 2x^2y$$
$$\dot{y} = -y$$
$$\dot{V} = -2(x^2 + y^2)$$

FIG. 75.

RUNGE-KUTTA METHOD.

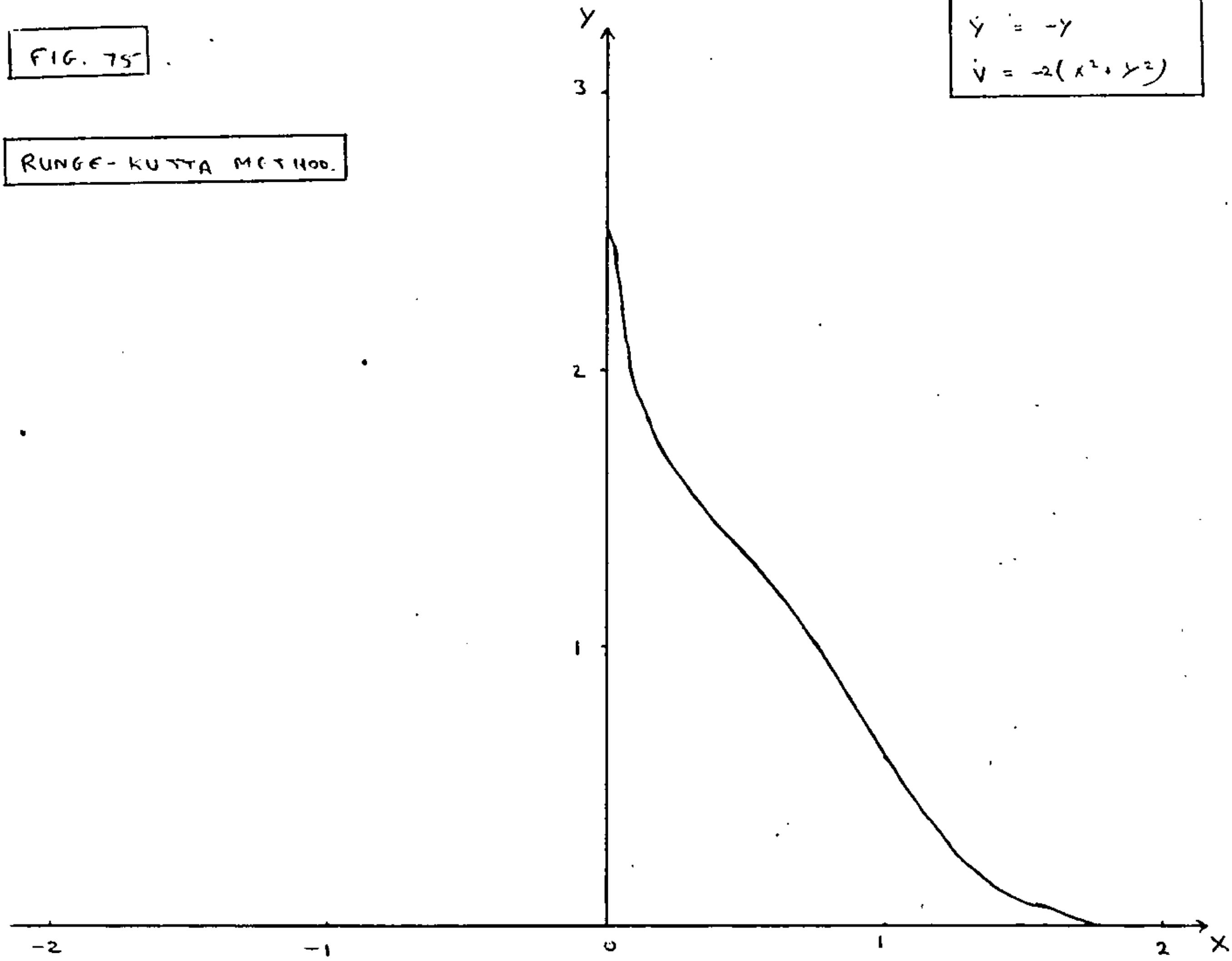
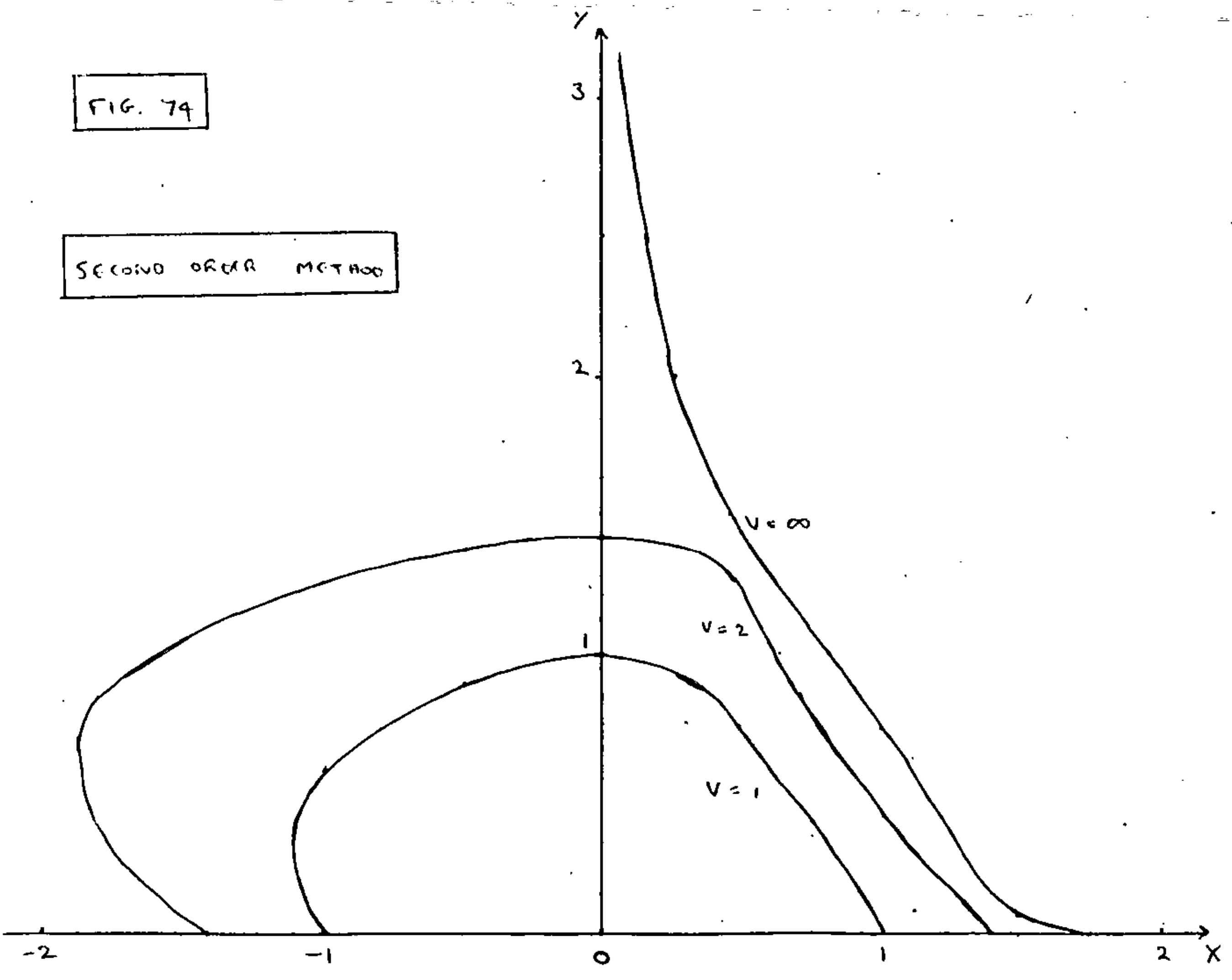


FIG. 74

SECOND ORDER METHOD



Chapter 6

Solution on Characteristics

1. Introduction

In previous chapters we have seen that various methods of solving Zubov's equation have been considered, and each have had problems of some variety. The series construction and the Lie series procedure each had convergence problems, while numerical methods have problems of stability and accuracy. In this chapter a numerical method is presented which overcomes the stability problems by computing solutions in a different way. We do not suggest that this method cannot be improved upon, and further possibilities are considered later in the chapter.

Previously considered methods attempt to use the given initial conditions for Lyapunov functions

$$V(\underline{0}) = 0 \qquad 6.1.1$$

or some close approximation to it. However when computing the Zubov equation

$$\underline{f}(\underline{x})^T \frac{\partial V}{\partial \underline{x}} = - \phi(\underline{x})(1 - dV(\underline{x})) \qquad 6.1.2$$

from the initial conditions 6.1.1 to try to obtain the contour where $V = \infty$ or $V = 1$ it has been seen that the computation becomes unstable especially on a grid where neighbouring values of $V(\underline{x})$ are large near the boundary of the D.O.A., and the accuracy of the numerical methods is not so good.

In this chapter we develop a method of computing which is initiated near the boundary of the D.O.A.. The method then computes trajectories which either tend to the origin or away from it, depending on where the computation is initiated. The problems of numerical instability are largely eliminated.

To illustrate this particular type of situation we consider the numerical computation of the quantity y_n which satisfies

$$y_{n+1} - 10\cdot 1 y_n + y_{n-1} = -1\cdot 35n$$

$$n = 1,2,3,4 \qquad 6.1.3$$

$$y_0 = 0, \; y_5 = 0\cdot 8333.$$

Fox and Mayers (45) solve 6.1.3 in two ways, one of which is unstable and the other is stable. The unstable method involves computing two series while the stable method for this problem involves simultaneous equations. There are other examples of this phenomenon, but this one illustrates the point.

First of all, in section 2, justification of this method of computation is attempted. The details of the algorithm are explained in sections 3 to 7 and the actual computer program is explained in section 8.

This algorithm is developed in 2 dimensions and in section 9 the possibility of applying it to higher orders is considered although generalisation is not as easy as with finite difference methods.

The chapter is concluded with various examples in section 10 to show how far the algorithm is developed, and then there are conclusions and further possibilities in section 11.

2. Justification

The algorithm presented in this chapter is different from other methods and requires some justification. There are seven areas where justification for this method can be made.

2.1 D.O.A. Inside a Bounded Set

The methods based on series ideas compute stability regions which are bounded but the D.O.A. may be unbounded. The grid methods also have a problem when trying to compute unbounded D.O.A.s in that we have to decide when to stop computation.

In either method we have a problem of deciding when we have computed the D.O.A. to sufficient accuracy, particularly if the D.O.A. is unbounded. In this chapter no attempt is made to compute complete D.O.A.s. The stability regions indicated by this method are an approximation to $S_R \cap D(\underline{f})$ where

$$S_R = \left\{ \underline{x} : |\underline{x}| \leq R \right\}. \qquad 6.2.1$$

There are examples for which the D.O.A.s are much further from the origin on one side than the other, but the indicated regions of asymptotic stability are nearly circular. The example

$$\dot{x} = y$$
$$\dot{y} = -x(1 + y) - y(1 - y^2)$$

from Shields (28) is such a case. The R.A.S.s computed by Shields fall short of the D.O.A. due to the boundary being much closer to the origin on one side than the other side.

However in this chapter, the algorithm used computes stability regions inside a bounded region and does not have this convergence problem.

2.2 Convergence and Conservativeness

The algorithm is conservative in that any points outside the D.O.A. will not be taken to be inside it. However it is still the case that some areas inside the D.O.A. will be outside the R.A.S. computed. However letting certain parameters of this algorithm approach zero or infinity will enable the boundary of the D.O.A. to be approached arbitrarily by the computed R.A.S.. In particular by letting step-sizes and accuracy parameters of numerical computation become zero, and the computed boundary to be obtained on V = p where p becomes large.

For convergence we shall consider the Hahn example

$$\dot{x} = -x + 2x^2y$$
$$\dot{y} = -y \qquad 6.2.2$$

Using $\phi(x,y) = 2x^2 + 2y^2$ 6.2.3

and d = 0, the analytical solution of the Zubov equation is given by $V(x,y) = y^2 + x^2/(1-xy)$ showing that the D.O.A. is given by $xy < 1$. 6.2.4

However the series construction gives

$$V^{(2N)}(x,y) = y^2 + x^2\sum_{i=0}^{N-1}(xy)^i \qquad 6.2.5$$

The region of convergence $R_c(\phi,\underline{f})$ of 6.2.5 is in fact

$$|xy| < 1 \qquad 6.2.6$$

and if $R_c(\phi,\underline{f}) \subset D(\underline{f})$ then a conjecture of Shields and Storey (38) suggests that $R_N(\phi,\underline{f})$ does not converge uniformly to $D(\underline{f})$ as $N \rightarrow \infty$. Shields (28) shows that the regions $R_{2N}(\phi,\underline{f})$ indicated by the Lyapunov functions 6.2.5 with $\phi,\underline{f}$ given by 6.2.3 and 6.2.2 respectively lie inside 6.2.6 for even N and increase slowly for odd N.

Numerical methods, however, have no such problems as convergence depends on being able to make the algorithm parameters small or large as appropriate, which may be achieved up to the limits of computer capability.

2.3 Arbitrary S_R

Attempts to compute unbounded D.O.A.s accurately have been seen to be possibly a difficult task from aspects such as the shape of the D.O.A. and convergence. However it is unnecessary to determine the whole boundary of an unbounded D.O.A., since in practical situations the initial conditions will be within a finite domain on which we can place a bound.

Certain physical limitations mean that the initial conditions must be constrained to some extent and so there must exist an upper bound for R in 6.2.1 beyond which we would not be interested because such initial conditions will not arise. This bounded value is different for each problem, and while having justified computation of $S_R \cap D(\underline{f})$ rather than $D(\underline{f})$ we can make R arbitrarily large (subject again to computer limitations).

2.4 Efficiency and Speed

This algorithm is relatively quick and straightforward to carry out as it is systematic in effectively reducing a 2-dimensional problem to two inter-linked one-dimensional problems. The grid methods of Chapter 4 are simpler to compute if values of V are required. But to attain the same accuracy as this algorithm, the grid methods become very complicated to actually locate the boundary of the D.O.A.. In concentrating on characteristics in this chapter we do not encounter the problem of deciding where the boundary intersects with a grid.

2.5 Stability

We have already seen in Chapter 4 that instability caused by trying to obtain the contour $V = \infty$ may be overcome by tackling the problem from the other direction. A similar technique is used here.

It was noted in Chapter 5 that to obtain the contour $V = p$ we have to try to find grid points such that $V_m^n < p < V_m^{n+1}$ and $V_m^n < p < V_{m+1}^n$ whilst hoping that neither $((m+1)h, \frac{2\pi n}{N})$ or $(mh, \frac{2\pi(n+1)}{N})$ are outside the D.O.A.. It is much more stable to select a point (r_0, θ_0) or (x_0, y_0) and compute in such a way as to decide whether (x_0, y_0) is inside the D.O.A. or not.

2.6 Choice of $\phi(x,y)$

We may write down the system equations as

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y) \qquad 6.2.7$$

and also write down the Zubov equation in the form

$$\frac{dV}{dt} = -\phi(x,y)(1 - dV).$$

The method of this chapter is based on solving the equations

$$\frac{dx}{dV} = \frac{-f(x,y)}{\phi(x,y)(1-dV)} \qquad 6.2.8$$

$$\frac{dy}{dV} = \frac{-g(x,y)}{\phi(x,y)(1-dV)}.$$

Now it may be possible to choose $\phi(x,y)$ in such a way as to make solution of 6.2.8 significantly easier than 6.2.7. This in fact will seldom be possible, but consider the example

$$\dot{x} = -x^3 - xy^2$$
$$\dot{y} = -x^2y - y^3 \qquad 6.2.9$$

using $\phi(x,y) = x^2 + y^2$ 6.2.10

we find that the corresponding version of 6.2.8 becomes

$$\frac{dx}{dV} = x$$

$$\frac{dy}{dV} = y$$

which is much easier to solve. However we may notice that for example 6.2.9, 6.2.10 $P(f) = P(g) = 3$, $P(\phi) = 2$ which by theorem 3.6.5 means that $V(x,y)$ is not a Lyapunov function.

2.7 Finite Computation

The question arises that if the method is based on characteristics then why not compute the solutions of 6.2.7 and forget all about $V(x,y)$? The answer lies in the fact that $x(t)$, $y(t) \to 0$ as $t \to \infty$. This means that it is possible to decide that a point $(x_0,y_0) \in D(f,g)$ because the characteristics through (x_0,y_0) appear to be approaching the origin if computation is terminated at some condition such as $t = T$ or $\|\underline{x}(t)\| < \varepsilon$ for some $\varepsilon > 0$, although they might eventually be found to be unstable as $t \to \infty$. Vice-versa we may mistakenly decide $(x_0,y_0) \notin D(f,g)$ as characteristics through (x_0,y_0) initially do not approach the origin and computation is terminated when $t = T$ or $\|\underline{x}\| > R$ for some large R.

However the use of V overcomes this problem since $x(V),y(V) \to 0$ as $V \to 0$, and $x(V),y(V)$ become complex for $V < 0$. These properties of $x(V),y(V)$ enable us to decide on limited computation whether $(x_0,y_0) \in D(f,g)$ or not.

3. Numerical Integration

The system equations in m dimensions are given by

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

or in scalar form as

$$\dot{x}_i = f_i(\underline{x}) \qquad 6.3.1$$

$i = 1,2,\ldots,m.$

We assume that the origin is a critical point when we consider the stability of the origin

$$\text{i.e.}\ f_i(\underline{0}) = 0 \qquad i = 1,2,\ldots,m. \tag{6.3.2}$$

It was noted in theorem 1.8.1 that Zubov's equation

$$\dot{V} = -\phi(\underline{x})(1 - dV) \qquad d = 0 \text{ or } 1 \tag{6.3.3}$$

may be solved by converting 6.3.3 to the auxiliary equations given by

$$\frac{dx_1}{f_1(\underline{x})} = \ldots = \frac{dx_m}{f_m(\underline{x})} = \frac{-dV}{\phi(\underline{x})(1-dV)} \qquad d = 0 \text{ or } 1. \tag{6.3.4}$$

Simple re-arrangement of 6.3.4 gives a new system of equations

$$\frac{dx_i}{dV} = \frac{-f_i(\underline{x})}{\phi(\underline{x})(1-eV)} \qquad i = 1,\ldots,m,\ e = 0 \text{ or } 1. \tag{6.3.5}$$

Hence we see that by including the denominator of 6.3.5 in the R.H.S. we have converted a system 6.3.1 of O.D.E.s for x_i in terms of t into a system 6.3.5 for x_i in terms of V. We wish to consider the differences between 6.3.1 and 6.3.5.

Suppose the origin of 6.3.1 is stable and the initial conditions $x_i(0)$, $i = 1,2,\ldots,m$, lie within the boundary of the D.O.A. That is $(\underline{x}(0)) \in D(\underline{f})$ where D is the domain of attraction for given $\underline{f}$. Then according to definition 1.4.6 the solutions $x_i(t)$, $i = 1,2,\ldots,m$, of 6.3.1 approach the origin as $t \to \infty$. Let us denote the solutions of 6.3.5 by $x_i(V)$, $i = 1,2,\ldots,m$. The m terms on the L.H.S. of 6.3.4 are the same as would be obtained from 6.3.1 by eliminating t. This shows that the trajectories followed by $x_i(V)$, $i = 1,\ldots,m$, are the same as those traversed by $x_i(t)$, $i = 1,\ldots,m$. It remains to consider the variation of $x_i(V)$ with resect to V.

Now by reference to theorem 3.6.5 we know that providing $P(\phi) > P(\underline{f}) - 1$ and the origin is asymptotically stable then the Lyapunov function $V(\underline{x})$ which by theorem 1.7.12 is the unique solution of 6.1.2 is positive definite in the D.O.A..

That is

$$V(\underline{x}) > 0 \quad \text{for } \underline{x} \in D(\underline{f}),\ \underline{x} \neq \underline{0}. \qquad V(\underline{0}) = 0. \tag{6.3.6}$$

Also we refer to theorem 1.7.6 which says that system trajectories $x_i(t)$, $i = 1,\dots,m$, cross contours $V(\underline{x}) = p$ once only as $t \to \infty$ and that these contours are crossed with V decreasing for an asymptotically stable system. Since we have already seen that the trajectories $x_i(V)$ are the same as those of $x_i(t)$, $i = 1,\dots,m$, then we see that the trajectories $x_i(V)$ cross the contours $V(\underline{x}) = p$ once only for decreasing V.

Hence for an asymptotically stable system $x_i(V) \to 0$ as V decreases. But by 6.3.6 $V(\underline{0}) = 0$ and hence $x_i(0) = 0$. Thus we have shown that $x_i(V) \to 0$ as $V \to 0+$. The question arises as to what happens when $V < 0$ as 6.3.6 holds for all $\underline{x}$ in a neighbourhood of the origin? $V(\underline{x})$ and $x_i(V)$, $i = 1,\dots,m$, are continuous functions and must be continuous as V passes from positive to negative.

The contour given by

$$V(\underline{x}) = p \qquad 6.3.7$$

may be solved for $\underline{x}$ given $p < 0$. By theorem 1.7.9 we know that V is a continuous function of $\underline{x}$. This implies that given V_2 in a neighbourhood of V_1 then there exists $\underline{x}_2$ in a neighbourhood of $\underline{x}_1$ where

$$V_1 = V(\underline{x}_1)$$
$$V_2 = V(\underline{x}_2)$$

for small $\delta > 0$, where $|V_2 - V_1| < \delta$, providing that $\underline{x}$ exists for such V. Now in the field of real numbers we see for p small and negative there is no $\underline{x}$ in the neighbourhood of $\underline{x} = \underline{0}$ satisfying 6.3.7 . However when we consider the complex numbers there is such $\underline{x}$ satisfying 6.3.7 in a neighbourhood of the origin for $p < 0$. It is a fundamental property of complex numbers that except for singularities where $\|x\| \to \infty$ as $V \to V_1$ or $\underline{x}$ is indeterminate then $\underline{x}$ is continuous in V and can be evaluated from a known function. The origin is not a singularity as $\underline{x}(0) = \underline{0}$ is quite well-defined and so $\underline{x}$ is continuous in V for negative V as well as for positive V.

Integrating 6.3.5 with respect to V gives

$$x_i(V) = x_i(V_o) - \int_{V_o}^{V} \frac{f_i(\underline{x})}{\phi(\underline{x})(1-dV)} dV \qquad 6.3.8$$

$i = 1,\dots,m.$

In this algorithm we are interested in evaluating 6.3.8 for decreasing V. Substituting $V = V_o - p$ in 6.3.8 gives

$$x_i(V_o - p) = x_i(V_o) - \int_{V_o}^{V_o - p} \frac{f_i(\underline{x})}{\phi(\underline{x})(1-dV)} dV \qquad 6.3.9$$

$i = 1,\dots,m.$

Now we have seen that if $V_o > p$ then $x_i(V_o - p)$ is real and continuous with respect to V_o, while if $V_o < p$ then $x_i(V_o - p)$ is complex and continuous in V_o.

However we are concerned with numerical integration of 6.3.5 or 6.3.8. This may be done by any of the finite difference, Runge-Kutta or other schemes (52), (53), (54), (55) but the results of numerical computation of real functions are always real. When $V_o < p$ then integrating 6.3.5 numerically for decreasing V requires evaluation of the R.H.S. of 6.3.5 when $\underline{x}$ is in a neighbourhood of the origin. The results of such computation are the subject of the next section.

4. Theory of Negative V

For consideration of numerical integration of 6.3.5 we shall confine ourselves to 2 dimensions, where we write the system equations as

$$\dot{x} = f(x,y) \qquad 6.4.1$$

$$\dot{y} = g(x,y) \qquad 6.4.2$$

and Zubov's equation as

$$\dot{V} = -\phi(x,y)(1 - dV) \qquad 6.4.3$$

$$d = 0,1.$$

Substituting 6.4.1, 6.4.2, 6.4.3 into 6.3.5 we obtain

$$\frac{dx}{dV} = \frac{-f(x,y)}{\phi(x,y)(1-dV)} \qquad 6.4.4$$

$$\frac{dy}{dV} = \frac{-g(x,y)}{\phi(x,y)(1-dV)} \qquad 6.4.5$$

6.4.4 and 6.4.5 are the equations we require to solve for decreasing V from the initial conditions

$$x_o = x(V_o) \qquad 6.4.6$$

$$y_o = y(V_o) \qquad 6.4.7$$

Let h be the step-size in V used in the numerical computation and define $\hat{x}^{(j)}$ as the computed value of $x(V_o - jh)$ with the initial condition $\hat{x}^{(0)} = x_o = x(V_o)$ and similarly for $\hat{y}^{(j)}$, $\hat{V}^{(j)}$, $j = 1,2,\ldots,n$, $nh = p$.

In section 3 we saw that $x(V_o - jh)$, $y(V_o - jh)$ are real for $V_o - jh > 0$. Hence $\hat{x}^{(j)}$, $\hat{y}^{(j)}$ will be computed as approximations to $x(V_o - jh)$, $y(V_o - jh)$ to the order of accuracy of the method used, while $V_o - jh > 0$.

Now using theorem 3.2.1 and theorem 3.6.5 as applied to 6.4.1 to 6.4.3 we have

$$P(\phi) > P(f,g) - 1$$
$$P(f) \geqslant P(f,g) \qquad 6.4.8$$
$$P(g) \geqslant P(f,g)$$

with equality in at least one of the equations 6.4.8. f/ϕ and g/ϕ are indeterminate at the origin, but 6.4.8 implies that system 6.4.4, 6.4.5 does not satisfy a Lipschitz condition for $(x,y) \in S_\varepsilon$ for all $\varepsilon > 0$ and so 6.4.4, 6.4.5 do not necessarily have unique solutions in S_ε for all $\varepsilon > 0$. If f/ϕ and g/ϕ tend to finite limits along some trajectory then $x(V), y(V)$ are continuous and the solutions $\hat{x}^{(j)}, \hat{y}^{(j)}$, $j = 1,..,n$, will appear reasonably smooth. This case is not considered as $\phi(x,y)$ may be chosen such that f/ϕ or g/ϕ become infinite at the origin. This is no great restriction as theorem 3.6.6 shows that problems for which the Zubov series construction is possible guarantees the above requirement. It will be seen that the analysis of integration can still hold if f/ϕ, g/ϕ remain finite as $(x,y) \rightarrow (0,0)$ but requires more care in distinguishing stable and unstable systems.

We now consider the stage of integration when V passes from positive to negative.

Let j be such that

$$V_o = jh + e \quad \text{where } 0 \leqslant e < h. \qquad 6.4.9$$

Define x_e, y_e such that

$$x_e = x(V_o - jh) = x(e) \qquad 6.4.10$$

$$y_e = y(V_o - jh) = y(e) \qquad 6.4.11$$

Let us assume that no truncation errors have occurred in the first j stages of integration.

i.e. $\hat{x}^{(j)} = x_e$

$$\hat{y}^{(j)} = y_e$$

$$\hat{V}^{(j)} = e.$$

We are interested in the (j+1)th step where $\hat{V}^{(j+1)} = e - h < 0$. To illustrate the behaviour of $\hat{x}^{(j+1)}$, $\hat{y}^{(j+1)}$ we look at Euler's method specifically. Applied to 6.4.4, 6.4.5 this become

$$\hat{x}^{(j+1)} = \hat{x}^{(j)} + \frac{hf(\hat{x}^{(j)}, \hat{y}^{(j)})}{\phi(\hat{x}^{(j)}, \hat{y}^{(j)})(1 - d\hat{V}^{(j)})} \qquad 6.4.12$$

$$\hat{y}^{(j+1)} = \hat{y}^{(j)} + \frac{hg(\hat{x}^{(j)},\hat{y}^{(j)})}{\phi(\hat{x}^{(j)},\hat{y}^{(j)})(1-d\hat{v}^{(j)})} \qquad 6.4.13$$

$$\hat{v}^{(j+1)} = \hat{v}^{(j)} - h. \qquad 6.4.14$$

Now as $e \to 0$, $x_e \to o$ and $y_e \to 0$. Hence either $\dfrac{f(x_e,y_e)}{\phi(x_e,y_e)}$ or $\dfrac{g(x_e,y_e)}{\phi(x_e,y_e)} \to \pm\infty$ or both as $e \to 0$ by the assumption on choice of $\phi(x,y)$.

Hence we see by letting $\hat{x}^{(j)},\hat{y}^{(j)} \to 0$ in 6.4.12, 6.4.13 that either $\hat{x}^{(j+1)} \to \pm\infty$ or $\hat{y}^{(j+1)} \to \pm\infty$ or both. By reference to equation 6.4.9 and the definition of j and e, we see that as V_o varies the computation of $\hat{x}^{(n)},\hat{y}^{(n)}$ will have discontinuities occurring for such V_o for which $e = 0$.

Throughout this section it has been assumed that 6.4.1, 6.4.2 is a stable system and that $(x_o,y_o) \in D(f,g)$. If however $(x_o,y_o) \notin D(f,g)$ or $D(f,g)$ does not exist then trajectories $x(V)$, $y(V)$ will not approach the origin. Since $\phi(x,y) \neq 0$ for $(x,y) \neq 0$, then there will be no such discontinuities in $\hat{x}^{(n)}$, $\hat{y}^{(n)}$.

Having shown that the computed values $\hat{x}^{(n)}$, $\hat{y}^{(n)}$ of $x(V_o - p)$, $y(V_o - p)$, $(nh = p)$, will be discontinuous when $V_o = jh$, $1 \leq j \leq n-1$, we consider instead variation of (x_o,y_o). For each (x_o,y_o) there will be a corresponding V_o (6.4.6, 6.4.7) but V_o will not generally be known. However the behaviour of $\hat{x}^{(n)}$, $\hat{y}^{(n)}$ as (x_o,y_o) vary can be analysed to establish V_o.

For this algorithm we consider (x_o,y_o) varying along a radial line from the origin.

We write $x_o = r_o\cos\theta$

$y_o = r_o\sin\theta$

where θ is fixed and r_o is allowed to be variable.
For given x_o,y_o,h,p 6.4.4, 6.4.5 may be integrated numerically to obtain $\hat{x}^{(n)},\hat{y}^{(n)}$ as approximations to $x(V_o - p)$, $y(V_o - p)$ where $nh = p$.

Let us define

$$W(r_o,\theta,h,p) = \left\| (\hat{x}^{(n)},\hat{y}^{(n)}) \right\| \qquad 6.4.15$$

It is the variation of W with respect to r_o for fixed θ,h,p that we analyse. We have shown that if $(r,\theta) \in D(f,g)$ and $V_o < p$ then for varying r_o (hence for varying V_o) W is expected to have discontinuities corresponding to $V_o = jh$ for the Euler method and to any V_o requiring computation of the R.H.S. of 6.4.4, 6.4.5 for small (x,y) for other methods. Since f/ϕ, g/ϕ are never actually computed at the origin, these theoretical discontinuities become high frequency oscillations in practice.

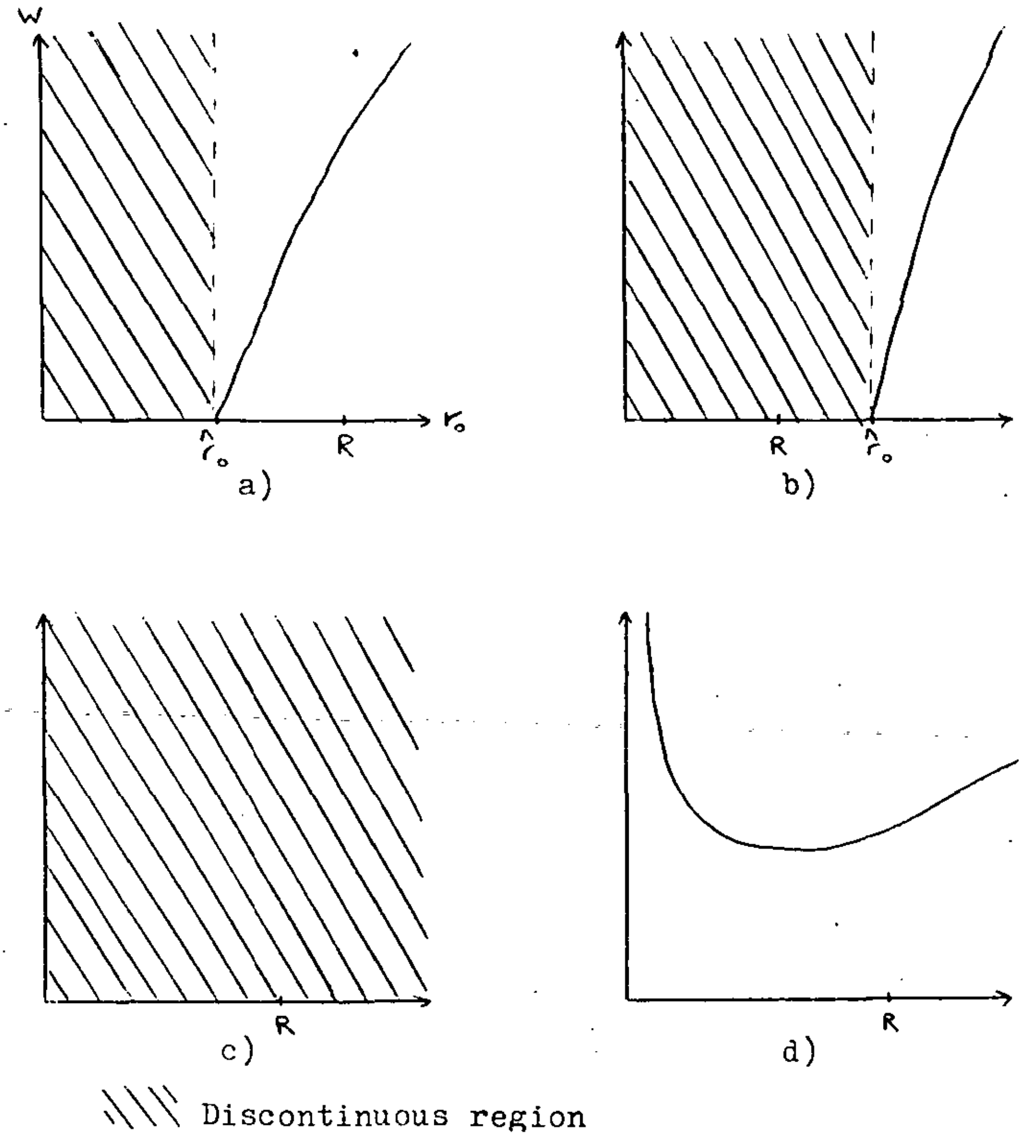


Fig. 76

Figs. 76a,b show typical variation of W for a system with asymptotically stable origin but for which there exists $(\hat{r}_o, \theta)$ such that

$$\hat{x}_o = \hat{r}_o \cos\theta$$
$$\hat{y}_o = \hat{r}_o \sin\theta$$
$$\text{and} \quad \hat{V}_o = p.$$

If 6.4.4, 6.4.5 could be integrated without error this would give

$$\hat{x}^{(n)} = x(V_o - p) = 0$$
$$\hat{y}^{(n)} = y(V_o - p) = 0$$

and $\quad W(\hat{r}_o,\theta,h,p) = 0.$

However numerical integration without error cannot be done and we have to try to obtain $\hat{r}_o$ numerically. That is the subject of the next section.

Fig. 76c shows the variation of W for a system where the origin is asymptotically stable in the whole and p is chosen large enough so that $V_o < p$ within the range of r_o chosen. Fig. 76d shows the smooth variation for an unstable system. Note: if $r_o = 0$ then $\hat{x}^{(j)}$, $\hat{y}^{(j)}$ should be computed to be at zero for all $j \leq n$, as $f(0,0) = 0$, $g(0,0) = 0$. But as would be expected if f/ϕ or g/ϕ become infinite at the origin we observe that $W(r_o,\theta,h,p) \to \infty$ as $r_o \to 0$. If in certain cases f/ϕ, g/ϕ are finite expressions for $(x,y) \in S_\varepsilon$ for all $\varepsilon > 0$, then fig. 76d is slightly amended so that $W(r_o,\theta,h,p)$ remains finite also. Likewise the discontinuous regions of fig. 76a,b,c are amended to indicate smooth variation of W with respect to r_o.

Before we leave this section a note should be made of the possible analysis of the behaviour of W with respect to r_o.

The system trajectories of 6.4.1, 6.4.2 are given by

$$S(x,y) = c \qquad 6.4.16$$

where c is arbitrary. Also the Lyapunov function $V(x,y)$ is given by solutions of 6.4.3 with $V(0,0) = 0$.

Substituting $V_o = V(r_o\cos\theta,\ r_o\sin\theta)$ into 6.4.9 we obtain

$$V(r_o\cos\theta,\ r_o\sin\theta) = jh + e, \qquad 6.4.17$$
$$0 \leq e < h.$$

Also from the definitions 6.4.10,6.4.11

$$e = V(x_e,y_e). \qquad 6.4.18$$

From 6.4.16 we have

$$S(r_o\cos\theta,\ r_o\sin\theta) = c \qquad 6.4.19$$

and $\quad S(x_e,y_e) = c. \qquad 6.4.20$

Equations 6.4.17 to 6.4.20 represent 4 equations for c,e,x_e,y_e in terms of θ,h,r_o.

Hence for fixed θ,h we see that c,e,x_e,y_e may be obtained as functions of r_o, then reference to 6.4.12, 6.4.13 or the corresponding equations for a different numerical method will yield information about $\hat{x}^{(j+1)}$, $\hat{y}^{(j+1)}$ and hence $\hat{x}^{(n)}$, $\hat{y}^{(n)}$ and W.

4.1 Example

Consider the Hahn system

$$\dot{x} = -x + 2x^2y \qquad 6.4.21$$

$$\dot{y} = -y \qquad 6.4.22$$

Solving Zubov's equation with d = 0, $\phi(x,y) = 2(x^2 + y^2)$

we obtain $$V(x,y) = y^2 + \frac{x^2}{1-xy} \qquad 6.4.23$$

as the analytic solution.

The system trajectories are given by

$$\frac{dy}{dx} = \frac{y}{x-2x^2y}$$

which has a solution $\frac{x}{y(1-xy)} = c$.

Hence the equations 6.4.17 to 6.4.20 become for this example

$$r_o^2\sin^2\theta + \frac{r_o^2\cos^2\theta}{1-r_o^2\sin\theta\cos\theta} = jh + e$$

$$y_e^2 + \frac{x_e^2}{1-x_ey_e} = e \qquad 0 \leq e < h \qquad 6.4.24$$

$$\frac{\cos\theta}{\sin\theta(1-r_o^2\sin\theta\cos\theta)} = c$$

$$\frac{x_e}{y_e(1-x_ey_e)} = c \qquad 6.4.25$$

The equations 6.4.4, 6.4.5 for this example become

$$\frac{dx}{dV} = \frac{x - 2x^2y}{2x^2 + 2y^2} \qquad 6.4.26$$

$$\frac{dy}{dV} = \frac{y}{2x^2 + 2y^2} \qquad 6.4.27$$

The behaviour of the system close to the origin was investigated. For h = 0·25, c = 5, 6.4.24 and 6.4.25 were solved to obtain x_e, y_e for different e. Then from 6.4.26, 6.4.27 the Fourth Order Runge-Kutta method was used to obtain $\hat{x}^{(j+1)}, \hat{y}^{(j+1)}$ given $\hat{x}^{(j)} = x_e$, $\hat{y}^{(j)} = y_e$. Fig. 88 shows the plot of sign $\hat{x}^{(j+1)} \| (\hat{x}^{(j+1)}, \hat{y}^{(j+1)}) \|^e$ against e. It is noticeable that there are 3 distinct breakpoints, and these are predictable by the method used.

The definition of the Fourth Order Ruge-Kutta method for a system

$$\frac{d\underline{x}}{dV} = \underline{F}(\underline{x}, V)$$

is $\hat{\underline{x}}^{(j+1)} = \hat{\underline{x}}^{(j)} + \frac{h}{6}(\underline{k}_1^{(j)} + 2\underline{k}_2^{(j)} + 2\underline{k}_3^{(j)} + \underline{k}_4^{(j)})$

$$\begin{aligned} \text{where}\quad \underline{k}_1^{(j)} &= \underline{F}(\hat{\underline{x}}^{(j)}, \hat{V}^{(j)}) \\ \underline{k}_2^{(j)} &= \underline{F}(\hat{\underline{x}}^{(j)} + h\underline{k}_1^{(j)}/2,\ \hat{V}^{(j)} + h/2) \\ \underline{k}_3^{(j)} &= \underline{F}(\hat{\underline{x}}^{(j)} + h\underline{k}_2^{(j)}/2,\ \hat{V}^{(j)} + h/2) \\ \underline{k}_4^{(j)} &= \underline{F}(\hat{\underline{x}}^{(j)} + h\underline{k}_3^{(j)},\ \hat{V}^{(j)} + h) \end{aligned} \qquad 6.4.28$$

$$j = 0,\ldots,n-1.$$

Applying 6.4.28 to 6.4.26, 6.4.27 the breakpoints must occur where any one of 6.4.28 becomes large, and this explains the nature of the fig. 88.

Another point of interest is the question of whether we can tie down regions of $\hat{x}^{(j)}$, $\hat{y}^{(j)}$ such that $\hat{x}^{(j+1)}$, $\hat{y}^{(j+1)}$ are of opposite sign. That is, for the Euler method given by 6.4.12, 6.4.13 we are interested to find (x,y) such that

$$x\,\phi(x,y) = -h\ f(x,y) \qquad 6.4.29$$

and (x,y) such that

$$y\,\phi(x,y) = -h\ g(x,y) \qquad 6.4.30$$

Substituting 6.4.21, 6.4.22, 6.4.23 into 6.4.29, 6.4.30 and re-arranging we obtain that $\hat{x}^{(j+1)}$ changes sign when $(\hat{x}^{(j)}, \hat{y}^{(j)})$ satisfy

$$x^2 + hxy + y^2 = h/2 \qquad 6.4.31$$

and $\hat{y}^{(j+1)}$ changes sign when $(\hat{x}^{(j)}, \hat{y}^{(j)})$ satisfy

$$x^2 + y^2 = h/2 \qquad 6.4.32$$

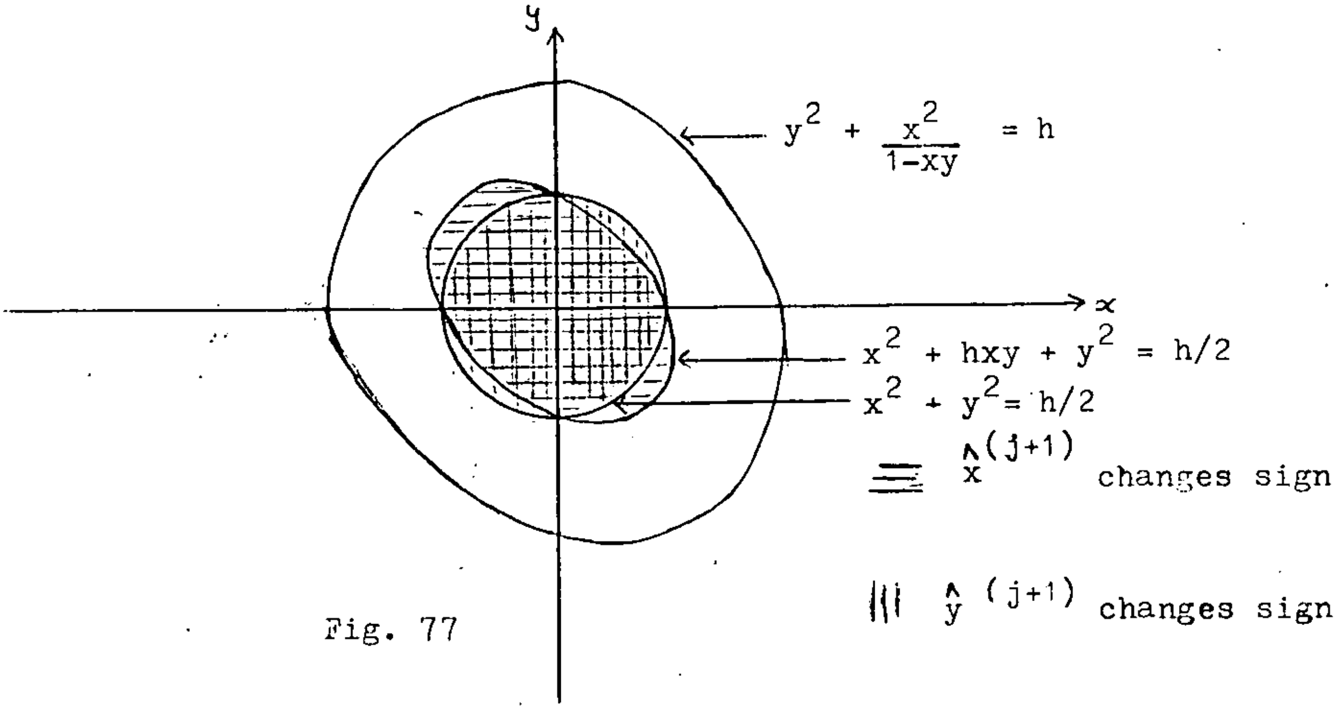


Fig. 77

Fig. 77 shows the regions given by 6.4.31 and 6.4.32. Also shown is the contour $V(x,y) = h$ which shows that Euler's method is not accurate.

The final point to note is that since we know that if $(\hat{x}^{(j)},\hat{y}^{(j)})$ is in a neighbourhood of the origin then $(\hat{x}^{(j+1)},\hat{y}^{(j+1)})$ is liable to be a long way from the origin. Now if $(\hat{x}^{(j+1)},\hat{y}^{(j+1)})$ are such that $(\hat{x}^{(j+1)},\hat{y}^{(j+1)}) \in D(f,g)$ then further computation to obtain $\hat{x}^{(n)}$, $\hat{y}^{(n)}$ and W proceeds along the system trajectory given by

$$S(x,y) = S(\hat{x}^{(j+1)},\hat{y}^{(j+1)}).$$

However if $(\hat{x}^{(j+1)},\hat{y}^{(j+1)}) \notin D(f,g)$ then computation proceeds along an unstable trajectory and $\hat{x}^{(n)}$, $\hat{y}^{(n)}$, W will be smooth with respect to r_o.

Thus the point of interest concerns whether $(\hat{x}^{(j+1)},\hat{y}^{(j+1)}) \in D(f,g)$.

For the Hahn example the D.O.A. is given by 6.2.4. Using the Euler method 6.4.12, 6.4.13 and the f,g given by 6.4.21, 6.4.22 we require

$$\hat{x}^{(j+1)}\hat{y}^{(j+1)} < 1$$

i.e. $$xy(x^2+y^2+hxy-h/2)(x^2+y^2-h/2) < (x^2+y^2)^2 \qquad 6.4.33$$

where (x,y) represent $(\hat{x}^{(j)},\hat{y}^{(j)})$.

Fig. 89 represents 6.4.33 for $h = 0{\cdot}1$. The region given by 6.4.33 is well within those given by 6.4.31 or 6.4.32. Thus we see that the regions of r_o such that $(\hat{x}^{(n)},\hat{y}^{(n)})$ lies outside the D.O.A. form a small proportion of the total variation of r_o.

5. Computation of $\hat{r}_o$.

The previous section showed how given a fixed r_o,θ,h,p we can integrate 6.4.4, 6.4.5 for n steps, where $nh = p$ to obtain $\hat{x}^{(n)}$, $\hat{y}^{(n)}$, $\hat{V}^{(n)}$ where $\hat{V}^{(n)} = V_o - p$ then defining $W(r_o,\theta,h,p)$ by 6.4.15 we can get a picture of the behaviour of W with respect to r_o as outlined in fig. 76. Now we have to obtain $\hat{r}_o(\theta,h,p)$ as the computed value of r_o where $V_o = p$, hence if the computation was precise

$$\hat{x}^{(n)} = x(V_o - p) = 0$$
$$\hat{y}^{(n)} = y(V_o - p) = 0$$
$$W(\hat{r}_o,\theta,h,p) = 0. \qquad 6.5.1$$

If W were plotted against r_o as in fig. 76 it should be possible to pick out $\hat{r}_o$, and so we require an efficient algorithm for picking out $\hat{r}_o$ from the computed values of W.

From 6.5.1 we expect that $W(\hat{r}_o,\theta,h,p) \rightarrow 0$ as $h \rightarrow 0$ if the numerical method of obtaining W converges to the true value. Now since $W(r_o,\theta,h,p) > 0$ for $V_o - p > 0$ (by 6.3.9) but $W(r_o,\theta,h,p)$ is discontinuous for $V_o - p < 0$, and $W \geqslant 0$ (by 6.4.15) we see that $\hat{r}_o$ is a local minimum of $W(r_o,\theta,h,p)$. In the discontinuous regions of fig. 76a,b,c there will be many local minima. $\hat{r}_o(\theta,h,p)$ is therefore defined as

$$\hat{r}_o(\theta,h,p) = \max r_o \in \left\{ r_o : \frac{\partial W}{\partial r_o}(r_o,\theta,h,p) = 0 \right\}. \qquad 6.5.2$$

The definition 6.5.2 is the criterion used to establish $\hat{r}_o$ from computed $W(r_o,\theta,h,p)$. We start by computing $W(R,\theta,h,p)$ where R is as in 6.2.1, then compute W for decreasing r_o until W increases again. The second stage is to interval halve to obtain $\hat{r}_o$ accurately.

Stage I

Fix R and δr as input parameters.
From computation of 6.4.4, 6.4.5 by method 6.4.12, 6.4.13 or otherwise and the definition 6.4.15 we obtain

$$W(R - i\delta r,\theta,h,p) \text{ for } i = -1,0,1,2,\ldots,I$$

terminating the procedure when

$$W(R - I\delta r,\theta,h,p) \geqslant W(R - (I-1)\delta r,\theta,h,p). \qquad 6.5.3$$

Then we define a minimal system to consist of r_l, r_m, r_u where

$$W(r_l,\theta,h,p) \geqslant W(r_m,\theta,h,p)$$
$$W(r_u,\theta,h,p) \geqslant W(r_m,\theta,h,p).$$

From the terminating criterion 6.5.3 we see that

$$\begin{aligned} r_u &= R - (I-2)\delta r \\ r_m &= R - (I-1)\delta r \\ r_l &= R - I\delta r \end{aligned} \qquad 6.5.4$$

providing $I \geqslant 1$. If 6.5.3 is true for $I = 0$
i.e. $W(R,\theta,h,p) \geqslant W(R + \delta r,\theta,h,p)$
then r_m, r_l are defined as in 6.5.4 but with

$$r_u = R + \delta r + \varepsilon. \qquad 6.5.5$$

Also we define

$$\begin{aligned} W_l &= W(r_l,\theta,h,p) \\ W_m &= W(r_m,\theta,h,p) \\ W_u &= W(r_u,\theta,h,p) \end{aligned} \qquad 6.5.6$$

with $W_u = \infty$ if r_u defined by 6.5.5.

Definition 6.5.1

A minimal system of W with respect to r_o is defined by 6.5.6 where

$$\begin{aligned} &r_l < r_m < r_u \\ &W_l \geqslant W_m \\ &W_u \geqslant W_m. \end{aligned} \qquad 6.5.7$$

At the end of Stage I we have tracked W for decreasing r_o until a minimum point has been reached and in Stage II an interval halving procedure finds $\hat{r}_o$ to a specified accuracy.

Stage II

Given a minimal system with the properties of 6.5.7 the purpose is now to find another system with the same properties but closer together and to continue doing so until

$$r_u - r_l < \varepsilon \qquad 6.5.8$$

where ε is input as a specified accuracy.

Define $r_o = \dfrac{r_m + r_u}{2}$ and compute $W(r_o, \theta, h, p)$. 6.5.9

There are three possible situations depending on the value of W:

a) If $W(r_o,\theta,h,p) \geqslant W_u$ (fig.78) then we know that r_m does not possess the required property of $\hat{r}_o$. The new system is defined as $r_l = r_o$, $r_m = r_u$, $r_u = r$ (taken from a store of previously used r_o). The process is repeated from 6.5.9.

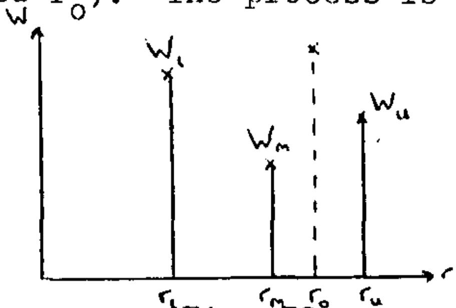


Fig. 78

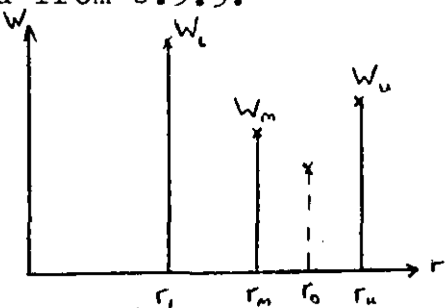


Fig. 79

b) If $W(r_o,\theta,h,p) \leqslant W_m$ (fig. 79) then the new minimal system is defined as $r_l = r_m$, $r_m = r_o$, $r_u = r_u$. The process is also repeated from 6.5.9.

c) If $W_m < W(r_o,\theta,h,p) < W_u$ (fig. 80) then we can define a new system as $r_l = r_l$, $r_m = r_m$, $r_u = r_o$.

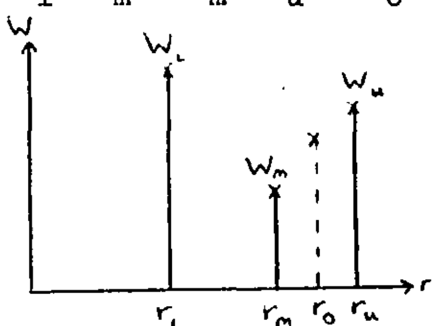


Fig. 80

However the minimum point $\hat{r}_o$ could lie between r_l and r_m and we must check this. Hence we define the new system as above and then put $r_o = \frac{r_l + r_m}{2}$.

There are now two possible cases:

i) If $W(r_o,\theta,h,p) \leq W_m$ then the new system is defined as $r_l = r_l$, $r_m = r_o$, $r_u = r_m$.

or ii) If $W(r_o,\theta,h,p) > W_m$ then the new system is defined as $r_l = r_o$, $r_m = r_m$, $r_u = r_u$.

Now the process is repeated from 6.5.9.

After each case a),b),c) criterion 6.5.8 is checked before r_o is re-defined by 6.5.9. If 6.5.8 is satisfied then the computation is ended and r_m has the property of $\hat{r}_o$ defined by 6.5.2.

Having computed $\hat{r}_o$ we require to decide whether we have a stable or unstable system, and where the boundary of the D.O.A.actually is. $\hat{r}_o$ is an approximation to a point on the contour $V = p$, and p must be chosen to be such that the contour $V = p$ is close enough to the contour $V = \infty$ or $V = 1$. This may be achieved by iteration on p.

There are two possible situations that can occur:

a) $\hat{r}_o(\theta,h,p) < R$ 6.5.10

or b) $r_o(\theta,h,p) \geqslant R$. 6.5.11

If 6.5.11 is true then the situations are as shown in figs.76b,c,d The first two are stable cases while fig. 76d is unstable. The deciding factor is that in figs.76b,c computation of $\hat{r}_o$ meets the discontinuous region. It is almost certain (though not completely) that during computation a "false" minimum arose. That is, case a) of Stage II occurred when a minimum point had to be rejected. Thus if 6.5.11 is true, an unstable system is one in which case a) of Stage II never occurs.

If 6.5.10 is true then the situations are as shown in figs. 76a,d. As before an unstable system never gives rise to case a) of Stage II. But a stable system almost always will give rise to case a).

Thus, in summary, we see that if case a) of Stage II does not arise then the system is almost certainly unstable. Otherwise it is stable and either $\hat{r}_o < R$ or the boundary of the D.O.A. is outside the R-circle.

The small possibilities of error in stability depend on

δr. The larger δr is set to, the less likely a mistake becomes as $R - I\delta r$ is well inside the discontinuous regions of fig. 76 for a stable system. Another approach not investigated here may be to replace Stage I by a system involving computing

$$W(i\delta r,\theta,h,p) \qquad i = 1,2,\ldots,I \qquad 6.5.12$$

terminating when it can be recognised that W is continuous with respect to i or when $i\delta r > R$.

Mention must be made of what may happen when f/ϕ and g/ϕ do not become infinite at the origin. $W(r_0,\theta,h,p)$ is a continuous function of r_0 in this case but since we are looking for r_0 to satisfy 6.5.1 then we may still use the definition 6.5.2. An unstable system could be picked by recognising that for a stable system $W(\hat{r}_0,\theta,h,p) \propto 0$ while for an unstable system $W(\hat{r}_0,\theta,h,p)$ is finite but not small. A criterion such as

$$W(\hat{r}_0,\theta,h,p) < W_0 \qquad 6.5.13$$

could possibly be used to decide stability.

It is recognised that this algorithm is designed to compute boundaries of the D.O.A.s of stable systems and not necessarily to indicate stability or instability, but as seen here and later the likelihood of stability being computed incorrectly is small and can be made arbitrarily small.

Having obtained $\hat{r}_0(\theta,h,p)$ for given θ,h,p we have other parameters to manipulate still and possible variations in h,p are the subject of the next section.

6. Variation of h,p

In section 5 $\hat{r}_0(\theta,h,p)$ was obtained according to the definition 6.5.2 for fixed θ,h,p. In this section we attempt to outline how h,p may be varied to obtain the boundary point more accurately.

It has been noticed that different h yield different values for $\hat{r}_0$. Now we require $\hat{r}_0$ to be a conservative estimate of the boundary point, so it is a reasonable idea to have some means of varying h so that we are sure that $(\hat{r}_0,\theta) \in D(\underline{f})$. Making h small reduces the errors involved in the numerical method but increases the opportunities for instability when computing $\hat{x}^{(j)}$, $\hat{y}^{(j)}$ near the origin. As $h \rightarrow 0$, $\hat{r}_0$ increases since a greater range of r_0 will involve computation near the origin.

To explain this, suppose that the spherical region

$$\hat{x}^{(j)2} + \hat{y}^{(j)2} < \delta^2 \qquad 6.6.1$$

δ small

defines in some way a region where computation of 6.4.4, 6.4.5 yields unstable results. i.e. R.H.S. of 6.4.4, 6.4.5 are large. Also suppose that r_o is such that $V_o = p$. Numerical integration of 6.4.4, 6.4.5 follows a trajectory which hopefully gives $\hat{x}^{(n)} = 0$, $\hat{y}^{(n)} = 0$ if the integration is accurate enough. However $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ is the crucial point. If $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ is inside the domain given by 6,6.1 then computation of $(\hat{x}^{(n)}, \hat{y}^{(n)})$ from $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ is unstable. If there are no errors up to computation of $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ and $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ is outside the δ-circle then $(\hat{x}^{(n)}, \hat{y}^{(n)})$ should be close to the origin. But as $h \to 0, n \to \infty$ and hence $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ tends to $(\hat{x}^{(n)}, \hat{y}^{(n)})$ and hence $(\hat{x}^{(n-1)}, \hat{y}^{(n-1)})$ must for some $h > 0$ lie inside the δ-circle, and hence $(\hat{x}^{(n)}, \hat{y}^{(n)})$ could be some distance from the origin. This analysis shows that as $h \to 0$ the region of discontinuity of fig. 76 becomes larger and $\hat{r}_o$ increases.

Now as h gets large errors occur in computation which also make $W(r_o, \theta, h, p)$ less predictable for a greater range of r_o. Hence $\hat{r}_o$ increases as $h \to 0$ and as $h \to \infty$. Hence there exists a minimum of $\hat{r}_o(\theta, h, p)$ with respect to h.

There are therefore 3 possible basic ways to vary h in the algorithm:

a) Fix h and compute $\hat{r}_o(\theta, h, p)$.

b) Increase n (n=p/h) until

$$\left|\hat{r}_o(\theta, p/n, p) - \hat{r}_o(\theta, p/n+1, p)\right| < \varepsilon.$$

c) Vary h to find the minimum of $\hat{r}_o$ with respect to h.

Now there is no reason why h should be constant. We have already seen that small h near the origin is undesirable. Therefore the best system incorporates variable step-sizes.

The method used in this algorithm is to compute $\hat{r}_o(\theta, \underline{h}, \sum_{j=1}^{i} p_j)$ by the analysis of section 5 of the behaviour of $W(r_o, \theta, \underline{h}, \sum_{j=1}^{i} p_j)$ with respect to r_o. It is the vectors $\underline{h}, \underline{p}$ where

$$\underline{h}^T = (h_1, \ldots, h_i)$$

$$\text{and} \qquad \underline{p}^T = (p_1, \ldots, p_i) \qquad 6.6.2$$

which are varied. Computation of W involves computing $(\hat{x}^{(n)}, \hat{y}^{(n)})$ by 6.4.15 and the numerical integration of 6.4.4, 6.4.5 by a method such as 6.4.12, 6.4.13, 6.4.14 or otherwise is carried out by varying the step-sizes using 6.6.2. i.e. 6.4.14 is replaced by

$$\hat{V}(k_i) = \hat{V}(k_i-1) - h_i \qquad k_i = 1,\dots,n_i \qquad 6.6.3$$
$$n_i = p_i/h_i$$

$$\hat{V}\left(\sum_{m=j+1}^{i} n_m+k_j\right) = \hat{V}\left(\sum_{m=j+1}^{i} n_m+k_j-1\right) - h_j \qquad 6.6.4$$

where $k_j = 1,\dots,n_j$

$n_j = p_j/h_j$

$j = i-1, i-2,\dots,1.$

From 6.6.3, 6.6.4 we see that

$$n = \sum_{j=1}^{i} n_j \qquad 6.6.5$$
$$p = \sum_{j=1}^{i} p_j$$

We now denote $\underline{h}^{(j)}$ by

$$\underline{h}^{(j)T} = (h_1,\dots,h_j), \quad j = 1,\dots,i.$$

The steps of the method involve varying h_j, $j = 1,\dots,i$, having been given p and hence a certain distribution $\underline{p}$ satisfying 6.6.5. The steps are as follows:

i) Given p_1 compute $\hat{r}_o(\theta,h_1,p_1)$ varying h_1 according to any of the rules a),b),c). Denote the value of h_1 chosen as the optimum by one of these rules as h_1^*.

ii) Given p_m, h_m^*, $m = 1,\dots,j-1$, and for given p_j compute $\hat{r}_o(\theta,\underline{h}^{(j)},\sum_{m=1}^{j} p_m)$ where

$$\underline{h}^{(j)T} = (h_1^*,\dots,h_{j-1}^*,h_j) \qquad 6.6.6$$

varying h_j according to any of the rules a),b),c) to find the optimum denoted as h_j^*, $j = 2,\dots,i$.

Thus we obtain a series of optimal step-sizes h_j^*, $j = 1,..i$, where $\hat{r}_o(\theta,\underline{h}^{(j)},\sum_{m=1}^{j} p_m)$ is the best obtainable r_o such that (r_o,θ) is a point on the contour $V = \sum_{m=1}^{j} p_m$ with $\underline{h}^{(j)}$ as 6.6.6 and varying h_j.

For the methods given in sections 3,4,5,6 we have obtained $r^*(\theta,p)$ as the optimal value of $\hat{r}_o(\theta,\underline{h},p)$ where $\underline{h}$ is varied as above according to the definitions of optimality given in a),b),c).

i.e. $r^*(\theta,p) = \text{optimum } \hat{r}_o(\theta,\underline{h},p)$ with respect to $\underline{h}$ 6.6.7
given p_j, $j = 1,\dots,i$.

Define $\underline{h}^{*T} = (h_1^*,\dots,h_i^*)$ where h_j^*, $j = 1,\dots,i$, is the optimum value of h_j.

Then $r^*(\theta,p) = r_o(\theta,\underline{h}^*,p)$.

7. Computation of the D.O.A.

Sections 3,4,5,6 have shown how we have obtained $r^*(\theta,p)$ for a particular value of θ and of p, where r^* is obtained by reference to the definition 6.6.7, methods a),b),c) of section 6, the definitions 6.5.2 and 6.4.15.

Define $J(\theta)$ as follows:

$J(\theta) = 0$ if the discontinuous region of fig. 76 not encountered.
$J(\theta) = 1$ if the discontinuous region of fig. 76 were encountered. 6.7.1

It was stated in section 5 that if a stable system is being investigated the discontinuous regions will be identified by the location of other minima of W than $\hat{r}_o$ as given by the definition 6.5.2. But it is possible, particularly if δr is small, that other minima may not be encountered and $J(\theta) = 0$ will result by 6.7.1. Clearly by reference to fig. 76d we see that for small δr only one minimum of W is located. Hence if an unstable system is investigated $J = 0$ results. If a stable system is looked at then $J = 1$ is most likely but $J = 0$ is possible.

To establish instability we need to compute $J(\theta)$ for various θ and then if $J(\theta) = 0$ for all θ then the system is unstable, while if $J(\theta) = 1$ for any θ then the system is stable. Using $J(\theta)$ and $r^*(\theta,p)$ we are ready to compute the boundary of the D.O.A..

Fix θ_o, θ' and $\delta\theta$ as input parameters. θ_o is the first value of θ for which we compute r^* and J. $\delta\theta$ is the accuracy to which we compute $\hat{\theta}$ where $\hat{\theta}$ is given by $r^*(\hat{\theta},p) = R$. θ' is is the increment step for θ.

The steps for obtaining $\hat{\theta}$ are as follows:

Stage I

1) If $J(\theta_o) = 1$ and $r^*(\theta_o,p) < R$ 6.7.2

we have established stability and a boundary point inside the R-circle. We put $\hat{\theta} = \theta_o$.

2) Otherwise compute $r^*(\theta_o+i\theta',p)$ and $J(\theta_o+i\theta')$, $i = 1,2,..,I$ until either

i) $r^*(\theta_o+I\theta',p) < R$
and some $J(\theta_o+i\theta') = 1$, $i = 1,...,I$ 6.7.3

or ii) $I\theta' \geq 2\pi$ 6.7.4

in which case the system is unstable if
$J(\theta_o+ i\theta') = 0$ for all $i = 1,2,...,I$ 6.7.5
or stable in S_R otherwise.

3) Given that from 6.7.3 the system is stable and the boundary is outside the R-circle for $\theta = \theta_0 + (I-1)\theta'$ but inside for $\theta = \theta_0 + I\theta'$, we interval halve to determine $\hat{\theta}$ where

$$r^*(\hat{\theta},p) < R \qquad 6.7.6$$

$$\text{but } r^*(\hat{\theta}-\delta\theta,p) > R.$$

At the completion of Stage I we either have

a) an unstable system indicated by 6.7.5,

or b) a system which is stable in the R-circle indicated by the negative of 6.7.5,

or c) a boundary point $(r^*(\hat{\theta},p),\hat{\theta})$ on the contour $V = p$, obtained by 6.7.2 or 6.7.6.

If a),b) exist then there is no further computation necessary, so let us assume c) is the situation reached. We now re-define $\theta_0 = \hat{\theta}$ where $\hat{\theta}$ is given by 6.7.2 or 6.7.6. Given $r^*(\theta,p)$ and θ_0 and $V_0 = p$ the next step is to integrate equations 6.4.4, 6.4.5 for increasing V as this will track closer to the boundary.

<u>Stage II</u>

Hence given

$$\begin{aligned} \hat{x}^{(0)} &= r^*(\hat{\theta},p)\cos\hat{\theta} \\ \hat{y}^{(0)} &= r^*(\hat{\theta},p)\sin\hat{\theta} \\ \hat{V}^{(0)} &= p \\ \hat{\theta}^{(0)} &= \hat{\theta} \end{aligned} \qquad 6.7.7$$

we integrate 6.4.4, 6.4.5 numerically with fixed step-size h' computing $\hat{r}^{(j)}$, $\hat{\theta}^{(j)}$, $\hat{x}^{(j)}$, $\hat{y}^{(j)}$, $j = 1,2,\ldots$ until one of three criteria is satisfied:

either i) $\hat{x}^{(j)2} + \hat{y}^{(j)2} > R$ 6.7.8

(boundary has left R-circle)

or ii) $\left|\hat{\theta}^{(j)} - \theta_0\right| \geq 2\pi$ 6.7.9

(boundary is completely traced)

or iii) computation of the boundary has gone on long enough- usually determined by

$$\hat{V}^{(j)} > p' \qquad 6.7.10$$

for some p'.

At the end of Stage II we have one of these 3 situations. If ii) is true then the boundary is traced and there is no further computation necessary. For the other situations define $\theta_1 = \hat{\theta}^{(j)}$. We now have the situation illustrated in fig. 81

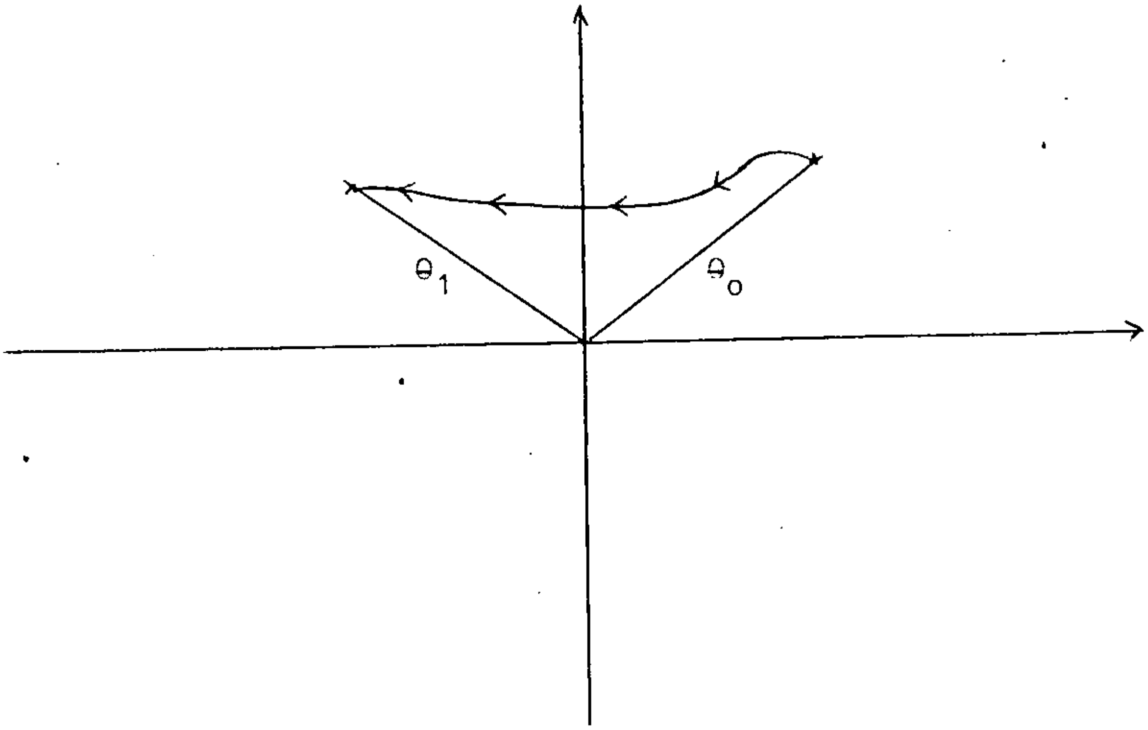


Fig. 81

or a mirror image of it. To proceed from here we must repeat Stage I again using θ_1 in rules 1),2),3) and equations 6.7.2, 6.7.3, 6.7.5 instead of θ_o. But we do not re-define θ_o the second time around. Using $\hat{\theta}$ we again proceed through Stage II for a second time and we arrive at one of the situations in figs. 82 or 83.

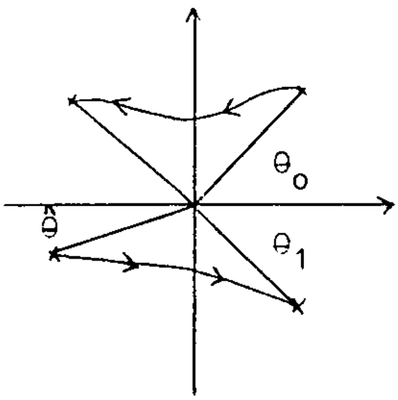


Fig.82

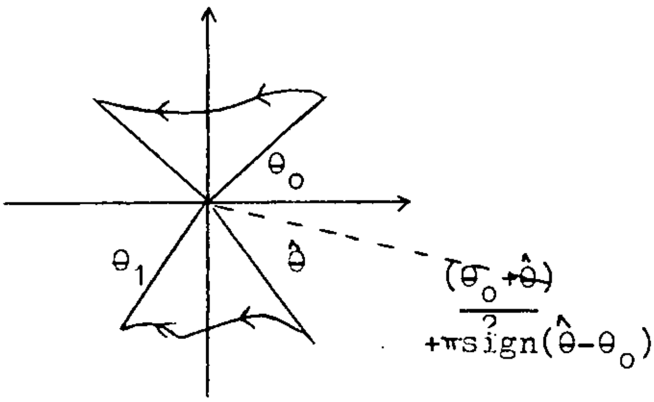


Fig. 83

If the situation is as in fig. 82 then the whole process is repeated until either 6.7.4 or 6.7.9 is satisfied, at which time the boundary of the D.O.A. is computed inside S_R.

The situation of fig. 83 is unusual but could theoretically happen. The boundary has been obtained for $\theta_o \leq \theta \leq \hat{\theta}$ (or $\hat{\theta} \leq \theta \leq \theta_o$ as the case may be) but for θ near to $\hat{\theta}$ or θ_o trajectories obtained by Stage II tend to return to the already known region. The procedure from here is to define

$$\theta_1 = \hat{\theta}$$

and $$\theta = \frac{\theta_0 + \hat{\theta}}{2} + \operatorname{sign}(\hat{\theta} - \theta_0)\,\pi \qquad 6.7.11$$

as in fig. 83.

i.e. θ is taken halfway between the θ_0 and $\hat{\theta}$ radials and we try to find a new $\hat{\theta}$ by stage I using 6.7.11 instead of θ_0.

By repeated analysing of which of the criteria of Stage I or Stage II are satisfied and which situations occur between that of figs. 82 and 83 we may obtain the entire boundary of the D.O.A. inside S_R.

This process when all put together in a computation algorithm is sufficient to define computation of the boundary of the D.O.A. inside S_R providing a method of obtaining $r^*(\theta,p)$ is available for each θ,p.

The case of where f/ϕ and g/ϕ are finite at the origin is not considered as mentioned in section 4 because we tend to find that differential equations with non-unique solutions arise. But the algorithm could be amended by replacing 6.7.1 by a definition such as

$$J(\theta) = 0 \text{ if } W(r^*(\theta,p),\theta,\underline{h}^*,p) > W_0$$

$$J(\theta) = 1 \text{ if } W(r^*(\theta,p),\theta,\underline{h}^*,p) < W_0$$

as suggested in 6.5.13.

7.1 Illustration

To illustrate the methods of this section we use the system 6.4.21, 6.4.22 and its solution 6.4.23. The D.O.A. is given by 6.2.4. If we put into the algorithm the fixed parameters of this section $\theta_0 = 0^o$, $\theta' = 30^o$, $\delta\theta = 5^o$, with $R = 3{\cdot}0$ the results go as follows:

1) $\theta = \theta_0 = 0^o$ $\qquad r^* > R.$

2) $\theta = \theta_0 + \theta' = 30^o$ $\qquad r^* = 1{\cdot}4686 < R.$

3) $\theta = 15^o$ $\qquad r^* = 1{\cdot}8595 < R.$

4) $\theta = 7\frac{1}{2}^o$ $\qquad r^* = 2{\cdot}4149 < R.$

5) $\theta = 3\frac{3}{4}^o$ $\qquad r^* > R.$

Thus we find that $\theta_0 = 7\frac{1}{2}^o$ to an accuracy of 5^o.

The extension is plotted (fig. 84) until $(x,y) \notin S_R$ at $\theta_1 \approx 83^o$

We return to find a new $\hat{\theta}$

6) $\theta = 113^o$ $\qquad r^* > R.$

7) $\theta = 143^{\circ}$ $r^{*} > R.$

8) $\theta = 173^{\circ}$ $r^{*} > R.$

9) $\theta = 203^{\circ}$ $r^{*} = 1\cdot5920 < R.$

10) $\theta = 188^{\circ}$ $r^{*} = 2\cdot3598 < R.$

11) $\theta = 180\frac{1}{2}^{\circ}$ $r^{*} > R.$

12) $\theta = 184^{\circ}$ $r^{*} > R.$

Thus $\theta_2 = 188^{\circ}$ to an accuracy of 5°.

The extension is plotted until 6.7.8 is satisfied at $\hat{\theta}^{(n)} = 263^{\circ}$.

We look for a new $\hat{\theta}$

13) $\theta = 293^{\circ}$ $r^{*} > R.$

14) $\theta = 323^{\circ}$ $r^{*} > R.$

15) $\theta = 353^{\circ}$ $r^{*} > R.$

16) $\theta = 383^{\circ}$ $r^{*} = 1\cdot5920 < R.$

17) $\theta = 368^{\circ}$ $r^{*} = 2\cdot3498 < R.$

18) $\theta = 360\frac{1}{2}^{\circ}$ $r^{*} > R.$

19) $\theta = 364^{\circ}$ $r^{*} > R.$

Thus $\theta_2 = 368^{\circ}$ to an accuracy of 5°.

But $\theta_2 - \theta_0 > 360^{\circ}$ and so the whole boundary has been plotted.

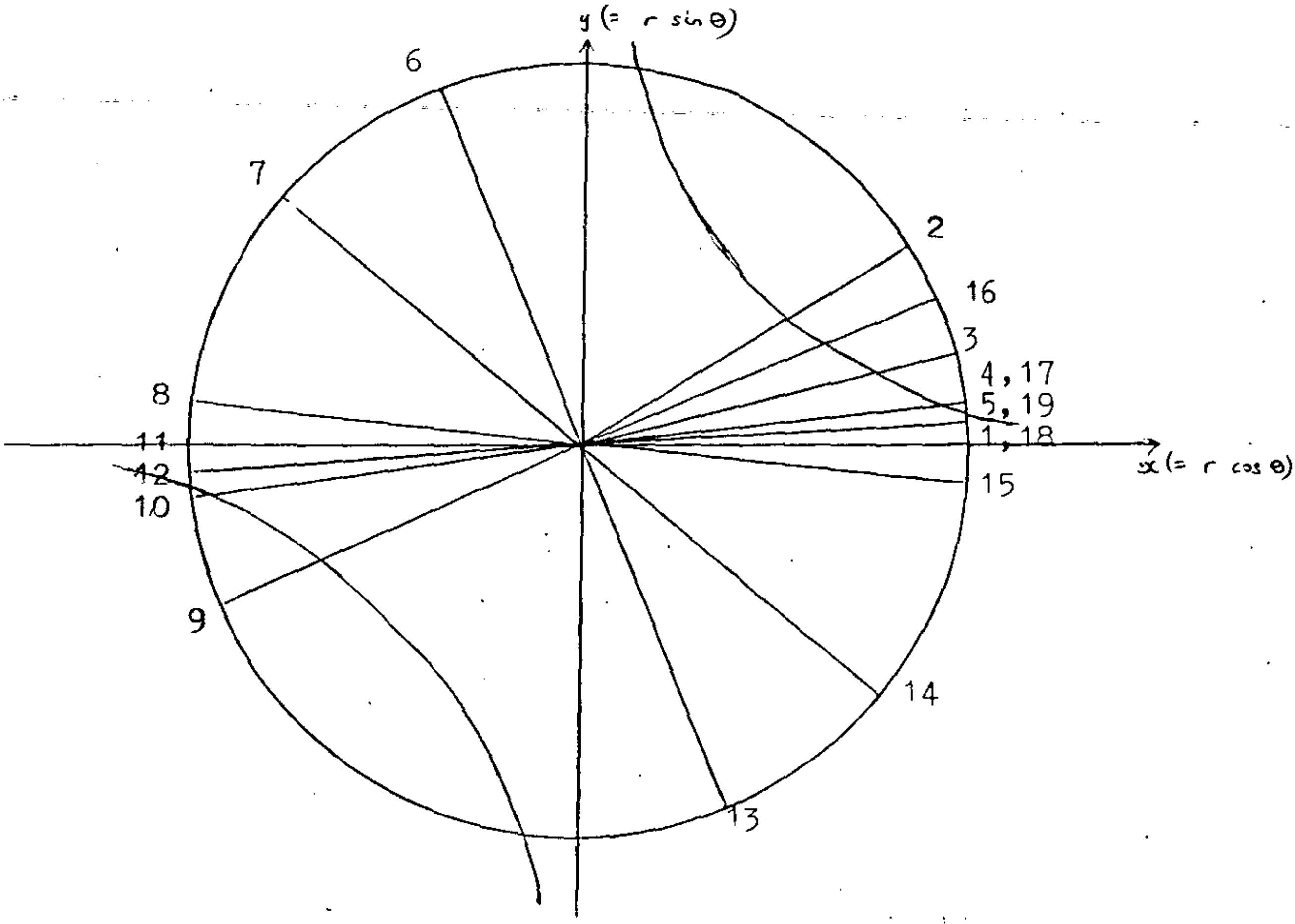


Fig. 84

8. Algorithm Details

8.1 Main Program

The main program has 12 parameters which are specified by the user on a data card for each run. The first card must contain a single integer which represents the number of times the algorithm is to run. Then for each run there is one card with 12 numbers on it followed by one or more cards with 3 numbers. Listed below is a list of the 12 parameters with computer name and the name used in the development of the theory.

Name	Description	Computer label
R	Radius of spherical domain S_R.	RØ
δr	Decrement of r_o in Stage I of section 5.	RINCRØ
ε	The accuracy of calculation of the minimal set to give $\hat{r}_o$.	EBSILON
θ_o	Initial value of θ in degrees.	ATHETHØ
$\delta\theta$	Accuracy of computation of $\hat{\theta}$.	DELTA1
θ'	Increase or decrease in θ.	DELTA2
u	Parameters to be used in variation of ϕ	U
q	or variation of f, g or anything else.	Q
h'	Fixed step-size for computing the boundary of the D.O.A..	HEX
d	Control parameter to decide if regular or modified Zubov equation.	MV
i	Number of different step-sizes in computation of $V = p$.	JM
	Print-out rate of the boundary curve.	NJEX

After this we require i cards with these parameters listed.

Name	Description	Computer label
p_j	Change in V over which step-size is h_j.	VØ(J)
$n_s^{(j)}$	Number of steps during integration i.e. $h_j = p_j / n_s^{(j)}$	NSTART(J)
$n_o^{(j)}$	Increment in numbers of steps. $j, J = 1, 2, \ldots, i$.	NØ(J)

The variation of the step-sizes discussed in section 6 is done by fixing p_j and during integration from V to $V-p_j$ we alter n_j where

$$h_j = \frac{p_j}{n_j}.$$

The main program does all the manipulation of sections 6 and 7. Subroutine CONTIN is called to calculate $\hat{r}_o(\theta,\underline{h},p)$, $J(\theta)$ given $\theta,\underline{h},p$. The main program carries out the manipulation of the step-sizes to compute $r^*(\theta,p)$. From $r^*(\theta,p)$ and $J(\theta)$ the analysis of section 7 is carried out to obtain the boundary of the D.O.A. calling subroutine EXTEND to perform the integration.

Given h_m, $m = 1,\ldots,j-1$, the method of varying h_j depends on the values of $n_s^{(j)}$, $n_o^{(j)}$. There are three cases for obtaining h_j^*, $j = 1,\ldots,i$, which correspond to the 3 suggested ways of obtaining h_j^* in section 6.

a) If $n_o^{(j)} = 0$ then $h_j^* = \dfrac{p_j}{n_s^{(j)}}$ fixed.

b) If $n_o^{(j)} = 1$ then we obtain a sequence $h_j^{(k)}$ letting

$$h_j = h_j^{(k)}, \quad k = 1,2,\ldots \qquad 6.8.1$$

successively in 6.6.6 obtaining $\hat{r}_o(\theta,\underline{h}^{(j,k)},\sum_{m=1}^{j} p_m)$ where we denote

$$\underline{h}^{(j,k)} = (h_1^*,\ldots,h_{j-1}^*,h_j^{(k)}), \quad j = 1,2,\ldots,i \qquad 6.8.2$$

The sequence $h_j^{(k)}$ is defined by

$$h_j^{(k)} = p_j/n_j^{(k)}, \quad k = 1,2,\ldots$$

where
$$n_j^{(1)} = n_s^{(j)}$$
$$n_j^{(2k+1)} = n_j^{(2k)} + 1, \quad k = 1,2,\ldots$$
$$n_j^{(2k)} = n_j^{(2k-1)} + n_s^{(j)}, \quad k = 1,2,\ldots$$

Let us denote for brevity

$$\hat{r}_o^{(k)} = \hat{r}_o(\theta,\underline{h}^{(j,k)},\sum_{m=1}^{j} p_m) \qquad 6.8.3$$

considering only $h_j^{(k)}$ as variable. Then we define

$$h_j^* = h_j^{(2k+1)} \qquad 6.8.4$$

where
$$\left|\hat{r}_o^{(2k+1)} - \hat{r}_o^{(2k)}\right| < \varepsilon \qquad 6.8.5$$

c) If $n_o^{(j)} > 1$ then we again define a sequence using the notation of 6.8.1 and 6.8.2.

This time the sequence $h_j^{(k)}$ is defined by

$$h_j^{(k)} = p_j/n_j^{(k)}, \qquad k = 1,2,\ldots$$

where
$$n_j^{(1)} = n_s^{(j)}$$

$$n_j^{(2k+1)} = n_j^{(2k)} + 1, \; k = 1,2,\ldots$$

$$n_j^{(2k)} = n_j^{(2k-1)} + n_o^{(j)}, \; k = 1,2,\ldots$$

Using the notation of 6.8.3 we again define h_j^* by 6.8.4 if 6.8.5 holds for some k. However if k is found such that

$$\hat{r}_o^{(k+1)} > \hat{r}_o^{(k)}$$
$$\hat{r}_o^{(k-1)} > \hat{r}_o^{(k)} \qquad 6.8.6$$

then we have obtained a minimal set according to definition 6.5.1 and an interval halving process establishes h_j^* where

$$h_j^* = p_j/_{n_j^*}$$

and n_j^* gives a smaller $\hat{r}_o$ than $n_j^* + 1$ or $n_j^* - 1$, $j = 1,\ldots,i$.

The rule a) is used when no optimization is required. Rules b),c) are safer in that they require $\hat{r}_o(\theta,\underline{h},p)$ to satisfy either 6.8.5 or 6.8.6. The steps of method c) have built in protection for any calculation of $\hat{r}_o$ which may be in error due to the particular $\hat{r}_o$ satisfying the definition 6.5.2 for given ε. For small ε there is more calculation involved to obtain $\hat{r}_o$ but more accuracy also. Method c) rejects inconsistent results for $\hat{r}_o$ along the way.

When h_j^* is calculated according to these rules then h_{j+1} may be varied and so on. When h_j^*, $j = 1,\ldots,i$, are all fixed then $r^*(\theta,p)$ has been computed as

$$r^*(\theta,p) = r_o(\theta,\underline{h}^*,p)$$

where $\underline{h}^* = (h_1^*,\ldots,h_i^*)$ the vector of optimum step-sizes.

8.2 Subroutine CONTIN

Subroutine CONTIN is called by the main program to obtain $\hat{r}_o(\theta,\underline{h},p)$ and $J(\theta)$. It has 14 arguments of which 12 are sent by the main program and 2 are returned.

RØ, RINCRØ, EBSILON, MV, U, Q are taken direct from the input list in section 8.1. ALPHA represents the value of θ in radians and AHPLA is θ in degrees for purposes of print-out.

The remaining 4 input parameters are associated with step-size alteration as follows:

VN represents p_j, $j = 1,\ldots,i$.

M is an array representing $(n_1^*,n_2^*,\ldots,n_{j-1}^*,n_j^{(k)})$ where $h_m^* = p_m/_{n_m^*}$, $m = 1,\ldots,j-1$, and $h_j^{(k)}$ is undetermined.

NLIFT represents $n_j^{(k)}$, k = 1,2,...

JV represents j.

The subroutine computes $\hat{r}_o(\theta,\underline{h},p)$ by the method of section 5 in which a minimum of $W(r_o,\theta,\underline{h},p)$ with respect to r_o is found. Subroutine RUNKUT is called to perform the integration of 6.4.4, 6.4.5 and compute $W(r_o,\theta,\underline{h},p)$ given r_o and CONTIN manipulates r_o to obtain $\hat{r}_o(\theta,\underline{h},p)$. When CONTIN is finished RMIN represents $\hat{r}_o(\theta,\underline{h},p)$ and JC is $J(\theta)$.

8.3 Subroutine RUNKUT

Subroutine RUNKUT is called by CONTIN to integrate 6.4.4, 6.4.5 and calculate $W(r_o,\theta,\underline{h},p)$. The integration is carried out by the Fourth Order Runge-Kutta method but any other numerical method may be substituted for this method. There are 9 inputs and 1 output to this subroutine.

1) RR is the value of r_o.
2) TT is θ in radians, same as ALPHA in CONTIN.
3) HH is the vector $(h_1^*,h_2^*,\ldots,h_{j-1}^*,h_j^{(k)})$, k = 1,2,...
4) NN is the vector $(n_1^*,n_2^*,\ldots,n_{j-1}^*,n_j^{(k)})$.
5) JJ represents j.
6) SS is the value of $d\sum_{m=1}^{j} p_m$.
7) MM is the value of d.
8) UU the system parameters.
9) QQ the system parameters.

The output is DD which represents $W(r_o,\theta,\underline{h},p)$.

8.4 Subroutine EXTEND

Subroutine EXTEND is called by the main program to integrate 6.4.4, 6.4.5 for increasing V to trace out the boundary of the D.O.A.. It has 13 arguments.

1) R1 is the same as RØ in the input list.
2) TH is the value of $\hat{\theta}$ in 6.7.7.
3) AT represents θ_o.
4) AT1 represents θ_1.
5) RØ represents $r^*(\theta,p)$.
6) UN the system parameters.
7) QN the system parameters.
8) VN represents p.
9) MN represents d.
10) W where W = 1 for θ increasing and W = -1 for θ decreasing
11) H represents h' in the input list - same as HEX.
12) NJ is the same as NJEX in the input list.

13) JB is a control variable to note the past history of W to see if the backtracking of fig. 83 occurs.

8.5 Specification of f, g, ϕ

f,g,ϕ are specified in subroutines RUNKUT and EXTEND in the form $\dfrac{-f(x,y)}{\phi(x,y)(1-dV)}$ and $\dfrac{-g(x,y)}{\phi(x,y)(1-dV)}$. The function arguments are X,Y,S,U,Q where X,Y are the variables, U,Q are the parameters to enable us to vary ϕ or f,g and S = dV.

9. Higher Orders

Mention should be made here of the possible extension of this algorithm to systems of higher orders. The numerical integration theory developed in section 3 has been worked out in m dimensions and is readily applicable to m > 2. The theory of negative V also generalises to m dimensions as the system trajectories are still given by $x_i(V)$ for i = 1,...,m.

It is when we consider the variation of the initial point $\underline{x}_o$ in m dimensions that further thought is required. We again consider variation of $\underline{x}_o$ along a radial line given by $(r_o,\underline{\theta})$ where $\underline{\theta}$ is a fixed (m-1)-dimensional vector

$$\underline{\theta}^T = (\theta_1,\ldots,\theta_{m-1})$$

and $(r_o,\underline{\theta})$ are given by

$$\begin{aligned} x_1 &= r_o\cos\theta_1 \\ x_i &= r_o\sin\theta_1 \sin\theta_2\ldots\sin\theta_{i-1}\cos\theta_i \qquad i = 2,\ldots,m-1 \\ x_m &= r_o\sin\theta_1 \sin\theta_2\ldots\sin\theta_{m-1} \end{aligned} \qquad 6.9.1$$

The set S_R is defined in m dimensions by

$$S_R = \left\{ \underline{x} : x_1^2 + x_2^2 + \ldots + x_m^2 < R^2 \right\}. \qquad 6.9.2$$

Hence the function $W(r_o,\underline{\theta},h,p)$ can still be computed for fixed $\underline{\theta}$,h,p and varying r_o to obtain the function $\hat{r}_o(\underline{\theta},h,p)$ defined by 6.5.2. Likewise the methods of section 6 for varying h and changing h during integration may be carried out to give $r^*(\underline{\theta},p)$.

Up to computation of $r^*(\underline{\theta},p)$ for given $\underline{\theta}$,p the method is the same, but the method of section 7 for varying scalar θ is not directly applicable to vector $\underline{\theta}$. There is certainly scope for further development of this algorithm to work out how to vary $\underline{\theta}$ to cover all dimensions.

A further difficulty is that the computation of 6.4.4, 6.4.5 for increasing V with step-size h' also yields one-dimensional characteristic curves and cannot be generalised.

It seems likely that to vary $\underline{\theta}$ usefully would involve varying θ_i for some i while keeping the others fixed, then computing $r^*(\underline{\theta},p)$ and then integrating 6.4.4, 6.4.5 for increasing V would give a system of neighbouring curves which would define a grid in (m-2) dimensions for the boundary of the D.O.A..

Such a grid system would be, for example, given by $\hat{r}(j,k)$, $\hat{\underline{\theta}}(j,k)$, $\hat{\underline{x}}(j,k)$ where the initial conditions are

$$\hat{r}(\bar{0},k) = r^*(\hat{\underline{\theta}}(0,k),p)$$

$$\hat{\underline{\theta}}(0,k) = (\frac{2\pi k}{N}, \theta_2', \theta_3', \ldots, \theta_m')$$

$$k = 1,2,\ldots,N$$

$$\theta_i' \quad \text{constants } i = 2,\ldots,m.$$

i.e. variation of $\underline{\theta}$ is carried out only on one component, θ_1 in this case.

Then, by referencing Stage II of tracing the boundary in section 7 we compute, by the same numerical methods used in 2 dimensions, the point

$$\hat{r}(j,k),\ \hat{\underline{\theta}}(j,k),\ \hat{\underline{x}}(j,k) \qquad 6.9.3$$

given $\hat{r}(m,k)$, $\hat{\underline{\theta}}(m,k)$, $\hat{\underline{x}}(m,k)$

$$m = 0,1,\ldots,j-1,$$

$$j = 1,2,\ldots$$

The quantities in 6.9.3 are related by their components as given by 6.9.1. The trajectories given by 6.9.3 for fixed k and varying j would again be terminated by 6.7.8, 6.7.10 or by a slightly amended version of 6.7.9 given by

$$\left|\hat{\theta}_1(j,k) - \hat{\theta}_1(0,k)\right| \geqslant 2\pi$$

where $\hat{\theta}_1(j,k)$ is the first component of $\hat{\underline{\theta}}(j,k)$.

The boundary of the D.O.A. may be built up by this grid method along characteristics by taking N large enough so that absolute differences

$$\left\|\hat{\underline{x}}(j,k) - \hat{\underline{x}}(j,k-1)\right\| \qquad k = 1,2,\ldots,N$$

are of the same order of magnitude as

$$\left\|\hat{\underline{x}}(j,k) - \hat{\underline{x}}(j-1,k)\right\| \qquad j = 1,2,\ldots$$

However to know whether all of S_R where S_R is given by 6.9.2 has been covered could be quite complicated.

10. Examples

Example 10.1

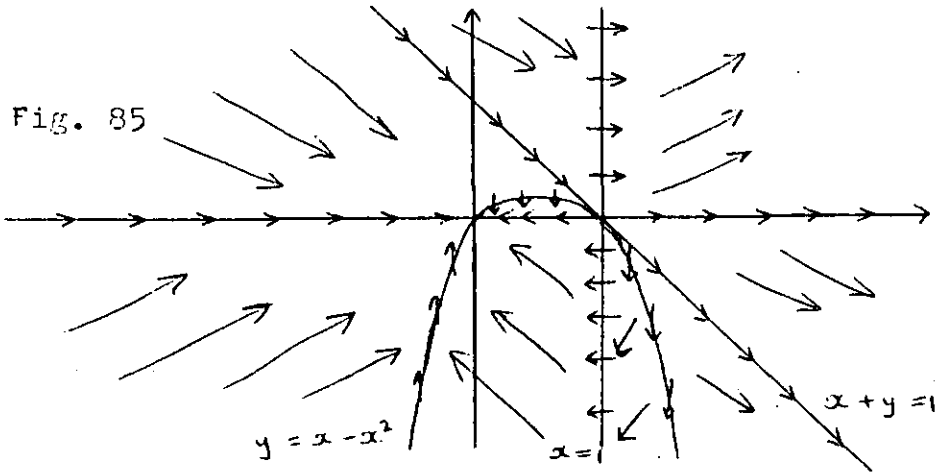
$$\dot{x} = -x + y + x^2$$
$$\dot{y} = -y + xy \qquad 6.10.1$$

This example from Texter (34) has an unbounded D.O.A.. The whole (x,y) plane is divided into two parts by the boundary of the D.O.A.. It will be noticed that

$$x + y = 1 \qquad 6.10.2$$

is a solution of 6.10.1 and it can be shown that 6.10.2 is the boundary of the D.O.A. for $x < 1$, $y > 0$. This is achieved by studying the direction field of the trajectories by considering $(\dot{x},\dot{y})$ at any point in the (x,y) plane.

Fig. 85



Considering 6.10.1 we see that $\dot{x} = 0$ at

$$y = x - x^2 \qquad 6.10.3$$

while $\dot{y} = 0$ at $y = 0$ or $x = 1$. 6.10.4

Thus dividing the (x,y) plane into regions where $\dot{x}$ and $\dot{y}$ are positive and negative separated by 6.10.3 and 6.10.4 we obtain the sketch of the direction field shown in fig. 85. From fig. 85 it is immediately apparent that the points in the region $x < 1$, $y > 0$ are stable if they cross 6.10.3 and unstable if they cross $x = 1$. Points on 6.10.2 tend towards $x = 1$, $y = 0$ and therefore 6.10.2 is the boundary in this part of the plane.

The region $x < 1$, $y < 0$ is seen to be stable while the region $x > 1$, $y > 0$ is unstable. To consider the remaining region $x > 1$, $y < 0$ we return to 6.10.2 and notice that this line is unstable and hence the region $x + y > 1$, $x > 1$, $y < 0$ is not in the D.O.A.. The remaining region given by $x + y < 1$, $x > 1$, $y < 0$ is inconclusive by inspection of fig. 85. Thus we have the situation of fig. 86.

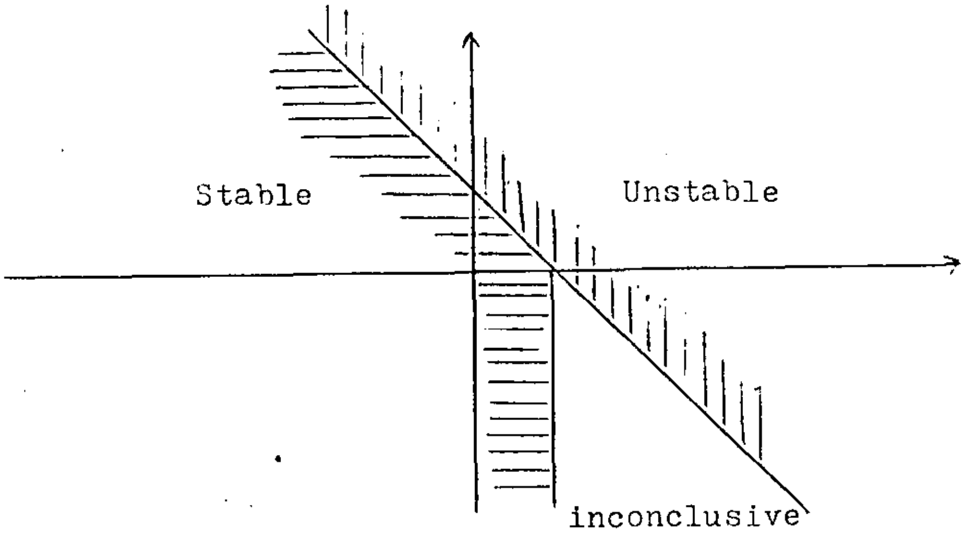


Fig. 86

The example was tested on the algorithm with the inputs given by

$$
\begin{aligned}
R &= 3\cdot 0 \\
\delta r &= 0\cdot 4 \\
\varepsilon &= 0\cdot 0001 \\
\theta_o &= 30\cdot 0 \\
\delta\theta &= 5\cdot 0 \\
\theta' &= 30\cdot 0 \\
h' &= 0\cdot 2 \\
d &= 0 \\
i &= 1 \\
p_1 &= 25\cdot 0 \\
n_s^{(1)} &= 40 \\
n_o^{(1)} &= 0
\end{aligned}
\qquad 6.10.5
$$

The results are shown on fig. 90 for $\phi(x,y) = x^2 + y^2$ and are in accordance with fig. 86.

<u>Example 10.2</u>

$$\dot{x} = -2x + y + x^2/8 + 3xy/8 + y^2/16$$
$$\dot{y} = -x - 2y + 3x^2/8 + xy/16 + 3y^2/4$$

This particular example was chosen so that the low order terms of the series solution would be fairly simple. The low order terms become

$$V_2(x,y) = x^2/4 + y^2/4$$
$$V_3(x,y) = x^2y/16 + y^3/16$$

where $\phi(x,y) = x^2 + y^2$.

The parameters used were as in 6.10.5 except for $h' = 0\cdot 01$. Various ϕ were used which were

$$\phi(x,y) = q(x^2 + y^2)$$

with $q = 1\cdot 0,\ 1\cdot 1,\ 1\cdot 2, \ldots, 1\cdot 8$.

The results are all very similar and are only shown on fig. 91 for $q = 1\cdot 0$. The boundary is shown where it lies inside $S_{3\cdot 0}$ but it seems that a greater value of R should have been used.

Example 10.3

$$\dot{x} = -x + y + x(x^2 + y^2)$$
$$\dot{y} = -x - y + y(x^2 + y^2) \qquad 6.10.6$$

This example from Shields (28) has a well-known D.O.A. with a boundary given by $x^2 + y^2 = 1$. It is interesting to consider the behaviour of the parametric representation of this example for $x(V)$, $y(V)$ as well as $x(t)$, $y(t)$. First let us convert 6.10.6 to polar co-ordinates, which gives

$$\dot{r} \cos \theta - r \dot{\theta} \sin \theta = -r \cos \theta + r \sin \theta + r^3 \cos \theta \qquad 6.10.7$$
$$\dot{r} \sin \theta + r \dot{\theta} \cos \theta = -r \sin \theta - r \cos \theta + r^3 \sin \theta.$$

From 6.10.7 we obtain the differential equations

$$\dot{r} = -r + r^3$$
$$\dot{\theta} = -1$$

the solutions of which

$$r^2(t) = \frac{r_o^2}{r_o^2 + (1-r_o^2)e^{2t}} \qquad 6.10.8$$
$$\theta(t) = \theta_o - t$$

where $r(t) = r_o$, $\theta(t) = \theta_o$ at $t = 0$.

From 6.10.8 we see that the trajectories spiral round the origin as $t \to \infty$ but are stable if $r_o < 1$. Now we investigate the representation obtained by considering $\frac{dx}{dV}$, $\frac{dy}{dV}$ where

$$\dot{V} = -q(x^2 + y^2). \qquad 6.10.9$$

Dividing 6.10.6 by 6.10.9 we obtain

$$\frac{dx}{dV} = \frac{x-y-x(x^2+y^2)}{q(x^2+y^2)}$$
$$\frac{dy}{dV} = \frac{x+y-y(x^2+y^2)}{q(x^2+y^2)}$$

or in polar co-ordinate form

$$\frac{dr}{dV} \cos \theta - r \frac{d\theta}{dV} \sin \theta = (r \cos \theta - r \sin \theta - r^3 \cos \theta)/qr^2 \qquad 6.10.10$$
$$\frac{dr}{dV} \sin \theta + r \frac{d\theta}{dV} \cos \theta = (r \cos \theta + r \sin \theta - r^3 \sin \theta)/qr^2$$

The solutions of 6.10.10 are given by

$$r^2(V) = 1 - k_1 e^{-\frac{2V}{q}}$$
$$2\theta(V) + k_2 = \log\left(\frac{1 - k_1 e^{-\frac{2V}{q}}}{k_1 e^{-\frac{2V}{q}}}\right). \qquad 6.10.11$$

The arbitrary constants in 6.10.11 are removed by the initial condition

$r(V) = 0, \ \theta(V) = \theta_0$ when $V = 0$.

The trajectories are thus given by

$$r^2(V) = 1 - e^{\frac{-2V}{q}}$$

$$\theta(V) = \theta_0 + \tfrac{1}{2}\log\left(\frac{r^2(1-r_0^2)}{r_0^2(1-r^2)}\right) \qquad 6.10.12$$

where $r(V) = r_0$ when $\theta(V) = \theta_0$.

6.10.12 correspond very well to the series construction of $V(x,y)$ in the form

$$V(x,y) = \sum_{n=2}^{\infty}\sum_{k=0}^{n} a_{n,k}x^k y^{n-k} \qquad 6.10.13$$

and substituting 6.10.13 into 6.10.9 using 6.10.6 gives the solution

$$a_{n,k} = \frac{q}{n} \frac{(n/2)!}{(k/2)!(n/2 - k/2)!}$$

if k,n are both even and $a_{n,k} = 0$ otherwise.

Hence $V(x,y) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{q}{2n} \, {}_nC_k \, (x^k y^{n-k})^2$

$$= \sum_{n=1}^{\infty} \frac{q}{2n} (x^2 + y^2)^n.$$

Therefore the closed form of V is given by

$$V(x,y) = \frac{q}{2} \log\left(\frac{1}{1-x^2-y^2}\right) \qquad 6.10.14$$

6.10.12a and 6.10.14 are the same expression.

This analysis shows an example of the behaviour of $x(V)$, $y(V)$ and how it is related to $x(t)$, $y(t)$ and also how beneficial it is to compute trajectories which reach the origin after finite computation.

The parameters used in the computation are as in 6.10.5. Fig. 92 shows all the points obtained for various attempts at computation for various values of h'. The trajectories near the boundary of the D.O.A. circle the origin very rapidly as V increases.

An analysis of results for $\hat{r}_0$ was attempted to see what effect θ,h,q have. The algorithm computed $\hat{r}_0$ at $\theta = 30^o$, 15^o, $7\frac{1}{2}^o$, $3\frac{3}{4}^o$ before plotting the boundary and these points were looked at. In accordance with 6.10.12a or 6.10.14 θ had negligible effect on $\hat{r}_o$. h had some effect on $\hat{r}_o$ seen by varying $n_s^{(1)}$, $n_o^{(1)}$ but the most significant effect was caused by the variation of q. Fig. 93 shows the variation of $\hat{r}_o$ against q at $\theta = 30^o$, $n_s^{(1)} = 40$, $n_o^{(1)} = 0$. This shows, in general, a trend of $\hat{r}_o \rightarrow 1$ as q increases which is in contradict: to 5.11.42 of example 11.7 of Chapter 5 where it was shown that when $\dot{V} = -\dot{q}(x^2 + y^2)$ then the contour V = p is nearer to the origin as q increases. However it does show that since $\frac{dx}{dV}$, $\frac{dy}{dV}$ decrease in magnitude as q increases, that computational accuracy is better preserved for smaller f/ϕ, g/ϕ.

Example 10.4

Uncoupled systems were also looked at.

The system

$$\dot{x} = -x + 2x^2 - 3x^3$$
$$\dot{y} = -y + 2y^2 - 3y^3$$

is stable in the whole and the algorithm showed this. But the system

$$\dot{x} = -2x + 3x^2 - x^3$$
$$\dot{y} = -2y + 3y^2 - y^3$$

has a D.O.A. given by $x < 1$, $y < 1$. This example was computed before the variation of θ developed in section 7 was introduced. Even so, fig. 94 shows the boundary is being traced towards the critical point (1,1) though not enough points were actually printed out.

Example 10.5

$$\dot{x} = y$$
$$\dot{y} = -x - \mu(x^2 - 1)y \qquad 6.10.15$$

This example is the well-known Van Der Pol equation with a known bounded D.O.A. for $\mu < 0$. Various interesting results were obtained from this example.. The parameters used were the usual ones given by 6.10.5 unless otherwise stated.

$$\phi(x) = x^2 + y^2$$

was used, and $\mu = -0{\cdot}5, -1{\cdot}0, -1{\cdot}5, -2{\cdot}0, -2{\cdot}5, -3{\cdot}0, -4{\cdot}0$ substituted in 6.10.15. The results are shown graphically in figs. 95 to 101 and show the correct pattern of the D.O.A. for changing μ. The convention is that crosses and circles represent computation of $\hat{r}_o(\theta, \underline{h}, p)$ and crosses are from the boundary print-out.

There are some points which occur due to the variation of $W(r_o, \Theta, \underline{h}, p)$ with respect to r_o being of the form in fig. 87.

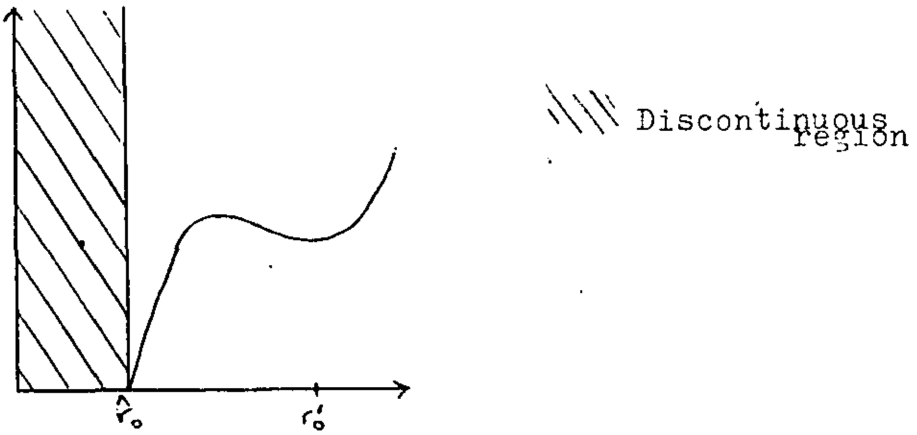


Fig. 87

In these circumstances $\hat{r}_o = r_o'$ may be evaluated satisfying the definition 6.5.2. But also we see that from 6.7.1 $J(\Theta) = 0$ and these points are neglected. As mentioned in section 5 this problem is resolved simply by increasing δr or possibly by using the alternative system 6.5.12. However in figs. 95 - 99 it is noticed that such incorrect values of $\hat{r}_o$ are rejected due to $J(\Theta) = 0$ and the boundary is computed when a correct point with $J = 1$ is obtained.

The "reverse-time" example

$$\dot{x} = -y$$
$$\dot{y} = x + \mu(x^2 - 1)y$$

was also considered. The algorithm showed this example to be unstable since $J(\Theta) = 0$ for all Θ used to compute $\hat{r}_o(\Theta, \underline{h}, p)$.

Example 10.6

$$\dot{x} = -6x + y + y^2$$
$$\dot{y} = y + x^2 \qquad 6.10.16$$

6.10.16 has two critical points at (0,0) and (2,-4). Consideration of the linear parts of 6.10.16 shows that the origin is unstable but the other critical point is stable. It is therefore best for computation to translate 6.10.16 to its stable critical point and it becomes

$$\dot{x} = -6x - 7y + y^2$$
$$\dot{y} = 4x + y + x^2 \qquad 6.10.17$$

The D.O.A. of 6.10.17 is unbounded but consists of a region around the origin and a narrow corridor which is unbounded in the third quadrant of the (x,y) plane. For this example $R = 8 \cdot 0$ was chosen and $h' = 0 \cdot 02$, $\delta\Theta = 1^o$, otherwise as 6.10.5. The results are obtained using

$$\phi(x,y) = q(x^2 + y^2)$$

with $q = 1, 1\frac{1}{4}$.

They are very impressive as shown in figs. 102, 103 respectively for obtaining the region $S_{8\cdot 0} \cap D(f,g)$.

Example 10.7

$$\dot{x} = 6y - 2y^2$$
$$\dot{y} = -10x - 2y + 4x^2 + 2xy + 4y^2$$

This example from Davies (46) has a D.O.A. given by

$$(x - \tfrac{1}{2})^2 + y^2 = 1.$$

Using $\phi(x,y) = q(x^2 + y^2)$

and the parameter values of 6.10.5 except for $h' = 0\cdot 01$, the boundary was computed for various q. Fig. 104 shows the boundary for $q = 1\cdot 8$.

Example 10.8

$$\dot{x} = -x(1 - x^2 - y^2)$$
$$\dot{y} = -y(1 - x^2 - y^2) \qquad 6.10.18$$

6.10.18 has a D.O.A. given by

$$x^2 + y^2 < 1, \qquad 6.10.19$$

but it is also noticeable from 6.10.18 that the entire boundary of the D.O.A. is a critical point of 6.10.18. The trajectories of 6.10.18 are lines of constant θ which tend towards the origin if 6.10.19 is satisfied. Hence any attempt to compute the boundary once $\hat{r}_0(\theta,\underline{h},p)$ is established will not result in increasing or decreasing θ. The results gave $\hat{r}_0(\theta,h,p)$ at various θ, $p = 25\cdot 0$, $h = 25/40$, and we find $\hat{r}_0$ slightly less than 1 each time and the trajectories are traced out giving constant θ, and $r \to 1$.

Example 10.9

$$\dot{x} = -x + 2x^2y$$
$$\dot{y} = -y$$

This example by Hahn (10) has been much used in the development of the theory in this thesis. It remains to show here how accurately the algorithm actually obtains its D.O.A..

The input parameters used are exactly as in 6.10.5 and $\phi(x,y) = 2x^2 + 2y^2$.

The results are shown for the first quadrant only in fig. 105 as they are symmetric with respect to a rotation through 180°.

11. Conclusions

In this chapter there has been developed what seems to be the method which best combines the safety of conservativeness of the estimate with accuracy of computation. The series method guarantees a conservative estimate absolutely but after some difficult algebra. The algebra involved in this method is that needed to integrate 6.4.4, 6.4.5 and the most difficult item is evaluation of f/ϕ and g/ϕ.

The finite difference methods are much simpler than this method but have been seen to suffer from instability of various kinds which result in inferior R.A.S.s and some of the problems cannot be overcome by reducing the grid so that the step-sizes tend to zero.

This method does not guarantee absolutely a conservative but accurate estimate of the D.O.A. since the computation of $\hat{r}_0$ could conceivably pick out the wrong minimum of $W(r,\theta,\underline{h},p)$. However we do claim that if δr is large, $\varepsilon \rightarrow 0$ then the correct value is obtained although as δr becomes large, $\varepsilon \rightarrow 0$ the computation inevitably increases. Likewise the boundary is obtained more accurately as $h', \delta\theta \rightarrow 0$. With the values of δr, ε, $h', \delta\theta$ used so far the results have been accurate enough in that correct $\hat{r}_0$ is nearly always obtained. It may be a possibility of further research to find a way to be sure of this other than by letting the input parameters in section 8.1 become zero or large respectively. The method of setting $n_0^{(j)} > 1$ and optimising $\hat{r}_0$ with respect to h does have included in it a facility for rejecting incorrect results by testing whether the graph of $\hat{r}_0(\theta,\underline{h},p)$ is smooth with respect to $\underline{h}$ or not.

The algorithm developed here is not perfect and does not calculate exact D.O.A.s for every example known. There is still room for improvement:

1) Maybe a better definition of $\hat{r}_0$ than 6.5.2.

2) Maybe a way of testing if $V_0 < p$ from the integration other than using $W(r,\theta,\underline{h},p)$.

3) Maybe $\hat{r}_0$ should be obtained by considering curves other than lines of constant θ.

4) There are many places where iteration could be incorporated to satisfy the user of convergence to the correct results.

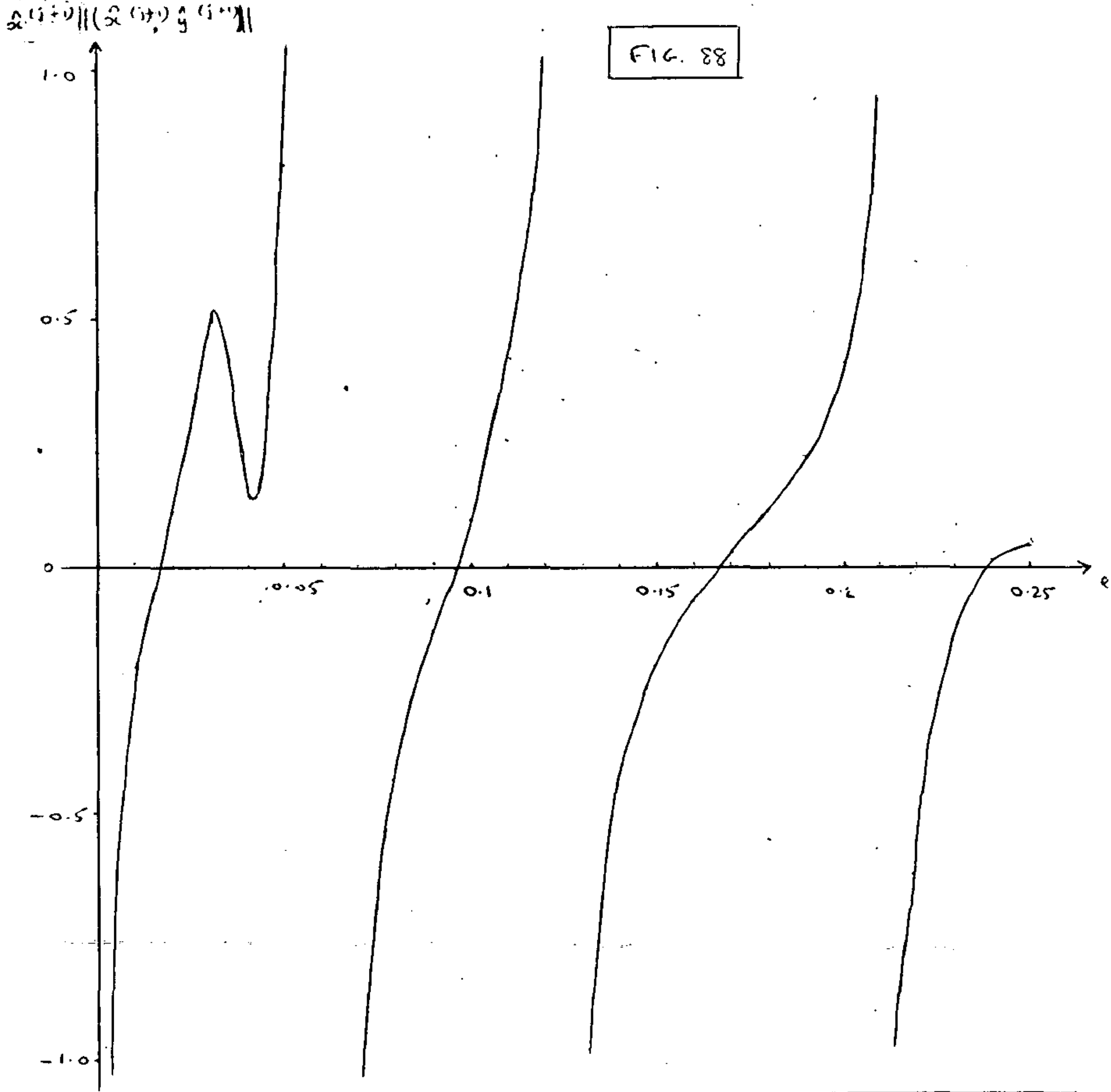

1·0
0·5
0
−0·5
−1·0
0·05
0·1
0·15
0·2
0·25
e

FIG. 88

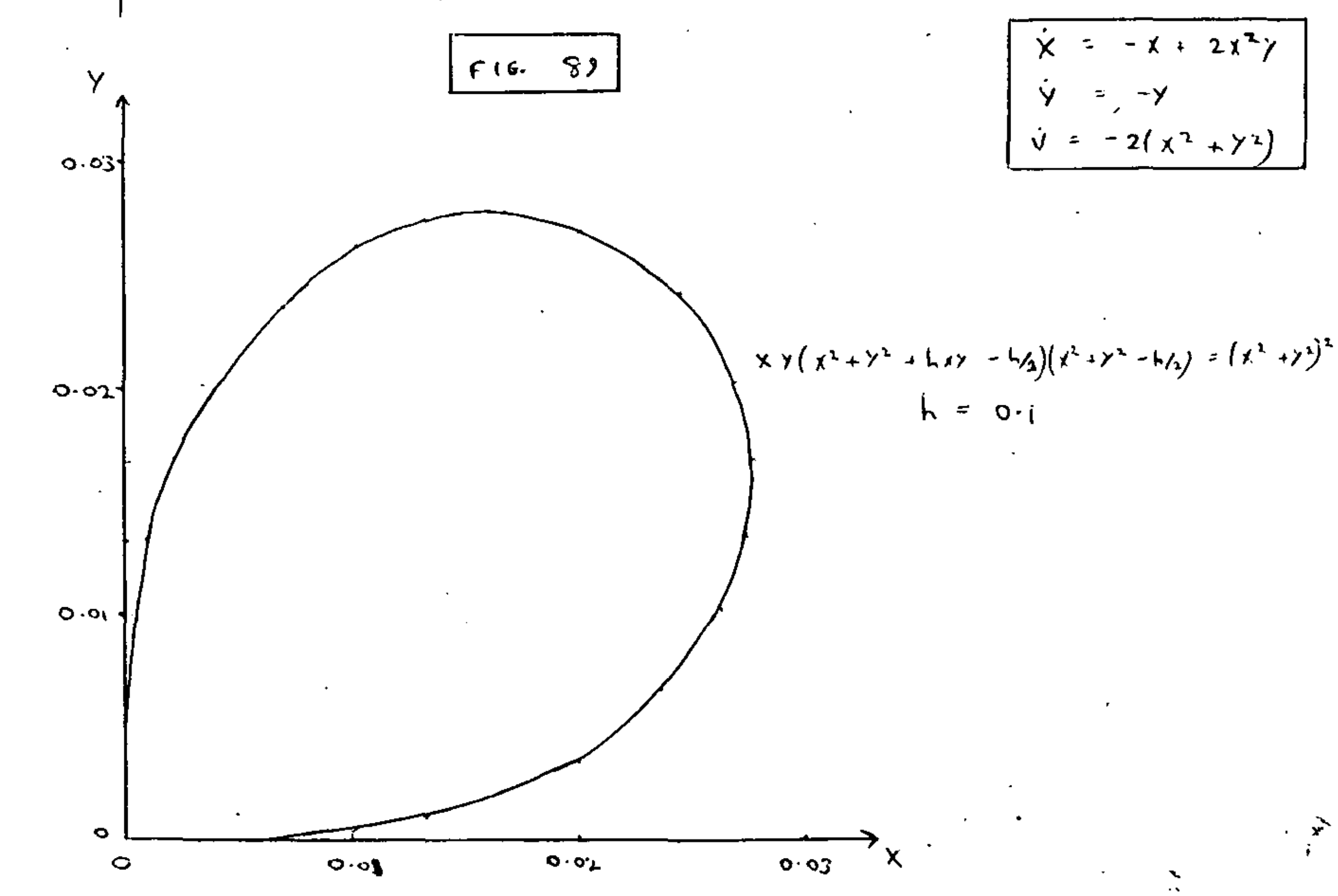

Y
X
0·03
0·02
0·01
0
0·01
0·02
0·03
$\dot{x} = -x + 2x^2y$
$\dot{y} = -y$
$\dot{V} = -2(x^2+y^2)$
$xy(x^2+y^2+hxy-h/3)(x^2+y^2-h/2) = (x^2+y^2)^2$
$h = 0{\cdot}1$

FIG. 89

FIG. 90

$$\dot{x} = -x + y + x^2$$
$$\dot{y} = -y + xy$$
$$\dot{V} = -(x^2 + y^2)$$

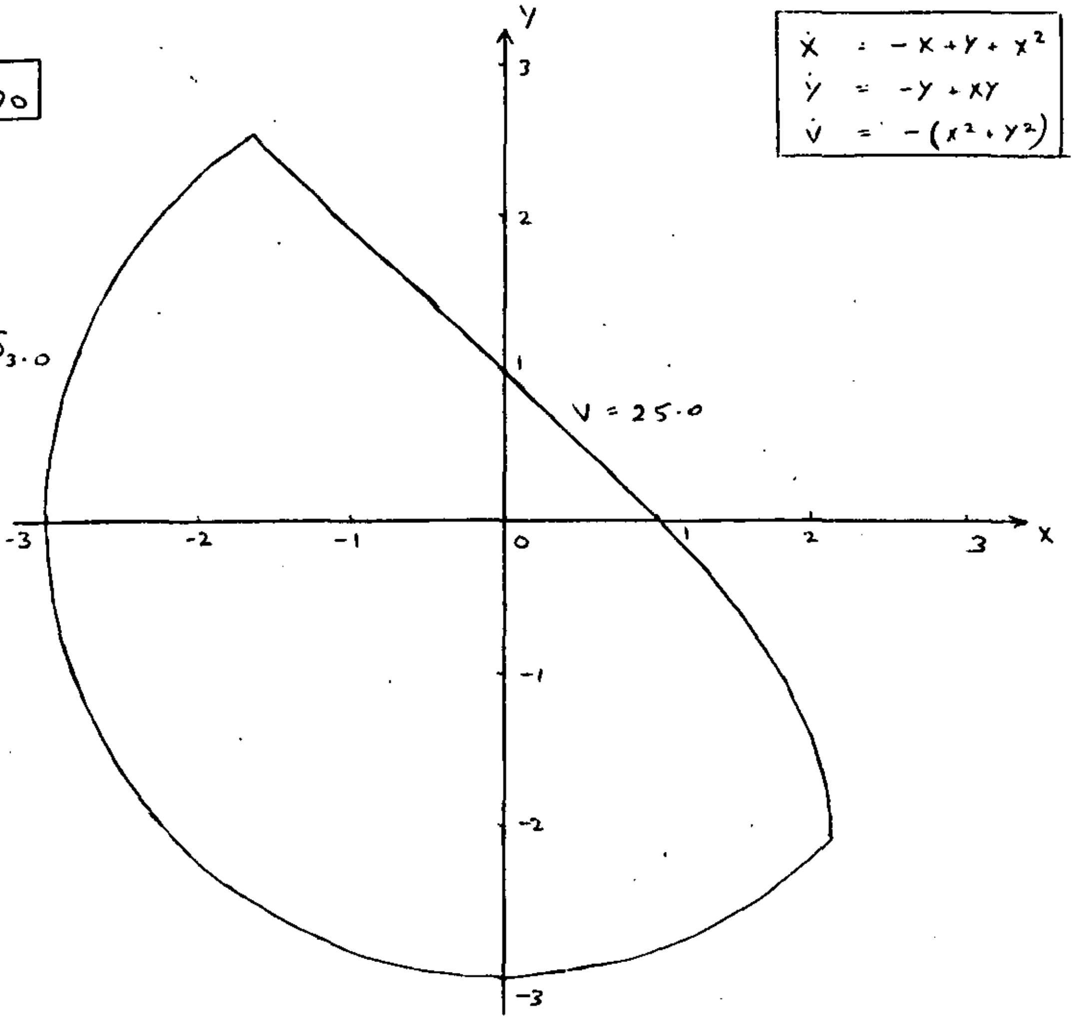


FIG. 91

$$\dot{x} = -2x + y + x^2/8 + 3xy/8 + y$$
$$\dot{y} = -x - 2y + 3x^2/8 + xy/16 + 3$$
$$\dot{V} = -(x^2 + y^2)$$

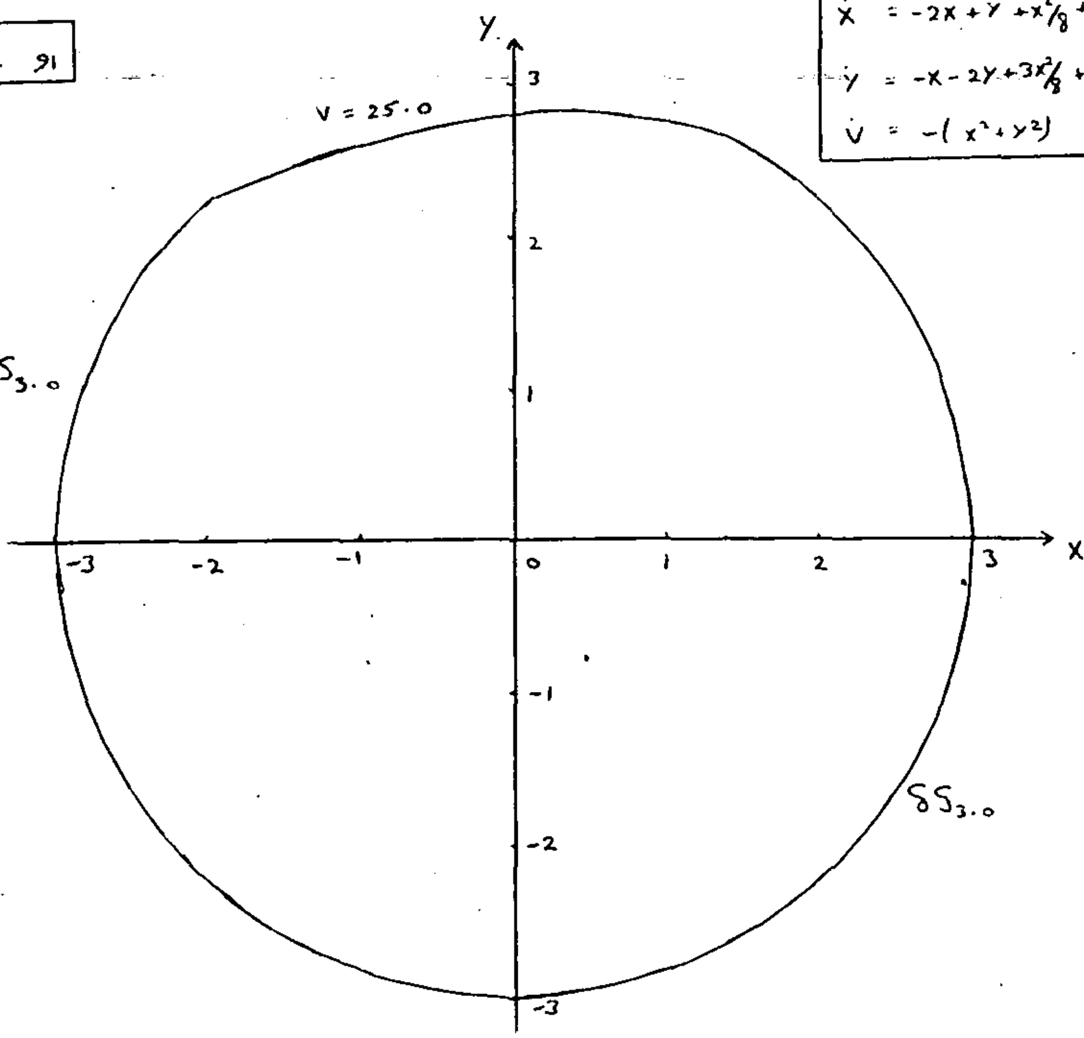


FIG. 92

$$\dot{x} = \cancel{-x} + y + x(x^2+y^2)$$
$$\dot{y} = -x - y + y(x^2+y^2)$$
$$\dot{v} = -(x^2+y^2)(1-v)$$

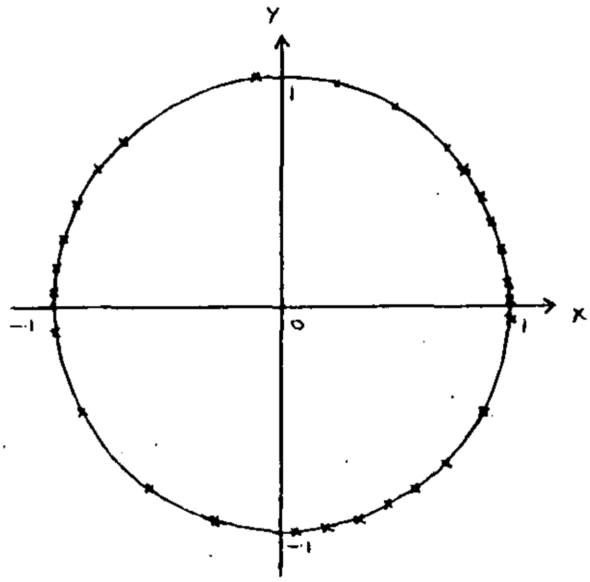


FIG. 94

$$\dot{x} = -2x + 3x^2 - x^3$$
$$\dot{y} = -2y + 3y^2 - y^3$$

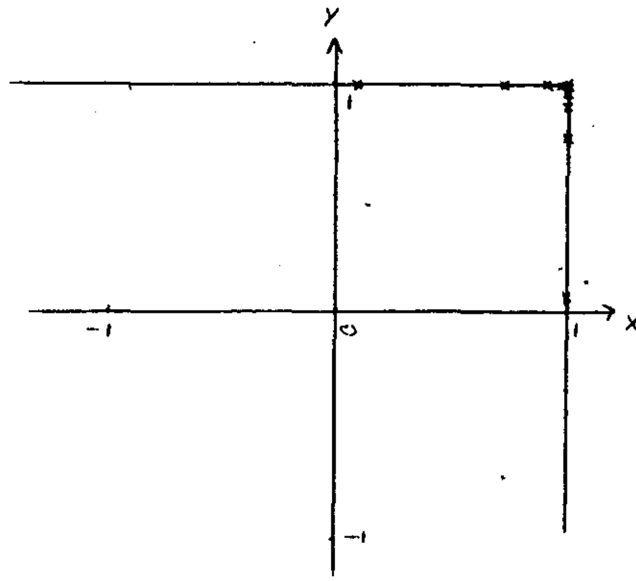


FIG. 93

$$\theta = 30^\circ$$
$$h = 0.625$$

$$\dot{x} = -x + y + x(x^2+y^2)$$
$$\dot{y} = -x - y + y(x^2+y^2)$$
$$\dot{v} = -q(x^2+y^2)(1-v)$$

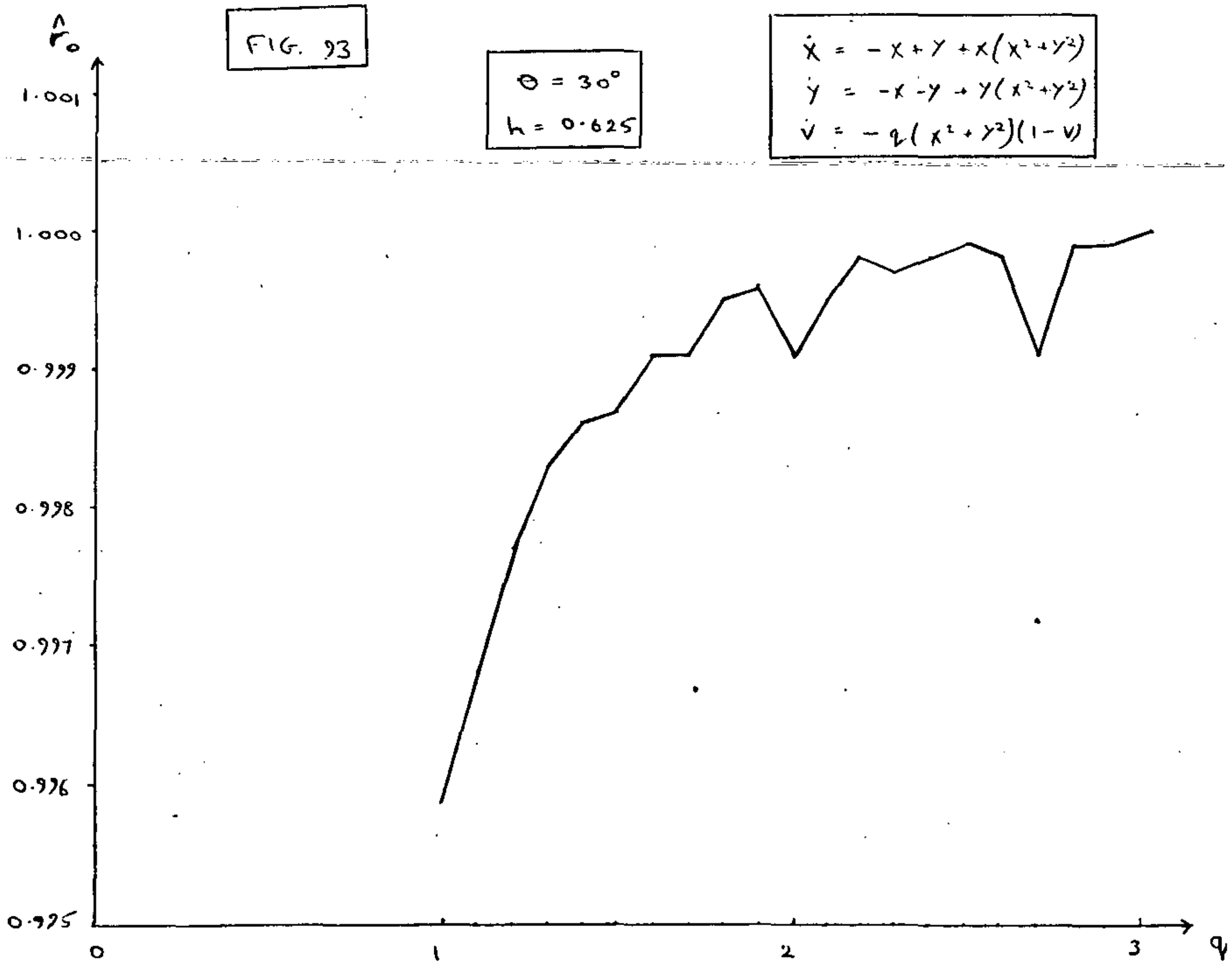


FIG. 95

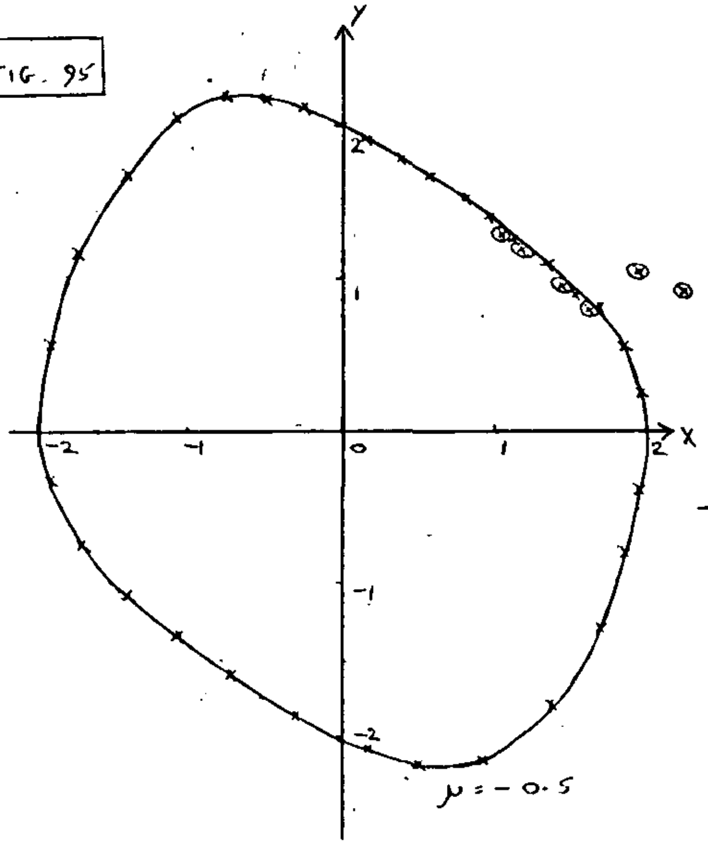
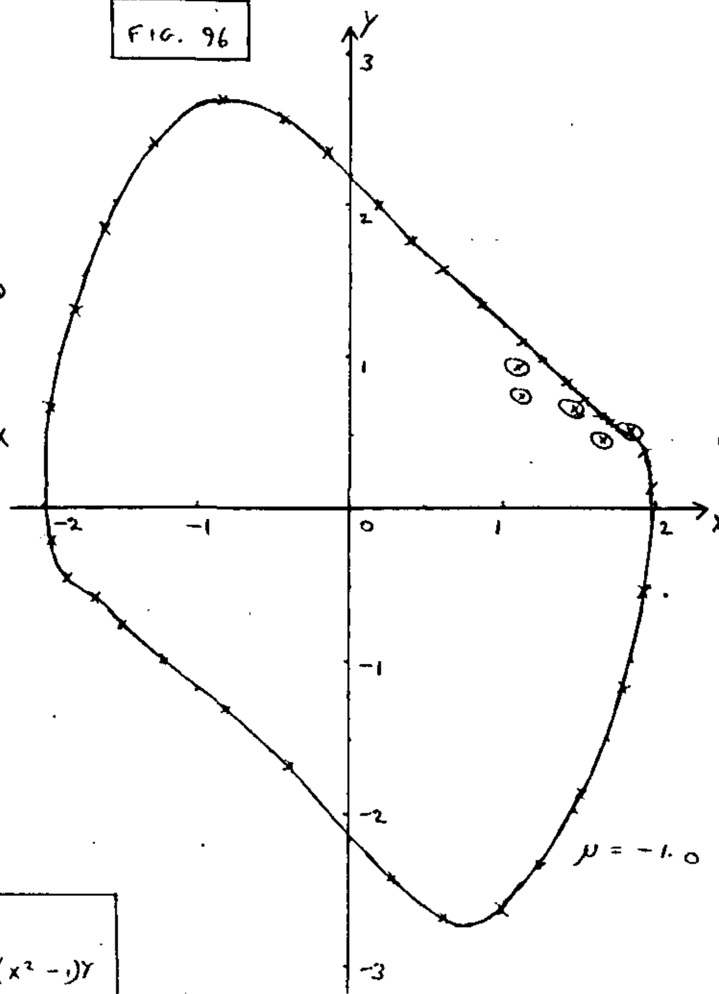


FIG. 96



$$\dot{X} = Y$$
$$\dot{Y} = -X - \mu(X^2 - 1)Y$$
$$\dot{V} = -x^2 - y^2$$

FIG. 97

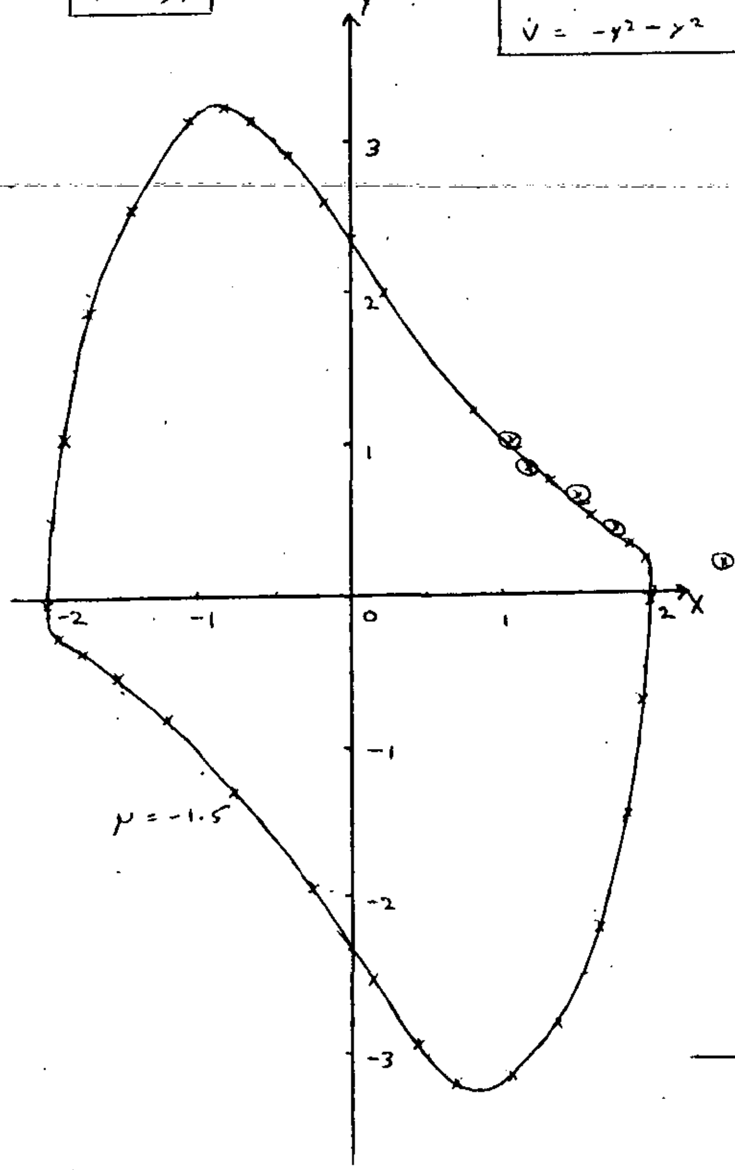


FIG. 101

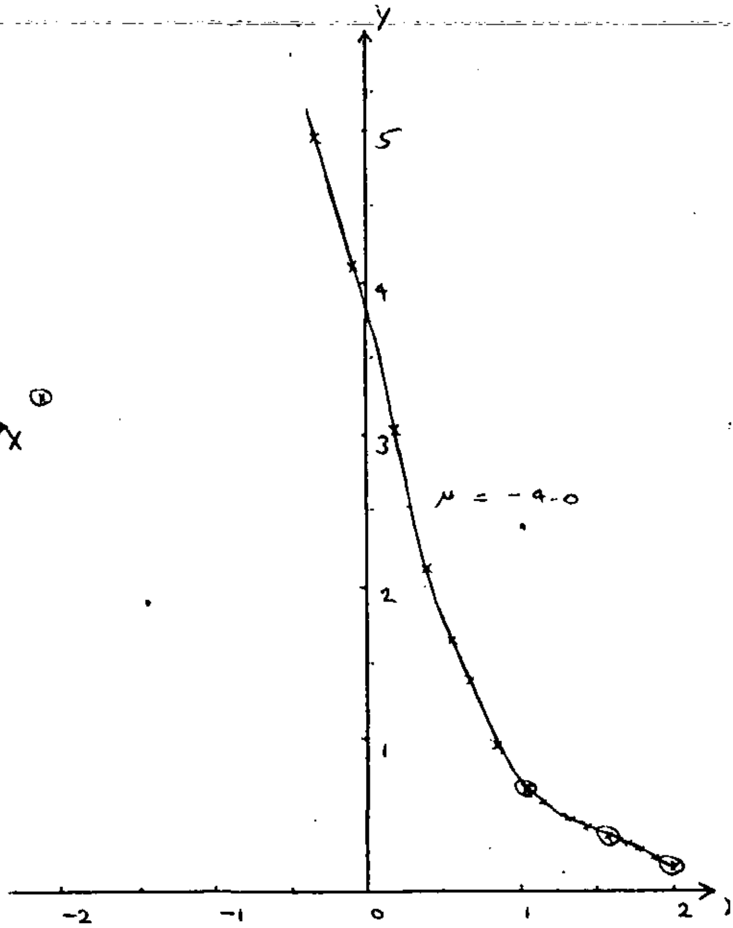


FIG. 98

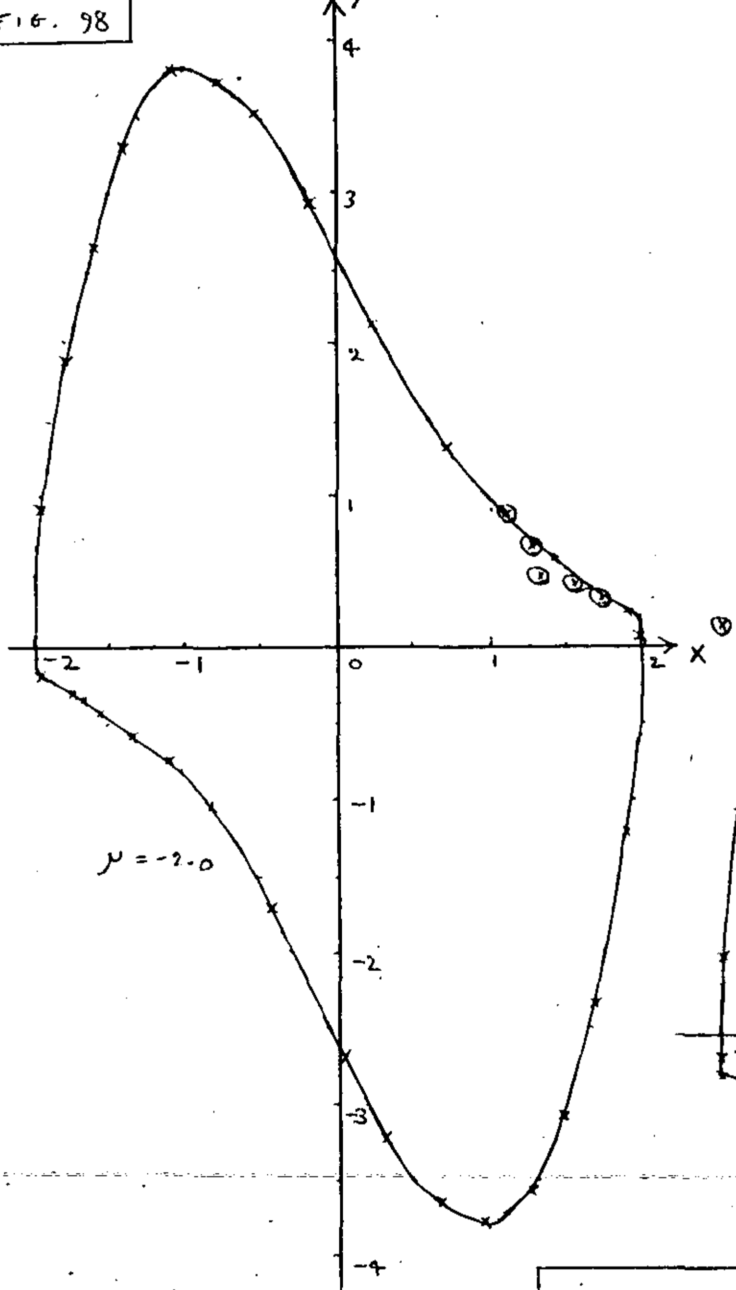
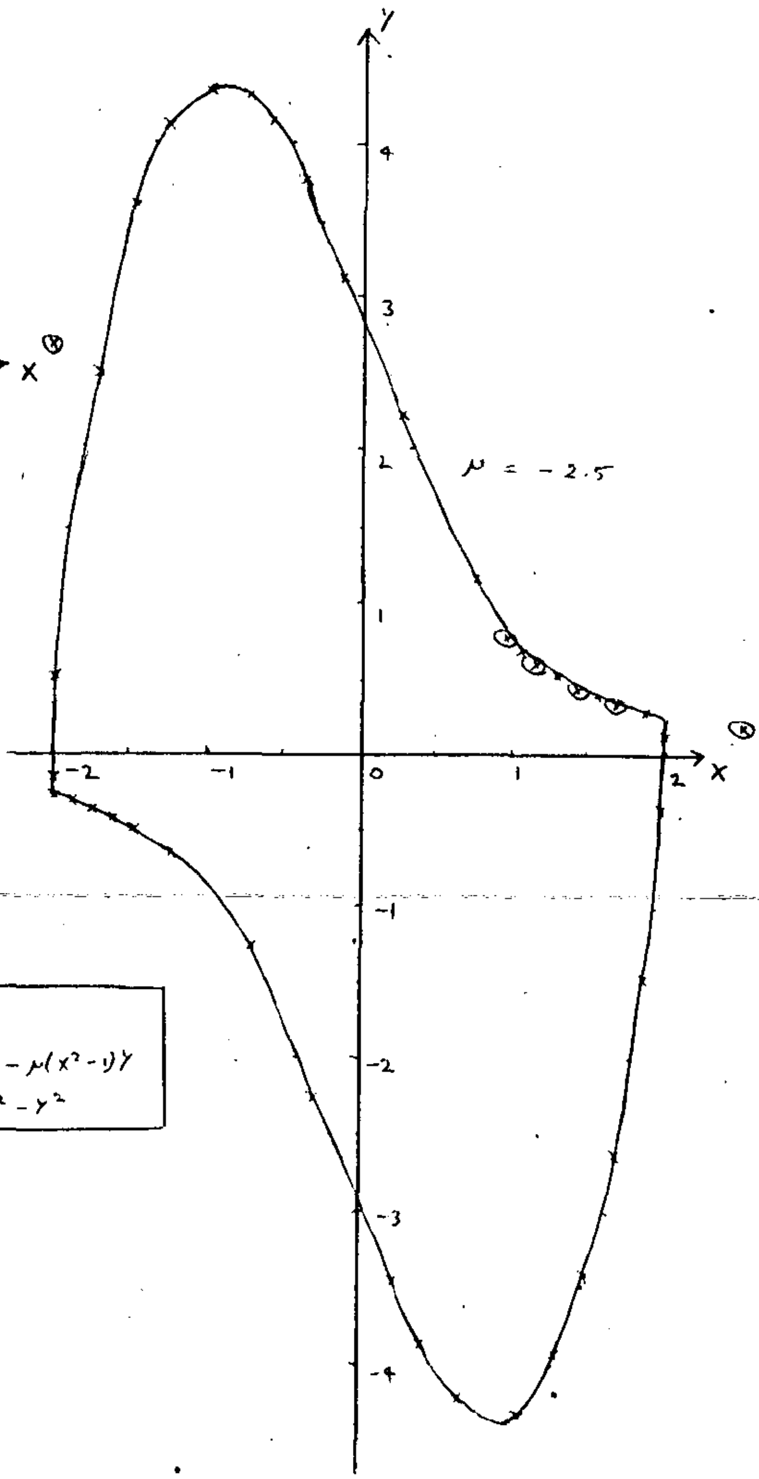


FIG. 99



$$\dot{X} = Y$$
$$\dot{Y} = -X - \mu(X^2-1)Y$$
$$\dot{V} = -X^2 - Y^2$$

FIG. 100

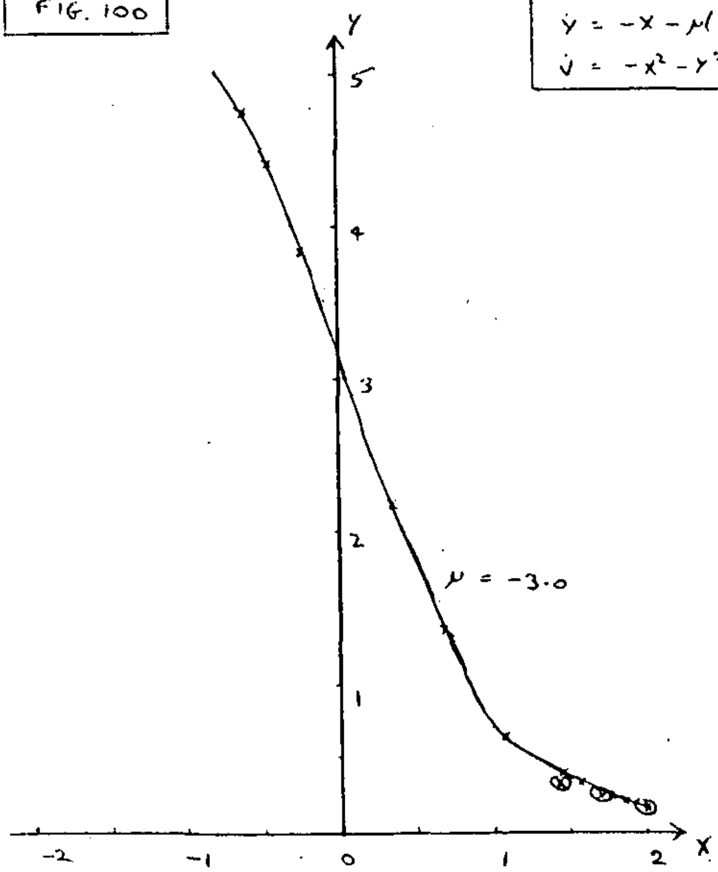
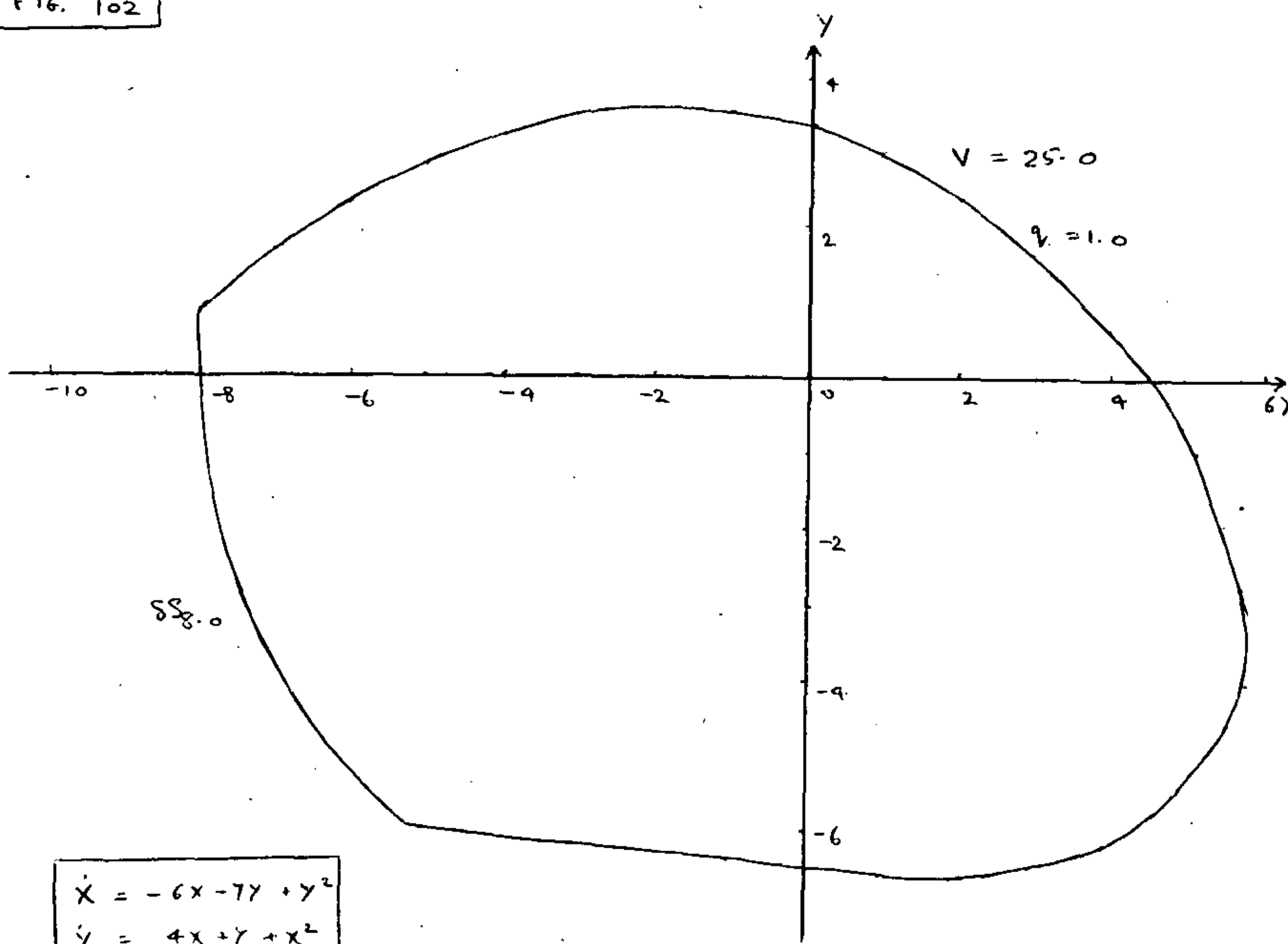


FIG. 102



$$\dot{x} = -6x - 7y + y^2$$
$$\dot{y} = 4x + y + x^2$$
$$\dot{v} = -q(x^2 + y^2)$$

FIG. 103

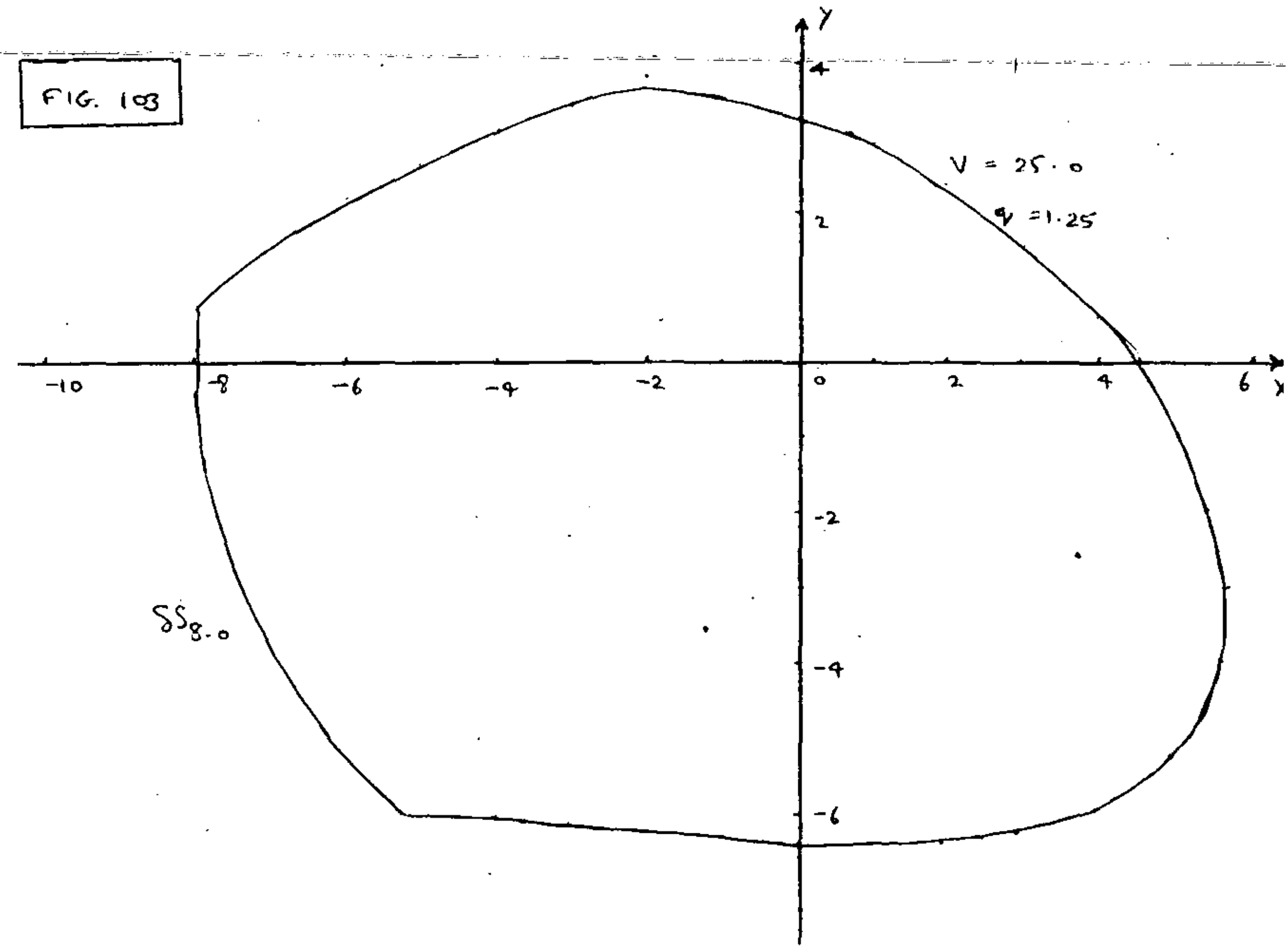


FIG. 104

$$\dot{x} = 6y - 2y^2$$
$$\dot{y} = -10x - y + 4x^2 + 2xy + 4y^2$$
$$\dot{V} = -1.8(x^2 + y^2)$$

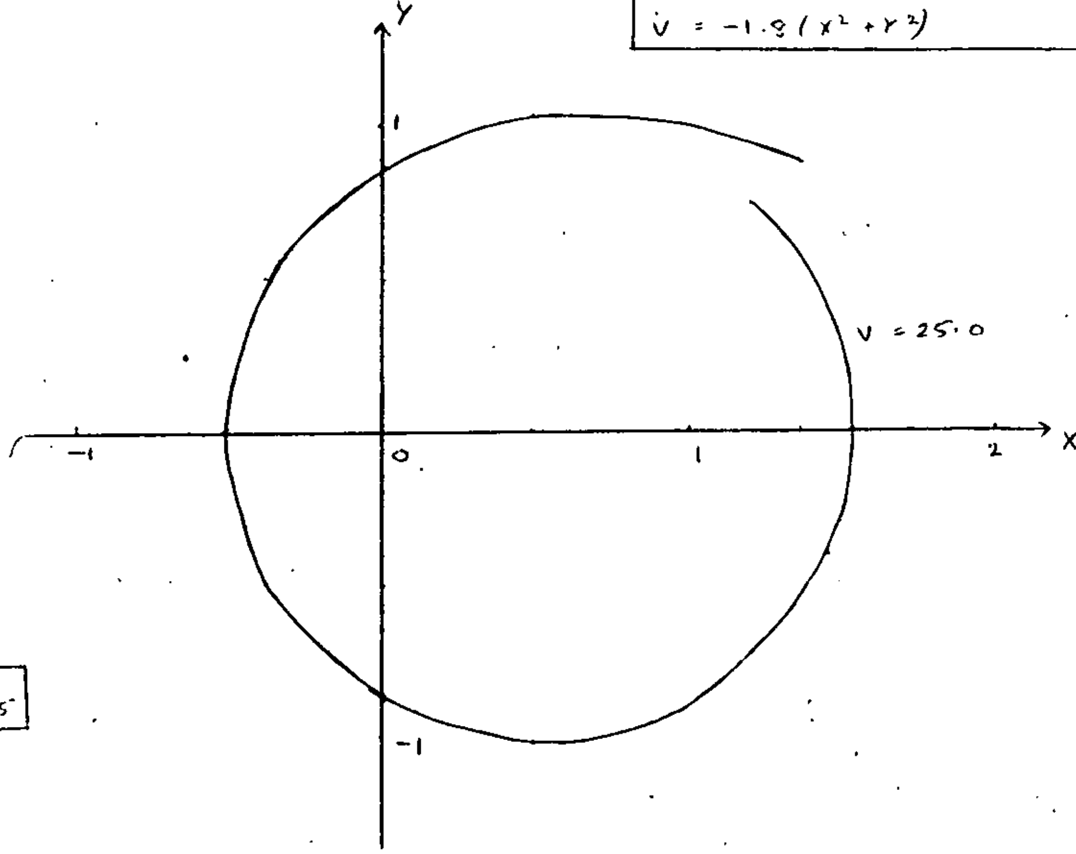
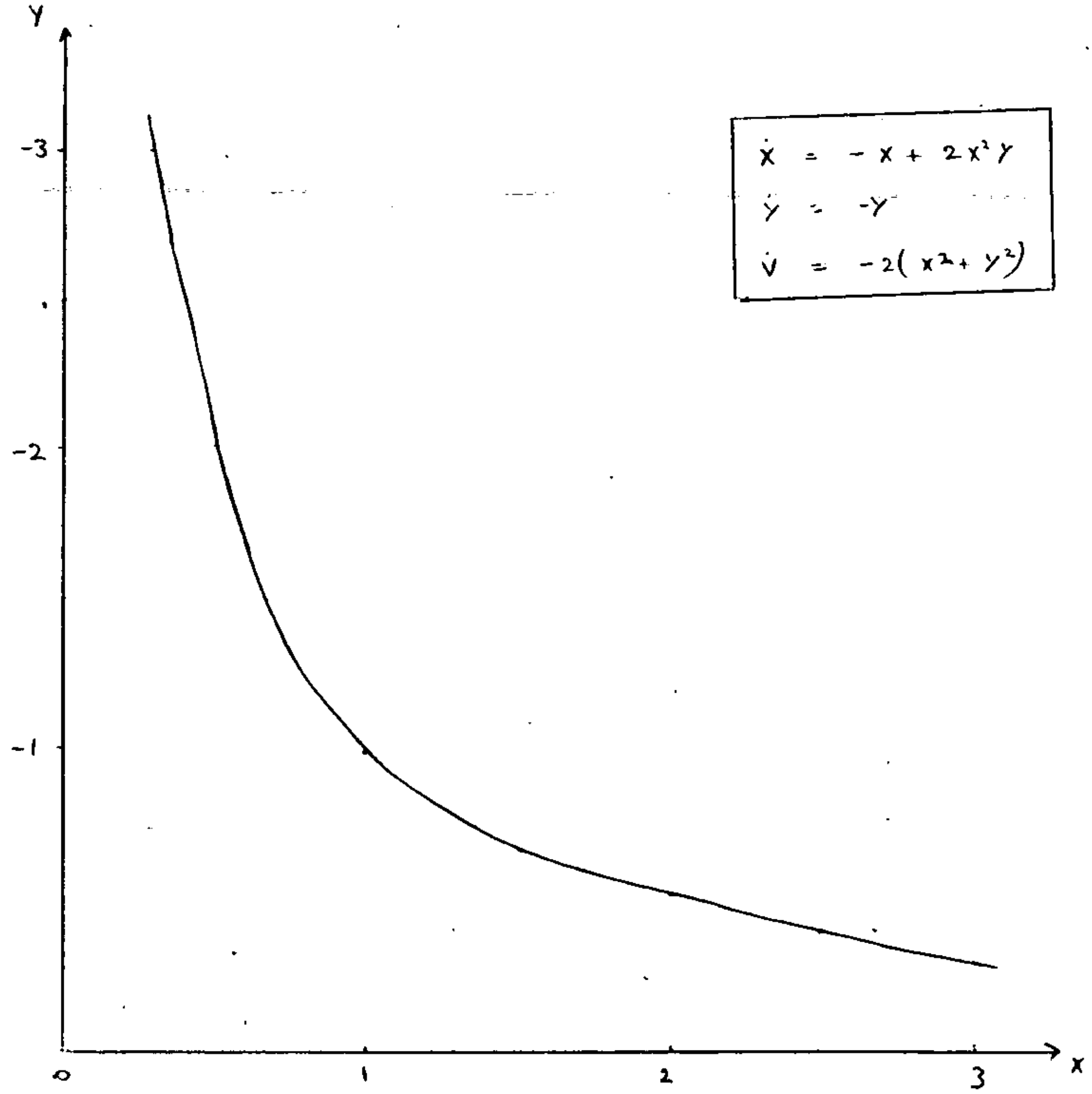


FIG. 105

$$\dot{x} = -x + 2x^2 y$$
$$\dot{y} = -y$$
$$\dot{V} = -2(x^2 + y^2)$$



Appendix A

Theorem

If the system equations are given by

$$\dot{x} = A_{1,1}x + A_{1,2}y + g_1(x,y)$$
$$\dot{y} = A_{2,1}x + A_{2,2}y + g_2(x,y)$$

where $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$ is a stability matrix and where

$g_1(x,y)$, $g_2(x,y)$ have power series expansions of homogeneous degree 2 and above then the series construction for $V(x,y)$ from Zubov's equation

$$(A_{1,1}x + A_{1,2}y + g_1(x,y))\frac{\partial V}{\partial x}(x,y) + (A_{2,1}x + A_{2,2}y + g_2(x,y))\frac{\partial V}{\partial y}$$
$$= -\phi(x,y)(1 - dV(x,y)) \qquad \text{A.1}$$

contains a unique quadratic part providing $\phi(x,y)$ has a series expansion of homogeneous degree 2 and above.

Proof

We may write $\phi(x,y)$, $V(x,y)$ in the form

$$\begin{aligned} \phi(x,y) &= q_{2,0}y^2 + q_{2,1}xy + q_{2,2}x^2 + Q(x,y) \\ V(x,y) &= V_{2,0}y^2 + V_{2,1}xy + V_{2,2}x^2 + W(x,y) \end{aligned} \qquad \text{A.2}$$

where $Q(x,y)$, $W(x,y)$ have terms of degree 3 or greater. Substituting A.2 into A.1 we may then isolate the quadratic terms which are the lowest powers of x,y in A.1.

$$(A_{1,1}x + A_{1,2}y)(V_{2,1}y + 2V_{2,2}x)$$
$$+ (A_{2,1}x + A_{2,2}y)(2V_{2,0}y + V_{2,1}x) \equiv -q_{2,0}y^2 - q_{2,1}xy - q_{2,2}x^2$$
$$\text{A.3}$$

A.3 represents an identity in x and y. Hence the terms in x^2, xy, y^2 must each be zero giving the relationships

$$\begin{aligned} A_{1,2}V_{2,1} + 2A_{2,2}V_{2,0} &= -q_{2,0} \\ A_{1,1}V_{2,1} + 2A_{1,2}V_{2,2} + 2A_{2,1}V_{2,0} + A_{2,2}V_{2,1} &= -q_{2,1} \\ 2A_{1,1}V_{2,2} + A_{2,1}V_{2,1} &= -q_{2,2} \end{aligned} \qquad \text{A.4}$$

Writing A.4 in matrix form

$$\begin{bmatrix} 2A_{2,2} & A_{1,2} & 0 \\ 2A_{2,1} & A_{1,1}+A_{2,2} & 2A_{1,2} \\ 0 & A_{2,1} & 2A_{1,1} \end{bmatrix} \begin{bmatrix} V_{2,0} \\ V_{2,1} \\ V_{2,2} \end{bmatrix} \quad \begin{bmatrix} -q_{2,0} \\ -q_{2,1} \\ -q_{2,2} \end{bmatrix} \qquad \text{A.5}$$

For A.5 to yield a unique solution it is necessary that the determinant of the matrix in A.5 is non-zero. So we need the result

$$(A_{1,1} + A_{2,2})(A_{1,1}A_{2,2} - A_{1,2}A_{2,1}) \neq 0. \qquad \text{A.6}$$

Now we need the properties of a stability matrix. By definition 1.4.7 this means the eigenvalues of A have negative real parts. The eigenvalues of A satisfy

$$\lambda^2 - (A_{1,1}+A_{2,2})\lambda + A_{1,1}A_{2,2} - A_{1,2}A_{2,1} = 0 \qquad \text{A.7}$$

Solving A.7 for λ gives

$$\lambda = \frac{(A_{1,1}+A_{2,2}) \pm \sqrt{(A_{1,1}+A_{2,2})^2 - 4(A_{1,1}A_{2,2}-A_{1,2}A_{2,1})}}{2}.$$

There are two cases depending on the sign of the discriminant $\Delta = (A_{1,1}+A_{2,2})^2 - 4(A_{1,1}A_{2,2}-A_{1,2}A_{2,1})$:

a) If $\Delta < 0$ then we have

$$A_{1,1}A_{2,2}-A_{1,2}A_{2,1} > (A_{1,1}+A_{2,2})^2/4 \geqslant 0$$

and the real parts are $(A_{1,1}+ A_{2,2})/2$ thus showing that $A_{1,1}+ A_{2,2} < 0$ for a stability matrix.

b) If $\Delta > 0$ the roots are both real and since they must be negative we have

$$A_{1,1} + A_{2,2} < 0$$

and $\Delta^2 < (A_{1,1}+ A_{2,2})^2$.

Both cases when put together satisfy equation A.6. Hence A.1 has a unique solution for $V_{2,0}$, $V_{2,1}$, $V_{2,2}$.

End of proof.

Appendix B

Theorem

Given (x_m, y_m) satisfying

$$x_m^2 + y_m^2 = r^2m^2 \qquad \text{B.1}$$

then the straight line through (x_m, y_m) tangential to

$$x^2 + y^2 = r^2m^2 \qquad \text{B.2}$$

passes through the circle

$$x^2 + y^2 = r^2(m+1)^2 \qquad \text{B.3}$$

at (x_{m+1}, y_{m+1}) where

$$x_{m+1} = x_m - \frac{y_m(2m+1)^{\frac{1}{2}}}{m}$$
$$y_{m+1} = y_m + \frac{x_m(2m+1)^{\frac{1}{2}}}{m}. \qquad \text{B.4}$$

Proof.

The straight line through (x_m, y_m) tangential to B.2 is given by

$$xx_m + yy_m = x_m^2 + y_m^2. \qquad \text{B.5}$$

Now using B.4 we see that

$$x_{m+1}x_m + y_{m+1}y_m = x_m^2 - \frac{x_my_m(2m+1)^{\frac{1}{2}}}{m} + y_m^2 + \frac{x_my_m(2m+1)^{\frac{1}{2}}}{m}$$

which means (x_{m+1}, y_{m+1}) satisfies B.5. Also from B.4 we have

$$x_{m+1}^2 + y_{m+1}^2 = x_m^2 - \frac{2x_my_m(2m+1)^{\frac{1}{2}}}{m} + \frac{y_m^2(2m+1)}{m^2} + y_m^2 + \frac{2x_my_m(2m+1)^{\frac{1}{2}}}{m}$$
$$+ \frac{x_m^2(2m+1)}{m^2}.$$

Therefore
$$x_{m+1}^2 + y_{m+1}^2 = \frac{(x_m^2 + y_m^2)(m+1)^2}{m^2}. \qquad \text{B.6}$$

Substituting B.1 into B.6 gives the result that (x_{m+1}, y_{m+1}) satisfies B.3. End of proof.

Appendix C

Theorem

Let $A_i\underline{x} = \underline{0}$, $i = 1,..,n-1$, C.1

be n-1 independent planes in R^n containing the origin. The constants λ_i, $i = 1,\dots,n-1$, can always be found such that any point $\underline{x} \in R^n$ satisfies

$$\lambda_1 A_1\underline{x} + \lambda_2 A_2\underline{x} + \dots + \lambda_{n-1}A_{n-1}\underline{x} = \underline{0}. \qquad \text{C.2}$$

Proof

Writing C.2 in the form

$$\left(\sum_{i=1}^{n-1} \lambda_i A_i\right)\underline{x} = \underline{0} \qquad \text{C.3}$$

we see that C.2 is a set of n-2 linear equations. If we fix $\underline{x}$ we see that C.3 consists of n-2 equations in the unknowns λ_i, $i = 1,\dots,n-1$. Now if we fix λ_{n-1} where we assume $A_{n-1}\underline{x} \neq \underline{0}$ we have

$$\left(\sum_{i=1}^{n-2} \lambda_i A_i\right)\underline{x} = -\lambda_{n-1}A_{n-1}\underline{x}. \qquad \text{C.4}$$

C.4 is a set of n-2 equations in the unknowns λ_i, $i = 1,\dots,n-2$, which can, by the definition of independence of the planes C.1, be solved uniquely. If $A_{n-1}\underline{x} = \underline{0}$ then we may put $\lambda_i = 0$, $i = 1,\dots,n-2$, and $\lambda_{n-1} = 1$. Hence we may find λ_i, $i = 1,\dots,n-1$ such that a particular $\underline{x}$ satisfies C.2. End of proof.

Appendix D

The generalised transformation for rectangular co-ordinates $x_1,\ldots,x_n$ to polar co-ordinates $r,\Theta_1,\ldots,\Theta_{n-1}$ is given by

$$x_1(r,\underline{\Theta}) = r\cos\Theta_1$$

$$x_i(r,\underline{\Theta}) = r\sin\Theta_1\ldots\ldots\sin\Theta_{i-1}\cos\Theta_i \qquad i = 2,\ldots,n-1 \qquad \text{D.1}$$

$$x_n(r,\underline{\Theta}) = r\sin\Theta_1\ldots\ldots\sin\Theta_{n-2}\sin\Theta_{n-1}$$

where $\underline{\Theta}^T = (\Theta_1,\ldots,\Theta_{n-1})$.

Some results follow immediately from D.1.

1) $\sum_{i=1}^{n} x_i(r,\underline{\Theta})^2 = r^2$

2) $x_i(r,\underline{\Theta}) = x_i(r,\underline{\Theta} + 2\pi\underline{I}_j)$ for all $i,j = 1,\ldots,n$, where the jth element of $\underline{I}_j$ is 1 and all the others are zero.

Appendix E

Theorem

The Zubov equation in rectangular co-ordinates

$$\sum_{i=1}^{n} f_i(\underline{x})\frac{\partial V}{\partial x_i}(\underline{x}) = -\phi(\underline{x})(1-dV(\underline{x})) \qquad \text{E.1}$$

is transformed to the Zubov equation in polar co-ordinates

$$F(r,\underline{\theta})\frac{\partial V}{\partial r}(r,\underline{\theta}) + \sum_{i=1}^{n-1} G_i(r,\underline{\theta})\frac{\partial V}{\partial \theta_i}(r,\underline{\theta}) = -\phi(r,\underline{\theta})(1-dV(r,\underline{\theta})) \qquad \text{E.2}$$

by the transformation D.1. The connection between the terms of E.1 and E.2 is given by

$$F(r,\underline{\theta}) = c_1 f_1 + \sum_{k=2}^{n-1}\left(\prod_{j=1}^{k-1} s_j\right)c_k f_k + \left(\prod_{j=1}^{n-1} s_j\right)f_n \qquad \text{E.3}$$

$$rG_1(r,\underline{\theta}) = -s_1 f_1 + c_1 c_2 f_2 + \sum_{k=3}^{n-1} c_1\left(\prod_{j=2}^{k-1} s_j\right)c_k f_k + c_1\left(\prod_{j=2}^{n-1} s_j\right)f_n \qquad \text{E.4}$$

$$r\left(\prod_{j=1}^{i-1} s_j\right)G_i(r,\underline{\theta}) = -s_i f_i + c_i c_{i+1} f_{i+1} + \sum_{k=i+2}^{n-1} c_i\left(\prod_{j=i+1}^{k-1} s_j\right)c_k f_k + c_i\left(\prod_{j=i+1}^{n-1} s_j\right)f_n \qquad \text{E.5}$$

$i = 2,\ldots,n-3.$

$$r\left(\prod_{j=1}^{n-3} s_j\right)G_{n-2}(r,\underline{\theta}) = -s_{n-2}f_{n-2} + c_{n-2}c_{n-1}f_{n-1} + c_{n-2}s_{n-1}f_n \qquad \text{E.6}$$

$$r\left(\prod_{j=1}^{n-2} s_j\right)G_{n-1}(r,\underline{\theta}) = -s_{n-1}f_{n-1} + c_{n-1}f_n \qquad \text{E.7}$$

where $s_j = \sin\theta_j$, $c_j = \cos\theta_j$ for $n \geqslant 4$.

Proof

Using the chain rule of differentiation we have

$$\frac{\partial V}{\partial r} = \sum_{k=1}^{n}\frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial r} \qquad \text{E.8}$$

$$\frac{\partial V}{\partial \theta_i} = \sum_{k=1}^{n}\frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial \theta_i} \qquad i = 1,\ldots,n-1. \qquad \text{E.9}$$

Using the transformation D.1 and differentiating with respect to r and θ_i gives

$$\frac{\partial x_1}{\partial r} = c_1 \qquad \text{E.10}$$

$$\frac{\partial x_i}{\partial r} = \left(\prod_{j=1}^{i-1} s_j\right)c_i \qquad i = 2,\ldots,n-1. \qquad \text{E.11}$$

$$\frac{\partial x_n}{\partial r} = \left(\prod_{j=1}^{n-1} s_j\right) \qquad \text{E.12}$$

$$\frac{\partial x_i}{\partial \theta_i} = -r\left(\prod_{j=1}^{i} s_j\right) \qquad i = 1,\ldots,n-1. \qquad \text{E.13}$$

$$\frac{\partial x_2}{\partial \theta_1} = rc_1 c_2 \qquad \text{E.14}$$

$$\frac{\partial x_i}{\partial \theta_1} = rc_1\left(\prod_{j=2}^{i-1} s_j\right)c_i \qquad i = 3,\ldots,n-1. \qquad \text{E.15}$$

$$\frac{\partial x_n}{\partial \theta_1} = rc_1\left(\prod_{j=2}^{n-1} s_j\right) \qquad \text{E.16}$$

$$\frac{\partial x_i}{\partial \theta_{i-1}} = r\left(\prod_{j=1}^{i-2} s_j\right)c_{i-1}c_i \qquad i = 3,\ldots,n-1. \qquad \text{E.17}$$

$$\frac{\partial x_i}{\partial \theta_k} = r\left(\prod_{j=1}^{k-1} s_j\right)c_k\left(\prod_{j=k+1}^{i-1} s_j\right)c_i \qquad i = 4,\ldots,n-1. \quad k = 2,\ldots,i-2. \qquad \text{E.18}$$

$$\frac{\partial x_n}{\partial \theta_k} = r\left(\prod_{j=1}^{k-1} s_j\right)c_k\left(\prod_{j=k+1}^{n-1} s_j\right) \qquad k = 2,\ldots,n-2. \qquad \text{E.19}$$

$$\frac{\partial x_n}{\partial \theta_{n-1}} = r\left(\prod_{j=1}^{n-2} s_j\right)c_{n-1} \qquad \text{E.20}$$

$$\frac{\partial x_i}{\partial \theta_k} = 0 \qquad i = 1,\ldots,n-2. \quad k = i+1,\ldots,n-1. \qquad \text{E.21}$$

Substituting E.10 to E.21 into E.8 and E.9 gives a set of linear equations for $\frac{\partial V}{\partial r}$, $\frac{\partial V}{\partial \theta_k}$, $k = 1,\ldots,n-1$, in terms of $\frac{\partial V}{\partial x_i}$, $i = 1,\ldots,n$. These may be inverted to establish $\frac{\partial V}{\partial x_i}$ in terms of $\frac{\partial V}{\partial r}$, $\frac{\partial V}{\partial \theta_k}$ to give

$$\frac{\partial V}{\partial x_1} = c_1\frac{\partial V}{\partial r} - \frac{s_1}{r}\frac{\partial V}{\partial \theta_1} \qquad \text{E.22}$$

$$\frac{\partial V}{\partial x_2} = s_1c_2\frac{\partial V}{\partial r} + \frac{c_1c_2}{r}\frac{\partial V}{\partial \theta_1} - \frac{s_2}{rs_1}\frac{\partial V}{\partial \theta_2} \qquad \text{E.23}$$

$$\frac{\partial V}{\partial x_3} = s_1s_2c_3\frac{\partial V}{\partial r} + \frac{c_1s_2c_3}{r}\frac{\partial V}{\partial \theta_1} + \frac{c_2c_3}{rs_1}\frac{\partial V}{\partial \theta_2} - \frac{s_3}{rs_1s_2}\frac{\partial V}{\partial \theta_3} \qquad \text{E.24}$$

$$\frac{\partial V}{\partial x_i} = \left(\prod_{j=1}^{i-1} s_j\right)c_i\frac{\partial V}{\partial r} + \frac{c_1}{r}\left(\prod_{j=2}^{i-1} s_j\right)c_i\frac{\partial V}{\partial \theta_1} + \sum_{k=2}^{i-2}\frac{c_k}{r}\frac{\left(\prod_{j=k+1}^{i-1} s_j\right)c_i}{\left(\prod_{j=1}^{k-1} s_j\right)}\frac{\partial V}{\partial \theta_k}$$

$$+ \frac{c_{i-1}c_i}{r\left(\prod_{j=1}^{i-2} s_j\right)}\frac{\partial V}{\partial \theta_{i-1}} - \frac{s_i}{r\left(\prod_{j=1}^{i-1} s_j\right)}\frac{\partial V}{\partial \theta_i} \qquad i = 4,\ldots,n-1. \qquad \text{E.25}$$

$$\frac{\partial V}{\partial x_n} = \left(\prod_{j=1}^{n-1} s_j\right)\frac{\partial V}{\partial r} + \frac{c_1}{r}\left(\prod_{j=2}^{n-1} s_j\right)\frac{\partial V}{\partial \theta_1} + \sum_{k=2}^{n-2}\frac{c_k}{r}\frac{\left(\prod_{j=k+1}^{n-1} s_j\right)}{\left(\prod_{j=1}^{k-1} s_j\right)}\frac{\partial V}{\partial \theta_k} + \frac{c_{n-1}}{r\left(\prod_{j=1}^{n-2} s_j\right)}\frac{\partial V}{\partial \theta_{n-1}} \qquad \text{E.26}$$

Substituting E.22 to E.26 into E.1 we may collect together the the terms in $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial \theta_i}$, $i = 1,\ldots,n-1$, using E.2 to establish the results E.3 to E.7. End of proof.

The results for n = 2,3 are similarly proved but limits and variation of subscripts are less complicated than those above.

Appendix F

Define $Y_n(x) = -\sum_{i=1}^{n} \frac{(-x)^i}{i!}$ F.1

Theorem

1) For even n, $Y_n(x)$ possesses just one zero x_n in $x > 0$ and $x_n \leq n$,

$$Y_n(x) > 0 \text{ for } x < x_n, \qquad \text{F.2}$$

$$Y_n(x) < 0 \text{ for } x > x_n. \qquad \text{F.3}$$

2) For odd n, $Y_n(x) > 0$ for all $x > 0$.

Proof

The proof is by induction. We assume that for even n, $Y_{n-2}(x)$ satisfies part 1) of the theorem.

Now from F.1 we know that

$$Y_{n-1}(x) = Y_{n-2}(x) + \frac{x^{n-1}}{(n-1)!}. \qquad \text{F.4}$$

Now $\frac{x^{n-1}}{(n-1)!} > 0$ for $x > 0$ and from F.2 we see that

$$Y_{n-2}(x) > 0 \text{ for } 0 < x < x_{n-2} \qquad \text{F.5}$$

and hence by F.4 $Y_{n-1}(x) > 0$ for $0 < x < x_{n-2}$. F.6

Differentiating F.1 with respect to x we obtain the relationship

$$Y'_{n-1}(x) = 1 - Y_{n-2}(x) \qquad \text{F.7}$$

and from F.3 we see that

$$Y'_{n-1}(x) > 0 \text{ for } x > x_{n-2}. \qquad \text{F.8}$$

Hence upon integrating F.8 with respect to x we see that

$$Y_{n-1}(x) > Y_{n-1}(x_{n-2}) = \frac{x_{n-2}^{n-1}}{(n-1)!} > 0 \qquad \text{F.9}$$

for $x > x_{n-2}$.

Combining F.6 and F.9 we have proved that $Y_{n-1}(x) > 0$ for $x > 0$. To prove the theorem for $Y_n(x)$ also we require the relation

$$Y_n(x) = Y_{n-2}(x) + \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!}. \qquad \text{F.10}$$

For $0 < x < x_{n-2}$ we know F.5 holds and since $x_{n-2} < n$ we observe that

$$\frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} > 0 \qquad \text{F.11}$$

for $0 < x < x_{n-2}$.

Hence from F.5 and F.11 we see that substitution in F.10 shows that

$$Y_n(x) > 0 \text{ for } 0 < x < x_{n-2}. \qquad \text{F.12}$$

Also from F.10

$$Y_n(n) = Y_{n-2}(n) < 0$$
$$Y_n(x) < Y_{n-2}(x) < 0 \text{ for } x > n. \qquad \text{F.13}$$

Hence $Y_n(x)$ has at least one zero in $x_{n-2} < x < n$. We have to prove there is only one zero. For this we take another version of F.7

$$Y_n'(x) = 1 - Y_{n-1}(x). \qquad \text{F.14}$$

From F.8 we know that $Y_{n-1}(x)$ is an increasing function for $x > x_{n-2}$. Hence F.14 shows that $Y_n'(x)$ has at most one zero for $x > x_{n-2}$. If there existed more than one x such that $Y_n(x) = 0$, $x_{n-2} < x < n$, there would exist at least two values of x for which $Y_n'(x) = 0$. Hence there is only one zero of $Y_n(x)$ for $x_{n-2} < x < n$ and by F.12 and F.13 there is only one zero of $Y_n(x)$ for $x > 0$. To complete the theorem we need to show that

$$Y_2(x) = x - \frac{x^2}{2}$$

satisfies the theorem. Now $Y_2(x) > 0$ for $0 < x < 2$ and $Y_2(x) < 0$ for $x > 2$. Although $x_2 = 2$ we see by F.13 that $Y_4(4) = Y_2(4) < 0$ and $x_n < n$ for $n > 2$.

This completes the proof.

Conclusions

Throughout this thesis the emphasis has been on how to solve the Zubov equation, whether it can be solved, and whether the results of solution are meaningful. The Zubov equation is extremely useful in theory on account of its providing a Lyapunov function which indicates the complete D.O.A.. The trouble has always been deeply rooted in being able to obtain this function either analytically or numerically. This has led to the series construction which is well known as having problems of convergence of the R.A.S.s to the D.O.A.. The numerical construction attempted by Shields (28) was found in that work to break down. However, the main results are based on numerical construction procedures and an in-depth look at the problems and ways around them has been attempted. It has been shown that when the Zubov equation is tackled in a stable manner it is possible to obtain an algorithm which gives estimates of the D.O.A..

An attempt to compare methods of solution of Zubov's equation and finding D.O.A.s has been made in Chapter 2. This has served to bring together the methods and problems before proceeding to solve the Zubov equation in a numerical way.

The series construction is the most popular method for obtaining approximations to $V(\underline{x})$ and $D(\underline{f})$. Its problems include those of non-uniform convergence, complicated equations and even possible breakdown of the construction. Usual examples on which the series method is based are ones in which $\underline{f}(\underline{x})$ has a linear part and $\phi(\underline{x})$ has a quadratic part, as well as both $\underline{f}$ and ϕ having a series expansion. The series construction then has no problems as far as obtaining $V^{(N)}(\underline{x})$ is concerned. It has been found by several authors to give non-uniformly convergent R.A.S.s. Even when a $V^{(N)}(\underline{x})$ is found the difficulties of finding $p^{*}(\underline{f},V)$ analytically are immense. The series construction for systems without linear parts is found to be not necessarily absolutely determined. Examples showed that some ϕ enable V to be determined and not others in this case. A possible topic for research is to consider the relationship of ϕ to $\underline{f}$ such that $V(\underline{x})$ can be obtained in series form.

The Lie series method has slightly different but similar problems. It, too, is seen to indicate non-uniformly convergent R.A.S.s, and it has a bigger drawback than the series constructio For, whereas the series method is at least conservative, the Lie series construction does not guarantee this. Any truncation of a series such as the Lie series involves a Local Truncation Error. This error increases without limit as t becomes large and negative and there is no equivalent restriction to $\dot{V} = 0$ holding the computation back. This computational instability renders the method hazardous. The only answer to computational instability is to reverse the whole problem and compute from the unstable end. The other obvious drawback to Lie series is the reliability on complete differentiability of $\underline{f}(\underline{x})$.

Transformations represented a possible field of study to find solutions in terms of other variables which may simplify the problem. We have yet to find a transformation which helps in any way in general.

The geometric view of Lyapunov's second method is an interesting possibility. Infante and Clark do not directly use Zubov's approach but obtain the quantity $\hat{W}$ as the magnitude of a vector W which is akin to setting $\dot{V}$ as $-\phi$. However the $\hat{W}$ so obtained is not in general strictly positive definite. It may be a research topic to modify this method round to Zubov's approach by arranging for $\hat{W}$ to be strictly positive definite.

Rodden's computational algorithm has a lot of advantages in its incorporating iteration to track along curves. It seems to present possible problems, though not insurmountable ones, in higher orders.

Davidson and Cowan, and Texter each attempted to define a way of deciding if a trajectory was stable. They are not generalisable to 3 or more dimensions, and even in two dimensions require subjective decisions on trajectories which are not in D.O.A.s bounded by limit cycles.

At the end of all these comparisons and studies, which are admittedly not a complete comparison covering all aspects, it seemed that the Zubov equation still required a method of solution which is convegent, accurate, conservative. Obtaining $R^*(\underline{f},V)$ for given $\underline{f}$ and V would seem to be best done by Rodden, but as yet we still require to obtain $D(\underline{f})$ rather than $R_N(\phi,\underline{f})$.

by Zubov's method.

The numerical results in Chapters 4,5,6 form the important part of this thesis. In the course of computation certain aspects of the theory of the Zubov equation came to light which seem to be hitherto unconsidered. As mentioned previously, the Zubov equation has a unique solution if $\underline{f}$ has a linear part and ϕ has a quadratic part but not necessarily otherwise.

In Chapter 3 the question of which ϕ enable the Zubov equation to be solved given $\underline{f}$ seems to have been answered. Zubov himself states that ϕ must be chosen relative to the rate of decrease of $\underline{x}(t)$. However the rate of decrease of $\underline{x}(t)$ is fundamentally tied in with the behaviour of $\underline{f}(\underline{x})$ near the origin. Since we do not need the Zubov equation if we can obtain $\underline{x}(t)$ we can see that inspection of $\underline{f}(\underline{x})$ is the more logical way to go to choose $\phi(\underline{x})$. The result of theorem 3.6.5 ties down the choice absolutely except in the exceptional circumstances which are mentioned in theorem 3.6.4. No example has been met satisfying this exceptional property but it may be a matter for future research to investigate whether such cases can happen and what happens to the Zubov construction if they do.

The author's definition of asymptotic degree is probably not new but acts as a very useful tool in this theory. One thing that maybe could be a difficulty in choosing $\phi(\underline{x})$ by this method is that of actually obtaining $P(\underline{f})$. It has been shown that only radial lines need be considered but no mention is made of how to go about choosing radial lines to establish the asymptotic degree of a function. It is felt that this is a field of its own and only the actual result is of relevance to this thesis.

The one-dimensional Zubov equation is altogether simpler to solve as it becomes an O.D.E.. Correspondingly the system equation can also be directly integrated and the one-dimensional Zubov equation is found to possess no great advantages over finding $x(t)$. The results of finding x as $t \to -\infty$ or as $V \to +\infty$ are much the same and $\phi(x)$ can only possibly make computing a little easier. The later chapters show enormous advantages in finding $\underline{x}(V)$ rather than $\underline{x}(t)$ but these do not apply to one dimension. Theorem 4.2.1 explained that the sign of $\dot{x}$ is the only thing that matters and to find the sign of $\frac{dx}{dV}$

instead is no real improvement.

The significant result of Chapter 4 is the asymptotic analysis near the critical point $x' \neq 0$. This again can be done in terms of either t or V, but shows how successive computed values of x(t) or x(V) can be used to obtain an estimate for x' based on the theoretical known behaviour of x(t) or x(V) when f(x) can be approximated near x'. It is based on recognising the pattern of the computed x's as being from a theoretical differential equation which the actual D.E. approximates near the singularity. Such recognition has to be slightly subjective but results have shown that the pattern is often quite obvious.

No detail has been gone into about whether the estimate of x' is always conservative and here, too, is a topic for further work, but we may be able to iterate on estimates of x' from different values of V and $\delta(p,V)$ to see what happens, although for the examples considered the estimate of x' is almost constant after a certain point.

In Chapter 5 the finite difference methods have been extensively considered. Shields (28) consideration of the finite difference scheme was found to be disappointing and there is a fairly obvious reason why. Namely that implicit methods cannot obtain values of V on any circle where V is infinite. Thus other finite difference shemes have been considered and moulded in various ways to try to establish a method which will indicate when $V = \infty$.

Unfortunately there is a jungle of problems to hack through to get to the boundary of D($\underline{f}$). First, it may not be possible to get off the ground at all unless initial conditions are chosen arbitrarily. The more accurate the method chosen the greater seems to be the problem near the origin, except for implicit schemes. There seems little doubt that an implicit scheme such as the Shields' Method is the best way to begin computation.

The need to revert to an explicit method as successive computation of V on concentric circles approaches the boundary is clearly seen. Also clearly seen is the necessity of a method such as that in section 5.8 for guaranteeing that all points used in the difference scheme are actually within the D.O.A..

Having shaped the method around the obvious problems there are the hidden ones of accuracy, convergence and stability. Theoretical results have been established for these properties and the method becomes considerably complicated to take them into account. At this stage it seems that all considered methods have strengths and weaknesses and it is difficult to decide which to use especially if only the grid numbers are known. An "optimum" has been suggested which can probably be considerably improved.

Finally in this method is the one problem which seems to sabotage the explicit schemes considered. This is caused by zero coefficients of the P.D.E. being solved. This is a difficult problem to get around which must involve a fundamental re-think of order of computation. The philosophy of computing all points on a circle is a good one, but in the case of explicit methods is bound to involve the coefficient of the unknown term being dependent on F and possibly Φ as well. It may be necessary if such methods are to be pursued to think how the computation could be done in a different order so the coefficient of the unknown term is never small.

It has been realised already that the only way to compute problems which are basically unstable is to turn them around. This is what the algorithm of Chapter 6 does. An attempt has been made to compute from a point near the boundary to determine if it is inside the D.O.A. or not. Texter, and Davidson and Cowan do this, but they use t as the independent variable. This is where the use of V to compute $\underline{x}(V)$ rather than $\underline{x}(t)$ or $V(\underline{x})$ has the advantage. Determination of the behaviour of a trajectory with time is to some extent subjective as $\underline{x}(t) \rightarrow 0$ as $t \rightarrow \infty$. However it has been shown that $\underline{x}(V) \rightarrow 0$ as $V \rightarrow 0$ providing that ϕ is chosen correctly and the "finiteness" of computation seems to be a great help. Thus from limited computation we see that it can be determined whether an initial point is inside the D.O.A. or not. It then remains to vary that initial point and to play around with step-sizes, accuracy parameters etc., to obtain the complete D.O.A..

This algorithm goes some way to satisfying the need for an algorithm which finds D.O.A.s accurately and shows that the Zubov equation can be used to find good estimates of $D(\underline{f})$. It can still be improved and some suggestions were made at the end of Chapter 6.

In conclusion it has been shown that the Zubov approach, although not in its original form, can lead to a method of finding accurate R.A.S.s. The approach of computing $\underline{x}(V)$ which has come out of this thesis is a different parameterisation and could be arrived at without reference to the Zubov equation. But the basic theory of Lyapunov and Zubov provided a great help in devising a parameterisation which is useful.

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      SEND TO (ED,SEMICOMPUSER.AXXX)
      DUMP ON (ED,PROGRAM USER)
      WORK (ED,WORKFILEUSER)
      RUN

      PROGRAM (PROG)
      COMPRESS INTEGER AND LOGICAL
      INPUT 1,5=CR0
      OUTPUT 2,6=LP0
      TRACE 2
      END

      MASTER DCHALL
      DIMENSION NPUT(200),ERMIN(200),H(4),VO(4),NSTART(4),NO(4),NSPUR(1
     10),JD(200)
  146 FORMAT(I0)
    1 FORMAT(9F0.0,3I0)
    7 FORMAT(1H0,10HRADIUS   = ,F7.4,3X,10HDELTA R = ,F7.4,3X,10HEBSILON
     1= ,F7.4,3X,10H1ST ANGLE ,F7.2,3X,10HANG. ACC. ,F7.2,3X,10HANG. IN
     2. ,F7.2)
   59 FORMAT(1H ,3HMU ,F5.2,3X,3HQU ,F5.2,3X,10HSTEP SIZE ,F7.4,3X,11HT
     1Y NUMBER ,I1,3X,9HEQUATION ,I1,3X,18HNO. OF STEP SIZES ,I1,3X,15H
     2RINT OUT RATE ,I2)
  153 FORMAT(F0.0,I0,I0)
    4 FORMAT(1H ,20X,15HINCREMENT OF V ,F7.4,3X,14HSTARTING STEP ,I3,3X
     115HINCREMENT STEP ,I3)
   57 FORMAT(1H0,3X,5HTHETA,3X,10HVALUE OF V,8X,9HINITIAL X,14X,9HINITI
     1 Y,11X,14HFINAL DISTANCE,3X,9HITERATION,3X,11HSTEP NUMBER,3X,1HJ
  136 FORMAT(1H ,8HUNSTABLE)
      PI=3.1415926538
      READ(1,146)JN
      DO 152 L152=1,JN,1
      READ(1,1)RO,RINCRO,EBSILON,ATETHO,DELTA1,DELTA2,U,Q,HEX,MV,JM,NJ
      JC=0
      JE=0
      JB=1
      W=1.0
      WRITE(2,7)RO,RINCRO,EBSILON,ATETHO,DELTA1,DELTA2
      WRITE(2,59)U,Q,HEX,L152,MV,JM,NJEX
      ATETH=ATETHO
      DO 181 L181=1,JM,1
      READ(1,153)VO(L181),NSTART(L181),NO(L181)
      WRITE(2,4)VO(L181),NSTART(L181),NO(L181)
```

```
181 CONTINUE
    JDIS=1
 30 WRITE(2,57)
    THETA=ATETH*PI/180.0
    VV=0.0
    JC=0
    DO 147 L147=1,JM,1
    VV=VV+V0(L147)
    NPUT(1)=0
    ERMIN(1)=1.0E+60
    DIFFMAX=1.0E+60
    TIFFMAX=1.0E+60
    J=0
    K=2
    NPUT(2)=NSTART(L147)
 12 CALL CONTIN(R0,RINCR0,THETA,ATETH,EBSILON,V0(L147),MV,N,NPUT(K),L1
   147,U,Q,ERMIN(K),JC)
    IF(ERMIN(K)-ERMIN(K-1)+(NPUT(K)-NPUT(K-1))*TIFFMAX)25,24,24
 25 J=J+1
    NSPUR(J)=NPUT(K)
    KJS=0
 74 K=K+KJS
    IF(NO(L147)-1)63,63,119
 63 NPUT(K)=NPUT(K-KJS)+NSTART(L147)
    GO TO 135
119 NPUT(K)=NPUT(K-KJS)+NO(L147)
135 IF(NPUT(K)-250)12,21,21
 24 IF(NO(L147)-1)52,10,70
 52 NMAX2=NPUT(K)
    EMAX2=ERMIN(K)
    GO TO 21
 10 IF(ABS(ERMIN(K)-ERMIN(K-1))-EBSILON)52,52,72
 70 IF(ERMIN(K)-ERMIN(K-1)+EBSILON)72,11,11
 72 DIFFMAX=TIFFMAX
    TIFFMAX=ABS((ERMIN(K-1)-ERMIN(K))/(NPUT(K)-NPUT(K-1))*EBSILON)
    KJS=1
    GO TO 74
124 K=K+1
    NPUT(K)=NMAX2+NO(L147)
    GO TO 12
 11 NMAX1=NPUT(K-2)
    NMAX2=NPUT(K-1)
    NMAX3=NPUT(K)
    EMAX1=ERMIN(K-2)
    EMAX2=ERMIN(K-1)
    EMAX3=ERMIN(K)
    JMAX1=2
    JMAX2=2
 17 GO TO(13,14),JMAX1
 13 GO TO(21,18),JMAX2
 14 K=K+1
    NMID=(NMAX1+NMAX2)/2
    IF(J)46,46,103
103 NMID1=NMID
    NMID2=NMID+1
    DO 37 L37=1,NMID-NMAX1,1
    DO 26 L26=1,J,1
    IF(NMID1-NSPUR(L26))26,27,26
 26 CONTINUE
    NPUT(K)=NMID1
    GO TO 104
 27 IF(NMID2-NMAX2)120,55,55
```

```
120 DO 79 L79=1,J,1
    IF(NMID2-NSPUR(L79))79,80,79
 79 CONTINUE
    NPUT(K)=NMID2
    GO TO 104
 80 NMID1=NMID1-1
    NMID2=NMID2+1
 37 CONTINUE
    GO TO 55
 46 NPUT(K)=NMID
104 CALL CONTIN(RO,RINCRO,THETA,ATETH,EBSILON,VO(L147),MVIN,NPUT(K),L
   147,U,Q,ERMIN(K),JC)
    IF(ERMIN(K)-EMAX1+(NPUT(K)-NMAX1)*DIFFMAX)35,34,34
 35 J=J+1
    NSPUR(J)=NPUT(K)
    K=K-1
    GO TO 62
 55 JMAX1=1
    K=K-1
    GO TO 62
 34 IF(ERMIN(K)-EMAX2-EBSILON)15,15,16
 15 NMAX3=NMAX2
    EMAX3=EMAX2
    NMAX2=NPUT(K)
    EMAX2=ERMIN(K)
123 IF(NMAX2-NMAX1-1)89,89,90
 89 JMAX1=1
    GO TO 121
 90 JMAX1=2
121 IF(NMAX3-NMAX2-1)91,91,97
 91 JMAX2=1
    GO TO 17
 97 JMAX2=2
    GO TO 17
 16 IF(ERMIN(K)-EMAX1-EBSILON)2,2,5
  2 TIFFMAX=(EMAX1-ERMIN(K))/(NPUT(K)-NMAX1)+EBSILON
    IF(EMAX2-ERMIN(K)+(NMAX2-NPUT(K))*TIFFMAX)108,107,107
108 J=J+1
    NSPUR(J)=NMAX2
    IF(ERMIN(K)-EMAX3-EBSILON)109,109,110
109 NMAX2=NPUT(K)
    EMAX2=ERMIN(K)
    GO TO 123
110 DIFFMAX=TIFFMAX
    TIFFMAX=(ERMIN(K)-EMAX3)/(NMAX3-NPUT(K))+EBSILON
    NMAX1=NPUT(K)
    EMAX1=ERMIN(K)
133 NMAX2=NMAX3
    EMAX2=EMAX3
    NMAX3=10000
    M3=0
    DO 111 L111=1,K,1
    IF(NPUT(L111)-NMAX3)112,111,111
112 IF(NPUT(L111)-NMAX2)111,111,113
113 NMAX3=NPUT(L111)
    M3=L111
111 CONTINUE
    IF(M3)124,124,125
125 EMAX3=ERMIN(M3)
    IF(EMAX2-EMAX3-EBSILON)123,123,132
132 DIFFMAX=TIFFMAX
    TIFFMAX=(EMAX2-EMAX3)/(NMAX3-NMAX2)+EBSILON
```

```
      NMAX1=NMAX2
      EMAX1=EMAX2
      GO TO 133
  107 DIFFMAX=TIFFMAX
      NMAX1=NPUT(K)
      EMAX1=ERMIN(K)
      IF(NMAX2-NMAX1-1)92,92,62
   92 JMAX1=1
   62 GO TO(17,18),JMAX2
   18 K=K+1
      NMID=(NMAX2+NMAX3)/2
      IF(J)67,67,105
  105 NMID1=NMID
      NMID2=NMID+1
      DO 64 L64=1,NMID-NMAX2,1
      DO 65 L65=1,J,1
      IF(NMID1-NSPUR(L65))65,66,65
   65 CONTINUE
      NPUT(K)=NMID1
      GO TO 106
   66 IF(NMID2-NMAX3)126,86,86
  126 DO 84 L84=1,J,1
      IF(NMID2-NSPUR(L84))84,85,84
   84 CONTINUE
      NPUT(K)=NMID2
      GO TO 106
   85 NMID1=NMID1-1
      NMID2=NMID2+1
   64 CONTINUE
   67 NPUT(K)=NMID
  106 CALL CONTIN(RO,RINCRO,THETA,ATETH,EBSILON,VO(L147),MV,N,NPUT(K),L1
     147,U,Q,ERMIN(K),JC)
      IF(ERMIN(K)-EMAX1+(NPUT(K)-NMAX1)*DIFFMAX)78,77,77
   78 J=J+1
      NSPUR(J)=NPUT(K)
      K=K-1
      GO TO 17
   86 JMAX2=1
      K=K-1
      GO TO 17
   77 IF(ERMIN(K)-EMAX2+EBSILON)19,20,20
   19 DIFFMAX=(EMAX1-EMAX2)/(NMAX2-NMAX1)+EBSILON
      NMAX1=NMAX2
      EMAX1=EMAX2
      NMAX2=NPUT(K)
      EMAX2=ERMIN(K)
      GO TO 123
   20 NMAX3=NPUT(K)
      EMAX3=ERMIN(K)
      IF(NMAX3-NMAX2-1)96,96,17
   96 JMAX2=1
      GO TO 17
    5 NMAX2=NMAX1
      EMAX2=EMAX1
      NMAX3=NPUT(K)
      EMAX3=ERMIN(K)
      NMAX1=-100
      DO 9 L9=1,K,1
      IF(NPUT(L9)-NMAX1)9,9,22
   22 IF(NPUT(L9)-NMAX2)23,9,9
   23 NMAX1=NPUT(L9)
      M2=L9
```

```
  9 CONTINUE
    EMAX1=ERMIN(M2)
    NM=-100
    M4=0
    DO 127 L127=1,K,1
    IF(NPUT(L127)-NM)127,127,128
128 IF(NPUT(L127)-NMAX1)129,127,127
129 NM=NPUT(L127)
    M4=L127
127 CONTINUE
    IF(M4)130,130,131
130 DIFFMAX=1.0E+60
    GO TO 123
131 DIFFMAX=(ERMIN(M4)-EMAX1)/(NMAX1-NM)+EBSILON
    GO TO 123
 21 N(L147)=NMAX2
147 CONTINUE
    IF(JE)217,217,218
217 JE=JC
218 GO TO(171,172),JDIS
171 IF(EMAX2-R0)173,183,183
173 RTH=ATETH-W*DELTA2
    JDIS=2
    GO TO 176
172 IF(EMAX2-R0)176,177,177
176 STH=ATETH
    EMAX2=EMAX2
    IF(JC)134,134,175
134 JDIS=1
183 IF((ATETH-ATETH0)*W-360.0)174,182,182
182 GO TO(301,301,302),JB
302 ATETH0=ATETH1
    ATETH1=ATETH2-W*360.0
    ATETH=ATETH1
    W=-W
    JB=2
    GO TO 174
301 IF(JE)184,184,152
184 WRITE(2,136)
    GO TO 152
177 RTH=ATETH
    ATETH=STH
175 IF(ABS(STH-RTH)-DELTA1)178,178,179
179 ATETH=(RTH+STH)/2.0
    GO TO 30
174 ATETH=ATETH+W*DELTA2
    GO TO 30
178 GO TO(303,304,305),JB
303 IF((ATETH-ATETH0)*W-360.0)320,152,152
304 IF((ATETH-ATETH0)*W-360.0)180,152,152
305 IF((ATETH-ATETH0)*W-360.0)180,302,302
320 ATETH0=ATETH
180 ATETH2=ATETH
    W1=W
    JDIS=1
    THETA=ATETH*PI/180.0
    CALL EXTEND(R0,THETA,ATETH0,ATETH1,EMAX2,U,Q,VV,MV,W1,HEX,NJEX,JE
    ATETH=THETA*180.0/PI
    GO TO(306,307,308),JB
308 IF(W*W1)309,309,310
309 ATETH1=ATETH2
    GO TO 311
```

```
310 ATETHO=ATETH2-W*360.0
311 ATETH=(ATETHO+ATETH1+W*360.0)/2.0
    ATETH2=ATETH
    IF(ABS((ATETH1-ATETH0)*W-360.0)-DELTA1)152,152,30
306 JB=2
    W=W1
    ATETHO=ATETH2
    GO TO 312
307 IF(W+W1)313,313,312
313 JB=3
    ATETH1=ATETH2
    GO TO 311
312 IF((ATETH-ATETHO)*W-360.0)225,152,152
225 ATETH1=ATETH
    ATETH=ATETH+W*DELTA2
    GO TO 30
152 CONTINUE
    END
```

, LENGTH 2030, NAME DCHALL

```
      SUBROUTINE CONTIN(RO,RINCRO,ALPHA,AHPLA,EBSILON,VN,MV,M,NLIFT,JV,U
     1,Q,RMIN,JC)
      DIMENSION RS(200),RF(200),V(4),M(4),H(4)
   50 FORMAT(1H ,F8.4,5X,F8.4,5X,E15.8,8X,E15.8,8X,E15.8,5X,I3,10X,I3,7
     1,I1)
      VV=0.0
      M(JV)=NLIFT
      V(JV)=VN
      DO 60 L60=1,JV,1
      VV=VV+V(L60)
      H(L60)=-V(L60)/M(L60)
   60 CONTINUE
      DV=VV*MV
      I=2
      DMIN=1.0E+60
      RS(1)=RO+RINCRO+EBSILON/2.0
      RF(1)=1.0E+60
      RS(2)=RO+RINCRO
   54 CALL RUNKUT(RS(I),ALPHA,H,M,JV,DV,MV,U,Q,RF(I))
      IF(RF(I)-DMIN)53,36,36
   53 DMINH=DMIN
      DMIN=RF(I)
      I=I+1
      RS(I)=RS(I-1)-RINCRO
      GO TO 54
   36 RMIN=RS(I-1)
      RMINH=RS(I-2)
      RINCR=RMINH-RMIN
      RINCRMINUS=RMIN-RS(I)
   38 CONTINUE
   31 IF(RINCR-EBSILON)32,40,40
   40 RINCR=RINCR/2.0
      IF(I-200)56,39,39
   56 I=I+1
      RS(I)=RMIN+RINCR
      CALL RUNKUT(RS(I),ALPHA,H,M,JV,DV,MV,U,Q,RF(I))
      IF(RF(I)-DMIN)41,42,42
   41 RMIN=RS(I)
```

```
      DMIN=RF(I)
      RINCRMINUS=RINCR
      GO TO 38
   42 IF(RF(I)-DMINH)43,44,44
   43 DMINH=RF(I)
      RMINH=RS(I)
   32 IF(RINCRMINUS-EBSILON)215,216,216
  215 IF(RINCR-EBSILON)39,38,38
  216 I=I+1
      RINCRMINUS=RINCRMINUS/2.0
      RS(I)=RMIN-RINCRMINUS
      CALL RUNKUT(RS(I),ALPHA,H,M,JV,DV,MV,U,Q,RF(I))
      IF(RF(I)-DMIN)45,38,38
   45 RMINH=RMIN
      DMINH=DMIN
      RMIN=RS(I)
      DMIN=RF(I)
      RINCR=RINCRMINUS
      GO TO 38
   44 RINCRMINUS=RINCR
      RMIN=RMINH
      DMIN=DMINH
      RMINH=RO+RINCRO+EBSILON
      JC=1
      DO 47 L47=1,I,1
      IF(RS(L47)-RMIN)47,47,48
   48 IF(RS(L47)-RMINH)49,47,47
   49 RMINH=RS(L47)
      M1=L47
   47 CONTINUE
      DMINH=RF(M1)
      RINCR=RMINH-RMIN
      GO TO 38
   39 XMIN=RMIN*COS(ALPHA)
      YMIN=RMIN*SIN(ALPHA)
      WRITE(2,50)AHPLA,VV,XMIN,YMIN,DMIN,I,NLIFT,JC
      RETURN
      END

, LENGTH   624, NAME  CONTIN


      SUBROUTINE RUNKUT(RR,TT,HH,NN,JJ,SS,MM,UU,QQ,DD)
      DIMENSION HH(4),NN(4)
      F(X,Y,S,U,Q)=X*(1.0-X*X-Y*Y)/(2.0*(X*X+Y*Y))/(1.0-S)
      G(X,Y,S,U,Q)=Y*(1.0-X*X-Y*Y)/(2.0*(X*X+Y*Y))/(1.0-S)
      SN=SS
      X0=COS(TT)*RR
      Y0=SIN(TT)*RR
      XN=X0
      YN=Y0
      DO 33 L33=1,JJ,1
      L8=JJ+1-L33
      DH=HH(L8)*MM
      DO 3 L3=1,NN(L8),1
      P11=F(XN,YN,SN,UU,QQ)
      P12=G(XN,YN,SN,UU,QQ)
      X=XN+HH(L8)*P11/2.0
      Y=YN+HH(L8)*P12/2.0
      S=SN+DH/2.0
```

```
      P21=F(X,Y,S,UU,QQ)
      P22=G(X,Y,S,UU,QQ)
      X=XN+HH(L8)*P21/2.0
      Y=YN+HH(L8)*P22/2.0
      P31=F(X,Y,S,UU,QQ)
      P32=G(X,Y,S,UU,QQ)
      X=XN+HH(L8)*P31
      Y=YN+HH(L8)*P32
      SN=SN+DH
      P41=F(X,Y,SN,UU,QQ)
      P42=G(X,Y,SN,UU,QQ)
      XN=XN+HH(L8)*(P11+2.0*P21+2.0*P31+P41)/6.0
      YN=YN+HH(L8)*(P12+2.0*P22+2.0*P32+P42)/6.0
    3 CONTINUE
   33 CONTINUE
      DD=SQRT(XN*XN+YN*YN)
      RETURN
      END

, LENGTH   416, NAME  RUNKUT


      SUBROUTINE EXTEND(R1,TH,AT,AT1,RO,UN,QN,VN,MN,W,H,NJ,JB)
      F(X,Y,S,U,Q)=X*(1.0-X*X-Y*Y)/(2.0*(X*X+Y*Y))/(1.0-S)
      G(X,Y,S,U,Q)=Y*(1.0-X*X-Y*Y)/(2.0*(X*X+Y*Y))/(1.0-S)
    6 FORMAT(1H0,2X,12HX-COORDINATE,13X,12HY-COORDINATE,14X,10HVALUE OF
     1V,13X,5HANGLE)
    8 FORMAT(1H ,E15.8,10X,E15.8,10X,E15.8,10X,F8.4,10X,F4.1)
      WRITE(2,6)
      SN=VN*MN
      DH=H*MN
  141 PI=3.1415926538
      V0=VN
      XN=COS(TH)*RO
      YN=SIN(TH)*RO
      J=0
      JD=JB
      VTEST=(VN+50.0-1.0*MN)/(1.0+MN*49.0)
  214 IF(J-NJ)150,151,151
  151 J=0
      IF(XN)202,203,203
  202 SIXN=-1.0
      GO TO 204
  203 SIXN=1.0
  204 IF(YN)205,206,206
  205 SIYN=-1.0
      GO TO 207
  206 SIYN=1.0
  207 BETA=ATAN(YN/XN)+(1.0-SIYN*(1.0+SIXN)/2.0)*PI
  231 IF(ABS(BETA-TH)-PI)232,232,233
  233 IF(BETA-TH)234,234,235
  234 BETA=BETA+2.0*PI
      GO TO 231
  235 BETA=BETA-2.0*PI
      GO TO 231
  232 TEBA=BETA*180.0/PI
      WRITE(2,8)XN,YN,V0,TEBA,W
      GO TO(314,314,314,315,316),JD
  314 IF((BETA-TH)*W+1.0E-10)317,318,318
  317 W=-1.0*W
```

```
      GO TO(318,319,319),JD
  318 JD=4
  315 TH=BETA
      IF((TEBA-AT)*W-360.0)167,165,165
  319 JD=5
  316 TH=BETA
      IF((TEBA-AT1)*W)167,165,165
  167 IF(XN*XN+YN*YN-R1*R1)166,165,165
  166 IF(V0-VTEST)150,165,165
  150 J=J+1
      V0=V0+H
      P11=F(XN,YN,SN,UN,QN)
      P12=G(XN,YN,SN,UN,QN)
      X=XN+H*P11/2.0
      Y=YN+H*P12/2.0
      S=SN+DH/2.0
      P21=F(X,Y,S,UN,QN)
      P22=G(X,Y,S,UN,QN)
      X=XN+H*P21/2.0
      Y=YN+H*P22/2.0
      P31=F(X,Y,S,UN,QN)
      P32=G(X,Y,S,UN,QN)
      X=XN+H*P31
      Y=YN+H*P32
      SN=SN+DH
      P41=F(X,Y,SN,UN,QN)
      P42=G(X,Y,SN,UN,QN)
      XN=XN+H*(P11+2.0*P21+2.0*P31+P41)/6.0
      YN=YN+H*(P12+2.0*P22+2.0*P32+P42)/6.0
      IF(SN+DH-0.999999)214,168,168
  168 H=H/10.0
      DH=DH/10.0
      GO TO 214
  165 CONTINUE
      RETURN
      END

, LENGTH   635, NAME  EXTEND

      FINISH

TION -  NO ERRORS

         65 BUCKETS USED

/ XPCK 12K     DATE     13/02/79   TIME 19/17/38

22AM)
M (DBM)
   9728

 SEG      DCHALL
 SEG      CONTIN
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