# The General Pole Placement Problem in Singular Systems

by

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## Abstract.

Over the last decade infinite poles and zeros have been recognised as having fundamental relevance to the analysis of the dynamical behaviour of a system. Indeed even the classical theory of characteristic root loci alludes to the existence of infinite zeros without defining them as such whilst the significance of the infinite poles has more recently emerged in the study of non-proper systems.

In the first part of the work a method for examining the infinite pole and zero structure of a rational matrix based on the Laurent expansion of that matrix about the point at infinity is described. The method consequently leads to a test for the absence of infinite zeros in a rational matrix and certain relationships for polynomial matrices.

The poles, both finite and infinite, of a linear time invariant system are determined from the zeros of the invariant polynomials of certain matrices. The pole positions may be changed using constant gain feedback from a set of generalised states or from the system outputs. The conditions under which arbitrary pole placement can be achieved in this way are well understood particularly in regard to the finite poles. A more general problem is that of assigning the pole structure as determined by the invariant polynomials rather than simply the set of zeros of these polynomials.

The general problem is first considered for the case of constant output feedback. The properties of a minimal factorisation of a rational matrix are exploited to give necessary conditions on the simultaneous placement of both the finite and infinite pole structures. These conditions are subsequently interpreted for the case of systems represented in generalised state space form under constant gain feedback from a set of generalised states. Further conditions are obtained by considering the infinite frequency structure of such systems. In particular necessary and sufficient conditions for the placement of the infinite pole structure are presented which are seen to have a direct relationship with the various notions of controllability associated with generalised state space systems.

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Contents.

## Chapter 1. Introduction.

Polynomial systems matrix theory has increasingly developed into an important tool in the investigation of physical control systems. The theory developed and described by people such as Rosenbrock [1970], Wolovich [1974] and Kailath [1980] adapts the theory of linear algebra and matrix theory to study the behaviour of physical systems. This theory is applicable to systems described by a linear model. The model, given as a set of differential equations, can be transformed using a suitable transformation into a set of algebraic equations. The transformation usually employed is the Laplace transformation and the resulting domain is referred to as the frequency domain. All the information describing the system's behaviour may be encoded into a single partitioned matrix, then, by using the theory of linear algebra and the relevant algebraic interpretation of the physical, important properties of the system can be deduced.

Early interest has been focused on systems described in state space form, i.e. systems of the form,

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + D\left(\tfrac{d}{dt}\right) u(t) \end{aligned}\right\} \tag{1.1.1}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ is an $l$-vector of inputs and $y(t)$ is an $m$-vector of outputs and where $A, B, C, D$ are matrices of the appropriate dimensions whose elements are taken from a general field, $F$, which is usually taken to be the field of real numbers, $\Re$. The properties of these systems have been widely investigated and their behaviour is well understood. However, such a description can not adequately describe what is termed as the impulsive behaviour of a system (i.e. significant behaviour attributable to the point at infinity in the frequency (transformed) domain). This has led to the investigation of systems of the form

$$\left.\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1.1.2}$$

with $x(t), u(t), y(t), A, B, C$ as above and where $E$ is a constant matrix. Systems of this form are called generalised state space systems. When $E$ is non-singular, (1.1.2) reduces to the familiar state space description. Otherwise the system consists of a combination of first order differential equations and algebraic equations whose infinite frequency behaviour is displayed in a convenient manner. Properties of such systems have only recently begun to be investigated, although they arise quite commonly for example in the study of composite systems, switched capacitor networks and in certain cases of component failure.

The poles of a system essentially determine the dynamical properties of the response of the system. By employing suitable feedback it is possible to relocate these poles (and so tailor the system's response to some desired requirement) and group them together in a certain manner. Investigation of the conditions under which the pole structure (locations and groupings) can be assigned is referred to as the general pole assignment problem. This problem has been solved (see Rosenbrock and Rowe, 1970) for the case of state space systems and the case of constant state feedback of the form

$$u(t) = Kx(t) + v(t) \tag{1.1.3}$$

where $K$ is a constant matrix of the appropriate dimensions. It is only in recent years that the general pole assignment problem has been approached for generalised state space systems using constant generalised state feedback (Cobb 1981, Armentano 1984, Kucera and Zagalak, 1988).

This is the main problem considered in this thesis where, unlike previous work, both the finite and infinite pole structures are taken into account.

A detailed discussion of the pole placement problem is presented in chapter 2 together with a critical analysis of the work previously undertaken which includes identification of the problems that remain unsolved and the areas which have not been investigated. The problems considered in this thesis are placed in this context.

The underlying feature of this thesis is the investigation of the assignment of the infinite pole structure. For this reason chapter 3 considers the infinite frequency structure associated with the rational matrix. The first part of the chapter discusses the various equivalent definitions of the infinite frequency structure of a rational matrix and highlights a particular systematic method of obtaining this structure form a Laurent expansion of the given matrix. New results eminating from this discussion are displayed in the second part of the chapter. The results include new conditions for the absence of infinite zeros in a rational matrix and certain relationships which reveal further properties concerning the structure of polynomial matrices.

Chapter 4 explores the general pole placement problem when constant gain output feedback is applied around a system and where the resulting system may possess both finite and infinite poles. The problem is approached by adopting a minimal factorisation description of the open loop transfer function matrix. The properties of such a factorisation enable both the finite and infinite pole structures of the resulting closed loop transfer function matrix to be displayed in a convenient manner. This gives rise to new separate necessary conditions for the finite pole structure and the infinite pole structure of the closed loop system. Further analysis produces necessary conditions for the simultaneous placement of the two structures which are stronger than any yet obtained. These results are displayed in a neat, graphical manner by the introduction of a suitable step function.

Chapters 5 and 6 consider systems described in generalised state space form. For such systems the assignment of the poles by suitable feedback is closely associated with the controllability properties of the representation. The various definitions of controllability for generalised state space systems are discussed in chapter 5 from which it is concluded that two main notions of controllability exist. The algebraic conditions associated with each notion are presented so providing an analogy with the algebraic conditions given by Rosenbrock [1970] for the conventional notion of controllability in regular state space systems. The algebraic conditions include a combination of existing results and some new results. The polynomial matrix approach adopted provides a means of treating the results in a unified manner and yields simpler proofs of the existing results. The differences between the two notions of controllability are reflected in the role of the so called non-dynamic variables and this is illustrated by introducing a new time domain definition. A comparison of the two notions of controllability is presented together with some further new conditions for a system to be controllable in each case.

Chapter 6 investigates the general pole placement problem in generalised state space systems. An initial result is obtained by interpreting the work presented in chapter 4 for systems described in generalised form. This gives rise to new necessary conditions for the simultaneous placement of both the finite and infinite pole structures in the generalised state space case. The specific assignment of the infinite pole structure is then considered by exploiting the detailed structure of a canonical form associated with the system. This approach produces new necessary conditions for the multiplicity of the closed loop infinite poles. Supplementing this result with the result derived earlier in the chapter leads to new necessary and sufficient conditions for the closed loop infinite pole structure. This result provides a complete characterisation for such achievable structures. Finally, this result is used to update the initial result concerning the simultaneous assignment of both the finite and infinite pole structures.

Chapter 7 discusses three other approaches to the pole placement problems. A two stage method for generalised state space systems is first described where the infinite pole structure is first assigned followed at the second stage by the finite pole structure. The result obtained provides a partial solution to the general pole placement problem for such systems. The closed loop infinite pole structure is also investigated by considering the Laurent expansion about the point at infinity of the closed loop transfer function matrix. This method is of more relevance to individual systems and gives rise to a simple condition for testing if the closed loop system is proper. The third approach involves employing a bilinear transformation so that the infinite pole structure can be investigated in the same way as the finite pole structure. This method generally enables results concerning proper closed loop systems to be generalised to the non-proper case but for generalised state space systems this does not follow. The reasons for this are subsequently explained.

Finally, chapter 8 contains some concluding remarks and highlights areas for further research.

Chapter 2. Pole Placement Problems.

§1. Introduction.

The pole placement problem is concerned with investigating the conditions under which the poles of a system can be relocated by means of a suitable feedback. The poles of a system are determined by the zeros of a certain matrix so that relocating the poles is equivalent to altering the zeros of that particular matrix. If the groupings of these poles are also considered then the problem is concerned with assigning the invariant polynomials of the matrix which determines the poles of the system. This form of the problem is referred to as the general pole placement problem. A detailed description of these problems is given in section 2 together with a discussion of the difficulties that arise when infinite poles are considered.

A critical appraisal of previous work undertaken on the pole placement problems is presented in sections 3 and 4. The pole placement problems using output feedback are first considered where the review is divided into three sections; namely the pole placement problem under dynamic output feedback, the pole placement problem under constant output feedback and the general pole placement problem. In section 4 the case of state feedback, which can be employed when the system is described in state space or generalised state space form, is considered. Finally, section 5 places in context the problems investigated in this dissertation.

§2. Description of pole placement problems.

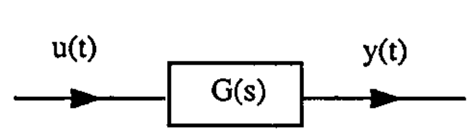
Consider a system with $(r+m)\times(r+\ell)$ system matrix (see Rosenbrock, 1970)

$$P(s)=\left[\begin{array}{c|c} T(s) & U(s) \\ \hline -V(s) & W(s) \end{array}\right] \tag{2.2.1}$$

and corresponding transfer function matrix

$$G(s)=V(s)T^{-1}(s)U(s)+W(s).$$

A pictorial description of this system is given as follows




and this is referred to as the open loop system. Let a feedback of the form

$$u(t)=v(t)-K(s)y(t) \tag{2.2.2}$$

where $K(s)$ is a proper compensator be applied to the system. Let the resulting closed loop system have system matrix

$$P_C(s) = \left[\begin{array}{c|c} T_C(s) & U_C(s) \\ \hline -V_C(s) & W_C(s) \end{array}\right]. \qquad (2.2.3)$$

When $G(s)$ is a proper matrix the open loop poles are all located at finite locations. The system poles are given by the zeros of $\mid T(s) \mid= 0$. If the system has least order, i.e. has no finite input or output decoupling zeros, then the system poles will correspond to the poles of the transfer function matrix which in turn are given by the zeros of the denominator polynomials of the McMillan form of $G(s)$ (see chapter 3 for further details).

The poles of the closed loop system are, in general, different from those of the open loop system. In fact the closed loop poles will possess an additional set of poles equal in number to the order of the compensator $K(s)$. When $K(s)$ is constant the closed loop system has the same number of poles as the open loop system. Employing feedback of the form (2.2.2) thus enables the poles to be relocated in more desirable locations. The pole placement problem can be defined in the following manner.

> Given the open loop system (2.2.1) and a monic polynomial $\phi(s)$, find suitable conditions under which the matrix $T_C(s)$ in (2.2.3) has determinant $\alpha\phi(s)$, $\alpha \neq 0$.

$T_C(s)$ does not have to be of the form $d\left(\phi_1\ \phi_2\ \ldots\ \phi_n\right)$

A more general form of the pole placement problem is concerned with assigning the invariant polynomials of $T_C(s)$ and defined as follows.

> Let the open loop system be given by (2.2.1) and $\phi_1(s),\ \phi_2(s), \ldots, \phi_q(s)$ be $q$ non-zero monic polynomials with $\phi_i(s) \mid \phi_{i+1}(s),\ i = 1, 2, \ldots, q-1$. Then, find suitable conditions under which the Smith form of the matrix $T_C(s)$ in (2.2.3) is equal to $\mathrm{diag}[\phi_1(s), \phi_2(s), \ldots, \phi_q(s)]$.

This is referred to as the general pole assignment problem.

If $G(s)$ is non-proper the open loop system will possess infinite poles. The infinite pole structure cannot be investigated in the same way as the finite structure. For system poles the infinite pole structure is defined (see Verghese, 1978) using the normalised polynomial system matrix, $P_N(s)$, associated with the system. Specifically,

$$P_N(s) = \left[\begin{array}{ccc|c} T(s) & U(s) & 0 & 0 \\ -V(s) & W(s) & I_m & 0 \\ 0 & -I_\ell & 0 & I_\ell \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} T_N(s) & U_N \\ \hline -V_N & 0 \end{array}\right]$$

and the infinite system poles are subsequently defined as the infinite zeros of $T_N(s)$ (for further discussion of infinite zeros see chapter 3). In a similar way the infinite poles of the transfer function matrix are obtained by considering a different matrix than that used to investigate the finite poles. A detailed analysis of the infinite poles and zeros of a rational matrix is presented in chapter 3. As in the finite case the system poles at infinity become equivalent to the transfer function matrix poles at infinity if the system does not possess infinite input or output decoupling zeros.

For the case where the open loop system is proper or non-proper the pole placement problem therefore consists of two problems; the first concerns the assignment of the finite poles and the second the assignment of the infinite poles. For the general pole placement problem the problem thus involves assigning the invariant polynomials of two different matrices. The problem is further complicated by the fact that the matrices that give rise to the pole structures are interrelated so that applying a certain feedback may produce the required finite pole structure but will not give rise to the required infinite pole structure. Previous work on the pole placement problem has in general concentrated on strictly proper systems though recently more attention has been given to proper and non-proper systems especially with the emergence of the generalised state space description. It is this area that will be the main consideration of this thesis.

## §3. Pole placement using output feedback.

The survey of previous work on pole placement problems using output feedback is presented in three parts. The first part deals with the pole placement problem using dynamic feedback whilst the special case of constant feedback is considered in the second part. Finally, the general pole placement problem is discussed.

First recall that any system with a strictly proper transfer function matrix can be represented in state space form (see Rosenbrock, 1970). As a consequence of this fact many of the results concerning pole placement using output feedback are given with reference to this state space form. The results presented in the following discussion will be given in terms of the state space description

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2.3.1}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ an $\ell$-vector of control inputs, $y(t)$ an $m$-vector of outputs and $A, B, C$ are matrices of the appropriate dimensions.

### (i) Dynamic output feedback.

The early work on the pole placement problem using output feedback concentrated on finding a proper compensator with the smallest order that will assign to arbitrary locations in the complex plane all the poles of the closed loop system.

The first results were developed for strictly proper systems. Pearson [1969] considered a controllable and observable single input, multi-output system with observability index $\mu_m$. It was shown that arbitrary pole placement (subject to the usual condition that complex poles occur in conjugate pairs) can be achieved for such a system with a proper compensator of order $\mu_m - 1$.

Pearson and Ding [1969] generalised this result to a least order multi-input, multi-output system with strictly proper transfer function matrix

$$G(s) = C[sI - A]^{-1}B \tag{2.3.2}$$

and observability index $\mu_m$. If $q$ is the smallest number of inputs which control $A$ then a proper compensator $K(s)$ of order $q(\mu_m - 1)$ can be found such that $n + q(\mu_m - 1)$ closed loop poles can be arbitrarily assigned.

Brasch and Pearson [1970] improved this result by showing that a proper compensator of order $\min(\mu_m - 1, \lambda_\ell - 1)$ can be chosen to achieve arbitrary pole placement for the system described above where $\mu_m$ is the observability index and $\lambda_\ell$ the controllability index of the system.

This result was generalised to proper systems by Wolovich [1971] who used a combination of feedforward and feedback control. Chen and Hsu [1971] subsequently gave the same result using feedback only.

Kimura [1975] produced a result for strictly proper systems which is mutually independent of the results given by Brasch and Pearson [1970]. Kimura [1975] showed that for a controllable and observable system a dynamic compensator of order $p = n - m - \ell + 1$ can assign almost arbitrary poles for the overall closed loop system provided the poles to be assigned are all distinct. When $m, \ell$ are both large the result due to Kimura [*ibid.*] tends to be a superior result to the one given by Brasch and Pearson [1970] in most cases. Consider the following example.

**(2.3.3) Example.** For a system of dimension four with two inputs and two outputs the result due to Kimura [1975] indicates that a compensator of order 1 is sufficient for arbitrary pole placement. Consider the specific system $(A, B, C)$ given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is both controllable and observable. To find the order of the compensator required under the result given by Brasch and Pearson [1970] the controllability index of the system must be first obtained. Now

$$[B, AB, A^2B] = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $B$ has rank 2, $[B, AB]$ has rank 3 and $[B, AB, A^2B]$ has rank 4. Thus, by definition, the controllability index, $\lambda_\ell$, of the system is equal to 3. Similarly,

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

so that $C$ has rank 2, $[C^T, A^TC^T]^T$ has rank 3 and $[C^T, A^TC^T, (A^2)^TC^T]^T$ has rank 4. Hence the observability index, $\mu_m$, of the system is also equal to 3. Therefore, the result due to Brasch and Pearson [1970] requires a compensator of order 2 for arbitrary pole assignment in this particular system. Hence, the result due to Kimura [1975] is superior in this case.

A more general approach to the problem of pole assignment by dynamic compensators is to determine the maximum number of poles of the closed loop system that can be arbitrarily assigned by a proper compensator of fixed order, $p$ say. Then, it is possible to determine the least order proper compensator which will assign all the closed loop poles. This approach was adopted by Ahmari and Vacroux [1973] who gave the following result.

**(2.3.4) Theorem (Ahmari and Vacroux, 1973).** Given a system $(A, B, C)$ which is both controllable and observable and where $A$ is cyclic and the matrices

$$[B, AB, \ldots, A^p B], \quad [C^T, A^T C^T, \ldots, (A^p)^T C^T]^T$$

have ranks $u_p, v_p$ respectively. Then, there exists a compensator of order $p$ such that

$$\max(u_p, v_p) + p$$

poles of the closed loop system can be placed arbitrarily close to $\max(u_p, v_p)+p$ preassigned values.

It follows that the least order proper compensator that will arbitrarily assign all the closed loop poles has order

$$p = \min(\lambda_\ell - 1, \mu_m - 1)$$

which is in agreement with the result derived by Brasch and Pearson [1970].

Kimura [1978] showed that the result given by Ahmari and Vacroux [1973] could be improved if additional conditions are met. If $G(s) = C[sI_n - A]^{-1}B$ is the transfer function matrix of the system then let $G(s_1, s_2)$, $v_i, u_i$ be defined as follows

$$G(s_1, s_2) \triangleq \frac{G(s_1) - G(s_2)}{s_1 - s_2}$$

$$v_i \triangleq \operatorname{rank}[B, AB, \ldots, A^i B]$$

$$u_i \triangleq \operatorname{rank}[C^T, A^T C^T, \ldots, (A^i)^T C^T]^T.$$

The main result due to Kimura [*ibid.*] now follows.

**(2.3.5) Theorem (Kimura, 1978).** Let the strictly proper system given by (2.3.1) satisfy the following conditions

(i) the system is both controllable and observable,

(ii) $G(s_1, s_2) \neq 0$ except for a finite number of pairs $(s_1, s_2)$, $\quad s_1, s_2 \in \mathbb{C}$,

(iii) $m \geq \ell - 1$.

Then, there exists a dynamic compensator of order $p$ such that

$$\min[n, \ell - 1 + u_p] + p$$

closed loop poles can be placed arbitrarily close to any prescribed symmetric set which contains a symmetric subset of $\ell - 1$ numbers.

If $\ell \geq m - 1$ then the above theorem is valid if $m, \ell, u_p$ are replaced by $\ell, m, v_p$ respectively. Therefore, if $| \ell - m | \leq 1$ and conditions (i), (ii) of theorem (2.3.5) hold, then the number of poles assignable by a dynamic compensator of order $p$ is

$$\min\{\max[\ell - 1 + u_p, m - 1 + v_p], n\} + p$$

(the restriction that the symmetric set of poles to be assigned must contain a symmetric subset of $\ell - 1$ numbers no longer applies).

It is of interest to know what type of transfer function matrix satisfies condition (ii) of theorem (2.3.5). Kimura [1978] notes that if $\ell$ or $m$ is greater than 1 only a very exceptional system fails to satisfy condition (ii). If $n \leq m + \ell - 1$ then condition (ii) is always satisfied.

Under the additional conditions (ii) and (iii) theorem (2.3.5) improves the result of Ahmari and Vacroux [1973]. These conditions make it possible to assign $\ell - 1$ additional poles. The theorem also improves on the result of Brasch and Pearson [1970].

Under the result given by Kimura [1978] for all $n$ poles to be arbitrarily assigned it is necessary that

$$\ell - 1 + u_p \geq n \quad \text{or} \quad m - 1 + v_p \geq n$$

i.e. the order $p$ of the dynamic compensator $K(s)$ that will assign all $n + p$ closed loop poles is

$$p = \{\min p : u_p \geq n - \ell + 1 \quad \text{or} \quad v_p \geq n - m + 1\}$$

The result due to Brasch and Pearson [1970] indicates that the order $p$ is given by

$$p = \min\{p : u_p \geq n \quad \text{or} \quad v_p \geq n\}.$$

Hence, if conditions (ii) and (iii) of theorem (2.3.5) are satisfied a lower order compensator can be found which arbitrarily assigns all the closed loop poles.

Williams and Hesselink [1978] produce a further necessary condition for a compensator of order $p$ to generically assign all the poles of the closed loop system, i.e. assign the closed loop poles arbitrarily close to the preassigned values. This necessary condition is

$$p(m + \ell - 1) + m\ell \geq n$$

but since it does not refer to the controllability or the observability indices it is not possible to make a direct comparison with the previous results.

Djaferis [1983] also considered the problem of finding the number of poles that can be arbitrarily assigned using a proper compensator of order $p$. It was shown that

$$\min(n+p,(p+1)\ell+p) \qquad m \leq \ell$$

$$\min(n+p,(p+1)m+p) \qquad \ell \leq m$$

closed loop poles can be assigned arbitrarily close to the preassigned values using an output feedback compensator of order $p$, and where $n$ in this case is the McMillan degree of $G(s)$.

A further result was given by Djaferis and Narayana [1985]. For a generic system with $m \geq \ell$ and McMillan degree $n$ and with controllability indices

$$\lambda_{G1} \geq \lambda_{G2} \geq \ldots \geq \lambda_{G\ell} > 0$$

then

$$\min((p+1)m+p+b(\ell-1),n+p)$$

closed loop poles can be assigned arbitrarily close to the preassigned values using a proper compensator of order $p$, where

$$b = \min\left\{\left[\frac{m}{\ell}\right], \lambda_{G\ell}\right\}$$

and $\left[\frac{m}{\ell}\right]$ = largest integer smaller than or equal to $\frac{m}{\ell}$.

A dual result holds when $\ell \geq m$ with $\ell$ replacing $m$ and $\mu_{Gm}$, the smallest observability index, replacing $\lambda_{G\ell}$. In many cases this leads to a lower bound on the dynamic compensator required to assign all the closed loop poles.

**(2.3.6) Example.** Let $n = 10, m = \ell = 3$. For all $n$ poles to be arbitrarily assigned the result due to Djaferis and Narayana [1985] requires

$$(p+1)m+p+b(\ell-1) \geq n+p$$

which in this case implies

$$3p \geq 5.$$

Therefore, a compensator of order 2 will suffice. The result given by Djaferis [1983] requires

$$(p+1)\ell+p \geq n+p$$

which in this case implies

$$3p \geq 7$$

and the compensator must have order 3 at least. Thus, for systems of the above dimension the result due to Djaferis and Narayana [1985] is superior to the one given by Djaferis [1983].

**(ii) Constant output feedback.**

The use of dynamic feedback may possibly be impractical from an engineering or economic viewpoint, and it would be hard to justify such an approach if the use of the available outputs with constant feedback gains would meet the design requirements in a much simpler manner. Results concerning the case when the feedback matrix is constant can of course be deduced from the work on dynamic feedback, but in general, this case has been treated separately.

For the case where the open loop transfer function matrix is strictly proper the closed loop poles can be immediately identified. Adopting the state space description (2.3.1) and output feedback of the form

$$u(t) = v(t) - Ky(t)$$

results in the following closed loop state equation

$$\dot{x}(t) = [A - BKC]x(t) + Bv(t).$$

Therefore, the closed loop system poles are given by the zeros of

$$\mid sI_n - A + BKC \mid .$$

An early result was given by Davison [1970] and independently by Jameson [1970], which states that if rank $C = m$, $(A, B)$ is controllable with $A$ cyclic then a linear feedback of the output can always be found so that $m$ poles of the system can be placed arbitrarily close to $m$ preassigned values (chosen in complex conjugate pairs).

Davison and Chatteridge [1971] extended this result to non-cyclic matrices by using the results of Brasch and Pearson [1970]. An improved result was also given which may be expressed as follows.

**(2.3.7) Theorem (Davison and Chatteridge, 1971).** If the system given by (2.3.1) is both controllable and observable with rank $C = m(\leq n)$, rank $B = \ell(\leq n)$ then a linear feedback of the output variables $u(t) = Ky(t)$ can always be found such that $\max(m, \ell)$ poles of the system can be placed arbitrarily close to $\max(m, \ell)$ preassigned values (chosen in complex conjugate pairs).

Sridhar and Lindorff [1973] gave an alternative proof of this result and showed that in certain cases more than $\max(m, \ell)$ poles can be arbitrarily placed. Davison and Wang [1975] subsequently improved the earlier result of Davison and Chatteridge [1971] with the following result.

**(2.3.8) Theorem (Davison and Wang, 1975).** Given a controllable and observable system with rank $B = \ell$, rank $C = m$ then for almost all $(B, C)$ pairs there exists an output gain matrix $K$ such that $\min(n, m+\ell-1)$ poles of the closed loop system can be assigned arbitrarily close to $\min(n, m+\ell-1)$ specified values (subject to the usual complex conjugate condition).

This theorem contains two notions of genericity. The first is seen in the fact that the result is true for almost all $(B, C)$ pairs, i.e. for generic $(B, C)$ pairs. Thus, given a specific plant matrix $A$ the result could break down for some particular choice of $(B, C)$ but perturbing this choice should give the result. This is illustrated in the example below. The second notion of genericity is contained in the fact that the closed loop system poles can be assigned arbitrarily close to the specific values, i.e. generic pole placement.

**(2.3.9) Example.** Let $A, B, C$ be given as

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $B, C$ have full rank and the system is both controllable and observable. The closed loop poles are given by the solution of

$$\mid sI - A + BKC \mid = 0.$$

Let $K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$ then, in this example,

$$sI - A + BKC = \begin{bmatrix} s+1 & 0 & k_3 & k_4 \\ 0 & s+2 & k_1 & k_2 \\ 0 & -1 & s+1 & 0 \\ 1 & 0 & 0 & s+2 \end{bmatrix}$$

and

$$\mid sI - A + BKC \mid = (s+1)^2(s+2)^2 + (s+2)(s+1)(k_1 - k_4) + k_3k_2 - k_1k_4. \quad (2.3.10)$$

The coefficients of $s^4$ and $s^3$ in (2.3.10) are not affected by the choice of $K$ whilst the coefficients of $s^2$ and $s$ are inter dependent by choice of $k_1 - k_4$. Hence it is apparent that there are only two degrees of freedom in the specification of the four closed loop poles so that only two poles can be arbitrarily assigned.

Since $n = 4$ and $m + \ell - 1 = 3$ in this example the result of theorem (2.3.8) suggests that it should be possible to assign at least three poles arbitrarily. This is not the case, as noted above, which implies that the choice of $(B, C)$ must be non-generic for the given plant matrix $A$.

Consider a slight perturbation of the pair $(B, C)$. Let

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \epsilon & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where $\epsilon$ is some parameter. Now

$$sI - A + BKC = \begin{bmatrix} s+1 & 0 & k_3 & k_4 \\ 0 & s+2 & k_1 & k_2 \\ 0 & -1 & s+1+\epsilon k_1 & \epsilon k_2 \\ 1 & 0 & 0 & s+2 \end{bmatrix}. \qquad (2.3.11)$$

The determinant of (2.3.11) is given by

$$(s+2)^2(s+1)^2+\epsilon k_1(s+1)(s+2)^2+(s+1)(s+2)(k_1-k_4)+\epsilon(s+2)(k_3k_2-k_1k_4)+k_3k_2-k_1k_4$$

which on inspection reveals that the coefficient of $s^3$ can be determined by an appropriate choice of $k_1$, the coefficient of $s^2$ by appropriate choice of $k_4$ and the coefficient of $s$ by appropriate choice of $k_3k_2$. The coefficient of $s^0$ is determined by the choice of the higher degree coefficients. Thus there are three degrees of freedom in the specification of the four poles. Hence, when $\epsilon \neq 0$ it is possible to arbitrarily assign three poles to the closed loop system so demonstrating the genericity of $(B, C)$ in theorem (2.3.8).

A specific description of the possible $(B, C)$ pairs that satisfy theorem (2.3.8) can be obtained from the work of Kimura [1978] on dynamic feedback. When constant feedback is employed the result due to Kimura [*ibid.*], described in theorem (2.3.5), becomes equivalent to that of theorem (2.3.8) but instead of stating the result for generic $(B, C)$ Kimura [*ibid.*] places certain conditions on the open loop system. These conditions are seen to be satisfied by systems in general so confirming that the statement of genericity for $(B, C)$ pairs is justified in theorem (2.3.8). Consider example (2.3.9) once again.

**(2.3.12) Example.** Let $A, B, C$ be given as in example (2.3.9) with $B$ in its unperturbed form. It was seen that it is not possible to arbitrarily assign the number of closed loop poles predicted by theorem (2.2.8). This is explained by Kimura's result stated in theorem

(2.3.5). This equivalent result requires the open loop system to satisfy certain conditions. In particular, $G(s_1, s_2) \neq 0$ except for a finite number of pairs $(s_1, s_2)$. In this example

$$G(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \frac{1}{(s+1)(s+2)}$$

and

$$G(s_1, s_2) = \begin{bmatrix} -s_1 - s_2 - 3 & 0 \\ 0 & s_1 + s_2 + 3 \end{bmatrix} \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}.$$

Now $G(s_1, s_2) = 0$ when $s_2 + s_1 + 3 = 0$. Thus, $G(s_1, s_2) = 0$ for an infinite number of pairs $(s_1, s_2)$ which implies that the result due to Kimura [*ibid.*] does not hold for this particular system.

Further when $B$ is of the form

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \epsilon & 0 \\ 0 & 0 \end{bmatrix}$$

then $G(s)$ and $G(s_1, s_2)$ are given by

$$G(s) = \begin{bmatrix} 1 + (s+2)\epsilon & 0 \\ 0 & -1 \end{bmatrix} \times \frac{1}{(s+1)(s+2)}$$

$$G(s_1, s_2) = \begin{bmatrix} -s_1 - s_2 - 3 - \epsilon(s_2+2)(s_1+2) & 0 \\ 0 & s_1 + s_2 + 3 \end{bmatrix}$$

$$\times \frac{1}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}.$$

Now $G(s_1, s_2)$ may equal 0 only when $(s_1, s_2)$ is equal to $(-1, -2)$ and $(-2, -1)$ so that the perturbed system satisfies condition (ii) of theorem (2.3.5). It also satisfies the other conditions which means that the predicted number of closed loop poles can be assigned. This was shown to be the case in example (2.3.9).

From theorem (2.3.8) it is seen that for a generic system if $m + \ell - 1 \geq n$ then all $n$ closed loop poles can be generically assigned. Kimura [1975] in fact showed that all systems satisfying this condition and which are both observable and controllable can be generically assigned $n$ arbitrary poles as long as the poles are distinct. Kimura [1977] subsequently improved these results to show that almost arbitrary pole placement is possible for almost all systems if

$$\text{(i)} \quad n < \ell + m + \lambda_{G1}$$

$$\text{(ii)} \quad \ell > \mu_{G1}$$

$$\text{(iii)} \quad m \geq \lambda_{G1}$$

where $\lambda_{G1}, \mu_{G1}$ are the controllability index and observability index respectively of the open loop transfer function matrix $G(s)$.

Using techniques from modern algebraic geometry Hermann and Martin [1977] showed that $m\ell \geq n$ is a necessary and sufficient condition for generic pole placement by complex constant output feedback applied to a strictly proper system. For real constant output feedback the condition $m\ell \geq n$ is only a necessary condition. This necessary condition was established by Williams and Hesselink [1978]. They also proved that the condition is not sufficient. Brockett and Byrnes [1981] also noted this result and produced a sufficient condition for generic pole placement when $m\ell = n$. They showed that if $d(m, \ell)$ defined as

$$d(m,\ell) \triangleq \frac{1!2!\ldots(\ell-1)!1!2!\ldots(m-1)!(m\ell)!}{1!2!\ldots(m+\ell-1)!} \tag{2.3.13}$$

is odd, then generic pole placement is guaranteed.

More recently Giannakopoulos and Karcanias [1985] considered the problem of pole placement using non-dynamic output feedback by using tools from exterior algebra and classical algebraic geometry. Their work includes alternative proofs of previous results, extensions of these results to the proper case and subsequently some new results. These results include new sufficient conditions such as the following which generalises, in particular, the result given by Brockett and Byrnes [1981].

**(2.3.14) Theorem (Giannakopoulos and Karcanias, 1985).** Let $G(s) \in \Re^{m\times\ell}(s)$ be a generic strictly proper transfer function matrix with rank $\{G(s)\} = \min(m, \ell) \neq 1$ and $m\ell \geq n$. If the number

$$g(a_0, a_1, \ldots, a_{\ell-1}) \triangleq \frac{n!}{a_0!a_1!\ldots a_{\ell-1}!}\prod_{i>j}(a_i - a_j)$$

is odd for some set $(a_0, a_1, \ldots, a_{\ell-1})$ where

$$n = \sum_{i=0}^{\ell-1} a_i - \frac{1}{2}\ell(\ell-1) \tag{2.3.15}$$

and

$$0 \leq a_0 < a_1 < \ldots < a_{\ell-1} \leq m + \ell - 1 \tag{2.3.16}$$

then $G(s)$ is generically pole assignable by real output feedback.

To illustrate this result consider the following example.

**(2.3.17) Example.** Let

$$G(s) = \begin{bmatrix} \frac{-1}{s^2} & \frac{1}{s} \\ \frac{1}{s} - \frac{2}{s^2} & \frac{2}{s} \end{bmatrix}$$

where $G(s)$ is strictly proper, rank $G(s) = 2$ and $m = 2, \ell = 2, n = 2$. Choose

$$a_0 = 1, a_1 = 2$$

to satisfy (2.3.15) and (2.3.16). Then

$$\begin{aligned} g(a_0, a_1) &= \frac{n!}{a_0! a_1!} \cdot (a_1 - a_0) \\ &= 1 \end{aligned}$$

which is odd. Hence, by theorem (2.3.14), $G(s)$ is generically pole assignable by real output feedback.

To confirm this consider a state space realisation of $G(s)$ of the form

$$G(s) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \triangleq C[sI - A]^{-1} B.$$

Then, if $K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$ is the constant output feedback matrix, the closed loop poles are given by the zeros of the determinant of the matrix

$$\begin{aligned} sI - A + BKC &= \begin{bmatrix} s & 0 \\ 1 & s \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} s + k_2 & k_1 + 2k_2 \\ 1 + k_4 & s + k_3 + 2k_4 \end{bmatrix} \end{aligned}$$

i.e.

$$| sI - A + BKC | = s^2 + s(k_2 + k_3 + 2k_4) + k_2 k_3 - k_4 k_1 - k_1 - 2k_2. \qquad (2.3.18)$$

The coefficients of $s^1$ and $s^0$ in (2.3.18) are mutually independent by the choice of $k_1$. Hence, it is possible to assign the closed loop poles at arbitrary locations as predicted.

When $m\ell = n$ in theorem (2.3.14) the number $g(a_0, a_1, \ldots, a_{\ell-1})$ becomes equal to $d(m, \ell)$ in (2.3.13) so that the result due to Brockett and Byrnes [1981] is a special case of theorem (2.3.14). Other results presented by Giannakopoulos and Karcanias [1985] involve the recently introduced invariant, the Plücker matrix $P_n$, which is constructed from the exterior product of the columns of $\begin{bmatrix} T_G(s) \\ V_G(s) \end{bmatrix}$ where $G(s) = V_G(s) T_G^{-1}(s)$. For brevity these results are not described here but, in general, involve conditions on the rank of $P_n$ which reinforce existing conditions.

### (iii) The general pole placement problem.

The general pole placement problem using output feedback was first considered by Rosenbrock and Hayton [1978], who assumed that the open loop system has a strictly proper transfer function matrix. The results obtained are superior and contain many of the earlier results, notably that of Brasch and Pearson [1970] and results concerning state feedback. The main results obtained by Rosenbrock and Hayton [1978] are presented below. Recall that the general problem is concerned with assigning the invariant polynomials of $T_C(s)$ in (2.3.3). It was shown that $T_C(s)$ can have at most $\ell$ non-unit invariant polynomials so that its Smith form can be expressed as

$$\mathrm{diag}[I, \phi_\ell(s), \phi_{\ell-1}(s), \ldots, \phi_1(s)] \tag{2.3.19}$$

where $\phi_i(s) \mid \phi_{i-1}(s),\ i = 2, \ldots, \ell$. Necessary conditions on the degrees of the $\phi_i(s)$ are given by the following.

**(2.3.20) Theorem (Rosenbrock and Hayton, 1978).** Let $G(s)$ be an $m \times \ell$ strictly proper transfer function matrix with controllability indices $\lambda_{G1} \geq \lambda_{G2} \geq \ldots \geq \lambda_{G\ell}$ and let $K(s)$ be proper with observability indices $\mu_{K1} \geq \mu_{K2} \geq \ldots \geq \mu_{K\ell}$. Let the Smith form of the resulting matrix $T_C(s)$ be of the form (2.3.19). Then, the degrees, $\delta(\phi_i(s))$, of the $\phi_i(s),\ i = 1, 2, \ldots, \ell$, must satisfy the necessary conditions

$$\sum_{i=1}^{k} \delta(\phi_i(s)) \geq \max\{\sum_{i=1}^{k}(\lambda_{Gi} + \mu_{K,\ell+1-i}), \sum_{i=1}^{k}(\lambda_{G,\ell+1-i} + \mu_{Ki})\} \qquad k = 1, 2, \ldots, \ell \tag{2.3.21}$$

with equality when $k = \ell$.

Rosenbrock and Hayton [1978] also produced a sufficient condition for the assignment of the invariant polynomials of $T_C(s)$ and this is expressed in the following theorem.

**(2.3.22) Theorem (Rosenbrock and Hayton, 1978).** Let $G(s)$ be a strictly proper $m \times \ell$ matrix with $\ell \leq m$ and controllability indices $\lambda_{G1} \geq \lambda_{G2} \geq \ldots \geq \lambda_{G\ell}$ and observability indices $\mu_{G1} \geq \mu_{G2} \geq \ldots \geq \mu_{Gm}$. Let $\phi_i(s)$, $i = 1, 2, \ldots, \ell$, be as described above. Then, sufficient conditions for the existence of a proper $\ell \times m$ rational matrix $K(s)$ such that the Smith form of $T_C(s)$ is (2.3.19), are

$$\sum_{i=1}^{k} \delta(\phi_i(s)) \geq \sum_{i=1}^{k} (\lambda_{Gi} + \mu_{G1} - 1) \qquad k = 1, 2, \ldots, \ell \tag{2.3.23}$$

with equality holding when $k = \ell$.

Note that in both (2.3.20) and (2.3.22) there corresponds a "dual" result in which the roles of $\ell$ and $m$, $\lambda_{Gi}$ and $\mu_{Gi}$, $\lambda_{Ki}$ and $\mu_{Ki}$ are reversed. Taking this into account it is seen that the proper compensator $K(s)$ described in theorem (2.3.22) will have order

$$\min[\ell(\mu_{G1} - 1),\ m(\lambda_{G1} - 1)].$$

Rosenbrock and Hayton [1978] note that the sufficient conditions of theorem (2.3.22) could be improved. It is conjectured that the sufficiency conditions might be replaced by sharper and more symmetric conditions

$$\sum_{i=1}^{k} \delta(\phi_i) \geq \sum_{i=1}^{k} (\lambda_{Gi} + \mu_{Gi} - 1) \qquad k = 1, 2, \ldots, \min(\ell, m)$$

with equality holding when $k = \min(\ell, m)$. No proof or counter example of this condition was given in the paper, but it was proved (without equality when $k = \min(\ell, m)$) by Koussiouris [1979]. The result obtained by Koussiouris [*ibid.*] imposed additional conditions on the invariant polynomials but leads to a much lower order for the compensator $K(s)$ than that needed under the sufficient conditions presented by Rosenbrock and Hayton [1978].

Hammer [1983] also produced improved sufficient conditions. The new conditions are dependent on certain invariants which are basically determined by the unstable poles and zeros and the zeros at infinity of the open loop transfer function matrix. These invariants can be directly compared with the controllability and observability indices and as a result lead to improved sufficient conditions. Details of the background to this work can be found in Hammer [1981, 1983*a*].

Alternative proofs to the sufficient conditions presented by Rosenbrock and Hayton [1978] are given by Emre [1980] and Zagalak and Kucera [1985]. Zagalak and Kucera [*ibid.*] also obtain necessary conditions which are an improvement on the necessary conditions of theorem (2.3.20). These improved necessary conditions are presented in the following theorem.

**(2.3.24) Theorem (Zagalak and Kucera, 1985).** Consider the strictly proper system and proper feedback as described in theorem (2.3.20). Let the Smith form of the matrix

$T_C(s)$ formed when this feedback is applied around this system be described by (2.3.19). Then, the degrees, $\delta(\phi_i(s))$, of the $\phi_i(s)$'s must satisfy the necessary conditions

$$\sum_{i=1}^{k} \delta(\phi_i(s)) \geq \max_{1 \leq j \leq \binom{\ell}{k}} S_j^k \qquad k = 1, 2, \ldots, \ell \tag{2.3.25}$$

with equality holding when $k = \ell$ and where

$$S_j^k = \sum_{\alpha=1}^{k} (\lambda_{Gj_\alpha} + \mu_{K,\ell-j_\alpha+1}) \qquad j_1 \neq j_2 \neq \cdots \neq j_k.$$

To illustrate these stronger necessary conditions consider the following example.

**(2.3.26) Example.** Consider a system with controllability indices

$$\lambda_{G1} = 2,\ \lambda_{G2} = 2,\ \lambda_{G3} = 1$$

and a proper compensator with observability indices

$$\mu_{K1} = 3,\ \mu_{K2} = 3,\ \mu_{K3} = 2.$$

The degrees of the invariant polynomials of $T_C(s)$ must satisfy the necessary conditions (2.3.21) under the result given by Rosenbrock and Hayton [1978] which in this case result in the following conditions

$$\delta(\phi_1(s)) \geq 4, \quad \delta(\phi_1(s)) + \delta(\phi_2(s)) \geq 9, \quad \delta(\phi_1(s)) + \delta(\phi_2(s)) + \delta(\phi_3(s)) = 13.$$

The result due to Zagalak and Kucera [1985] requires the degrees of the invariant polynomials to satisfy the following necessary conditions in this case.

$$\delta(\phi_1(s)) \geq 5, \quad \delta(\phi_1(s)) + \delta(\phi_2(s)) \geq 9, \quad \delta(\phi_1(s)) + \delta(\phi_2(s)) + \delta(\phi_3(s)) = 13.$$

Thus, in this case, the result due to Zagalak and Kucera [1985] provides stronger necesary conditions than the result due to Rosenbrock and Hayton [1978].

The review presented above reflects the fact that the pole placement problem for the case of output feedback has been extensively considered. The diversity and complexity of some of the results indicate the level of difficulty encountered in the problem. In some instances it is not possible to directly compare the results because they are given in terms of different characteristics of the system but the variety of results provide suitable indications of the restrictions on the pole placement for a particular system.

For the general pole placement problem necessary and sufficient conditions have not been obtained. The problem therefore has not been completely solved and provides a suitable area for further research. The general pole placement problem has also been restricted to the strictly proper case. Indeed, the pole placement problem has largely been concerned with such systems and the assignment of poles at infinite locations using output feedback has not been considered. In chapter 4 this aspect of the problem is investigated by extending the general pole placement problem to include systems which may possess proper or non-proper transfer function matrices. The problem will be restricted to constant output feedback but the placement of both the finite and infinite pole structures will be taken into account. This aspect of the problem has not been previously investigated.

## §4. Pole placement using state feedback.

By definition the output of a system is always accessible and hence output feedback can always be employed around a system. On the other hand the output does not necessarily reflect the behaviour of the internal states of the system. This can be illustrated if one considers an $n$-state system which has just one output so that detailed information about the internal states may be difficult to obtain from the output or maybe even lost.

It is therefore of greater benefit to be able to feedback the internal states of the system directly thus resulting in greater flexibility in assigning the poles of the system.

This type of feedback can be undertaken for systems described in state space form and in generalised state space form and a brief historical review of this work is now presented.

Consider first, state space systems represented in the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.4.1}$$

where again $x(t)$ is an $n$-vector of internal states, $u(t)$ is an $\ell$-vector of control inputs and $A, B$ are real matrices of the appropriate dimensions. If state feedback of the form

$$u(t) = -Kx(t) + v(t) \tag{2.4.2}$$

where $K$ is a constant real $\ell \times n$ matrix is applied to the system then the closed loop system poles are given by solutions to the equation

$$\det[sI - A + BK] = 0.$$

An early result on pole placement was given by Rissanen [1960] who considered systems having one input. He showed that if such a system is controllable then all the poles of the closed loop system may be assigned arbitrarily. Kalman [1963] gave an alternative proof of this result and he later pointed out that this result had been given by Bertram in 1959 and by Bass in 1961 in unpublished lecture notes. Popov [1964] generalised this result to multi-input, multi-output systems. He showed that given an arbitrary monic polynomial $\phi(s)$ of degree $n$ there exists an $\ell \times n$ matrix $K$, possibly complex, such that the characteristic polynomial of $A-BK$ is $\phi(s)$ if and only if the pair $(A, B)$ is controllable and where $A, B$ have real entries. Wonham [1967] showed that it is possible to choose $K$ with all its entries real. This result was also proved by Luenberger and Anderson [1967] and alternative proofs were given by Davison [1968] and Heymann [1968]. Thus, it has been shown that in the case of state feedback all the poles of the closed loop system may be assigned arbitrarily (subject to complex poles occurring in complex pairs) if and only if the system is controllable.

The general pole placement problem in state space systems was considered by Rosenbrock and Rowe [1970] who produced necessary and sufficient conditions for the invariant polynomials of $sI - A + BK$ to satisfy. These conditions are given in terms of the controllability indices of the system which is the natural expression of such conditions in view of the direct relationship between the controllability of the system and pole placement. The result given by Rosenbrock and Rowe [1970] is presented below.

**(2.4.3) Theorem (Rosenbrock and Rowe, 1970).** Let the state space system described by (2.4.1) be a controllable system with controllability indices, $\lambda_i$, ordered $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_\ell$. Then, there exists a suitable constant feedback matrix, $K$, such that the Smith form of $sI_n - A + BK$ is diag $\{I_{n-\ell}, \phi_1(s), \phi_2(s), \ldots, \phi_\ell(s)\}$ provided the monic polynomials $\{\phi_i(s)\}$ satisfy the following necessary and sufficient conditions
(i) $\phi_i(s) \mid \phi_{i+1}(s) \qquad i = 1, 2, \ldots, \ell - 1$
(ii) $\sum_{i=1}^{k} \delta(\phi_i) \leq \sum_{i=1}^{k} \lambda_i \qquad k = 1, 2, \ldots, \ell$
with equality when $k = \ell$.

Rosenbrock and Rowe [1970] adopt a state space approach to prove this theorem. Other authors have produced alternative proofs. Dickenson [1974] obtained the same result also using a state space approach, whilst Flamm [1980] produced a geometric proof for the necessity part. Flamm [*ibid.*] proved the sufficiency part by means of an explicit algorithm for the construction of a feedback control which performs the required task. This was done in three stages:

(i) transform the system into a suitable canonical form,
(ii) construct a $K$ such that the degrees of the invariant polynomials of $A - BK$ are as desired,
(iii) construct a $K$ to change the invariant polynomials as desired without changing their degrees.

This leads to a complex algorithm requiring a change of basis at each step which means having to find suitable transformation matrices. Munzer and Prätzel-Wolters [1979] gave a module theoretic description of the problem and a subsequent proof of the theorem whilst Kailath [1980] presented a simpler proof using transfer function methods directly.

The pole placement problems as described in section 2 have therefore been satisfactorily solved for state space systems using constant state feedback. The results obtained reflect the powerful nature of such a feedback in state space systems which means that the investigation of the effect of dynamic state feedback on pole placement has not proved necessary.

For generalised state space systems the pole placement problem is also related to the controllability properties of the system. The notions of controllability for such systems are fully discussed in chapter 5 and as a consequence a description of the pole placement problems in generalised state space systems will be given in chapter 6 rather than in this chapter.

## §5. Conclusions.

The pole placement problems are important problems in polynomial systems matrix theory due to the dynamic properties associated with the poles of the system. This is reflected in the number of authors who have considered such problems. The review presented in the previous sections discusses the various results and highlights aspects of the problems which remain unsolved. It was noted that a major area that has not been considered in detail is that concerning the assignment of the infinite poles. This is due to the fact that an adequate understanding of the infinite frequency structure of a system and means of investigating such structure have only recently been developed.

This thesis will investigate in detail the assignment of the infinite pole structure, concentrating in particular on systems represented in generalised state space form. The aim of the work is to generalise existing results to include the assignment of the infinite poles and thus provide complete solutions to the problems under consideration.

## Chapter 3. Infinite Frequency Structure of a Rational Matrix.

### §1. Introduction.

In this chapter the infinite frequency structure of a rational matrix is investigated. The definition of infinite poles and zeros eminates from the corresponding definitions of finite poles and zeros of a rational matrix. This development is discussed in section 2 with particular reference to the role of a minimal factorisation of a rational matrix (see Pugh and Ratcliffe, 1980) and its Smith McMillan form at infinity (see Vardulakis *et al.*, 1982). In section 3 the relationship between the Smith McMillan form at infinity of a rational matrix and the Laurent expansion about the point at infinity of that matrix is described (see Demianczuk, 1990). This relationship provides a means of investigating the infinite poles and zeros of a rational matrix. This is undertaken in sections 3 and 4 to produce some original results concerning the absence of infinite zeros and the particular case of polynomial matrices.

### §2. Definition of infinite poles and zeros.

The Smith McMillan form of a rational matrix provides an appropriate means of defining the finite poles and zeros of that particular matrix. This definition of the finite poles and zeros provides a satisfactory extension of the definition from a single rational function to the matrix case from the point of view of the dynamic interpretation of the finite poles and zeros. Formally this definition is given below.

Let the Smith McMillan form of an $m \times \ell$ rational matrix , $G(s)$, be represented by

$$S(G(s)) = \begin{cases} \operatorname{diag} \begin{pmatrix} \dfrac{\epsilon_i(s)}{\psi_i(s)} & 0_{m,\ell-m} \end{pmatrix} & \ell > m \\ \operatorname{diag} \begin{pmatrix} \dfrac{\epsilon_i(s)}{\psi_i(s)} \end{pmatrix} & \ell = m \\ \operatorname{diag} \begin{pmatrix} \epsilon_i(s)/\psi_i(s) \\ 0_{m-\ell,\ell} \end{pmatrix} & \ell < m \end{cases}$$

where $\epsilon_i(s), \psi_i(s),\ i = 1, 2, \ldots, h = \min(\ell, m)$, are monic polynomials and if $\epsilon_{q+1}(s) = \epsilon_{q+2}(s) = \ldots = \epsilon_h(s) \equiv 0$ for some $q$ then $\psi_{q+1}(s) = \psi_{q+2}(s) = \ldots = \psi_h(s) \equiv 1$. Also $\psi_i(s) \mid \psi_{i-1}(s),\ i = 2, 3, \ldots, h$, and $\epsilon_i(s) \mid \epsilon_{i+1}(s),\ i = 1, 2, \ldots, q-1$. Then, the finite zeros and poles are defined as follows.

**(3.2.1) Definition.** The FINITE ZEROS of $G(s)$ are defined as the roots of the (non-zero) numerator polynomials $\{\epsilon_i(s)\}$ of $S(G(s))$ and the FINITE POLES of $G(s)$ are defined as the roots of the denominator polynomials $\{\psi_i(s)\}$ of $S(G(s))$.

Smith Form

An equivalent definition can be given in terms of a coprime factorisation of $G(s)$. First introduce the following definition.

**(3.2.2) Definition.** Let $D(s)$ be an $m \times \ell$ polynomial matrix. Then, $s_0 \in \mathbb{C}$ is said to be a ZERO OF DEGREE $k$ of $D(s)$ in case $(s - s_0)^k$ is an elementary divisor of $D(s)$. The set of zeros of $D(s)$ is the set of all such numbers $s_0$, a zero of degree $k$ being included $k$ times. Further, the MULTIPLICITY OF A ZERO at $s_0 \in \mathbb{C}$ is said to be equal to the total number of elementary divisors of the form $(s - s_0)^k, k > 0$.

Recall that if $G(s)$ is an $m \times \ell$ rational matrix then it may be decomposed into relatively prime factors,

$$G(s) = N_1(s)\, D_1^{-1}(s) = D_2^{-1}(s)\, N_2(s) \tag{3.2.3}$$

where $D_1(s)$, $N_1(s)$ are relatively (right) prime and $D_2(s), N_2(s)$ are relatively (left) prime. Any $m \times \ell$ polynomial matrix $N_1(s), N_2(s)$ satisfying (3.2.3) is referred to as a numerator of $G(s)$ whilst any $\ell \times \ell$ polynomial matrix $D_1(s)$ or $m \times m$ polynomial matrix $D_2(s)$ satisfying (3.2.3) is referred to as a denominator of $G(s)$. Pugh and Ratcliffe [1979] showed that all numerators of $G(s)$ are unimodular equivalent and all denominators are extended unimodular equivalent (Pugh and Shelton, 1976). These observations subsequently gave rise to the following definition.

**(3.2.4) Definition.** $s_0 \in \mathbb{C}$ is a ZERO (POLE) OF DEGREE $k$ of the rational matrix $G(s)$ if it is a zero of degree $k$ of any numerator (denominator). Also the MULTIPLICITY OF A ZERO (POLE) at $s_0 \in \mathbb{C}$ is equal to the multiplicity of the zero at $s_0$ of the numerator (denominator).

The equivalence of this definition of finite poles and zeros of a rational matrix to that given by definition (3.2.1) is seen by noting that the poles and zeros of the rational matrix are not affected by transforming the matrix to Smith McMillan form. Thus, for the case $\ell = m$,

give zeros

$$\begin{aligned} S(G(s)) &= [\text{diag}\,\{\epsilon_i(s)\}]\ [\text{diag}\,\{\psi_i(s)\}]^{-1} \\ &\triangleq S(N_1(s))\,(S(D_1(s)))^{-1} \end{aligned}$$

give poles.

where $S(D_1(s))$, $S(N_1(s))$ correspond to the respective Smith forms of any denominator and any numerator of $G(s)$. Since the matrices $S(D_1(s))$, $S(N_1(s))$ are relatively (right) prime due to the properties of the $\epsilon_i(s), \psi_i(s)$ it follows that the two definitions, (3.2.1) and (3.2.4), coincide.

The above definitions do not provide an immediate extension to the case of infinite poles and zeros. This is due to the fact that the unimodular transformations inherent in both definitions lead, in general, to the destruction of the infinite frequency structure.

The infinite frequency structure of the rational transfer function matrix associated with a system has an important bearing on the behaviour of that system. For instance, the properties of the infinite poles determine the high gain and high frequency behaviour whilst the infinite zeros are directly related to the decouplability properties of the system. Thus, a suitable means of investigating the infinite frequency structure is seen to be necessary. The development of such methods was based on the standard technique of employing a transformation which takes the point at infinity to a finite point so that the infinite frequency structure can be analysed using techniques associated with investigation of the finite frequency. The simplest such transformation takes the point at infinity to zero giving rise to the following definition.

**(3.2.5) Definition (Pugh and Ratcliffe 1979, Verghese 1978).** An $m \times \ell$ rational matrix $G(s)$ is said to have an INFINITE ZERO (POLE) OF DEGREE $k$ in case $w = 0$ is a finite zero (pole) of degree $k$ for the rational matrix $G\left(\frac{1}{w}\right)$. Further, the MULTIPLICITY OF AN INFINITE ZERO (POLE) is equal to the multiplicity of the zero (pole) at $w = 0$ of the rational matrix $G\left(\frac{1}{w}\right)$.

It then follows that the investigation of the pole and zero structures at $w = 0$ of $G\left(\frac{1}{w}\right)$ can proceed by employing a suitable coprime factorisation. The disadvantage of this approach is that a coprime factorisation of $G(s)$ has no direct relation to a coprime factorisation of $G\left(\frac{1}{w}\right)$ , so that if both the finite and infinite frequency structure is to be investigated two seperate factorisations must be employed. This disadvantage can be overcome if a minimal factorisation (Forney, 1975) is adopted for $G(s)$. This particular factorisation enables both the finite and infinite pole and zero structures to be deduced from the same factorisation. In particular, the following lemma holds.

P56.

**(3.2.6) Lemma (Pugh and Ratcliffe, 1980).** Let $G(s)$ be an $m \times \ell$ rational matrix factorised as

$$G(s) = N_1(s)\, D_1^{-1}(s) \tag{3.2.7}$$

where the columns of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ constitute a minimal basis with column degrees $c_i, i = 1, 2, \ldots, \ell$. Let $\Lambda_1(s) = \text{diag}\,[s^{c_1}, s^{c_2}, \ldots, s^{c_\ell}]$. Then,

(i) the finite pole structure of $G(s)$ is the finite zero structure of $D_1(s)$ and the infinite pole structure of $G(s)$ is the zero structure at $w = 0$ of the polynomial matrix

$$D_1\left(\tfrac{1}{w}\right)\Lambda_1(w),$$

(ii) the finite zero structure of $G(s)$ is the finite zero structure of $N_1(s)$ and the infinite zero structure of $G(s)$ is the zero structure at $w = 0$ of the polynomial matrix

$$N_1\left(\tfrac{1}{w}\right)\Lambda_1(w).$$

A factorisation of the form (3.2.7) where the columns of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ constitute a minimal basis will be referred to as a right minimal factorisation of $G(s)$.

A dual result of lemma (3.2.6) holds when a left minimal factorisation of $G(s)$ is employed, i.e.

$$G(s) = D_2^{-1}(s) N_2(s) \tag{3.2.8}$$

where the rows of $[D_2(s) \quad N_2(s)]$ constitute a minimal basis. Again the matrices giving rise to the finite and infinite pole and zero structure from different minimal factorisations are appropriately related by unimodular or extended unimodular equivalence transformations so that the pole and zero structures can be investigated by adopting any left or right minimal factorisation. The relationships are formally characterised below where the above notation has been adopted.

**(3.2.9) Theorem (Pugh and Ratcliffe, 1979).** Matrices of the form $N_1(s), N_2(s)$ are unimodular equivalent whilst matrices of the form $D_1(s), D_2(s)$ are extended unimodular equivalent.

**(3.2.10) Theorem.** Matrices of the form $N_1\left(\frac{1}{w}\right)\Lambda_1(w)$ and $\Lambda_2(w)\, N_2\left(\frac{1}{w}\right)$ are unimodular equivalent whilst matrices of the form $D_1\left(\frac{1}{w}\right)\Lambda_1(w)$ and $\Lambda_2(w)\, D_2\left(\frac{1}{w}\right)$ are extended unimodular equivalent.

**Proof.** Consider first the case of two minimal right factorisations, i.e.

$$N_1(s)\,(D_1(s))^{-1} = N_1^*(s)\,(D_1^*(s))^{-1}.$$

Then,

$$N_1\left(\tfrac{1}{w}\right)\Lambda_1(w)\,\Lambda_1(w)^{-1}\left(D_1\left(\tfrac{1}{w}\right)\right)^{-1} = N_1^*\left(\tfrac{1}{w}\right)\Lambda_1^*(w)\Lambda_1^{*-1}(w)\left(D_1^*\left(\tfrac{1}{w}\right)\right)^{-1}$$

or, alternatively,

$$\left[N_1\left(\tfrac{1}{w}\right)\Lambda_1(w)\right]\left[D_1\left(\tfrac{1}{w}\right)\Lambda_1(w)\right]^{-1} = \left[N_1^*\left(\tfrac{1}{w}\right)\Lambda_1^*(w)\right]\left[D_1^*\left(\tfrac{1}{w}\right)\Lambda_1^*(w)\right]^{-1}. \tag{3.2.11}$$

The matrix $\begin{bmatrix} D_1\left(\frac{1}{w}\right) & \Lambda_1(w) \\ N_1\left(\frac{1}{w}\right) & \Lambda_1(w) \end{bmatrix}$ has full rank when $w \neq 0$ since $D_1(s), N_1(s)$ are relatively (right) prime. When $w = 0$

$$\begin{bmatrix} D_1\left(\frac{1}{w}\right) & \Lambda_1(w) \\ N_1\left(\frac{1}{w}\right) & \Lambda_1(w) \end{bmatrix}_{w=0} = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}_{hc}$$

where $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}_{hc}$ is the high order coefficient matrix with respect to the columns of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$.

Since $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ is a minimal basis the matrix $\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}_{hc}$ has full rank. Hence, it follows that $D_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$, $N_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ are relatively (right) prime. Similarly for $D_1^*\left(\frac{1}{w}\right)\,\Lambda_1^*(w)$, $N_1^*\left(\frac{1}{w}\right)\,\Lambda_1^*(w)$. Thus, it follows, from (3.2.11) (see Rosenbrock 1970, p139), that $D_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $D_1^*\left(\frac{1}{w}\right)\,\Lambda_1^*(w)$, $N_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $N_1^*\left(\frac{1}{w}\right)\,\Lambda_1^*(w)$ are unimodular equivalent. A similar argument holds for the case of two left factorisations.

Finally consider the case where

$$D_2^{-1}(s)\,N_2(s) = N_1(s)\,D_1^{-1}(s).$$

Then,

$$\Lambda_2(w)\,N_2\left(\tfrac{1}{w}\right)\,\left[D_1\left(\tfrac{1}{w}\right)\,\Lambda_1(w)\right] = \left[\Lambda_2(w)\,D_2\left(\tfrac{1}{w}\right)\right]\,N_1\left(\tfrac{1}{w}\right)\,\Lambda_1(w)$$

where, employing a similar argument to that adopted above, it is seen that $\Lambda_2(w)\,N_2\left(\frac{1}{w}\right)$ and $\Lambda_2(w)\,D_2\left(\frac{1}{w}\right), D_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $N_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ are relatively (left) prime. Hence, it follows by definition that $N_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $\Lambda_2(w)\,N_2\left(\frac{1}{w}\right), D_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $\Lambda_2(w)\,D_2\left(\frac{1}{w}\right)$ are extended unimodular equivalent. Since $N_1\left(\frac{1}{w}\right)\,\Lambda_1(w)$ and $\Lambda_2(w)\,N_2\left(\frac{1}{w}\right)$ are both $m \times \ell$ the relationship reduces to one of unimodular equivalence , so completing the proof. $\square$

The above proof demonstrates quite clearly the important role of the minimal factorisations in establishing the unimodular and extended unimodular equivalence relationships and thus making the result of lemma (3.2.6) meaningful.

In a similar way the infinite frequency structure of a rational matrix can be investigated by performing a bilinear transformation and then obtaining the Smith McMillan form of the subsequent rational matrix. Adopting a general bilinear transformation of the form

$$s = \frac{\alpha w + \beta}{\gamma w + \delta} \qquad \gamma \neq 0, \quad \alpha\delta - \beta\gamma \neq 0, \quad \alpha, \beta, \gamma, \delta, \in \Re$$

results in the point $s = \frac{\alpha}{\gamma}$ being transformed to the point $w = \infty$ and $s = \infty$ being transfered to the point $s = \frac{-\delta}{\gamma}$. The real numbers $\alpha, \gamma$ may be chosen in such a way that the point $s = \frac{\alpha}{\gamma}$ does not correspond to a pole or zero of the matrix $G(s)$ thus ensuring that all the poles and zeros of $G\left(\dfrac{\alpha w + \beta}{\gamma w + \delta}\right)$ are located at finite locations. As a result the Smith McMillan form of the rational matrix $G\left(\dfrac{\alpha w + \beta}{\gamma w + \delta}\right)$ reveals both the finite and infinite pole and zero structure of the original matrix, $G(s)$. This approach becomes difficult to use if the dimensions of $G(s)$ are greater than $3 \times 3$ or if the individual entries in $G(s)$ contain high powers of $s$ (Vardulakis *et al.*, 1982). The method also seems cumbersome if it is only the infinite frequency structure that is of interest. For this case the emergence of the Smith McMillan form at infinity (Vardulakis *et al.*, 1982) is of relevance. This form is obtained by pre and post multiplying the rational matrix $G(s)$ by a series of rational matrices, known as biproper matrices, which have no poles or zeros at infinity. Employing such transformations ensures that the infinite poles and zeros of the original matrix $G(s)$ remain unchanged. If $\Re_{pr}(s)$ is the ring of proper rational functions then a biproper matrix is defined as follows.

**(3.2.12) Definition.** The $m \times m$ rational matrix $W(s) \in \Re_{pr}^{m\times m}(s)$ is said to be BIPROPER if and only if

(i) $\lim_{s \to \infty} W(s) = W_\infty \in \Re^{m\times m}$

(ii) $\det W_\infty \neq 0$.

The definition of a biproper matrix leads to an equivalence relationship referred to as equivalence at infinity.

**(3.2.13) Definition.** The $m \times \ell$ rational matrices $G_1(s)$ and $G_2(s)$ are said to be EQUIVALENT AT INFINITY if there exist biproper rational matrices $W(s) \in \Re_{pr}^{m\times m}(s)$, $V(s) \in \Re_{pr}^{\ell\times\ell}(s)$ such that

$$W(s)\,G_1(s)\,V(s) = G_2(s).$$

A canonical form under equivalence at infinity is the Smith McMillan form at infinity described by the following lemma.

**(3.2.14) Lemma (Vardulakis *et al.*, 1982).** Let $G(s) \in \Re^{m\times\ell}(s)$ with rank $G(s) = r$. Then, there exist biproper rational matrices, $W(s) \in \Re_{pr}^{m\times m}(s)$ and $V(s) \in \Re_{pr}^{\ell\times\ell}(s)$, such that

$$W(s)\,G(s)\,V(s) = S^\infty(G(s))$$

where

$$S^\infty(G(s)) = \begin{cases} [Q(s) \quad 0_{m,\ell-m}] & \ell > m \\ Q(s) & \ell = m \\ \begin{bmatrix} Q(s) \\ 0_{m-\ell,m} \end{bmatrix} & \ell < m \end{cases}$$

and

$$Q(s) = \mathrm{diag}\{s^{\eta_1}, s^{\eta_2}, \ldots, s^{\eta_r}, 0, 0, \ldots, 0\} \tag{3.2.15}$$

with $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_k \geq 0 \geq \eta_{k+1} \geq \ldots \geq \eta_r$. $S^\infty(G(s))$ is called the SMITH McMILLAN FORM AT INFINITY of $G(s)$.

The Smith McMillan form at infinity of $G(s)$ provides an alternative means of defining the infinite poles and zeros of $G(s)$.

**(3.2.16) Definition.** If $p_\infty$ is the number of $\eta_i$'s in (3.2.15) with $\eta_i > 0$ then $G(s)$ has $p_\infty$ POLES AT INFINITY, each having degree $\eta_i$. Similarly, if $z_\infty$ is the number of $\eta_i$'s in (3.2.15) with $\eta_i < 0$ then $G(s)$ has $z_\infty$ ZEROS AT INFINITY, each having degree $| \eta_i |$.

The definition of infinite poles and zeros is equivalent to the earlier definition (3.2.5) and provides a straightforward characterisation of the infinite frequency structure. A neat and convenient way of representing the infinite frequency structure is by means of a step function, which is of particular relevance to the contents of section 3. The adoption of a step function is also seen to be an appropriate way of illustrating results in subsequent chapters. In this instance make the following definition.

**(3.2.17) Definition.**

$$S^\infty(i) \triangleq \begin{cases} \eta_i & i = \text{integer} \\ \eta_{i+} & i \neq \text{integer} \end{cases}$$

where $i+$ denotes the upwards rounded version of $i$.

Since the $\eta_i$'s are ordered in a decreasing manner it follows that $S^\infty(i)$ is a decreasing staircase as shown by figure (3.2.18).

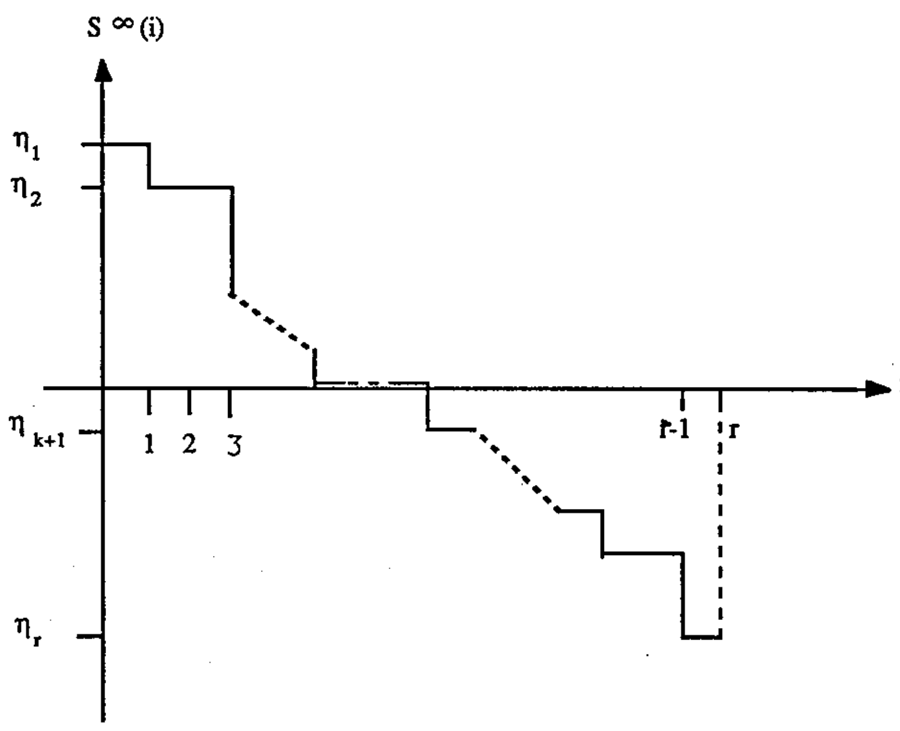



fig. (3.2.18)

$S^{\infty}(i)$ has been defined in such a way that it contains all the infinite frequency information concerning $G(s)$ in a completely irredundant way.

The Smith McMillan form at infinity can obviously be obtained for a particular rational matrix by applying a sequence of appropriate biproper transformations. This, though, is a rather cumbersome method since the sequence of transformations is not unique. An alternative and more systematic method is obtained by exploiting the Laurent expansion about the point at infinity of the original rational matrix. Van Dooren *et al.* [1979] develop this theory for finite frequencies to obtain the Smith McMillan form about a certain point, and suggest that it could be employed for the infinite case. The subsequent definition of the Smith McMillan form at infinity makes the extension of this theory to the infinite case more relevant. A description of the way this theory has been extended to the infinite case is given in the next section. This then leads to an alternative method of obtaining the Smith McMillan form at infinity and hence of investigating the infinite poles and zeros of a rational matrix.

## §3. The Laurent expansion and Toeplitz matrices of a rational matrix.

Van Dooren *et al.* [1979] use the Laurent expansion of $G(s)$ about a finite point, $s_0$, and the corresponding Toeplitz matrices to determine the Smith McMillan form at $s_0$ of $G(s)$. In an analogous way the Smith McMillan form at infinity of $G(s)$ can be determined by considering the Laurent expansion about the point at infinity of $G(s)$ and the corresponding Toeplitz matrices as described by Demianczuk [1990].

Suppose the Laurent expansion about the point at infinity of $G(s)$ is of the following form

$$\begin{aligned} G(s) &= \sum_{i=-\infty}^{\ell} G_i s^i \\ &= G_\ell\, s^\ell + G_{\ell-1}\, s^{\ell-1} + \ldots + G_0 + G_{-1}s^{-1} + \ldots \end{aligned}$$

Let $G(s)$ have rank $r$. Then, the Toeplitz matrices at infinity are defined as follows.

**(3.3.1) Definition.** The TOEPLITZ MATRICES AT INFINITY, $T_i^{\infty}(G)$, associated with $G(s)$ are defined as

$$T_i^{\infty}(G) = \begin{bmatrix} G_\ell & G_{\ell-1} & \ldots & G_{-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & G_{\ell-1} \\ & 0 & & G_\ell \end{bmatrix}, \qquad i \geq -\ell.$$

The information concerning the ranks of the $T_i^{\infty}(G)$ will determine the rank indices at infinity of $G(s)$ which are defined in the following manner.

**(3.3.2) Definition.** The RANK INDICIES AT INFINITY of $G(s)$ are defined as

$$\rho_i^\infty = \text{rank } (T_i^\infty(G)) - \text{rank } (T_{i-1}^\infty(G)) \quad i = -\ell, -\ell+1, \ldots$$

where it is assumed that rank $(T_{-\ell-1}^\infty(G)) = 0$.

It is now shown that these rank indices at infinity are invariant under the transformation of equivalence at infinity given by definition (3.2.13).

**(3.3.3) Theorem.** Let $G(s)$, $H(s)$ be two $m \times \ell$ rational matrices. If $G(s)$, $H(s)$ are equivalent at infinity then they have the same rank indices at infinity.

**Proof.** Since $G(s)$, $H(s)$ are equivalent at infinity there exist biproper rational matrices $M(s)$, $N(s)$ of dimensions $m \times m$ and $\ell \times \ell$ respectively, such that

$$M(s)G(s)N(s) = H(s). \tag{3.3.4}$$

Since $M(s)$, $N(s)$ are biproper they have no infinite poles or zeros and so their Laurent expansions about the point at infinity take the form

$$M(s) = M_0 + M_{-1}\ s^{-1} + M_{-2}\ s^{-2} + \ldots$$

$$N(s) = N_0 + N_{-1}\ s^{-1} + N_{-2}\ s^{-2} + \ldots$$

where $M_0 = M(s = \infty)$, $N_0 = N(s = \infty)$ are non-singular. Let the Laurent expansion about the point at infinity of $G(s), H(s)$ be given by

$$G(s) = \sum_{i=-\infty}^{g} G_i s^i$$

$$H(s) = \sum_{i=-\infty}^{h} H_i s^i.$$

Substituting these expressions into (3.3.4), and comparing coefficients of $s$ gives rise to the following relationship.

$$\begin{bmatrix} M_0 & \cdots & \cdots & M_{-\ell-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & M_0 \end{bmatrix} \begin{bmatrix} G_\ell & \cdots & \cdots & G_{-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & G_\ell \end{bmatrix} \begin{bmatrix} N_0 & \cdots & \cdots & N_{-\ell-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & N_0 \end{bmatrix}$$

$$= \begin{bmatrix} H_\ell & \cdots & \cdots & H_{-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & 0 & & H_\ell \end{bmatrix} \tag{3.3.5}$$

where $\ell = \min\,(g, h)$.

Since $M_0$, $N_0$ are non-singular it follows that the Toeplitz matrices built on $M(s)$, $N(s)$ are also non-singular. Therefore, from (3.3.5), it follows that

$$\operatorname{rank} T_i^\infty(G) = \operatorname{rank} T_i^\infty(H)$$

as required. □

As a consequence of the above result it follows that a rational matrix, $G(s)$, has the same rank indices at infinity as its Smith McMillan form at infinity, $S^\infty(G(s))$. Therefore, the properties of the rank indices at infinity, $\rho_i^\infty$, of $G(s)$ can be deduced from the Toeplitz matrices at infinity of $S^\infty(G(s))$, i.e. $T_i^\infty(S^\infty(G))$, where the variable $s$ has been dropped from the notation for convenience. These Toeplitz matrices have a particularly simple structure because of the special form of $S^\infty(G(s))$. Specifically, note that

(i) all the rows of $T_i^\infty(S^\infty(G))$ are either zero or have one non-zero entry (a "one"),

(ii) the non-zero rows of $T_i^\infty(S^\infty(G))$ are linearly independent.

From the second property it follows that

$$\begin{aligned} \rho_i^\infty &= \operatorname{rank} T_i^\infty\,(S^\infty(G)) - \operatorname{rank} T_{i-1}^\infty\,(S^\infty(G)) \\ &= \operatorname{rank}\,[S_\ell(G),\ S_{\ell-1}(G),\ \ldots,\ S_{-i}(G)]. \end{aligned} \tag{3.3.6}$$

where $S_j(G)$ is the $j^{\text{th}}$ coefficient in the Laurent expansion at infinity of $S^\infty(G(s))$. Further, it can be seen, using the above properties, that

$$\operatorname{rank}\,[S_\ell(G), S_{\ell-1}(G), \ldots, S_{-i}(G)]$$

is equal to the number of 1's in $[S_\ell(G), S_{\ell-1}(G), \ldots, S_{-i}(G)]$, which in turn equals the number of powers, $\eta_j$, greater than or equal to $i$ in $S^\infty(G)$. It should also be noted due to the properties of the $s_i$'s, that

$$\operatorname{rank}\,[S_\ell(G), S_{\ell-1}(G), \ldots, S_{-i}(G)]$$

will at some stage equal $r$, the normal rank of $G(s)$, but that $\operatorname{rank}\,[S_\ell(G), S_{\ell-1}(G), \ldots, S_{-i}(G)]$ can not exceed $r$.

Thus, a direct relationship between the rank indices at infinity of $G(s)$ and its Smith McMillan form at infinity has been established which makes it possible to deduce the Smith McMillan form at infinity of $G(s)$ from the rank differences of its Toeplitz matrices at infinity. To derive this relationship define the rank index function at infinity, $R^{\infty}(i)$, associated with the rank indices at infinity, $\rho_i^{\infty}$, as follows.

**(3.3.7) Definition.**

$$R^{\infty}(i) = \begin{cases} \rho_i^{\infty} & i = \text{integer} \\ \rho_{i-}^{\infty} & i = \text{non-integer} \end{cases}$$

*where $i-$ is the downward rounded version of $i$.*

Again, using (3.3.6), it is seen that $R^{\infty}(i)$ is an increasing staircase as illustrated by figure (3.3.8).

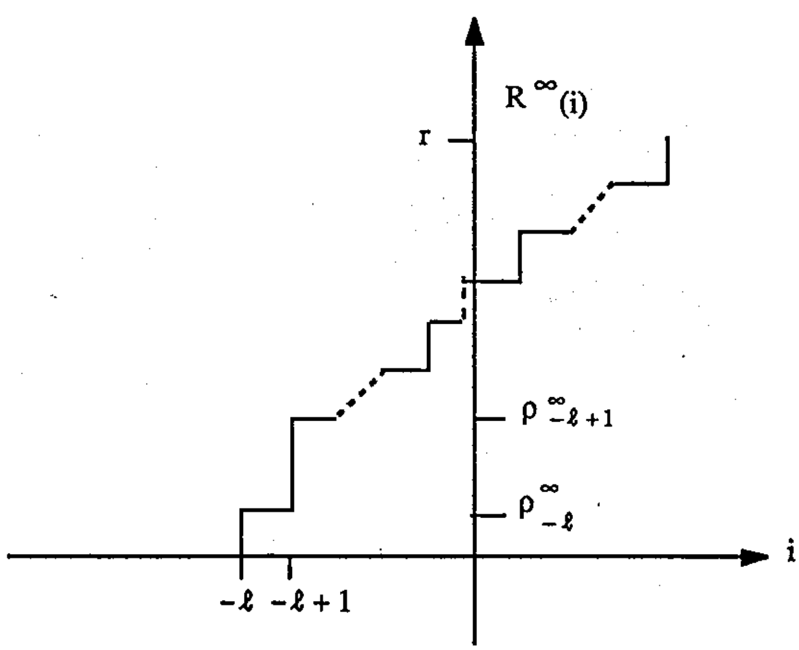



fig. (3.3.8)

The $R^{\infty}(i)$ staircase is in fact a $90^o$ rotation of the $S^{\infty}(i)$ staircase defined in (3.2.17) and so the Smith McMillan form at infinity of $G(s)$ can be deduced directly from the $R^{\infty}(i)$ staircase as follows.

**(3.3.9) Theorem.** If, in the notation of lemma (3.2.14), $S^{\infty}(G(s))$ denotes the Smith McMillan form at infinity of the rational matrix $G(s)$, and $\rho_i^{\infty}$ denote the rank indices of $G(s)$ constructed on the basis of its Laurent expansion about the point at infinity, then

$$S^{\infty}(G(s)) \triangleq \text{block diag } \{Q_i(s)\} \tag{3.3.10}$$

where $Q_i(s)$ is the $(\rho_i^\infty - \rho_{i-1}^\infty) \times (\rho_i^\infty - \rho_{i-1}^\infty)$ matrix given by

$$Q_i(s) \triangleq \begin{pmatrix} s^{-i} & 0 & \dots & 0 \\ 0 & s^{-i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s^{-i} \end{pmatrix}$$

for $i = -\ell,\ -\ell+1,\ \dots,$ and if $\rho_i^\infty - \rho_{i-1}^\infty = 0$ then the corresponding matrix $Q_i(s)$ is not present in (3.3.10).

**Proof.** If $\rho_i^\infty - \rho_{i-1}^\infty \neq 0$ then from (3.3.6) and the diagonal structure of $S^\infty(G(s))$ it follows that

$$\operatorname{rank} S_{-i}(G) = \rho_i^\infty - \rho_{i-1}^\infty$$

where $S_{-i}(G)$ is the coefficient of $s^{-i}$ in the Laurent expansion at infinity of $S^\infty(G(s))$. This, in turn, implies that $S^\infty(G(s))$ possesses $\rho_i^\infty - \rho_{i-1}^\infty$ diagonal elements of the form $s^{-i}$. If $\rho_i^\infty - \rho_{i-1}^\infty = 0$ then

$$\operatorname{rank} S_{-i}(G) = 0$$

and $S^\infty(G(s))$ does not possess any diagonal elements of the form $s^{-i}$. Hence $S^\infty(G(s))$ is as described by (3.3.10), as required. □

In particular the pole/zero structure at infinity may then be deduced as follows.

**(3.3.11) Corollary.** If, in theorem (3.3.9) $\rho_i^\infty - \rho_{i-1}^\infty \neq 0$, then
(i) $G(s)$ will have $\rho_i^\infty - \rho_{i-1}^\infty$ poles at infinity of degree $|i|$, if $i < 0$,
(ii) $G(s)$ will have $\rho_i^\infty - \rho_{i-1}^\infty$ zeros at infinity of degree $i$, if $i > 0$.

**Proof.** Follows directly from theorem (3.3.9) and definition (3.2.16). □

To illustrate the way the infinite pole and zero structure of a rational matrix can be obtained from its Laurent expansion about the point at infinity consider the following example.

**(3.3.12) Example.** Let

$$\frac{1}{s^2(1-\frac{2}{s^2})}$$

$$\frac{1}{s^2}\left(1-\frac{2}{s^2}\right)^{-1}$$

$$G(s) = \begin{bmatrix} s^3 & 0 & \dfrac{1}{s^2-2} & 0 \\ 0 & 2s & 0 & \dfrac{1}{s(s-1)} \\ 0 & 0 & \dfrac{s-1}{s^3} & 0 \\ 0 & s-1 & 0 & \dfrac{1}{s^3} \end{bmatrix}.$$

The Laurent expansion of $G(s)$ about the point at infinity is given by

$$G(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

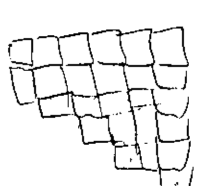
$$+ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} s^{-2} + 0(s^{-3})$$

Since a particular Toeplitz matrix is formed by adding a block column to the previous Toeplitz matrix, the Toeplitz matrices associated with this expansion can be expressed in the composite form

$$\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc}
\multicolumn{4}{c}{T^{\infty}_{-3}(G)} & \multicolumn{4}{c}{T^{\infty}_{-2}(G)} & \multicolumn{4}{c}{T^{\infty}_{-1}(G)} & \multicolumn{4}{c}{T^{\infty}_{0}(G)} & \multicolumn{4}{c}{T^{\infty}_{1}(G)} & \multicolumn{4}{c}{T^{\infty}_{2}(G)} \\
1&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&1&0 \\
0&0&0&0 & 0&0&0&0 & 0&2&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&1 \\
0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&1&0 \\
0&0&0&0 & 0&0&0&0 & 0&1&0&0 & 0&-1&0&0 & 0&0&0&0 & 0&0&0&0 \\
\hline
 & & & & 1&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & 0&0&0&0 & 0&0&0&0 & 0&2&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & 0&0&0&0 & 0&0&0&0 & 0&1&0&0 & 0&-1&0&0 & 0&0&0&0 \\
\hline
 & & & & & & & & 1&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&2&0&0 & 0&0&0&0 \\
 & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&1&0&0 & 0&-1&0&0 \\
\hline
 & & & & & & & & & & & & 1&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&2&0&0 \\
 & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 & 0&1&0&0 \\
\hline
 & & & & & & & & & & & & & & & & 1&0&0&0 & 0&0&0&0 \\
 & & & & & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 \\
 & & & & & & & & & & & & & & & & 0&0&0&0 & 0&0&0&0 \\
\hline
 & & & & & & & & & & & & & & & & & & & & 1&0&0&0 \\
 & & & & & & & & & & & & & & & & & & & & 0&0&0&0 \\
 & & & & & & & & & & & & & & & & & & & & 0&0&0&0 \\
 & & & & & & & & & & & & & & & & & & & & 0&0&0&0
\end{array}$$

Rank $T_{-3}$

Rank $T_{-2}$ etc.



$$T^{\infty}_{i}(G) = \begin{pmatrix} G_l & G_{l-1} & \cdots & G_{-i} \\ & \ddots & & \\ 0 & & & G_l \end{pmatrix}$$

$G_3 \quad G_2 \quad G_1 \quad G_0 \quad G_{-1} \quad G_{-2}$

$G_2 \downarrow 0$, $G_{-1} \downarrow 0$

$\therefore T_{-3} = (G_l) \quad T_{-2} = \begin{pmatrix} G_3 & G_2 \\ 0 & G_3 \end{pmatrix}$

The ranks of the Toeplitz matrices are

$$\text{rank} \quad T_{-3}^{\infty}(G) = 1 \qquad \rho_0 = 1$$

$$\text{rank} \quad T_{-2}^{\infty}(G) = 2$$

$$\text{rank} \quad T_{-1}^{\infty}(G) = 4$$

$$\text{rank} \quad T_{0}^{\infty}(G) = 6$$

$$\text{rank} \quad T_{1}^{\infty}(G) = 8$$

$$\text{rank} \quad T_{2}^{\infty}(G) = 12$$

and the corresponding rank indices at infinity of $G(s)$ are given by

$$\rho_{-3}^{\infty} = 1$$

$$\rho_{-2}^{\infty} = 1$$

$$\rho_{-1}^{\infty} = 2$$

$$\rho_{0}^{\infty} = 2$$

$$\rho_{1}^{\infty} = 2$$

$$\rho_{2}^{\infty} = 4.$$

Note that when $\rho_k^{\infty}(G) = \text{rank } G(s)$ for some $k$ the search can be terminated since $\rho_j^{\infty}(G) - \rho_{j-1}^{\infty}(G) = 0$ for $j = k+1, k+2, \ldots$. It therefore follows from corollary (3.3.11) that $G(s)$ possesses two infinite poles, one of degree three and one of degree one, and two infinite zeros both of degree two. This is confirmed by investigating the infinite frequency structure via a minimal factorisation of $G(s)$. A suitable minimal factorisation of $G(s)$ is given as

$$G(s) = \begin{bmatrix} s^3 & 0 & s^3 & 0 \\ 0 & 2s & 0 & s^2 \\ 0 & 0 & (s^2-2)(s-1) & 0 \\ 0 & s-1 & 0 & (s-1) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (s^2-2)s^3 & 0 \\ 0 & 0 & 0 & s^3(s-1) \end{bmatrix}^{-1}$$

$$s^5 - 2s^3$$

$$= N_1(s)\, D_1^{-1}(s)$$

where the column degrees of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ are $c_1 = 3, c_2 = 1, c_3 = 5, c_4 = 4$.

Let $\Lambda_1(s) = \text{diag } [s^3, s, s^5, s^4]$ then the infinite pole structure of $G(s)$ is given by the zero structure at $w = 0$ of

$$D_1\left(\tfrac{1}{w}\right)\Lambda_1(w) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{w^2}-2\right)\frac{1}{w^3} & 0 \\ 0 & 0 & 0 & \frac{1}{w^3}\left(\frac{1}{w}-1\right) \end{bmatrix} \begin{bmatrix} w^3 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w^5 & 0 \\ 0 & 0 & 0 & w^4 \end{bmatrix}$$

$$= \begin{bmatrix} w^3 & 0\ {-w} & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & 1-2w^2 & 0 \\ 0 & 0 & 0 & 1-w \end{bmatrix}.$$

Now $D_1\left(\frac{1}{w}\right)\Lambda_1(w)$ has Smith form

$$\operatorname{diag}\{1, 1, w, w^3(1-2w^2)(1-w)\}$$

which confirms that $G(s)$ has the stipulated infinite pole structure. Similarly, the infinite zero structure of $G(s)$ is given by the zero structure at $w = 0$ of

$$N_1\left(\tfrac{1}{w}\right)\Lambda_1(w) = \begin{bmatrix} \frac{1}{w^3} & 0 & \frac{1}{w^3} & 0 \\ 0 & \frac{2}{w} & 0 & \frac{1}{w^2} \\ 0 & 0 & \frac{1}{w^3}-\frac{1}{w^2}-\frac{2}{w}+2 & 0 \\ 0 & \frac{1}{w}-1 & 0 & \frac{1}{w}-1 \end{bmatrix} \begin{bmatrix} w^3 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w^5 & 0 \\ 0 & 0 & 0 & w^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & w^2 & 0 \\ 0 & 2 & 0 & w^2 \\ 0 & 0 & w^2-w^3-2w^4+2w^5 & 0 \\ 0 & 1-w & 0 & w^3-w^4 \end{bmatrix}.$$

The Smith form of $N_1\left(\frac{1}{w}\right)\Lambda_1(w)$ is

$$\operatorname{diag}\{1, 1, w^2(w-1), w^2(w-1)(2w^2-1)(1-2w)\}$$

which indicates that $G(s)$ possesses two infinite zeros of degree two as predicted.

## §4. A test for the absence of infinite zeros in a rational matrix.

The investigation of the changes in the rank indices at infinity of the Toeplitz matrices at infinity associated with a rational matrix provides a method of determining the McMillan structure of the infinite poles and zeros of that matrix. When all the infinite poles and zeros have been found the subsequent differences between successive rank indicies will be zero. Interpreting this for the case of a rational matrix which does not possess any infinite zeros leads to the following result.

**(3.4.1) Theorem.** The $m \times \ell$ rational matrix $G(s)$ of normal rank $r$ will possess no infinite zeros if and only if

$$\text{rank } (T_0^\infty(G)) = \text{rank } (T_{-1}^\infty(G)) + r. \tag{3.4.2}$$

**Proof.** The rank difference of two successive Toeplitz matrices of $G(s)$ can not exceed $r$, so that if

$$\text{rank } (T_k^\infty(G)) - \text{rank } (T_{k-1}^\infty(G)) \triangleq \rho_k^\infty = r \quad \text{for some } k \geq -\ell$$

then

$$\rho_{k+i}^\infty = r \qquad i = 1, 2, \ldots \tag{3.4.3}$$

and

$$\rho_{k+i}^\infty - \rho_{k+i-1}^\infty = 0 \qquad i = 1, 2, \ldots \tag{3.4.4}$$

Now $G(s)$ will possess no infinite zeros if and only if

$$\rho_i^\infty - \rho_{i-1}^\infty = 0 \qquad i = 1, 2, \ldots$$

which by (3.4.3), (3.4.4) holds if and only if

$$\rho_0^\infty = r.$$

By definition

$$\rho_0^\infty = \text{rank } T_0^\infty(G) - \text{rank } T_{-1}^\infty(G)$$

to give result. □

**(3.4.5) Corollary.** The $m \times \ell$ polynomial matrix $P(s)$ of normal rank $r$ will possess no infinite zeros if and only if

$$\text{rank } (T_0^\infty\ (P)) = \delta + r \tag{3.4.6}$$

where $\delta$ is the McMillan degree of $P(s)$.

**Proof.** Let the highest power of $s$ in $P(s)$ be $n$, i.e.

$$P(s) = P_n s^n + P_{n-1}s^{n-1} + \ldots + P_1 s + P_0$$

where $P_0, P_1, \ldots, P_n$ are constant matrices and $P_n \neq 0$. Then,

$$T_{-1}^{\infty}(P) = \begin{bmatrix} P_n & P_{n-1} & \ldots & P_2 & P_1 \\ & \ddots & \ddots & & P_2 \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & P_{n-1} \\ & & & & P_n \end{bmatrix}.$$

Now the rank of $T_{-1}^{\infty}(P)$ is equal to $\delta$, the McMillan degree of $P(s)$ (Pugh, 1976). Thus, from the result of theorem (3.4.1), the polynomial matrix $P(s)$ will possess no infinite zeros if and only if

$$\operatorname{rank}(T_0^{\infty}(P)) = \delta + r$$

as required. □

To illustrate the result of theorem (3.4.1) and corollary (3.4.5) consider the following example.

**(3.4.7) Example.** Let

$$P(s) = \begin{bmatrix} s^2 & s^3 & 0 \\ s & 0 & 1 \end{bmatrix}$$

and, since $P(s)$ is a polynomial matrix, its Laurent expansion about the point at infinity is immediate. The resulting Toeplitz matrices can be obtained from the single structure

$$\begin{array}{ccc|ccc|ccc|ccc} \multicolumn{3}{c}{T_{-3}^{\infty}(P)} & \multicolumn{3}{c}{T_{-2}^{\infty}(P)} & \multicolumn{3}{c}{T_{-1}^{\infty}(P)} & \multicolumn{3}{c}{T_{0}^{\infty}(P)} \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \cline{4-12} & & & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \cline{7-12} & & & & & & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{10-12} & & & & & & & & & 0 & 1 & 0 \\ & & & & & & & & & 0 & 0 & 0 \end{array}$$

and the corresponding ranks are

$$\text{rank}\ \ T_{-3}^{\infty}(P) = 1$$

$$\text{rank}\ \ T_{-2}^{\infty}(P) = 2$$

$$\text{rank}\ \ T_{-1}^{\infty}(P) = 4$$

$$\text{rank}\ \ T_{0}^{\infty}(P) = 6.$$

It follows that

$$\text{rank}\ \ T_0^{\infty}(P) = \text{rank}\ \ T_{-1}^{\infty}(P) + r$$

where $r = \text{rank}\ \ P(s)$. Hence, by theorem (3.4.1), $P(s)$ does not possess any infinite zeros. This can be confirmed by considering a minimal factorisation of $P(s)$. A suitable factorisation of $P(s)$ is in fact immediate, i.e.

$$\begin{aligned} P(s) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} s^2 & s^3 & 0 \\ s & 0 & 1 \end{bmatrix} \\ &= D_2^{-1}(s)\, N_2(s). \end{aligned}$$

If $\Lambda_2(s) = \begin{bmatrix} s^3 & 0 \\ 0 & s \end{bmatrix}$ then the infinite zero structure of $P(s)$ is given by the finite zero structure at $w = 0$ of

$$\begin{aligned} \Lambda_2(w)\, N_2\left(\tfrac{1}{w}\right) &= \begin{bmatrix} w^3 & 0 \\ 0 & w \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{w^2} & \frac{1}{w^3} & 0 \\ \frac{1}{w} & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} w & 1 & 0 \\ 1 & 0 & w \end{bmatrix} \end{aligned}$$

which has Smith form equal to $[I_2 \quad 0]$. Hence, $P(s)$ has no infinite zeros.

Note that the McMillan degree, $\delta$, of a polynomial matrix, $P(s)$, is equal to the highest degree amongst all minors of $P(s)$. In this example $\delta = 4$ so that $P(s)$ satisfies the test for the absence of infinite zeros given in corollary (3.4.5), as expected.

The necessary and sufficient condition (3.4.6) of corollary (3.4.5) is equivalent to a condition concerning the minors of $P(s)$ which is expressed by the following.

**(3.4.8) Theorem.** Let $P(s)$ be an $m \times \ell$ polynomial matrix of normal rank $r$ and whose highest power of $s$ is $n$. Let $T_0^{\infty}(P)$ and $\delta$ be as defined previously. Then, rank $(T_0^{\infty}(P)) = \delta + r$ if and only if $P(s)$ has an $r \times r$ minor of degree $\delta$.

**Proof.** Let

$$P(s) = P_n s^n + P_{n-1} s^{n-1} + \ldots + P_1 s + P_0$$

where $P_0, P_1, \ldots, P_n$ are constant matrices and $P_n \neq 0$.
Assume that

$$\text{rank } (T_0^\infty(P)) = \delta + r$$

i.e.

$$\text{rank} \begin{bmatrix} P_n & P_{n-1} & \ldots & P_1 & P_0 \\ & \ddots & \ddots & & P_1 \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & P_n \end{bmatrix} = \delta + r. \tag{3.4.9}$$

Define $P'(s)$ as

$$P'(s) = P_n s^{n+1} + P_{n-1} s^n + \ldots + P_1 s^2 + P_0 s$$

then the McMillan degree of $P'(s)$ indicated by $\delta'$ is equal to $\delta + r$ by (3.4.9).

Now $P'(s) = \text{diag } \{s, s, \ldots, s\} P(s)$ and since $\delta' = \delta + r$ there exists a minor of order $q$ say of $P'(s)$ with degree $\delta + r$. Let this minor be denoted by

$$P'(s) \begin{pmatrix} i_1, i_2, \ldots, i_q \\ j_1, j_2, \ldots, j_q \end{pmatrix}$$

which, by the Binet-Cauchy theorem, can be expressed as

$$P'(s) \begin{pmatrix} i_1, i_2, \ldots, i_q \\ j_1, j_2, \ldots, j_q \end{pmatrix} =$$

$$\sum_{1 \leq k_1 < k_2 < \ldots < k_q \leq \min(m,\ell)} \begin{bmatrix} s & & & & 0 \\ & s & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & & s \end{bmatrix}^{i_1, i_2, \ldots, i_q}_{k_1, k_2, \ldots, k_q} P(s) \begin{pmatrix} k_1, k_2, \ldots, k_q \\ j_1, j_2, \ldots, j_q \end{pmatrix}.$$

Now the highest degree of any minor of $P(s)$ is, by definition, equal to $\delta$. Thus, for $P'(s)$ to have a minor of degree $\delta + r$, it follows that

$$\begin{bmatrix} s & & & & 0 \\ & s & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & & s \end{bmatrix}$$

has to have a minor of order $r$ in the above summation. This is only possible by taking $q = r$ which implies that $P(s)$ has a minor of order $r$ with degree $\delta$.

Conversely, assume that $P(s)$ has an $r \times r$ minor of degree $\delta$. Let this minor be

$$P(s)\begin{pmatrix} i_1, i_2, \ldots, i_r \\ j_1, j_2, \ldots, j_r \end{pmatrix}.$$

Consider the following $r \times r$ minor of $P'(s)$

$$P'(s)\begin{pmatrix} i_1, i_2, \ldots, i_r \\ j_1, j_2, \ldots, j_r \end{pmatrix} =$$

$$\sum_{1 \le k_1 < k_2 < \ldots < k_r \le \min(m,\ell)} \begin{bmatrix} s & & & & 0 \\ & s & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \ddots \\ & & & & & s \end{bmatrix}^{i_1, i_2, \ldots, i_r}_{k_1, k_2, \ldots, k_r} P(s)\begin{pmatrix} k_1, k_2, \ldots, k_r \\ j_1, j_2, \ldots, j_r \end{pmatrix}. \tag{3.4.10}$$

When the set $(k_1, k_2, \ldots, k_r)$ becomes equal to the set $(i_1, i_2, \ldots, i_r)$ then the series contains a term of degree $\delta + r$. All the other terms in the series have degree less than $\delta + r$ so that $P'(s)$ has an $r \times r$ minor of degree $\delta + r$. By definition of the McMillan degree $\delta$ and by (3.4.10) it follows that $P'(s)$ does not have a minor of degree greater than $\delta + r$. Thus, the McMillan degree of $P'(s)$ is equal to $\delta + r$, and so

$$\operatorname{rank} \begin{bmatrix} P_n & P_{n-1} & \ldots & P_1 & P_0 \\ & \ddots & \ddots & \ddots & P_1 \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & P_{n-1} \\ & & & & P_n \end{bmatrix} = \delta + r$$

which implies rank $T_0^\infty(P) = \delta + r$, as required. □

The result of theorem (3.4.8) provides confirmation of the test given by Hayton *et al.* [1988] for the absence of infinite zeros in a polynomial matrix. The new test presented in corollary (3.4.5) is a simpler test than that produced by Hayton *et al.* [*ibid.*] since it is generally easier to calculate the rank $(T_0^\infty(P))$ than all the $r \times r$ minors. Note that in both tests the McMillan degree of $P(s)$ must be calculated although this again merely involves the computation of the rank of a constant matrix, i.e. $T_0^\infty(P)$.

### §5. Infinite poles and zeros of a polynomial matrix.

For the particular case of a polynomial matrix, $P(s)$, the previous definitions and discussions concerning the infinite poles and zeros obviously apply. In addition there exist other definitions of infinite poles and zeros specifically defined for the polynomial case. Such a definition was introduced by Hayton *et al.* [1988] who extended the notion of homogenising a matrix pencil to a general polynomial matrix and subsequently defined the associated infinite elementary divisors. It was seen that the degrees of the infinite divisors have a direct relationship to the infinite poles and zeros as defined by (3.2.4), so extending in a neat way the theory of infinite elementary divisors from the matrix pencil to the general matrix case.

Tan and Vandewalle [1988, 1988*a*] also adopt a homogenising technique to define the infinite poles and zeros. In contrast to Hayton *et al.* [1988] who homogenise the whole matrix Tan and Vandewalle [1988, 1988*a*,] homogenise each element on an individual basis, and also generalise the notion of the degree of a polynomial so that a polynomial of the form

$$p_n s^n + p_{n-1} s^{n-1} + \ldots + p_1 s + p_0 \qquad p_n \neq 0$$

can have degree greater than or equal to $n$.

This leads to confusion in defining the subsequent zeros. For instance, consider the matrices

$$\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \tag{3.5.1}$$

$\underline{\underline{\delta = 1}}$ ⇒ 1 infinite pole

No infinite zeros

and

$$\begin{bmatrix} s & s \\ 0 & 1 \end{bmatrix}. \tag{3.5.2}$$

If these matrices are regarded as column based polynomial matrices, i.e. the entries in each column being of equal degrees, then under the definition adopted by Tan and Vandewalle [*ibid.*] (3.5.1) has no infinite zeros whereas (3.5.2) has one infinite zero. This is not satisfactory since (3.5.2) is obtained from (3.5.1) by a constant transformation which should have no effect on the dynamic properties of the original matrix. Another disadvantage of the approach adopted by Tan and Vandewalle [*ibid.*] is that the degrees of the infinite zeros and poles are not defined.

A further definition of infinite poles and zeros of a polynomial matrix, $P(s)$ , is provided by Bosgra and Van der Weiden [1981] in terms of the degrees of the minors of the polynomial matrix. This definition has been shown to be equivalent to the definition of infinite poles and zeros given in terms of the structure at $w = 0$ of $P\left(\frac{1}{w}\right)$ and in terms of the infinite elementary divisors (see Hayton *et al.*, 1988). Formally the characterisation of the infinite poles and zeros as presented by Bosgra and Van der Weiden [1981] is defined below.

**(3.5.3) Definition.** Let $P(s)$ be an $m \times \ell$ polynomial matrix of normal rank $r$, and let $\delta_i$ be the highest degree occurring among the $i \times i$ minors of $P(s)$. Let $\delta$ (the McMillan degree of $P(s)$) denote the largest of the $\delta_i$, $i = 1, 2, \ldots, r$, and let $k_1$ (respectively $k_2$) denote the smallest (respectively largest) order of minors for which $\delta_i = \delta$. Then, $P(s)$ is said to have $k_1$ INFINITE POLES with degrees $\delta_1, \delta_2 - \delta_1, \ldots, \delta - \delta_{k_1 - 1}$ and $r - k_2$ INFINITE ZEROS with degrees $\delta - \delta_{k_2+1}, \delta_{k_2+1} - \delta_{k_2+2}, \ldots, \delta_{r-1} - \delta_r$.

By definition, the $\delta_i$ satisfy the relationships

$$\delta_1 \geq \delta_2 - \delta_1 \geq \ldots \geq \delta_{k_1} - \delta_{k_1-1}$$

and (3.5.4)

$$\delta_{r-1} - \delta_r \geq \delta_{r-2} - \delta_{r-1} \geq \ldots \geq \delta_{k_2} - \delta_{k_2+1}$$

so that if $S^{\infty}(P(s))$ is the Smith McMillan form at infinity of $P(s)$ described by lemma (3.2.14) then the corresponding matrix $Q(s)$ is given by

$$Q(s) = \text{diag } \{s^{\delta_1}, s^{\delta_2 - \delta_1}, \ldots, s^{\delta_{k_1} - \delta_{k_1 - 1}}, 1, 1, \ldots, 1, s^{-(\delta_{k_2} - \delta_{k_2+1})}, s^{-(\delta_{k_2+1} - \delta_{k_2+2})}, \ldots, s^{-(\delta_{r-1} - \delta_r)}, 0, 0, \ldots, 0\}$$

where the number of 1's equals $k_2 - k_1$.

It was seen in section 3 that the Smith Mcmillan form at infinity and hence the pole and zero structure at infinity of a rational matrix can be obtained from its Laurent expansion about the point at infinity. It therefore follows, for a polynomial matrix $P(s)$, that there exists a relationship between the rank indices at infinity of $P(s)$, as defined by (3.3.2), and the highest degree of minors of $P(s)$. This relationship is formally characterised by the following theorems.

**(3.5.5) Theorem.** Let $P(s)$ be an $m \times \ell$ polynomial matrix with the highest power of $s$ equal to $n$. Assume that the rank indices at infinity of $P(s), \rho_i^\infty$, defined as in (3.3.2) are known and let $\delta_i$'s be defined as in (3.5.3). Also, let

$$v_i = \rho_i^\infty - \rho_{i-1}^\infty \qquad i = -n, -n+1, \ldots$$

and define

$$d_0 = 0$$

$$d_i = \sum_{j=0}^{i-1} v_{-n+j} \qquad i = 1, 2, \ldots, n$$

$$d'_0 = 0$$

$$d'_i = \sum_{j=1}^{i} v_j \qquad i = 1, 2, \ldots$$

Then,

$$k_1 = \sum_{i=0}^{n-1} v_{-n+i} \tag{3.5.6}$$

and, for $i = 0, 1, \ldots, n-1$

$$\delta_{d_i+j} = j. \mid -n+i \mid + \delta_{d_i} \qquad j = 0, 1, \ldots, d_{i+1} - d_i. \tag{3.5.7}$$

Similarly

$$k_2 = r - \sum_{i=1}^{\infty} v_i \tag{3.5.8}$$

$$\delta_{k_2} = \delta_{d_n} \tag{3.5.9}$$

and, for $i = 1, 2, \ldots$

$$\delta_{k_2+d'_{i-1}+j} = \delta_{k_2+d'_{i-1}} - j.i \qquad j = 0, 1, \ldots, d'_i - d'_{i-1}. \tag{3.5.10}$$

**Proof.** From corollary (3.3.11) the multiplicity of the infinite poles is equal to

$$\sum_{i=0}^{n-1} v_{-n+i}$$

whilst, from definition (3.5.3), the multiplicity is given by $k_1$. Hence

$$k_1 = \sum_{i=0}^{n-1} v_{-n+i}$$

to give (3.5.6).

From corollary (3.3.11) if $v_{-n+i}$, $i = 0, 1, \ldots, n-1$, is non-zero then $P(s)$ has $v_{-n+i}$ infinite poles each of degree $|-n+i|$. Thus, with the $d_i$'s defined as above and $\delta_0 \triangleq 0$, if $v_{-n+i} \neq 0$ for some $i = 0, 1, \ldots, n-1$ then

$$\left.\begin{aligned} \delta_{d_i+1} - \delta_{d_i} &= |-n+i| \\ \delta_{d_i+2} - \delta_{d_i+1} &= |-n+i| \\ &\vdots \\ \delta_{d_{i+1}} - \delta_{d_{i+1}-1} &= |-n+i| \end{aligned}\right\}. \tag{3.5.11}$$

Now $\delta_{d_i}$ is known from previous relationships so that (3.5.11) can be rewritten as

$$\left.\begin{aligned} \delta_{d_i+1} &= |-n+i| + \delta_{d_i} \\ \delta_{d_i+2} &= 2\,|-n+i| + \delta_{d_i} \\ &\vdots \\ \delta_{d_{i+1}} &= (d_{i+1} - d_i)\ |-n+i| + \delta_{d_i} \end{aligned}\right\}. \tag{3.5.12}$$

If $v_{-n+i} = 0$ then $d_{i+1} = d_i$ and (3.5.12) gives rise to the identity

$$\delta_{d_{i+1}} = \delta_{d_i}$$

which provides no further information concerning the $\delta_i$'s. Hence, summarising the relationships (3.5.12) into one expression yields (3.5.7).

Similarly, by corollary (3.3.11), the multiplicity of the infinite zeros is given by

$$\sum_{i=1}^{\infty} v_i$$

whilst, by definition (3.5.3), the multiplicity is also equal to $r - k_2$. Hence

$$k_2 = r - \sum_{i=1}^{\infty} v_i$$

to give (3.5.8).

Also, by definition (3.5.3), the McMillan degree, $\delta$, of $P(s)$ is equal to both $\delta_{k_1}$ and $\delta_{k_2}$. From (3.5.6),

$$\begin{aligned} k_1 &= \sum_{i=0}^{n-1} v_{-n+i} \\ &= d_n. \end{aligned}$$

Hence,

$$\delta = \delta_{k_1} = \delta_{d_n}$$

and, since $\delta = \delta_{k_2}$, it follows that

$$\delta_{k_2} = \delta_{d_n}$$

to give (3.5.9).

Finally, by corollary (3.3.11) and the definition of the $d_i'$'s, if $v_i \neq 0,\ i = 1, 2, \ldots$, then

$$\left.\begin{aligned} \delta_{k_2+d'_{i-1}} - \delta_{k_2+d'_{i-1}+1} &= i \\ \delta_{k_2+d'_{i-1}+1} - \delta_{k_2+d'_{i-1}+2} &= i \\ &\vdots \\ \delta_{k_2+d'_i-1} - \delta_{k_2+d'_i} &= i \end{aligned}\right\}. \tag{3.5.13}$$

Now $\delta_{k_2+d'_{i-1}}$ is known from previous relationships so that (3.5.13) can be rewritten as

$$\left.\begin{aligned} \delta_{k_2+d'_{i-1}+1} &= \delta_{k_2+d'_{i-1}} - i \\ \delta_{k_2+d'_{i-1}+2} &= \delta_{k_2+d'_{i-1}} - 2i \\ &\vdots \\ \delta_{k_2+d'_i} &= \delta_{k_2+d'_{i-1}} - (d'_i - d'_{i-1}).i \end{aligned}\right\} \tag{3.5.14}$$

which, for similar reasons given for the case of infinite poles gives rise to (3.5.10), as required. □

Note that it is not possible to deduce

$$\delta_i, \qquad i = k_1 + 1, k_1 + 2, \ldots, k_2 - 1 \tag{3.5.15}$$

from the rank indices at infinity of $P(s)$. This is due to the fact that the $\delta_i$'s listed in (3.5.15) do not contribute any information concerning the infinite poles and zeros of $P(s)$ and hence can not be related to the rank indices at infinity by considering the equivalent definitions of infinite poles and zeros.

**(3.5.16) Theorem.** Let $P(s)$ be an $m \times \ell$ polynomial matrix with the highest power of $s$ equal to $n$ and normal rank $r$. Assume that the $\delta_i$'s defined in (3.5.3) are known and let the rank indices at infinity of $P(s)$ be defined as in (3.3.2). Also define, for $i = 0, 1, \ldots, n-1$,

$$\begin{aligned} q_{n+i} \triangleq \quad & \text{number of times } |-n+i| \text{ occurs in the set} \\ & \delta_1, \delta_2 - \delta_1, \ldots, \delta - \delta_{k_1-1} \end{aligned}$$

and, for $i = 1, 2, \ldots$,

$$q'_i \triangleq \text{ number of times } i \text{ occurs in the set}$$
$$\delta - \delta_{k_2+1}, \delta_{k_2+1} - \delta_{k_2+2}, \ldots, \delta_{r-1} - \delta_r.$$

Then,

$$\rho^{\infty}_{-n+i} = \rho^{\infty}_{-n+i-1} + q_{-n+i} \qquad i = 0, 1, \ldots, n-1 \tag{3.5.17}$$

where

$$\rho^{\infty}_{-n-1} = 0$$

and

$$\rho^{\infty}_0 = k_2 \tag{3.5.18}$$

$$\rho^{\infty}_i = \rho^{\infty}_{i-1} + q'_i \qquad i = 1, 2, \ldots \tag{3.5.19}$$

**Proof.** By definition $q_{-n+i}$ is the number of infinite poles of $P(s)$ with degree $|-n+i|$. Hence,

$$\rho^{\infty}_{-n+i} - \rho^{\infty}_{-n+i-1} = q_{-n+i} \qquad i = 0, 1, \ldots, n-1$$

where $\rho^{\infty}_{-n-1} = 0$ since the highest power of $s$ in $P(s)$ is equal to $n$, to give (3.5.17).

From (3.5.4) and the definition of $\rho^{\infty}_i$ it follows that

$$\rho^{\infty}_{|\delta_r - \delta_{r-1}|} = r.$$

Also,

$$\sum_{i=1}^{|\delta_r - \delta_{r-1}|} (\rho^{\infty}_i - \rho^{\infty}_{i-1}) \equiv \rho^{\infty}_{|\delta_r - \delta_{r-1}|} - \rho^{\infty}_0 = r - k_2.$$

Hence, $\rho^{\infty}_0 = k_2$ to give (3.5.18). Now $q'_i$ is the number of infinite zeros of $P(s)$ of degree $i$. Thus,

$$\rho^{\infty}_i - \rho^{\infty}_{i-1} = q'_i \qquad i = 1, 2, \ldots$$

to give (3.5.19), as required. □

The relationships between the rank indices at infinity of a polynomial matrix and the degrees of its minors described by theorems (3.5.5) and (3.5.16) are not straightforward. Simpler but more general relationships can be deduced by exploiting the fact that the total number of infinite poles and zeros must be the same under each definition. This gives rise to the following.

**(3.5.20) Theorem.** Let $P(s)$ be an $m \times \ell$ polynomial matrix of normal rank $r$ and let $n$ denote the highest power of $s$ occurring in elements of $P(s)$. Suppose the rank indices

$\rho_i^\infty$, $i = -n, -n+1, \ldots, -1, 0, 1, \ldots, h$, of $P(s)$ are known, where $h$ is the smallest integer for which

$$\rho_h^\infty - \rho_{h-1}^\infty \neq 0, \quad \rho_i^\infty - \rho_{i-1}^\infty = 0 \quad \forall\, i > h.$$

If $\delta$ denotes the McMillan degree of $P(s)$ and $\delta_r$ the highest degree amongst all $r \times r$ minors of $P(s)$, then

$$\delta = \sum_{i=-n}^{-1} \rho_i^\infty \tag{3.5.21}$$

$$\delta_r = \sum_{i=-n}^{h-1} \rho_i^\infty - h\rho_h^\infty. \tag{3.5.22}$$

**Proof.** Let $\delta_i$ be the highest degree for $i \times i$ minors of $P(s)$ and let $k_1$, $k_2$ be as defined previously. Let $p_\infty$ (respectively $z_\infty$) denote the total number of poles (respectively zeros) at infinity counted according to multiplicity and degree. Now, if $p_\infty$ is computed from the $\delta_i$ then, from definition (3.5.3),

$$p_\infty = \sum_{i=1}^{k_1} (\delta_i - \delta_{i-1}) \quad (\delta_0 \triangleq 0)$$

i.e.,

$$p_\infty = \delta_{k_1} \equiv \delta \tag{3.5.23}$$

by definition of $k_1$. On the other hand, if $p_\infty$ is computed from the $\rho_i^\infty$'s then, from corollary (3.3.11),

$$p_\infty = \sum_{i=-n}^{-1} (\rho_i^\infty - \rho_{i-1}^\infty) .\ |i| \quad (\rho_{-n-1}^\infty \triangleq 0)$$

$$= (\rho_{-n}^\infty - \rho_{-n-1}^\infty)n + (\rho_{-n+1}^\infty - \rho_{-n}^\infty)\,(n-1) + \ldots + (\rho_{-1}^\infty - \rho_{-2}^\infty)$$

i.e.,

$$p_\infty = \sum_{i=-n}^{-1} \rho_i^\infty. \tag{3.5.24}$$

Equations (3.5.23) and (3.5.24) together then yield (3.5.21).

Proceeding similarly with the computation of $z_\infty$ gives, from the $\delta_i$'s,

$$z_\infty = \sum_{i=k_2}^{r-1} (\delta_i - \delta_{i+1}) = \delta_{k_2} - \delta_r. \tag{3.5.25}$$

Alternatively, from the $\rho_i^\infty$'s,

$$\begin{aligned} z_\infty &= \sum_{i=1}^{h} (\rho_i^\infty - \rho_{i-1}^\infty).i \\ &= (\rho_1^\infty - \rho_0^\infty) + 2(\rho_2^\infty - \rho_1^\infty) + \ldots + (h-1)\,(\rho_{h-1}^\infty - \rho_{h-2}^\infty) + h(\rho_h^\infty - \rho_{h-1}^\infty) \\ &= h\rho_h^\infty - \sum_{i=0}^{h-1} \rho_i^\infty. \end{aligned} \tag{3.5.26}$$

Equating (3.5.25) and (3.5.26) gives

$$\delta_r - \delta_{k_2} = \sum_{i=0}^{h-1} \rho_i^\infty - h\rho_h^\infty. \tag{3.5.27}$$

However, by the definition of $k_2$,

$$\delta_{k_2} = \delta$$

and, in view of (3.5.21), the relationship (3.5.27) reduces to (3.5.22), as required. □

To illustrate the result of theorem (3.5.20) consider the following example.

**(3.5.28) Example.** Let

$$P(s) = \begin{bmatrix} 1 & 1 & 0 \\ s^3 & 1+s^3 & s \\ 0 & s & 1+s \end{bmatrix}.$$

Then, the associated Toeplitz matrices are obtained from

$$
\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
\multicolumn{3}{c}{T^{\infty}_{-3}(G)} & \multicolumn{3}{c}{T^{\infty}_{-2}(G)} & \multicolumn{3}{c}{T^{\infty}_{-1}(G)} & \multicolumn{3}{c}{T^{\infty}_{0}(G)} & \multicolumn{3}{c}{T^{\infty}_{1}(G)} & \multicolumn{3}{c}{T^{\infty}_{2}(G)} \\
0&0&0 & 0&0&0 & 0&0&0 & 1&1&0 & & & & & & \\
1&1&0 & 0&0&0 & 0&0&1 & 0&1&0 & &0& & &0& \\
0&0&0 & 0&0&0 & 0&1&1 & 0&0&1 & & & & & & \\
\hline
 & & & 0&0&0 & 0&0&0 & 0&0&0 & 1&1&0 & & & \\
 & & & 1&1&0 & 0&0&0 & 0&0&1 & 0&1&0 & &0& \\
 & & & 0&0&0 & 0&0&0 & 0&1&1 & 0&0&1 & & & \\
\hline
 & & & & & & 0&0&0 & 0&0&0 & 0&0&0 & 1&1&0 \\
 & & & & & & 1&1&0 & 0&0&0 & 0&0&1 & 0&1&0 \\
 & & & & & & 0&0&0 & 0&0&0 & 0&1&1 & 0&0&1 \\
\hline
 & & & & & & & & & 0&0&0 & 0&0&0 & 0&0&0 \\
 & & & & & & & & & 1&1&0 & 0&0&0 & 0&0&1 \\
 & & & & & & & & & 0&0&0 & 0&0&0 & 0&1&1 \\
\hline
 & & & & & & & & & & & & 0&0&0 & 0&0&0 \\
 & & & & & & & & & & & & 1&1&0 & 0&0&0 \\
 & & & & & & & & & & & & 0&0&0 & 0&0&0 \\
\hline
 & & & & & & & & & & & & & & & 0&0&0 \\
 & & & & & & & & & & & & & & & 1&1&0 \\
 & & & & & & & & & & & & & & & 0&0&0
\end{array}
$$

and

$$\text{rank} \quad T^{\infty}_{-3}(P) = 1$$

$$\text{rank} \quad T^{\infty}_{-2}(P) = 2$$

$$\text{rank} \quad T^{\infty}_{-1}(P) = 4$$

$$\text{rank} \quad T^{\infty}_{0}(P) = 6$$

$$\text{rank} \quad T^{\infty}_{1}(P) = 8$$

$$\text{rank} \quad T^{\infty}_{2}(P) = 11.$$

The corresponding rank indices at infinity are given by

$$\rho_{-3}^{\infty} = 1$$

$$\rho_{-2}^{\infty} = 1$$

$$\rho_{-1}^{\infty} = 2$$

$$\rho_{0}^{\infty} = 2$$

$$\rho_{1}^{\infty} = 2$$

$$\rho_{2}^{\infty} = 3$$

with

$$\rho_{i}^{\infty} = 3, \quad i \geq 3.$$

Hence $h = 2$. Now, from theorem (3.5.20), $\delta$ and $\delta_r$ are given by

$$\delta = \sum_{i=-3}^{-1} \rho_i^{\infty} = 1 + 1 + 2 = 4$$

$$\begin{aligned} \delta_r &= \sum_{i=-3}^{1} \rho_i^{\infty} - 2\rho_2^{\infty} \\ &= 1 + 1 + 2 + 2 + 2 - 2.3 \\ &= 2 \end{aligned}$$

which is confirmed on inspection.

Notice that this example demonstrates that the difference between successive rank indices can be non-zero long after the last term in the Laurent expansion has been introduced into the corresponding Toeplitz matrix. The search will only terminate when $\rho_i^{\infty} = \operatorname{rank} P(s)$ for some $i$.

The result of theorem (3.5.20) leads to the following corollary.

**(3.5.29) Corollary.** $P(s)$ will possess no infinite zeros if and only if there exists an $r \times r$ minor of $P(s)$ with degree $\delta$.

**Proof.** From (3.5.21) and (3.5.22)

$$\begin{aligned}\delta_r &= \delta + \sum_{i=0}^{h-1} \rho_i^\infty - h\rho_h^\infty \\ &= \delta + \rho_0^\infty + \rho_1^\infty + \ldots + \rho_{h-1}^\infty - h\rho_h^\infty \\ &= \delta - (\rho_1^\infty - \rho_0^\infty) - 2(\rho_2^\infty - \rho_1^\infty) - \ldots - h(\rho_h^\infty - \rho_{h-1}^\infty) \\ &= \delta - \sum_{i=1}^{h} i.(\rho_i^\infty - \rho_{i-1}^\infty) \end{aligned} \tag{3.5.30}$$

Now, $P(s)$ will have no infinite zeros if and only if

$$\rho_i^\infty - \rho_{i-1}^\infty = 0 \qquad i = 1, 2, \ldots$$

Hence it follows, from (3.5.30), that $P(s)$ has no infinite zeros if and only if

$$\delta_r = \delta$$

as required. □

This test for the absence of infinite zeros in a polynomial matrix is the test given by Hayton *et al.* [1988] and discussed previously in section 4.

The relationships in theorem (3.5.20) can be refined further if instead of the rank indices, $\rho_i^\infty$, the actual ranks of the Toeplitz matrices formed from $P(s)$ are used which consequently give rise to the following corollaries.

**(3.5.31) Corollary.** Let $T_i^\infty(P)$, $i = -n, \; -n+1, \; \ldots$ denote the successive Toeplitz matrices formed from $P(s)$ viewed as a matrix polynomial.
Then,

$$\delta = \text{rank } T_{-1}^\infty(P) \tag{3.5.32}$$

and

$$\delta_r = (h+1).\text{ rank } T_{h-1}^\infty(P) - h.\text{rank } T_h^\infty(P). \tag{3.5.33}$$

**Proof.** This follows directly from (3.5.21) and (3.5.22) on noting that

$$\rho_i^\infty = \text{rank } T_i^\infty(P) - \text{rank } T_{i-1}^\infty(P).$$ □

The result (3.5.32) is of course well-known (Pugh, 1976) and provides a simple computational scheme for evaluating the McMillan degree of a polynomial matrix. The result

(3.5.33) is new and could be used computationally to evaluate the highest degree of $r \times r$ minors of $P(s)$. There is however one difficulty surrounding the formula (3.5.33) and that lies in the requirement that $h$ be known a priori. There is thus in (3.5.33) more than just a requirement that the ranks of two successive Toeplitz matrices be known.

**(3.5.34) Corollary.** If $P(s)$ is a square non-singular matrix then

$$\begin{aligned}\deg\,(\det\ P(s)) &= \sum_{i=-n}^{h-1} \rho_i^\infty - h\rho_h^\infty \\ &= (h+1).\text{rank } T_{h-1}^\infty(P) - h.\text{rank } T_h^\infty(P).\end{aligned}$$

**Proof.** If $P(s)$ is square then $m = \ell$ and since it is non-singular then $r = m$. Thus, $\delta_r \equiv \deg(\det P(s))$ and the result follows. □

The above result suggests a method by which the degree of a determinant may be computed without recourse to evaluation of the determinant itself. The need for such a method can be illustrated by considering the insertion of output feedback as represented by the constant matrix $K$ around the open loop transfer function matrix $G(s)$. If $D(s)$ denotes the non strictly proper part of $G(s)$ (i.e. the polynomial part of $G(s)$) then a necessary and sufficient condition for the closed loop system to be proper is (Pugh, 1984)

$$\deg.\ \det(I + KD(s)) = \delta(D(s)). \tag{3.5.35}$$

A result of the form of corollary (3.5.34) is clearly required to evaluate the left hand side of this relationship. Note that on the right hand side of (3.5.35), $\delta(D(s))$ denotes the McMillan degree of $D(s)$ and this may be evaluated quite readily from (3.5.32) of corollary (3.5.31).

## §6. Conclusions.

In this chapter the infinite frequency structure of a rational matrix has been considered. A discussion of the various definitions of infinite poles and zeros was presented in section 2 with particular reference to the definition based on a minimal factorisation and to the definition via the Smith McMillan form at infinity. A method of obtaining the Smith McMillan form at infinity of a rational matrix was described in section 3 and this method was subsequently exploited to produce new results concerning the absence of infinite zeros and the infinite structure of polynomial matrices. Specifically a new test for the absence of infinite zeros in a rational matrix was presented in section 4 which, when adopted to the case of polynomial matrices, provides a simpler test than that provided by Hayton *et al.* [1988]. Section 5 further considers the particular case of polynomial matrices by considering two equivalent definitions of infinite poles and zeros. This leads to the characterisation of certain relationships between the rank indices at infinity of a matrix and the highest degrees of some of its minors and provides an alternative means of evaluating certain features associated with a polynomial matrix.

# Chapter 4. The General Pole Placement Problem using Constant Output Feedback.

## §1. Introduction.

In this chapter the general pole placement problem using constant output feedback is investigated. The system under consideration may possess a proper or non-proper transfer function matrix so that both the finite and infinite pole structures must be taken into account. The problem is approached by exploiting the properties of a certain factorisation of the open loop transfer function matrix, and this theory is discussed in section 2. Using this approach some new necessary conditions are obtained for the closed loop finite pole structure and infinite pole structure, and these are presented in section 3. The conditions presented in section 3 relate separately to the finite pole and infinite pole structures. The subsequent refinement of these conditions into an overall condition on the total structure is developed in section 4. Finally the connection of these new results with the previous results obtained in the special case of systems with strictly proper transfer function matrices is investigated.

## §2. Preliminaries.

It was seen in chapter 3 that a minimal factorisation of a rational matrix, $G(s)$, provides a straightforward characterisation of both the finite and infinite pole and zero structure of $G(s)$. A further property of a minimal factorisation can be exploited when constant output feedback is applied to the system.

Consider a system with an $m \times \ell$ rational transfer function matrix $G(s)$. Let $G(s)$ have a right minimal factorisation

$$G(s) = N_1(s) D_1^{-1}(s).$$

Further, let $G_K(s)$ denote the transfer function matrix of the system formed when constant output feedback is applied to the original system as described by figure (4.2.1).

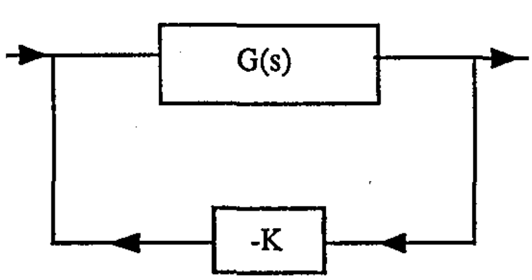



fig. (4.2.1)

Then, $G_K(s)$ is given by

$$G_K(s) = G(s)[I + KG(s)]^{-1}$$

where it is assumed that $| I + KG(s) | \neq 0$. A right minimal factorisation of $G_K(s)$ can be immediately obtained from the corresponding right minimal factorisation of $G(s)$, as described by the following lemma.

**(4.2.2) Lemma (Pugh and Ratcliffe, 1980).** Let

$$G(s) = N_1(s)\, D_1^{-1}(s) \tag{4.2.3}$$

be a right minimal factorisation of $G(s)$ where $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ has column degrees, $c_i, i = 1, 2, \ldots, \ell$. If $G_K(s)$ is the resulting closed loop transfer function matrix obtained by applying constant output feedback $K$ around $G(s)$, then

$$G_K(s) = N_1(s)[D_1(s) + K\, N_1(s)]^{-1} \tag{4.2.4}$$

is a right minimal factorisation of $G_K(s)$. Further the column degrees of $\begin{bmatrix} D_1(s) + KN_1(s) \\ N_1(s) \end{bmatrix}$ are identical to the column degrees of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$.

A straightforward right minimal factorisation of $G_K(s)$ therefore exists and, using the result of lemma (3.2.6), the finite and infinite pole structure of the closed loop system under constant output feedback can be investigated. This is undertaken in the next section. A dual result to lemma (4.2.2) exists when a left minimal factorisation of $G(s)$ is employed of the form

$$G(s) = D_2^{-1}(s)\; N_2(s) \tag{4.2.5}$$

where the rows of $[D_2(s) \quad N_2(s)]$ constitute a minimal basis with row degrees $r_i, i = 1, 2, \ldots, m$, and where $\Lambda_2(s)$ is defined as $\Lambda_2(s) = \text{diag}\,[s^{r_1}, s^{r_2}, \ldots, s^{r_m}]$. It then follows that the closed loop pole structure can also be investigated by considering a left minimal factorisation of $G(s)$.

## §3. Necessary conditions for the separate placement of a finite and infinite pole structure by output feedback.

Consider the $m \times \ell$ transfer function matrix $G(s)$ factorised as in (4.2.3). From the results described in the previous section and in chapter 3 the finite and infinite pole structures of the closed loop system, factorised as in (4.2.4), are given by the zero structure of $D_1(s) + KN_1(s)$ and the zero structure at $w = 0$ of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\Lambda_1(w)$ respectively. The zero structures of $D_1(s) + K\,N_1(s)$ and $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\Lambda_1(w)$ in turn are given by their respective sets of invariant polynomials. Let the invariant polynomials of $D_1(s) + K\,N_1(s)$ be $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ where

$$\alpha_i(s) \mid \alpha_{i-1}(s) \qquad i = 2, 3, \ldots, \ell \tag{4.3.1}$$

and

$$\deg \alpha_i(s) = a_i \qquad i = 1, 2, \ldots, \ell. \tag{4.3.2}$$

Let the invariant polynomials of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\Lambda_1(w)$ be $\beta_1(w), \beta_2(w), \ldots, \beta_\ell(w)$ where

$$\beta_i(w) \mid \beta_{i-1}(w) \qquad i = 2, 3, \ldots, \ell. \tag{4.3.3}$$

The zero structure at $w = 0$ of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\Lambda_1(w)$ is given by factors of the form $w^{b_i}$ of $\beta_i(w)$ $(b_i \geq 0)$, $i = 1, 2, \ldots, \ell$. Hence,

$$b_i \leq \deg(\beta_i(w)). \tag{4.3.4}$$

as $\beta_i$ can contain other factors of $w$, say $(w-2)$ which would increase the degree of $\beta_i$ part $w^{b_i}$

It therefore follows that the finite and infinite pole structures of the closed loop transfer function matrix can be described in terms of the $a_i$'s and $b_i$'s.

Necessary conditions for the closed loop finite pole structure to satisfy are now presented in terms of the $c_i$'s defined in the previous section.

**(4.3.5) Theorem.** Let $G(s)$ be an $m \times \ell$ rational transfer function matrix factorised as

$$G(s) = N_1(s)\,D_1^{-1}(s)$$

where $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ forms a minimal basis with column degrees, $c_i$, ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Let $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ be monic polynomials with real coefficients which satisfy (4.3.1), (4.3.2). Then, for there to exist a constant matrix $K$ such that $D_1(s) + K\,N_1(s)$ has invariant polynomials $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ it is necessary that

$$\sum_{i=k+1}^{\ell} a_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1. \tag{4.3.6}$$

**Proof.** By definition $\alpha_{k+1}(s)\alpha_{k+2}(s)\ldots\alpha_\ell(s)$ is the greatest common divisor of all $(\ell-k)\times(\ell-k)$ minors in $D_1(s)+K\,N_1(s)$ for $k=0,1,\ldots,\ell-1$. Let $e_i$, $i=1,2,\ldots,\ell$, be the column degrees of $D_1(s)+K\,N_1(s)$ taken to correspond with the $c_i$. Thus,

$$e_i \le c_i, \qquad i=1,2,\ldots,\ell.$$

It then follows that

$$\deg\left[\alpha_{k+1}(s)\alpha_{k+2}(s)\ldots\alpha_\ell(s)\right] \le \sum_{i=k+1}^{\ell} e_i \le \sum_{i=k+1}^{\ell} c_i \qquad k=0,1,\ldots,\ell$$

i.e.,

$$\sum_{i=k+1}^{\ell} a_i \le \sum_{i=k+1}^{\ell} c_i \qquad k=0,1,\ldots,\ell-1$$

as required. □

A similar necessary condition for the degrees of the infinite poles can also be given.

**(4.3.7) Theorem.** Let $G(s)$ be an $m\times\ell$ rational transfer function matrix and $D_1(s)$, $N_1(s), c_1,\ldots,c_\ell$ be as described in theorem (4.3.5). Let $\beta_1(w),\beta_2(w),\ldots,\beta_\ell(w)$ be monic polynomials with real coefficients which satisfy (4.3.3) and let

$$\beta_i(w) = w^{b_i}\beta_i'(w) \qquad i=1,2,\ldots,\ell \tag{4.3.8}$$

where $\beta_i'(0)\neq 0$ and take $\Lambda_1(w)$ to be $\mathrm{diag}[w^{c_1},w^{c_2},\ldots,w^{c_\ell}]$. Then, for there to exist a constant matrix $K$ such that $[D_1(\frac{1}{w})+K\,N_1(\frac{1}{w})]\Lambda_1(w)$ has invariant polynomials $\beta_1(w),\beta_2(w),\ldots,\beta_\ell(w)$ it is necessary that

$$\sum_{i=k+1}^{\ell} b_i \le \sum_{i=k+1}^{\ell} c_i \qquad k=0,1,\ldots,\ell-1.$$

**Proof.** By definition $\beta_{k+1}(w)\beta_{k+2}\ldots\beta_\ell(w)$ is the greatest common divisor of all $(\ell-k)\times(\ell-k)$ minors in $[D_1(\frac{1}{w})+K\,N_1(\frac{1}{w})]\Lambda_1(w)$ for $k=0,1,\ldots,\ell-1$. Let $f_i$, $i=1,2,\ldots,\ell$, be the column degrees of $[D_1(\frac{1}{w})+K\,N_1(\frac{1}{w})]\Lambda_1(w)$ taken to correspond with the $c_i$, so that

$$f_i \le c_i, \qquad i=1,\ldots,\ell.$$

It then follows that

$$\deg\left[\beta_{k+1}(w)\beta_{k+2}(w)\ldots\beta_\ell(w)\right] \le \sum_{i=k+1}^{\ell} f_i \le \sum_{i=k+1}^{\ell} c_i \qquad i=0,1,\ldots,\ell-1$$

i.e.,

$$\sum_{i=k+1}^{\ell} \deg(\beta_i(w)) \le \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1$$

and, from (4.3.4) and (4.3.8),

$$\sum_{i=k+1}^{\ell} b_i \le \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1$$

as required. □

The results of theorems (4.3.5), (4.3.7) can be most conveniently illustrated by means of a step function, defined as follows.

**(4.3.9) Definition.**

$$C_k \triangleq \begin{cases} \sum_{j=k+1}^{\ell} c_j & k = 0, 1, \ldots, \ell - 1 \\ \sum_{j=k_-+1}^{\ell} c_j & k = \text{non-integer} \end{cases}$$

where $k_-$ is the downward rounded version of $k$.

Pictorially it can be seen that $C_k$ is a decreasing staircase as illustrated by figure (4.3.10).

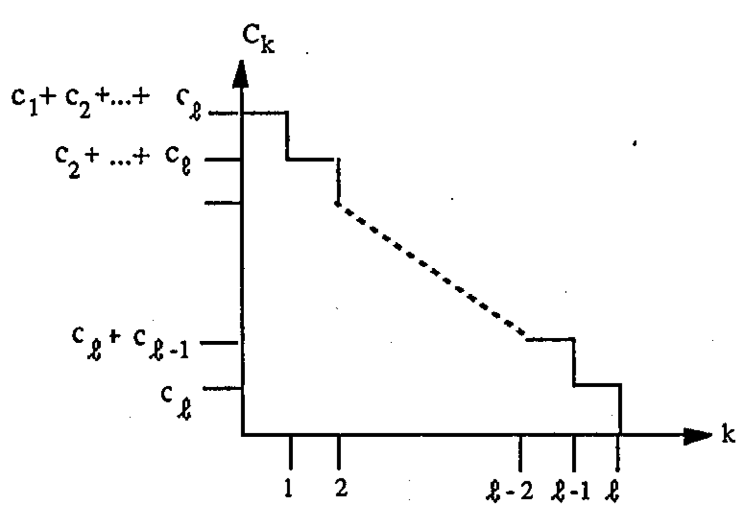



fig. (4.3.10)

It therefore follows, from theorems (4.3.5) and (4.3.7), that the $a_i, b_i$ must be chosen such that the staircases corresponding to $\sum_{i=k+1}^{\ell} a_i, \sum_{i=k+1}^{\ell} b_i,\ k = 0, 1, \ldots, \ell - 1$ lie below the staircase given by figure (4.3.10). Note that if the $c_i$'s had been ordered in any other way the corresponding staircase would either lie on or above the staircase pictured above. Thus, the ordering $c_1 \geq c_2 \geq \ldots \geq c_\ell$ can be regarded as a minimal ordering in the sense that the associated staircase provides the lowest, of this type, of upper bound for the $\sum a_i, \sum b_i$.

As was noted in section 2 the pole structure of $G(s)$ could as easily be investigated by considering a left minimal factorisation of $G(s)$ as represented by (4.2.5), thus leading to analogous necessary conditions to those of theorems (4.3.5) and (4.3.7) in terms of the left factorisation. Combining the necessary conditions from each factorisation leads to stronger necessary conditions for the separate assignments of the finite and infinite pole structures.

Let $G(s)$ be an $m \times \ell$ rational matrix with right and left minimal factorisations

$$N_1(s)\, D_1^{-1}(s) \quad \text{and} \quad D_2^{-1}(s)\, N_2(s)$$

respectively, and where the column degrees, $c_i$, of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$ and the row degrees, $r_i$, of $[D_2(s) \quad N_2(s)]$ are ordered $r_1 \geq r_2 \geq \ldots \geq r_m$. Let $t_1 = \min(m, \ell)$, $t_2 = \max(m, \ell)$ and let $\alpha_1(s), \alpha_2(s), \ldots, \alpha_{t_1}(s)$ be monic polynomials such that

$$\alpha_i(s) \mid \alpha_{i-1}(s) \qquad i = 2, 3, \ldots, t_1$$

and $\deg \alpha_i(s) = a_i$, $i = 1, 2, \ldots, t_1$. Also, let $\beta_1(w), \beta_2(w), \ldots, \beta_{t_1}(w)$ be monic polynomials such that

$$\beta_i(w) \mid \beta_{i-1}(w) \qquad i = 2, 3, \ldots, t_1$$

and where

$$\beta_i(w) = w^{b_i} \quad \beta_i'(w) \qquad i = 1, 2, \ldots, t_1$$

in which $\beta_i'(0) \neq 0$. Let $\Lambda_1(s) = \operatorname{diag}[s^{c_1}, s^{c_2}, \ldots, s^{c_\ell}]$ and $\Lambda_2(s) = \operatorname{diag}[s^{r_1}, s^{r_2}, \ldots, s^{r_m}]$.

Combining the necessary conditions obtained by using a right minimal factorisation with the necessary conditions obtained by using a left factorisation results in the following tighter necessary conditions on the closed loop finite pole structure.

**(4.3.11) Theorem.** Consider an $m \times \ell$ transfer function matrix, $G(s)$, described above and let $D_1(s), N_1(s), D_2(s), N_2(s), c_i, r_i, a_i, \alpha_i, t_i$, also be defined as above. Then, for there to exist a constant matrix $K$ such that the non-unit invariant polynomials of $D_1(s) + K\, N_1(s)$ and $D_2(s) + N_2(s)\, K$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_{t_1}(s)$ it is necessary that

$$\sum_{i=k+1}^{t_1} a_i \leq \sum_{i=k+1}^{t_2} d_i \qquad k = 0, 1, \ldots, t_1 - 1 \tag{4.3.12}$$

where

$$\sum_{i=k+1}^{t_2} d_i = \min \left[ \sum_{i=k+1}^{\ell} c_i, \quad \sum_{i=k+1}^{m} r_i \right].$$

**Proof.** By construction the non-zero invariant polynomials of $D_1(s) + K\,N_1(s)$ and $D_2(s) + N_2(s)\,K$ are equivalent, so that there can be at most $t_1 = \min(\ell, m)$ non-zero invariant polynomials. By theorem (4.3.5) the degrees, $a_i$, of these non-zero invariant polynomials must satisfy the necessary conditions

$$\sum_{i=k+1}^{t_1} a_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, t_1 - 1. \tag{4.3.13}$$

Similarly, by considering a left minimal factorisation of $G(s)$ the $a_i$, $i = 1, 2, \ldots, t_1$, must also satisfy the necessary conditions

$$\sum_{i=k+1}^{t_1} a_i \leq \sum_{i=k+1}^{m} r_i \qquad k = 0, 1, \ldots, t_1 - 1. \tag{4.3.14}$$

Combining (4.3.13) with (4.3.14) gives rise to the necessary conditions (4.3.12), as required. □

In a similar way stricter necessary conditions are obtained for the closed loop infinite pole structure.

**(4.3.15) Theorem.** Consider an $m \times \ell$ transfer function matrix, $G(s)$, described above and let $D_1(s), N_1(s), D_2(s), N_2(s), c_i, r_i, b_i, \beta_i, t_i, \Lambda_i(s)$ also be defined as above. Then, for there to exist a constant matrix $K$ such that the non-unit invariant polynomials of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\,\Lambda_1(w)$ and $\Lambda_2(w)[D_2(\frac{1}{w}) + N_2(\frac{1}{w})\,K]$ are $\beta_1(w), \beta_2(w), \ldots, \beta_{t_1}(w)$ it is necessary that

$$\sum_{i=k+1}^{t_1} b_i \leq \sum_{i=k+1}^{t_2} d_i \qquad k = 0, 1, \ldots, t_1 - 1 \tag{4.3.16}$$

where

$$\sum_{i=k+1}^{t_2} d_i = \min\left[\sum_{i=k+1}^{\ell} c_i, \sum_{i=k+1}^{m} r_i\right].$$

**Proof.** By construction the non-zero invariant polynomials of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\Lambda_1(w)$ and $\Lambda_2(w)[D_2(\frac{1}{w}) + N_2(\frac{1}{w})K]$ are equivalent, so that there can be at most $t_1 = \min(\ell, m)$ non-zero invariant polynomials. By theorem (4.3.7) the non-zero polynomials must satisfy the following necessary conditions expressed in terms of the $b_i$, $i = 1, 2, \ldots, t_1$,

$$\sum_{i=k+1}^{t_1} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad i = 0, 1, \ldots, t_1 - 1. \tag{4.3.17}$$

Similarly, by considering a left minimal factorisation of $G(s)$ the $b_i$, $i = 1, 2, \ldots, t_1$, must also satisfy the necessary conditions

$$\sum_{i=k+1}^{t_1} b_i \leq \sum_{i=k+1}^{m} r_i \qquad i = 0, 1, \ldots, t_1 - 1. \tag{4.3.18}$$

Combining (4.3.17) with (4.3.18) gives rise to the necessary conditions (4.3.16), as required. □

The necessary conditions of theorems (4.3.11) and (4.3.15) can be described in a more straightforward fashion by employing the staircase description. Without loss of generality let $m \geq \ell$ and let the staircase function corresponding to each minimal factorisation be constructed in a similar way to that shown previously. Combining both staircases on the same diagram results in figure (4.3.19).

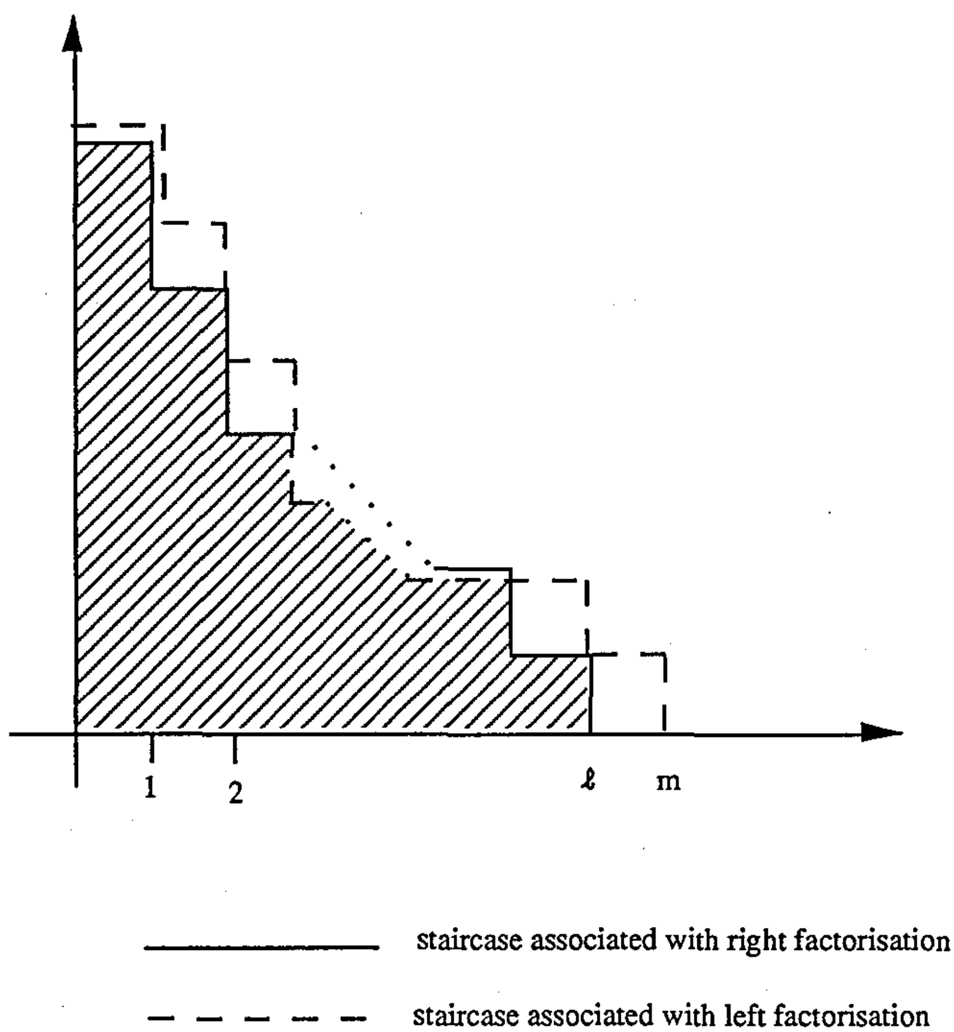



fig. (4.3.19)

Note that the two staircases might not intersect at all or might intersect at more than one point. The necessary conditions of theorem (4.3.11) then state that the closed loop finite pole structure must be such that the staircase corresponding to $\sum_{i=k+1}^{t_1} a_i$, $k = 0, 1, \ldots, t_1 - 1$ lies in the shaded area of figure (4.3.19). A similar interpretation can be made for theorem (4.3.15). Theorems (4.3.11) and (4.3.15) obviously provide stronger necessary conditions than those contained in theorems (4.3.5) and (4.3.7) respectively.

The necessary conditions of theorem (4.3.11) and (4.3.15) are not sufficient conditions as is demonstrated by the following example.

**(4.3.20) Example.** Let

$$G(s) = \frac{1}{s(s^2-1)} \begin{bmatrix} s^4 & -s^3 \\ s(1-s) & s^3-1 \end{bmatrix}.$$

Right and left minimal factorisations of $G(s)$ are respectively given by

$$N_1(s)\, D_1^{-1}(s) = \begin{bmatrix} s^3 & 0 \\ 1 & s \end{bmatrix} \begin{bmatrix} s^2 & 1 \\ s & s \end{bmatrix}^{-1},$$

$$D_2^{-1}(s)\, N_2(s) = \begin{bmatrix} s-1 & s \\ 0 & s^2+s \end{bmatrix}^{-1} \begin{bmatrix} s^2-s & 1 \\ -s & s^2+s+1 \end{bmatrix}.$$

It therefore follows that

$$c_1 = 3, \quad c_2 = 1$$

$$r_1 = 2, \quad r_2 = 2$$

so that the closed loop finite pole structure, as described by $a_1$ and $a_2$, must satisfy the necessary conditions

$$\left.\begin{array}{r} a_2 \leq 1 \\ a_1 + a_2 \leq 4 \end{array}\right\}.$$

Similarly, the closed loop infinite pole structure, as described by $b_1$ and $b_2$, must satisfy the necessary conditions

$$\left.\begin{array}{r} b_2 \leq 1 \\ b_1 + b_2 \leq 4 \end{array}\right\}.$$

The closed loop finite pole structure $a_1 = 0, a_2 = 0$ satisfies the necessary conditions of theorem (4.3.11). For the closed loop system to have this pole structure it is necessary that

$$| D_1(s) + K\, N_1(s) | = \alpha \tag{4.3.21}$$

where $\alpha$ is a non-zero constant. If

$$K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

then

$$|D_1(s) + K\,N_1(s)| = [k_1(1+k_4) - k_2\,k_3]\,s^4 + [1+k_4-k_3]\,s^3 - k_2\,s^2 + [k_2-1]s - k_4.$$

For (4.3.21) to hold, it follows that

$$k_2 = 0 \quad \text{and} \quad k_2 - 1 = 0.$$

This is clearly not possible indicating that the pole structure $a_1 = 0, a_2 = 0$ can not be assigned by constant output feedback and so demonstrating that the necessary conditions of theorem (4.3.11) are not sufficient.

By implication it follows that it is not possible to assign all the poles at infinite location. In particular the infinite pole structure $b_1 = 4, b_2 = 0$ can not be assigned so demonstrating that the necessary conditions of theorem (4.3.15) are not sufficient.

## §4. Necessary conditions for the simultaneous placement of a finite pole structure and an infinite pole structure by output feedback.

When designing the closed loop system it is of greater interest to know whether a finite and infinite pole structure can be assigned simultaneously rather than separately. This has therefore lead to the investigation of necessary conditions for such an assignment.

Initial necessary conditions can be deduced from theorems (4.3.5) and (4.3.7), namely that the $a_i$'s and $b_i$'s as defined in those aforementioned theorems must satisfy the conditions that

$$\sum_{i=k+1}^{\ell} a_i + \sum_{i=k+1}^{\ell} b_i \leq 2 \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1. \tag{4.4.1}$$

This is a very crude bound on $\sum a_i + \sum b_i$ since it is known, for example, that

$$\sum_{i=1}^{\ell} a_i + \sum_{i=1}^{\ell} b_i = \sum_{i=1}^{\ell} c_i. \tag{4.4.2}$$

Stricter necessary conditions are now presented which include the condition (4.4.2) and which generalise the results of theorems (4.3.11) and (4.3.15) to the case when the finite and infinite pole structures are assigned simultaneously. The result is first given in terms of a right minimal factorisation of the associated transfer function matrix.

**(4.4.3) Theorem.** Let $G(s)$ be an $m \times \ell$ transfer function matrix factorised as in theorems (4.3.5) and (4.3.7). Let $\alpha_i(s), a_i$ be given as in theorem (4.3.5) and $\beta_i(w), b_i, \Lambda_1(w)$ as in theorem (4.3.7). Then, for there to exist a constant matrix $K$ such that $D_1(s)+KN_1(s)$ has invariant polynomials $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ and $[D_1(\frac{1}{w}) + KN_1(\frac{1}{w})]\,\Lambda_1(w)$ has invariant polynomials $\beta_1(w), \beta_2(w), \ldots, \beta_\ell(w)$ it is necessary that

$$\sum_{i=k+1}^{\ell} a_i + \sum_{i=k+1}^{\ell} b_i \le \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1 \tag{4.4.4}$$

with equality holding when $k = 0$.

**Proof.** By definition $\alpha_\ell(s)\ \alpha_{\ell-1}(s)\ \ldots\ \alpha_{\ell-k+1}(s)$ is the greatest common divisor of all $k \times k$ minors in $D_1(s) + K\,N_1(s)$. Let

$$v_k \triangleq \sum_{i=\ell-k+1}^{\ell} a_i \quad \text{and} \quad \mu_k \triangleq \sum_{i=\ell-k+1}^{\ell} c_i \quad k = 1, 2, \ldots, \ell$$

then

$$\alpha_\ell(s)\ \alpha_{\ell-1}(s)\ \ldots\ \alpha_{\ell-k+1}(s) = t_{v_k} s^{v_k} + \ldots + t_1 s + t_0$$

where $v_k \le \mu_k$ and $t_{v_k} \ne 0$.

Now each $k \times k$ minor of $D_1\left(\frac{1}{w}\right) + KN_1\left(\frac{1}{w}\right)$ will be of the form

$$\alpha_\ell\left(\tfrac{1}{w}\right)\ \alpha_{\ell-1}\left(\tfrac{1}{w}\right) \ldots \alpha_{\ell-k+1}\left(\tfrac{1}{w}\right)\ f\left(\tfrac{1}{w}\right) \tag{4.4.5}$$

for some polynomial $f(s)$. Further, among all $k \times k$ minors of $D_1(s) + K\,N_1(s)$ the corresponding polynomials $f(s)$ are coprime for finite $s$. Thus, all $k \times k$ minors of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\,\Lambda_1(w)$ will be of the form

$$\alpha_\ell\left(\tfrac{1}{w}\right)\ \alpha_{\ell-1}\left(\tfrac{1}{w}\right) \ldots \alpha_{\ell-k+1}\left(\tfrac{1}{w}\right)\ f\left(\tfrac{1}{w}\right) w^{\eta} \tag{4.4.6}$$

where $\eta \ge \mu_k$.

The greatest common divisor of all $k \times k$ minors in $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\,\Lambda_1(w)$ will therefore be

$$\alpha_\ell\left(\tfrac{1}{w}\right)\ \alpha_{\ell-1}\left(\tfrac{1}{w}\right) \ldots \alpha_{\ell-k+1}\left(\tfrac{1}{w}\right)\ w^{\mu'_k}$$

where

$$\mu'_k \le \mu_k \tag{4.4.7}$$

i.e.,

$$\left[t_{v_k}\left(\tfrac{1}{w}\right)^{v_k} + \ldots + t_1\left(\tfrac{1}{w}\right) + t_0\right]\ w^{\mu'_k}$$

$$= t_{v_k}\ w^{\mu'_k - v_k} + \ldots + t_1\ w^{\mu'_k - 1} + t_0 w^{\mu'_k}$$

$$= w^{\mu'_k - v_k}\ \left[t_{v_k} + \ldots + t_1 w^{v_k - 1} + t_0 w^{v_k}\right]$$

where $\mu'_k \geq v_k$ since $\alpha_\ell\left(\frac{1}{w}\right)\alpha_{\ell-1}\left(\frac{1}{w}\right)\ldots\alpha_{\ell-k+1}\left(\frac{1}{w}\right)w^{\mu'_k}$ must be a polynomial. It therefore follows by definition that

$$b_\ell + b_{\ell-1} + \ldots + b_{\ell-k+1} = \mu'_k - v_k$$

i.e.,

$$\sum_{i=\ell-k+1}^{\ell} b_i + \sum_{i=\ell-k+1}^{\ell} a_i = \mu'_k.$$

Hence, from (4.4.7)

$$\sum_{i=k+1}^{\ell} b_i + \sum_{i=k+1}^{\ell} a_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell.$$

When $k = \ell$, $f(\frac{1}{w}) = 1$ in (4.4.5) and $\eta = \mu_\ell$ in (4.4.6). This means that the greatest common divisor of all $\ell \times \ell$ minors in $\left[D_1\left(\frac{1}{w}\right) + KN_1\left(\frac{1}{w}\right)\right]\Lambda_1(w)$ is of the form

$$w^{\mu_\ell - v_\ell} \cdot \phi(w)$$

where $\phi(w)$ has no factors of the form $w^\alpha, \alpha > 0$.
Hence,

$$\sum_{i=1}^{\ell} b_i + \sum_{i=1}^{\ell} a_i = \sum_{i=1}^{\ell} c_i$$

as required. □

Again similar necessary conditions can be obtained by using a left minimal factorisation. Combining the necessary conditions from each factorisation leads to the following theorem. The notation for this theorem is as described in the previous section.

**(4.4.8) Theorem.** Consider an $m \times \ell$ rational transfer function matrix, $G(s)$, described above and let $N_1(s), D_1(s), N_2(s), D_2(s), c_i, r_i, \beta_i(s), \alpha_i(s), a_i, b_i, t_i, \Lambda_1(s), \Lambda_2(s)$ also be defined as above. Then, for there to exist a constant matrix $K$ such that the non-unit invariant polynomials of $D_1(s) + K\,N_1(s)$ and $D_2(s) + N_2(s)K$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_{t_1}(s)$ and the non-unit invariant polynomials of $[D_1(\frac{1}{w}) + K\,N_1(\frac{1}{w})]\,\Lambda_1(w)$ and $\Lambda_2(w)\;[D_2(\frac{1}{w}) + N_2(\frac{1}{w})K]$ are $\beta_1(w), \beta_2(w), \ldots, \beta_{t_1}(w)$ it is necessary that

$$\sum_{i=k+1}^{t_1} a_i + \sum_{i=k+1}^{t_1} b_i \leq \sum_{i=k+1}^{t_2} d_i \qquad k = 0, 1, \ldots t_1 - 1 \tag{4.4.9}$$

where

$$\sum_{i=k+1}^{t_2} d_i = \min\left[\sum_{i=k+1}^{\ell} c_i \quad , \sum_{i=k+1}^{m} r_i\right]$$

and with equality holding when $k = 0$ in (4.4.9).

**Proof.** The result follows by combining the separate necessary conditions obtained by considering a left and a right minimal factorisation of $G(s)$ in the same way that the results of theorems (4.3.11) and (4.3.15) were derived. $\square$

Theorem (4.4.8) obviously provides stronger necessary conditions than the ones obtained by considering each factorisation separately. This is demonstrated by the following example.

**(4.4.10) Example.** Let

$$G(s) = \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ s & \frac{1}{s^3} & s^2 \\ 0 & 0 & \frac{1}{s^5} \end{bmatrix}$$

$$= \begin{bmatrix} -s^5 & 0 & 1 \\ 0 & 1 & s^2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -s^6 & 0 & s \\ 0 & s^3 & 0 \\ s^5 & 0 & 0 \end{bmatrix}^{-1} \triangleq N_1(s)\, D_1^{-1}(s)$$

$$= \begin{bmatrix} 0 & 0 & s^5 \\ 0 & s^3 & 0 \\ s & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ s^4 & 1 & s^5 \\ 1 & 0 & 0 \end{bmatrix} \triangleq D_2^{-1}(s)\, N_2(s)$$

where $N_1(s)\, D_1^{-1}(s)$ is a right minimal factorisation and $D_2^{-1}(s)\, N_2(s)$ is a left minimal factorisation. It therefore can be seen that

$$c_1 = 6, \ c_2 = 3, \ c_3 = 2,$$

$$r_1 = 5, \ r_2 = 5, \ r_3 = 1.$$

The necessary conditions obtained by considering the right factorisation requires $a_i$, $b_i$ to satisfy

$$a_3 + b_3 \leq 2$$

$$a_3 + a_2 + b_3 + b_2 \leq 5$$

$$a_3 + a_2 + a_1 + b_3 + b_2 + b_1 = 11$$

whilst the necessary conditions obtained by considering the left factorisation requires $a_i$, $b_i$ to satisfy

$$a_3 + b_3 \leq 1$$

$$a_3 + a_2 + b_3 + b_2 \leq 6$$

$$a_3 + a_2 + a_1 + b_3 + b_2 + b_1 = 11.$$

Let

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{bmatrix}$$

and $\Lambda_1(w) = \text{diag }[w^6, w^3, w^2], \Lambda_2(w) = \text{diag }[w^5, w^5, w]$.

First, consider the right factorisation whose necessary conditions require $a_3, b_3$ to satisfy

$$a_3 + b_3 \leq 2.$$

The closed loop infinite pole structure is given by the zero structure at $w = 0$ of

$$\left[D_1\left(\tfrac{1}{w}\right) + K\,N_1(\tfrac{1}{w})\right]\,\Lambda_1(w) = \begin{bmatrix} -1 - k_1 w + k_3 w^6 & k_2 w^3 & w + k_1 w^2 + k_2 \\ -k_4 w + k_6 w^6 & 1 + k_5 w^3 & k_4 w^2 + k_5 \\ w - k_7 w + k_9 w^6 & k_8 w^3 & k_7 w^2 + k_8 \end{bmatrix}. \tag{4.4.11}$$

It follows from element (2,2) of (4.4.11), that $w$ cannot be a common factor of all $1 \times 1$ minors of (4.4.11), i.e. $b_3 = 0$. Now the finite pole structure of the closed loop system is given by the invariant polynomials of

$$D_1(s) + KN_1(s) = \begin{bmatrix} -s^6 - k_1 s^5 + k_3 & k_2 & k_2 s^2 + s + k_1 \\ -k_4 s^5 + k_6 & s^3 + k_5 & k_5 s^2 + k_4 \\ s^5 - s^5 k_7 + k_9 & k_8 & k_8 s^2 + k_7 \end{bmatrix}. \tag{4.4.12}$$

If the greatest common divisor of all $1 \times 1$ minors is to have degree greater than zero then the element in position (1,2) of (4.4.12) indicates that $k_2 = 0$. In this event it follows from the (1,3) element of (4.4.12) that the highest possible degree for this divisor is 1. Hence $a_3 \leq 1$. Thus, by investigating the closed loop pole structure via the right minimal factorisation it is seen that $a_3, b_3$ must satisfy the necessary condition

$$a_3 + b_3 \leq 1$$

which confirms the necessary condition obtained by considering the left factorisation.

Similarly consider the pole structure obtained by using the left minimal factorisation which requires $a_3, a_2, b_3, b_2$ to satisfy the necessary condition

$$a_3 + a_2 + b_3 + b_2 \leq 6.$$

The infinite pole structure of the closed loop system is given by the zero structure at $w = 0$ of

$$\Lambda_2(w)\left[D_2\left(\tfrac{1}{w}\right) + N_2\left(\tfrac{1}{w}\right)K\right] =$$

$$\begin{bmatrix} k_7 w^5 & k_8 w^5 & k_9 w^5 + 1 \\ k_4 w^5 + k_1 w + k_7 & k_5 w^5 + w^2 + k_2 w + k_8 & k_6 w^5 + k_3 w + k_9 \\ k_1 w + 1 & k_2 w & k_3 w \end{bmatrix}. \tag{4.4.13}$$

The $2 \times 2$ minor formed by deleting the second row and second column of (4.4.13) is given by

$$\begin{vmatrix} k_7 w^5 & k_9 w^5 + 1 \\ k_1 w + 1 & k_3 w \end{vmatrix} = -1 + w^6 (k_7 k_3 - k_9 k_1) - k_1 w - k_9 w^5$$

which is not divisible by $w$ regardless of the choice of $k_1, k_7, k_9, k_3$. Hence $b_3 + b_2 = 0$. The finite pole structure of the closed loop system is given by the invariant polynomials of

$$D_2(s) + N_2(s)K = \begin{bmatrix} k_7 & k_8 & s^5 + k_9 \\ k_1 s^4 + k_7 s^5 + k_4 & k_2 s^4 + k_8 s^5 + s^3 + k_5 & k_9 s^5 + k_3 s^4 + k_6 \\ s + k_1 & k_2 & k_3 \end{bmatrix}. \tag{4.4.14}$$

Suppose that there exists a $K$ such that $a_3 + a_2 > 5$. Then, all $2 \times 2$ minors of (4.4.14) must have at least degree 6. Consider the minor formed by deleting the third column and second row of (4.4.14), i.e.

$$\begin{vmatrix} k_7 & k_8 \\ s + k_1 & k_2 \end{vmatrix} = k_2 k_7 - k_8 k_1 - k_8 s.$$

For the above assumption to hold it follows that $k_8 = 0 = k_2$ or $k_8 = 0 = k_7$. If $k_8 = 0 = k_7$ consider the minor formed by deleting the first column and second row of (4.4.14), i.e.

$$\begin{vmatrix} 0 & s^5 + k_9 \\ k_2 & k_3 \end{vmatrix} = -k_2 k_9 - k_2 s^5$$

which implies $k_2 = 0$ for the above assumption to hold. Thus, it is necessary that $k_2 = 0$ and $k_8 = 0$. Now the minor formed by deleting the third column and first row of (4.4.14) is given by

$$\begin{vmatrix} k_1 s^4 + k_7 s^5 + k_4 & s^3 + k_5 \\ s + k_1 & 0 \end{vmatrix} = -s^4 - k_1 s^3 - k_5 s - k_1 k_5$$

which shows that it is not possible to find a $K$ such that all $2 \times 2$ minors of (4.4.14) have at least degree 6. Hence, the original assumption is false and it is deduced that $a_3 + a_2 \leq 5$. It then follows that

$$a_3 + a_2 + b_3 + b_2 \leq 5$$

which confirms the necessary condition obtained from the left factorisation.

Using the staircase description of figure (4.4.15)

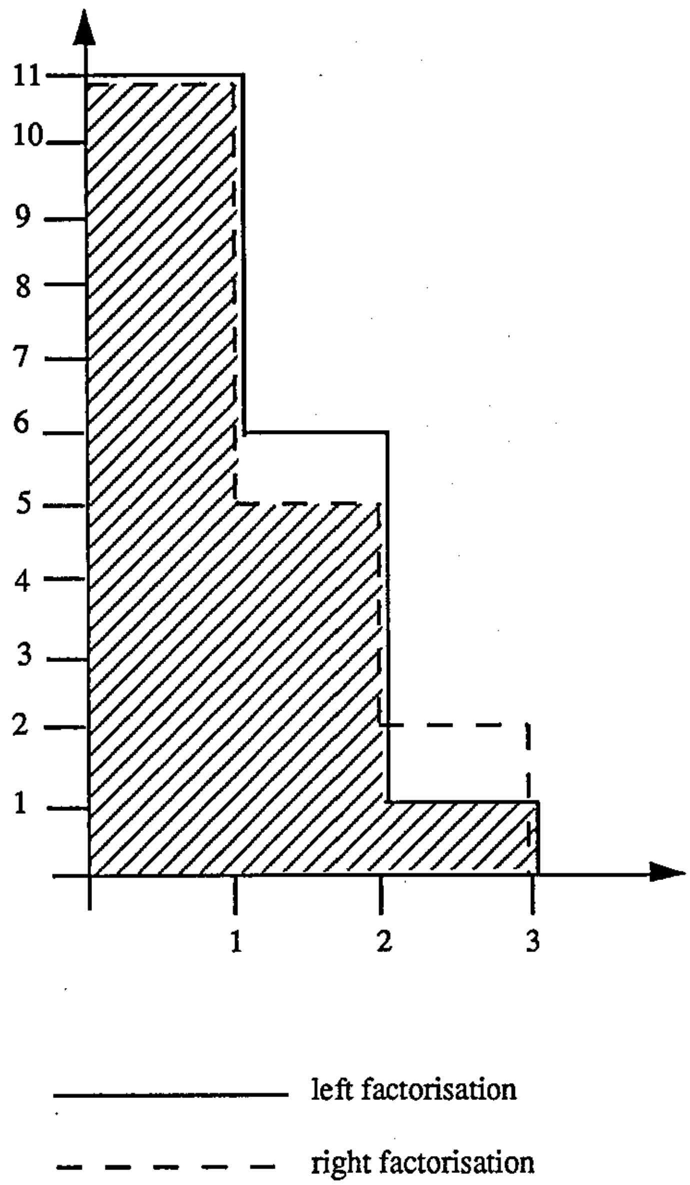



fig. (4.4.15)

it is seen that the desired closed loop system pole structure must give rise to a staircase function which must lie within the shaded area.

The above example also demonstrates that the necessary conditions of theorem (4.4.8) are not sufficient conditions. In particular, the pole structure $b_3 = 1,\ b_2 = 1,\ b_1 = 1,$ $a_3 = 0,\ a_2 = 3,\ a_1 = 5$ satisfies the conditions (4.4.9) of theorem (4.4.8) but it was seen that $b_3$ must satisfy $b_3 = 0$ in the above example.

## §5. Relationship with previous results in the case of strictly proper transfer function matrices.

For the case when the system has a strictly proper transfer function matrix, $G(s)$, all the open loop poles will be situated at finite locations. Further, when constant output feedback of the form described in figure (4.2.1) is applied around this system the resulting closed loop system also has all its poles at finite locations (Rosenbrock and Pugh, 1974). The results of theorem (4.3.5) and (4.3.11) can be interpreted for this situation with the necessary conditions (4.3.6) and (4.3.12) modified to include equality when $k = 0$. The necessary conditions (4.3.6) are in agreement with the necessary conditions obtained by Rosenbrock and Hayton [1978] for this particular case. Rosenbrock and Hayton [*ibid.*] considered the general pole placement problem for systems with strictly proper transfer function matrices using dynamic output feedback but their result can be interpreted for constant output feedback.

The result given by Rosenbrock and Hayton [*ibid.*] corresponding to theorem (4.3.5) is presented below with a slight alteration of notation so that the result can be directly compared with that of theorem (4.3.5).

**(4.5.1) Theorem (Rosenbrock and Hayton, 1978).** Let $G(s) = T_G^{-1}(s)\,U_G(s)$ be $m \times \ell$ and strictly proper with $T_G(s), U_G(s)$ relatively (left) prime. Let $\lambda_{G1} \geq \lambda_{G2} \geq \ldots, \geq \lambda_{G\ell}$ be the controllability indices of $G(s)$. Let $K(s) = T_K^{-1}(s)\,U_K(s)$ be the proper dynamic feedback with observability indices $\mu_{K1} \geq \mu_{K2} \geq \ldots \geq \mu_{K\ell}$. Then, the closed loop pole structure as defined by $a_i$, $i = 1, 2, \ldots, \ell$, must satisfy the necessary conditions

$$\sum_{i=1}^{k} a_i \geq \max\left[\sum_{i=1}^{k}(\lambda_{Gi} + \mu_{K,\ell+1-i}),\quad \sum_{i=1}^{k}(\lambda_{G,\ell+1-i} + \mu_{Ki})\right] \quad k = 1, 2, \ldots, \ell \qquad (4.5.2)$$

with equality holding when $k = \ell$ if and only if $T_k(s), U_k(s)$ are relatively (left) prime.

Now if $K(s)$ is taken to be a constant matrix the associated observability indices, $\mu_{Ki}$, are all equal to zero. Hence, the necessary conditions of theorem (4.5.1) become

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} \lambda_{Gi} \qquad k = 1, 2, \ldots, \ell \qquad (4.5.3)$$

with equality holding when $k = \ell$. It also follows, since $G(s)$ is strictly proper, that the controllability indices are equivalent to the $c_i$'s defined earlier (see Forney, 1975). Thus, replacing $\lambda_{Gi}$ by $c_i$ in (4.5.3) and reordering gives (4.3.6).

Hence, the necessary conditions given by Rosenbrock and Hayton [1978] and the necessary conditions of theorem (4.3.5) are equivalent for the case when constant output feedback is applied around a system with a strictly proper transfer function matrix. The necessary conditions of theorem (4.3.11), obtained by considering both a left and right minimal factorisation of $G(s)$, are stronger conditions than those of theorem (4.3.5) and hence are stronger than the necessary conditions obtained by Rosenbrock and Hayton [*ibid.*] for this particular case. This suggests that the general result presented by Rosenbrock and Hayton [*ibid.*] could be improved. Indeed Kucera and Zagalak [1985] subsequently derived stricter necessary conditions than those obtained by Rosenbrock and Hayton [1978] for dynamic feedback.

For the special case of constant output feedback these new necessary conditions presented by Kucera and Zagalak [1985] are equivalent to those obtained by Rosenbrock and Hayton [*ibid.*] and hence are in agreement with the necessary conditions of theorem (4.3.5) as applied to systems with strictly proper transfer function matrices.

For the strictly proper case Rosenbrock and Hayton [1978] also present sufficient conditions for the placement of a pole structure using dynamic compensators. The proof of this sufficient condition given by Rosenbrock and Hayton [*ibid.*] requires the compensator to have order equal to $\sum_{i=1}^{\ell}(\lambda_{Gi} - 1)$. Thus, for the compensator to be constant, i.e. have order zero, the largest controllability index of $G(s)$ must be 1. In this instance the sufficient conditions are equivalent to the necessary conditions, so that it is possible to find a constant feedback matrix such that the closed loop pole structure satisfies conditions (4.5.3). The requirement that $\lambda_{G1} = 1$ is a very restrictive one so that the result for constant feedback applies only to certain systems.

## §6. Conclusions.

The general pole placement problem using constant output feedback has been considered in this chapter. The treatment is novel since it allows for the possibility that the open loop system and the closed loop system possess a proper or a non-proper transfer function matrix. As a result the assignment of both the finite and infinite pole structures have been investigated .

The problem was approached by exploiting the properties of a minimal factorisation of the open loop transfer function matrix. In section 3 separate necessary conditions were presented for the finite and infinite closed loop structure to satisfy. These new conditions were given naturally in terms of the column and row indices of certain minimal factorisations associated with the open loop transfer function matrix. The results were neatly illustrated by means of suitable step functions. Further original necessary conditions were presented in section 4 which generalise, in an appropriate manner, the results presented in section 3 to the case where the finite and infinite pole structures are considered simultaneously. Finally, in section 5 the relationship with previous work concerning systems with strictly proper transfer function matrices was investigated. For the specific case of constant output feedback the necessary conditions presented in this chapter were seen to be stricter than those obtained previously by both Rosenbrock and Hayton [1978] and Zagalak and Kucera [1985].

## Chapter 5. Notions of Controllability in Generalised State Space Systems.

### §1. Introduction.

Consider linear time invariant systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1.1}$$

$$y(t) = Cx(t) \tag{5.1.2}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ is an $\ell$-vector of control inputs and $y(t)$ is a $m$-vector of outputs. $E, A, B, C$ are real constant matrices of the appropriate dimensions.

The properties of such systems can be divided into two cases depending on the non-singularity or singularity of the matrix $E$. When $E$ is non-singular the equations are said to be in regular state space form and the properties of such systems have been widely investigated.

If $E$ is singular the behaviour of the system with defining equations (5.1.1), (5.1.2) differs considerably. The system now consists of algebraic as well as first order differential equations which can lead to what is termed impulsive motion occurring in the system. In the frequency domain this corresponds to the presence of infinite frequency behaviour. The system is then said to be in generalised state space form. The terms general (or generalised) and regular will be adopted to distinguish between the two cases.

The difference between the regular and generalised cases is clearly reflected by the transfer function matrix, $G(s)$, associated with (5.1.1), (5.1.2), i.e.

$$G(s) = C[sE - A]^{-1} B.$$

When $E$ is non-singular, $G(s)$ will be a strictly proper matrix whilst when $E$ is singular $G(s)$ could additionally be either proper or non-proper. It should be noted that early investigations of systems with proper or non-proper transfer function matrices were undertaken (see Rosenbrock, 1970) by adapting the regular system to incorporate a term $D\left(\frac{d}{dt}\right) u(t)$ on the right hand side of (5.1.2). The $D\left(\frac{d}{dt}\right)$ term gives rise to the polynomial part of the transfer function matrix. It was usually assumed that the system had also existed for $t < 0$. This implies that the initial state of the system, $x(0-)$, satisfies the equations (5.1.1) so that the system can not display an impulsive response and its behaviour mirrors that of a regular system. This approach proved satisfactory under these assumptions.

When it became apparent that the presence of impulsive motions was inevitable and even desirable in some systems, e.g. component failure at $t = 0$ or in switching, it was found that the existing approach of adapting the regular case was inadequate. This was mainly due to the fact that the infinite frequency behaviour of the system was inadequately displayed and that the transformation theory associated with the regular case only preserved the infinite frequency behaviour of $D(s)$. It became apparent that a more complete theory was needed to further encompass the generalised case. Naturally the development of such theory has been closely associated with the regular theory with the specific properties of the generalised case taken into account.

In this chapter the concepts of controllability in generalised state space systems are considered. These concepts emanate from the concepts of controllability in regular state space systems which are described in section 3. Associated with these notions of controllability are a set of algebraic conditions which are derived and described by Rosenbrock [1970]. The extension of the concepts of controllability to generalised state space systems has been developed in both the frequency and time domain and a discussion of this development is presented in section 4. Two main definitions of controllability have emerged. In the frequency domain the difference between the two definitions is reflected in the role of what are termed non-dynamic variables. A new definition is presented which subsequently illustrates, in a novel manner, the important role of the non-dynamic variables in the time-domain.

In section 5 analogous algebraic conditions to those associated with controllability in regular state space systems are presented for the two main concepts of controllability in generalised state space systems. Certain of these conditions have been previously established (see Lewis, 1986) but others have not. A polynomial system approach is adopted to provide a unified treatment and also simpler proofs to the existing results. New results are presented which together with the existing results provide a complete analogy to the conditions presented by Rosenbrock [1970] for the regular case. In section 6 the roles of the non-dynamic and dynamic variables in generalised state space systems are further discussed.

Before considering the notions of controllability a description of a canonical form known as the Kronecker form associated with the generalised state space system is first presented in section 2. This form will be required for subsequent chapters as well as this present one.

## §2. The Kronecker form.

The polynomial system matrix, $P(s)$, associated with the system (5.1.1), (5.1.2) is given by

$$P(s) = \left[\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right] \tag{5.2.1}$$

where it is assumed that $| sE - A | \neq 0$.

The matrix $sE - A$ is therefore a regular pencil and thus can be transformed by means of pre and post multiplication by constant non-singular matrices, $M$ and $N$, into the following form

$$M[sE - A]N = \begin{bmatrix} sI_{n_1} - A_1 & 0 \\ 0 & sJ - I_{n_2} \end{bmatrix} \tag{5.2.2}$$

where $n_1 = \deg | sE - A |$, $n_2 = n - n_1$ and $J$ is nilpotent. In addition $J$ may be chosen to be in Jordan canonical form with say $p$ Jordan blocks each of order $q_i$ and without loss of generality ordered

$$q_1 \geq q_2 \geq \ldots \geq q_{p-t} > q_{p-t+1} = q_{p-t+2} = \ldots = q_p = 1. \tag{5.2.3}$$

The nilpotency index of $J$ is therefore equal to $q_1$. This canonical form was first described by Wierestrass [1867] but is a special case of the canonical form derived by Kronecker [1890] for more general matrix pencils [see Gantmacher, 1959, for full description]. For this reason the canonical form (5.2.2) with $J$ in Jordan form (5.2.3) will be referred to as the Kronecker form of the matrix pencil $sE - A$ despite the fact that $A_1$ might not be in the strict form required by Kronecker [1890].

Adapting the transformation (5.2.2) to the system matrix $P(s)$ gives rise to the following.

$$\left[\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} sE - A & B \\ \hline -C & 0 \end{array}\right]\left[\begin{array}{c|c} N & 0 \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{cc|c} sI_{n_1} - A_1 & 0 & B_1 \\ 0 & sJ - I_{n_2} & B_2 \\ \hline -C_1 & -C_2 & 0 \end{array}\right]. \tag{5.2.4}$$

The transformation represented by (5.2.4) is a restricted system equivalence transformation (Rosenbrock, 1974) and thus preserves the fundamental characteristics of the system at all frequencies $s$. The system (5.1.1), (5.1.2) is then said to be in Kronecker form when its system matrix is represented as

$$\left[\begin{array}{cc|c} sI_{n_1} - A_1 & 0 & B_1 \\ 0 & sJ - I_{n_2} & B_2 \\ \hline -C_1 & -C_2 & 0 \end{array}\right] \tag{5.2.5}$$

where $J$ is in Jordan canonical form (5.2.3). Similarly the pencil $[sE - A \quad B]$ is said to be in Kronecker form when $sE - A$ is given by (5.2.2) and $J$ is again in Jordan normal form (5.2.3). Further, with reference to the system described in Kronecker form, let $b_i^T$, $i = 1, 2, \ldots, n$, represent the rows of $B$ and let the term last position rows of $B$ refer to the rows of $B$ which correspond to the last position rows of the Jordan blocks of $J$, i.e. rows $n_1 + q_1, n_1 + q_1 + q_2, \ldots, n_1 + q_1 + \ldots + q_p$.

The transfer function matrix, $G(s)$, associated with (5.2.1) can now be written as

$$G(s) = C_1[sI_{n_1} - A_1]^{-1}B_1 + C_2[sJ - I_{n_2}]^{-1}B_2 \tag{5.2.6}$$

where $C_1[sI_{n_1} - A_1]^{-1}B_1$ is strictly proper and $C_2[sJ - I_{n_2}]^{-1}B_2$ is polynomial. Similarly the defining equations for the system can be partitioned as

$$\dot{x}_1(t) = A_1x_1(t) + B_1u(t) \tag{5.2.7}$$

$$J\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{5.2.8}$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{5.2.9}$$

The state variables $x_1(t)$ are governed by regular state space equations and can be regarded as determining the finite frequency behaviour of the system. The state variables $x_2(t)$ contribute the polynomial part of the transfer function matrix and therefore can be regarded as determining the infinite frequency behaviour of the system. Cobb [1981] describes the respective subsystems as slow and fast to reflect the enforced motion associated with each subsystem, the first set of equations producing exponential responses whilst the second set producing impulsive responses. This can be seen from the solution of (5.2.7) and (5.2.8) which are readily given as

$$x_1(t) = e^{A_1t}x_1(0-) + \int_0^t e^{A_1(t-\tau)}B_1u(\tau)\,d\tau \tag{5.2.10}$$

$$x_2(t) = -\sum_{i=1}^{q_1-1} \delta^{(i-1)}J^ix_2(0-) - \sum_{i=0}^{q_1-1} J^iB_2u^{(i)}(t). \tag{5.2.11}$$

## §3. Regular state space systems.

The regular state space system associated with (5.1.1), (5.1.2) when $E$ is non-singular can equivalently be written as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.3.1}$$

$$y(t) = Cx(t) \tag{5.3.2}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ an $\ell$-vector of control inputs, $y(t)$ is an $m$-vector of outputs and $A, B, C$ are constant real matrices of the appropriate dimensions.

The fundamental notion of controllability of a system considers how the internal states of a system can be affected by applying a suitable control. It then follows that the controllability characteristics of a regular state space system as described above are completely determined by considering (5.3.1). The definitions of controllability most commonly associated with such systems are now presented.

**(5.3.3) Definition.** The system given by (5.3.1) is said to be CONTROLLABLE TO THE ORIGIN if, given any state $x(0) = \xi$, there exists a time $\tau > 0$ and control $u(t)$ defined on $[0, \tau]$ such that $x(\tau) = 0$.

**(5.3.4) Definition** The system given by (5.3.1) is said to be CONTROLLABLE FROM THE ORIGIN (OR REACHABLE) if, given any state $\xi$, there exists a time $\tau > 0$, and a control $u(t)$ defined on $[0, \tau]$ such that if $x(0) = 0$ then $x(\tau) = \xi$.

**(5.3.5) Definition.** The system given by (5.3.1) is said to be CONTROLLABLE if, given any two states $\xi_1, \xi_2$, there exists a $\tau > 0$ and a control $u(t)$ defined on $[0, \tau]$ such that $x(0) = \xi_1$ and $x(\tau) = \xi_2$.

The particular attraction of the above definitions lies in the algebraic properties associated with a system satisfying such definitions. As a result of the time invariance of the matrix coefficients in (5.3.1) the three notions of controllability defined are equivalent. It is therefore generally accepted that a system satisfying such definitions be called a controllable system.

The algebraic properties associated with a controllable system were investigated by Rosenbrock [1970] who showed that the system represented by (5.3.1) is controllable if and only if the matrices $sI - A, B$ are relatively (left) prime. Thus, the advantage of adopting such a definition is obvious because of the direct association with the absence of decoupling zeros and the connection with a minimal realisation of a system. A summary of the characteristics associated with a controllable system is given by the following theorem.

**(5.3.6) Theorem (Rosenbrock, 1970).** For a system described by (5.3.1) the following conditions are equivalent

(a) the system is controllable,

(b) the polynomial matrices $sI_n - A, B$ are relatively (left) prime,

(c) the $n \times q\ell$ matrix

$$[B, AB, \ldots, A^{q-1}B]$$

has rank n, where $q$ is any integer not less than the degree of the minimal polynomial of $A$,

(d) the $qn \times [(q-1)n + q\ell]$ matrix

$$R = \begin{bmatrix} I_n & & & & & & & B \\ -A & I_n & & & 0 & & \iddots & \\ & -A & \ddots & & & \iddots & & \\ 0 & & \ddots & I_n & 0 & B & & 0 \\ & & & -A & B & & & \end{bmatrix},$$

with $q$ as in (c), has rank $qn$,

(e) given any polynomial $n$-vector $\bar{c}(s)$ with elements of degree $q-1$ or less and $q$ as in (c), there exist a polynomial $n$-vector $\bar{x}(s)$ with elements of degree $q-2$ or less and a polynomial $\ell$-vector $\bar{y}(s)$ with elements of degree $q-1$ or less such that

$$(sI_n - A)\bar{x}(s) + B\bar{y}(s) = \bar{c}(s),$$

(f) there exists an $n \times n$ polynomial matrix $X(s)$ with elements of degree $q-2$ or less with $q$ as in (c) and an $\ell \times \ell$ polynomial matrix $Y(s)$ with elements of degree $q-1$ or less, such that

$$(sI_n - A)X(s) + BY(s) = I_n,$$

(g) let $A$ be in Jordan normal form and let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the distinct eigenvalues of $A$ with $\lambda_i$ having multiplicity $q_i$, $i = 1, 2, \ldots, p$. Let $b_i^T$, $i = 1, 2, \ldots, n$, be the rows of $B$. Then, the rows $b_{q_1}^T, b_{q_1+q_2}^T, \ldots, b_{q_1+q_2+\ldots+q_p}^T$ are linearly independent,

(h) the matrix pencil $[sI_n - A \quad B]$ does not possess any finite zeros.

The notion of controllability can alternatively be viewed in terms of the natural modes of a system which in the case of state space systems are exponential modes. A state space system of the form (5.3.1), (5.3.2) would then be controllable if all the exponential modes could be individually excited from zero initial conditions by means of an input that contains no component at the modal frequency. Natural modes that cannot be excited in this way are termed uncontrollable modes and are associated with the zeros of the matrix pencil $[sI_n - A \quad B]$. Rosenbrock [1970] termed such zeros input-decoupling zeros and the absence of these zeros indicate that the system is controllable. The connection with theorem (5.3.6) is therefore immediate.

## §4. Generalised state space systems.

The notions of controllability for generalised state space systems have been derived as natural extensions of the notions of controllability described in the previous section for regular state space systems. In particular the frequency domain notion of controllability has ben concerned with extending the concept of input decoupling zeros to include infinite zeros as well as finite zeros, whilst the time domain approach has been concerned with incorporating the impulsive motion associated with such systems. Each approach has revealed important characteristics of generalised state space systems and together have yielded notions of controllability which best reflect the properties of such systems. In the light of the results from regular state space systems it is not surprising that the notions of controllability defined in the frequency domain are related to the notions defined in the time domain.

Rosenbrock [1974] was the first to seriously consider the properties of generalised state space systems using the frequency domain approach. His definition of infinite input decoupling zeros was based on Kronecker's work on infinite elementary divisors. Despite yielding some neat results the definition was regarded by Rosenbrock [1974] as being unsatisfactory. Verghese *et al.* [1981] pointed out that the deficiency was due to the fact that Rosenbrock [1974] had not taken into sufficient account the dynamical properties of the system. The definition of infinite input decoupling zeros adopted by Verghese *et al.* [1981] overcame the deficiencies associated with the corresponding definition of Rosenbrock [1974]. The superiority of the new definition subsequently lead to the natural reconciliation of McMillan degree theory with the generalised case, an extension Rosenbrock [*ibid.*] failed to achieve under his definition.

A time domain notion of controllability was given by Yip and Sincovec [1981] in terms of the reachable states that could be attained from a particular set of initial states. These initial states were constrained to satisfy the defining equations (5.1.1) of the system thus ruling out any impulsive motion. This definition of controllability therefore failed to incorporate a characteristic of generalised state space systems which distinguishes it from the regular case. The definition is seen to be equivalent to the absence of finite decoupling zeros and infinite decoupling zeros as defined by Rosenbrock [1974].

Cobb [1984] accounted for the possible impulsive motion and basing his ideas on the frequency domain work by Verghese *et al.* [1981] produced an explicit mathematically precise time domain formulation of controllability which proves to be equivalent to the absence of infinite input decoupling zeros as defined by Verghese *et al.* [*ibid.*]. This definition is given in terms of the system's ability to generate a maximal class of impulses using non-impulsive controls and is not directly related to the ability of the system to reach certain states. Lewis and Ozcaldiran [1984] later introduced definitions of controllability and reachability in terms of the states of the system which proved to be equivalent to the absence of infinite input decoupling zeros as defined by Rosenbrock [1974] and by Verghese *et al.* [1981] respectively.

A more detailed description of this development is now presented.

**(i) The frequency domain.**

Rosenbrock [1974] was the first to formulate a new theory for generalised state space systems. This was based on extending the theory associated with the regular case to cover the infinite frequency behaviour associated with the generalised system. To this end Rosenbrock [*ibid.*] introduced the transformation of restricted system equivalence which preserves the infinite frequency behaviour of the system (as well as the finite frequency behaviour) and defined the decoupling zeros at infinity. In particular the infinite input decoupling zeros are defined as follows.

**(5.4.1) Definition.** The INFINITE INPUT DECOUPLING ZEROS of (5.1.1), (5.1.2) are given by the finite zeros at $s = 0$ of $[E - sA \quad B]$.

Note that when the system is represented in Kronecker form (5.2.5) the infinite input decoupling zeros are given by the finite zeros at $s = 0$ of $[J - sI_{n_2} \quad B_2]$. Using these definitions Rosenbrock [*ibid.*] was able to extend previous results for the regular case to the generalised case; notably results concerning the Kalman decomposition and equivalence theorems.

Verghese *et al.* [1981] identified the weaknesses in Rosenbrock's definition and proposed new definitions based on the dynamic properties of the system which provided a greater extension of results from the regular case to the generalised system. Verghese *et al.* [*ibid.*] distinguished between what are termed dynamic and non-dynamic variables. The non-dynamic variables are referred to as the generalised state variables associated with the trivial Jordan blocks of $J$ in (5.2.8) and are regarded as non-dynamic in the sense that the initial conditions on these variables have no affect on the future response of the system and the behaviour of these variables are instantly determined by the input alone. The only significant contribution of these non-dynamic variables is apparently their contribution to a constant feedthrough term in the associated transfer function matrix. The approach adopted by Rosenbrock [1974] makes no distinction between the dynamic and non-dynamic variables and so is thus seen to be too restrictive. This is illustrated by considering a trivial augmentation of (5.1.1) with equations of the form

$$x_j(t) = 0 \qquad j = n+1, n+2, \ldots, n'$$

resulting in a system matrix given by

$$\left[\begin{array}{cc|c} sE - A & 0 & B \\ 0 & I_{n'-n} & 0 \\ \hline -C & 0 & 0 \end{array}\right]. \tag{5.4.2}$$

The new system matrix is not restricted system equivalent to the original matrix despite the fact that it possesses the same dynamical properties and is considered under the definition given by Rosenbrock [1974] to have, in particular, $n' - n$ more infinite input decoupling zeros than the original system.

By taking these dynamical observations into consideration Verghese *et al.* [1981] modified the theory presented by Rosenbrock [1974]. A new equivalence relationship, strong equivalence, was defined which, in general terms, permits transformations in addition to those of restricted system equivalence which eliminate or add non-dynamic variables to the system (provided of course that these operations do not modify the constant term in the system transfer function). A closed form expression of strong equivalence directly in terms of the system matrices involved was subsequently presented by Pugh *et al.* [1987]. The transformation implies that the systems (5.4.3), (5.4.4) are equivalent and overcomes one of the inherent deficiencies of the restricted system equivalence definition.

$$\left[\begin{array}{cc|c} sE-A & 0 & B \\ 0 & I & B' \\ \hline -C & -C' & 0 \end{array}\right] \tag{5.4.3}$$

$$\left[\begin{array}{c|c} sE-A & B \\ \hline -C & C'B' \end{array}\right] \tag{5.4.4}$$

The notion of controllability for generalised state space was approached by Verghese *et al.* [1981] by considering the interpretation of controllability in terms of the excitation of the natural modes of the system. In contrast to the regular state space system the generalised state space system possesses impulsive natural modes, and is defined to be controllable at infinity by Verghese *et al.* [*ibid.*] if all these impulsive modes can be individually excited from zero initial conditions by means of a non-impulsive input. The uncontrollable impulsive modes are associated with the infinite zeros as defined by (3.2.5) of the matrix pencil

$$[sE - A \quad B]$$

(or of $[sJ - I \quad B_2]$ if the system is represented in Kronecker form) and are termed infinite input decoupling zeros.

This definition of infinite input decoupling zeros differs from the definition of infinite input decoupling zeros presented by Rosenbrock [1974] in the sense that it ignores the non-dynamic variables of the system. Thus, the system represented by (5.4.2) would have the same number of infinite input decoupling zeros as the original system (5.1.1).

The term strongly controllable system is used by Verghese *et al.* [1981] to describe a system which is both controllable at infinity and controllable in the finite sense, i.e. a system which does not possess either finite input decoupling or infinite input decoupling zeros as defined by Verghese *et al.* [*ibid.*].

**(ii) Time Domain.**

Lewis and Ozcaldiran [1984] adopted for generalised state space systems the definitions of controllability and reachability associated with the regular case.

**(5.4.5) Definition.** The system (5.1.1) is said to be REACHABLE if for all $\xi \in \Re^n$ there exists a control such that the solution $x(t)$ is continuously differentiable and satisfies $x(0-) = 0,\ x(\tau) = \xi$ for some $\tau > 0$.

**(5.4.6) Definition.** The system (5.1.1) is said to be CONTROLLABLE if for all $\xi \in \Re^n$ there exists a control $u(t)$ such that the solution $x(t)$ is continuously differentiable and satisfies $x(0-) = \xi,\ x(\tau) = 0$ for some $\tau > 0$.

The above definitions lead to appropriate physical interpretation of the absence of input decoupling zeros. In particular, reachability as defined by (5.4.5) is equivalent to the absence of both finite input decoupling zeros and infinite input decoupling zeros as defined by Rosenbrock [1974], whilst controllability as defined by (5.4.6) is equivalent to the absence of both finite input decoupling zeros and infinite input decoupling zeros as defined by Verghese *et al.* [1981]. The terms REACHABILITY AT INFINITY and CONTROLLABILITY AT INFINITY refer to the reachability and controllability (as defined above) respectively for the subsystem (5.2.8) when (5.1.1) is represented in Kronecker form. It then follows that the system is reachable at infinity if it has no infinite input decoupling zeros as defined by Rosenbrock [1974] and is controllable at infinity if it has no infinite input decoupling zeros as defined by Verghese *et al.* [1981].

Other notions of reachability and controllability for systems in generalised state space form are, in general, equivalent to the notions defined by (5.4.5) and (5.4.6). For instance the definition of $R$-controllability introduced by Yip and Sincovec [1981] is equivalent to the notion of reachability as defined by (5.4.5). In their discussion Yip and Sincovec [*ibid.*] assumed that the initial conditions on the internal states satisfied the defining equations (5.1.1), and called the set of such initial conditions the set of admissible conditions. The notion of $R$-controllability is then associated with the ability of the system to transfer from any admissible initial condition to any final state in a set of states (this set is in fact equivalent to the set of admissible initial conditions). The requirement that the initial conditions satisfy the defining equations imply that the solution to (5.1.1) does not contain any impulses. Thus, a distinguishing characteristic of generalised state space systems, namely that the system can give rise to impulsive motion, has in a sense been ignored in the definition of $R$-controllability.

Cobb [1984], on the other hand, takes into account the impulsive behaviour of the system. His definition is not based on the ability of the system to reach certain states but on the ability of the system to generate a maximal class of impulses using piecewise smooth non-impulsive controls. Systems which satisfy this condition are said to be impulse controllable. The definition of impulse controllability emanates from the idea of modal controllability introduced by Verghese *et al.* [1981] in the frequency domain. Cobb [1984] gives the time domain definition of these ideas so that impulse controllability is seen to be equivalent to the absence of infinite input decoupling zeros as defined by Verghese *et al.* [1981] and hence to the notion of controllability at infinity.

The notions of controllability and reachability discussed have been seen to fall into two categories which have a direct relationship with the absence of infinite input decoupling zeros as defined by Rosenbrock [1974] and by Verghese *et al.* [1981]. The previous discussion of the frequency domain approach highlighted the differences between the dynamic and non-dynamic variables. This distinction is not reflected in the respective time domain definitions of controllability and reachability associated with the absence of infinite input decoupling zeros. To illustrate the difference between the dynamic and non-dynamic variables in the time domain the following alternative definition of reachability is introduced. This definition is given in terms of the subsystem (5.2.8) when the system is represented in Kronecker form.

**(5.4.7) Definition.** The system represented in Kronecker form (5.2.5), is said to be SYSTEM STATE REACHABLE AT INFINITY if given any $\xi \in \Re^{n_2 - t}$ there exists a suitable control such that $x_2(0-) = 0$, $x_2(\tau) = (\xi^T, \eta^T)^T$ for some $\tau > 0$ and where $\eta \in \Re^t$ is completely arbitrary.

The definition of system state reachability at infinity will be seen in the following section to be equivalent to the definition of controllability at infinity. Comparison of this definition of system state reachability at infinity with the definition of reachability at infinity illustrates clearly the role of the dynamic and non-dynamic variables. This is further discussed in section 6.

## §5. Algebraic results associated with the controllability notions in generalised state space systems.

In this section a generalisation of the algebraic conditions contained in theorem (5.3.6) for the regular case is presented for the notions of reachability at infinity and system state reachability at infinity. Certain of these conditions have been previously obtained and are summarised by Lewis [1986]. A polynomial system approach is adopted to provide new proofs to these existing results. This is followed, for both notions, by new results which together with the existing results provide a complete analogy to the algebraic results associated with the regular case.

In the light of the discussion in section 2 it will be sufficient to consider the system

$$J\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{5.5.1}$$

with $n_2 \times n_2$ matrix $J$ in Jordan normal form (5.2.3). First consider the notion of reachability at infinity.

**(5.5.2) Theorem (Lewis, 1986).** For the system represented by (5.5.1) the following conditions are equivalent

(a) the system is reachable at infinity,
(b) rank $[B_2, JB_2, \ldots, J^{q_1-1}B_2] = n_2$,
(c) $v^T[sJ - I]^{-1}B_2 = 0$ for constant $v$ implies $v = 0$,
(d) the last position rows of $B_2$ are all linearly independent,
(e) the system has no infinite input decoupling zeros as defined by Rosenbrock [1974],
(f) rank $[J \quad B_2] = n_2$.

**Proof.** The proof of this theorem will follow the indicated scheme which best illustrates the relationships between the various conditions.

$$\begin{tikzcd}
(a) & & & (e) \arrow[d, Rightarrow] \\
& (b) \arrow[ul, Leftrightarrow] \arrow[r, Leftrightarrow] \arrow[dl, Leftrightarrow] & (d) \arrow[ur, Rightarrow] & (f) \arrow[l, Rightarrow] \\
(c) & & &
\end{tikzcd}$$

**(a) $\Longleftrightarrow$ (b).** The solution to equations (5.5.1) is given by

$$x_2(t) = -\sum_{i=1}^{q_1-1} \delta^{(i-1)} J^i x_2(0-) - \sum_{i=0}^{q_1-1} J^i B_2 u^{(i)}(t).$$

Thus, the system will be reachable at infinity if and only if given $\alpha \in \Re^{n_2}$ and $\tau > 0$ there exists a suitable control such that

$$\alpha = -\sum_{i=0}^{q_1-1} J^i B_2 u^{(i)}(\tau)$$

i.e.

$$\alpha = -[B_2, JB_2, \ldots, J^{q_1-1}B_2] \begin{bmatrix} u(\tau) \\ u^{(1)}(\tau) \\ \vdots \\ u^{(q_1-1)}(\tau) \end{bmatrix}.$$

The system will therefore be reachable at infinity if and only if

$$\mathrm{rank}[B_2, JB_2, \ldots, J^{q_1-1}B_2] = n_2$$

as required.

**(b) $\Longleftrightarrow$ (c).**

$$\begin{aligned} & \mathrm{rank}[B_2, JB_2, \ldots, J^{q_1-1}B_2] = n_2 \\ \Longleftrightarrow\ & v^T B_2 = v^T JB_2 = \ldots = v^T J^{q_1-1} B_2 = 0 \text{ implies } v = 0 \\ \Longleftrightarrow\ & v^T B_2 + s v^T JB_2 + \ldots + v^T s^{q_1-1} J^{q_1-1} B_2 = 0 \text{ implies } v = 0 \\ \Longleftrightarrow\ & v^T [sJ - I]^{-1} B_2 = 0 \text{ implies } v = 0, \text{ as required.} \end{aligned}$$

**(b) $\Longleftrightarrow$ (d).** Consider in detail the structure of $JB_2$, where $\eta_i = \sum_{j=1}^{i} q_j$, $i = 1, 2, \ldots, p$, and $b_i^T$, $i = 1, 2, \ldots, n_2$, are the rows of $B_2$.

$$JB_2 = \begin{bmatrix} 0 & 1 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & \ddots & 1 & & & & \mathbf{0} & & \\ & & & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & 1 & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & \ddots & 1 & \\ & \mathbf{0} & & & & & & & 0 & \\ & & & & & & & & & 0 \\ & & & & & & & & & & \ddots \\ & & & & & & & & & & & 0 \end{bmatrix} \begin{bmatrix} b_1^T \\ \vdots \\ b_{\eta_1}^T \\ \vdots \\ b_{\eta_{p-t-1}+1}^T \\ \vdots \\ b_{\eta_{p-t}}^T \\ b_{\eta_{p-t+1}}^T \\ \vdots \\ b_{\eta_p}^T \end{bmatrix} = \begin{bmatrix} b_2^T \\ \vdots \\ b_{\eta_1}^T \\ 0 \\ \vdots \\ b_{\eta_{p-t-1}+2}^T \\ \vdots \\ b_{\eta_{p-t}}^T \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly

$$J^2 B_2 = \begin{bmatrix} b_3^T \\ \vdots \\ b_{\eta_1}^T \\ 0 \\ 0 \\ | \\ | \\ | \\ b^T_{\eta_{p-t-1}+3} \\ \vdots \\ b^T_{\eta_{p-t}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

and for $J^i B_2,\ i = 3, 4, \ldots, q_1 - 1$, so that

$$[B_2.\ JB_2, \ldots,\ J^{q_1-1} B_2] = \begin{bmatrix} b_1^T & b_2^T & b_3^T & b^T_{\eta_1-1} & b^T_{\eta_1} \\ \vdots & \vdots & \vdots & b^T_{\eta_1} & \\ \vdots & \vdots & b^T_{\eta_1} \ \ddots & & \\ b^T_{\eta_1-1} & b^T_{\eta_1} & 0 & & \\ b^T_{\eta_1} & 0 & 0 & & \\ b^T_{\eta_1+1} & b^T_{\eta_1+2} & b^T_{\eta_1+3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & & b^T_{\eta_2} & & \\ \vdots & b^T_{\eta_2} & 0 & & \\ b^T_{\eta_2} & 0 & 0 & & \\ | & | & | & & \\ b^T_{\eta_{p-t-1}+1} & b^T_{\eta_{p-t-1}+2} & b^T_{\eta_{p-t-1}+3} & & \\ & \vdots & \vdots & \ddots & \\ \vdots & & b^T_{\eta_{p-t}} & & \\ & b^T_{\eta_{p-1}} & 0 & & \\ b^T_{\eta_{p-t}} & 0 & 0 & \cdots & 0 \\ b^T_{\eta_{p-t+1}} & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ b^T_{\eta_p} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{5.5.3}$$

with the number of non-zero rows of $J^{q_1-1}B_2$ being equal to the number of Jordan blocks of $J$ of size $q_1$. Reorder the rows of (5.5.3) such that the first $t_1$ say rows are the rows of (5.5.3) which have non-zero entries in block $J^{q_1-1}B_2$, the next $t_1+t_2$ rows are the rows of (5.5.3) which have non-zero entries in block $J^{q_1-2}B_2$ and so on down to the last $p$ rows which are the rows of (5.5.3) with non-zero entries in block $B_2$ only, i.e.

$$\begin{bmatrix}
b_1^T & \cdots\cdots & b^T_{\eta_1-1} & b^T_{\eta_1} \\
\vdots & & \vdots & \vdots \\
\vdots & & b^T_{\eta_{t_1}-1} & b^T_{\eta_{t_1}} \\
\vdots & & b^T_{\eta_1} & 0 \\
\vdots & & \vdots & \vdots \\
\vdots & & b^T_{\eta_{t_1}} & \vdots \\
\vdots & & b^T_{\eta_{t_1+1}} & \vdots \\
\vdots & & \vdots & \vdots \\
\vdots & & b^T_{\eta_{t_1+t_2}} & \vdots \\
\vdots & & 0 & \vdots \\
\vdots & & \vdots & \vdots \\
b^T_{\eta_1} & 0 \cdots\cdots & 0 & 0 \\
b^T_{\eta_2} & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
b^T_{\eta_{p-t}} & \vdots & \vdots & \vdots \\
b^T_{\eta_{p-t+1}} & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
b^T_{\eta_p} & 0 \cdots\cdots & 0 & 0
\end{bmatrix} . \qquad (5.5.4)$$

Now assume (b) holds then, since $[B_2, JB_2, \ldots, J^{q_1-1}B_2]$ is an $n_2 \times \ell q_1$ matrix of rank $n_2$, it follows that all its rows must be linearly independent. In particular the last $p$ rows of (5.5.4) must be linearly independent, which gives (d). Conversely, assume (d) holds then, since the vectors $b^T_{\eta_1}, b^T_{\eta_2}, \ldots, b^T_{\eta_p}$ are linearly independent, it follows from (5.5.4) that $[B_2, JB_2, \ldots, J^{q_1-1}B_2]$ has rank $n_2$ , to give (b) as required.

**(d) $\Rightarrow$ (e).** The infinite input decoupling zeros of the system as defined by Rosenbrock [1974] are given by the zeros at $s = 0$ of $[J - sI \quad B_2]$, where

$$
[J - sI \quad B_2]_{s=0} = \left[\begin{array}{cccccccccc|c}
0 & 1 & & & & & & & & & b_1^T \\
 & \ddots & \ddots & & & & & & & & \vdots \\
 & & \ddots & 1 & & & & & & & b_{\eta_1-1}^T \\
 & & & 0 & & & \mathbf{0} & & & & b_{\eta_1}^T \\
 & & & & \ddots & & & & & & \vdots \\
 & & & & & 0 & 1 & & & & b_{\eta_{p-t-1}+1}^T \\
 & & & & & & \ddots & \ddots & & & \vdots \\
 & & & & & & & \ddots & 1 & & b_{\eta_{p-t-1}}^T \\
 & \mathbf{0} & & & & & & & 0 & & b_{\eta_{p-t}}^T \\
 & & & & & & & & & 0 & b_{\eta_{p-t+1}}^T \\
 & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & 0 & b_{\eta_p}^T
\end{array}\right] \qquad (5.5.5)
$$

and $\eta_i = \sum_{j=1}^{i} q_j, \; i = 1, 2, \ldots, p$.

Now since the last position rows of $B_2$ corresponding to all the Jordan blocks of $J$ are assumed to be linearly independent, i.e. rows $b_{\eta_1}^T, b_{\eta_2}^T, \ldots, b_{\eta_p}^T$, it follows by inspection of (5.5.5) that $[J - sI \quad B_2]_{s=0}$ has full rank. Thus, the system has no infinite input decoupling zeros as defined by Rosenbrock [1974], as required.

**(e) $\Rightarrow$ (f).** Since the system has no infinite input decoupling zeros as defined by Rosenbrock [*ibid.*] it follows that $[J - sI \quad B_2]$ has full rank when $s = 0$, i.e. rank $[J \quad B_2] = n_2$, as required.

**(f) $\Rightarrow$ (d).** Consider the matrix $[J \quad B_2]$ as represented by (5.5.5). Since $[J \quad B_2]$ has rank $n_2$ all the rows of (5.5.5) are linearly independent. In particular rows $q_1, q_1 + q_2, \ldots, q_1 + q_2 + \ldots + q_p$ are linearly independent which are the last position rows of $B_2$ corresponding to all the Jordan blocks of $J$, as required to complete the proof. □

The adoption of a polynomial matrix approach is therefore seen to provide a unified treatment of the algebraic conditions summarised by Lewis [1986]. New algebraic conditions associated with a system that is reachable at infinity are now presented. These conditions are analogous to conditions (d) and (e) of theorem (5.3.6).

**(5.5.6) Theorem.** If $q_1$ is the index of nilpotency of the matrix $J$, then the condition that

$$\operatorname{rank}[B_2, JB_2, \ldots, J^{q-1}B_2] = n_2$$

where $q \geq q_1$ is equivalent to

(a) the $qn_2 \times [(q-1)n_2 + q\ell]$ matrix $R$ has rank $qn_2$ where

$$R = \begin{bmatrix} J & & & & & & B_2 \\ -I & \ddots & & 0 & & \cdot^{\cdot^{\cdot}} & \\ & \ddots & \ddots & & \cdot^{\cdot^{\cdot}} & & \\ 0 & & J & & B_2 & & 0 \\ & & -I & B_2 & & & \end{bmatrix}, \tag{5.5.7}$$

(b) given any polynomial $n_2$-vector $d(s)$ with elements of degree $q-1$ or less there exists a polynomial $n_2$-vector $x(s)$ with elements of degree $q-2$ or less and a polynomial $\ell$-vector $y(s)$ with elements of degree $q-1$ or less such that

$$[sJ - I]x(s) + B_2 y(s) = d(s). \tag{5.5.8}$$

[Note that for the case when $q_1 = 1$ and $q$ is taken to be such that $q = q_1$ then the degree of $x(s)$ in (b) is equal to 0.]

**Proof.** (a) In the matrix $R$, add $J$ times the first (block) column to the second, then $J$ times the second (block) column to the third and so on to give

$$\begin{bmatrix} J & J^2 & \cdots & J^{q-1} & & & B_2 \\ -I & & & & & \cdot^{\cdot^{\cdot}} & \\ & \ddots & & 0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & -I & & B_2 & & 0 \\ & & & -I & B_2 & & \end{bmatrix}. \tag{5.5.9}$$

Next, add $J$ times the second row of (5.5.9) to the first row, then $J^2$ times the third row of (5.5.9) to the first row and so on to give

$$\begin{bmatrix} 0 & \cdots & 0 & J^{q-1}B_2 & \cdots & JB_2 & B_2 \\ -I & & & & & B_2 & \\ & \ddots & & 0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & \ddots & & \cdot^{\cdot^{\cdot}} & & 0 \\ & & -I & B_2 & & & \end{bmatrix}. \tag{5.5.10}$$

The above operations preserve the rank of $R$ so that from (5.5.10) $R$ will have rank $qn_2$ if and only if $[J^{q-1}B_2, \ldots, JB_2, B_2]$ has rank $n_2$, as required.

(b) Let

$$x(s) = x_0 + x_1 s + \ldots + x_{q-2}s^{q-2}$$

$$y(s) = y_0 + y_1 s + \ldots + y_{q-1}s^{q-1}$$

$$d(s) = d_0 + d_1 s + \ldots + d_{q-1}s^{q-1}$$

and substitute into (5.5.8) to give

$$(sJ - I)(x_0 + x_1 s + \ldots + x_{q-2}s^{q-2}) + B_2(y_0 + y_1 s + \ldots + y_{q-1}s^{q-1})$$
$$= d_0 + d_1 s + \ldots + d_{q-1}s^{q-1}.$$

Multiplying out the products, and equating powers of $s$ results in the following set of equations.

$$\begin{array}{lllllll}
Jx_{q-2} & & & & + & B_2 y_{q-1} & = & d_{q-1} \\
-x_{q-2} & + Jx_{q-3} & & & + & B_2 y_{q-2} & = & d_{q-2} \\
& & \ddots & & & \vdots & & \vdots \\
& & & -x_1 + Jx_0 & + & B_2 y_1 & = & d_1 \\
& & & -x_0 & + & B_2 y_0 & = & d_0 .
\end{array}$$

Rewriting the above set of equations in matrix form

$$R \begin{bmatrix} x_{q-2} \\ \vdots \\ x_0 \\ y_0 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} d_{q-1} \\ \vdots \\ d_0 \end{bmatrix} \tag{5.5.11}$$

where $R$ is of the form (5.5.7). The equations (5.5.11) will have a solution for all $d_i \in \Re$, $i = 0, 1, \ldots, q-1$, if and only if $R$ has rank $qn_2$, as required. □

In the regular state space case the analogous conditions to those in theorem (5.5.6) lead to a third necessary and sufficient condition (Rosenbrock, 1968). However, in the generalised case the corresponding condition is only sufficient. This condition is presented below followed by an example to illustrate that the condition is not necessary.

**(5.5.12) Theorem.** If rank $[B_2, JB_2, \ldots, J^{q-1}B_2] = n_2$ where $q \geq q_1$ then there exists an $n_2 \times n_2$ polynomial matrix $X(s)$ with elements of degree $q-2$ or less and an $\ell \times n_2$ polynomial matrix $Y(s)$ with elements of degree $q-1$ or less such that

$$(sJ - I)X(s) + B_2Y(s) = I_{n_2}. \tag{5.5.13}$$

**Proof.** If $e_i$ is the $i^{\text{th}}$ column of $I_{n_2}$, put $d(s) = e_i$ in (5.5.8). Let the corresponding solution be $x^{(i)}(s)$ and $y^{(i)}(s)$. If $X(s), Y(s)$ are the matrices having $x^{(i)}(s)$ and $y^{(i)}(s)$ as their respective columns then $X(s), Y(s)$ satisfy (5.5.13) as required. □

**(5.5.14) Example.** Take

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then

$$[B_2 \quad JB_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which has rank 1.

Let

$$X(s) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Y(s) = [1 \quad s]$$

where both $X(s)$, $Y(s)$ satisfy the degree condition of theorem (5.5.12). Then,

$$[sJ - I]X(s) + B_2Y(s) = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad s]$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, there exist suitable $X(s), Y(s)$ satisfying (5.5.13) despite rank $[B_2, JB_2, \ldots, J^{q-1}B_2]$ being less then $n_2$ in this case.

The results of theorems (5.5.6) and (5.5.12) together with theorem (5.5.2) provide an analogy of the algebraic conditions of theorem (5.3.6) for the notion of reachability at infinity in generalised state space systems.

A similar generalisation is now undertaken for the notion of system state reachability at infinity. Lewis [1986] summarised the existing algebraic conditions associated with systems that are controllable at infinity. These algebraic conditions are now shown to be equivalent to the notion of system state reachability at infinity.

**(5.5.15) Theorem.** For the system represented by (5.5.1) the following conditions are equivalent

(a) the system is system state reachable at infinity,

(b) the system has no input decoupling zeros at infinity as defined by Verghese *et al.* [1981],

(c) the last position rows of $B_2$ corresponding to the non-trivial Jordan blocks of $J$ are linearly independent,

(d) rank $[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2] = n_2 - p$,

(e) dim $\{\text{span } v \in \Re^{n_2};\ v^T J[sJ - I]^{-1} B_2 = 0\} = n_2 - p$.

**Proof.** The proof of this theorem will follow the indicated scheme.

$$\begin{tikzcd}
(a) \arrow[dr, Leftrightarrow] & & & \\
& (c) \arrow[r, Leftrightarrow] & (d) \arrow[r, Leftrightarrow] & (e) \\
(b) \arrow[ur, Leftrightarrow] & & &
\end{tikzcd}$$

**(a) $\Longleftrightarrow$ (c).** The states of the system at time $\tau$ with zero initial conditions are given by

$$x_2(\tau) = -[B_2 u(\tau) + JB_2 u^{(1)}(\tau) + \ldots + J^{q_1-1} B_2 u^{(q_1-1)}(\tau)]$$

or, in matrix form,

$$x_2(\tau) = -[B_2, JB_2, \ldots, J^{q_1-1}B_2] \begin{bmatrix} u(\tau) \\ u^{(1)}(\tau) \\ \vdots \\ u^{(q_1-1)}(\tau) \end{bmatrix}.$$

The detailed structure of $[B_2, JB_2, \ldots, J^{q_1-1}B_2]$ was seen in (5.5.3) to be given by

$$
\begin{bmatrix}
b_1^T & b_2^T & \cdots & b_{\eta_1-1}^T & b_{\eta_1}^T \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & b_{\eta_1}^T & \\
 & b_{\eta_1}^T & & & \\
b_{\eta_1}^T & 0 & & & \\
b_{\eta_1+1}^T & b_{\eta_1+2}^T & & & \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & b_{\eta_2}^T & & & \\
b_{\eta_2}^T & 0 & & & \\
\vdots & \vdots & & & \\
b_{\eta_{p-t-1}+1}^T & b_{\eta_{p-t-1}+2}^T & & & \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & b_{\eta_{p-t}}^T & & & \\
b_{\eta_{p-t}}^T & 0 & & & \\
b_{\eta_{p-t+1}}^T & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
b_{\eta_p}^T & 0 & \cdots & \cdots & 0
\end{bmatrix}
\tag{5.5.16}
$$

where $\eta_i = \sum_{j=1}^{i} q_j, \ i = 1, 2, \ldots, p$.

The system will then be system state reachable at infinity if and only if there exists a suitable control input such that for all $\alpha \in \Re^{n_2 - t}$

$$
\alpha = F \begin{bmatrix} u(\tau) \\ u^{(1)}(\tau) \\ \vdots \\ u^{(q_1-1)}(\tau) \end{bmatrix}
$$

where

$$F = \begin{bmatrix}
b_1^T & b_2^T & & b_{\eta_1 -1}^T & b_{\eta_1}^T \\
 & \vdots & & b_{\eta_1}^T & \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & b_{\eta_1}^T & & & \\
b_{\eta_1}^T & 0 & & & \\
b_{\eta_1 +1}^T & b_{\eta_1 +2}^T & & & \\
 & \vdots & & \cdot^{\cdot^{\cdot}} & \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & b_{\eta_2}^T & & & \\
b_{\eta_2}^T & 0 & & & \\
\vdots & \vdots & & & \\
b_{\eta_{p-t-1}+1}^T & b_{\eta_{p-t-1}+2}^T & & & \\
 & \vdots & & \cdot^{\cdot^{\cdot}} & \\
\vdots & \vdots & \cdot^{\cdot^{\cdot}} & & \\
 & b_{\eta_{p-t}}^T & & & \\
b_{\eta_{p-t}}^T & 0 & & &
\end{bmatrix}.$$

This will be the case if and only if rank $F = n_2 - t$. Reordering the rows of $F$ to form a matrix $F'$ where the first $t_1$ say rows of $F'$ are the rows of $F$ which have a non-zero entry in any of the last $\ell$ positions, the next $t_1 + t_2$ rows of $F'$ are the rows of $F$ which have zero entries in the last $\ell$ positions but a non-zero entry in any of the preceding $\ell$ positions, and so on down to the last $p - t$ rows which correspond to the rows of $F$ which only have a non-zero entry in one of the first $\ell$ positions, i.e.

$$
F' = \begin{bmatrix}
b_1^T & \cdots & \cdots & \cdots & b_{\eta_1 -1}^T & b_{\eta_1}^T \\
\vdots & & & & \vdots & \vdots \\
\vdots & & & & b_{\eta_{t_1}-1}^T & b_{\eta_{t_1}}^T \\
\vdots & & & & b_{\eta_1}^T & 0 \\
\vdots & & & & \vdots & \vdots \\
\vdots & & & & b_{\eta_{t_1}}^T & \vdots \\
\vdots & & & & b_{\eta_{t_1+1}}^T & \vdots \\
\vdots & & & & \vdots & \vdots \\
\vdots & & & & b_{\eta_{t_1+t_2}}^T & \vdots \\
\vdots & & & & 0 & \vdots \\
\vdots & & & 0 & \vdots & \vdots \\
b_{\eta_1}^T & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & & & \vdots & \vdots \\
b_{\eta_{p-t}}^T & 0 & \cdots & \cdots & 0 & 0
\end{bmatrix}.
$$

Now if rank $F' = n_2 - t$ then since $F'$ is an $(n_2 - t) \times \ell q_1$ matrix it follows that all its rows must be linearly independent. In particular the last $p - t$ rows of $F'$ must be linearly independent which implies that the last position rows of $B_2$ corresponding to the non-trivial Jordan blocks of $J$ are linearly independent. If on the other hand the last position rows of $B_2$ corresponding to the non-trivial Jordan blocks of $J$ are linearly independent then, from the structure of $F'$, it follows that rank $F' = n_2 - t$. Since $F'$ has the same rank as $F$ it is therefore concluded that the system will be system state reachable at infinity if and only if the last position rows of $B_2$ corresponding to the non-trivial Jordan blocks of $J$ are linearly independent, as required.

**(b)** $\Longleftrightarrow$ **(c)**. The infinite input decoupling zeros of a system are defined as the infinite zeros of the pencil $[sJ - I \quad B_2]$ which in turn are given by the zeros at $w = 0$ of $[\frac{1}{w}J - I \quad B_2]$ [see definition (3.2.5)]. Now

$$
\left[\tfrac{1}{w}J - I \quad B_2\right] = \begin{bmatrix}
-1 & \frac{1}{w} & & & & & & & & & b_1^T \\
 & \ddots & \ddots & & & & & & & & \vdots \\
 & & \ddots & \frac{1}{w} & & & & \mathbf{0} & & & \vdots \\
 & & & -1 & & & & & & & b_{\eta_1}^T \\
 & & & & \ddots & & & & & & \vdots \\
 & & & & & -1 & \frac{1}{w} & & & & b_{\eta_{p-t-1}+1}^T \\
 & & & & & & \ddots & \ddots & & & \vdots \\
 & & \mathbf{0} & & & & & \ddots & \frac{1}{w} & & \vdots \\
 & & & & & & & & -1 & & b_{\eta_{p-t}}^T \\
 & & & & & & & & & -1 & b_{\eta_{p-t+1}}^T \\
 & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & -1 & b_{\eta_p}^T
\end{bmatrix}
$$

which can be factorised as

$$[\tfrac{1}{w}J - I \quad B_2] = D^{-1}(w)N(w)$$

$$
= \begin{bmatrix}
w & & & & & & & & & \\
 & \ddots & & & & & & & & \\
 & & w & & & & \mathbf{0} & & & \\
 & & & 1 & & & & & & \\
 & & & & \ddots & & & & & \\
 & & & & & w & & & & \\
 & & & & & & \ddots & & & \\
 & & \mathbf{0} & & & & & w & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1 \\
 & & & & & & & & & & \ddots \\
 & & & & & & & & & & & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
-w & 1 & & & & & & & & & & b_1^T w \\
 & \ddots & \ddots & & & & & & & & & \vdots \\
 & & -w & 1 & & & & & & & & b_{\eta_1-1}^T w \\
 & & & -1 & & & & \mathbf{0} & & & & b_{\eta_1}^T \\
 & & & & \ddots & & & & & & & \vdots \\
 & & & & & -w & 1 & & & & & b_{\eta_{p-t-1}+1}^T w \\
 & & & & & & \ddots & \ddots & & & & \vdots \\
 & & & & & & & -w & 1 & & & b_{\eta_{p-t}-1}^T w \\
 & & \mathbf{0} & & & & & & -1 & & & b_{\eta_{p-t}}^T \\
 & & & & & & & & & -1 & & b_{\eta_{p-t+1}}^T \\
 & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & -1 & b_{\eta_p}^T
\end{bmatrix}.
$$

The factorisation is a minimal one and so, by lemma (3.2.6), the finite zero structure of $[\frac{1}{w}J - I \quad B_2]$ is given by the finite zero structure of $N(w)$. Now, since rank $N(w) = n_2$, it follows that $N(w)$ does not possess a zero at $w = 0$ if and only if rank $N(0) = n_2$, where

$$N(0) = \left[\begin{array}{ccccccccc}
0 & 1 & & & & & & & 0 \\
 & \ddots & \ddots & & & & & & \vdots \\
 & & 0 & 1 & & & & & 0 \\
 & & & -1 & & \mathbf{0} & & & b_{\eta_1}^T \\
 & & & & \ddots & & & & \vdots \\
 & & & & & 0 & 1 & & 0 \\
 & & & & & & \ddots & \ddots & \vdots \\
 & & & & & & & 1 & 0 \\
 & & \mathbf{0} & & & & & -1 & b_{\eta_{p-t}}^T \\
 & & & & & & & -1 & b_{\eta_{p-t+1}}^T \\
 & & & & & & & \ddots & \vdots \\
 & & & & & & & -1 & b_{\eta_p}^T
\end{array}\right]. \tag{5.5.17}$$

Adding row $q_1 + q_2 + \ldots + q_j - 1$ to row $q_1 + q_2 + \ldots + q_j$ for $j = 1, 2, \ldots, p - t$ in (5.5.17) results in

$$\left[\begin{array}{ccccccccc}
0 & 1 & & & & & & & 0 \\
 & \ddots & \ddots & & & & & & \vdots \\
 & & & 1 & & & & & 0 \\
 & & & 0 & & \mathbf{0} & & & b_{\eta_1}^T \\
 & & & & \ddots & & & & \vdots \\
 & & & & & 0 & 1 & & 0 \\
 & & & & & & \ddots & \ddots & \vdots \\
 & & & & & & & 1 & 0 \\
 & & \mathbf{0} & & & & & 0 & b_{\eta_{p-t}}^T \\
 & & & & & & & -1 & b_{\eta_{p-t+1}}^T \\
 & & & & & & & \ddots & \vdots \\
 & & & & & & & -1 & b_{\eta_p}^T
\end{array}\right].$$

Therefore, rank $N(0) = n_2$ if and only if the rows $b^T_{\eta_1}, b^T_{\eta_2}, \ldots, b^T_{\eta_{p-t}}$ are linearly independent, i.e. the last position rows of $B_2$ corresponding to the non-trivial Jordan blocks of $J$ are linearly independent , as required.

(c) $\Longleftrightarrow$ (d). From (5.5.16)

$[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2] =$

$$\begin{bmatrix}
b_2^T & b_3^T & & b_{\eta_1-1}^T & b_{\eta_1}^T \\
\vdots & \vdots & \iddots & b_{\eta_1}^T & \\
 & b_{\eta_1}^T & \iddots & & \\
b_{\eta_1}^T & 0 & & & \\
0 & 0 & & & \\
b_{\eta_1+2}^T & b_{\eta_1+3}^T & & & \\
\vdots & \vdots & \iddots & & \\
 & b_{\eta_2}^T & \iddots & & \\
b_{\eta_2}^T & 0 & & & \\
0 & 0 & & & \\
\vdots & \vdots & & & \\
b_{\eta_{p-t-1}+2}^T & b_{\eta_{p-t-1}+3}^T & & & \\
\vdots & \vdots & \iddots & & \\
 & b_{\eta_{p-t}}^T & \iddots & & \\
b_{\eta_{p-t}}^T & 0 & & & \\
0 & 0 & & & \\
0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
0 & 0 & \cdots & \cdots & 0
\end{bmatrix}. \tag{5.5.18}$$

First assume (c) holds. Then, from the structure of (5.5.18) it follows that the rank of $[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2]$ is equal to $n_2$ less the number of zero rows. On inspection the number of zero rows equals the number of Jordan blocks in $J$, i.e. $p$. Hence

$$\text{rank } [JB_2, J^2B_2, \ldots, J^{q_1-1}B_2] = n_2 - p.$$

Conversely assume (d) holds. Now since $[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2]$ is an $n_2 \times \ell(q_1 - 1)$ matrix of rank $n_2 - p$ it must posses $n_2 - p$ linearly independent rows. Again it is seen on inspection of (5.5.18) that $[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2]$ has $p$ zero rows which implies that the remaining $n_2 - p$ rows must be linearly independent. In particular rows $q_1 - 1, q_1 + q_2 - 1, \ldots, q_1 + q_2 + \ldots + q_{p-t} - 1$ are linearly independent to give (c). Hence (c), (d) are equivalent as required.

**(d) $\Longleftrightarrow$ (e).**

$$\text{rank}\,[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2] = n_2 - p$$

$$\Longleftrightarrow \dim\,\{\text{span } v \in \Re^{n_2}; v^T[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2] = 0\} = n_2 - p$$

$$\Longleftrightarrow \dim\,\{\text{span } v \in \Re^{n_2}; v^TJB_2 = v^TJ^2B_2 = \ldots = v^TJ^{q_1-1}B_2 = 0\} = n_2 - p$$

$$\Longleftrightarrow \dim\,\{\text{span } v \in \Re^{n_2};$$

$$0 = v^T\{J[I + sJ + \ldots + s^{q_1-2}J^{q_1-2} + s^{q_1-1}J^{q_1-1}]B_2\} = n_2 - p$$

$$\Longleftrightarrow \dim\,\{\text{span } v \in \Re^{n_2}; 0 = v^TJ[sJ - I]^{-1}B_2\} = n_2 - p.$$

Hence (d), (e) are equivalent as required to complete the proof. □

As a result of theorem (5.5.15) it clearly follows that the notion of system state reachability at infinity is equivalent to the notion of controllability at infinity.

New algebraic conditions associated with a system that is system state reachable at infinity are now presented. These are analogous to conditions (d) and (e) of theorem (5.3.6).

**(5.5.19) Theorem.** Let $q_1$, the index of nilpotency of the matrix $J$, be taken such that $q_1 \geq 2$ and let $q \geq q_1$. Then, the condition that

$$\text{rank}\,[JB_2, J^2B_2, \ldots, J^{q-1}B_2] = n_2 - p$$

is equivalent to

(a) the $(q-1)n_2 \times [(q-2)n_2 + (q-1)\ell]$ matrix $R$ having rank $(q-1)n_2 - p$ where

$$R = \begin{bmatrix} J & & & & & JB_2 \\ -I & \ddots & & 0 & \cdot^{\cdot^{\cdot}} & \\ & \ddots & \ddots & \cdot^{\cdot^{\cdot}} & & \\ 0 & & \ddots \; J & JB_2 & & 0 \\ & & -I & JB_2 & & \end{bmatrix}. \qquad (5.5.20)$$

(b) given any polynomial $n_2$-vector $d(s)$ with elements of degree $q-3$ or less there exists a polynomial $n_2$-vector $x(s)$ with elements of degree $q-3$ or less and a polynomial $\ell$-vector $y(s)$ with elements of degree $q-2$ or less such that

$$[sJ - I]x(s) + JB_2y(s) = d(s). \qquad (5.5.21)$$

[Note that when $q_1 = 2$ then $q$ is taken such that $q \geq q_1 + 1$ in (b).]

**Proof.** (a) In the matrix $R$, add $J$ times the first (block) column to the second, then $J$ times the second to the third and so on to give

$$\begin{bmatrix} J & J^2 & \cdots & J^{q-2} & & & JB_2 \\ -I & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & \ddots & 0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & -I & & JB_2 & & 0 \\ & & & -I & JB_2 & & \end{bmatrix}. \qquad (5.5.22)$$

Next, add $J$ times the second (block) row of (5.5.22) to the first row, then $J^2$ times the third row of (5.5.22) to the first row and so on to give

$$\begin{bmatrix} 0 & \cdots & 0 & J^{q-1}B_2 & \cdots & J^2B_2 & JB_2 \\ -I & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & \ddots & 0 & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \ddots & JB_2 & & 0 \\ & & -I & JB_2 & & & \end{bmatrix}. \qquad (5.5.23)$$

The matrix (5.5.23) has the same rank as $R$ which implies that $R$ has rank $(q-1)n_2 - p$ if and only if rank $[J^{q-1}B_2, \ldots, J^2B_2, JB_2] = n_2 - p$, as required.

(b) Let

$$x(s) = x_0 + x_1 s + \ldots + x_{q-3}s^{q-3}$$

$$y(s) = y_0 + y_1 s + \ldots + y_{q-2}s^{q-2}$$

$$d(s) = d_0 + d_1 s + \ldots + d_{q-3}s^{q-3}$$

and substitute into (5.5.21) to give

$$(sJ - I)(x_0 + x_1 s + \ldots + x_{q-3}s^{q-3}) + JB(y_0 + y_1 s + \ldots + y_{q-2}s^{q-2})$$
$$= d_0 + d_1 s + \ldots + d_{q-3}s^{q-3}.$$

Multiplying out the products, and equating powers of $s$ gives rise to the following set of equations.

$$
\begin{array}{lllllll}
J x_{q-3} & & & & + \; JB_2\, y_{q-2} & = & 0 \\
-x_{q-3} & + \; J x_{q-4} & & & + \; JB_2\, y_{q-3} & = & d_{q-3} \\
& & \ddots & & \vdots & \vdots & \vdots \\
& & & -x_1 \; + \; Jx_0 & + \; JB_2\, y_1 & = & d_1 \\
& & & -x_0 & + \; JB_2\, y_0 & = & d_0 \; .
\end{array}
$$

Rewriting the above set of equations in matrix form results in

$$
R \begin{bmatrix} x_{q-3} \\ \vdots \\ x_0 \\ y_0 \\ \vdots \\ y_{q-2} \end{bmatrix} = \begin{bmatrix} 0 \\ d_{q-3} \\ \vdots \\ d_0 \end{bmatrix} \tag{5.5.24}
$$

where $R$ is given by (5.5.20).

The equations (5.5.24) will have a solution for any set $d_i$, $i = 0, 1, \ldots, q-3$, if and only if

$$
\operatorname{rank} R = \operatorname{rank} \left\{ R \left| \begin{bmatrix} 0 \\ d_{q-3} \\ \vdots \\ d_0 \end{bmatrix} \right. \right\}
$$

where

$$
\left\{ R \left| \begin{bmatrix} 0 \\ d_{q-3} \\ \vdots \\ d_0 \end{bmatrix} \right. \right\} = \begin{bmatrix} J & & & & & JB_2 & 0 \\ -I & J & & 0 & & \cdot^{\cdot^{\cdot}} & d_{q-3} \\ & \ddots & \ddots & & \cdot^{\cdot^{\cdot}} & & \vdots \\ 0 & & \ddots & J & 0 & 0 & \vdots \\ & & & -I & JB_2 & & d_0 \end{bmatrix} . \tag{5.5.25}
$$

As in (a), a series of column operations will transform (5.5.25) into

$$\begin{bmatrix} J & J^2 & \cdots & J^{q-2} & 0 & \cdots & 0 & JB_2 & 0 \\ -I & \ddots & & & & & \cdot^{\cdot^{\cdot}} & & d_{q-3} \\ & \ddots & & 0 & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & 0 & \ddots & & \cdot^{\cdot^{\cdot}} & & & 0 & \vdots \\ & & & -I & JB_2 & & & & d_0 \end{bmatrix} \tag{5.5.26}$$

Next, add $J$ times the second row of (5.5.26) to the first, then $J^2$ times the third row of (5.5.26) to the first and so on to give

$$\begin{bmatrix} 0 & \cdots & 0 & J^{q-1}B_2 & \cdots & J^2B_2 & JB_2 & J\,d_{q-3} + J^2\,d_{q-4} + \cdots + J^{q-2}d_0 \\ -I & \ddots & & & & \cdot^{\cdot^{\cdot}} & & d_{q-3} \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & & \ddots & \cdot^{\cdot^{\cdot}} & & & & \vdots \\ & & -I & JB_2 & & & & d_0 \end{bmatrix}. \tag{5.5.27}$$

It is seen from (5.5.27) that the augmented matrix (5.5.25) has the same rank as $R$ for any set $d_i$, $i = 0, 1, \ldots, q-3$, if and only if rank $[J^{q-1}B_2, \ldots, J^2B_2, JB_2] = n_2 - p$, as required to complete the proof. □

A comparison of the necessary and sufficient conditions of the previous theorem with the corresponding conditions associated with the notion of reachability at infinity reveals that the major difference between the two sets of conditions is that the matrix $B_2$ is premultiplied by $J$ in the conditions presented in theorem (5.5.19). The effect of this is to remove the influence of the last position rows of $B_2$ corresponding to the trivial blocks of $J$, which reflects the role of the non-dynamic variables in the definitions of the two notions of reachability. This aspect will be further discussed in section 6.

In an analogous way to the case of reachability at infinity a sufficient condition can be obtained as a direct consequence of condition (b) of theorem (5.5.19). This condition is presented below and is followed by an example which demonstrates that it is not a necessary condition.

**(5.5.28) Theorem.** If rank $[JB_2, J^2B_2, \ldots, J^{q-1}B_2] = n_2 - p$ where $q \geq q_1$, then there exists an $n_2 \times n_2$ polynomial matrix $X(s)$ with elements of degree $q-3$ or less and an $\ell \times n_2$ polynomial matrix $Y(s)$ with elements of degree $q-2$ or less such that

$$[sJ - I]X(s) + JB_2Y(s) = I_{n_2}. \tag{5.5.29}$$

**Proof.** If $e_i$ is the $i^{\text{th}}$ column of $I_{n_2}$, put $d(s) = e_i$ in (5.5.21). Let the corresponding solution be $x^{(i)}(s)$ and $y^{(i)}(s)$. If $X(s), Y(s)$ are the matrices having $x^{(i)}(s)$ and $y^{(i)}(s)$ as their respective columns then $X(s), Y(s)$ satisfy (5.5.29) as required. □

**(5.5.30) Example.** Take

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$[JB_2\ J^2B_2\ J^3B_2] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 2 ($< 3 = n_2 - p$ in this case).

Let

$$X(s) = \begin{bmatrix} -1 & -s & 0 & 0 \\ 0 & -1 & -s & 0 \\ 0 & 0 & -1 & -s \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad Y(s) = \begin{bmatrix} 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & s^2 \end{bmatrix}$$

where both $X(s)$ and $Y(s)$ satisfy the degree conditions of theorem (5.5.28). Then,

$$[sJ - I]X(s) + JB_2Y(s) =$$

$$\begin{bmatrix} -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -s & 0 & 0 \\ 0 & -1 & -s & 0 \\ 0 & 0 & -1 & -s \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & s^2 \end{bmatrix} = I_4$$

Thus there exist suitable $X(s), Y(s)$ satisfying (5.5.29) despite rank $[JB_2, J^2B_2, \ldots, J^{q-1}B_2]$ being less than $n_2 - p$ in this case.

The result of theorems (5.5.19) and (5.5.28) together with (5.5.15) provide an analogy of theorem (5.3.6) for the notion of system state reachability at infinity. Hence, the generalisation of theorem (5.3.6) has been obtained in the cases of both reachability at infinity and system state reachability at infinity (or equivalently controllability at infinity).

## §6. Further discussion of the notions of controllability in generalised state space systems.

The two definitions of reachability, namely reachability at infinity and system state reachability at infinity, demonstrate in a straightforward manner the role of the non-dynamic variables in determining the controllability properties of a generalised state space system. To illustrate the concepts behind these definitions of reachability consider the following example.

**(5.6.1) Example.** Consider the system represented by

$$J\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{5.6.2}$$

where

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad b_i \in \Re,\ i = 1,2,3.$$

The solution to (5.6.2) with zero initial conditions is given by

$$\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \\ x_{23}(t) \end{bmatrix} = - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) - \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix} u^{(1)}(t).$$

It is clear that it is not possible to control all three states from the origin to any arbitrary point at $t = \tau$ since the state $x_{22}(\tau)$ is directly related to $x_{23}(\tau)$. Thus, the system is not reachable at infinity; a fact confirmed by the rank of $[B_2 \ \ JB_2]$. It is though possible to control two of the three states to arbitrary positions at $t = \tau$ if and only if $b_2 \neq 0$. In particular it is possible to control the states $x_{21}(t)$ and $x_{22}(t)$, i.e. the system is system state reachable at infinity. Again, this is confirmed by the fact that the last position row of $B_2$ corresponding to the non-trivial block of $J$ is linearly independent if and only if $b_2 \neq 0$.

Now consider a system consisting only of dynamic variables. Let $J$ and $B_2$ in (5.6.2) be given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where $b_i \in \Re,\ i = 1,2,3$. The solution of (5.6.2) under zero initial conditions is now given by

$$\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \\ x_{23}(t) \end{bmatrix} = - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u(t) - \begin{bmatrix} b_2 \\ b_3 \\ 0 \end{bmatrix} u^{(1)}(t) - \begin{bmatrix} b_3 \\ 0 \\ 0 \end{bmatrix} u^{(2)}(t).$$

In this case it is possible to find a suitable control that will transfer all three states to any arbitrary position at $t = \tau$ if and only if $b_3 \neq 0$. Thus, the system is both reachable at infinity and system state reachable at infinity if and only if $b_3 \neq 0$.

In this case

$$\text{rank } [B_2, JB_2, J^2B_2] = 3$$

and

$$\text{rank } [JB_2, J^2B_2] = 2$$

if and only if $b_3 \neq 0$ to confirm the previous observations.

It has been shown that if a generalised state space system possesses no infinite input decoupling zeros as defined by Rosenbrock [1974] it also possesses no infinite input decoupling zeros as defined by Verghese *et al.* [*ibid.*]. This implies that the set of infinite input decoupling zeros as defined by Verghese *et al.* [1981] is a subset of the set of infinite decoupling zeros as defined by Rosenbrock [1974]. Hence, the requirement that the system has no infinite input decoupling zeros as defined by Rosenbrock [*ibid.*] is a stronger requirement than the one requiring that the system has no infinite input decoupling zeros as defined by Verghese *et al.* [1981]. This is clearly reflected in the notions of reachability associated with the respective definitions of the infinite input decoupling zeros. In particular it is seen that system state reachability at infinity is only concerned with obtaining knowledge concerning the dynamic variables whilst reachability at infinity requires in addition knowledge concerning the non-dynamic variables. This distinction is not immediate from comparisons of other time domain definitions. For instance it follows that the notion of controllability as defined by Lewis and Ozcaldiran [1984] imposes less stringent conditions on the system than the notion of $R$-controllability as defined by Yip and Sincovec [1981] but this is not immediate from first inspection. This confusion is due to the fact that Lewis and Ozcaldiran [1984] allow for impulsive motion in the system while Yip and Sincovec [1981] do not. The inclusion of impulsive motion is therefore seen in a sense to increase the capability of the system to achieve the required objective.

In general the non-dynamic variables have no significant bearing on the system behaviour although in chapter 6 they are seen to provide additional characteristics to the system. Since the notion of reachability is concerned with the dynamic properties of a system it seems that requiring knowledge of the non-dynamic variables serves no purpose. It is therefore concluded that the definition of system state reachability at infinity is a more suitable definition than that of reachability at infinity.

This observation reinforces the discussion presented by Verghese *et al.* [1981] which points out that the deficiency of the definition of infinite input decoupling zeros as given by Rosenbrock [1974] lies in the way it treats both the dynamic and non-dynamic variables in the same manner. The definition of infinite input decoupling zeros presented by Verghese *et al.* [*ibid.*] takes into account the differences between the two types of variables and is regarded as the most appropriate definition. In a similar way the time domain definitions of controllability associated with the absence of infinite input decoupling zeros as defined by Verghese *et al.* [1981] seem to be the most appropriate. These definitions take into account the possible impulsive motion associated with generalised state space systems thus reflecting the dynamic properties of such systems whereas the definitions associated with the absence of infinite input decoupling zeros as defined by Rosenbrock [1974] ignore the impulsive motion.

To avoid confusion when discussing the various controllability concepts for generalised state space systems in subsequent chapters the following definition is made.

**(5.6.3) Definition.** Let the generalised state space system be represented as in (5.1.1). Then, the system is said to be CONTROLLABLE if it has no finite input decoupling zeros and CONTROLLABLE AT INFINITY if it has no infinite input decoupling zeros as defined by Verghese *et al.* [1981]. Further, if the system is both controllable and controllable at infinity then it is said to be STRONGLY CONTROLLABLE. Also, the system will be termed REACHABLE AT INFINITY if it does not possess any infinite input decoupling zeros as defined by Rosenbrock [1974] and STRONGLY REACHABLE if, in addition, it does not possess any finite input decoupling zeros.

Finally some new necessary conditions are presented for a system to be strongly controllable and strongly reachable.

**(5.6.4) Theorem.** The generalised state space system represented in Kronecker form (5.2.5) is strongly controllable only if

$$(q_1 - 1)\ell \geq n_2 - p \tag{5.6.5}$$

and is strongly reachable only if

$$q_1 \ell \geq n_2. \tag{5.6.6}$$

**Proof.** The system is strongly controllable only if it is controllable at infinity. If the system is controllable at infinity then it is necessary that the $n_2 \times (q_1 - 1)\ell$ matrix $[JB_2, J^2B_2, \ldots, J^{q_1-1}B_2]$ has rank $n_2 - p$ which implies the necessary condition (5.6.5). Similarly the system is strongly reachable only if it is reachable at infinity. If the system is reachable at infinity then it is necessary that the $n_2 \times q_1\ell$ matrix $[B_2, JB_2, \ldots, J^{q_1-1}B_2]$ has rank $n_2$ which implies the necessary condition (5.6.6), as required. □

**(5.6.7) Theorem.** The generalised state space system represented in Kronecker from (5.2.5) is strongly controllable only if

$$\ell \geq p - t \tag{5.6.8}$$

and is strongly reachable only if

$$\ell \geq p. \tag{5.6.9}$$

**Proof.** Again the system is strongly controllable only if it is controllable at infinity. For the system to be controllable at infinity it is necessary that the last position rows of the $n_2 \times \ell$ matrix $B_2$ corresponding to the non-trivial blocks of $J$ are linearly independent. Now the system has $p - t$ non trivial blocks which implies $B_2$ must have rank at least equal to $p - t$ which implies $\ell \geq p - t$ to give (5.6.8). Similarly the system is completely reachable only if it is reachable at infinity. For the system to be reachable at infinity it is necessary that the $p$ last position rows of $B_2$ are linearly independent. Since $B_2$ is $n_2 \times \ell$ this implies $\ell \geq p$ to give (5.6.9), as required. □

A comparison of the respective necessary conditions presented in theorems (5.6.4) and (5.6.7) can be made as a result of the following theorem.

**(5.6.10) Theorem.** If a generalised state space system is represented in Kronecker form (5.2.5) then

$$p - t \geq \frac{n_2 - p}{q_1 - 1} \tag{5.6.11}$$

and

$$p \geq \frac{n_2}{q_1}. \tag{5.6.12}$$

**Proof.** The $q_i$, $i = 1, 2, \ldots, p$, are defined such that

$$q_1 \geq q_2 \geq \ldots \geq q_{p-t} > q_{p-t+1} = q_{p-t+2} = \ldots = q_p \equiv 1$$

and

$$q_1 + q_2 + \ldots + q_{p-t} + q_{p-t+1} + \ldots + q_p = n_2. \tag{5.6.13}$$

If each $q_i$, $i = 1, 2, \ldots, p$, is replaced by $q_1$ in (5.6.13) it follows that

$$pq_1 \geq n_2.$$

Hence,

$$p \geq \frac{n_2}{q_1}$$

to give (5.6.12).

Similarly if each $q_i$, $i = 1, 2, \ldots, p-t$ is replaced by $q_1$ in (5.6.13) it follows that

$$(p-t)q_1 + q_{p-t+1} + \ldots + q_p \geq n_2$$

and, since $q_{p-t+1} = q_{p-t+2} = \ldots = q_p \equiv 1$, it follows that

$$(p-t)q_1 + t \geq n_2.$$

Subtracting $p$ from each side gives

$$(p-t)q_1 + t - p \geq n_2 - p$$

or

$$p(q_1 - 1) - t(q_1 - 1) \geq n_2 - p.$$

Hence,

$$p - t \geq \frac{n_2 - p}{q_1 - 1}$$

to give (5.6.11), as required. □

The result of theorem (5.6.10) indicates that the necessary conditions of theorem (5.6.7) are stricter than the corresponding necessary conditions of theorem (5.6.4). The necessary conditions in turn indicate the minimum number of control inputs required for a system to be strongly controllable or strongly reachable. As expected the minimum number of control inputs for a system to be strongly reachable is always greater than or equal to the minimum number required for a system to be strongly controllable, so reflecting the previous discussion.

## §7. Conclusions.

In this chapter the concepts of controllability associated with a generalised state space system have been considered. The historical background was discussed in section 4 from which is was concluded that there exist two main notions of controllability in generalised state space systems. In section 5 algebraic conditions associated with these two notions of controllability were presented which provide an analogy of the algebraic conditions associated with a controllable regular state space system given by Rosenbrock [1970] and described in section 3. These algebraic conditions consist of both existing and original results. The polynomial matrix approach adopted in this work provides a way of treating these results in a unified manner as well as introducing simpler proofs of the existing conditions.

The role of the non-dynamic variables in the controllability properties of a system is clearly reflected in the frequency domain. The introduction of a new definition enables this connection to be established in the time domain. The importance of the non-dynamic variables were further discussed in section 6 where new necessary conditions were presented for a system to be controllable under the two main definitions of controllability.

# Chapter 6. The General Pole Placement Problem in Generalised State Space Systems.

## §1. Introduction.

The main distinguishing feature between state space and generalised state space systems as far as pole placement problems are concerned lies in the fact that infinite poles might arise in the generalised case. The infinite poles possess different characteristics to finite poles. This is reflected in the way that infinite poles give rise to impulsive responses in the system which is in sharp contrast to the exponential responses produced by finite poles. Also, as was seen in chapter 3, the infinite poles are defined in a different manner to finite poles. This therefore means that the presence of infinite poles contributes an additional dimension to the problems of pole assignment. Previous work has mainly concentrated on the case where the closed loop poles are all located at finite locations. A summary of existing results is presented in section 2. The remaining sections of this chapter will consider in detail the case where the closed loop system may possess infinite poles.

In section 3 the main result from chapter 4 is interpreted for systems in generalised state space form to produce new necessary conditions for the simultaneous placement of both the finite and infinite pole structures in such systems. The closed loop infinite pole structure is specifically considered in section 4. The detailed structure of the Kronecker form of the system is exploited to produce a necessary condition for the multiplicity of the closed loop infinite poles. In section 5 necessary and sufficient conditions are presented for the closed loop infinite pole structure. This provides a complete description of the infinite pole structure that can be assigned using constant feedback around a generalised state space system. The simultaneous placement of both finite and infinite pole structures is reconsidered in section 6 where the necessary conditions presented in section 3 are updated by results from subsequent sections. Finally in section 7 the relationship between the results of section 5 and the recent work of Fahmy and O'Reilly [1989] is investigated.

## §2. Pole placement problems in generalised state space systems.

Recall that the finite and infinite frequency behaviour of a generalised state space system can be separated by transforming the system into Kronecker form. The separation of the system into two subsystems makes it possible to feedback only the finite states of the system or only the infinite states of the system. Cobb [1981] calls such feedback slow and fast feedback respectively. It was shown by Cobb [*ibid.*] that slow feedback can be employed to arbitrarily assign a finite pole if and only if that pole is controllable (in the finite sense). Not surprisingly therefore the finite poles of the system can be arbitrarily assigned using slow feedback with reference to the results on pole placement in state space systems. The use of such feedback does not affect the positioning of the infinite poles.

Cobb [*ibid.*] showed that the infinite poles could be relocated at finite locations by employing fast feedback if and only if the system is controllable at infinity. The finite poles subsequently formed are seen to be controllable (in the finite sense) and can therefore be arbitrarily relocated by employing slow feedback. Armentano [1984] proved in fact that controllability at infinity is equivalent to the existence of a constant state feedback (i.e. a feedback which incorporates both the finite and infinite states) which assigns the infinite poles to prespecified finite locations. Thus, the two stage method employed by Cobb [1981] is equivalent to employing a single feedback matrix so making the distinction between fast feedback and state feedback redundant. For this reason it will only be necessary to distinguish between the case where only the finite states are fed back and the case where the finite and infinite states are simultaneously fed back. These two types of feedback will be referred to as pure state feedback and generalised state feedback respectively from now on.

The results presented by Cobb [1981] and Armentano [1984] have therefore shown that all the poles (both finite and infinite) can be relocated at arbitrary finite locations if and only if the system is strongly controllable. This further emphasises the desirability of this notion of controllability in comparison with that of strong reachability. This result leads to a neat generalisation, from the regular to the general case, of the result due to Wonham [1967].

The general pole placement problem was considered by Kucera and Zagalak [1988] for the case when all the closed loop poles are placed at finite locations, i.e. the resulting system is proper. The system under consideration is of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{6.2.1}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ an $\ell$-vector of control inputs and $E, A, B$ are constant real matrices of the appropriate dimensions. Let the feedback be given by

$$u(t) = -Kx(t) + v(t) \tag{6.2.2}$$

where $K$ is a constant real matrix. The main result presented by Kucera and Zagalak [*ibid.*] is given in terms of the minimal column indices of a right minimal matrix fraction description of the open loop transfer function, $G(s) = [sE - A]^{-1}B$, defined as follows.

**(6.2.3) Definition.** Let $G(s) = N(s)D^{-1}(s)$ be a right minimal factorisation of $G(s)$ then the right minimal indices of $G(s)$ are defined as the column degrees of

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

and denoted by $c_i$, $i = 1, 2, \ldots, \ell$.

Kucera and Zagalak [*ibid.*] adopt the term complete controllability indices to describe the $c_i$'s. This terminology seems appropriate for the problem under consideration but in the wider context it is inappropriate since a right factorisation is usually associated with the observability properties of a system. The term right minimal indices seems more suitable and will be used in this work.

**(6.2.4) Theorem (Kucera and Zagalak, 1988).** Let the generalised state space system (6.2.1) be strongly controllable with right minimal indices, $c_i$, ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Further, let $a_1(s), a_2(s), \ldots, a_\ell(s)$ be arbitrary monic polynomials subject to

(i) $a_{i+1}(s) \mid a_i(s)$, $i = 1, 2, \ldots, \ell - 1$,

(ii) $\sum_{i=1}^{\ell} \deg(a_i(s)) = r$ where $r = \operatorname{rank} E$.

Then, there exists a constant feedback matrix $K$ such that $sE - A + BK$ has non-unit invariant polynomials $a_1(s), a_2(s), \ldots, a_\ell(s)$ if and only if

$$\sum_{i=1}^{k} \deg a_i(s) \geq \sum_{i=1}^{k} c_i \qquad k = 1, 2, \ldots, \ell.$$

The proof of the sufficiency part of the above theorem given by Kucera and Zagalak [1988] is incomplete since a particular step is quoted without a full proof or reference to a proof. This step is formally justified by a result presented in section 5 where a complete proof is offered.

Kucera and Zagalak [1988] adopt a minimal factorisation to prove the above result but do not make the explicit connection with the properties of a minimal factorisation. Further Kucera and Zagalak [*ibid.*] do not recognise that a minimal factorisation carries the infinite structure in a particularly simple way. This property will be exploited in this chapter to investigate the infinite pole structure that can be assigned using constant generalised state feedback in generalised state space systems. This aspect of pole placement has not been generally investigated although some authors such as Lewis and Ozcaldiran [1984] and Fahmy and O'Reilly [1989] have considered the infinite eigenstructure assignment problem. Most of the previous work has concentrated on the case where the closed loop system is proper but the desirability of assigning both finite and infinite poles has been noted by Chu [1987] and the need to consider such problems has been illustrated by Dai [1988] in his work on the design of observers for discrete time descriptor systems. The investigation of the general infinite pole placement problem is therefore seen to be of physical significance as well as being important in extending the result due to Kucera and Zagalak [1988] to the case where the closed loop system might possess infinite poles.

### §3. Necessary conditions for the simultaneous placement of both the finite and infinite pole structures.

Consider the generalised state space system represented by

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t) \tag{6.3.1}$$

where $x(t) \in \Re^n$ is the generalised state of the system and $u(t) \in \Re^\ell$ is the input vector with $n \geq \ell$. $E, A, B$ are matrices of the appropriate dimensions with $E$ assumed singular of rank $r$, and $\mid sE - A \mid \neq 0$. It is assumed that the system is strongly controllable and that the output equation is given by

$$y(t) = x(t). \tag{6.3.2}$$

Thus, when constant generalised state feedback of the form

$$u(t) = -K\,x(t) + v(t) \tag{6.3.3}$$

is applied to (6.3.1) this is equivalent to output feedback of the form

$$u(t) = -Ky(t) + v(t).$$

Therefore the new results concerning constant output feedback which were developed in chapter 4 can be interpreted for the general pole placement problem using generalised state feedback in a generalised state space system of the form (6.3.1), (6.3.2). Note that it is the transfer function poles that are investigated by using a minimal factorisation but, since the system is assumed to be strongly controllable, this is equivalent to investigating the system poles given via the invariant polynomials of certain matrices.

Recall that the strongest necessary conditions on the closed loop pole structure using constant output feedback were obtained by considering both the left and right minimal factorisation of the associated transfer function matrix. For generalised state space systems the strongest necessary conditions are always obtained by considering a right minimal factorisation associated with the transfer function matrix since the staircase associated with the right minimal factorisation always lies on or below the staircase associated with the left factorisation. This is a direct consequence of the following results.

**(6.3.4) Lemma.** Let the strongly controllable system described by (6.3.1), (6.3.2) be given in Kronecker form (5.2.5). Then,

$$G(s) = [sE - A]^{-1} B \tag{6.3.5}$$

is a left minimal factorisation of the transfer function matrix $G(s)$.

**Proof.** $[sE - A]^{-1} B$ is a left minimal factorisation of $G(s)$ if and only if

(i) rank $[sE - A \quad B] = n$ for all $s \in \mathbb{C}$

(ii) rank $[sE - A \quad B]_{hr} = n$

where $[sE - A \quad B]_{hr}$ is the high order coefficient matrix of $[sE - A \quad B]$ with respect to the rows.

Condition (i) is equivalent to the system having no finite decoupling zeros which is an immediate consequence of the fact that the system is strongly controllable. For condition (ii)

$[sE - A \quad B]_{hr}$

$$
\left[\begin{array}{cccccccccccccccccccc}
1 &&&&&&&&&&&&&&&&&&& 0 \\
& 1 &&&&&&&&&&&&&&&&&& 0 \\
&& \ddots &&&&&&&&&&&&&&&&& \vdots \\
&&& 1 &&&&&&&&&&&&&&&& 0 \\
&&&& 0 & 1 &&&&&&& \text{\Large 0} &&&&&&& 0 \\
&&&&& \ddots & \ddots &&&&&&&&&&&&& \vdots \\
&&&&&& 0 & 1 &&&&&&&&&&&& 0 \\
&&&&&&& -1 &&&&&&&&&&&& b_{v_1}^T \\
&&&&&&&& \ddots &&&&&&&&&&& \vdots \\
&&&&&&&&& \ddots &&&&&&&&&& \vdots \\
&&&&&&&&&& \ddots &&&&&&&&& \vdots \\
&&&&&&&&&&& 0 & 1 &&&&&&& 0 \\
&&&&&&&&&&&& \ddots & \ddots &&&&&& \vdots \\
&&&&&&&&&&&&& 0 & 1 &&&&& 0 \\
&&&&&& \text{\Large 0} &&&&&&&& -1 &&&&& b_{v_{p-t}}^T \\
&&&&&&&&&&&&&&& -1 &&&& b_{v_{p-t+1}}^T \\
&&&&&&&&&&&&&&&& -1 &&& b_{v_{p-t+2}}^T \\
&&&&&&&&&&&&&&&&& \ddots && \vdots \\
&&&&&&&&&&&&&&&&&& -1 & b_{v_p}^T
\end{array}\right]
\tag{6.3.6}
$$

where $v_j = n_1 + \sum_{k=1}^{j} q_k,\ j = 1, 2, \ldots, p$, and $b_i^T$ is the $i^{\text{th}}$ row of $B$.

Adding row $i-1$ to row $i$ for $i = n_1 + q_1, n_1 + q_1 + q_2, \ldots,\ n_1 + q_1 + \ldots + q_{p-t}$ in (6.3.6) gives rise to

$$
\left[\begin{array}{ccccccccccccccccccc}
1 & & & & & & & & & & & & & & & & & & 0 \\
 & 1 & & & & & & & & & & & & & & & & & 0 \\
 & & \ddots & & & & & & & & & & & & & & & & \vdots \\
 & & & 1 & & & & & & & & & & & & & & & 0 \\
 & & & & 0 & 1 & & & & & & & \mathbf{0} & & & & & & 0 \\
 & & & & & \ddots & \ddots & & & & & & & & & & & & \vdots \\
 & & & & & & \ddots & 1 & & & & & & & & & & & 0 \\
 & & & & & & & 0 & & & & & & & & & & & b_{v_1}^T \\
 & & & & & & & & \ddots & & & & & & & & & & \vdots \\
 & & & & & & & & & \ddots & & & & & & & & & \vdots \\
 & & & & & & & & & & \ddots & & & & & & & & \vdots \\
 & & & & & & & & & & & 0 & 1 & & & & & & 0 \\
 & & & & & & & & & & & & \ddots & \ddots & & & & & \vdots \\
 & & & & & & & & & & & & & \ddots & 1 & & & & 0 \\
 & & & & & & & \mathbf{0} & & & & & & & 0 & & & & b_{v_{p-t}}^T \\
 & & & & & & & & & & & & & & & -1 & & & b_{v_{p-t+1}}^T \\
 & & & & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & & & & -1 & b_{v_p}^T
\end{array}\right].
\tag{6.3.7}
$$

It then follows from (6.3.7) that (6.3.6) has full rank if $b_i^T$, $i = n_1+q_1,\ n_1+q_1+q_2, \ldots, n_1+q_1+\ldots+q_{p-t}$, are linearly independent. This is guaranteed by the fact that the system is strongly controllable, so completing the proof. □

**(6.3.8) Lemma.** Let $G(s)$ be the transfer function matrix associated with the strongly controllable system described by (6.3.1), (6.3.2). Then, the row degrees $r_i$, $i = 1, 2, \ldots, n$, associated with a left minimal factorisation of $G(s)$ are

$$r_1 = 1, r_2 = 1, \ldots, r_{n-p} = 1, r_{n-p+1} = 0, \ldots, r_n = 0$$

where $p$ is the rank deficiency of $E$.

**Proof.** Without loss of generality take the system to be in Kronecker form. Then, since the system is assumed to be strongly controllable, it follows from lemma (6.3.4) that

$[sE-A]^{-1}B$ is a left minimal factorisation. The row degrees of $[sE-A \quad B]$ are either 0 or 1 with the number of rows with zero degrees equal to the rank deficiency of $E$. Reordering these row degrees therefore gives the result. □

The results of lemmas (6.3.4) and (6.3.8) provide a means of proving the hypothesis stated earlier concerning the properties of the staircases associated with the respective minimal factorisations of the open loop transfer function matrix.

**(6.3.9) Lemma.** Let $G(s)$ be the transfer function matrix associated with a strongly controllable system described by (6.3.1), (6.3.2). Let $c_1 \geq c_2 \geq \ldots \geq c_\ell$ be the right minimal indices associated with $G(s)$ and $r_1 \geq r_2 \geq \ldots \geq r_n$ be the corresponding left minimal indices. Then, the staircase function, defined as in (4.3.9), constructed from the right minimal indices lies on or below the corresponding staircase formed from the left minimal indices.

**Proof.** From lemma (6.3.8) it follows that the staircase associated with the left minimal factorisation is as described in figure (6.3.10).

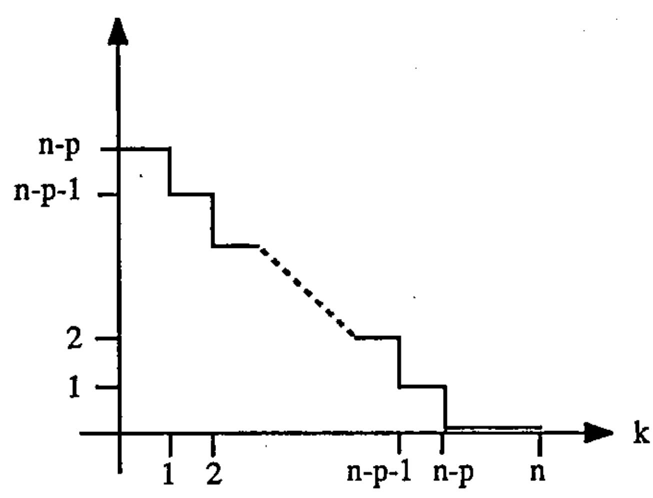



fig. (6.3.10)

For a right minimal factorisation, $G(s) = N_1(s)\,D_1^{-1}(s)$, the column degrees $c_i$, $i = 1,2,\ldots,\ell$, of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ must satisfy

$$\sum_{i=1}^{\ell} c_i = n - p.$$

Then, if the staircase associated with the right factorisation intersects the staircase associated with the left factorisation at some point then $c_i = 0$ for some $i$. But, since the $c_i$'s are ordered in decreasing fashion, this means that the two staircases can only intersect when $k = n - p$ indicating that the staircase associated with the right minimal factorisation lies on or below the staircase associated with the left factorisation, as requried. $\square$

Recall that $\alpha_i(s),\ i = 1, 2, \ldots, \ell$, are monic polynomials such that

$$\alpha_i(s) \mid \alpha_{i-1}(s) \qquad i = 2, 3, \ldots, \ell$$

and

$$\deg \alpha_i(s) = a_i \qquad i = 1, 2, \ldots, \ell.$$

Also, $\beta_i(w),\ i = 1, 2, \ldots, \ell$, are monic polynomials with

$$\beta_i(w) \mid \beta_{i-1}(w) \qquad i = 2, 3, \ldots, \ell$$

and $b_i,\ i = 1, 2, \ldots, \ell$, are defined by

$$\beta_i(w) = w^{b_i}\ \beta_i'(w) \qquad i = 1, 2, \ldots, \ell \qquad \beta_i(0) \neq 0.$$

Then, interpreting the result of theorem (4.5.8) for a system in generalised state space form gives rise to the following.

**(6.3.11) Theorem.** Let $G(s)$ be the transfer function matrix associated with the strongly controllable system represented by (6.3.1), (6.3.2), i.e. $G(s) = [sE - A]^{-1} B$. Let $G(s)$ have a right minimal factorisation

$$G(s) = N_1(s)\ D_1^{-1}(s)$$

where the column degrees of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Finally, let $\Lambda_1(s) = \text{diag}\ [s^{c_1}, s^{c_2}, \ldots, s^{c_\ell}]$. Then, for there to exist a constant matrix $K$ such that the invariant polynomials of $D_1(s) + KN_1(s)$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ and the invariant polynomials of $[D_1\left(\frac{1}{w}\right) + KN_1\left(\frac{1}{w}\right)]\Lambda_1(w)$ are $\beta_1(w), \beta_2(w), \ldots, \beta_\ell(w)$ it is necessary that

$$\sum_{i=k+1}^{\ell} a_i + \sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1 \tag{6.3.12}$$

with equality holding when $k = 0$.

**Proof.** From theorem (4.4.8), the $a_i, b_i,\ i = 1, 2, \ldots, \ell$, must satisfy the necessary conditions that

$$\sum_{i=k+1}^{\ell} a_i + \sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{n} d_i \qquad k = 0, 1, \ldots, \ell - 1 \tag{6.3.13}$$

with equality when $k = 0$, and where

$$\sum_{i=k+1}^{n} d_i = \min\left[\sum_{i=k+1}^{\ell} c_i, \sum_{i=k+1}^{n} r_i\right]$$

and $r_i, i = 1, 2, \ldots, n$, are the left minimal indices of $G(s)$. From lemma (6.3.9) it follows that

$$\sum_{i=k+1}^{n} d_i = \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1$$

which on substitution into (6.3.13) gives rise to (6.3.12) as required. □

The result of theorem (6.3.11) provides necessary conditions for the simultaneous placement of both the finite and infinite pole structures using constant generalised state feedback around a generalised state space system of the form (6.3.1), (6.3.2). The following example demonstrates that the necessary conditions are not sufficient ones.

**(6.3.14) Example.** Let

$$G(s) = \begin{bmatrix} 1 & -\frac{1}{s} \\ s-1 & -1 \\ 0 & \frac{1}{s} \end{bmatrix}$$

whose right and left minimal factorisations are respectively

$$N_1(s) \quad D_1^{-1}(s) = \begin{bmatrix} 1 & -1 \\ s-1 & -s \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1}$$

$$D_2^{-1}(s) \quad N_2(s) = \begin{bmatrix} s & -1 & 0 \\ 0 & 0 & s \\ 1 & 0 & 1 \end{bmatrix}^{-1} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and where the left minimal factorisation is of the form $[sE - A]^{-1}B$. It therefore follows that the necessary conditions are obtained from the column degrees of the right minimal factorisation, i.e. $c_1 = 1, c_2 = 1$. Hence $a_i, b_i$ must satisfy the necessary conditions

$$\left.\begin{aligned} a_2 + b_2 &\leq 1 \\ a_2 + a_1 + b_2 + b_1 &= 2 \end{aligned}\right\}. \tag{6.3.15}$$

Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix}$ and $\Lambda_1(w) = \mathrm{diag}\,[w, w]$. Consider a pole structure with two poles at infinity both of order one and no finite poles, i.e.

$$b_2 = 1, \; b_1 = 1, \; a_2 = 0, \; a_1 = 0. \tag{6.3.16}$$

This pole structure satisfies the conditions (6.3.12). Now, the closed loop pole structure at infinity is given by the zero structure at $w = 0$ of

$$\left[D_1\left(\tfrac{1}{w}\right) + KN_1\left(\tfrac{1}{w}\right)\right]\Lambda_1(w) = \begin{bmatrix} k_2 + w(1 + k_1 - k_2) & -k_2 + w(k_3 - k_1) \\ k_5 + w(k_4 - k_5) & (1 - k_5) + w(k_6 - k_4) \end{bmatrix} \quad (6.3.17)$$

For the above pole structure (6.3.16) to be assigned it is necessary that all $1 \times 1$ minors of (6.3.17) possess a common factor $w$ which in the case of the $(2,1)$ and $(2,2)$ elements implies that

$$k_5 = 0 \quad \text{and} \quad 1 - k_5 = 0$$

so indicating a clear contradiction. Thus, it is not possible to assign the pole structure (6.3.16) to the closed loop system which illustrates that condition (6.3.12) of theorem (6.3.11) is not a sufficient one.

The result of theorem (6.3.11) generalises the necessary conditions obtained by Kucera and Zagalak [1988] to the case where the closed loop system possesses both finite and infinite poles. For the case where the closed loop system is proper, i.e. $b_i = 0$, $i = 1, 2, \ldots, \ell$, the necessary conditions are equivalent to those obtained by Kucera and Zagalak [*ibid.*] (see theorem (6.2.4)). Unfortunately the sufficient conditions do not generalise in the same way.

## §4. Necessary conditions for the placement of the infinite pole structure.

The Kronecker form of a generalised state space system plays a crucial role in the investigation of the closed loop infinite pole structure. The special structure of this form makes it possible to deduce properties concerning certain minors of particular matrices which can then be translated into properties of the invariant polynomials of that matrix and hence the pole structure. The most significant result as far as investigating the closed loop infinite pole structure is concerned is presented in the following theorem.

**(6.4.1) Theorem.** Given $[sE-A \quad B]$ in Kronecker form (i.e. $sE-A$ as in (5.2.2)) with the last position rows of $B$ corresponding to the non-trivial blocks linearly independent and $\Lambda(w)$ defined by

$$\left.\begin{aligned} &\Lambda(w) = \text{diag}\,\{w^{i_1}, w^{i_2}, \ldots, w^{i_n}\} \\ &i_j = 0 \quad \text{if } j = n_1+q_1, n_1+q_1+q_2, \ldots, n_1+q_1+q_2+\ldots+q_p \\ &i_j = 1 \quad \text{otherwise} \end{aligned}\right\}. \tag{6.4.2}$$

Then, the matrix

$$\Lambda(w)\ [\tfrac{1}{w}E - A + BK]$$

possesses a non-zero $(n-\delta)\times(n-\delta)$ minor which is not divisible by $w$, where $K$ is a constant $\ell \times n$ matrix and $\delta$ is the number of linearly independent last position rows of $B$.

**Proof.** Let

$$K = [\,k_1 \quad k_2 \quad \ldots \quad k_n\,],\ B = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix}$$

and $A_1 = (a_{ij}),\ i,j = 1,2,\ldots,n_1$. Then, the matrix $\Lambda(w)\left[\frac{1}{w}E - A + BK\right]$ will be of the form

$$\left[\begin{array}{c|ccc} A_{11} & & A_{12} & \\ \hline A_{v_1 1} & B_{v_1 v_1} & \cdots & B_{v_1 v_p} \\ A_{v_2 1} & \vdots & B_{v_2 v_2} & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{v_p 1} & B_{v_p v_1} & \cdots & B_{v_p v_p} \end{array}\right] \tag{6.4.3}$$

where $v_i = n_1 + \sum_{j=1}^{i} q_j,\ i = 1, 2, \ldots, p,\ v_0 = n_1$ and

$$A_{11} = \begin{bmatrix} 1 - a_{11}w + b_1^T k_1 w & \ldots & -a_{1n_1}w + b_1^T k_{n_1} w \\ \vdots & \ddots & \vdots \\ -a_{n_1 1}w + b_{n_1}^T k_1 w & \ldots & 1 - a_{n_1 n_1}w + b_{n_1}^T k_{n_1} w \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} b_1^T k_{n_1+1} w & \ldots & b_1^T k_{v_p} w \\ \vdots & & \vdots \\ b_{n_1}^T k_{n_1+1} w & \ldots & b_{n_1}^T k_{v_p} w \end{bmatrix}$$

$$A_{v_i 1} = \begin{bmatrix} b_{v_{i-1}+1}^T k_1 w & \ldots & b_{v_i-1}^T k_{n_1} w \\ \vdots & & \vdots \\ b_{v_i-1}^T k_1 w & \ldots & b_{v_i-1}^T k_{n_1} w \\ b_{v_i}^T k_1 & \ldots & b_{v_i}^T k_{n_1} \end{bmatrix}$$

which reduces to the last row when $q_i = 1$,

$$B_{v_i v_i} = \begin{bmatrix} w(-1 + b_{v_{i-1}+1}^T k_{v_{i-1}+1}) & 1 + b_{v_{i-1}+1}^T k_{v_{i-1}+2} w & \ldots & b_{v_{i-1}+1}^T k_{v_i} w \\ b_{v_{i-1}+2}^T k_{v_{i-1}+1} & w(-1 + b_{v_{i-1}+2}^T k_{v_{i-1}+2}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ b_{v_i-1}^T k_{v_{i-1}+1} w & \ldots & w(-1 + b_{v_i-1}^T k_{v_i-1}) & 1 + b_{v_i-1}^T k_{v_i} w \\ b_{v_i}^T k_{v_{i-1}+1} & \ldots & \ldots & -1 + b_{v_i}^T k_{v_i} \end{bmatrix}$$

which reduces to $B_{v_i v_i} = \left[-1 + b_{v_i}^T k_{v_i}\right]$ when $q_i = 1$, and

$$B_{v_i v_j} = \begin{bmatrix} b_{v_{i-1}+1}^T\ k_{v_{j-1}+1} w & \ldots & b_{v_{i-1}+1}^T\ k_{v_j} w \\ \vdots & & \vdots \\ b_{v_i-1}^T\ k_{v_{j-1}+1} w & \ldots & b_{v_i-1}^T\ k_{v_j} w \\ b_{v_i}^T\ k_{v_{j-1}+1} & \ldots & b_{v_i}^T\ k_{v_j} \end{bmatrix}$$

which reduces to the last row when $q_i = 1$.

Alternatively replacing the $-a_{ij} + b_i^T k_j$ and $b_i^T k_j$ by $*$ for all $i, j$ the general structure of (6.4.3) can be seen in the following simpler form.

$$
\left[\begin{array}{cccc|ccccc|ccccc|cccc}
1+*w & *w & \cdots & *w & *w & & \cdots & & & \cdots & & & & \cdots & & & & *w \\
*w & 1+*w & \ddots & \vdots & \vdots & & & & & & & & & & & & & \vdots \\
\vdots & \ddots & \ddots & *w & \vdots & & & & & & & & & & & & & \vdots \\
*w & \cdots & *w & 1+*w & *w & & \cdots & & & \cdots & & & & \cdots & & & & *w \\
\hline
*w & \cdots & \cdots & *w & w(-1+*) & 1+*w & \cdots & & *w & *w & \cdots & & & & \cdots & & & *w \\
\vdots & & & \vdots & *w & \ddots & \ddots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & & & & & \vdots \\
*w & \cdots & \cdots & *w & *w & \cdots\ *w & w(-1+*) & & 1+*w & *w & & & & & & & & *w \\
* & \cdots & \cdots & * & * & \cdots\ \cdots & * & & -1+* & * & \cdots & & & & \cdots & & & * \\
& & & & & & & & & \ddots & & & & & & & & \\
& & & & & & & & & & \ddots & & & & & & & \\
& & & & & & & & & & & \ddots & & & & & & \\
\hline
*w & & \cdots & & \cdots & & & & *w & w(-1+*) & 1+*w & \cdots & & *w & *w & \cdots & \cdots & *w \\
\vdots & & & & & & & & \vdots & *w & \ddots & \ddots & & \vdots & \vdots & & & \vdots \\
\vdots & & & & & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
*w & & \cdots & & \cdots & & & & *w & *w & \cdots\ *w & w(-1+*) & & 1+*w & *w & \cdots & \cdots & *w \\
* & & \cdots & & \cdots & & & & * & * & \cdots\ \cdots & * & & -1+* & * & \cdots & \cdots & * \\
\hline
* & & \cdots & & & & \cdots & & & & \cdots & & & * & -1+* & * & \cdots & * \\
\vdots & & & & & & & & & & & & & \vdots & * & -1+* & \ddots & \vdots \\
\vdots & & & & & & & & & & & & & \vdots & \vdots & \ddots & \ddots & * \\
* & & \cdots & & & & \cdots & & & & \cdots & & & * & * & \cdots & * & -1+*
\end{array}\right].
\tag{6.4.4}
$$

First, take $\delta = p$ and consider the $n-p$ minor formed by deleting rows $n_1+q_1, n_1+q_1+q_2, \ldots, n_1+q_1+q_2+\ldots+q_p$ and columns $n_1+1, n_1+q_1+1, \ldots, n_1+q_1+\ldots+q_{p-1}+1$ in (6.4.3). Using the notation of (6.4.4) this minor is of the form

$$\begin{vmatrix} 1+*w & *w & & \cdots & \cdots & *w \\ *w & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & 1+*w & \ddots & & \vdots \\ \vdots & & *w & 1+*w & \ddots & \vdots \\ \vdots & & & w(-1+*) & \ddots & *w \\ & & & & \ddots & \ddots \\ *w & \cdots & \cdots & \cdots & w(-1+*) & 1+*w \end{vmatrix}$$

and equals $1+g(w)$ where $g(0)=0$. Thus, the theorem is proved for the case when $\delta = p$.

To complete the proof it is sufficient to show that if $\eta$ of the last position rows corresponding to the trivial blocks are linearly dependent on the other last positions rows then there exists a $n-p+\eta$ non-zero minor not divisible by $w$. Without loss of generality let the last $\eta$ last position rows corresponding to the trivial blocks be linearly dependent on previous last position rows, i.e.

$$b^T_{n_1+q_1+\ldots+q_{p-\eta+i}} = \sum_{j=1}^{p-\eta} \xi_j \, b^T_{n_1+q_j} \qquad i = 1,2,\ldots,\eta, \quad \xi_j \in \Re.$$

Take the $n_1+q_1+\ldots+q_{p-\eta+1}$ row of (6.4.3) and subtract suitable multiples of rows $n_1+q_j$, $j=1,2,\ldots,p-\eta$, so that row $n_1+q_1+\ldots+q_{p-\eta+1}$ consists of a "-1" in position $n_1+q_1+\ldots+q_{p-\eta+1}$ with all other entries zero except possibly the entries in position $n_1+q_1, n_1+q_1+q_2, \ldots, n_1+q_1+\ldots+q_{p-\eta}$. The "-1" can be used to remove the other non-zero elements in this row by suitable column operations so that row $n_1+q_1+\ldots+q_{p-\eta+1}$ is now of the form

$$[0 \ \ldots \ 0 \ \ -1 \ \ 0 \ \ldots \ 0]$$

where the "-1" is at position $n_1+q_1+\ldots+q_{p-\eta+1}$. Similarly for rows $n_1+q_1+\ldots+q_{p-\eta+i}$, $i=2,3,\ldots,\eta$, where all the elements of the $i^{\text{th}}$ row are zero except for a "-1" in position $n_1+q_1+\ldots+q_{p-\eta+i}$. These row and column operations do not destroy the zero structure at $w=0$ of (6.4.3).

Then, the $n-p+\eta$ minor formed by deleting rows $n_1+q_1, n_1+q_1+q_2, \ldots, n_1+q_1+\ldots+q_{p-\eta}$ and columns $n_1+1, n_1+q_1+1, \ldots, n_1+q_1+\ldots+q_{p-\eta-1}+1$ of the resulting matrix is of the form

$$
\left|\begin{array}{ccccccccccc}
1+*w & *w & & \cdots & & \cdots & & \cdots & & & *w \\
*w & \ddots & \ddots & & & & & & & & \\
& \ddots & 1+*w & \ddots & & & & & & & \vdots \\
\vdots & & *w & 1+*w & \ddots & & & & & & \vdots \\
& & & (-1+*)w & \ddots & \ddots & & & & & \\
\vdots & & & & \ddots & \ddots & \ddots & & & & \\
*w & & \cdots & & (-1+*)w & & 1+*w & *w & & \cdots & *w \\
0 & & \cdots & & \cdots & & 0 & -1 & 0 & \cdots & 0 \\
\vdots & & & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & & & & \ddots & \ddots & 0 \\
0 & & \cdots & & \cdots & & & \cdots & & 0 & -1
\end{array}\right|
$$

where some of the $*$ differ from those given in (6.4.4). This minor is equal to $\pm 1 + f(w)$ where $f(0) = 0$. Thus, there exists an $n - p + \eta$ non-zero minor of (6.4.3) which is not divisible by $w$, so completing the proof. $\square$

To illustrate the result of theorem (6.4.1) consider the following example.

**(6.4.5) Example.** Let

$$
[sE - A \quad B] = \begin{bmatrix} s & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & s & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & b & 1 \end{bmatrix}
$$

which is in Kronecker form with $b \in \Re$. In this case $p = 2, q_1 = 2, q_2 = 1$ so that if $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$ and $\Lambda(w) =$ diag$[w, w, 1, 1]$ then

$$
\Lambda(w)\ \left[\tfrac{1}{w}E - A + BK\right] = \begin{bmatrix} 1 + wk_1 & wk_2 & wk_3 & wk_4 \\ wk_1 & -w + wk_2 & 1 + wk_3 & wk_4 \\ k_5 & k_6 & -1 + k_7 & k_8 \\ k_1 b + k_5 & k_2 b + k_6 & k_3 b + k_7 & -1 + k_4 b + k_8 \end{bmatrix} \qquad (6.4.6)
$$

If $b \neq 0$ then $\delta = 2$ and theorem (6.4.1) states that (6.4.6) possesses a $2 \times 2$ minor which is not divisible by $w$. This is confirmed by considering the $2 \times 2$ minor formed by deleting the third and fourth rows and second and fourth columns of (6.4.6), i.e.

$$\begin{vmatrix} 1 + wk_1 & wk_3 \\ wk_1 & 1 + wk_3 \end{vmatrix} = 1 + w(k_1 + k_3)$$

This minor is clearly not divisible by $w$.

If $b = 0$ then $\delta = 1$ and theorem (6.4.1) states that (6.4.6) possesses a $3 \times 3$ non-zero minor which is not divisible by $w$. Assume the contrary, then the particular minor

$$\begin{vmatrix} 1 + wk_1 & wk_3 & wk_4 \\ wk_1 & 1 + wk_3 & wk_4 \\ k_5 & -1 + k_7 & k_8 \end{vmatrix} = k_8 + g_1(w), \quad g_1(0) = 0$$

formed by deleting the fourth row and second column of (6.4.6) must be divisible by $w$. This implies $k_8 = 0$. Next, consider the minor

$$\begin{vmatrix} 1 + wk_1 & wk_3 & wk_4 \\ wk_1 & 1 + wk_3 & wk_4 \\ k_5 & k_7 & -1 + k_8 \end{vmatrix} = -1 + k_8 + g_2(w), \quad g_2(0) = 0$$

formed by deleting the second column and the third row of (6.4.6). If this minor is divisible by $w$ then $k_8 = 1$ which leads to a contradiction. Hence, there exists a non-zero $3 \times 3$ minor of (6.4.6) which is not divisible by $w$ as predicted by theorem (6.4.1).

The result of theorem (6.4.1) provides detailed information concerning the minors of certain matrices associated with the generalised state space system (6.3.1), (6.3.2). This information can subsequently be interpreted in terms of the infinite pole structure of the system formed by applying constant generalised state feedback around (6.3.1), (6.3.2) which gives rise to the following corollary.

**(6.4.7) Corollary.** Let $G(s) = [sE - A]^{-1}B$ be a strongly controllable system. Then the multiplicity of the closed loop infinite poles under constant generalised state feedback of the form (6.3.3) can not exceed $\delta$ where $\delta$ is the number of linearly independent last position rows of $B$ associated with the Kronecker form of the system, i.e.

$$b_{\delta+1} = b_{\delta+2} = \ldots = b_\ell = 0.$$

**Proof.** Since the open loop system is assumed to be strongly controllable it follows from lemma (6.3.4) that for the system represented in Kronecker form

$$G(s) = [sE - A]^{-1}B$$

is a left minimal factorisation of $G(s)$. If $G_K(s)$ is the closed loop transfer function matrix then, by the dual of lemma (4.2.2),

$$G_K(s) = [sE - A + BK]^{-1}B$$

is also a left minimal factorisation and $[sE - A + BK \quad B]$ has the same row degrees as $[sE - A \quad B]$. Therefore, by lemma (3.2.6), the infinite pole structure is given by the zero structure at $w = 0$ of

$$\Lambda(w)\left[\tfrac{1}{w}E - A + BK\right] \tag{6.4.8}$$

where $\Lambda(w)$ is given by (6.4.2). Let $\beta_i(w)$, $i = 1, 2, \ldots, n$ be the invariant polynomials of (6.4.8) such that $\beta_i(w) \,\big|\, \beta_{i-1}(w)$, $i = 2, 3, \ldots, n$ and $D_j(w)$, $1 \le j \le n$, be the monic polynomials which are the greatest common divisors of all $j \times j$ minors of (6.4.8). Then

$$\beta_i(w) = \frac{D_{n+1-i}(w)}{D_{n-i}(w)} \qquad i = 1, 2, \ldots, n, \; D_0 \triangleq 1.$$

It then follows from theorem (6.4.1) that $\beta_i(0) \neq 0$, $i = \delta + 1, \delta + 2, \ldots, n$. Hence

$$b_i = 0 \qquad i = \delta + 1, \delta + 2, \ldots, n$$

as required. □

Note that since the open loop system is assumed to be strongly controllable, the last position rows of $B$ corresponding to the non-trivial blocks of $J$ form a linearly independent set so that $\delta$ will always be greater than or equal to $p - t$.

The result of corollary (6.4.7) indicates that the possible multiplicities of the infinite poles of the closed loop system are dependent on how "controllable" the system is. It follows that in general the possible multiplicities will be greater if the system is assumed to be strongly reachable than if it is assumed to be strongly controllable. Therefore, as far as assigning the infinite pole structure is concerned it is more advantageous for the system to be strongly reachable.

Recall that the difference between the strong reachability and strong controllability definitions lies in the way they deal with the infinite frequency behaviour of the system. It should also be recalled that if a system is strongly reachable then it is also strongly controllable. In the light of these facts it is therefore not surprising that requiring the system to be strongly reachable will result, in general, in a greater flexibility in the placement of the infinite poles.

Note that in the regular case the multiplicity of the closed loop finite poles can obtain a maximum value equal to rank B. In the generalised case it has now been shown that the multiplicity of closed loop infinite poles can obtain a maximum value equal to rank of the last position rows of $B$ when the system is represented in Kronecker form.

Consider again example (6.4.5) to illustrate the result of corollary (6.4.7).

**(6.4.9) Example.** Let

$$G(s) = [sE - A]^{-1}B$$

$$= \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ b & 1 \end{bmatrix}$$

where $sE - A$ is in Kronecker form and $b$ is an arbitrary constant. A right minimal factorisation of $G(s)$ is given by

$$G(s) = N(s)\,D^{-1}(s)$$

$$= \begin{bmatrix} 1 & 0 \\ -s & s \\ 0 & 1 \\ -sb & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & -1 \end{bmatrix}^{-1}.$$

It then follows that $c_1 = 1, c_2 = 1$ and $\Lambda(w) = \text{diag}[w, w]$. Let

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$$

be the constant feedback matrix so that the closed loop infinite pole structure is given by the zero structure at $w = 0$ of

$$\left[D\left(\tfrac{1}{w}\right) + KN\left(\tfrac{1}{w}\right)\right] \Lambda(w) = \begin{bmatrix} k_1 w + 1 - k_2 - k_4 b & k_2 + k_3 w + k_4 w \\ k_5 w - k_6 - k_8 b & k_6 - w + k_7 w + k_8 w \end{bmatrix}. \qquad (6.4.10)$$

Now the closed loop infinite pole structure will have multiplicity 2 if and only if each element of (6.4.10) is divisible by $w$. An investigation of elements in position (1, 1) and (1, 2) indicate that this will be so if and only if

$$1 - k_2 - k_4 b = 0 \quad \text{and} \quad k_2 = 0.$$

This in turn implies that the closed loop infinite pole structure has multiplicity 2 if and only if the last position rows of $B$ have rank 2. Thus, it is seen in this example how the linear independence properties of the last position rows of $B$ influence the multiplicity of the closed loop infinite poles.

## §5. Necessary and sufficient conditions for the placement of the infinite pole structure.

In this section the placement of the infinite pole structure for a generalised state space system is further considered. Stronger necessary conditions are obtained by combining the necessary conditions of corollary (6.4.7) together with the results of theorem (6.3.11) for the case when only the infinite poles are of concern. The resulting conditions are shown to be also sufficient conditions. Before presenting this new result, two important theorems are considered which are crucial in establishing the new necessary and sufficient conditions.

The first theorem is concerned with the properties of a right minimal factorisation of the transfer function matrix $G(s) = [sE - A]^{-1}B$. The theorem also provides a formal justification of a crucial result assumed without proof by Kucera and Zagalak [1988].

The result is given for the system (6.3.1), (6.3.2) when it is represented in Kronecker form, so that

$$[-B \quad sE - A] = \begin{bmatrix} -B_1 & sI_{n_1} - A_1 & 0 \\ -B_2 & 0 & sJ - I_{n_2} \end{bmatrix} \tag{6.5.1}$$

where $\begin{bmatrix} sI_{n_1} - A_1 & 0 \\ 0 & sJ - I_{n_2} \end{bmatrix}$ is as described by (5.2.2) and (5.2.3). Also, $B_2$ is taken to be in column echelon form so that $B_2$ consists of, from left to right, $\delta$ columns of the form

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the "1" is at position $q_1 + q_2 + \ldots + q_i$, $i = 1, 2, \ldots, \delta$, and $\delta$ is the number of linearly independent last position rows of $B$, then $p - \delta$ zero columns and finally $\ell - p$ columns whose elements are irrelevant. With the system represented in this canonical form the following result can now be stated.

**(6.5.2) Theorem.** Consider a strongly controllable generalised state space system (6.3.1), (6.3.2) represented in Kronecker form (6.5.1) with $B_2$ in column echelon form. Let $G(s) = N(s)D^{-1}(s)$ be a right minimal factorisation of the associated transfer function matrix and let $\begin{bmatrix} D \\ N \end{bmatrix}_{hc}$ denote the high order coefficient matrix with respect to the columns of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$. Then, when $p-t<\ell$, rows

$$p-t+1, p-t+2, \ldots, \ell, \quad \ell+n_1+1, \ell+n_1+q_1+1, \ldots, \ell+n_1+q_1+\ldots+q_{p-t-1}+1$$

and when $p-t=\ell$ rows

$$\ell+n_1+1, \ell+n_1+q_1+1, \ldots, \ell+n_1+q_1+\ldots+q_{p-t-1}+1$$

of $\begin{bmatrix} D \\ N \end{bmatrix}_{hc}$ and of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ are linearly independent where $q_0 \triangleq 0$.

**Proof.** By theorem (5.6.7) and the fact that the system is strongly controllable it follows that

$$p-t \leq \ell.$$

Since $G(s)=[sE-A]^{-1}B=N(s)\,D^{-1}(s)$ it also follows, with $[-B \quad sE-A]$ in Kronecker form and $N(s)$ suitably partitioned, that

$$\begin{bmatrix} -B_1 & sI_{n_1}-A_1 & 0 \\ -B_2 & 0 & sJ-I_{n_2} \end{bmatrix} \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = 0 \tag{6.5.3}$$

Let $-B=(b_{ij})$, $i=1,2,\ldots,n$, $j=1,2,\ldots,\ell$, and $A_1=(a_{ij})$, $i,j=1,2,\ldots,n_1$, then (6.5.3) can be written as

$$
\left[\begin{array}{*{25}{c}}
b_{11} & \cdots & \cdots & \cdots & & & b_{1,p+1} & \cdots & b_{1,\ell} & s-a_{11} & \cdots & -a_{1n_1} \\
\vdots & & & & & & \vdots & & \vdots & \vdots & \ddots & \vdots \\
b_{n_1 1} & \cdots & \cdots & \cdots & & & b_{n_1,p+1} & \cdots & b_{n_1,\ell} & -a_{n_1 1} & \cdots & s-a_{n_1 n_1} \\
0 & & & & & & b_{n_1+1,p+1} & \cdots & b_{n_1+1,\ell} & & & & -1 & s & & & & & 0 \\
\vdots & & & & & & \vdots & & \vdots & & & & & \ddots & \ddots \\
0 & & 0 & & & & \vdots & & \vdots & & & & & & \ddots & s \\
-1 & & & & & & b_{n_1+\eta_1,p+1} & \cdots & b_{n_1+\eta_1,\ell} & & & & & & & -1 \\
\ddots & & & & & & | & & | & & & & & & & & \ddots \\
 & \ddots & & & & & | & & | & & & & & & & & & \ddots \\
 & -1 & & & & & b_{n_1+\eta_{p-t},p+1} & \cdots & b_{n_1+\eta_{p-t},\ell} & & & & & & & & & & -1 \\
 & & -1 & & & & b_{n_1+\eta_{p-t+1},p+1} & \cdots & b_{n_1+\eta_{p-t+1},\ell} & & & & & & & & & & & -1 \\
 & & & \ddots & & & \vdots & & \vdots & & & & 0 & & & & & & & & -1 \\
 & & & -1 & & & b_{n_1+\eta_s,p+1} & & b_{n_1+\eta_s,\ell} & & & & & & & & & & & & & \ddots \\
 & 0 & & & 0 & & \vdots & & \vdots & & & & & & & & & & & & & & \ddots \\
 & & & & \ddots & & \vdots & & \vdots & & & & & & & & & & & & & & & \ddots \\
 & & & & & 0 & b_{n_1+\eta_p,p+1} & \cdots & b_{n_1+\eta_p,\ell} & & & & & & & & & & & & & & & & -1
\end{array}\right]
$$

$$
\times \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = 0
$$

where $\eta_j = \sum_{k=1}^{j} q_1 + q_2 + \ldots + q_k, \quad j = 1, 2, \ldots p.$

Let $U \in \Re^{n \times n}$ represent suitable column operations on $[-B \quad sE - A]$ such that

$[-B \quad sE-A]\,U =$

$$
\left[\begin{array}{ccccccccccccccccccccccc}
b_{11} & \cdots & \cdots & \cdots & & & b'_{1,p+1} & \cdots & b'_{1,\ell} & s-a_{11} & \cdots & -a_{1n_1} \\
\vdots & & & & & & \vdots & & \vdots & \vdots & \ddots & \vdots \\
b_{n_1 1} & \cdots & \cdots & \cdots & & & b'_{n_1,p+1} & \cdots & b'_{n_1,\ell} & -a_{n_1 1} & \cdots & s-a_{n_1 n_1} \\
0 & & & & & & b_{n_1+1,p+1} & \cdots & b_{n_1+1,\ell} & & & & -1 & s \\
\vdots & & & & & & \vdots & & \vdots & & & & \ddots & \ddots \\
0 & & & & & & b_{n_1+\eta_1-1,p+1} & & b_{n_1+\eta_1-1,\ell} & & & & & \ddots & s \\
-1 & & & & & & 0 & \cdots & 0 & & & & & & -1 \\
\ddots & & & & & & \vert & & \vert & & & & & & \ddots \\
& \ddots & & & & & \vert & & \vert & & & & & & & \ddots \\
& & \ddots & & & & \vert & & \vert & & & & & & & & \ddots \\
& & 0 & & & & b_{n_1+\eta_{p-t-1}+1,p+1} & \cdots & b_{n_1+\eta_{p-t-1}+1,\ell} & & & & & & & & & -1 & s \\
& & \vdots & & & & \vdots & & \vdots & & & & & & & & & \ddots & \ddots \\
& & 0 & & & & b_{n_1+\eta_{p-t}-1,p+1} & \cdots & b_{n_1+\eta_{p-t}-1,\ell} & & & & & & & & & & \ddots & s \\
& & -1 & & & & 0 & \cdots & 0 & & & & & & & & & & & -1 \\
& & & 0 & & & 0 & \cdots & 0 & & & & & & & & & & & & -1 \\
& & & & \ddots & & \vdots & & \vdots & & & & & & & & & & & & & \ddots \\
& & & & & 0 & 0 & & 0 & & & & & & & & & & & & & & -1
\end{array}\right]
$$

where

$$
b'_{i,j} = \sum_{\xi=1}^{\delta} \alpha_{i\xi}\, b_{i,\xi} + b_{i,j} \qquad i = 1,2,\ldots n_1,\ j = p+1, p+2, \ldots, \ell,\ \alpha_{i\xi} \in \Re.
$$

and let

$$
U^{-1}\begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} D'(s) \\ N_1(s) \\ N_2'(s) \end{bmatrix}
$$

so that from (6.5.3)

$$[-B \quad sE - A]\, U U^{-1} \begin{bmatrix} D(s) \\ N_1(s) \\ N_2'(s) \end{bmatrix} = 0.$$

If $[d_1', \ldots, d_t', \beta_1, \ldots, \beta_{n_1}, \gamma_1, \ldots, \gamma_{n_2}]^T$ be a column of $\begin{bmatrix} D'(s) \\ N_1(s) \\ N_2'(s) \end{bmatrix}$ then it must satisfy the following set of equations.

$$\begin{array}{cccccccccccccc}
b_{11}d_1' & +b_{12}d_2'+ & \cdots\cdots\cdots & +b_{1t}'d_t' & +(s-a_{11})\beta_1 & -a_{12}\beta_2 & \cdots & -a_{1n_1}\beta_{n_1} & & & & & & =0 \\
\vdots & & & & & & & \vdots & & & & & & \vdots \\
b_{n_1 1}d_1' & +b_{n_1 2}d_2'+ & \cdots\cdots\cdots & +b_{n_1 t}'d_t' & -a_{n_1 1}\beta_1 & -a_{n_1 2}\beta_2- & \cdots & +(s-a_{n_1 n_1})\beta_{n_1} & & & & & & =0 \\
 & & b_{n_1+1,p+1}d_{p+1}'+ & \cdots & +b_{n_1+1,t}d_t' & -\gamma_1+\gamma_2 s & & & & & & & & =0 \\
 & & b_{n_1+2,p+1}d_{p+1}'+ & \cdots & b_{n_1+2,t}d_t' & & -\gamma_2+\gamma_3 s & & & & & & & =0 \\
 & & \vdots & & \vdots & & \ddots & & & & & & & \vdots \\
 & & b_{n_1+q_1-1,p+1}d_{p+1}'+ & \cdots & +b_{n_1+q_1-1,t}d_t' & & & -\gamma_{q_1-1}+\gamma_{\eta_1}s & & & & & & =0 \\
-d_1' & & & & & & & -\gamma_{\eta_1} & & & & & & =0 \\
 & & & | & & & & & \ddots & & & & & \vdots \\
\ddots & & & | & & & & & \ddots & & & & & \vdots \\
 & & & | & & & & & \ddots & & & & & \vdots \\
 & & b_{n_1+\eta_{p-t-1}+1,p+1}d_{p+1}'+ & \cdots & +b_{n_1+\eta_{p-t-1}+1,t}d_t'+ & & \cdots & & -\gamma_{\eta_{p-t-1}+1} & +s\gamma_{\eta_{p-t-1}+2} & & & & =0 \\
 & \ddots & \vdots & & \vdots & & & & & \ddots & & & & \vdots \\
 & & b_{n_1+\eta_{p-t}-1,p+1}d_{p+1}'+ & \cdots & +b_{n_1+\eta_{p-t}-1,t}d_t' & & & & & -\gamma_{\eta_{p-t}-1} & +s\gamma_{\eta_{p-t}} & & & =0 \\
 & -d_{p-t}' & & & & & & & & & -\gamma_{\eta_{p-t}} & & & =0 \\
 & & & & & & & & & & & -\gamma_{\eta_{p-t+1}} & & =0 \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & -\gamma_{\eta_p} & =0
\end{array} \tag{6.5.4}$$

Now consider in which positions the highest degree can occur in $[d'_1, \ldots, d'_\ell, \beta_1, \ldots, \beta_{n_1}, \gamma_1, \ldots, \gamma_{n_2}]^T$. From the first $n_1$ equations of (6.5.4) it follows that the highest degree element does not occur in positions $\beta_1, \beta_2, \ldots, \beta_{n_1}$. Also, from the last $t$ equations of (6.5.4) $\gamma_{\eta_{p-t+1}} = \gamma_{\eta_{p-t+2}} = \ldots = \gamma_p \equiv 0$. Next, consider the equations associated with the first Jordan block where it is seen that the highest degree can only occur in $d'_{p+1}, d'_{p+2}, \ldots, d'_\ell, \gamma_1$. Similarly for the equations associated with the other non-trivial blocks. Also, it is possible for the highest degree to occur in elements

$$d'_{p-t+1}, d'_{p-t+2}, \ldots, d'_p$$

so that the set of elements where the highest degree can occur is

$$d'_{p-t+1}, \ldots, d'_p, d'_{p+1}, \ldots, d'_\ell, \gamma_1, \gamma_{q_1+1}, \ldots, \gamma_{q_1+\ldots+q_{p-t-1}+1}$$

with suitable interpretation when $p - t = \ell, p > \ell$ and $t = 0$. This set consists of $\ell$ elements so that $\begin{bmatrix} D' \\ N_1 \\ N'_2 \end{bmatrix}_{hc}$, the high order coeffeicient matrix with respect to the columns of $\begin{bmatrix} D'(s) \\ N_1(s) \\ N'_2(s) \end{bmatrix}$, will only have non-zero elements in rows

$$p - t + 1, \ldots, \ell, \ell + n_1 + 1, \ell + n_1 + q_1 + 1, \ldots, \ell + n_1 + q_1 + \ldots + q_{p-t-1} + 1. \quad (6.5.5)$$

Since $\begin{bmatrix} D'(s) \\ N_1(s) \\ N'_2(s) \end{bmatrix}$ is also a minimal basis its follows that the rows given by (6.5.5) of $\begin{bmatrix} D' \\ N_1 \\ N'_2 \end{bmatrix}_{hc}$ and also of $\begin{bmatrix} D'(s) \\ N_1(s) \\ N'_2(s) \end{bmatrix}$ are linearly independent. Transforming the minimal basis $\begin{bmatrix} D'(s) \\ N_1(s) \\ N'_2(s) \end{bmatrix}$ back to the form $\begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}$ does not affect the rows given by (6.5.5). Hence the result. □

By definition, if the $(n+\ell)\times\ell$ matrix $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ forms a minimal basis then both $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ and $\begin{bmatrix} D \\ N \end{bmatrix}_{hc}$ possess $\ell$ rows which are linearly independent. The result of theorem (6.5.2) identifies such a set of rows for the case when $G(s) = N(s)\,D^{-1}(s) = [sE-A]^{-1}B$. To illustrate the result consider the following example.

**(6.5.6) Example.** Let

$$[-B \quad sE-A] = \begin{bmatrix} 0 & -1 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & s & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & s \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

which is in Kronecker form with $B_2$ in column echelon form. In this example

$$p=2,\ t=0,\ q_1=2,\ q_2=2,\ \ell=2,\ n_1=1.$$

If $G(s) = N(s)\,D^{-1}(s)$ is a right minimal factorisation of the transfer function matrix associated with this system then theorem (6.5.2) states that the fourth and sixth rows of $\begin{bmatrix} D \\ N \end{bmatrix}_{hc}$ and $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ are linearly independent. A right minimal factorisation is given by

$$G(s) = \begin{bmatrix} -1 & 0 \\ 0 & s \\ 0 & 1 \\ s^2 & 0 \\ s & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -s & 0 \end{bmatrix}^{-1}$$

which on inspection of

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \left[\begin{array}{cc} 0 & -1 \\ -s & 0 \\ \hline -1 & 0 \\ 0 & s \\ 0 & 1 \\ s^2 & 0 \\ s & 0 \end{array}\right] \quad \text{and} \quad \begin{bmatrix} D \\ N \end{bmatrix}_{hc} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

confirms the result of theorem (6.5.2) in this case.

Next consider the following theorem which provides a means of constructing a suitable polynomial matrix required in a subsequent proof.

**(6.5.7) Theorem.** Let $\Lambda(w) = \text{diag}\,[w^{c_1},\ w^{c_2}, \ldots, w^{c_\ell}]$ where the $c_i$'s are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Let $D_{hc}$ be an $\ell \times \ell$ non-singular matrix and let $b_i,\ i = 1, 2, \ldots, \ell$, satisfy the conditions

$$\left.\begin{aligned} & b_1 \geq b_2 \geq \ldots \geq b_\ell, \\ \text{and} \quad & \sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1 \end{aligned}\right\} \tag{6.5.8}$$

Then, there exists a polynomial matrix $C(s)$ such that
(i) $C\left(\frac{1}{w}\right)\Lambda(w)$ has Smith form

$$\text{diag}\,[w^{b_1}, w^{b_2}, \ldots, w^{b_\delta}, 1, 1, \ldots, 1] \qquad 1 \leq \delta \leq \ell$$

(ii) the last $\ell - \delta$ columns of $C\left(\frac{1}{w}\right)\Lambda(w)\Big|_{w=0} D_{hc}^{-1}$ are linearly independent.

**Proof.** Let $H(w) = \text{diag}\,[w^{b_1}, w^{b_2}, \ldots, w^{b_\delta}, 1, 1, \ldots, 1]$ and define $h_i$ to be the column degree of the $i^{\text{th}}$ column of $H(w)$.

If $h_i \leq c_i,\ i = 1, 2, \ldots, \ell$, then a polynomial $C(s)$ exists to satisfy (i). Otherwise there exists a $k$ such that $h_k > c_k$ and, by (6.5.8), it follows that there exists a $j(> k)$ such that $h_j < c_j$. Then, take $w$ times row $j$ and add to row $k$. Let $\alpha = h_k - h_j - 1$ $(> 0)$ and subtract $w^\alpha$ times column $j$ from column $k$. Finally, interchange rows $k$ and $j$ to give the matrix $H'(w)$ with column degrees $h'_i,\ i = 1, 2, \ldots, \ell$, where

$$h'_i = h_i \qquad i \neq k, j$$

$$h'_k = h_k - 1$$

$$h'_j = h_j + 1$$

and where the element in position $i$ of column $i$ is of higher degree than any other element in the same column. Employing similar transformations gives rise to a matrix $H'(w)$ in which

$$h'_1 \geq h'_2 \geq \ldots \geq h'_\ell$$

and

$$h'_i \leq c_i \qquad i = 1, 2, \ldots, \ell.$$

Also, the Smith form of $H'(w)$ is equivalent to the Smith form of $H(w)$. Since the transformations only involve adding $\beta w^{\alpha}, \alpha > 0, \beta \in \Re$ times a row (column) to another row (column) and row interchanges it follows that the number of 1's present in $H'(w)$ is the same as the number in $H(w)$ and that they occur in the same columns. Therefore it is possible to employ further row interchanges to give the matrix $H''(w)$ where

$$H''(0) = H(0).$$

These additional row operations do not affect the column degrees so that $H''(w)$ has the same column degrees as $H'(w)$, and is also column proper.

If $H''(w)$ is such that the last $\ell - \delta$ columns of $H''(0)\, D_{hc}^{-1}$ are linearly independent then an appropriate $C(s)$ to satisfy condition (i), (ii) can be deduced from $H''(w)$. Otherwise partition $D_{hc}^{-1}$ as

$$D_{hc}^{-1} = \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] \tag{6.5.9}$$

where $D_{12}$ is a $\delta \times (\ell - \delta)$ matrix and $D_{22}$ a $(\ell - \delta) \times (\ell - \delta)$ matrix. Now since $D_{hc}^{-1}$ is non-singular the matrix $\begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix}$ will have $\ell - \delta$ linearly independent rows and if $D_{22}$ has rank $\eta$ there exists $\ell - \delta - \eta$ rows of $D_{12}$ which are linearly independent of the $\eta$ linearly independent rows of $D_{22}$. Let $Q$ be the matrix

$$Q = \left[\begin{array}{c|c} I_{\delta} & 0 \\ \hline \Psi & I_{\ell-\delta} \end{array}\right]$$

where

$$(\psi)_{ij} = \begin{cases} 1 & \text{if row } j \text{ of } D_{12} \text{ is linearly independent and row } i \text{ of } D_{22} \\ & \text{is linearly dependent on the } \delta \text{ linearly independent rows of } D_{22} \\ 0 & \text{otherwise.} \end{cases}$$

Then, if

$$QD_{hc}^{-1} = \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D'_{21} & D'_{22} \end{array}\right]$$

the matrix $D'_{22}$ is non-singular and therefore the last $\ell - \delta$ columns of $H''(0)QD_{hc}^{-1}$ will be linearly independent. Further, the effect of $Q$ on $H''(w)$ is to add certain of the last $\ell - \delta$ columns to the first $\delta$ columns. In view of the fact that the column degrees, $h'_i$, of $H''(w)$ are ordered $h'_1 \geq h'_2 \geq \ldots \geq h'_\ell$ and that $H''(w)$ is column proper the column degrees of the resulting matrix $H'''(w)$ are equivalent to those of $H''(w)$. Hence, a suitable $C(s)$ to satisfy conditions (i) and (ii) can be deduced from $H'''(w)$ to complete the proof. $\square$

To illustrate the constructive nature of the proof of theorem (6.5.7) consider the following example.

**(6.5.10) Example.** Let $c_1 = 1,\ c_2 = 1,\ c_3 = 1$ and

$$D_{hc}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Suppose that $C\left(\frac{1}{w}\right)\Lambda(w)$ has Smith form

$$\text{diag}\,[w^3, 1, 1].$$

Then, to construct a suitable $C(s)$ such that conditions (i) and (ii) of theorem (6.5.7) are satisfied first define

$$H(w) = \begin{bmatrix} w^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Following the procedure described by Rosenbrock [1970] construct the matrix $H'(w)$ in the following manner.

$$\text{row } 1 \Rightarrow \text{row } 1 + \text{w}\cdot \text{ row } 2 \qquad \begin{bmatrix} w^3 & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{col. } 1 \Rightarrow \text{col. } 1 - w^2\cdot \text{ col. } 2 \qquad \begin{bmatrix} 0 & w & 0 \\ -w^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{row } 1 \Leftrightarrow \text{row } 2 \qquad \begin{bmatrix} -w^2 & 1 & 0 \\ 0 & w & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{row } 1 \Rightarrow \text{row } 1 + \text{w}\cdot \text{ row } 3 \qquad \begin{bmatrix} -w^2 & 1 & w \\ 0 & w & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{col. } 1 \Rightarrow \text{col. } 1 + \text{w}\cdot \text{ col. } 3 \qquad \begin{bmatrix} 0 & 1 & w \\ 0 & w & 0 \\ w & 0 & 1 \end{bmatrix}$$

$$\text{row } 1 \Leftrightarrow \text{row } 3 \qquad \begin{bmatrix} w & 0 & 1 \\ 0 & w & 0 \\ 0 & 1 & w \end{bmatrix} \triangleq H'(w).$$

Further row interchanges result in the matrix

$$H''(w) = \begin{bmatrix} 0 & w & 0 \\ 0 & 1 & w \\ w & 0 & 1 \end{bmatrix}$$

which is of the form described in theorem (6.5.7).

Now

$$H''(0)\, D_{hc}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

where the last $\ell - \delta = 2$ columns are not linearly independent.

Next, partition $D_{hc}^{-1}$ as in (6.5.9)

$$D_{hc}^{-1} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right].$$

Let

$$Q = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$$

then

$$Q\, D_{hc}^{-1} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

where

$$D'_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is non-singular. Also

$$H''(0) Q\, D_{hc}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

where the last $\ell - \delta = 2$ columns of $H''(0)Q\, D_{hc}^{-1}$ are linearly independent, as required.

Finally

$$H'''(w) \triangleq H''(w)Q = \begin{bmatrix} 0 & w & 0 \\ w & 1 & w \\ w+1 & 0 & 1 \end{bmatrix}$$

where the column degrees of $H'''(w)$ are equal to the corresponding $c_i$'s. Hence, an appropriate $C(s)$ can be deduced by noting that

$$H'''(w) = C\left(\tfrac{1}{w}\right)\Lambda(w)$$

to give

$$C(s) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & s & 1 \\ 1+s & 0 & s \end{bmatrix}.$$

Return now to the problem of assigning the infinite pole structure in a generalised state space system using constant generalised state feedback. The results of theorems (6.5.2) and (6.5.7) enable the following theorem to be proved.

**(6.5.11) Theorem.** Let $G(s)$ be the transfer function matrix associated with the strongly controllable generalised state space system (6.3.1), (6.3.2) and let

$$G(s) = N(s)\,D^{-1}(s)$$

be a right minimal factorisation of $G(s)$ where the column degrees, $c_i$, of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Then, the infinite pole structure of the closed loop system under generalised state feedback of the form (6.3.3) and represented in terms of the $b_i$, $i = 1, 2, \ldots, \ell$, must satisfy the necessary and sufficient conditions that

(i) $b_{\delta+1} = b_{\delta+2} = \ldots = b_\ell = 0$

where $\delta$ is the number of linearly independent last position rows of $B$ when the system is represented in Kronecker form,

(ii) $\displaystyle\sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell-1.$ (6.5.12)

**Proof.** The necessity part of the result follows as a direct consequence of corollary (6.4.7) and by taking $a_i = 0,\ i = 1, 2, \ldots, \ell$ in theorem (6.3.11).

For the sufficiency part, let the matrix polynomial $C(s)$ be chosen such that $C\left(\frac{1}{w}\right)\Lambda(w)$ , where $\Lambda(w)=\mathrm{diag}[w^{c_1},w^{c_2},\ldots,w^{c_\ell}]$ has Smith form

$$\mathrm{diag}\left[w^{b_1}f_1(w),w^{b_2}f_2(w),\ldots,w^{b_\delta}f_\delta(w),f_{\delta+1}(w),\ldots,f_\ell(w)\right] \tag{6.5.13}$$

where the $b_i$'s satisfy condition (ii) of the theorem and where $f_{i+1}(w) \mid f_i(w)$, $i = 1,2,\ldots,\ell-1$, with $f_i(0)\neq 0$, $i=1,2,\ldots,\ell$. Without loss of generality the $f_i(w)$'s can be taken to be

$$f_i(w)=1 \qquad i=1,2,\ldots,\ell$$

since only the structure of (6.5.13) at $w=0$ will be of relevance.

Then, the sufficiency part is proved if there exists a constant matrix $K$ such that

$$\left[D\left(\tfrac{1}{w}\right)+K\,N\left(\tfrac{1}{w}\right)\right]\Lambda(w)=C\left(\tfrac{1}{w}\right)\Lambda(w) \tag{6.5.14}$$

or equivalently

$$D(s)+K\,N(s)=C(s). \tag{6.5.15}$$

Now since $D(s),N(s)$ are relatively (right) prime it follows (see Rosenbrock, 1970) that there exist polynomial matrices $X(s),Y(s)$ such that

$$X(s)\,D(s)+Y(s)\,N(s)=C(s). \tag{6.5.16}$$

Thus, the sufficiency part of the theorem will be proved if there exist $X(s),Y(s)$ satisfying (6.5.16) where $X(s),Y(s)$ are constant matrices with $X(s)$ non-singular.

Without loss of generality assume that the system is represented in Kronecker form with $B_2$ in column echelon form.

Then, the fundamental relationships can be written as

$$\begin{bmatrix} X(s) & Y_1(s) & Y_2(s) \\ -B_1 & sI_{n_1}-A_1 & 0 \\ -B_2 & 0 & sJ-I_{n_2} \end{bmatrix} \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} C(s) \\ 0 \\ 0 \end{bmatrix}. \tag{6.5.17}$$

Let $T$ be the transformation which interchanges column $q_1+q_2+\ldots+q_{i-1}+1$ of $sJ-I_{n_2}$ with column $i$ of $-B_2$ for $i=1,2,\ldots,p-t$ and where $q_0\triangleq 0$. Then,

$$\begin{bmatrix} X(s) & Y_1(s) & Y_2(s) \\ -B_1 & sI_{n_1}-A_1 & 0 \\ -B_2 & 0 & sJ-I_{n_2} \end{bmatrix} T = \begin{bmatrix} \hat{X}(s) & Y_1(s) & \hat{Y}_2(s) \\ -\hat{B}_1 & sI_{n_1}-A_1 & \hat{H} \\ -\hat{B}_2 & 0 & \hat{J}(s) \end{bmatrix} \tag{6.5.18}$$

where $\hat{H}$ is a constant matrix and $\hat{J}_2(s)$ is a column reduced polynomial matrix with column degrees

$$\begin{cases} 0 & i=1,q_1+1,q_1+q_2+1,\ldots,q_1+q_2+\ldots+q_{p-1}+1 \\ 1 & \text{otherwise.} \end{cases}$$

The inverse transformation $T^{-1}$ acts on $\begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}$ by interchanging row $\ell + n_1 + q_1 + \ldots + q_{i-1} + 1$ with row $i$ for $i = 1, 2, \ldots, p - t$. Thus, if

$$T^{-1} \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} \hat{D}(s) \\ N_1(s) \\ \hat{N}_2(s) \end{bmatrix} \tag{6.5.19}$$

it follows from theorem (6.5.2) that $\hat{D}(s)$ is column reduced with column degrees $c_1, c_2, \ldots, c_\ell$. Also, as a consequence of the proof of theorem (6.5.2) the $i^{\text{th}}$ column degree of $N_1(s)$ is less than $c_i$, $i = 1, 2, \ldots, \ell$.

Finally, consider the structure of $\hat{N}_2(s)$. Combining (6.5.18) and (6.5.19) with (6.5.17) results, in particular, in

$$\begin{bmatrix} -\hat{B}_2 & 0 & \hat{J}(s) \end{bmatrix} \begin{bmatrix} \hat{D}(s) \\ N_1(s) \\ \hat{N}_2(s) \end{bmatrix} = 0$$

i.e.

$$\begin{bmatrix} -\hat{B}_2 & \hat{J}(s) \end{bmatrix} \cdot \begin{bmatrix} \hat{D}(s) \\ \hat{N}_2(s) \end{bmatrix} = 0.$$

Let $\begin{bmatrix} -B_1 \\ -B_2 \end{bmatrix} = (b_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, \ell$. If $[\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_\ell, \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_{n_2}]^T$ is a column of $\begin{bmatrix} D_2(s) \\ \hat{N}_2(s) \end{bmatrix}$ then it must satisfy

$$
\left[\begin{array}{*{26}{c}}
-1 &&&&&&&& b_{n_1+1,p+1} & \cdots & b_{n_1+1,\ell} & 0 & s \\
0 &&&&&&&& \vdots && \vdots & \vdots & -1 & \ddots \\
\vdots &&&&&&&& \vdots && \vdots & \vdots && \ddots & \ddots \\
\vdots &&&&&&&& \vdots && \vdots & \vdots &&& \ddots & \ddots \\
0 &&&&&&&& b_{v_1-1,p+1} & \cdots & b_{v_1-1,\ell} & 0 &&&& -1 & s \\
0 && \mathbf{0} &&&&&& b_{v_1,p+1} & \cdots & b_{v_1,\ell} & -1 & 0 & \cdots & \cdots & 0 & -1 &&&&& \mathbf{0} \\
\ddots &&&&&&&& | && | &&&&&&& \ddots \\
& \ddots &&&&&&& | && | &&&&&&& \ddots \\
& \ddots &&&&&&& | && | &&&&&&& \ddots \\
& -1 &&&&&&& b_{v_{p-t-1}+1,p+1} & \cdots & b_{v_{p-t-1}+1,\ell} &&&&&&&& 0 & s \\
& 0 &&&&&&& \vdots && \vdots &&&&&&&& \vdots & -1 & \ddots \\
& \vdots &&&&&&& \vdots && \vdots &&&&&&&& \vdots && \ddots & \ddots \\
& \vdots &&&&&&& \vdots && \vdots &&&&&&&& 0 &&& \ddots & s \\
& 0 &&&&&&& b_{v_{p-t},p+1} & \cdots & b_{v_{p-t},\ell} &&&&&&&& -1 & 0 & \cdots & 0 & -1 \\
&& -1 &&&&&& b_{v_{p-t+1},p+1} & \cdots & b_{v_{p-t+1},\ell} &&&&&&&&&&&&& -1 \\
&&& \ddots &&&&& \vdots && \vdots &&&&&&&&&&&&&& \ddots \\
&&&& -1 &&&& \vdots && \vdots &&&&&&&&&&&&&& \ddots \\
&& \mathbf{0} &&& 0 &&& \vdots && \vdots &&&&&&&& \mathbf{0} &&&&&& \ddots \\
&&&&&& \ddots &&&&&&&&&&&&&&&&&& \ddots \\
&&&&&&& 0 & b_{v_p,p+1} & \cdots & b_{v_p,\ell} &&&&&&&&&&&&&&& -1
\end{array}\right]
\left[\begin{array}{c}
\hat{d}_1 \\ \vdots \\ \hat{d}_\ell \\ \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_{n_2}
\end{array}\right] = 0
$$

where $v_i = n_1 + \sum_{j=1}^{i} q_j,\ i = 1, 2, \ldots, p$.

Adopting a similar argument to that employed in the proof of theorem (6.5.2) it follows from the first block corresponding to the non-trivial Jordan blocks that the degrees of $\hat{\gamma}_2, \hat{\gamma}_3, \ldots, \hat{\gamma}_{q_1}$ are all less than the highest degree amongst $\hat{d}_1, \hat{d}_{p+1}, \ldots, \hat{d}_\ell$. Similarly for the other non-trivial blocks. Thus,

$$\hat{N}_2\left(\tfrac{1}{w}\right)\Lambda(w)\Big|_{w=0}$$

can only have non-zero elements in rows

$$1, q_1+1, q_1+q_2+1, \ldots, q_1+q_2+\ldots+q_{p-1}+1.$$

Applying the transformations $T, T^{-1}$ to (6.5.17) results in

$$\begin{bmatrix} \hat{X}(s) & Y_1(s) & \hat{Y}_2(s) \\ -\hat{B}_1 & sI_{n_1}-A_1 & \hat{H} \\ -\hat{B}_2 & 0 & \hat{J}(s) \end{bmatrix} \begin{bmatrix} \hat{D}(s) \\ N_1(s) \\ \hat{N}_2(s) \end{bmatrix} = \begin{bmatrix} C(s) \\ 0 \\ 0 \end{bmatrix}. \tag{6.5.20}$$

Now there exist polynomial matrices $Q_1(s), Q_2(s)$ such that

$$Y_1(s) = Q_1(s)\,(sI_{n_1}-A_1) + \bar{Y}_1$$

and

$$\hat{Y}_2(s) - Q_1(s)\,\hat{H} = Q_2(s)\,\hat{J}(s) + \bar{Y}_2$$

where $\bar{Y}_1[sI_{n_1}-A_1]^{-1}$ and $\bar{Y}_2\,\hat{J}^{-1}(s)$ are strictly proper and $\bar{Y}_1, \bar{Y}_2$ are constant matrices. Further since $\hat{J}^{-1}(s)$ is of the form

$$\text{diag}\,[\psi_1(s), \psi_2(s), \ldots, \psi_{p-t}(s), -I_t]$$

where

$$\psi_i(s) = \begin{bmatrix} & -1 \\ \theta_i(s) & 0 \\ & \vdots \\ & 0 \end{bmatrix} \qquad \theta_i(s) \in \Re^{q_i\times(q_i-1)}(s), \qquad i = 1, 2, \ldots, p-t$$

and since $\bar{Y}_2\,\hat{J}^{-1}(s)$ is strictly proper it follows that columns $1, q_1+1, q_1+q_2+1, \ldots, q_1+q_2+\ldots, q_{p-t-1}+1, q_1+q_2+\ldots+q_{p-t+1}, \ldots, q_1+q_2+\ldots+q_p$ of $\bar{Y}_2$ are zero.

The premultiplication of (6.5.20) by the unimodular matrix

$$\begin{bmatrix} I_\ell & -Q_1(s) & -Q_2(s) \\ 0 & I_{n_1} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}$$

gives rise to

$$\begin{bmatrix} \bar{X}(s) & \bar{Y}_1 & \bar{Y}_2 \\ -\hat{B}_1 & sI_{n_1} - A_1 & \hat{H} \\ -\hat{B}_2 & 0 & \hat{J}(s) \end{bmatrix} \begin{bmatrix} \hat{D}(s) \\ N_1(s) \\ \hat{N}_2(s) \end{bmatrix} = \begin{bmatrix} C(s) \\ 0 \\ 0 \end{bmatrix} \tag{6.5.21}$$

where $\bar{X}(s) = \hat{X}(s) + Q_1(s)\,\hat{B}_1 + Q_2(s)\,\hat{B}_2$.

Make the transformation $s = \frac{1}{w}$ in (6.5.21) and post multiply by $\Lambda(w)$ resulting, in particular, in

$$\left[\bar{X}\left(\tfrac{1}{w}\right) \quad \bar{Y}_1 \quad \bar{Y}_2\right] \begin{bmatrix} \hat{D}\left(\frac{1}{w}\right) \\ N_1\left(\frac{1}{w}\right) \\ \hat{N}_2\left(\frac{1}{w}\right) \end{bmatrix} \Lambda(w) = C\left(\tfrac{1}{w}\right)\Lambda(w). \tag{6.5.22}$$

Now $C\left(\frac{1}{w}\right)\Lambda(w)\Big|_{w=0}$ is a constant matrix and has rank deficiency $\delta$ by (6.5.13). Also

$$\bar{Y}_1 N_1\left(\tfrac{1}{w}\right)\Lambda(w)\Big|_{w=0} = 0$$

since the $i^{\text{th}}$ column degree of $N_1(s)$ is less than $c_i$, $i = 1, 2, \ldots, \ell$. From the structure of $\hat{N}_2\left(\frac{1}{w}\right)\Lambda(w)\Big|_{w=0}$ and the arrangement of the zero columns of $\bar{Y}_2$ it is concluded that

$$\bar{Y}_2 \hat{N}_2\left(\tfrac{1}{w}\right)\Lambda(w)\Big|_{w=0} = 0.$$

Let $\hat{D}_{hc}$ be the high order coefficient matrix with respect to the columns of $\hat{D}(s)$. Then,

$$\hat{D}\left(\tfrac{1}{w}\right)\Lambda(w)\Big|_{w=0} = \hat{D}_{hc}$$

which is non-singular by construction. Thus, (6.5.22) gives rise to

$$X\left(\tfrac{1}{w}\right)\Big|_{w=0} = \left[C\left(\tfrac{1}{w}\right)\Lambda(w)\right]_{w=0}\hat{D}_{hc}^{-1}$$

which implies $\bar{X}(s)$ is a constant matrix with rank deficiency $\delta$.

Applying the inverse transformation results in

$$\begin{bmatrix} X' & \bar{Y}_1 & Y_2' \\ -B_1 & sI_{n_1} - A_1 & 0 \\ -B_2 & 0 & sJ - I_{n_2} \end{bmatrix} \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} C(s) \\ 0 \\ 0 \end{bmatrix} \tag{6.5.23}$$

where $X', Y_2'$ are constant matrices. The inverse transformation replaces column $i$ of $\bar{X}$ by the zero column $q_1 + q_2 + \ldots + q_{i-1} + 1$ of $\bar{Y}_2$ for $i = 1, 2, \ldots, p - t$. The matrix $C(s)$ was originally chosen such that $C\left(\frac{1}{w}\right) \Lambda(w)$ has Smith form

$$\operatorname{diag}\left[w^{b_1}, w^{b_2}, \ldots, w^{b_\delta}, 1, 1, \ldots, 1\right]$$

By theorem (6.5.7), $C(s)$ can be chosen such that the further condition that the last $\ell - \delta$ columns of $C\left(\frac{1}{w}\right) \Lambda(w) \Big|_{w=0} \hat{D}_{hc}^{-1}$ are linearly independent is also satisfied. Thus, the rank deficiency of $X'$ can be restored by adding suitable multiples of rows $q_1 + q_2 + \ldots + q_i$, $i = 1, 2, \ldots, \delta$, of $[-B_2 \quad 0 \quad sJ - I_{n_2}]$ to the first $\ell$ rows of (6.5.23). This operation does not destroy the constancy of $\bar{Y}_1$ or $Y_2'$ so demonstrating the existence of constant $X(s), Y(s)$ with $X(s)$ non-singular such that (6.5.16) holds. Hence, the sufficiency part of the theorem is proved as required. □

The result of theorem (6.5.11) provides a complete characterisation of the infinite pole structure that can be assigned by employing constant generalised state feedback around a generalised state space system (6.3.1), (6.3.2). If all the closed loop poles are located at infinity then equality holds when $k = 0$ in condition (ii). The resulting theorem provides an analogy to the result of Kucera and Zagalak [1988] for the case where all the poles are assigned at infinite locations.

Note also that the result of theorem (6.5.11) indicates that it is not possible to assign poles at infinite locations in the closed loop system if and only if $\delta = 0$. This condition is equivalent to the condition that the system is in regular state space form and confirms the result that when constant output feedback is applied to such systems the closed loop system poles are all still located at finite locations (see Rosenbrock and Rowe, 1974). For proper systems the result of theorem (6.5.11) indicates that it is always possible to place poles at infinite locations using constant output feedback.

## §6. Further necessary conditions for the simultaneous placement of both the finite and infinite pole structures.

In section 3 necessary conditions were obtained for the simultaneous placement of both the finite and infinite pole structures using constant generalised state feedback in generalised state space systems. Stronger necessary conditions are produced by combining the conditions described in section 3 with the subsequent results concerning the assignment of the infinite pole structure, giving rise to the following.

**(6.6.1) Theorem.** Let $\alpha_i(s), \beta_i(w), a_i, b_i,\ i = 1, 2, \ldots, \ell$, be given as in theorem (6.3.11). Let $G(s)$ be the transfer function matrix associated with the strongly controllable system represented by (6.3.1), (6.3.2), i.e.

$$G(s) = [sE - A]^{-1} B$$

and let $G(s)$ have a right minimal factorisation

$$G(s) = N_1(s)\, D_1^{-1}(s)$$

where the column degrees, $c_i$, of $\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$ are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Let $\Lambda_1(s) = \text{diag}\,[s^{c_1}, s^{c_2}, \ldots, s^{c_\ell}]$. Then, for there to exist a constant matrix $K$ such that the invariant polynomials of $D_1(s) + K\, N_1(s)$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_\ell(s)$ and the invariant polynomials of $[D_1\left(\frac{1}{w}\right) + K\, N_1\left(\frac{1}{w}\right)]\, \Lambda_1(w)$ are $\beta_1(w), \beta_2(w), \ldots, \beta_\ell(w)$ it is necessary that

(i) $b_{\delta+1} = b_{\delta+2} = \ldots = b_\ell = 0$

where $\delta$ is the number of linearly independent last position rows of $B$ when the system is represented in Kronecker form,

(ii) $\displaystyle\sum_{i=k+1}^{\ell} a_i + \sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1$

with equality when $k = 0$.

**Proof.** Conditions (i) follow from the necessary conditions of corollary (6.4.7) whilst conditions (ii) follow from the necessary conditions presented in theorem (6.3.11), as required. □

In example (6.3.14) it was demonstrated that the necessary conditions of theorem (6.3.11) are not sufficient. The additional conditions included in theorem (6.6.1) explain clearly why the pole structure $b_1 = 1,\ b_2 = 1$ cannot be assigned by constant generalised state feedback for that particular system since, under condition (i) of theorem (6.6.1), $b_2$ must equal zero in this case.

Efforts to find an example to demonstrate that the necessary conditions of (6.6.1) are not sufficient have been unsuccessful. Equally it has not been possible to prove the sufficiency of the conditions. Thus, the question of whether or not the necessary conditions of theorem (6.6.1) are sufficient still remains unanswered.

## §7. Discussion of infinite pole assignment problem.

In a recent paper Fahmy and O'Reilly [1989] consider the pole placement problem in generalised state space systems for the case where all the poles are placed at infinite locations. A procedure to find a suitable matrix $K$ that will achieve this goal is presented.

Fahmy and O'Reilly [*ibid.*] assume that the system is strongly reachable. This implies that when the system is represented in Kronecker form all the last position rows of $B$ are linearly independent. In this case theorem (6.5.11) states that the multiplicity of the closed loop infinite poles can achieve its maximum, i.e. $\delta = p$, the number of Jordan blocks of $J$. This is in agreement with the possible multiplicities assumed by Fahmy and O'Reilly [*ibid.*].

Fahmy and O'Reilly [*ibid.*] do not examine the closed loop infinite pole structure in detail but note that the degrees of the closed loop infinite poles are related to the lengths of the generalised eigenvector chains associated with the $p$ infinite eigenvalues of $E$. The infinite eigenvectors and generalised eigenvectors are defined as follows.

**(6.7.1) Definition (Fahmy and O'Reilly, 1989).** Let $v_{i0}^{\infty} \in \Re^n$, $i = 1, 2, \ldots, p$, be defined by

$$E v_{i0}^{\infty} = 0 \qquad i = 1, 2, \ldots, p.$$

Then $v_{i0}^{\infty}$, $i = 1, 2, \ldots, p$, are called the $p$ INFINITE EIGENVECTORS of $E$. Further, if $K$ is the feedback matrix that places all the finite eigenvalues at infinite locations (i.e. assigns all the poles at infinite locations) then the GENERALISED INFINITE EIGENVECTOR CHAINS associated with the infinite eigenvectors are defined by

$$E v_{ij}^{\infty} = [A + BK] v_{i,j-1}^{\infty} \qquad j = 1, 2, \ldots, \alpha_i - 1, \ i = 1, 2, \ldots, p.$$

The lengths, $\alpha_i$, of the eigenvector chains satisfy

$$\sum_{i=1}^{p} \alpha_i = n$$

and are non-unique due to the freedom in choosing $K$. The degrees of the closed loop poles are equal to $\alpha_i - 1$, $i = 1, 2, \ldots, p$, and the relationship between the $\alpha_i$'s and the previously defined $b_i$'s is given by

$$b_i = \alpha_i - 1 \qquad i = 1, 2, \ldots, p. \tag{6.7.2}$$

The result of theorem (6.5.11) therefore gives an immediate characterisation of the possible chain lengths associated with a particular system and this is described by the following.

**(6.7.3) Theorem.** Consider the generalised state space system (6.3.1), (6.3.2) which is assumed to be strongly reachable. Let the transfer function matrix, $G(s)$, associated with this system have a right minimal factorisation of the form

$$G(s) = N(s)\,D^{-1}(s)$$

where the column degrees, $c_i$, of $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ are ordered $c_1 \geq c_2 \geq \ldots \geq c_\ell$. Let constant generalised state feedback, $K$, be applied to this system in such a way that all the poles are placed at infinite locations. Also, let the infinite eigenvectors and generalised infinite eigenvectors be defined by (6.7.1) and let $\alpha_i$, $i = 1, 2, \ldots, p$, be the associated chain lengths. Then, if the $\alpha_i$'s are ordered $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_p$ then the $\alpha_i$'s satisfy the necessary and sufficient conditions that

$$\sum_{i=k+1}^{p} (\alpha_i - 1) \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, p-1 \tag{6.7.4}$$

with equality when $k = 0$.

**Proof.** Since the system is strongly reachable then

$$\delta = p$$

where $\delta$ is the number of linearly independent last position rows of $B$ when the system is represented in Kronecker form. Then, by theorem (6.5.11), the closed loop infinite pole structure represented by $b_i$, $i = 1, 2, \ldots, \ell$, must satisfy the necessary and sufficient conditions

$$\text{(i)} \quad b_{p+1} = b_{p+2} = \ldots = b_\ell = 0$$

and

$$\text{(ii)} \quad \sum_{i=k+1}^{\ell} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, \ell - 1$$

or equivalently

$$\sum_{i=k+1}^{p} b_i \leq \sum_{i=k+1}^{\ell} c_i \qquad k = 0, 1, \ldots, p-1 \tag{6.7.5}$$

with equality when $k = 0$ since all the closed loop poles are located at infinite locations. From (6.7.2)

$$b_i = \alpha_i - 1 \qquad i = 1, 2, \ldots, p$$

which on substituting into (6.7.5) gives rise to (6.7.4) as required. □

The above characterisation of the infinite chain lengths is illustrated by the following example.

**(6.7.6) Example.** Let

$$G = [sE - A]^{-1}B$$

$$= \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

which is in Kronecker form and which is also strongly reachable. A right minimal factorisation of $G(s)$ is given by

$$G(s) = N(s)\,D^{-1}(s)$$

$$= \begin{bmatrix} 1 & 0 \\ -s^2 & 0 \\ -s & 0 \\ -s & -1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

so that $c_1 = 2,\ c_2 = 0$. Hence, the orders of the infinite poles of the closed loop system must satisfy

$$b_1 + b_2 \leq 2 \tag{6.7.7}$$

$$b_2 \leq 0 \tag{6.7.8}$$

with equality in (6.7.7) if all the poles are placed at infinity. Theorem (6.7.3) indicates that the chain lengths of the infinite eigenvectors and generalised eigenvectors as defined by (6.7.1) are

$$\alpha_1 = 3,\ \alpha_2 = 1$$

and that it is not possible for the chain lengths to be $\alpha_1 = 2,\ \alpha_2 = 2$. This can be seen by the following investigation.

The infinite eigenvectors are

$$v_{10}^{\infty} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_{20}^{\infty} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$ be the feedback so that

$$A + BK = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ 0 & 1 & 0 & 0 \\ k_1 & k_2 & 1+k_3 & k_4 \\ k_1+k_5 & k_2+k_6 & k_3+k_7 & 1+k_4+k_8 \end{bmatrix}.$$

Now $v_{21}^{\infty}$ is defined by

$$E v_{21}^{\infty} = [A + BK]\, v_{20}^{\infty}$$

i.e.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ 0 & 1 & 0 & 0 \\ k_1 & k_2 & 1+k_3 & k_4 \\ k_1+k_5 & k_2+k_6 & k_3+k_7 & 1+k_4+k_8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_4 \\ 0 \\ k_4 \\ 1+k_4+k_8 \end{bmatrix}$$

which implies $v_{21}^{\infty} = 0$, and $\alpha_2 = 1$ as predicted.

The results presented in this chapter on the placement of the infinite pole structure compliments the results given by Fahmy and O'Reilly [1989] by describing concisely the precise structure that can be assigned. More importantly the result of theorem (6.5.11) is concerned with the more general case when the system is assumed to be strongly controllable which, in the light of the discussion in chapter 5, is of more relevance to the design of closed loop systems.

## §8. Conclusions.

The general pole placement problem in generalised state space systems has been investigated in this chapter. The treatment of the problem is novel since it considers the assignment of the infinite pole structure as well as the finite pole structure. The problem was first approached by adopting a minimal factorisation of the open loop transfer function matrix. It was seen in chapters 2 and 3 that both the finite and infinite pole structures of a closed loop system, formed as a result of constant output feedback, can both be displayed in a particularly simple way in terms of a minimal factorisation of the open loop transfer function matrix. In section 3 the original results presented in chapter 4 were interpreted for systems in generalised form. This gave rise to a set of new necessary conditions for the simultaneous placement of both the finite and infinite pole structures in generalised state feedback.

In sections 4 and 5 the assignment of the infinite pole structure was specifically investigated by exploiting the detailed structure of the Kronecker canonical form of a system. New necessary conditions on the multiplicity of the closed loop infinite pole structure were presented in section 4. These conditions were given in terms of the last position rows of $B$ which indicates the existence of a close relationship between the placement of infinite poles and the notions of controllability in generalised state space systems.

Results from sections 3 and 4 were combined in section 5 to produce stronger necessary conditions on the closed loop infinite pole structure. These new necessary conditions were shown to be sufficient so providing a complete characterisation of the closed loop infinite pole structure and a generalisation of the result due to Kucera and Zagalak [1988] for the case when all the poles are placed at infinite locations. A stronger set of necessary conditions for the simultaneous placement of the finite and infinite pole structures was presented in section 6 by supplementing the conditions given in sections 3 with subsequent results. Section 7 discussed the relationship between the results presented in this chapter and the paper recently published by Fahmy and O'Reilly [1989].

# Chapter 7. Further Discussion of the General Pole Placement Problem.

## §1. Introduction.

The general pole placement problem is further discussed in this chapter.

In section 2 generalised state space systems are specifically considered and the problem is approached by assigning the finite and infinite pole structures in two separate stages. This gives rise to necessary and sufficient conditions for both the finite and infinite closed loop pole structures but the result is incomplete since the conditions on the respective pole structures are not directly related.

In section 3 a description of how a Laurent expansion about the point at infinity of a rational matrix can be used to investigate the pole structure under constant output feedback is presented. For certain systems this method provides a straightforward means of investigating the closed loop infinite pole structure and gives rise to a new condition for testing whether the closed loop system is proper.

Section 4 considers a bilinear transformation approach to the problem. It was seen that employing a bilinear transformation enables both the finite and infinite pole structures of a rational matrix to be simultaneously considered. Taking this as a basis, a possible minimal factorisation of a transformed matrix is investigated and the subsequent effect of constant output feedback on a transformed matrix is considered. The results provide a means of simultaneously investigating the finite and infinite closed loop pole structures. The theory is subsequently applied to the case of systems with transfer function matrices of the form $[sE - A]^{-1} B$.

## §2. Two stage approach.

The general pole placement problem for both finite and infinite pole structures in generalised state space systems may be approached by first assigning the infinite pole structure followed, using a second feedback, by the finite pole structure. This approach is made possible by the previously stated result that pure state feedback does not alter the infinite pole structure (Pugh *et al.*, 1988). A simpler proof of this result, more in spirit with the current work, is first presented.

**(7.2.1) Theorem.** Consider the strongly controllable generalised state space system (6.3.1), (6.3.2) represented in Kronecker form. Then, if constant pure state feedback is applied around this system the infinite poles of the system remain unchanged.

**Proof.** Let

$$G(s) = [sE - A]^{-1} B \tag{7.2.2}$$

be the open loop transfer function matrix where $sE - A$ is in Kronecker form. Then, by lemma (6.3.4), it follows that (7.2.2) is a left minimal factorisation of $G(s)$ and so the open loop infinite pole structure is given by the zero structure at $w = 0$ of

$$\Lambda(w) \begin{bmatrix} \frac{1}{w} I_{n_1} - A_1 & 0 \\ 0 & \frac{1}{w} J - I_{n_2} \end{bmatrix} \tag{7.2.3}$$

where

$$\Lambda(w) = \text{diag } [w^{i_1}, w^{i_2}, \ldots, w^{i_n}]$$

and

$$i_j = \begin{cases} 0 & j = n_1 + q_1, n_1 + q_1 + q_2, \ldots, n_1 + q_1 + \ldots + q_p \\ 1 & \text{otherwise.} \end{cases}$$

(7.2.3) can be written as

$$\begin{bmatrix} I_{n_1} - A_1 w & 0 \\ 0 & \Lambda'(w)\,(\frac{1}{w} J - I_{n_2}) \end{bmatrix}$$

where

$$\Lambda'(w) = \text{diag } [w^{i_1}, w^{i_2}, \ldots, w^{i_{n_2}}]$$

and

$$i_j = \begin{cases} 0 & j = q_1, q_1 + q_2, \ldots, q_1 + q_2 + \ldots + q_p \\ 1 & \text{otherwise.} \end{cases}$$

Since $I_{n_1} - A_1 w$ has full rank at $w = 0$ it follows that the infinite pole structure of the system is given by the zero structure at $w = 0$ of $\Lambda'(w)[\frac{1}{w} J - I_{n_2}]$.

Let pure state feedback of the form

$$u(t) = -K_1 x_1(t) + v(t)$$

be applied to the system. If the closed loop transfer function matrix is given by $G_K(s)$ then

$$G_K(s) = \begin{bmatrix} sI_{n_1} - A_1 + B_1 K_1 & 0 \\ B_2 K_1 & sJ - I_{n_2} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{7.2.4}$$

Since (7.2.2) is a minimal factorisation of $G(s)$ it follows that (7.2.4) is a minimal factorisation of $G_K(s)$ and the closed loop infinite pole structure is given by the zero structure at $w = 0$ of

$$\Lambda(w) \begin{bmatrix} \frac{1}{w} I_{n_1} - A_1 + B_1 K_1 & 0 \\ B_2 K_1 & \frac{1}{w} J - I_{n_2} \end{bmatrix}$$

or

$$\begin{bmatrix} I_{n_1} - A_1 w + B_1 K_1 w & 0 \\ \Lambda'(w) B_2 K_1 & \Lambda'(w) \left(\frac{1}{w} J - I_{n_2}\right) \end{bmatrix}. \tag{7.2.5}$$

At $w = 0$ (7.2.5) reduces to

$$\begin{bmatrix} I_{n_1} & 0 \\ \Lambda'(0) B_2 K_1 & \left\{\Lambda'(w) \left[\frac{1}{w} J - I_{n_2}\right]\right\}_{w=0} \end{bmatrix}$$

which indicates that the zero structure of (7.2.5) at $w = 0$ is given by

$$\Lambda'(w) \left[\tfrac{1}{w} J - I_{n_2}\right].$$

Thus, the closed loop infinite pole structure is identical to the open loop infinite pole structure as required. □

The result of theorem (7.2.1) thus enables the pole placement to be approached in two steps. First, the infinite pole structure can be assigned with reference to the necessary and sufficient conditions of theorem (6.5.11) then, secondly, pure state feedback can be employed to assign the finite pole structure. The necessary and sufficient conditions for the finite pole placement are supplied by theorem (2.4.3) (due to Rosenbrock and Rowe, 1970). Thus, if $G_K(s)$ is the closed loop transfer function matrix obtained from the first stage, then

$$G_K(s) = [sE - A + BK]^{-1} B.$$

If the closed loop system in Kronecker form is represented by

$$[sE - A + BK \quad B] = \begin{bmatrix} sI_{\hat{n}_1} - \hat{A}_1 & 0 & \hat{B}_1 \\ 0 & s\hat{J} - I_{\hat{n}_2} & \hat{B}_2 \end{bmatrix}.$$

Then, applying pure state feedback around this system is equivalent to considering the regular state space system

$$\dot{\hat{x}}_1(t) = \hat{A}_1 \hat{x}_1(t) + \hat{B}_1 u(t) \tag{7.2.6}$$

with state feedback of the form

$$u(t) = -\hat{L}_1 \hat{x}_1(t) + v(t). \tag{7.2.7}$$

The general pole placement problem for such systems was considered by Rosenbrock and Rowe [1970] and their result is given in theorem (2.4.3). The theorem is recalled here with special reference to the system (7.2.6) and feedback (7.2.7).

**(7.2.8) Theorem (Rosenbrock and Rowe, 1970).** Consider the state space system given by (7.2.6) and where $sI_{\hat{n}_1} - \hat{A}_1$ and $\hat{B}_1$ are relatively (left) prime. Let the minimal indices of $[sI_{\hat{n}_1} - \hat{A}_1 \quad \hat{B}_1]$ be given by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$. Let constant state feedback of the form (7.2.7) be applied to the system and let the non-unit invariant polynomials of $sI_{\hat{n}_1} - \hat{A}_1 + \hat{B}_1\hat{L}_1$ be given by $\alpha_i(s)$, $i = 1, 2, \ldots, \sigma = \min(\hat{n}_1, \ell)$, where $\alpha_i(s) \mid \alpha_{i-1}(s)$, $i = 2, 3, \ldots, \sigma$, and $\deg \alpha_i(s) = a_i$, $i = 1, 2, \ldots, \sigma$. Then, the $a_i$, $i = 1, 2, \ldots, \sigma$, must satisfy the necessary and sufficient conditions that

$$\sum_{i=k+1}^{\sigma} a_i \leq \sum_{i=k+1}^{\ell} \lambda_i \qquad k = 0, 1, \ldots, \sigma - 1$$

with equality when $k = 0$.

A partial solution to the general pole placement problem has therefore been found with the necessary and sufficient conditions given in terms of the $\lambda_i$'s and $c_i$'s but this result is incomplete since the $\lambda_i$'s and $c_i$'s are not directly related. A satisfactory solution to the problem would be achieved if the relationship between the $c_i$'s and $\lambda_i$'s could be fully characterised.

The above discussion lead to the following example which gives a further insight into this two stage approach to the problem.

**(7.2.9) Example.** Consider a strongly controllable generalised state space system with $[sE - A \quad -B]$ in Kronecker form

$$\left[\begin{array}{c|c|c} sI_{n_1} - A_1 & 0 & B_1 \\ \hline 0 & sJ - I_{n_2} & B_2 \end{array}\right] = \left[\begin{array}{cc|ccc|cc} s & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & s & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right].$$

Let a right minimal factorisation of the open loop transfer function be

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \equiv \begin{bmatrix} D(s) \\ N_1(s) \\ N_2(s) \end{bmatrix} = \left[\begin{array}{cc} s^2 & 0 \\ 0 & -1 \\ \hline -s & 0 \\ 1 & 0 \\ \hline 0 & s^2 \\ 0 & s \\ 0 & 1 \end{array}\right]$$

so that $c_1 = 2, c_2 = 2$. Note that $\begin{bmatrix} D(s) \\ N_1(s) \end{bmatrix}$ is a right minimal factorisation of the strictly proper subsystem $[sI_{n_1} - A_1]^{-1} B_1$ which therefore has controllability indices $\lambda_1 = 2$, $\lambda_2 = 0$.

Assume that this system has been formed as a result of employing generalised state feedback to assign the infinite pole structure $b_1 = 2, b_2 = 0$. If pure state feedback is applied around this system then theorem (7.2.8) requires the closed loop finite pole structure to satisfy the necessary conditions that

$$\left.\begin{aligned} a_1 + a_2 &= 2 \\ a_2 &\leq 0 \end{aligned}\right\}$$

which obviously implies that the pole structure $a_1 = 1, a_2 = 1$ cannot be assigned by employing constant pure state feedback.

It is important to interpret these observations correctly. On the surface the example seems to illustrate that the necessary conditions of theorem (6.6.1) are not sufficient conditions since the pole structure $b_1 = 2, b_2 = 0, a_1 = 1, a_2 = 1$ satisfies the necessary conditions but cannot be assigned by adopting the above approach. However, this pole structure can be assigned to the closed loop system if constant generalised state feedback is applied directly. If $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 \\ k_6 & k_7 & k_8 & k_9 & k_{10} \end{bmatrix}$ is the generalised state feedback matrix then the closed loop finite and infinite pole structures are given respectively by

$$D(s) + K\,N(s) = \begin{bmatrix} s^2 - k_1 s + k_2 & k_3 s^2 + k_4 s + k_5 \\ -k_6 s + k_7 & k_8 s^2 + k_9 s + k_{10} - 1 \end{bmatrix}$$

and

$$\left[D\left(\tfrac{1}{w}\right) + K\,N\left(\tfrac{1}{w}\right)\right]\Lambda(w) = \begin{bmatrix} k_2 w^2 - k_1 w + 1 & k_5 w^2 + k_4 w + k_3 \\ k_7 w^2 - k_6 w & (k_{10} - 1) w^2 + k_9 w + k_8 \end{bmatrix}$$

where $\Lambda(w) = \text{diag}\ \left[w^2, w^2\right]$.

Choosing $k_1 = 0,\ k_2 = -\alpha^2,\ k_3 = 0, k_4 = 1,\ k_5 = \alpha,\ k_6 = -1,\ k_7 = \alpha,\ k_8 = 0,$ $k_9 = 0,\ k_{10} = 1$ where $\alpha \in \Re$ gives rise to

$$D(s) + K\,N(s) = \begin{bmatrix} s^2 - \alpha^2 & s + \alpha \\ s + \alpha & 0 \end{bmatrix}$$

which has Smith form

$$\begin{bmatrix} s + \alpha & 0 \\ 0 & s + \alpha \end{bmatrix}$$

and

$$\left[D\left(\tfrac{1}{w}\right) + K\,N\left(\tfrac{1}{w}\right)\right]\Lambda(w) = \begin{bmatrix} -\alpha^2 w^2 + 1 & \alpha w^2 + w \\ \alpha w^2 + w & 0 \end{bmatrix}$$

which has Smith form

$$\begin{bmatrix} w^2(1+\alpha w) & 0 \\ 0 & (1+\alpha w) \end{bmatrix}.$$

Hence the closed loop system has a finite pole structure where $a_1 = 1,\ a_2 = 1$ and an infinite pole structure where $b_1 = 2,\ b_2 = 0$.

The explanation as to why generalised state feedback is able to assign the desired pole structure in the above example is that under this feedback one of the finite poles is relocated at infinity whilst one of the infinite poles is placed at the desired finite location. Employing pure state feedback cannot achieve this since this feedback cannot influence the infinite poles. This therefore implies that when adopting the two stage approach there exists a possible degree of freedom in the first stage which can be exploited to broaden the choice of pole placement in the second stage. In other words the controllability indices of the strictly proper part of the system are dependent on the feedback matrix adopted in the first stage. The necessary and sufficient conditions obtained via the two stage method are still valid but care must be taken in practice in obtaining the controllability indices, $\lambda_i$.

The above observations reinforce the earlier remark that the necessary and sufficient conditions obtained via the two stage method are unsatisfactory and also leaves open the question of whether or not the necessary conditions of theorem (6.6.1) are also sufficient.

## §3. Toeplitz matrix approach.

In chapter 3 a method for obtaining the infinite pole and zero structure of a rational matrix was described in terms of the Laurent expansion about the point at infinity of the rational matrix. This method can be exploited to investigate the pole placement problems associated with applying constant output feedback around a system. Let $G(s)$ be the open loop rational transfer function matrix and if $G_K(s)$ is the closed loop transfer function matrix formed as a result of applying constant output feedback, $K$, around $G(s)$ then

$$G_K(s) = G(s)[I + K\,G(s)]^{-1}.$$

Expanding $G_K(s)$ as a Laurent expansion about the point at infinity means that the infinite pole structure of $G(s)$ can be investigated as described in chapter 3.

It is difficult to make a general statement about the infinite pole structure that can be assigned by using the Toeplitz matrix method but this approach could be effectively employed in individual cases to determine the effect of a certain feedback matrix or in determining the freedom in designing the infinite pole structure. For instance, it is known that almost all constant output feedback matrices give rise to a closed loop system having a proper transfer function matrix (Anderson and Scott, 1976). This method can therefore be used to characterise the set of feedbacks that do in fact give rise to a closed loop system with a non-proper transfer function matrix.

The main difficulty with this approach lies in obtaining the Laurent expansion at infinity of the relevant transfer function matrix. This problem can be overcome to some extent in some special cases. Consider the case of a square $n \times n$ transfer function matrix which is of full rank. It follows (see Verghese, 1978) that the pole structure of such a matrix is isomorphic to the zero structure of its inverse. Thus, the pole structure of $G_K(s)$ can be investigated by considering the zero structure of

$$G_K^{-1}(s) = G^{-1}(s) + K.$$

The separation of $K$ from $G(s)$ means that $K$ appears as a whole in just one term in the corresponding Laurent expansion about the point at infinity so making it easier to investigate the effect of $K$ on the closed loop infinite pole structure.

The investigation is further simplified if the $n \times n$ matrix $G(s)$ is of the form

$$G(s) = [sE - A]^{-1}\,B \tag{7.3.1}$$

where $sE - A, B$ are relatively (left) prime and $B$ is of full rank. Then

$$\begin{aligned} G^{-1}(s) &= B^{-1}[sE - A] \\ &= s\,B^{-1}E - B^{-1}A \end{aligned}$$

and

$$G^{-1}(s) + K = sB^{-1}E - B^{-1}A + K$$

which is the Laurent expansion about the point at infinity of $G^{-1}(s) + K$. A simple test to determine whether or not the resulting closed loop system is proper can now be stated as a result of this immediate characterisation of the Laurent expansion about the point at infinity.

**(7.3.2) Theorem.** Let $G(s)$ be as described by (7.3.1). Then, if constant generalised state feedback $K$ is applied to this system the resulting closed loop system will be proper if and only if

$$\operatorname{rank}\begin{bmatrix} B^{-1}E & -B^{-1}A + K \\ 0 & B^{-1}E \end{bmatrix} = 2n - p$$

or, alternatively,

$$\operatorname{rank}\begin{bmatrix} E & -A + BK \\ 0 & E \end{bmatrix} = 2n - p$$

where $p$ is the rank deficiency of $E$.

**Proof.** The closed loop pole structure at infinity is isomorphic to the zero structure at infinity of $G_K^{-1}(s)$. Employing the test for the absence of infinite zeros stated in theorem (3.4.1) leads to the following condition for the closed loop system to be proper, namely that

$$\operatorname{rank}\begin{bmatrix} B^{-1}E & -B^{-1}A + K \\ 0 & B^{-1}E \end{bmatrix} = 2n - p.$$

Since

$$\begin{bmatrix} B^{-1}E & -B^{-1}A + K \\ 0 & B^{-1}E \end{bmatrix} = B^{-1}\begin{bmatrix} E & -A + BK \\ 0 & E \end{bmatrix}$$

and $B$ is of full rank, this condition is equivalent to

$$\operatorname{rank}\begin{bmatrix} E & -A + BK \\ 0 & E \end{bmatrix} = 2n - p$$

as required. □

When $E = I$ and hence $p = 0$ (i.e. system is in regular state space form) the rank conditions of theorem (7.3.2) are always satisfied. Thus, the closed loop system cannot possess any infinite poles. This is in agreement with the result given by Rosenbrock and Pugh [1974] which states that when constant output feedback is applied around a system with a strictly proper transfer function matrix then the transfer function matrix of the closed loop system will also be strictly proper.

For the case where the closed loop system does possess infinite poles the closed loop infinite pole structure can be investigated by considering the Toeplitz matrices associated with such systems which, for the system described in theorem (7.3.2), are given by the following.

$$
\begin{array}{ccccc}
T_{-1}^{\infty} & T_{0}^{\infty} & T_{1}^{\infty} & T_{2}^{\infty} & T_{3}^{\infty} \\
\hline
B^{-1}E & -B^{-1}A+K & 0 & 0 & 0 \\
 & B^{-1}E & -B^{-1}A+K & 0 & 0 \\
 & & B^{-1}E & -B^{-1}A+K & 0 \\
 & & & B^{-1}E & -B^{-1}A+K \\
 & & & & B^{-1}E
\end{array}
$$

The effect of the feedback matrix $K$ on the closed loop infinite pole structure can be easily investigated since $K$ is displayed in its complete form in the above structure. The approach can therefore be adopted to examine, for instance, the effect of a particular feedback matrix or to investigate, in a straightforward manner, the possible closed loop infinite pole structures that can be assigned for a particular system of this form.

## §4. Bilinear transformation methods.

It was seen in chapter 3 that it is possible to investigate the infinite frequency structure of a rational matrix by employing a bilinear transformation such that the infinite poles and zeros are relocated at finite locations in the resultant matrix and can then be investigated in the same way as the original finite poles and zeros. This approach can also be adopted in the study of the pole placement problems and a discussion of such an approach is presented in this section.

Consider the $n \times \ell$ transfer function matrix $G(s)$ with a right minimal factorisation of the form

$$G(s) = N(s)\, D^{-1}(s) \tag{7.4.1}$$

where $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ has column degrees $c_i$, $i = 1, 2, \ldots, \ell$.

Let

$$\Lambda(s) = \text{diag } [s^{c_1}, s^{c_2}, \ldots, s^{c_\ell}]. \tag{7.4.2}$$

Applying a bilinear transformation of the form

$$s = \tfrac{p}{p-\alpha} \qquad \alpha \neq 0 \tag{7.4.3}$$

transforms the point $s = \infty$ to $p = \alpha$. The transformed transfer function matrix $G\left(\frac{p}{p-\alpha}\right)$ is a rational matrix and a right minimal factorisation for $G\left(\frac{p}{p-\alpha}\right)$ can be immediately deduced from a right minimal factorisation of $G(s)$ as described by the following theorem.

**(7.4.4) Theorem.** Consider the rational transfer function matrix $G(s)$ factorised as in (7.4.1) and let $\Lambda(s)$ be defined as in (7.4.2). Then, the transformed rational matrix $G\left(\frac{p}{p-\alpha}\right)$ has a right minimal factorisation of the form

$$G\left(\tfrac{p}{p-\alpha}\right) = \left[N\left(\tfrac{p}{p-\alpha}\right)\ \Lambda(p-\alpha)\right] \quad \left[D\left(\tfrac{p}{p-\alpha}\right)\ \Lambda(p-\alpha)\right]^{-1}$$

**Proof.** Apply the transformation (7.4.3) to the minimal basis $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ so that

$$\begin{aligned}
\begin{bmatrix} D\left(\frac{p}{p-\alpha}\right) \\ N\left(\frac{p}{p-\alpha}\right) \end{bmatrix} &= \begin{bmatrix} \tilde{D}(p) \\ \tilde{N}(p) \end{bmatrix} \Lambda\left(\tfrac{1}{p-\alpha}\right) \\
&= \begin{bmatrix} \tilde{D}(p) \\ \tilde{N}(p) \end{bmatrix} \Lambda^{-1}(p-\alpha)
\end{aligned}$$

where $\tilde{D}(p), \tilde{N}(p)$ are polynomial matrices in $p$.

Then,

$$\tilde{D}(p) = D\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha)$$

$$\tilde{N}(p) = N\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha)$$

so that

$$\begin{aligned}\tilde{N}(p)\,\tilde{D}^{-1}(p) &= N\left(\tfrac{p}{p-\alpha}\right)\,\Lambda(p-\alpha)\,\Lambda^{-1}(p-\alpha)\,D^{-1}\left(\tfrac{p}{p-\alpha}\right)\\ &= N\left(\tfrac{p}{p-\alpha}\right)\,D^{-1}\left(\tfrac{p}{p-\alpha}\right)\\ &= G\left(\tfrac{p}{p-\alpha}\right).\end{aligned}$$

It now remains to show that $\begin{bmatrix}\tilde{D}(p)\\ \tilde{N}(p)\end{bmatrix}$ forms a minimal basis, i.e. $\begin{bmatrix}\tilde{D}(p)\\ \tilde{N}(p)\end{bmatrix}$ satisfies

(i) rank $\begin{bmatrix}\tilde{D}(p)\\ \tilde{N}(p)\end{bmatrix} = \ell$ for all $p \in \mathbb{C}$,

(ii) rank $\begin{bmatrix}\tilde{D}\\ \tilde{N}\end{bmatrix}_{hc} = \ell$ where $\begin{bmatrix}\tilde{D}\\ \tilde{N}\end{bmatrix}_{hc}$ denotes the high order coefficient matrix with respect to the columns of $\begin{bmatrix}\tilde{D}(p)\\ \tilde{N}(p)\end{bmatrix}$.

Take $p = \beta$ where $\beta \neq \alpha$ then

$$\begin{bmatrix}\tilde{D}(\beta)\\ \tilde{N}(\beta)\end{bmatrix} = \begin{bmatrix}D\left(\frac{\beta}{\beta-\alpha}\right)\\ N\left(\frac{\beta}{\beta-\alpha}\right)\end{bmatrix}\Lambda(\beta-\alpha) \tag{7.4.5}$$

and since $\begin{bmatrix}D(s)\\ N(s)\end{bmatrix}$ is a minimal basis and $\Lambda(\beta-\alpha)$ has full rank it follows that (7.4.5) has full rank for all $\beta \neq \alpha$. When $p = \alpha$

$$\begin{bmatrix}\tilde{D}(\alpha)\\ \tilde{N}(\alpha)\end{bmatrix} = \begin{bmatrix}D\\ N\end{bmatrix}_{hc}\Lambda(\alpha)$$

which will have full rank since $\alpha \neq 0$. Hence, condition (i) is satisfied.

For condition (ii), it follows that

$$\begin{bmatrix}\tilde{D}\\ \tilde{N}\end{bmatrix}_{hc} = \begin{bmatrix}D(1)\\ N(1)\end{bmatrix}$$

and since $\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$ forms a minimal basis then $\begin{bmatrix} D(1) \\ N(1) \end{bmatrix}$ has full rank and, hence, $\begin{bmatrix} \tilde{D} \\ \tilde{N} \end{bmatrix}_{hc}$ has full rank. Condition (ii) is thus satisfied to complete the proof. □

The result of theorem (7.4.4) leads to the following result concerning output feedback around $G(s)$.

**(7.4.6) Theorem.** Let a constant output feedback matrix $K$ be applied to the transfer function matrix factorised as in (7.4.1), followed by a bilinear transformation of the form (7.4.3). Then, the resulting transfer function matrix is equivalent to the transfer function matrix obtained by first employing the identical bilinear transformation and then implementing the constant output feedback $K$.

**Proof.** Applying constant output feedback $K$ around (7.4.1) results in the closed loop system having transfer function matrix

$$G_K(s) = N(s)\,[D(s) + K\,N(s)]^{-1}$$

where $\begin{bmatrix} D(s) + K\,N(s) \\ N(s) \end{bmatrix}$ forms a minimal basis. Next, employ the bilinear transformation $s = \frac{p}{p-\alpha}$. Then, by theorem (7.4.4) the matrix $G_K\left(\frac{p}{p-\alpha}\right)$ may be factorised as

$$G_K\left(\frac{p}{p-\alpha}\right) = N_1(p)\,D_1^{-1}(p)$$

where

$$D_1(p) = \left[D\left(\frac{p}{p-\alpha}\right) + K\,N\left(\frac{p}{p-\alpha}\right)\right]\Lambda(p-\alpha),$$

$$N_1(p) = N\left(\frac{p}{p-\alpha}\right)\Lambda(p-\alpha).$$

Conversely, first apply the bilinear transformation $s = \frac{p}{p-\alpha}$ to $G(s)$ so that, again by theorem (7.4.4), $G\left(\frac{p}{p-\alpha}\right)$ may be factorised as

$$G\left(\frac{p}{p-\alpha}\right) = N_2(p)\,D_2^{-1}(p)$$

where

$$D_2(p) = D\left(\frac{p}{p-\alpha}\right)\Lambda(p-\alpha), \tag{7.4.7}$$

$$N_2(p) = N\left(\frac{p}{p-\alpha}\right)\Lambda(p-\alpha). \tag{7.4.8}$$

Also, $\begin{bmatrix} D_2(p) \\ N_2(p) \end{bmatrix}$ forms a minimal basis with column degrees $c_i$, $i = 1, 2, \ldots, \ell$, so that when constant output feedback $K$ is employed around $G\left(\frac{p}{p-\alpha}\right)$, the closed loop transfer function matrix $G'_K\left(\frac{p}{p-\alpha}\right)$ is of the form

$$G'_K\left(\tfrac{p}{p-\alpha}\right) = N_2(p)[D_2(p) + K\,N_2(p)]^{-1}.$$

Substituting the expressions for $D_2(p)$ and $N_2(p)$ from (7.4.7) and (7.4.8) results in

$$\begin{aligned} G'_K\left(\tfrac{p}{p-\alpha}\right) &= N\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha)\left\{D\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha) + K\,N\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha)\right\}^{-1} \\ &= N\left(\tfrac{p}{p-\alpha}\right)\Lambda(p-\alpha)\left\{\left[D\left(\tfrac{p}{p-\alpha}\right) + K\,N\left(\tfrac{p}{p-\alpha}\right)\right]\Lambda(p-\alpha)\right\}^{-1}. \end{aligned}$$

Hence, $G_K\left(\frac{p}{p-\alpha}\right) \equiv G'_K\left(\frac{p}{p-\alpha}\right)$, as required. □

The result of theorem (7.4.6) is of interest in determining the effect of output feedback on the poles of a system. The result enables the system to be first transformed to one where the poles are situated in more favourable locations before the effect of constant output feedback is investigated. In particular for a system with a non-proper transfer function matrix, i.e. a system that possesses infinite poles, a suitable transformation can be employed so that the resulting system is proper, i.e. all the poles are located at finite locations. The pole placement problem is therefore reduced to considering the influence of output feedback on finite poles only.

This idea seemed appropriate for the general pole placement problem for generalised state space systems as described in chapter 6. Transforming the system into one with a proper transfer function matrix would mean that the result given by Kucera and Zagalak [1988] could then be applied and the general pole placement problem for both finite and infinite poles would be solved. Unfortunately this argument breaks down in this case since it might not be possible to realise the transformed system in the form $[sE - A]^{-1}B$ as demonstrated by the following example.

**(7.4.9) Example.** Let

$$G(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying the bilinear transformation (7.4.3) results in

$$G\left(\tfrac{p}{p-\alpha}\right) = \begin{bmatrix} 1 & -\frac{p}{p-\alpha} \\ 0 & 1 \end{bmatrix}.$$

If $G\left(\frac{p}{p-\alpha}\right)$ can be realised in the form $[pE-A]^{-1}B$ then

$$[pE-A]\,G\left(\tfrac{p}{p-\alpha}\right)=B \tag{7.4.10}$$

and if $pE-A=\begin{bmatrix} a_1(p) & a_2(p) \\ a_3(p) & a_4(p) \end{bmatrix}$, $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where $a_i(p)\in\Re[p]$, the ring of polynomials in the real field, $i=1,2,3,4$, and $b_i\in\Re$, $i=1,2,3,4$, then (7.4.10) becomes

$$\begin{bmatrix} a_1(p) & -a_1(p)\frac{p}{p-\alpha}+a_2(p) \\ a_3(p) & -a_3(p)\frac{p}{p-\alpha}+a_4(p) \end{bmatrix}=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}. \tag{7.4.11}$$

From (7.4.11)

$$a_1(p)=b_1 \qquad \text{and} \qquad -a_1(p)\frac{p}{p-\alpha}+a_2(p)=b_2.$$

which gives

$$a_2(p)=b_2+b_1.\frac{p}{p-\alpha}.$$

Now $a_2(p)$ is a polynomial in $p$ which implies $b_1=0$, and hence $a_1(p)=0$. Similarly it follows that $b_3=0$ and $a_3(p)=0$. Thus, $pE-A$ is singular so that $G\left(\frac{p}{p-\alpha}\right)$ can not be realised in the form $[pE-A]^{-1}B$.

The result due to Kucera and Zagalak [*ibid.*] only holds for systems with transfer function matrix of the form $[sE-A]^{-1}B$ so that if the transfer function matrix of the transformed system cannot be realised in this manner it is not possible to solve the general pole placement problem in generalised state space systems using the above argument. Despite this the above approach has highlighted certain properties which might be exploited in future investigations. For example, the reasoning could certainly be employed for the case of a general open loop transfer function matrix and where the general pole placement problem has been solved for the case where the closed loop transfer function matrix is proper.

The previous discussion raises the interesting problem of finding the set of rational matrices, $G(s)$, that can be realised in the form $[sE-A]^{-1}B$. The properties of such a realisation provide a partial solution to this problem, as described in the following theorem.

**(7.4.12) Theorem.** Consider the strongly controllable system with $n\times\ell$ transfer function matrix

$$G(s)=[sE-A]^{-1}B \tag{7.4.13}$$

where $B$ has full rank. Then $G(s)$ has no finite zeros.

**Proof.** Without loss of generality assume that the system is represented in Kronecker form. Then, since the system is strongly controllable, the factorisation (7.4.13) is a minimal factorisation. Hence, the finite zeros of $G(s)$ are given by the finite zeros of $B$. Now $B$ is a constant matrix which implies that it does not possess any finite zeros. Hence $G(s)$ has no finite zeros to complete the proof. □

The result of theorem (7.4.12) provides an explanation as to why it is not possible to realise certain transformed systems in the form $[sE - A]^{-1}B$. If the original system possesses an infinite zero then this zero will be relocated at a finite position under the transformation. Thus, it follows from the result of theorem (7.4.12) that the transformed system can not be realised as $[sE-A]^{-1}B$ in such cases. This explains why the transformed system in example (7.4.9) could not be realised in the form $[sE - A]^{-1}B$.

**(7.4.14) Example.** Consider again the matrix $G(s)$ of example (7.4.9), i.e.

$$G(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$$

with a left minimal factorisation of the form

$$G(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which indicates that $G(s)$ possesses an infinite zero of degree 1. Under the transformation (7.4.3), $G(s)$ becomes

$$G\left(\frac{p}{p-\alpha}\right) = \begin{bmatrix} 1 & \frac{-p}{p-\alpha} \\ 0 & 1 \end{bmatrix}.$$

A right minimal factorisation of $G\left(\frac{p}{p-\alpha}\right)$ is given by

$$G\left(\frac{p}{p-\alpha}\right) = \begin{bmatrix} 1 & -p \\ 0 & p-\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & p-\alpha \end{bmatrix}^{-1}$$

which indicates that $G\left(\frac{p}{p-\alpha}\right)$ has a zero at $p = \alpha$, i.e. the infinite zero of $G(s)$ has been relocated at $p = \alpha$ in the transformed matrix. Hence, by theorem (7.4.12), it is not possible to realise $G\left(\frac{p}{p-\alpha}\right)$ in the form $[pE - A]^{-1}B$ which confirms the earlier observation.

## §5. Conclusions.

Three further approaches to the general pole placement problem have been discussed in this chapter.

For generalised state space systems a two stage approach in which the infinite and finite pole structures are assigned by two separate feedbacks was described in section 2. This approach provided necessary and sufficient conditions for both the finite and infinite closed loop pole structures by suitable interpretation of the result given by Rosenbrock and Rowe [1970] for the general pole placement problem in the regular case and a result concerning the assignment of the infinite pole structure presented in chapter 6. The conditions on the finite and infinite pole structures are not directly related so the result does not provide a complete solution to the problem. The deficiencies of this approach were highlighted by means of a suitable example which demonstrated the importance of choosing the right combination of feedbacks.

It was seen in chapter 3 how the infinite pole structure of a rational matrix can be investigated by using the Laurent expansion about the point at infinity of that matrix. For the problem of investigating the infinite pole structure of a closed loop system this method was seen in section 3 to be of more relevance for individual systems rather than providing a general solution. The approach provides a straightforward means of investigating the effect of a certain feedback on the infinite pole structure in certain systems. In the case of generalised state space systems with a non-singular transfer function matrix a new condition for testing whether the closed loop system is proper was presented.

The infinite pole structure of a rational matrix can also be investigated by employing a bilinear transformation which relocates the infinite poles at finite locations. In section 4 it was shown how a minimal factorisation of a transformed rational matrix can be obtained from a minimal factorisation of the original matrix. Further, the effect of constant output feedback around a transformed transfer function matrix was seen to be equivalent to first applying the constant feedback and then transforming the resulting transfer function matrix. These results enable the finite and infinite pole structures to be investigated simultaneously. For systems with transfer function matrix $G(s) = [sE - A]^{-1}B$ it was shown that the transformed system cannot always be realised in this form and hence the general pole placement for both finite and infinite poles cannot be solved by interpreting the result due to Kucera and Zagalak [1988].

## Chapter 8. Conclusions.

The poles of a system essentially determine the dynamic response of the system with the finite poles giving rise to exponential responses and the infinite poles giving rise to impulsive responses.

The dynamic response of the system can be altered by relocating the poles using suitable feedback. The pole placement problem is concerned with investigating the conditions under which these poles can be relocated. If in addition to the location the pole groupings are also considered then the problem is referred to as the general pole placement problem. Previous work on both the pole placement and general pole placement problem has mainly concentrated on open loop systems that only possess finite poles or where all the poles of the closed loop system are located at finite locations. The work presented in this thesis has been concerned with considering the cases where either or both of the open loop and closed loop systems may possess infinite poles as well as finite ones. In particular the specific case of the general pole placement problem using constant gain feedback in generalised state space systems has been investigated in this context.

The problem was first approached by considering the general pole placement problem for the system formed by applying constant output feedback around an arbitrary transfer function matrix. Exploiting the properties associated with a minimal factorisation of a rational matrix enables both the finite and infinite pole structures to be investigated in a straightforward manner. This investigation gave rise to the results presented in chapter 4 which provide new necessary conditions for the placement of the finite pole structure and the placement of the infinite pole structure but more importantly for the simultaneous placement of the two structures. The conditions were given in terms of the right minimal indices of the open loop transfer function matrix which are equivalent to the controllability indices of the system if the transfer function matrix is proper or strictly proper. The results therefore extend the work of Rosenbrock and Hayton [1978] who considered the general pole placement problem for strictly proper systems using dynamic feedback to include the case of non-proper systems under constant output feedback.

The results presented in chapter 4 were subsequently interpreted for the general pole placement problem using constant generalised state feedback in generalised state space systems. The resulting necessary conditions for the simultaneous placement of both the finite and infinite pole structures provide a generalisation of the necessary conditions presented by Kucera and Zagalak [1988] who considered the case where all the closed loop poles are placed at finite locations. The infinite pole structure was then further investigated to first of all produce necessary conditions on the multiplicity of the closed loop poles. It was shown that the possible multiplicity is related to the number of linearly independent last position rows of $B$ when the system is represented in Kronecker form. The result provides an analogy to the condition that the possible multiplicity of the finite poles is related to the rank of $B$. The necessary conditions on the closed loop infinite pole structure were further strengthened by combining them with the earlier conditions on the simultaneous assignment of both the finite and infinite pole structures. The resulting conditions were shown to be also sufficient providing a complete characterisation of the achievable infinite pole structures for the closed loop system. The results neatly complement the recent work of Fahmy and O'Reilly [1989] who considered the assignment of all the closed loop poles at infinite locations. Indeed, when all the poles are placed at infinite locations the result presented in chapter 6 provides a direct analogy to the result due to Kucera and Zagalak [1988] for the case where all the poles are located at finite positions. The new necessary conditions on the infinite pole structure were also seen to give rise to stronger necessary conditions on the simultaneous placement of both finite and infinite pole structures. Efforts to prove that these condition are also sufficient have been unsuccessful but equally it has not been possible to find a suitable counter example. This indicates that these conditions could well be sufficient or close to being sufficient. This provides an obvious area for further research so that a necessary and sufficient condition for both the finite and infinite pole structures can be obtained.

Certain necessary and sufficient conditions were in fact obtained for the above problem by considering a two stage approach. The approach involved employing two separate feedbacks in which the first feedback assigns the infinite structure followed at the second stage by the finite structure. The subsequent necessary and sufficient conditions are given in terms of the right minimal indices of the original system and the controllability indices of the system formed as a result of applying the feedback. The controllability indices are in fact dependent on the first choice of feedback so that the result is not satisfactory and the solution cannot be regarded as being complete. The example accompanying this approach though does throw light on the mechanism that lies behind the assignment of the poles using constant generalised state feedback.

The other approaches to the general pole placement problem discussed here, although not providing complete solutions, do provide further insights into the problem and possible avenues for future work. The Toeplitz matrix approach is of value in considering the effect of certain feedbacks on individual systems. In this respect the method gave rise to a simple condition for a closed loop system to be proper. The bilinear transformation approach also provides an alternative means of investigating the problem. The results presented in this thesis can be used to extend existing results from the proper to the non-proper case. For generalised state space systems with transfer function matrices of the form $[sE - A]^{-1}B$ this does not follow and the reason for this failure was seen to highlight certain properties of such systems.

Two areas which are crucial to the problems considered in this thesis namely the infinite frequency structure of a rational matrix and the notions of controllability associated with generalised state space systems were also investigated. In chapter 3 the infinite frequency structure of a rational matrix was considered. The results presented in this chapter include a new condition to test for the absence of infinite zeros in a rational matrix. For the case of polynomial matrices this test was seen to be a simpler test than that presented by Hayton *et al.* [1988]. The infinite frequency structure of a polynomial matrix was studied in detail and the relationships between the degrees of the minors and the rank indices characterised. The accompanying results give rise to alternative means of calculating certain characteristics of the system.

Chapter 5 discussed the notions of controllability associated with systems in generalised state space form and concluded that there exists two main notions. New algebraic conditions were presented for the two notions which together with previous results provide an analogy to the algebraic conditions associated with the notion of controllability in regular state space systems presented by Rosenbrock [1970]. The polynomial system matrix approach was seen to provide a means of treating these results in a unified manner. The role of the non-dynamic variables was discussed and illustrated by introducing a new time domain definition. Finally new necessary conditions were presented for a system to be controllable under each notion.

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APPENDIX A + B.

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# Infinite-frequency structure and a certain matrix Laurent expansion

A. C. PUGH†, E. R. L. JONES†, O. DEMIANCZUK† and G. E. HAYTON‡

A method for determining the Smith–McMillan form at infinity of a rational matrix is derived by considering the Laurent expansion at infinity of the matrix. This method is used to provide a new test for the absence of infinite zeros in a rational matrix and a new formula for calculating the highest degree among the largest minors of a polynomial matrix.

## 1. Introduction

Van Dooren *et al.* (1979) presented a method for determining the Smith–McMillan form of a rational matrix from its Laurent expansion about a particular finite point $s_0 \in \mathbb{C}$. Alternatively the technique may be employed to determine the finite pole and zero structure of that matrix. Van Dooren *et al.* indicated how the theory might be modified to produce the infinite pole and zero structure of a rational matrix. However, this was not carried through completely since the concept of Smith–McMillan form at infinity was not available.

In this paper this simple modification is undertaken and a method is thereby developed that determines the infinite frequency structure of any rational matrix. The method is based on constructing the Smith–McMillan form at infinity (Vardulakis *et al.* 1982) of the given matrix from its Laurent expansion about the point at infinity. This technique proves to be fundamental to the study of the infinite-frequency structure of a rational matrix. In §§ 4, 5 and 6 three illustrations of this claim are presented. The first illustrates how the relationship between the decoupling invariants and the infinite zero structure of a decouplable system (Vardulakis 1980, Descusse and Dion 1982) can be established in a particularly simple fashion. The second illustration provides a new and computationally attractive test for the absence of infinite zeros, while the third illustration provides a new formula for calculating the highest degree occurring among the largest minors of a polynomial matrix.

## 2. The Smith–McMillan form at infinity

Vardulakis *et al.* (1982) introduced the concept of the Smith–McMillan form at infinity of a rational matrix. The main definitions are briefly presented here. In the following $\mathbb{R}[s]$ denotes the ring of polynomials in the indeterminate $s$ with coefficients in $\mathbb{R}$, while $\mathbb{R}(s)$ denotes the associated field of rational functions. Let $G(s) \in \mathbb{R}(s)^{m \times l}$. Then we make the following definitions.

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*Definition* 1

$G(s)$ will be called *proper* if

$$\lim_{s\to\infty} G(s)$$

exists. If the limit is zero then $G(s)$ will be called *strictly proper*, while if this limit is non-zero $G(s)$ will be called *exactly proper*.

Let $\mathbb{R}_{pr}(s)$ denote the ring of proper rational functions.

*Definition* 2

The $m \times m$ rational matrix $W(s) \in \mathbb{R}_{pr}^{m\times m}(s)$ is said to be *biproper* if and only if

(i) $$\lim_{s\to\infty} W(s) = W_\infty \in \mathbb{R}^{m\times m}$$

(ii) $$\det W_\infty \neq 0$$

where $\det(\cdot)$ denotes the determinant of the indicated matrix.

*Definition* 3

The $m \times l$ rational matrices $G_1(s)$ and $G_2(s)$ are said to be *equivalent at infinity* if there exist biproper matrices $W(s) \in \mathbb{R}_{pr}^{m\times m}(s)$, $V(s) \in \mathbb{R}_{pr}^{l\times l}(s)$ such that

$$W(s)G_1(s)V(s) = G_2(s)$$

Since $W(s)$ and $V(s)$ are biproper, it can be seen from Definition 2 that $W(s)$ and $V(s)$ possess neither poles nor zeros at infinity. It therefore follows from this that $G_1(s)$ and $G_2(s)$ have an identical pole–zero structure at infinity. A canonical form for a rational matrix under the equivalence relation of Definition 3 is its Smith–McMillan form at infinity, $S^\infty(G)$.

*Lemma* 1

Let $G(s) \in \mathbb{R}(s)^{m\times l}$ with rank $G(s) = r$. Then there exist biproper rational matrices $W(s)$ and $V(s)$ such that

$$W(s)G(s)V(s) = S^\infty(G) \tag{1}$$

where

$$S^\infty(G) = \begin{cases} [Q(s) \quad 0_{m,l-m}] & (l > m) \\ Q(s) & (l = m) \\ \begin{bmatrix} Q(s) \\ 0_{m-l,m} \end{bmatrix} & (l < m) \end{cases}$$

and

$$Q(s) = \operatorname{diag}\{s^{q_1}, s^{q_2}, \ldots, s^{q_r}, 0, 0, \ldots, 0\} \tag{2}$$

with $q_1 \geqslant q_2 \geqslant \ldots \geqslant q_k \geqslant 0 \geqslant q_{k+1} \geqslant \ldots \geqslant q_r$. $S^\infty(G)$ is called the *Smith–McMillan form at infinity of* $G(s)$.

Using the Smith–McMillan form at infinity of $G(s)$, the infinite poles and zeros of $G(s)$ may be defined as follows.

*Definition* 4

If $p_\infty$ is the number of $q_i$ in (2) with $q_i > 0$ then $G(s)$ has $p_\infty$ poles at infinity, each having degree $q_i$. Similarly, if $z_\infty$ is the number of $q_i$ in (2) with $q_i < 0$ then $G(s)$ has $z_\infty$ zeros at infinity, each having degree $|q_i|$.

This definition is equivalent to the earlier definitions of infinite poles and zeros and their degrees given by Verghese (1978) and Pugh and Ratcliffe (1979).

With reference to the Smith–McMillan form at infinity of $G(s)$, we make the following definition.

*Definition* 5

$$S^\infty(i) \triangleq \begin{cases} q_i & (i = \text{integer}) \\ q_{i+} & (i \neq \text{integer}) \end{cases}$$

where $i+$ denotes the upwards-rounded version of $i$.

Since the $q_i$ are ordered in a decreasing manner, it follows that $S^\infty(i)$ is a decreasing staircase, as shown in Fig. 1. $S^\infty(i)$ has been defined in such a way that it contains all the infinite-frequency information concerning $G(s)$ in a non-redundant way.

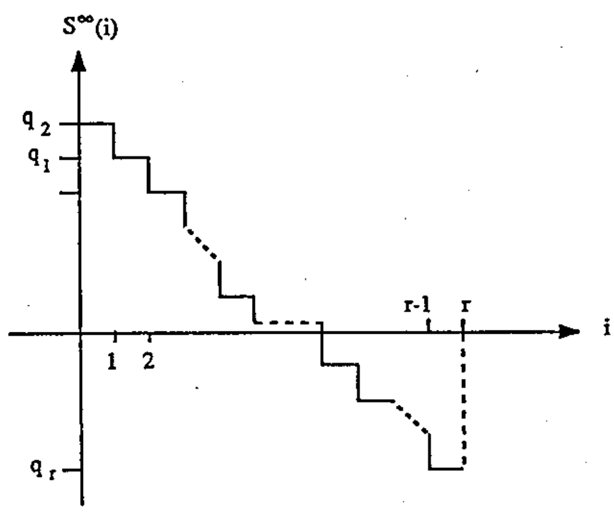


Figure 1.

### 3. The Laurent expansion and Toeplitz matrices of a rational matrix

Van Dooren *et al.* (1979) used the Laurent expansion of $G(s)$ about a finite point and the corresponding Toeplitz matrices to determine the Smith–McMillan form at $s_0$ of $G(s)$. In an analogous way the Smith–McMillan form at infinity of $G(s)$ can be determined by considering the Laurent expansion at infinity of $G(s)$ and the corresponding Toeplitz matrices.

Suppose that the Laurent expansion at infinity of $G(s)$ is of the form

$$G(s)=\sum_{i=-\infty}^{l} G_i s^i = G_l s^l + G_{l-1}s^{l-1} + \dots + G_0 + G_{-1}s^{-1} + \dots$$

The *Toeplitz matrices at infinity*, $T_i^\infty(G)$, associated with $G(s)$ are defined as

$$T_i^\infty(G)=\begin{bmatrix} G_l & G_{l-1} & \dots & G_{-i} \\ & \ddots & \ddots & \vdots \\ & 0 & \ddots & G_{l-1} \\ & & & G_l \end{bmatrix}, \quad i \geqslant -l \tag{3}$$

The information concerning the rank of the $T_i^\infty(G)$ will determine the rank indices at infinity of $G(s)$, which are defined in the following manner.

*Definition* 6

The *rank indices at infinity of* $G(s)$ are defined as

$$\rho_i^\infty(G)=\operatorname{rank}[T_i^\infty(G)]-\operatorname{rank}[T_{i-1}^\infty(G)], \quad i=-l,-l+1,\dots \tag{4}$$

where it is assumed that $\operatorname{rank}[T_{-l-1}^\infty(G)]=0$.

It is now shown that these rank indices at infinity are invariant under the transformation of equivalence at infinity given by Definition 3.

*Theorem* 1

Let $G(s)$ and $H(s)$ be two $m\times l$ rational matrices. If $G(s)$ and $H(s)$ are equivalent at infinity then they have the same rank indices at infinity.

*Proof*

Since $G(s)$ and $H(s)$ are equivalent at infinity, there exist biproper rational matrices $M(s)$ and $N(s)$ of dimensions $m\times m$ and $l\times l$ respectively such that

$$M(s)G(s)N(s)=H(s) \tag{5}$$

Since $M(s)$ and $N(s)$ are biproper at infinity, they have no infinite poles or zeros, and so their Laurent expansions about the point at infinity take the forms

$$M(s)=M_0+M_{-1}s^{-1}+M_{-2}s^{-2}+\dots$$

$$N(s)=N_0+N_{-1}s^{-1}+N_{-2}s^{-2}+\dots$$

where $M_0=M(s=\infty)$ and $N_0=N(s=\infty)$ are non-singular. Expand $G(s)$ and $H(s)$ in terms of their Laurent series at infinity:

$$G(s)=\sum_{i=-\infty}^{g} G_i s^i$$

$$H(s)=\sum_{i=-\infty}^{h} H_i s^i$$

On substituting these expressions into (5) and comparing coefficients of $s$, the following relationship is obtained:

$$\begin{bmatrix} M_0 & \cdots & \cdots & M_{-l-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & M_0 \end{bmatrix} \begin{bmatrix} G_i & \cdots & \cdots & G_{-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & G_i \end{bmatrix} \begin{bmatrix} N_0 & \cdots & \cdots & N_{-l-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & N_0 \end{bmatrix} = \begin{bmatrix} H_i & \cdots & \cdots & H_{-i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & H_i \end{bmatrix} \tag{6}$$

where $l = \min(g, h)$. Since $M_0$ and $N_0$ are non-singular, it follows from (6) that the Toeplitz matrices built on $M(s)$ and $N(s)$ are also non-singular. Therefore it follows from (5) that

$$\operatorname{rank}[T_i^\infty(G)] = \operatorname{rank}[T_i^\infty(H)]$$

as required. □

As a consequence of the above result, it follows that a rational matrix $G(s)$ has the same rank indices at infinity as its Smith–McMillan form at infinity, $S^\infty(G)$. Therefore the properties of the rank indices at infinity, $\rho_i^\infty$, of $G(s)$ can be deduced from the Toeplitz matrices at infinity of $S^\infty(G)$, i.e. $T_i^\infty(S^\infty(G))$. These Toeplitz matrices have a particularly simple structure because of the special form of $S^\infty(G)$. Specifically, note that

(i) all the rows of $T_i^\infty(S^\infty(G))$ are either zero or have one non-zero entry (a 'one');

(ii) the non-zero rows of $T_i^\infty(S^\infty(G))$ are linearly independent.

From the second property it follows that

$$\begin{aligned} \rho_i^\infty &= \operatorname{rank}[T_i^\infty(S^\infty(G))] - \operatorname{rank}[T_{i-1}^\infty(S^\infty(G))] \\ &= \operatorname{rank}[S_i(G) \quad S_{i-1}(G) \quad \dots \quad S_{-i}(G)] \end{aligned} \tag{7}$$

where $S_j(G)$ is the $j$th coefficient in the Laurent expansion at infinity of $S^\infty(G)$. Further, it can be seen, using the above properties, that rank $[S_i(G) \quad S_{i-1}(G) \quad \dots \quad S_{-i}(G)]$ is equal to the number of ones in $[S_i(G) \quad S_{i-1}(G) \quad \dots \quad S_{-i}(G)]$, which in turn equals the number of powers $q_j$ greater than or equal to $i$ in $S^\infty(G)$. It should also be noted that, owing to the properties of the $s_i$, rank $[S_i(G) \quad S_{i-1}(G) \quad \dots \quad S_{-i}(G)]$ will at some stage equal $r$, the normal rank of $G(s)$, but rank $[S_i(G) \quad S_{i-1}(G) \quad \dots \quad S_{-i}(G)]$ cannot exceed $r$.

Thus a direct relationship between the rank indices at infinity of $G(s)$ and its Smith–McMillan form at infinity has been established, which makes it possible to deduce the Smith–McMillan form at infinity of $G(s)$ from the rank differences of its Toeplitz matrices at infinity. To derive this relationship, define the rank index function at infinity $R^\infty(i)$, associated with the rank indices at infinity $\rho_i^\infty$, as follows.

*Definition 7*

$$R^{\infty}(i)=\begin{cases}\rho_i^{\infty} & (i=\text{integer})\\ \rho_{i-}^{\infty} & (i=\text{non-integer})\end{cases}$$

where $i-$ is the downward-rounded version of $i$.

Again, using (7), it is seen that $R^{\infty}(i)$ is an increasing staircase, as shown in Fig. 2. The $R^{\infty}(i)$ staircase is in fact a 90° rotation of the $S^{\infty}(i)$ staircase defined earlier, and so the Smith–McMillan form at infinity of $G(s)$ can be deduced directly from the $R^{\infty}(i)$ staircase as follows.

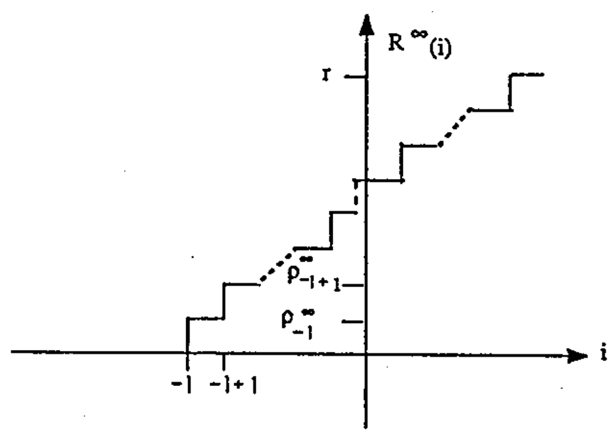


Figure 2.

*Theorem 2*

If, in the notation of Lemma 1, $S^{\infty}(G)$ denotes the Smith–McMillan form of the rational matrix $G(s)$, and $p_i^{\infty}$ denote the rank indices of $G(s)$ constructed on the basis of its Laurent expansion about the point at infinity, then

$$S^{\infty}(G)\triangleq \text{block diag }\{Q_i(s)\} \tag{8}$$

where $Q_i(s)$ is the $(\rho_i^{\infty}-\rho_{i-1}^{\infty})\times(\rho_i^{\infty}-\rho_{i-1}^{\infty})$ matrix given by

$$Q_i(s)\triangleq\begin{bmatrix} s^{-i} & 0 & \dots & 0\\ 0 & s^{-i} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & s^{-i}\end{bmatrix} \tag{9}$$

for $i=-l,-l+1,\dots,$ and if $\rho_i^{\infty}-\rho_{i-1}^{\infty}=0$ then the corresponding matrix $Q_i(s)$ is not present in (8).

In particular the pole/zero structure at infinity may then be deduced as follows.

*Corollary* 1

If, in Theorem 2, $\rho_i^\infty - \rho_{i-1}^\infty \neq 0$ then

(i) $G(s)$ will have $\rho_i^\infty - \rho_{i-1}^\infty$ poles at infinity of degree $|i|$ if $i<0$;

(ii) $G(s)$ will have $\rho_i^\infty - \rho_{i-1}^\infty$ zeros at infinity of degree $i$ if $i>0$.

## 4. Decoupling

A system is said to be decoupled when each output is controlled by a unique single input. A system can be decoupled by employing state feedback around the system. An algebraic condition for a system to be decoupled in this way was given by Falb and Wolovich (1967). Consider the system

$$\left.\begin{aligned} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{10}$$

where $x(t)$ is an $n$-vector of internal states, $u(t)$ an $m$-vector of control inputs and $y(t)$ an $m$-vector of outputs, and $A$, $B$ and $C$ are constant matrices of appropriate dimensions. Let the state feedback around (10) be given as

$$u(t) = Fx(t) + Gw(t) \tag{11}$$

where $F$ is a constant $m \times n$ matrix and $G$ is a non-singular constant $m \times m$ matrix.

*Definition* 8

Let $d_1, d_2, \ldots, d_m$ be given by

$$d_i = \min\{j : C_i A^j B \neq 0, j = 0, 1, \ldots, n-1\}$$

or

$$d_i = n-1 \quad \text{if } C_i A^j B = 0 \quad \text{for all } j$$

where $C^i$ is the $i$th row of $C$. Then the powers $d_i$, $i = 1, \ldots, m$, are known as the *decoupling invariants* of the system.

*Definition* 9

Define $B^*$ as

$$B^* = \begin{bmatrix} C_1 A^{d_1} B \\ C_2 A^{d_2} B \\ \vdots \\ C_m A^{d_m} B \end{bmatrix} \tag{12}$$

Then $B^*$ is known as the *decouplability matrix* of the system.

The Falb and Wolovich (1967) result can be expressed as follows.

*Lemma* 2

If $B^*$ is the decouplability matrix of the system (10) then there exists a pair of matrices $F$ and $G$ for (11) that decouple the system by state feedback if and only if $\det B^* \neq 0$.

For decouplable systems the decoupling invariants have been shown to be closely related to the orders of the infinite zeros of the system. Using the Toeplitz-matrix approach these results can be derived in a direct manner, as follows.

*Theorem* 3 (Vardulakis 1980, Descusse and Dion 1982)

The system represented by (10) is decouplable if and only if the associated transfer-function matrix $G(s)$ has $m$ infinite zeros each of order $w_i = d_i + 1$, $i = 1, 2, \ldots, m$, where the $d_i$ are the decoupling invariants of the system.

*Proof*

First assume that the system is decouplable. Then the decouplability matrix $B^*$ is non-singular. The transfer-function matrix $G(s)$ is given by

$$G(s) = C[sI - A]^{-1}B$$

which can be expanded as

$$G(s) = \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \ldots \tag{13}$$

Let the first non-zero matrix, $CA^{j-1}B$ say, in the series expansion of $G(s)$ have $k$ non-zero rows. Hence there are $k$ decoupling invariants of value $j-1$. The first non-zero Toeplitz matrix

$$T_j^\infty = [CA^{j-1}B]$$

will therefore have rank $k$, which indicates the presence of $k$ infinite zeros of order $j$. Now $C_iA^{d_i}B$ gives the first non-zero row coefficient of row $i$ in the series expansion (13). Thus the rank index at infinity of the Toeplitz matrices can only increase once a new row of the decouplability matrix $B^*$ appears in the Toeplitz matrix. Now, since the rows of $B^*$ are linearly independent and because of the special structure of the Toeplitz matrix, it follows that for every non-zero row of $B^*$ that corresponds to a decoupling invariant of value, say, $l-1$ there is an increase in the rank index at infinity of the Toeplitz matrix indicating an infinite zero of order $l$.

Now, for the converse, assume that $G(s)$ has $m$ infinite zeros each of order $w_i = d_i + 1$, $i = 1, 2, \ldots, m$. Then the change in the rank indices at infinity associated with a particular infinite zero will be caused by the introduction into the Toeplitz matrix $T_i^\infty$ of the first non-zero row coefficient $C_lA^{d_l+1}B$ for some row $l$ from the expansion (13). From the structure of the Toeplitz matrices it follows that for $T_i^\infty$ to have the appropriate rank, $C_lA^{d_l+1}B$ must be linearly independent of the other first non-zero row coefficient already present in $T_i^\infty$. These rows constitute the decouplability matrix, which is therefore non-singular, indicating the system is decouplable as required. □

The above theorem was originally presented in two separate parts. The necessity was established by Vardulakis (1980) using algebraic methods, while the sufficiency was proven by Descusse and Dion (1982) using geometric ideas.

It can be seen that the Toeplitz-matrix approach provides an alternative proof that unifies the two separate results in a much clearer and simpler way.

## 5. A test for the absence of infinite zeros in a rational matrix

The investigation of the changes in the rank indices of the Toeplitz matrices of a rational matrix provides a method of determining the McMillan structure of the infinite poles and zeros of the matrix. The process will terminate, i.e. all the infinite poles and zeros will have been found, when

$$\rho_k^\infty = r = \text{rank}\,[G(s)] \tag{14}$$

for some $k$. This is because, as noted earlier, the rank difference of two successive Toeplitz matrices cannot exceed $r$, which means that if (14) holds then

$$\rho_{k+i}^\infty = r, \quad i = 1, 2, \ldots \tag{15}$$

Thus in this case

$$\rho_{k+i}^\infty - \rho_{k+i-1}^\infty = 0, \quad i = 1, 2, \ldots$$

indicating that the search is complete.

This observation leads to the following test for the absence of infinite zeros in a rational matrix.

*Theorem* 4

The $m \times l$ rational matrix $G(s)$ of normal rank $r$ will possess no infinite zeros if and only if

$$\text{rank}\,[T_0^\infty(G)] = \text{rank}\,[T_{-1}^\infty(G)] + r \tag{16}$$

If $G(s)$ is taken as a matrix polynomial $P(s)$ whose highest power of $s$ is $n$, i.e.

$$P(s) = P_n s^n + P_{n-1} s^{n-1} + \ldots + P_1 s + P_0 \tag{17}$$

where $P_0, P_1, \ldots, P_n$ are constant matrices and $P_n \neq 0$, then

$$T_{-1}^\infty(P) = \begin{bmatrix} P_n & P_{n-1} & \cdots & P_{-2} & P_{-1} \\ & \ddots & \ddots & \cdots & P_{-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & P_{n-1} \\ & & & & P_n \end{bmatrix} \tag{18}$$

Now the rank of $T_{-1}^\infty(P)$ is equal to $\delta(P(s))$, the McMillan degree of $P(s)$ (Pugh 1976). This leads to the following corollary to Theorem 4.

*Corollary* 1

The $m \times l$ polynomial matrix $P(s)$ of normal rank $r$ will possess no infinite zeros if and only if

$$\text{rank}\,[T_0^\infty(P)] = \delta + r \tag{19}$$

where $\delta$ is the McMillan degree of $P(s)$.

Hayton *et al.* (1988) present a necessary and sufficient condition for the absence of infinite zeros in a polynomial matrix that involves investigating the degree of all the

$r \times r$ minors of $P(s)$. The above condition therefore provides a simpler test for the absence of infinite zeros in a polynomial matrix, since it is generally easier to calculate the rank of $T_0^\infty(P)$ than all the $r \times r$ minors. In both tests the McMillan degree of $P(s)$ must be calculated, although this again merely involves the computation of the rank of a constant matrix, i.e. $T_0^\infty(P)$.

## 6. Toeplitz structure and the minors of a polynomial matrix

For polynomial matrices Bosgra and Van der Weiden (1981) defined the infinite poles and zeros in terms of the highest degrees of minors of a certain order. Specifically, let $P(s)$ be an $m \times l$ polynomial matrix of normal rank $r$, and let $\delta_i$ be the highest degree occurring among the $i \times i$ minors of $P(s)$. Let $\delta$ denote the largest of the $\delta_i$, $i = 1, 2, \dots, r$; then $\delta$ is of course the McMillan degree of $P(s)$. Let $k_1$ (respectively $k_2$) denote the smallest (respectively largest) order of minors for which $\delta_i = \delta$.

*Definition* 10

With the above notation, $P(s)$ is said to have $k_1$ infinite poles with degrees $\delta_1, \delta_2 - \delta_1, \dots, \delta - \delta_{k_1 - 1}$, and $r - k_2$ infinite zeros with degrees $\delta - \delta_{k_2+1}$, $\delta_{k_2+1} - \delta_{k_2+2}, \dots, \delta_{r-1} - \delta_r$.

This definition has been shown by Bosgra and Van der Weiden (1981) and Hayton *et al.* (1988) to be entirely consistent with the definition of infinite poles and zeros and their degrees given by Pugh and Ratcliffe (1979), and hence with that obtained in Definition 4 via the Smith–McMillan form at infinity. It therefore follows that there exists a relationship between the $\delta_i$ as defined above and the rank indices $\rho_i^\infty$ as defined in § 3. Although this relationship is difficult to characterize in general, in two cases the characterization may be written down simply and exploited quite usefully.

*Theorem* 5

Let $P(s)$ be an $m \times l$ polynomial matrix of normal rank $r$ and let $n$ denote the highest power of $s$ occurring in elements of $P(s)$. Suppose that the rank indices $\rho_i^\infty$ ($i = -n, -n+1, \dots, -1, 0, 1, \dots, h$) of $P(s)$ are known, where $h$ is the smallest integer for which

$$\rho_h^\infty - \rho_{h-1}^\infty \neq 0, \quad \rho_i^\infty - \rho_{i-1}^\infty = 0 \quad \forall\, i > h$$

If $\delta$ denotes the McMillan degree of $P(s)$ and $\delta_r$ the highest degree amongst all $r \times r$ minors of $P(s)$ then

$$\delta = \sum_{i=-n}^{-1} \rho_i^\infty \tag{20}$$

$$\delta_r = \sum_{i=-n}^{h-1} \rho_i^\infty - h\rho_h^\infty \tag{21}$$

*Proof*

Let $\delta_i$ be the highest degree for $i \times i$ minors of $P(s)$ and let $k_1$ and $k_2$ be as defined previously. Let $p_\infty$ (respectively $z_\infty$) denote the total number of poles (respectively zeros) at infinity counted according to multiplicity and degree. Now, if $p_\infty$ is computed from the $\delta_i$ then from Definition 10

$$\left.\begin{aligned} p_\infty &= \sum_{i=1}^{k_1} (\delta_i - \delta_{i-1}) \quad (\delta_0 \triangleq 0) \\ \text{i.e.} \qquad & \\ p_\infty &= \delta_{k_1} \equiv \delta \end{aligned}\right\} \tag{22}$$

by definition of $k_1$. On the other hand, if $p_\infty$ is computed from the $\rho_i^\infty$ then from Corollary 1 of Theorem 2

$$\begin{aligned} p_\infty &= \sum_{i=-n}^{-1} (\rho_i^\infty - \rho_{i-1}^\infty)|i| \quad (\rho_{-n-1}^\infty \triangleq 0) \\ &= (\rho_{-n}^\infty - \rho_{-n-1}^\infty)n + (\rho_{-n+1}^\infty - \rho_{-n}^\infty)(n-1) + \dots + (\rho_{-1}^\infty - \rho_{-2}^\infty) \end{aligned} \tag{23}$$

i.e.

$$p_\infty = \sum_{i=-n}^{-1} \rho_i^\infty$$

Equations (22) and (23) together then yield (20) as required.

Proceeding similarly with the computation of $z_\infty$ gives from the $\delta_i$ that

$$z_\infty = \sum_{i=k_2}^{r-1} (\delta_i - \delta_{i+1}) = \delta_{k_2} - \delta_r \tag{24}$$

Alternatively, from the $\rho_i^\infty$,

$$\begin{aligned} z_\infty &= \sum_{i=1}^{h} (\rho_i^\infty - \rho_{i-1}^\infty)i \\ &= (\rho_1^\infty - \rho_0^\infty) + 2(\rho_2^\infty - \rho_1^\infty) + \dots + (h-1)(\rho_{h-1}^\infty - \rho_{h-2}^\infty) + h(\rho_h^\infty - \rho_{h-1}^\infty) \\ &= h\rho_h^\infty - \sum_{i=0}^{k-1} \rho_i^\infty \end{aligned} \tag{25}$$

Equating (24) and (25) gives

$$\delta_r - \delta_{k_2} = \sum_{i=0}^{h-1} \rho_i^\infty - h\rho_h^\infty \tag{26}$$

However, by the definition of $k_2$,

$$\delta_{k_2} = \delta$$

and, in view of (20), the relationship (26) reduces to (21), as required. □

The relationships in the above theorem can be refined further if instead of the rank indices $\rho_i^\infty$ the actual ranks of the Toeplitz matrices formed from $P(s)$ are used.

*Corollary* 1

Let $T_i^\infty(P)$, $i = -n, -n+1, \dots$, denote the successive Toeplitz matrices formed from $P(s)$ viewed as a matrix polynomial. Then

$$\delta = \operatorname{rank}\,[T_{-1}^\infty(P)] \tag{27}$$

and

$$\delta_r = (h+1)\operatorname{rank}\,[T_{h-1}^\infty(P)] - h\operatorname{rank}\,[T_h^\infty(P)] \tag{28}$$

*Proof*

This follows directly from (20) and (21) on noting that

$$\rho_i^\infty = \text{rank}\,[T_i^\infty(P)] - \text{rank}\,[T_{i-1}^\infty(P)]$$

□

The result (27) is of course well known (Pugh 1976) and provides a simple computational scheme for evaluating the McMillan degree of a polynomial matrix. The result (28) is new and could be used computationally to evaluate the highest degree of $r \times r$ minors of $P(s)$. There is, however, one difficulty surrounding the formula (28), and that lies in the requirement that $h$ be known *a priori*. There is thus in (28) more than just a requirement that the ranks of two successive Toeplitz matrices be known.

*Corollary* 2

If $P(s)$ is a square non-singular matrix then

$$\deg\,[\det P(s)] = \sum_{i=-n}^{h-1} \rho_i^\infty - h\rho_h^\infty \tag{29}$$

$$= (h+1)\,\text{rank}\,[T_{h-1}^\infty(P)] - h\,\text{rank}\,[T_h^\infty(P)] \tag{30}$$

*Proof*

If $P(s)$ is square then $m = l$, and since it is non-singular then $r = m$. Thus $\delta_r \equiv \deg\,[\det P(s)]$ and the result follows. □

The above result suggests a method by which the degree of a determinant may be computed without recourse to evaluation of the determinant itself. The need for such a method can be illustrated by considering the insertion of output feedback as represented by the constant matrix $F$ around the open-loop transfer-function matrix $G(s)$. If $D(s)$ denotes the non-strictly-proper part of $G(s)$ (i.e. the polynomial part of $G(s)$) then a necessary and sufficient condition for the closed loop system to be proper is (Pugh 1984)

$$\deg\,\{\det\,[I + FD(s)]\} = \delta(D(s)) \tag{31}$$

A result of the form of Corollary 2 is clearly required in order to evaluate the left-hand side of this relationship. Note that on the right-hand side of (31), $\delta(D(s))$ denotes the McMillan degree of $D(s)$, and this may be evaluated quite readily from (27) of Corollary 1.

## 7. Conclusions

In this paper the theory described by Van Dooren *et al.* (1979) has been modified and extended to produce a method of determining the infinite pole and zero structure of a rational matrix from its Laurent expansion about the point at infinity. In fact, taken together with the numerical refinements suggested by Van Dooren *et al.* (1979), a neat and numerically efficient algorithm is obtained (Demianczuk *et al.* 1986).

This particular method of identifying the infinite pole/zero structure is neat and quite powerful, as is evidenced in § 4, where the relationship between the degrees of the infinite zeros and the decoupling invariants of a decouplable system has been obtained in a most straightforward and much simpler way (Vardulakis 1980, Descusse and Dion 1982). In § 5 the theory has been utilized to produce a new and computationally attractive test for the absence of infinite zeros in a rational matrix. For polynomial matrices this results in a test that is more easily implementable than that given previously (Pugh and Ratcliffe 1979, Hayton *et al.* 1988).

Finally, in § 6 a new method for computing the highest degree of $r \times r$ minors of a polynomial matrix of normal rank $r$ has been suggested by the Toeplitz-matrix approach.

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# Necessary conditions for the general pole placement problem via constant output feedback

E. R. L. JONES†, A. C. PUGH† and G. E. HAYTON‡

A necessary condition is obtained for the simultaneous placement of a finite pole structure and an infinite pole structure of a linear system using constant output feedback. The result can be applied to the general pole placement problem in singular systems using constant generalized state feedback.

## 1. Introduction

The poles of a system play a fundamental role in determining the dynamical response of that system. How these poles can be relocated, by using suitable feedback, so that the dynamics of the system may be altered to ensure that the system responds in a particular desired manner has therefore long been of interest. The feedback under consideration in this paper is constant output feedback.

The conventional pole-assignment problem is concerned with the allocation of each pole on an individual basis. A more general version of this, referred to as the general pole-assignment problem, seeks to assign the pole structure in a more complete way by assigning the invariant polynomial structure to the particular matrix that determines the pole structure of the system.

In this paper the systems under consideration are assumed to be linear time-invariant with a transfer function that may be non-proper, i.e. they possess infinite poles. Thus the infinite pole structure must be assigned in addition to the finite pole structure, thus adding a further dimension to the problem. A necessary condition is obtained which provides an explanation as to why certain pole structures cannot be assigned to certain systems.

The above necessary condition can be extended to the general pole-assignment problem in singular systems using constant generalized state variable feedback, where it is assumed that the output is equal to the state of the system. A particular case of this problem was considered by Kucera and Zagalak (1988), who obtained necessary and sufficient conditions when the resulting closed-loop system is proper, i.e. possesses no infinite poles. The necessary conditions presented in this paper reduce to the necessary conditions obtained by Kucera and Zagalak in the case where the closed-loop system is proper.

## 2. Preliminaries

Consider a system with an $m \times l$ rational transfer function matrix $G(s)$. Let $G(s)$ be factorised as

$$G(s) = N_1(s)D_1^{-1}(s) \tag{1}$$

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where

$$\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix} \tag{2}$$

constitutes a minimal basis (Forney 1975). Let (1) be referred to as a right minimal factorization of $G(s)$. This minimal factorization provides a straightforward simultaneous characterization of the finite and infinite pole and zero structure of $G(s)$. In particular, the finite and infinite pole structure of $G(s)$ is given by the following lemma.

*Lemma* 1

Let the degree of the $i$th column of (2) be denoted by $c_i$ ($i = 1, 2, \ldots, l$) and define

$$\Lambda_1(s) \triangleq \operatorname{diag}[s^{c_1} \quad s^{c_2} \quad \ldots \quad s^{c_l}] \tag{3}$$

Then the finite pole structure of $G(s)$ corresponds to the finite zero structure of $D_1(s)$ and the infinite pole structure of $G(s)$ corresponds to the zero structure at $w = 0$ of the polynomial matrix

$$D_1(1/w)\Lambda_1(w)$$

*Proof*

For the proof see Pugh and Ratcliffe (1980)

Let $G_L(s)$ denote the transfer function matrix of the system formed when constant output feedback is applied to the original system as shown in Fig. 1. Then $G_L(s)$ is given by

$$G_L(s) = G(s)[I + LG(s)]^{-1}$$

where it is assumed that $|I + LG(s)| \neq 0$. The right minimal factorization of $G_L(s)$ is closely related to the right minimal factorization of $G(s)$, as is shown by the following lemma.

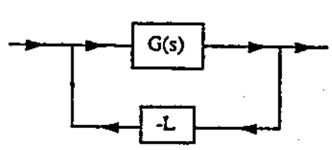


Figure 1.

*Lemma* 2

If $G_L(s)$ is the resulting closed-loop transfer function matrix obtained by applying a constant output feedback $L$ around $G(s)$, then

$$G_L(s) = N_1(s)[D_1(s) + LN_1(s)]^{-1} \tag{4}$$

is a right minimal factorization of $G_L(s)$. Further, the column degrees of

$$\begin{bmatrix} D_1(s) + LN_1(s) \\ N_1(s) \end{bmatrix}$$

are identical to the column degrees of

$$\begin{bmatrix} D_1(s) \\ N_1(s) \end{bmatrix}$$

*Proof*

For the proof see Pugh and Ratcliffe (1980)

A straightforward right minimal factorization of $G_L(s)$ therefore exists and, using the result of Lemma 1, the finite and infinite pole structure of the closed-loop system under constant output feedback can be investigated simultaneously. This is undertaken in the next section.

In a similar way, for a left minimal factorization of $G(s)$

$$G(s) = D_2^{-1}(s)N_2(s) \tag{5}$$

where $[D_2(s) \quad N_2(s)]$ forms a minimal basis with row degrees $r_i (i = 1, 2, \ldots, m)$. Let $\Lambda_2(s) \triangleq \mathrm{diag}\,[s^{r_1} \quad s^{r_2} \quad \ldots \quad s^{r_m}]$. Analogous results of Lemmas 1 and 2, with appropriate modifictions then follow for this factorization. The matrices $D_1(s)$ and $D_2(s)$ are extended unimodular equivalent (Pugh and Shelton 1978) so that their non-unit invariant polynomials are identical. Similarly $D_1(1/w)\Lambda_1(w)$ and $\Lambda_2(w)D_2(1/w)$ are also extended unimodular equivalent. It therefore follows that the pole structure of $G(s)$ can be deduced by considering either factorization.

## 3. Necessary conditions for general pole-assignment problem by output feedback

Consider the $m \times l$ transfer function matrix $G(s)$ factorized as in (1). From the results in the previous section, the finite and infinite pole structure of the closed-loop system, factorized as in (4), is given by the zero structure of $D_1(s) + LN_1(s)$ and the zero structure at $w = 0$ of $[D_1(1/w) + LN_1(1/w)]\Lambda_1(w)$, respectively. The zero structures of $D_1(s) + LN_1(s)$ and $[D_1(1/w) + LN_1(1/w)]\Lambda_1(w)$ in turn are given by their respective sets of invariant polynomials. Let the invariant polynomials of $D_1(s) + LN_1(s)$ be $\alpha_1(s)$, $\alpha_2(s), \ldots, \alpha_l(s)$, where

$$\alpha_i(s) | \alpha_{i-1}(s), \quad i = 2, 3, \ldots, l \tag{6}$$

and

$$\deg \alpha_i(s) = a_i, \quad i = 1, 2, \ldots, l \tag{7}$$

Let the invariant polynomials of $[D_1(1/w) + LN_1(1/w)]\Lambda_1(w)$ be $\beta_1(w), \beta_2(w),, \ldots, \beta_l(w)$ where

$$\beta_i(w) | \beta_{i-1}(w), \quad i = 2, 3, \ldots, l \tag{8}$$

The zero structure at $w = 0$ of $[D_1(1/w) + LN_1(1/w)]\Lambda_1(w)$ is given by factors of the form $w^{b_i}$ of $\beta_i(w)(b_i > 0)$, $i = 1, 2, \ldots, l$. Hence

$$b_i \leqslant \deg(\beta_i(w)), \quad \mathrm{i} = 1, 2, \ldots, l \tag{9}$$

It therefore follows from Lemma 1 that the finite and infinite pole structure of the closed-loop transfer function matrix can be described in terms of the $a_i$s and $b_i$s.

*Theorem* 1

For there to exist a constant matrix $L$ such that $D_1(s)+LN_1(s)$ has invariant polynomials $\alpha_1(s), \alpha_2(s), \ldots, \alpha_l(s)$ it is necessary that

$$\sum_{i=k+1}^{l} a_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1$$

*Proof*

By definition $\alpha_{k+1}(s)\alpha_{k+2}(s)\ldots\alpha_l(s)$ is the greatest common divisor of all $(l\text{-}k)\times(l\text{-}k)$ minors in $D_1(s)+LN_1(s)$ for $k=0,1,\ldots,l-1$. Let $e_i$, $i=1,2,\ldots,l$ be the column degrees of $D_1(s)+LN_1(s)$ taken to correspond with the $c_i$. Thus $e_i \leqslant c_i$, $i=1,\ldots,l$.

It follows that

$$\deg\,[\alpha_{k+1}(s)\alpha_{k+2}(s)\ldots\alpha_l(s)] \leqslant \sum_{i=k+1}^{l} e_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l$$

i.e.

$$\sum_{i=k+1}^{l} a_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1$$

as required. □

A similar necessary condition for the degrees of the infinite zeros can be given.

*Theorem* 2

Let $\beta_1(w), \beta_2(w), \ldots, \beta_l(w)$ be monic polynomials with real coefficients that satisfy (7) and let

$$\beta_i(w)=w^{b_i}\beta_i'(w), \quad i=1,2,\ldots,l \tag{10}$$

where $\beta_i'(0)\neq 0$, and take $\Lambda_1(w)$ to be diag $[w^{c_1} \quad w^{c_2} \quad \ldots \quad w^{c_l}]$. Then, for there to exist a constant matrix $L$ such that $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ has invariant polynomials $\beta_1(w), \beta_2(w), \ldots, \beta_l(w)$, it is necessary that

$$\sum_{i=k+1}^{l} b_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1$$

*Proof*

By definition $\beta_{k+1}(w)\beta_{k+2}(w)\ldots\beta_l(w)$ is the greatest common divisor of all $(l\text{-}k)\times(l\text{-}k)$ minors in $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ for $k=0,1,\ldots,l-1$. Let $f_i$, $i=1,2,\ldots,l$ be the column degrees of $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ taken to correspond with the $c_i$, so that $f_i\leqslant c_i$, $i=1,\ldots,l$.

Then it follows that

$$\deg\,[\beta_{k+1}(w)\beta_{k+2}(w)\ldots\beta_l(w)] \leqslant \sum_{i=k+1}^{l} f_i \leqslant \sum_{i=k+1}^{l} c_i, \quad i=0,1,\ldots,l-1$$

i.e.

$$\sum_{i=k+1}^{l} \deg\,(\beta_i(w)) \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1$$

and from (9) and (10)

$$\sum_{i=k+1}^{l} b_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1$$

as required. □

The results of Theorems 1 and 2 can be most conveniently illustrated by means of a step function. Define

$$C_k \triangleq \begin{cases} \sum_{j=k+1}^{l} c_j & k=0,1,\ldots,l-1 \\ \sum_{j=k_-+1}^{l} c_j & k=\text{non-integer} \end{cases}$$

where $k_-$ is the downward rounded version of $j$.

Pictorially it can be seen that $C_k$ is a decreasing staircase, as illustrated by Fig. 2. It therefore follows, from Theorems 1 and 2, that the $a_i$, $b_i$ must be chosen such that the staircases corresponding to $\sum_{i=k+1}^{l} a_i$, $\sum_{i=k+1}^{l} b_i$, $k=0,1,\ldots,l-1$ lie below the staircase given by Fig. 2. Note that if the $c_i$ had been ordered in any other way the corresponding staircase would either lie on or above the staircase pictured above. Thus, the ordering $c_1 \geqslant c_2 \geqslant \ldots \geqslant c_l$ can be regarded as a minimal ordering in the sense that the associated staircase provides the lowest, of this type, of upper bound for the $\sum a_i$, $\sum b_i$.

The theorems given above present a necessary condition for the placement of a finite pole structure and a separate necessary condition for the placement of an infinite pole structure. The main theorem of this paper presents a necessary condition for the simultaneous placement of a given finite pole structure and a given infinite pole structure. This will enable an explanation to be given as to why certain pole structures cannot be assigned to certain systems.

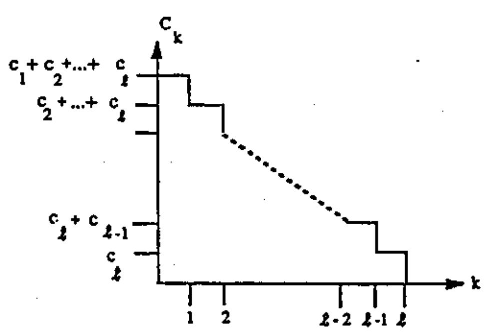


Figure 2.

*Theorem* 3

For there to exist a constant matrix $L$ such that $D_1(s)+LN_1(s)$ has invariant polynomials $\alpha_1(s), \alpha_2(s), \ldots, \alpha_l(s)$ and $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ to have invariant polynomials $\beta_1(w), \beta_2(w), \ldots, \beta_l(w)$ it is necessary that

$$\sum_{i=k+1}^{l} a_i + \sum_{i=k+1}^{l} b_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l-1 \tag{11}$$

with equality holding when $k=0$.

*Proof*

By definition $\alpha_l(s)\alpha_{l-1}(s)\ldots\alpha_{l-k+1}(s)$ is the greatest common divisor of all $k\times k$ minors in $D_1(s)+LN_1(s)$. Let

$$v_k \triangleq \sum_{i=l-k+1}^{l} a_i \quad \text{and} \quad \mu_k \triangleq \sum_{i=l-k+1}^{l} c_i, \quad k=1,2,\ldots,l$$

Then

$$\alpha_l(s)\alpha_{l-1}(s)\ldots\alpha_{l-k+1}(s) = t_{v_k}s^{v_k} + \ldots + t_1 s + t_0$$

where $v_k \leqslant \mu_k$ and $t_{v_k} \neq 0$.

Now each $k\times k$ minor of $D_1(1/w)+LN_1(1/w)$ will be of the form

$$\alpha_l(1/w)\alpha_{l-1}(1/w)\ldots\alpha_{l-k+1}(1/w)f(1/w) \tag{12}$$

for some polynomial $f(s)$. Further, among all $k\times k$ minors the corresponding polynomials $f(s)$ are coprime for finite $s$. Thus, all $k\times k$ minors of $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ will be of the form

$$\alpha_l(1/w)\alpha_{l-1}(1/w)\ldots\alpha_{l-k+1}(1/w)f(1/w)w^{\eta} \tag{13}$$

where $\eta \geqslant \mu_k$.

The greatest common divisor of all $k\times k$ minors in $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ will therefore be

$$\alpha_l(1/w)\alpha_{l-1}(1/w)\ldots\alpha_{l-k+1}(1/w)w^{\mu_k'}$$

where

$$\mu_k' \leqslant \mu_k \tag{14}$$

i.e.

$$\begin{aligned}[t_{v_k}(1/w)^{v_k} + \ldots + t_1(1/w) + t_0]w^{\mu_k'} &= t_{v_k}w^{\mu_k'-v_k} + \ldots + t_1 w^{\mu_k'-1} + t_0 w^{\mu_k'} \\ &= w^{\mu_k'-v_k}[t_{v_k} + \ldots + t_1 w^{v_k-1} + t_0 w^{v_k}]\end{aligned}$$

where $\mu_k' \geqslant v_k$ since $\alpha_l(1/w)\alpha_{l-1}(1/w)\ldots\alpha_{l-k+1}(1/w)w^{\mu_k'}$ must be a polynomial. It therefore follows by definition that

$$b_l + b_{l-1} + \ldots + b_{l-k+1} = \mu_k' - v_k$$

i.e.

$$\sum_{i=l-k+1}^{l} b_i + \sum_{i=l-k+1}^{l} a_i = \mu_k'$$

Hence, from (14)

$$\sum_{i=k+1}^{l} b_i + \sum_{i=k+1}^{l} a_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k=0,1,\ldots,l$$

When $k=l, f(1/w)=1$ in (12) and $\eta=\mu_l$ in (13). This means that the greatest common divisor of all $l\times l$ minors in $[D_1(1/w)+LN_1(1/w)]$ is of the form

$$w^{\mu_l-v_l}\phi(w)$$

where $\phi(w)$ has no factors of the form $w^{\alpha}$, $\alpha>0$. Hence

$$\sum_{i=1}^{l} b_i+\sum_{i=1}^{l} a_i=\sum_{i=1}^{l} c_i$$

as required. □

As was noted in § 2, the pole structure of $G(s)$ could as easily be investigated by considering a left minimal factorization of $G(s)$ as represented by (5). Thus, combining the necessary condition from each factorization leads to a stronger necessary condition. Let $t_1=\min(m,l)$, $t_2=\max(m,l)$ and let $\alpha_1(s), \alpha(s), \ldots, \alpha_{t_1}(s)$ be monic polynomials such that

$$\alpha_i(s)|\alpha_{i-1}(s), \quad i=2,3,\ldots,t_1$$

$$\deg \alpha_i(s)=a_i, \quad i=1,2,\ldots,t$$

Also, let $\beta_1(w), \beta_2(w), \ldots, \beta_{t_1}(w)$ be monic polynomials such that

$$\beta_i(w)|\beta_i(w), \quad i=2,3,\ldots,t_1$$

and where

$$\beta_i(w)=w^{b_i}\beta_i'(w), \quad i=1,2,\ldots,t_1$$

with $\beta_i'(0)\neq 0$. Let $\Lambda_1(s)=\operatorname{diag}[s^{c_1}\ s^{c_2}\ \ldots\ s^{c_l}]$ and $\Lambda_2(s)=\operatorname{diag}[s^{r_1}\ s^{r_2}\ \ldots\ s^{r_m}]$.

Combining the necessary condition obtained by using a right minimal factorization with the necessary condition obtained by using the left minimal factorization results in a much tighter necessary condition.

*Theorem* 4

Consider an $m\times l$ rational transfer function described above and let $N_1(s)$, $D_1(s)$, $N_2(s)$, $D_2(s)$, $c_i$, $r_i$, $\beta_i(s)$, $\alpha_i(s)$, $a_i$, $b_i$, $t_i$, $\Lambda_1(s)$, $\Lambda_2(s)$ also be defined as above. Then, for there to exist a constant matrix $L$ such that the non-unit invariant polynomials of $D_1(s)+LN_1(s)$ and $D_2(s)+N_2(s)L$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_{t_1}(s)$ and the non-unit invariant polynomials of $[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)$ and $\Lambda_2(w)[D_2(1/w)[D_2(1/w)+N_2(1/w)L]$ are $\beta_1(w), \beta_2(w), \ldots, \beta_{t_1}(w)$, it is necessary that

$$\sum_{i=k+1}^{t_1} a_i+\sum_{i=k+1}^{t_1} b_i\leqslant\sum_{i=k+1}^{t_2} d_i, \quad k=0,1,\ldots,t_1-1 \tag{15}$$

where

$$\sum_{i=k+1}^{t_2} d_i=\min\left[\sum_{i=k+1}^{l} c_i, \sum_{i=k+1}^{m} r_i\right]$$

and with equality holding when $k=0$ in (15).

The necessary condition of Theorem 4 can be described in a more straightforward fashion using the staircase description. Without loss of generality let $m\geqslant l$ and let the staircase function corresponding to each minimal factorization be constructed a similar way to that shown previously. Combining both staircases on the same diagram results in Fig. 3. Note that the two staircases might not intersect at all or might intersect at more than one point. The necessary condition of Theorem 4 then states that the closed-loop pole structure must be such that the staircase corresponding to $\sum_{i=k+1}^{t_1}(a_i+b_i)$, $k=0, 1, \ldots, t_1-1$, lies in the shaded area. Theorem 4 obviously provides a stronger necessary condition than that obtained by considering each factorization separately. This is demonstrated by the following example.

$$\begin{bmatrix} s & 0 & -s^6 \\ 0 & s^3 & 0 \\ 0 & 0 & s^5 \\ \hline 1 & 0 & -s^5 \\ s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Example* 1

Let

$$G(s)=\begin{bmatrix} 1/s & 0 & 0 \\ s & 1/s^3 & s^2 \\ 0 & 0 & 1/s^5 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & -s^5 \\ s^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s & 0 & -s^6 \\ 0 & s^3 & 0 \\ 0 & 0 & s^5 \end{bmatrix}^{-1} \triangleq N_1(s)D_1^{-1}(s)$$

1 ~~ama~~ pole
2 zero

$$=\begin{bmatrix} s & 0 & 0 \\ 0 & s^3 & 0 \\ 0 & 0 & s^5 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ s^4 & 1 & s^5 \\ 0 & 0 & 1 \end{bmatrix} \triangleq D_2^{-1}(s)N_2(s)$$

where $N_1(s)D_1^{-1}(s)$ is a right minimal factorization and $D_2^{-1}(s)N_2(s)$ is a left minimal factorization. It can therefore be seen that

$$c_3=2, \quad c_2=3, \quad c_1=6$$

$$r_3=1, \quad r_2=5, \quad r_1=5$$

The necessary condition obtained by considering the right factorization requires $a_i$ and $b_i$ to satisfy

$$\left.\begin{aligned} a_3+b_3 &\leqslant 2 \\ a_3+a_2+b_3+b_2 &\leqslant 5 \\ a_3+a_2+a_1+b_3+b_2+b_1 &= 11 \end{aligned}\right\} \tag{16}$$

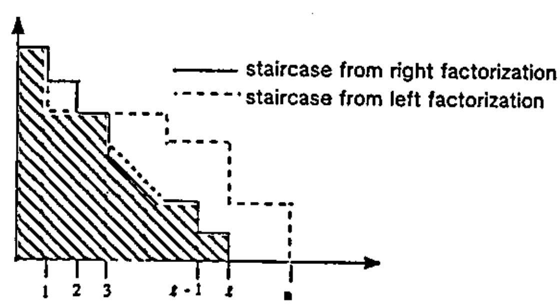


Figure 3.

while the necessary condition obtained by considering the left factorization requires $a_i$ and $b_i$ to satisfy

$$\left.\begin{aligned} a_3+b_3 &\leqslant 1 \\ a_3+a_2+b_3+b_2 &\leqslant 6 \\ a_3+a_2+a_1+b_3+b_2+b_1 &= 11 \end{aligned}\right\} \tag{17}$$

Let

$$L=\begin{bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{bmatrix}$$

and $\Lambda_1(w)=\text{diag}\,[w^2 \quad w^3 \quad w^6]$, $\Lambda_2(w)=\text{diag}\,[w \quad w^5 \quad w^5]$.

First, consider the right factorization whose necessary condition requires $a_3$ and $b_3$ to satisfy

$$a_3+b_3 \leqslant 2$$

The closed-loop infinite pole structure is given by the zero structure at $w=0$ of

$$[D_1(1/w)+LN_1(1/w)]\Lambda_1(w)=\begin{bmatrix} l_1w^2+w+l^2 & l_2w^3 & l_3w^6-l_1w-1 \\ l_4w^2+l_5 & l_5w^3+1 & l_6w^6-l_4w \\ l_7w^2+l_8 & l_8w^3 & l_9w^6+(1+l_7)w \end{bmatrix} \tag{18}$$

It follows from element (2, 2) of (18), that $w$ cannot be a common factor of all $1\times 1$ minors of (18), i.e. $b_3=0$. Now the finite pole structure of the closed-loop system is given by the invariant polynomials of

$$D_1(s)+LN_1(s)=\begin{bmatrix} l_2s^2+s+l_1 & l_2 & -s^6-l_1s^5+l_3 \\ l_5s^2+l_4 & s^3+l_5 & -l_4s^5+l_6 \\ l_8s^2+l_7 & l_8 & (1-l_7)s^5+l_9 \end{bmatrix} \tag{19}$$

If the greatest common divisor of $1\times 1$ minors is to have degree greater than zero, then in particular $l_2=0$. In that event it follows from the (1, 1) element that the highest possible degree for this divisor is 1. Hence $a_3\leqslant 1$. Thus, by investigating the closed-loop pole structure via the right minimal factorization it is seen that $a_3$ and $b_3$ must satisfy the necessary condition

$$a_3+b_3\leqslant 1$$

which confirms the necessary condition obtained by considering the left factorization.

Similarly, consider the pole structure obtained by using the left minimal factorization, which requires $a_3$, $a_2$, $b_3$, $b_2$ to satisfy the necessary condition

$$a_3+a_2+b_3+b_2\leqslant 6 \tag{20}$$

The infinite pole structure of the closed-loop system is given by the zero structure at

$w=0$ of

$$\Lambda_2(w)[D_2(1/w)+N_2(1/w)L]$$
$$=\begin{bmatrix} l_1w+1 & l_2w & l_3w \\ w^5l_4+wl_1+l_7 & l_5w^5+w^2+l_2w+l_8 & l_6w^5+l_3w+l_9 \\ l_7w^5 & l_8w^5 & l_9w+1 \end{bmatrix} \quad (21)$$

The $2\times 2$ minor formed by deleting the second row and second column is given by

$$\begin{vmatrix} 1+l_1w & l_3w \\ l_7w^5 & 1+l_9w^5 \end{vmatrix} = 1+l_1w+l_9w^5+(-l_3l_7+l_1l_9)w^6$$

which is not divisible by $w$ regardless of the choice of $l_3$, $l_7$, $l_1$ or $l_9$. Hence $b_3+b_2=0$. The finite pole structure of the closed-loop system is given by the invariant polynomials of

$$D_2(s)+N_2(s)L=\begin{bmatrix} l_2s^4+s+l_1 & l_2 & l_3 \\ l_7s^5+l_1s^4+l_4 & l_8s^5+l_2s^4+s^3+l_5 & l_9s^5+l_3s^4+l_6 \\ l_7 & l_8 & s^5+l_9 \end{bmatrix} \quad (22)$$

Suppose that there exists an $L$ such that $a_3+a_2>5$, then all non-zero $2\times 2$ minors of (22) must have at least degree 6. Consider the minor found by deleting the third column and second row of (22), i.e.

$$\begin{vmatrix} l_2s^4+s+l_1 & l_2 \\ l_7 & l_8 \end{vmatrix} = l_2l_8s^4+l_8s+l_1l_8-l_7l_2$$

For the above assumption to hold it follows that either $l_2=l_8=0$ or $l_7=l_8=0$. If $l_7=l_8=0$, consider the minor formed by deleting the first column and second row, i.e.

$$\begin{vmatrix} l_2 & l_3 \\ 0 & s^5+l_9 \end{vmatrix} = l_2s^5+l_2l_9$$

which implies $l_2=0$ for the above assumption to hold. Thus it is necessary that $l_2=0$ and $l_8=0$. Now the minor found by deleting the third column and third row of (2) is given by

$$\begin{vmatrix} s+l_1 & 0 \\ l_4l_7s^5+l_1s^4 & s^3+l_5 \end{vmatrix} = s^4+l_1s^3+l_5s+l_1l_5$$

which shows that it is not possible to find an $L$ such that all $2\times 2$ minors of (22) have at least degree 6. Hence the original assumption is false and it is deduced that $a_3+a_2\leqslant 5$. It then follows that

$$a_3+a_2+b_3+b_2\leqslant 5$$

which confirms the necessary condition obtained from the left factorization.

Using the staircase description of Fig. 4, it is seen that the desired closed-loop system pole structure must give rise to a staircase function which must lie within the shaded area.

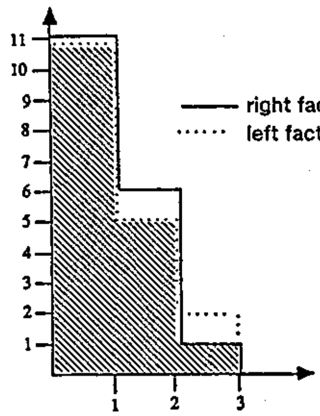


Figure 4.

The above example also demonstrates that the necessary conditions of Theorem 4 are not sufficient conditions. In particular, the pole structure $b_3 = 1$, $b_2 = 1$, $b_1 = 1$, $a_3 = 0$, $a_2 = 3$, $a_1 = 5$ satisfies the conditions (14) of Theorem 4, but it was seen that $b_3$ must satisfy $b_3 = 0$ in the above example.

## 4. General pole assignment in singular systems using generalized state feedback

Consider the singular system represented by

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{23}$$

where $x(t) \in \mathbb{R}^n$ is the generalized state of the system, $u(t) \in \mathbb{R}^l$ is the input vector and $n \geqslant l$. $E$, $A$, $B$ are constant matrices of the appropriate dimensions with $E$ assumed singular of rank $r$, and $|sE - A| \neq 0$. It is assumed that the system is strongly controllable, as defined by Verghese *et al.* (1981) and that the output equation is given by

$$y(t) = x(t) \tag{24}$$

Thus, when constant generalized state feedback of the form

$$u(t) = -Lx(t) + v(t)$$

is applied to (23) this is equivalent to output feedback of the form

$$u(t) = -Ly(t) + v(t)$$

Therefore the results of the previous section hold for the general pole-assignment problem using generalized state feedback in singular systems of the form (23) and (24).

For singular systems the strongest necessary conditions are always obtained by considering the right minimal factorization associated with the transfer function matrix because the staircase associated with the right minimal factorization always lies on or below the staircase associated with the left factorization. This is a direct consequence of the following lemma.

*Lemma* 3

Let $G(s)$ be the transfer function matrix of the system described by (23). Then the row degrees $r_i$, $i = 1, 2, \ldots, n$ associated with a left minimal factorization of $G(s)$ are

$$r_1 = 1, r_2 = 1, \ldots, r_{n-q} = 1, r_{n-q+1} = 0, \ldots, r_n = 0$$

where $q$ is the rank deficiency of $E$.

*Proof*

Without loss of generality the pencil $sE - A$ can be taken to be in Kronecker form, i.e.

$$sE - A = \begin{bmatrix} sI_{n_1} - A_1 & 0 \\ 0 & I_{n-n_1} - sJ \end{bmatrix}$$

where $n_1 = \deg|sE - A|$ and $J$ is in Jordan canonical form with all entries zero except perhaps for entries of 1 in certain positions in the first superdiagonal. Since the system is assumed to be controllable it follows that $[sE - A]^{-1}B$ forms a left minimal factorization. From the special form of $[sE - A]$ it follows that the row degrees of $[sE - A \quad B]$ are either 0 or 1 with the number of rows with zero degrees equal to the rank deficiency of $E$. Therefore reordering these row degrees gives the resuls.

□

From Lemma 3 it follows that the staircase associated with this left factorization is as in Fig. 5.

For the right factorization the $c_i$'s must satisfy

$$\sum_{i=1}^{l} c_i = n - q$$

If the staircase associated with the right factorization intersects the staircase associated with the left factorization at some point, then $c_i = 0$ for some $i$. But since the $c_i$s are in decreasing order this means that the two staircases can only intersect when $k = n - q$, confirming the claim that the staircase associated with the right minimal factorization lies on or below the staircase associated with the left factorization. For singular systems Theorem 4 therefore reduces to the following corollary.

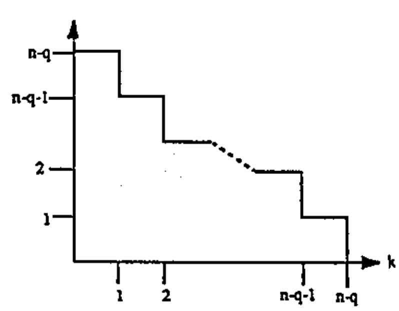


Figure 5.

*Corollary* 1

Let $G(s)$ be the transfer function matrix associated with the strongly controllable system represented by (23), i.e. $G(s) = [sE - A]^{-1}B$, and let $G(s)$ have a right minimal factorization.

$$G(s) = N_1(s)D_1^{-1}(s)$$

where the column degrees of

$$\begin{bmatrix} D(s) \\ N(s) \end{bmatrix}$$

are ordered $c_1 \geqslant c_2 \geqslant \ldots \geqslant c_l$. Let $\alpha_i$, $\beta_i$, $a_i$, $b_i$ and $\Lambda_1(s)$ be as defined previously. Then, for there to exist a constant matrix $L$ such that the invariant polynomials of $D_1(s) + LN_1(s)$ are $\alpha_1(s), \alpha_2(s), \ldots, \alpha_l(s)$ and the invariant polynomials of $[D_1(1/w) + LN_1(1/w)]\Lambda_1(w)$ are $\beta_1(w), \beta_2(w), \ldots, \beta_l(w)$, it is necessary that

$$\sum_{i=k+1}^{l} a_1 + \sum_{i=k+1}^{l} b_i \leqslant \sum_{i=k+1}^{l} c_i, \quad k = 0, 1, \ldots, l-1 \tag{25}$$

with equality holding when $k = 0$.

The necessary condition (25) in Corollary 1 is, also, not a sufficient condition, as is demonstrated by the following example.

*Example* 2

Let

$$G(s) = \begin{bmatrix} 1 & 1/s \\ s-1 & -1 \\ 0 & 1/s \end{bmatrix}$$

whose right and left minimal factorizations are, respectively

$$N_1(s)D_1^{-1}(s) = \begin{bmatrix} 1 & -1 \\ s-1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}^{-1}$$

$$D_2^{-1}(s)N_2(s) = \begin{bmatrix} s & -1 & 0 \\ 0 & 0 & s \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and where the left minimal factorization is of the form $[sE - A]^{-1}B$. It therefore follows that the necessary conditions are obtained from the column degrees of the right minimal factorization, i.e. $c_1 = 1$, $c_2 = 1$. Hence $a_i$ and $b_i$ must satisfy the necessary conditions

$$\left.\begin{aligned} a_2 + b_2 &\leqslant 1 \\ a_2 + a_1 + b_2 + b_1 &= 2 \end{aligned}\right\} \tag{26}$$

Let

$$L = \begin{bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{bmatrix} \quad \text{and} \quad \Lambda_1(w) = \operatorname{diag}\,[w \quad w]$$

Consider a pole structure with two poles at infinity, both of order one, and no finite poles, i.e.

$$b_2 = 1, \quad b_1 = 1, \quad a_2 = 0, \quad a_1 = 0 \tag{27}$$

This pole structure satisfies the conditions (25). Now the closed-loop pole structure at infinity is given by the zero structure at $w = 0$ of

$$[D_1(1/w) + LN_1(1/w)]\Lambda_1(w) = \begin{bmatrix} l_2 + w(1 + l_1 - l_2) & -l_2 + w(l_3 - l_1) \\ l_5 + w(l_4 - l_5) & (1 - l_5) + w(l_6 - l_4) \end{bmatrix} \tag{28}$$

For the above pole structure (27) to be assigned it is necessary that all $1 \times 1$ minors of (28) possess a common factor $w$, which in the case of the (1,2) and (2,2) elements implies that

$$l_5 = 0 \quad \text{and} \quad 1 - l_5 = 0$$

indicating a clear contradiction. Thus it is not possible to assign the pole structure (27) to the closed-loop system, which shows that condition (25) of Corollary 1 is not a sufficient one.

## 5. Conclusions

The properties of a minimal factorization of a transfer function matrix are exploited to obtain necesary conditions for the placement of the finite pole structure and the infinite pole structure simultaneously using constant output feedback. A neat way of presenting this result is given by the staircase idea. This method indicates clearly the restriction imposed on the closed-loop pole structure but gives no indication as to the least upper bound.

The result also holds for the general pole-placement problem in singular systems using constant generalized state feedback. This problem has been considered by Kucera and Zagalak (1988). They produced necessary and sufficient conditions for the case where the resulting closed-loop system is proper, i.e. all the closed-loop poles are located at finite positions. Corollary 1 generalizes Kucera and Zagalak's necessary condition to the case where the resulting closed-loop system may also be non-proper. Note that when $b_i = 0$ for all $i$ in (25), i.e. the closed-loop system is proper, the result reduces to the necessary conditions given by Kucera and Zagalak.

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