Extensions of Zubov's method for the determination of domains of attraction

EXTENSIONS OF ZUBOV'S METHOD FOR THE DETERMINATION OF DOMAINS OF ATTRACTION

BY

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To my wife, Noraini and our sons,

Fariz and Nazim with love.

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ABSTRACT

This thesis is concerned with the extension of Zubov's method for the determination of domains of attraction. The basic definitions and theorems of Liapunov and Zubov as well as a numerical algorithm (due to White) are given in the introductory chapter.

The application of the method of Zubov to some practical situations like power systems and control systems of order two is the subject of chapter two.

Chapter three describes the determination of the domains of attraction for scalar time varying systems The series solution has a similar problem of nonuniform convergence that occurs in autonomous systems.

Extension of the method to third order nonlinear autonomous systems is included in Chapter four so that it can be applied to a second order time varying system which is described in Chapter five. Results in the form of slices or cross-sections of the stability boundaries in the various principal planes are obtained.

Systems which have periodic solutions are examined and the domain of attraction of the stable limit cycle is determined in Chapter six. Approximate solutions are also used in trying to determine the domain of attraction of the periodic solutions.

In Chapter seven a technique for solving global optimization problems is presented. Several one-dimensional and two-dimensional minimization problems are solved and the results indicate the accuracy of this technique.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 STABILITY PROBLEM.

In man's search for methods that enable him to analyse physical processes, he is forced to approximate most of these processes by idealized mathematical models. These models will in general be systems of mathematical relationships which only approximate the original processes through some period of observation. Methods developed by Newton, Lagrange and others lead to a description of a physical process by a system of differential equations which is an adequate model for many applications. To understand the behaviour of the original process and to attempt to predict the behaviour in situations not observed, it becomes necessary to investigate the behaviour of solutions of these differential equations.

An important branch of the theory of differential equations dealing with the solution behaviour is the qualitative theory of differential equations. The qualitative theory is concerned with two major problems:

i) Classifying the solutions of differential equations by their behaviour.

ii) Searching for methods for determining the solution behaviour of a given system of differential equations, strictly on the basis of information supplied by the analytic properties of the right hand members of the differential equations.

A method used for tackling the second problem is the direct method of Liapunov, which is one of the main tools for the qualitative study of the solutions of differential equations. The application of the direct method of Liapunov to problems of practical significance has met with only limited success. However, there are notable exceptions such as the criterion given by Popov as a solution to Lure's problem. The Popov criterion is a necessary and sufficient condition for the existence of a positive definite Liapunov function of a 'quadratic plus integral form' which is capable of ensuring asymptotic stability in the large for a system of equations with a single nonlinearity. The Popov results are only valid for asymptotic stability in the large. If this is not the case, then more elaborate procedures than that of Popov must be found for estimating the domain of asymptotic stability.

In search of these techniques, several problems become obvious; first, it is difficult to construct a suitable Liapunov function and, second, the resulting approximation to the region of asymptotic stability, obtained by setting the Liapunov function equal to an appropriate constant does not guarantee a complete domain of asymptotic stability. These problems have been studied theoretically by Zubov [1] and are discussed further by Margolis and Vogt [2], Yu and Vongsuriya [3], Sarkar and Rao [4], Hewit [5] and many others. Great attention has been devoted to autonomous systems, however, the stability analysis of nonlinear nonautonomous systems is not a highly developed subject yet. It is the purpose of this thesis to investigate the stability properties and the domain of attraction of nonlinear nonautonomous systems, and also to extend the application of the method of Zubov to power and control systems. The stability region of third order nonlinear autonomous systems and periodic systems which still require further research will also be investigated. The terminology of the theory of stability will be borrowed to solve global optimization problems through the concept of the region of attraction of a minimum.

1.2 NOTATION.

All results are formulated in the n-dimensional Euclidean space R^n and the usual properties of this space will be assumed.

The norm of the vector, $|\underline{x}|$ will be given as

$$|\underline{x}| = (\underline{x}^T \underline{x})^{\frac{1}{2}} = d(\underline{x}, \underline{0}).$$

We shall denote the scalar quantity time by t, and the n+1 dimensional space of the variables ($\underline{x}$, t) will be represented as the Cartesian product $R^n \times R$. Let $S[r]$ denote the set of all $\underline{x}$ such that $|\underline{x}| \leqslant r$, $r \geqslant 0$. $S(r)$ denote the set of $\underline{x}$ such that $|\underline{x}| < r$, $r > 0$. I denote the set of all t such that $t \geqslant t_o$, $t_o \geqslant 0$ and $J = \{t:\bar{t} \leqslant t, \bar{t} \geqslant t_o\}$. In this notation, the subset of $R^n \times R$ for which $|\underline{x}| \leqslant r$, $t \geqslant t_o$ will be given as $S[r] \times I$.

A region is a set $U \subset R^n$ such that U is the union of an open connected set with some, none or all of its boundary points. The set U is an open region if none of its boundary points are included and is a closed region if all its boundary points are included. The complement of the set U with respect to R^n will be denoted by $R^n \setminus U$. In this terminology the boundary of an open set can be given by $\bar{U} \setminus U$.

The n-dimensional vector space with elements $\underline{x}$ will be called the phase space and the n+1 dimensional space composed of $(\underline{x}, t)$ will be referred to as the motion space. If the $x_i(t)$, $i = 1,\ldots,n$ are continuous functions of t, the segment of the curve in the motion space between t_1 and t_2, $t_1 \leq t \leq t_2$ will be called the motion of x(t) and its projection on the phase space $\underline{x}(t)$ will be called the trajectory of the motion.

The vector function $\underline{f}(\underline{x},t)$ is said to satisfy a Lipschitz condition on $S[r] \times I$ with respect to $\underline{x}$ if

$$|\underline{f}(\underline{x}_1,t) - \underline{f}(\underline{x}_2,t)| < m|\underline{x}_1-\underline{x}_2| \tag{1.2.1}$$

for $\underline{x}_1,\underline{x}_2 \in S[r]$, $t \in I$, $m \in (0,\infty)$

$\underline{f}(\underline{x},t)$ satisfies a local Lipschitz condition on $S[r] \times I$ with respect to $\underline{x}$ if for every $\underline{x}_o \in S[r]$, $t \in I$, there exist numbers $m > 0$, $\delta > 0$ such that

$$|\underline{f}(\underline{x}_1,t) - \underline{f}(\underline{x}_2,t)| < m|\underline{x}_1-\underline{x}_2| \quad , \qquad t \in I$$

whenever

$$|\underline{x}_1-\underline{x}_o| < \delta \quad , \quad |\underline{x}_2-\underline{x}_o| < \delta \quad .$$

The vector differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x},t) \tag{1.2.2}$$

is a system of n scalar differential equations of first order or in some cases a single scalar differential equation of n^{th} order. The point $(\underline{x}_o,t_o)$, where the solution originates, will be called the initial value, $\underline{x}_o$ the initial condition and t_o the initial instant. A solution $\underline{x}(t,\underline{x}_o,t_o)$ of the differential equation

(1.2.2) has the following properties

$$\frac{d}{dt}\underline{x}(t,\underline{x}_o,t_o) = \underline{f}(\underline{x}(t,\underline{x}_o,t_o))$$

$$\underline{x}(t_o,\underline{x}_o,t_o) = \underline{x}_o \quad .$$

If $\underline{f}(\underline{x},t)$ depends on t explicitly, it is said to be <u>nonautonomous</u>, if not, it is said to be <u>autonomous</u>. The <u>singular</u> points of (1.2.2) are defined as those values of $\underline{x}$ such that

$$\underline{f}(\underline{x},t) = 0 \quad \text{for all } t \in I.$$

Let $\overline{\underline{x}}$ be one such value of $\underline{x}$, then $\underline{x} = \overline{\underline{x}}$ will be a solution of (1.2.2) and $\underline{x} = \overline{\underline{x}}$ will be called an <u>equilibrium position</u> of the differential equations.

We shall assume that $\underline{f}(\underline{x},t)$ is smooth enough so as to ensure the existence of a solution for any finite initial condition $(\underline{x}_o,t_o)$. Several authors [6,7,8,13] have pointed out that the uniqueness of the solution of the given differential equation is not an essential requirement for the application of the direct method of Liapunov.

1.3 DEFINITIONS.

As a prelude to the development of the basic theory of the direct method of Liapunov, the concept of definiteness must be considered for scalar functions $V : R^n \to R$ and $V : R^n \times I \to R$.

<u>Definition 1.3.1.</u>

The real scalar single valued function $V(\underline{x})$ which is defined and continuous on $S[r]$ and such that $V(\underline{0}) = 0$ is positive definite on $S[r]$ if $V(\underline{x}) > 0$ for all $\underline{x} \in S[r] - \{\underline{0}\}$. $V(\underline{x})$ is negative definite on $S[r]$ if $-V(\underline{x})$ is positive definite on $S[r]$.

Definition 1.3.2.

The real scalar single valued function $V(\underline{x},t)$ which is defined and locally Lipschitzian on $S[r] \times I$ with $V(\underline{0},t) = 0$ for $t \in I$ and $V(\underline{x},t)$ is continuous on I, is positive definite [negative definite] on $S[r] \times I$ if there exists a function $W(\underline{x})$ which is positive definite on $S[r]$ such that

$$V(\underline{x},t) \geq W(\underline{x}) \quad [\leq -W(\underline{x})] \quad \text{for every } \underline{x} \in S[r] , \quad t \in I .$$

Definition 1.3.3.

The real scalar single valued function $V(\underline{x},t)$ is decrescent on $S[r] \times I$ if there exists a continuous function $W(\underline{x})$ defined on $S[r]$ such that $W(\underline{0}) = 0$ and

$$|V(\underline{x},t)| \leq W(\underline{x}) \quad \text{on} \quad S[r] \times I .$$

Definition 1.3.3a. (Alternative Definition).

A function $V(\underline{x},t)$ is decrescent if $\lim V(\underline{x},t) = 0$ uniformly with respect to t as $|\underline{x}| \to 0$.

Definition 1.3.4.

Let a real scalar single valued function $V(\underline{x},t)$ be defined and satisfy a local Lipschitz condition on some set $S[r] \times I$ and for any $\underline{x}$, let $V(\underline{x},t)$ be continuous on I, and $V(\underline{0},t) = 0$ on I. The total derivative of $V(\underline{x},t)$ along the integral curve $\underline{x}(t,\underline{x}_0,t_0)$ of $\dot{\underline{x}} = \underline{f}(\underline{x},t)$ is defined as

$$\dot{V}(\underline{x}(t,\underline{x}_o,t_o),t) = \limsup_{\Delta t \to 0^+} \frac{V(\underline{x}(t+\Delta t,\underline{x}_o,t_o),t+\Delta t) - V(\underline{x}(t,\underline{x}_o,t_o),t)}{\Delta t} \tag{1.3.1}$$

or

$$\dot{V}(\underline{x}(t,\underline{x}_o,t_o),t) = \limsup_{\Delta t \to 0^+} \frac{V(\underline{x}+\Delta t\underline{f}(\underline{x},t),t+\Delta t) - V(\underline{x},t)}{\Delta t} \quad .$$

If $V(\underline{x},t)$ is continuously differentiable on $S[r] \times I$ then (1.3.1) becomes

$$\dot{V}(\underline{x}(t,x_o,t_o),t) = \frac{\partial V}{\partial x_1} f_1(\underline{x},t) + \ldots + \frac{\partial V}{\partial x_n} f_n(\underline{x},t) + \frac{\partial V}{\partial t} \quad .$$

Let us assume for the given differential equation

$$\dot{\underline{y}} = \underline{g}(\underline{y},t)$$

that $\underline{Y}(t)$ is a particular solution. By changing the variable $\underline{x} = \underline{y} - \underline{Y}(t)$ the original equation becomes the equation of the perturbed motion,

$$\dot{\underline{x}} = \underline{g}(\underline{x} + \underline{Y}(t),t) - \underline{g}(\underline{Y}(t),t)$$

$$\left.\begin{aligned} &\text{or} && \dot{\underline{x}} = \underline{f}(\underline{x},t) \\ &\text{with} && \underline{f}(\underline{0},t) = \underline{0} \end{aligned}\right\} \tag{1.3.2}$$

The study of the behaviour of the solutions of the equation of perturbed motion (1.3.2) in the neighbourhood of the equilibrium $\underline{x} = \underline{0}$ is equivalent to considering the behaviour of the original differential equation in a neighbourhood of the particular solution $\underline{Y}(t)$. The stability behaviour of the solution of this differential equation is defined according to Liapunov [10] as follows:

Definition 1.3.5.

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_o)$ such that

$$|\underline{x}_o| < \delta \quad \text{implies} \quad |\underline{x}(t,\underline{x}_o,t_o)| < \varepsilon \ , \quad t \in I.$$

Definition 1.3.6.

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is asymptotically stable if

i) the equilibrium $\underline{x} = \underline{0}$ is stable

ii) there exists a $\gamma > 0$ such that $|\underline{x}_o| < \gamma$ implies

$$\lim_{t\to\infty} |\underline{x}(t,\underline{x}_o,t_o)| = 0 \ .$$

Definition 1.3.7.

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is unstable if there exists $\varepsilon > 0$ such that for every $\delta > 0$, there is an initial point $\underline{x}_o$ with $|\underline{x}_o| < \delta$ and the solution $\underline{x}(t,\underline{x}_o,t_o)$ is such that

$$|\underline{x}(t,\underline{x}_o,t_o)| \geq \varepsilon \quad \text{for some } t \in I \ .$$

Definition 1.3.8.

If the equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is asymptotically stable, then the set of initial values $(\underline{x}_o,t_o)$ such that

$$\lim_{t\to\infty} |\underline{x}(t,\underline{x}_o,t_o)| \to 0$$

forms the region of asymptotic stability, which will be denoted by $\Omega \times J$. Other terminologies for $\Omega \times J$ include "region of attraction of the point $\underline{x}$, $\underline{0}$" and "domain of attraction of the equilibrium".

Definition 1.3.9.

If the equilibrium $\underline{x} = \underline{0}$ is stable and

$$\lim_{t\to\infty} |\underline{x}(t,\underline{x}_o,t_o)| = 0$$

for all $(\underline{x}_o,t_o) \in R^n \times R$, then the equilibrium is said to be asymptotically stable in the large.

Definition 1.3.10.

The equilibrium $\underline{x} = \underline{0}$ is said to be uniformly stable if for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon)$ such that

$$|\underline{x}_o| < \delta \Rightarrow |\underline{x}(t,\underline{x}_o,t_o)| < \varepsilon \quad , \quad t \in I \ .$$

Definition 1.3.11.

The equilibrium $\underline{x} = \underline{0}$ is called uniformly asymptotically stable if

i) $\underline{x} = \underline{0}$ is uniformly stable

ii) for every $\eta > 0$ there exists a number $T(\eta)$ such that $|\underline{x}(t,\underline{x}_o,t_o)| < \eta$ for every $t \geq t_o + T(\eta)$ and for every $(\underline{x}_o,t_o)$, $t_o \geq 0$, $|\underline{x}_o| < \delta$.

Definition 1.3.12.

If the equilibrium $\underline{x} = \underline{0}$ is uniformly asymptotically stable for all $(\underline{x}_o, t_o) \in R^n \times R$, then the equilibrium is uniformly asymptotically in the large.

Definition 1.3.13.

An asymptotically stable equilibrium $\underline{x} = \underline{0}$ is called uniformly attracting if for every $h > 0$, $h < \delta$ and every $(\underline{x}_o, t_o)$ such that $h \leq |\underline{x}_o| \leq \delta$, $t_o \geq 0$ implies the existence of numbers $\alpha > 0$, $T > 0$ such that

$$|\underline{x}(t, \underline{x}_o, t_o)| > \alpha \quad \text{for every } t \in \left[t_o, t_o + T\right] .$$

1.4 MAIN THEOREMS OF LIAPUNOV.

The purpose of this section is to present the main theorems of Liapunov stability theory. Most of the material to be presented can be found in various places in the literature [8,9,11,12]. These theorems are included because Liapunov stability theory is the main tool used in this work but proofs are excluded since they are adequately reviewed in the literature. Liapunov theory has been applied to many practical problems such as power systems, chemical systems, control theory problems, networks, etc.; to determine whether the equilibria are stable or not, without actually solving the differential equations of the system. It is the beauty of this method that the stability as well as the estimates of the domain of attraction of many complicated problems can be inferred.

Theorem 1.4.1. (Stable)

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is stable if there exists a scalar function $V(\underline{x},t)$ with continuous first partial derivative with respect to $\underline{x}$ and t such that

i) $V(\underline{x},t)$ is positive definite on $S[r] \times I$

ii) The derivative $\dot{V}$ of V along the motion starting at $t,\underline{x}$ is at least negative semi-definite.

Theorem 1.4.2. (Asymptotic Stability)

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is asymptotically stable if there exists a scalar function $V(\underline{x},t)$ with continuous first partial derivatives with respect to $\underline{x}$ and t such that

i) $V(\underline{x},t)$ is positive definite on $S[r] \times I$

ii) $V(\underline{x},t)$ is decrescent on $S[r] \times I$

iii) $\dot{V}(\underline{x},t)$ is negative definite on $S[r] \times I$.

Theorem 1.4.3. (Instability)

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is unstable if there exists a scalar function $V(\underline{x},t)$ with continuous first partial derivatives with respect to $\underline{x}$ and t such that

i) $V(\underline{x},t)$ is decrescent on $S[r] \times I$

ii) $V(\underline{x},t)$ has a domain of negativity $V < 0$ for $t \geq t_0$

iii) $\dot{V}(\underline{x},t)$ is negative definite on $S[r] \times I$.

Theorem 1.4.4. (Asymptotic Stability in the Large).

The equilibrium $\underline{x} = \underline{0}$ of (1.3.2) is asymptotically stable in the large if there exists a scalar function $V(\underline{x},t)$ such that

i) $V(\underline{x},t)$ is positive definite

ii) $\dot{V}(\underline{x},t)$ is negative definite

iii) $V(\underline{x},t)$ is decrescent

iv) $V(\underline{x},t)$ is radially unbounded.

In many cases it is not sufficient to know merely that a system is asymptotically stable with respect to the undisturbed motion $\underline{x} = \underline{0}$. Some estimate of the allowed initial disturbances is required in addition.

Theorem 1.4.5. (The Region of Asymptotic Stability).

The region A containing the origin is a region of asymptotic stability of an asymptotically stable solution of system (1.3.2) if there exists a scalar function $V(\underline{x},t)$ possessing the properties

i) $V(\underline{x},t)$ is positive definite for $\underline{x} \in A$, $t \geq 0$

ii) $\dot{V}(\underline{x},t)$ is at least negative semidefinite in A

iii) $\dot{V}(\underline{x},t) \neq 0$ on any nontrivial trajectory in A

iv) $\underline{\nabla} V(\underline{x},t) \neq \underline{0}$ in A except at $\underline{x} = \underline{0}$

v) One of the curves $V(\underline{x},t) =$ constant defines the boundary of A.

1.5 THEOREMS OF ZUBOV.

In the process of seeking a suitable Liapunov function so that an estimate of the domain of attraction can be achieved, several methods like Ingwerson [14], Szego [15], Variable gradient [17], Nesbit [16] and many others have been developed. Since the Liapunov function is not unique, the stability domain defined by $V(\underline{x})$ = constant, may or may not be a good approximation to the actual domain of attraction. There has been no general method for generating a suitable Liapunov function which guarantees asymptotic stability and exact stability boundary. Zubov [1] to some extent develops a method for constructing a Liapunov function for asymptotically stable nonlinear systems which leads to the determination of the exact stability boundary. However not all Zubov's equations can be solved analytically. When the partial differential equation cannot be solved analytically, an approximate series solution is used and the stability boundary is only an approximation to the true stability boundary. If the equation can be solved analytically, the closed form Liapunov function defines an exact stability domain and hence the stability questions are completely answered.

The Zubov equation which is being studied may be written as

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\underline{x},t)\ f_i(\underline{x},t) + \frac{\partial V}{\partial t}(\underline{x},t) = -\phi(\underline{x},t)(1 - eV(\underline{x},t)) \tag{1.5.1}$$

for nonautonomous systems or

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x}(\underline{x})\ f_i(\underline{x}) = -\phi(\underline{x})(1 - eV(\underline{x})) \tag{1.5.2}$$

for autonomous systems, where ϕ is a positive definite function.

The equation given by $e = 1$ is known as Zubov's regular equation while $e = 0$ gives the Zubov's modified equation. Let us now state the Zubov Theorems for both stationary and non-stationary systems. Denote $V(\underline{x},t)$ as the solution to the regular Zubov equation and $W(\underline{x},t)$ as the solution to the modified equation.

Theorem 1.5.1.

The function $V(\underline{x},t)$, the solution to (1.5.1) is a Liapunov function which establishes the asymptotic stability of the unperturbed motion $\underline{x} = \underline{0}$ of system (1.3.2).

Theorem 1.5.2.

If a region A exists which includes the origin and within which $0 \leq V(\underline{x},t) \leq 1$, then any trajectory originated in this region will converge to the origin. Conversely, on any trajectory which converges to the origin,

$$0 \leq V(\underline{x},t) < 1 .$$

Corollary 1.5.1.

If the modified Zubov equation is considered then we will have the condition

$$0 \leq W(\underline{x},t) < \infty .$$

Proof of Theorem 1.5.2.

i) Sufficiency: $(0 \leq V(\underline{x}(0),0) < 1$ implies asymptotic stability). This follows immediately from (1.5.1) which can be written as

$$\frac{dV}{dt} = - \phi(1 - V) \quad . \tag{1.5.3}$$

By assumption, V is positive definite in A and has a negative definite time derivative given by (1.5.3). From (1.5.3) it is clear that V admits an infinitesimal upper bound. Hence, V satisfies all the conditions of Theorem 1.4.2. Therefore all solutions starting in A will remain in A and converge to the origin.

Necessity: (Asymptotic stability implies $0 \leqslant V(x(0), 0) < 1$).

Integrating (1.5.3) gives

$$V(x(0), 0) = 1 - (1-V(x(t), t)) \exp(- \int_0^t \phi(x(t'))dt'). \tag{1.5.4}$$

As $t \to \infty$, (1.5.4) becomes

$$V(x(0), 0) = 1 - \exp(- \int_0^\infty \phi(x(t'))dt'). \tag{1.5.5}$$

The integral in the exponential exists if and only if $x(t')$ originates in the asymptotic stability region.

If $\underline{x}(0) = \underline{0}$, we obtain the trivial solution of (1.3.2). Since $\phi(0) = 0$, then from (1.5.5) we have $V(\underline{0}, 0) = 0$ which is the initial condition on V. If the integral in (1.5.5) is infinite then V is equal to one. Hence from (1.5.5) V cannot be less than zero nor greater than one. This proves the theorem.

<u>Theorem 1.5.3.</u>

The curves $V(\underline{x},t) = 1$ or $W(\underline{x},t) = \infty$ if they exist are the integral curves of (1.3.2) which define the boundary of the domain of attraction.

<u>Definition 1.5.1.</u>

Let $\lambda \in (0,1)$ and $\beta \in (0,\infty)$.
$G(\lambda)$, $H(\beta)$ are the sets given by

$$G(\lambda) = \{(\underline{x},t): \ V(\underline{x},t) < \lambda\}$$

and

$$H(\beta) = \{(\underline{x},t): \ W(\underline{x},t) < \beta\} .$$

<u>Theorem 1.5.4.</u>

For any values of $\lambda \in (0,1)$, $\beta \in (0,\infty)$
$G(\lambda)$, $H(\beta)$ are bounded domains inside the domain A.

<u>Theorem 1.5.5.</u>

If $\lambda_1 < \lambda_2$ and $\beta_1 < \beta_2$, then

$$G(\lambda_1) \subset G(\lambda_2)$$

and

$$H(\beta_1) \subset H(\beta_2) \quad .$$

<u>Theorem 1.5.6.</u>

If $\lambda_1 = 1$ and $\beta = \infty$ then $G(1) = H(\infty) = A$.

Theorem 1.5.7.

For a fixed $\phi(\underline{x},t)$, the solution $V(\underline{x},t)$ of (1.5.1) is uniquely determined inside the domain A.

Theorem 1.5.8.

The limiting values of the function $V(\underline{x},t)$, $W(\underline{x},t)$ as $(\underline{x},t) \to (\underline{\xi}, \bar{t})$ are given by

$$\lim_{(\underline{x},t)\to(\underline{\xi},\bar{t})} V(\underline{x},t) = 1 \qquad \text{for all} \quad (\underline{\xi},\bar{t}) \in \bar{A}-A$$

and

$$\lim_{(\underline{x},t)\to(\underline{\xi},\bar{t})} W(\underline{x},t) = \infty \qquad \text{for all} \quad (\underline{\xi},\bar{t}) \in \bar{A}-A$$

where $(\underline{\xi},\bar{t})$ is a point on the boundary of region A.

Theorem 1.5.9.

In order for the equilibrium $\underline{x} = \underline{0}$ of system (1.3.2) to be asymptotically stable in the large it is necessary and sufficient that

$$V(\underline{x},t) < 1 \quad \text{or} \quad W(\underline{x},t) < \infty \quad \text{for all} \quad (\underline{x},t) \in R^n \times R .$$

Theorem 1.5.10.

In order for the region A which contains the origin to be the region of asymptotic stability of a uniformly asymptotically stable and uniformly attractive zero solution of the system (1.3.2), it is necessary and sufficient that there exists two functions $V(\underline{x},t)$, $\phi(\underline{x},t)$ possessing the following properties:

i) $V(\underline{x},t)$ is defined and continuous in A; and $\phi(\underline{x},t)$ is defined and continuous for all $\underline{x}$.

ii) $V(\underline{x},t)$ is positive definite in A and $\phi(\underline{x},t)$ is positive definite in R^n.

iii) V and $\phi \to 0$ uniformly with respect to t, $t \geq 0$ as $|\underline{x}| \to 0$;

iv) $\lim\limits_{(\underline{x},t)\to(\bar{\underline{x}},\bar{t})} V(\underline{x},t) = \infty$, where $(\bar{\underline{x}},\bar{t}) \in \bar{A}-A$ and is a point on the boundary of region A.

v) The total derivative of the function V, calculated by virtue of system (1.3.2), satisfies the relation

$$\frac{dV}{dt} = -\phi(\underline{x},t)$$

Not all of the proofs of the theorems are given. Theorem 1.5.2 is proved as an example. Other proofs of these theorems are readily available in the literature.

The theorems for autonomous systems follow in the same manner as that stated above for nonautonomous systems. Most of these theorems can be found in Zubov [1], Margolis [2] and other literatures.

1.6 OTHER METHODS.

It is convenient at this stage to describe other work carried out on Zubov's method in order to show its development and importance. Some have attempted to solve Zubov's equation by way of series solution as proposed by Zubov himself or other classical methods of solving differential equations. Burnand and Sarlos [18] employ Lie series of linear operators to solve Zubov's partial differential equation, while Kormanik and Li [19] apply this Lie series method to generate points on the stability boundary. Davidson and Cowan [20], Rodden [21], Texture [22] developed numerical techniques to analyze Liapunov functions and determine the stability region. Infante and Clark [23] present a different approach to the determination of asymptotic stability which utilizes the vector cross product to construct a Liapunov like function. Let us now discuss some of the methods which have been developed to generate a Liapunov function and study the stability analysis of the equilibrium.

i) Lagrange Charpit Method.

Recent work by Miyagi and Taniguchi [24] attempts to solve Zubov's equation by solving the characteristic equations by using the Lagrange Charpit method. They consider a nonlinear system represented by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \quad , \quad \underline{f}(\underline{0}) = \underline{0} \quad . \tag{1.6.1}$$

The construction of the Liapunov function is based on the solution of the partial differential equation

$$F(\underline{x},V,\underline{p}) = \underline{p}^T \underline{f}(x) + \phi(\underline{x}) = 0 \tag{1.6.2}$$

where $\underline{p} = \frac{\partial V}{\partial \underline{x}}$ and $\phi(\underline{x})$ is an arbitrary non-negative function. The characteristic equation for (1.6.2) is given by

$$\frac{dx_1}{\frac{\partial F}{\partial p_1}} = \cdots = \frac{dx_n}{\frac{\partial F}{\partial p_n}} = \frac{dV}{p_1 \frac{\partial F}{\partial p_1} + \cdots + p_n \frac{\partial F}{\partial p_n}} = \frac{-dp_1}{\frac{\partial F}{\partial x_1} + p_1 \frac{\partial F}{\partial V}} = \cdots = \frac{-dp_n}{\frac{\partial F}{\partial x_n} + p_n \frac{\partial F}{\partial V}} \tag{1.6.3}$$

where $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}$ include $\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}$ respectively.

The following n-1 equations containing at least one component of $\underline{p}$ are derived from equation (1.6.3):

$$\begin{aligned} &G_1(\underline{x}, V, \underline{p}, \frac{\partial \phi}{\partial \underline{x}}) = 0 \\ &G_2(\underline{x}, V, \underline{p}, \frac{\partial \phi}{\partial \underline{x}}) = 0 \\ &G_{n-1}(\underline{x}, V, \underline{p}, \frac{\partial \phi}{\partial \underline{x}}) = 0 \end{aligned} \tag{1.6.4}$$

F and G's have a common solution provided

$$\left[G_i, F\right] = \sum_{k=1}^{n} \left(\frac{\partial G_i}{\partial x_k} \frac{\partial F}{\partial p_k} - \frac{\partial F}{\partial x_k} \frac{\partial G_i}{\partial p_k}\right) = 0 \tag{1.6.5}$$

where $i = 1, 2, \ldots, n-1$.

When $[G_i, F]$ still give partial differential equations including $\underline{p}$, they also have a common solution with F. Letting $[G_i, F] = G$ once again, then all the unknown functions $\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}$ and ϕ will be determined from the conditions

$$\begin{aligned} \left[G_\ell, G_m\right] &= \sum_{k=1}^{n} \left(\frac{\partial G_\ell}{\partial x_k} \frac{\partial G_m}{\partial p_k} - \frac{\partial G_m}{\partial x_k} \frac{\partial G_\ell}{\partial p_k}\right) = 0 \\ \left[G_s, F\right] &= \sum_{k=1}^{n} \left(\frac{\partial G_s}{\partial x_k} \frac{\partial F}{\partial p_k} - \frac{\partial F}{\partial x_k} \frac{\partial G_s}{\partial p_k}\right) = 0 \end{aligned} \tag{1.6.6}$$

where ℓ, m = 1,2,...,max $\boxed{s}$, s > n-1, $\ell \neq m$ and

$$\frac{dG_\ell}{dx_k} = \frac{\partial G_\ell}{\partial x_k} + p_k \frac{\partial G_\ell}{\partial V} \quad . \tag{1.6.7}$$

Solving equations (1.6.2) and (1.6.4) gives $\underline{p}$ as a function of $\underline{x}$ and V

$$\underline{p} = \underline{p}(\underline{x},V) \ . \tag{1.6.8}$$

The possible Liapunov function is then given by

$$V(\underline{x}) = \int_o^{\underline{x}} \underline{p}^T \, d\underline{x} \tag{1.6.9}$$

with
$$-\dot{V}(\underline{x}) = \phi(\underline{x}) \ . \tag{1.6.10}$$

This method determines the arbitrary non-negative function $\phi(\underline{x})$ which allows the Liapunov function to be determined. However Zubov's theory is not employed in the determination of the stability boundary and thus the exact boundary is not found.

Let us extend the above method to a scalar time varying system. We consider

$$\dot{x} = f(x,t) \quad \text{with} \quad f(0,t) = 0 \ . \tag{1.6.11}$$

The partial differential equation will then be

$$F(x,t,V,p,r) = p\, f(x,t) + r + \phi(x,t) = 0 \tag{1.6.12}$$

where $p = \frac{\partial V}{\partial x}$, $r = \frac{\partial V}{\partial t}$, $\phi(x,t)$ is an arbitrary positive function.

The characteristic equations for (1.6.12) are given by

$$\frac{dx}{\frac{\partial F}{\partial p}} = dt = \frac{dV}{p\,\frac{\partial F}{\partial p} + r} = \frac{-\,dp}{\frac{\partial F}{\partial x} + p\,\frac{\partial F}{\partial V}} = \frac{-\,dr}{\frac{\partial F}{\partial t} + r\,\frac{\partial F}{\partial V}} \tag{1.6.13}$$

where F_x, F_t include ϕ_x, ϕ_t.

From equation (1.6.13) we derive two equations containing either p or r such that

$$G_1(x,t,V,r, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}) = 0$$

$$G_2(x,t,V,r, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}) = 0 \tag{1.6.14}$$

For F and G to be compatible, they must satisfy

$$\left[G_i,F\right] = \frac{\partial(G_i,F)}{\partial(x,p)} + \frac{\partial(G_i,F)}{\partial(t,r)}$$

$$= \frac{\partial G_i}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial G_i}{\partial p} + \frac{\partial G_i}{\partial t}\frac{\partial F}{\partial r} - \frac{\partial F}{\partial t}\frac{\partial G_i}{\partial r} = 0 \tag{1.6.15}$$

for $i = 1,2$.

The unknown functions ϕ_x, ϕ_t, ϕ are determined from the conditions

$$\left[G_1,G_2\right] = 0 \quad \text{and} \quad \left[G_2,F\right] = 0 \ . \tag{1.6.16}$$

Solving (1.6.12) and (1.6.14) we get p, r as a function of x, t, V.

Then

$$V(x,t) = \int_o^x p\,dx + \int_o^t r\,dt \tag{1.6.17}$$

with $$-\dot{V}(x,t) = \phi(x,t) \ . \tag{1.6.18}$$

Let us consider as an example a scalar time varying equation

$$\dot{x} = -\,tx(1-x) \tag{1.6.19}$$

$$F = -ptx(1-x) + r + \phi(x,t) = 0 \quad . \tag{1.6.20}$$

The characteristic equations are

$$\frac{dx}{-tx(1-x)} = dt = \frac{dV}{-ptx(1-x)+r} = \frac{-dp}{-pt(1-2x)+\phi_x} = \frac{-dr}{-px(1-x)+\phi_t} \, .$$

Suppose $$G_1 = \alpha(t)\ \beta(x) - r = 0 \tag{1.6.21}$$

then

$$G_2 = \left[G_1, F\right] = -\frac{\partial\beta}{\partial x}\alpha tx(1-x) + \frac{\beta\partial\alpha}{\partial t} + \frac{\partial\phi}{\partial t} - px(1-x) = 0 \tag{1.6.22}$$

$$\left[G_1, G_2\right] = -\frac{\alpha\partial\beta}{\partial x}x(1-x) + \left[-\frac{\partial\beta}{\partial x}\frac{\partial\alpha}{\partial t}tx(1-x) - \frac{\partial\beta}{\partial x}\alpha x(1-x)\right.$$

$$\left. + \frac{\beta\partial^2\alpha}{\partial t^2} + \frac{\partial^2\phi}{\partial t^2}\right] = 0 \tag{1.6.23}$$

$$\left[G_2, F\right] = \frac{\partial^2\beta}{\partial x^2}\alpha t^2 x(1-x) + \frac{\partial\beta}{\partial x}\alpha t^2(1-2x) - \frac{\partial\beta}{\partial x}\frac{\partial\alpha t}{\partial t} - \frac{t\partial^2\phi}{\partial t\partial x}$$

$$+ \frac{\partial\phi}{\partial x} + \frac{\alpha\partial\beta}{\partial x} = 0 \quad . \tag{1.6.24}$$

Integrating (1.6.23) with respect to t twice gives

$$\frac{\partial\phi}{\partial t} = \frac{\partial\beta}{\partial x}x(1-x)\int\alpha(t)dt + \frac{\partial\beta}{\partial x}x(1-x)\alpha(t)t - \frac{\beta\partial\alpha}{\partial t} + \Phi(x) \tag{1.6.25}$$

$$\phi(x,t) = \frac{\partial\beta}{\partial x}x(1-x)t\int\alpha(t)dt - \beta(x)\ \alpha(t) + \Phi(x)t + \Psi(x) \tag{1.6.26}$$

where $\Phi(x)$, $\Psi(x)$ are arbitrary functions.

From (1.6.24), (1.6.25), (1.6.26) we get

$$\Psi(x) = 0 \, .$$

Therefore

$$\phi(x,t) = \frac{\partial\beta}{\partial x}\, x(1-x)t \int \alpha(t)dt - \beta(x)\alpha(t) + \Phi(x)t \quad . \tag{1.6.27}$$

Now $\qquad r = \alpha(t)\beta(x)$

$$p = \frac{\Phi(x)}{x(1-x)} + \frac{\partial\beta(x)}{\partial x} \int \alpha(t)dt$$

therefore

$$V(x,t) = \int_0^x p dx + \int_0^t r dt$$

$$= \int_0^x \frac{\Phi(x)}{x(1-x)} dx + \beta(x) \int \alpha(t)dt$$

and

$$-\dot{V}(x,t) = \frac{\partial\beta}{\partial x}\, x(1-x)t \int \alpha(t)dt - \beta(x)\alpha(t) + \Phi(x)t \quad .$$

Suppose

$$\Phi(x) = \beta(x) = x^2, \quad \alpha(t) = 0$$

then $\qquad V(x,t) = -x - \ell n(1-x), \quad -\dot{V}(x,t) = tx^2$

for $\qquad \Phi(x) = x^2, \quad \beta(x) = -x - \ell n(1-x), \quad \alpha(t) = -e^{-t}$

$$V(x,t) = \left[-x - \ell n(1-x)\right]\left[1 + e^{-t}\right]$$

and $\qquad -\dot{V}(x,t) = x^2 t\, e^{-t} - \left[x + \ell n(1-x)\right] e^{-t} + tx^2 \quad .$

From this example if we apply Zubov theory we will get the domain of attraction as $(-\infty,1)$ for all $t \geq 0$ which is the actual domain of the problem.

ii) Format Method.

Peczkowski and Liu [25] introduce a method which generates a scalar V-function and has the property that, along the trajectories of a system under consideration, the scalar functions $\dot{V}$ or $\dot{V} - V_t$ take on a preassigned or desired form. It is called a format method since it is based upon a fundamental vector matrix equation or format $\underline{v} = [D + P]\underline{f}$, which mathematically represents every vector $\underline{v}$ which satisfies the scalar product $\underline{v}.\underline{f} = \dot{V} - V_t$. The Liapunov functions are generated by the format method in the following way.

Given system (1.3.2), a form

$$\underline{\nabla V}.\underline{f} = L(x,t)f_i^2 , \qquad i = 1,\ldots, \text{ or } n \tag{1.6.28}$$

is chosen where L is sign definite.

The matrix format is given as

$$\underline{v} = [D + P]\underline{f} \tag{1.6.29}$$

where D is an unspecified diagonal matrix and P an undetermined, arbitrary skew symmetric matrix.

The curl equations

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} , \qquad i \neq j , \qquad i,j = 1,\ldots,n \tag{1.6.30}$$

are solved to find the elements of the matrices P and D. The matrix format is then written as

$$\underline{\nabla V} = [D + P]\underline{f} . \tag{1.6.31}$$

Then

$$V(\underline{x},t) = \int_0^{\underline{x}} \underline{\nabla V}\, d\underline{x} \tag{1.6.32}$$

and

$$\dot{V}(\underline{x},t) = \underline{\nabla V}.\underline{f} + V_t = L(x,t)f_i^2 + V_t . \tag{1.6.33}$$

In this method we have to restrict system $\underline{f}$ so that V is positive definite, decrescent and $\dot{V}$ is negative semi-definite. The Liapunov function generated gives the sufficient conditions for stability or uniform asymptotic stability of the origin.

Consider a system

$$\ddot{x} + f(x,\dot{x},t) = 0$$

or

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - f(x_1,x_2,t) \ .$$

To generate the V-function which gives

$$\dot{V} = L(x)\ f^2$$

we choose $D = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}$, where L is assumed to be constant

$P = \begin{bmatrix} 0 & -p \\ p & 0 \end{bmatrix}$, a constant skew symmetric matrix.

The format is

$$\underline{v} = [D + P] f$$

$$= \begin{bmatrix} pf(x_1,x_2,t) \\ px_2 - Lf(x_1,x_2,t) \end{bmatrix} \ .$$

The curl equation gives

$$\frac{\partial f}{\partial x_2} \Big/ \frac{\partial f}{\partial x_1} = - L/p = k \quad \text{a positive constant.}$$

Hence $\underline{v} = \underline{\nabla} V = p \begin{bmatrix} f(x_1,x_2,t) \\ x_2 + kf(x_1,x_2,t) \end{bmatrix}$

and

$$V = \int_0^{x_1} f(x_1,0,t)dx_1 + \frac{x_2^2}{2} + k\int_0^{x_2} f(x_1,x_2,t)dx_2$$

is positive definite for all $t \geq 0$, with

$$\dot{V} = -k\{f(x_1,x_2,t)\}^2 \quad .$$

Suppose
$$f(x_1,x_2,t) = x_1^2 x_2(1+e^{-t})$$

then
$$V = \frac{x_2^2}{2} + \frac{x_1^2 x_2^2}{2} k(1+e^{-t})$$

and

$$\dot{V} = -kx_2^2 x_1^4 (1+e^{-t})^2.$$

This function is a candidate for a Liapunov function which gives sufficient conditions for the stability of the origin.

iii) Metric Algorithm [26,27]

The algorithm is described as follows

a) A time varying nonlinear differential equation is written as

$$\dot{x}_i = F_i(x,t) \qquad 1 \leq i \leq n\,, \quad t \geq 0 \tag{1.6.34}$$

b) A set of $n(n-1)/2$ differential equations of integral curves is formed by eliminating dt

$$\frac{dx_i}{dx_j} = \frac{F_i(x,t)}{F_j(x,t)}\,, \qquad j > i\ . \tag{1.6.35}$$

c) The equations are converted to the same number of differential one-forms by cross multiplying and rearranging

$$F_j(x,t)dx_i - F_i(x,t)dx_j = 0 , \qquad j > i \tag{1.6.36}$$

d) Reduce (1.6.36) to a single one-form by addition and subtraction

$$w = w_1(x,t)dx_1 + \dots + w_n(x,t)dx_n . \tag{1.6.37}$$

e) A line integration is taken with t held constant and the result is taken as a Liapunov function

$$V = \int w = \int_0^{x_1} w_1(y_1,0,\dots,t)dy_1 + \int_0^{x_2} w_2(x,y_2,0,\dots,t)dy_2$$

$$+ \dots + \int_0^{x_n} w_n(x_1,x_2,\dots,x_{n-1},y_n,t)dy_n \quad . \tag{1.6.38}$$

f) The total derivative of V with respect to t is taken and the V and $\dot{V}$ are restricted so that the stability theorems are satisfied.

To illustrate the method, let us consider a damped Mathieu equation

$$\ddot{x} + b(t)\,\dot{x} + a(t)x = 0$$

where b(t) and a(t) are continuous, $a_2 \geq a(t) \geq a_1 > 0$ and $b_2 \geq b(t) \geq b_1 > 0$.

In state variable form the equation is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - b(t)x_2 - a(t)x_1$$

therefore

$$\frac{dx_1}{dx_2} = \frac{x_2}{-b(t)x_2-a(t)x_1} \quad .$$

The differential form is

$$\left[b(t)x_2 + a(t)x_1\right]dx_1 + x_2dx_2 = 0$$

with
$$V = a(t)\,\frac{x_1^2}{2} + \frac{x_2^2}{2}$$

and
$$\dot{V} = \frac{\dot{a}(t)}{2}\,x_1^2 - b(t)\,x_2^2 \quad .$$

For $\dot{V}$ to be negative definite $\dot{a}(t)$ should be negative i.e. $a(t)$ is a decreasing function.

V is positive definite and decrescent.

Therefore if $a(t)$ is a decreasing function then the equilibrium of the Mathieu system will be asymptotically stable.

This algorithm does not provide a detailed theoretical discussion which leads to the formulation of (1.6.36) and (1.6.37). Each of the equations, (1.6.36) could possibly be multiplied by arbitrary constants and added as required by (1.6.37). Wall and Moe have multiplied these equations by unity but multiplication by arbitrary constants could have resulted in improved conditions of stability.

1.7 WHITE'S METHOD.

White [28] presents a numerical method which overcomes the problem of nonuniform convergence of Zubov's method and produces a better estimate of the domain of asymptotic stability.

The method initiates near the boundary of the stability domain and computes trajectories which either tend to the origin or away from itdepending on where the computation is initiated. The stability region considered is an approximation to $S_R \cap D(\underline{f})$ where $D(\underline{f})$ is the domain of asymptotic stability and

$$S_R = \{\underline{x} : |\underline{x}| < R\} \tag{1.7.1}$$

with R a positive number.

His approach of using V to compute $\underline{x}(V)$ is different from Texter [22] and Davidson and Cowan [20] where $\underline{x}(t)$ is being computed. The numerical method is briefly described as follows:

The system equations in m dimensions are given by

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

or equivalently

$$\dot{x}_i = f_i(\underline{x}) \qquad i = 1,\dots,m \tag{1.7.2}$$

with the origin assumed to be an equilibrium, i.e.

$$f_i(\underline{0}) = 0 \qquad i = 1,\dots,m \ . \tag{1.7.3}$$

Zubov's equation is given by

$$\dot{V} = -\,\phi(\underline{x})(1-eV)$$

or

$$\sum_{i=1}^{m} f_i(\underline{x}) \frac{\partial V}{\partial x_i} = -\,\phi(\underline{x})(1-eV) \tag{1.7.4}$$

where e = 0 or 1 for the modified or regular form respectively.

The auxiliary equations of (1.7.4) are

$$\frac{dx_1}{f_1(\underline{x})} = \dots\dots = \frac{dx_m}{f_m(\underline{x})} = -\frac{dV}{\phi(\underline{x})(1-eV)} \ . \tag{1.7.5}$$

Rearranging (1.7.5) gives

$$\frac{dx_i}{dV} = \frac{-f_i(\underline{x})}{\phi(\underline{x})(1-eV)} \qquad i = 1,\dots,m \,. \tag{1.7.6}$$

(1.7.2) and (1.7.6) are two systems of ordinary differential equations for x_i in terms of t and V respectively. The latter is obtained from (1.7.2) through the above procedure. It is assumed that the initial conditions $x_i(\underline{0})$, $i = 1,\dots,m$ lie in the domain of asymptotic stability. It is clearly seen from (1.7.5) that the trajectories $x_i(V)$, $i = 1,\dots,m$, are the same as those traversed by $x_i(t)$, $i = 1,\dots,m$. As with $x_i(t)$, $i = 1,\dots,m$, the trajectories $x_i(V)$, $i = 1,\dots,m$ cross the contours $V(\underline{x}) = p$, p = constant, once only for decreasing V in an asymptotically stable system and it follows that

$$x_i(V) \to 0 \quad \text{as} \quad V \to 0^+ \,.$$

Integrating (1.7.6) w.r.t. V gives

$$x_i(V) = x_i(V_o) - \int_{V_o}^{V} \frac{f_i(\underline{x})}{\phi(\underline{x})(1-eV)}\, dV \tag{1.7.7}$$

$i = 1,\dots,m$.

For the purpose of numerical integration, we restrict our study to systems of second order given by

$$\dot{x} = f(x,y) \tag{1.7.8}$$

$$\dot{y} = g(x,y) \quad . \tag{1.7.9}$$

Zubov's equation is then written as

$$\dot{V} = -\,\phi(x,y)(1-eV) \tag{1.7.10}$$

and (1.7.6) becomes

$$\frac{dx}{dV} = - \frac{f(x,y)}{\phi(x,y)(1-eV)} \qquad (1.7.11)$$

$$\frac{dy}{dV} = - \frac{g(x,y)}{\phi(x,y)(1-eV)} \quad . \qquad (1.7.12)$$

The initial point (x_o, y_o) is now considered to vary along the radial line from the origin and is written as

$$\begin{aligned} x_o &= r_o \cos\theta \\ y_o &= r_o \sin\theta \end{aligned} \qquad (1.7.13)$$

where θ is fixed and r_o is allowed to vary. A fourth order Runge Kutta method or any other numerical method (e.g. Fox [29]) will be used to integrate (1.7.11) and (1.7.12) to obtain $\hat{x}^{(n)}$, $\hat{y}^{(n)}$ as approximations to $x(V_o-p)$, $y(V_o-p)$ for given x_o, y_o, p and stepsize h where $p = nh$.

Define

$$W(r_o,\theta,h,p) = \|(\hat{x}^{(n)}, \hat{y}^{(n)})\| \quad . \qquad (1.7.14)$$

The variation of W with respect to r_o for fixed θ,h,p is then analysed. Having obtained the behaviour of W, we compute $\hat{r}_o(\theta,h,p)$ defined as

$$\hat{r}_o(\theta,h,p) = \max r_o \in \left\{ r_o \; : \; \frac{\partial W}{\partial r_o}(r_o,\theta,h,p) = 0 \right\} \quad . \qquad (1.7.15)$$

The computation starts by computing $W(R,\theta,h,p)$ where R is as in (1.7.1) and then computing W for decreasing r_o until W increases again, after which we interval half to obtain $\hat{r}_o$ accurately.

$\hat{r}_o$ is an approximation to a point on the contour V = p, and p is chosen such that the contour V = p is close enough to the contour V = ∞ or V = 1 (for e =0 or 1 respectively). In order to obtain the boundary point more accurately we use

$$r^*(\theta,p) = \text{optimum } \hat{r}_o(\theta,\underline{h},p) \text{ w.r.t. } \underline{h} \qquad (1.7.16)$$

for a particular value of θ and p.

To test whether the point obtained is stable or unstable J(θ) is defined as follows:

$$J(\theta) = \begin{cases} 0 & \text{if the discontinuous region of the curve W Vs } r_o \text{ is not encountered} \\ 1 & \text{if the discontinuous region of the curve W Vs } r_o \text{ is encountered} \end{cases} \qquad (1.7.17)$$

If $J(\theta) = 0$ for all θ, then the system is unstable, while $J(\theta) = 1$ for any θ gives a stable system. Using J(θ) and $r^*(\theta,p)$ the boundary of the domain of asymptotic stability can be computed.

White applies his algorithm to a limited number of nonlinear second order systems and the results are found to be satisfactory. He illustrates the method on the example of Hahn [8] in great detail and obtains the stability boundary in the first quadrant. He uses circles only as the region of approximation but the use of an ellipse as the region of approximation could be an advantage. It will be very worthwhile if the application of White's algorithm can be extended to some practical situations such as power and control engineering. Also the extension of White's method to time varying systems, nonlinear third order autonomous systems and periodic systems will greatly enhance the usefulness of the technique.

1.8 MOTIVATIONS.

The subject of stability theory has grown considerably in recent years and one of its branches, the determination of the domain of attraction of autonomous systems has been investigated by a number of researchers. Many attempts have been made to find the actual domain of attraction or to improve conservative estimates of the domain obtained by known methods. The main objective of this thesis is to apply the method of Zubov to some practical problems and to extend it to time varying and periodic systems and also to attempt to solve the global optimization problem.

In Chapter II White's method is used to determine the domain of attraction of power systems taking into account constant damping and no saliency, constant damping and saliency and variable damping and saliency. The results are compared with the works of Prabhakara et al. [30], Prusty [31], Miyagi and Taniguchi [24]. A control system containing a single nonlinearity, $\Phi(\sigma)$, which satisfies the sector condition only in the interval (m_1,m_2) and leaves the sector at $\sigma \leqslant m_1 < 0$ and $\sigma \geqslant m_2 > 0$ is included. The application of the method to scalar time varying systems can be found in Chapter III. The comparison between second order nonlinear autonomous systems and scalar time varying systems is outlined in this chapter. The Zubov series solution for scalar time varying systems is also presented. Extension of the method to third order nonlinear autonomous systems is included so that it can be used to compute the domain of attraction of second order time varying systems. In the third order system, cross-sections of the stability surface for the principal planes are given and compared with other known work on this subject. An attempt is also made towards computing the cross-sections of the stability boundaries at different heights of a particular axis. From these cross-sections, a solid shape is built by using the graphics package which is available from the Loughborough University of Technology Computer Centre.

Systems which have periodic solutions are examined and the domain of attraction of the stable limit cycle is determined in Chapter VI.

Finally, the task of solving the global optimization problem is presented in Chapter VII. The interactive graphical approach will hopefully ease the location of all the minima of a function and the determination of all the domains of attraction.

CHAPTER II

Second Order Autonomous Systems

2.1 INTRODUCTION.

In this chapter we will study the stability region of second order power systems and other autonomous systems. In the past several techniques have been proposed to determine the domain of asymptotic stability of such systems. Gless [32], El-Abiad and Nagappan [33] used Liapunov functions which are similar to the total system energy. Fallside and Patel [34] used the variable gradient method for studying synchronous machine stability problems, while Prusty and Sharma [31] employed the optimized Szego's Liapunov function for a single machine considering saliency and nonlinear damping. These authors obtained the stability region bounded by the Liapunov function $V(x) = V(x_c)$ and the line segment at the point $x = x_c$ where x_c indicates the unstable equilibrium state. These methods give only estimates of the asymptotic stability region which are well inside the actual region of asymptotic stability.

White's method [28] overcomes the problem of nonuniform convergence of Zubov's method and computes a much better estimate of the domain of asymptotic stability. Most of the work done in this chapter can be found in [35] where White's method is applied to the single machine systems which take into account constant damping, constant damping and saliency and also variable damping and saliency. The applications of the method to a control system which has a single nonlinearity and to a well known autonomous system are also included in this chapter. The stability regions are then compared with the known works on these systems.

2.2 PRACTICAL APPLICATION TO SECOND ORDER TIME INVARIANT SYSTEMS.

It is common for nonlinear dynamic systems such as electric power systems, chemical systems, reactors, etc. to have multiple equilibrium states, some of which have only a limited region of stability. We shall apply White's method to some existing power systems, control systems and autonomous systems and see whether the regions obtained are as accurate as the actual ones or better than those found by other conventional methods.

2.3 BRIEF DESCRIPTION OF POWER SYSTEM.

The notations used will be given in Appendix A. A typical power system is shown in Figure 2.3(i).

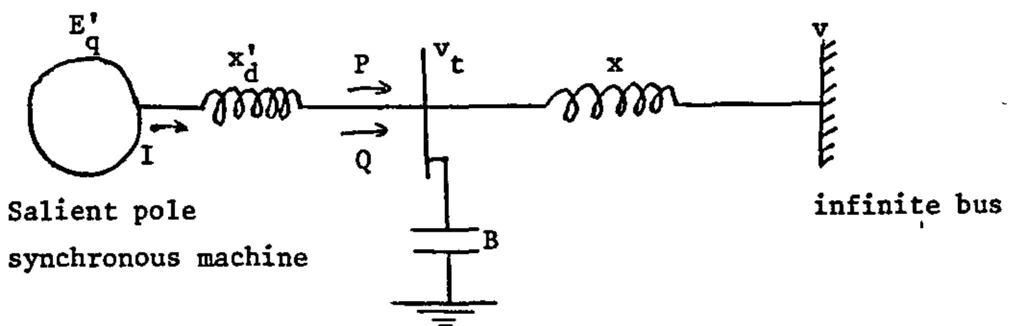


Fig. 2.3(i). Typical Power System.

A salient pole generator is connected to the infinite bus through a high voltage, long distance transmission system. The voltage of the infinite bus is constant while the machine voltage, being proportional to the field flux linkage, is variable during the transient period due to the field decay. All resistances are neglected.

The equation of motion of the synchronous machine [36] can be written as

$$M\ddot{\delta} = P_{mi}(\dot{\delta}) - P_e(E_q',\delta) - D(\delta)\dot{\delta} \quad . \qquad (2.3.1)$$

The variable damping is given by

$$D(\delta) = v^2 \left\{ \frac{x_d' - x_d''}{(x_e + x_d')^2} T_{do}'' \sin^2\delta + \frac{x_q' - x_q''}{(x_e + x_q')^2} T_{qo}'' \cos^2\delta \right\}$$

$$= a_1 \sin^2\delta + a_2 \cos^2\delta \quad . \qquad (2.3.2)$$

The electrical power output considering transient saliency is

$$P_e(E_q',\delta) = \frac{E_q' \, v}{x_e + x_d'} \sin\delta - \frac{v^2(x_q - x_d')}{2(x_e + x_q)(x_e + x_d')} \sin 2\delta$$

$$= b \sin\delta - c \sin 2\delta \quad . \qquad (2.3.3)$$

The flux decay equation is

$$\frac{dE_q'}{dt} = \frac{E_{ex}}{T_{do}'} - \frac{E_q'}{T_d'} + \frac{v(x_d - x_d')}{(x_e + x_d')T_{do}'} \cos\delta$$

$$= \frac{E_{ex}}{T_{do}'} - \eta_1 E_q' + \eta_2 \cos\delta. \qquad (2.3.4)$$

The governor action is described by

$$T_e \frac{dP_{mi}}{dt} + P_{mi} = P_{mio} - k\,\dot{\delta}$$

$$\frac{dP_{mi}}{dt} = -\gamma_1(P_{mi} - P_{mio}) - \gamma_2\,\dot{\delta} \qquad (2.3.5)$$

By transforming the coordinates so that the origin is the stable equilibrium point of the new state space, i.e. substituting $\delta = x_1 + \delta_o$, $\dot{\delta} = x_2$, $E'_q = x_3 + E'_{qo}$ and $x_4 = P_{mio} - P_{mi}$, the above equations will be reduced to

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\left\{a_1 \sin^2(x_1+\delta_o) + a_2 \cos^2(x_1+\delta_o)\right\} x_2$$

$$- b\left\{\sin(x_1+\delta_o) - \sin \delta_o\right\} + c\left\{\sin 2(x_1+\delta_o) - \sin 2\delta_o\right\}$$

$$- gx_3 \sin(x_1+\delta_o) - x_4$$

$$\dot{x}_3 = - \eta_1 x_3 - \eta_2\left\{\cos \delta_o - \cos(x_1+\delta_o)\right\}$$

$$\dot{x}_4 = - \gamma_1 x_4 + \gamma_2 x_2 \quad . \tag{2.3.6}$$

2.4 EXAMPLES AND RESULTS.

i) Synchronous generator with constant damping [37].

The differential equations of a single machine with constant damping, constant field linkage and constant input power are given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = - ax_2 - b\left\{\sin(x_1+\delta_o) - \sin \delta_o\right\} \tag{2.4.1}$$

where $a = 0.2$, $b = 1$ and $\delta_o = \frac{\pi}{4}$.

(2.4.1) has two critical points at (0,0) and (1.571,0) with the origin being an asymptotically stable point and the other critical point an unstable saddle point. The domain of attraction of (2.4.1) is unbounded but consists of a region around the origin which is unbounded only in the fourth quadrant. The result obtained by White's method using $\phi = x_1^2 + x_2^2$ is shown in figure 2.4(i). The figure clearly shows that this method gives a better estimate of the stability domain than Szego's method used by Prusty [31].

ii) Synchronous machine with constant damping and saliency [37].

The system equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - ax_2 - b\left\{\sin(x_1+\delta_o) - \sin \delta_o\right\} + c\left\{\sin 2(x_1+\delta_o) - \sin 2\delta_o\right\} \tag{2.4.2}$$

where $a = 0.2$, $b = 1$, $c = 0.2$ and $\delta_o = \frac{\pi}{4}$.

Here the origin is a stable point and (1.977,0) is an unstable saddle point. The difference between the abscissa of the two equilibrium points is used as a guide for choosing the value of R. In (2.4.2) the domain of asymptotic stability is also unbounded in the fourth quadrant.

Figure 2.4(ii) shows the stability regions obtained by this method and by Szego's method [31], illustrating its superiority. Comparing figures 2.4(i) and 2.4(ii), it is found that the stability region of a machine system with saliency is larger than the one without saliency. Again the domain obtained shows that this method gives a better approximation of the true stability domain.

iii) Synchronous machine with variable damping and saliency [30].

In this example the first two equations of (2.3.6) are used where the flux decay and governor action are neglected. The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\left\{a_1 \sin^2(x_1+\delta_o) + a_2 \cos^2(x_1+\delta_o)\right\}x_2$$

$$- b\left\{\sin(x_1+\delta_o) - \sin \delta_o\right\} + c\left\{\sin 2(x_1+ \delta_o) - \sin 2\delta_o\right\} \qquad (2.4.3)$$

where $a_1 = 0.1, \quad a_2 = 0.2, \quad b = 1, \quad c = 0.2, \quad \delta_o = \frac{\pi}{4}$.

Figure 2.4(iii) shows the stability regions obtained by this method, Szego [31] and Miyagi [24]. The stability regions of Prusty and Miyagi are all included in the region of this method showing the superiority of the method. There is a slight difference between the stability domain of systems (2.4.3) and (2.4.2) owing to the effect of variable damping introduced in (2.4.3) and this can be seen from figures 2.4(ii) and 2.4(iii).

The power systems considered have an unbounded domain in the fourth quadrant. As the stability regions obtained were only approximations to $S_R \cap D(\underline{f})$, then for bigger R, we will get a larger region in the fourth quadrant and hence give a good estimate of the stability domain. This can be seen from example ii) illustrated in figure 2.4(iv).

iv) Second order autonomous system.

Consider the system

$$\dot{x} = x^2 - y$$

$$\dot{y} = x - y \quad . \tag{2.4.4}$$

(2.4.4) has two critical points at (0,0) and (1,1). The origin is an asymptotically stable point while the point (1,1) is an unstable saddle point. The sketch of the behaviour of the trajectories is shown in figure 2.4(v). The thick curve which passes through (1,1) is the separatrix.

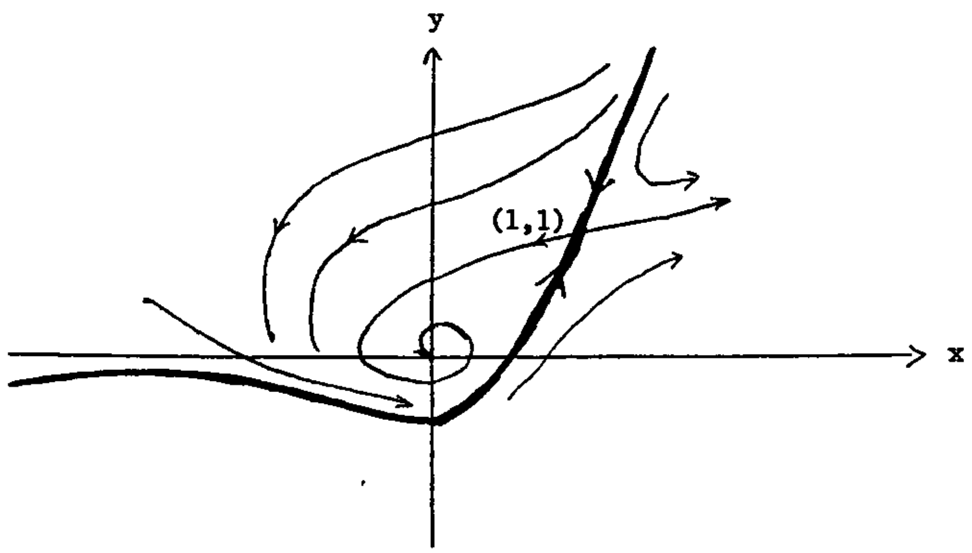


Figure 2.4(v)

This particular example is chosen to show the effect of $S_R \cap D(f)$ for different values of R on the domain of asymptotic stability. We can also infer from it the convergence of the stability boundary.

White's method has no convergence problem and this is illustrated in figure 2.4(vi). Here we will see that as R increases the stability region increases in the first three quadrants and the stability boundary in the fourth quadrant is always fixed. Various values of R were tested and clearly show that the equilibrium has a bigger stability region especially in the second quadrant. The boundaries obtained always pass through the saddle point (1,1).

2.5 CONTROL SYSTEMS.

The work concerning the existence and construction of Liapunov functions for the problem of Lurê in the form 'quadratic plus integral of the nonlinearity' is well known. Pai, Mohan and Rao [38] use Kalman's construction procedure [39] with a single nonlinearity to construct the Lurê type Liapunov function and compute the stability regions by the technique proposed by Walker and McClamroch [40]. However, their stability region drawn from the largest Liapunov function V(x) = constant which is contained in a finite interval (m_1,m_2) does not give a complete domain of attraction.

In this section we will compute the region of attraction of a control system with a single nonlinearity using the method discussed in Section 1.7. The nonlinearity $\Phi(\sigma)$ satisfies the Popov sector condition $0 < \frac{\Phi(\sigma)}{\sigma} < K$, for $\sigma \neq 0$, $K > 0$ in the interval (m_1,m_2) and hence there exists a region of stability around the origin of the state space. The region of attraction is compared with the region obtained by Walker and McClamroch.

Consider a system defined in the problem of Luŕe

$$\underline{\dot{x}} = A\,\underline{x} + \underline{b}\,\Phi(\sigma) \qquad (2.5.1)$$

$$\sigma = \underline{c}^T \underline{x}$$

where A is an n×n stable matrix (i.e. all its eigenvalues have negative real parts); $\underline{x}$ and $\underline{b}$ are column vectors, $\underline{c}^T$ is a row vector; σ is a scalar and $\Phi(\sigma)$ is assumed to satisfy the conditions

a) $\Phi(\sigma)$ is continuous for all σ

b) $0 < \frac{\Phi(\sigma)}{\sigma} < K$ for $\sigma \neq 0$ (2.5.2)

c) $\Phi(0) = 0$.

Luŕe uses the Liapunov function

$$V = \underline{x}^T B\,\underline{x} + \int_0^{\sigma} \Phi(\sigma)d\sigma \qquad (2.5.3)$$

where B is a positive definite symmetric matrix. He states that the sufficient conditions for global asymptotic stability of the zero solution of system (2.5.1) with Φ satisfying (2.5.2) are

i) $\lim\limits_{|\sigma|\to\infty} \int_0^{\sigma} \Phi(\sigma)d\sigma = \infty$ (2.5.4)

ii) $\underline{u}^T C^{-1}\,\underline{u} + \underline{b}^T\underline{c} < 0,$ where $\underline{u} = B\,\underline{b} + \frac{1}{2}A^T\underline{c}$

and $-C = A^T B + BA$.

A well known condition which is used for complete stability is the Popov frequency criterion:

$$\mathrm{Re}\left[(1+j\omega q)\;G(j\omega)\right] + \frac{1}{K} > 0 \qquad (2.5.5)$$

for some non-negative q, K and all real ω.

G(s) is the transfer function of the linear part of (2.5.1) obtained as

$$G(s) = -\underline{c}^T(sI - A)^{-1}\,\underline{b} \quad . \tag{2.5.6}$$

It is known that a nonlinearity $\Phi(\sigma)$ which satisfies the sector condition (2.5.2) for all $\sigma \neq 0$ will result in asymptotic stability in the large. In many practical problems it is very rare to have such a nonlinearity. In this section we will also consider a system in which the nonlinearity violates the sector condition (2.5.2) at some interval $\sigma \leqslant m_1 < 0$ and $\sigma \geqslant m_2 > 0$ (i.e. the sector condition is satisfied in the interval (m_1, m_2)). For this type of system, the region of asymptotic stability is not a global one. Here the sector condition is relaxed so that the system is asymptotically stable but not in the large.

Next we consider the following example given by Walker and McClamroch

$$\ddot{x} + a\dot{x} + bx + \Phi(x) = 0 \qquad (a > 0,\ b > 0) \quad . \tag{2.5.7}$$

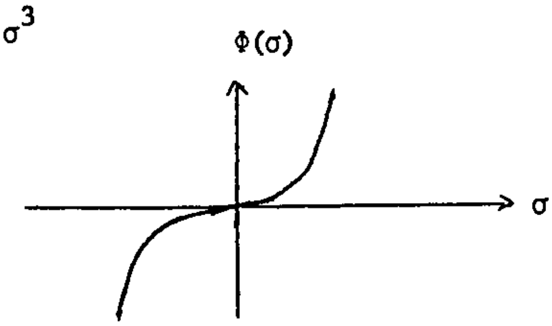
The system equation in the form of (2.5.1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Phi(\sigma) \tag{2.5.8}$$

$$\sigma = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad .$$

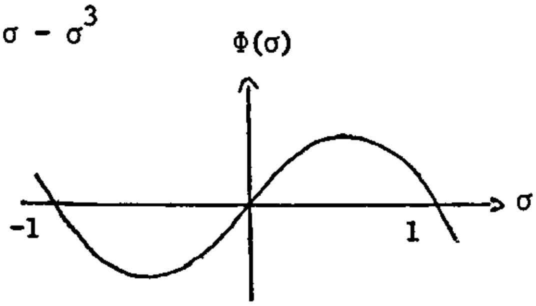
We will study this system with different types of nonlinearity for $a = 1$ and $b = 1$.

<u>Type (i)</u>. $\Phi(\sigma) = \sigma^3$



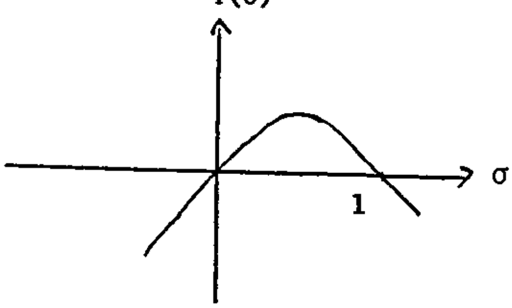
This nonlinearity satisfies the sector condition (2.5.2) for all $\sigma \neq 0$. Hence for this $\Phi(\sigma)$ we will have a global asymptotic stability and the region of attraction is the whole space. Figure 2.5(i) shows the region of stability by the numerical method for $R = 3.2$ and $R = 4.9$. Obviously for a system which is asymptotically stable in the large, White's method gives the full circle for each R as R is increased. This behaviour is not, of course, a proof of asymptotically stable in the large.

<u>Type (ii).</u> $\phi(\sigma) = \sigma - \sigma^3$



Here the nonlinearity satisfies the sector condition in the interval $(-1,1)$ and leaves the sector at $\sigma \leqslant -1$ and $\sigma \geqslant 1$. When this Φ is substituted in (2.5.8) we find that the system has three critical points, namely $(0,0)$, $(-\sqrt{2},0)$, $(\sqrt{2},0)$. The origin is the stable equilibrium point and the rest are unstable saddle points. This system will therefore have a domain of attraction which is not the whole space. Figure 2.5(ii) shows the domain of attraction obtained by the numerical method and the method used by Walker and McClamroch. The marked difference in areas produced from these two methods show that the numerical method gives a better estimate of the domain of attraction.

Type (iii). $\phi(\sigma) = \sigma - \sigma^2$



In this case we have a nonlinearity which satisfies the sector condition in the interval $(-\infty,1)$ and leaves the sector at $\sigma \geq 1$. The critical points are the origin which is asymptotically stable and (2,0) which is the unstable saddle point. The domain of attraction computed by White's method is shown in Figure 2.5(iii) together with that obtained by Walker and McClamroch. The result clearly shows the superiority of the former method.

2.6 CONCLUSION.

In this chapter, White's method is applied to second order autonomous systems including the power systems to determine the region of stability. The results are compared to those of Prusty and Miyagi. Although Miyagi attempts to solve the Zubov's partial differential equation using the classical Lagrange-Charpit method, the exact stability boundary is never achieved. This is due to the fact that the method determines the arbitrary non-negative function ϕ which allows the Liapunov function to be determined and the Liapunov function $V(\underline{x})$ = constant is used to plot the stability boundary. The results show that White's algorithm produces a better estimate of the stability region than that of Prusty and Miyagi. Different values of R were also used to show the various sizes of the domain and the convergence of the stability boundary is shown in Figure 2.4(vi). In this figure fixed boundary points are obtained in the fourth quadrant which show that White's method does not suffer from the nonuniform convergence.

A control system containing a single nonlinearity which satisfies the sector condition in the interval (m_1,m_2) and leaves the sector at $\sigma \leq m_1 < 0$ and $\sigma \geq m_2 > 0$ is also studied. The sector condition has been weakened so that the domain of attraction produced will not be the whole space. Numerical result shows that this method gives a good approximation of the true stability boundaries.

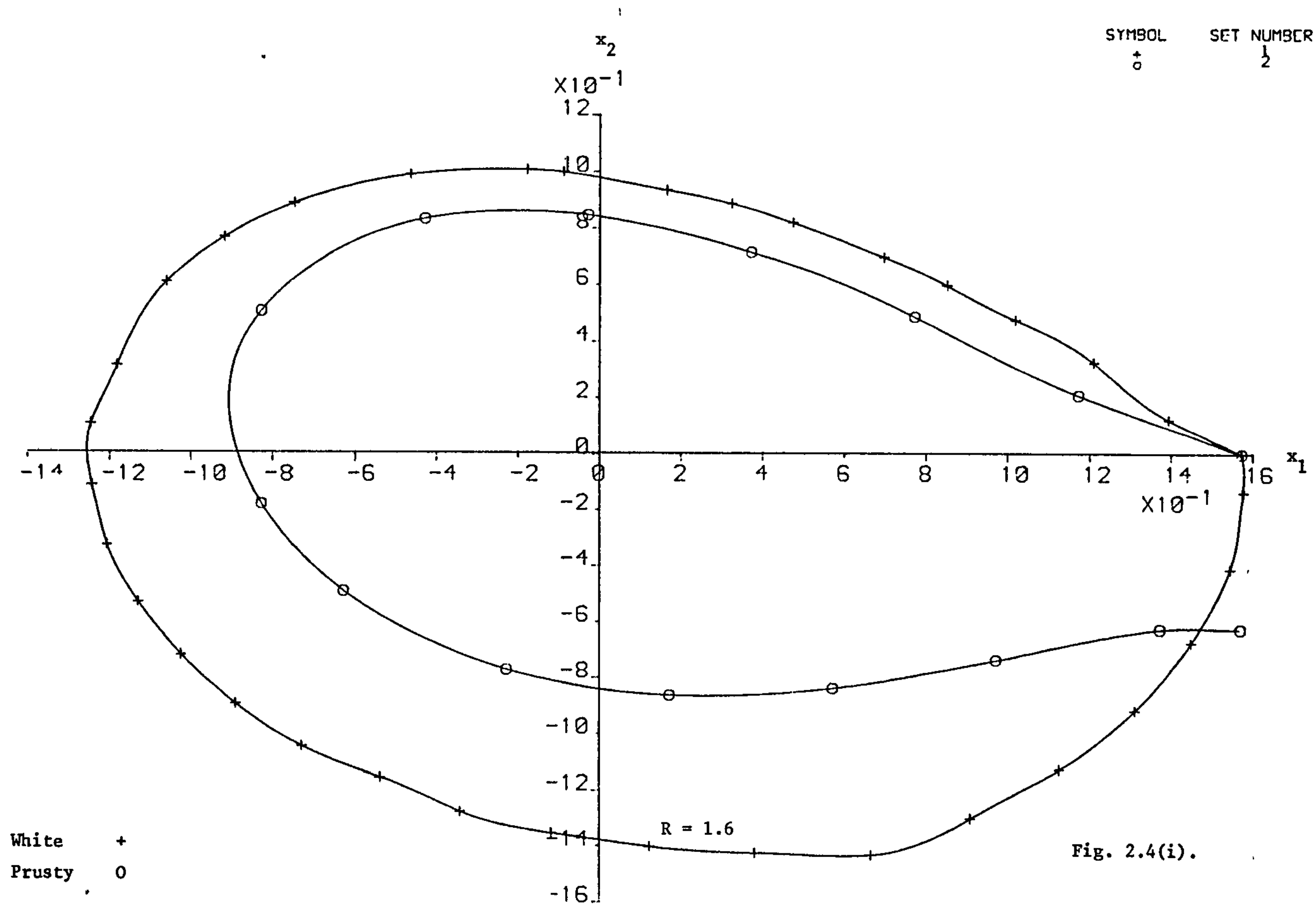
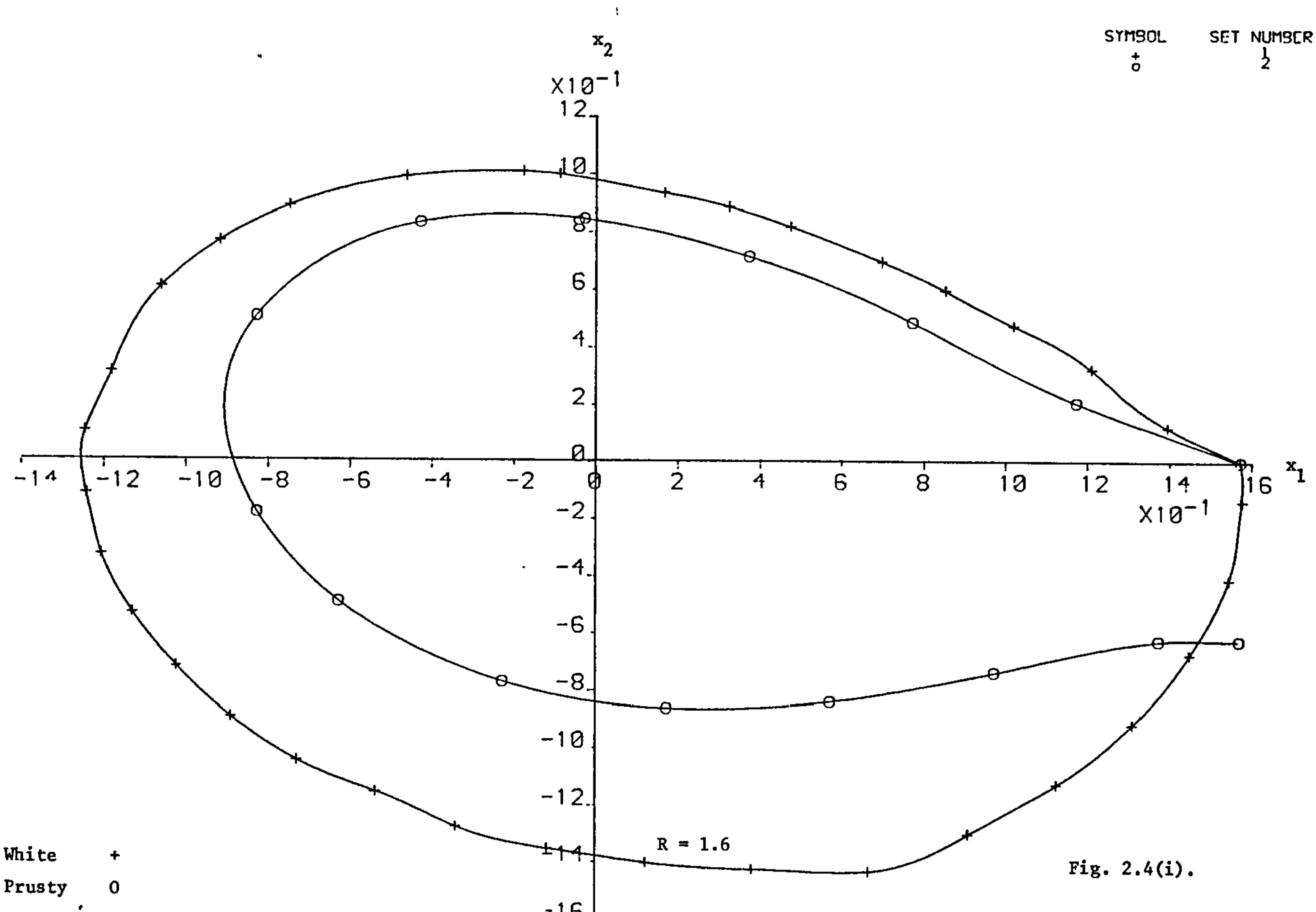


Fig. 2.4(i).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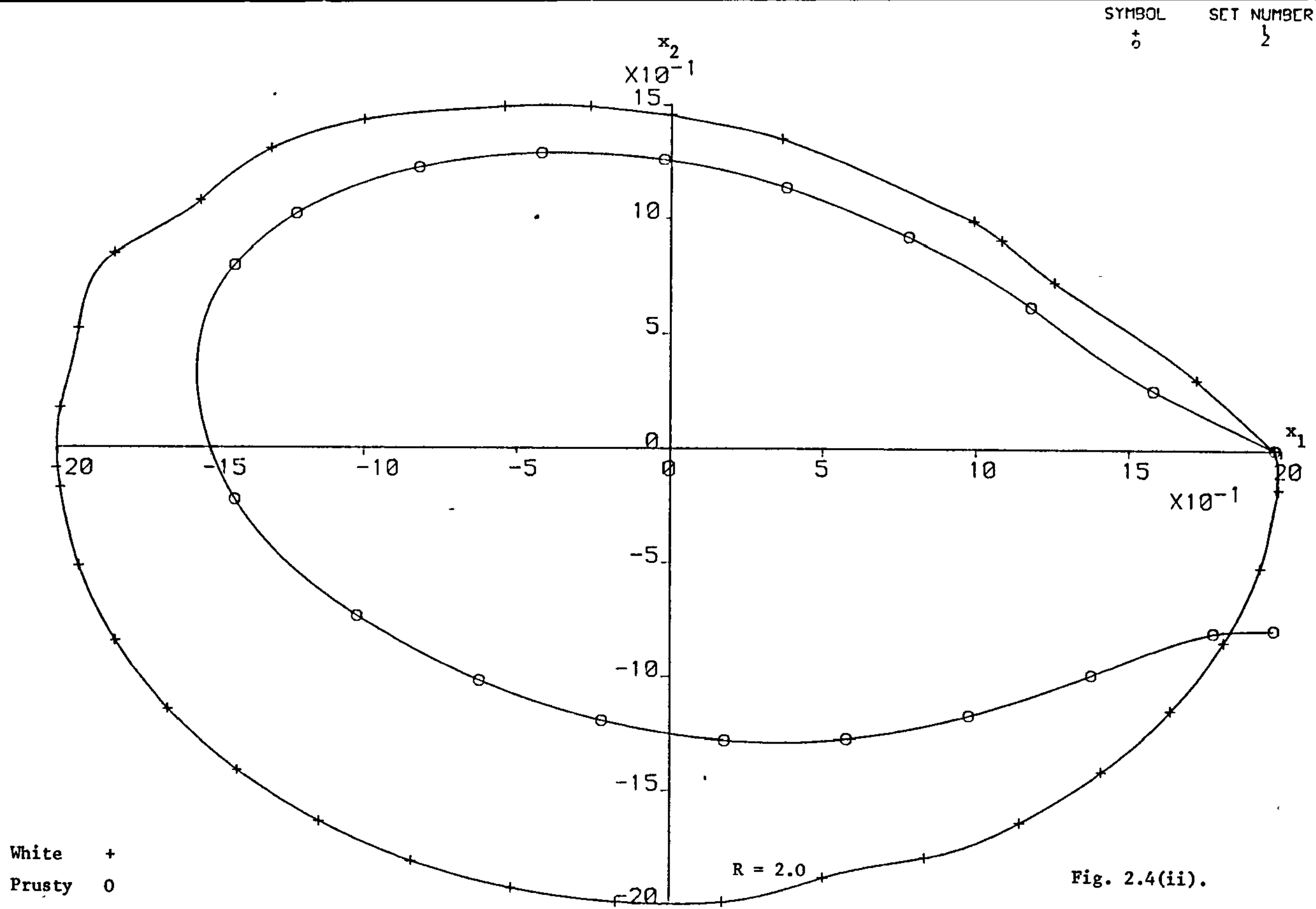
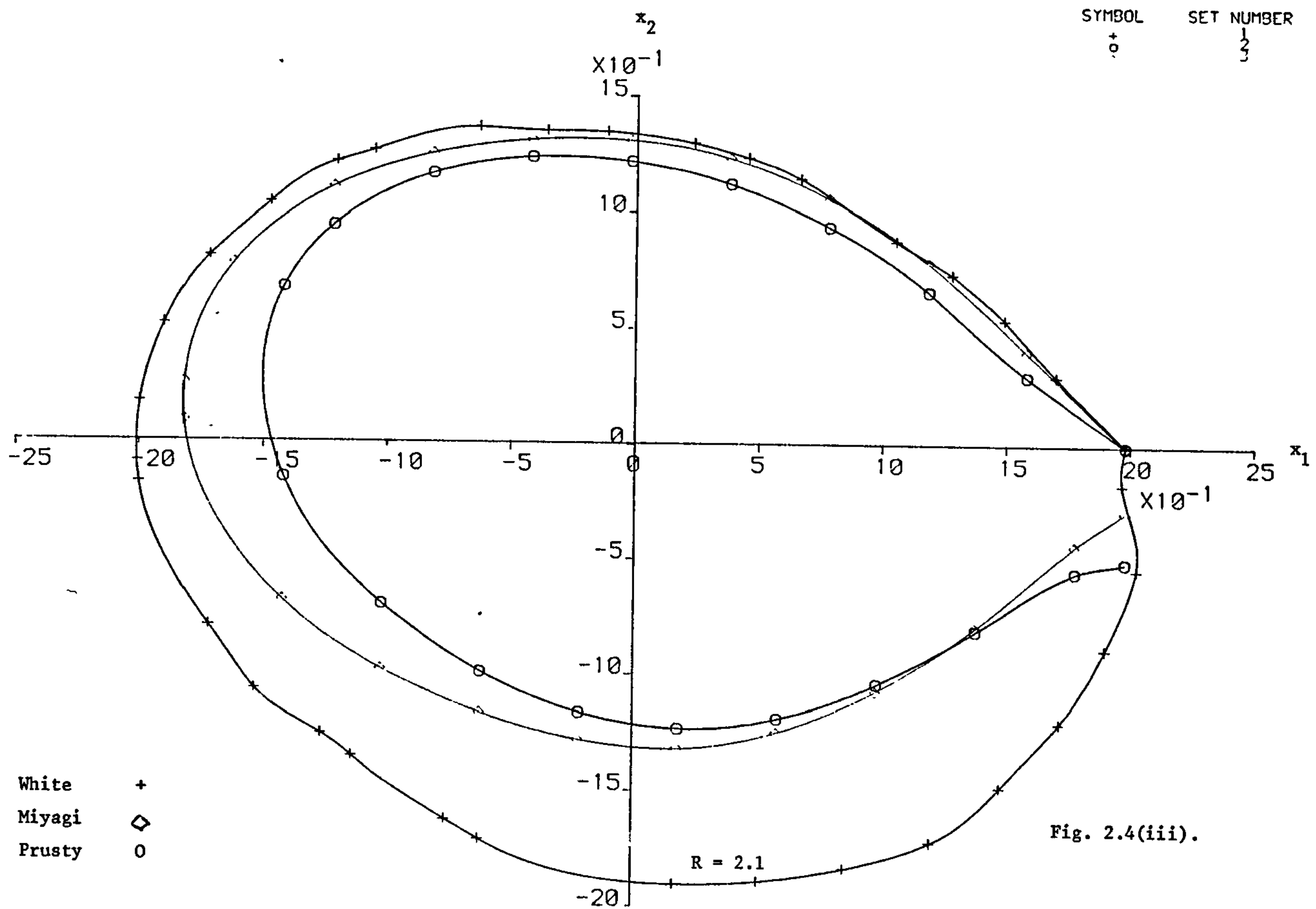
x_2
X10^-1
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14
-16
-14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16
X10^-1
x_1
R = 1.6
White +
Prusty 0

Fig. 2.4(i).



SYMBOL SET NUMBER
+ 1
o 2
x_2
X10^{-1}
15
10
5
0
-5
-10
-15
-20
-20
-15
-10
-5
0
5
10
15
20
x_1
X10^{-1}
White +
Prusty 0
R = 2.0

Fig. 2.4(ii).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+ 1
o 2
3
x_2
X10^{-1}
15
10
5
0
-5
-10
-15
-20
-25 -20 -15 -10 -5 0 5 10 15 20 25
x_1
X10^{-1}
White +
Miyagi ◇
Prusty O
R = 2.1

Fig. 2.4(iii).

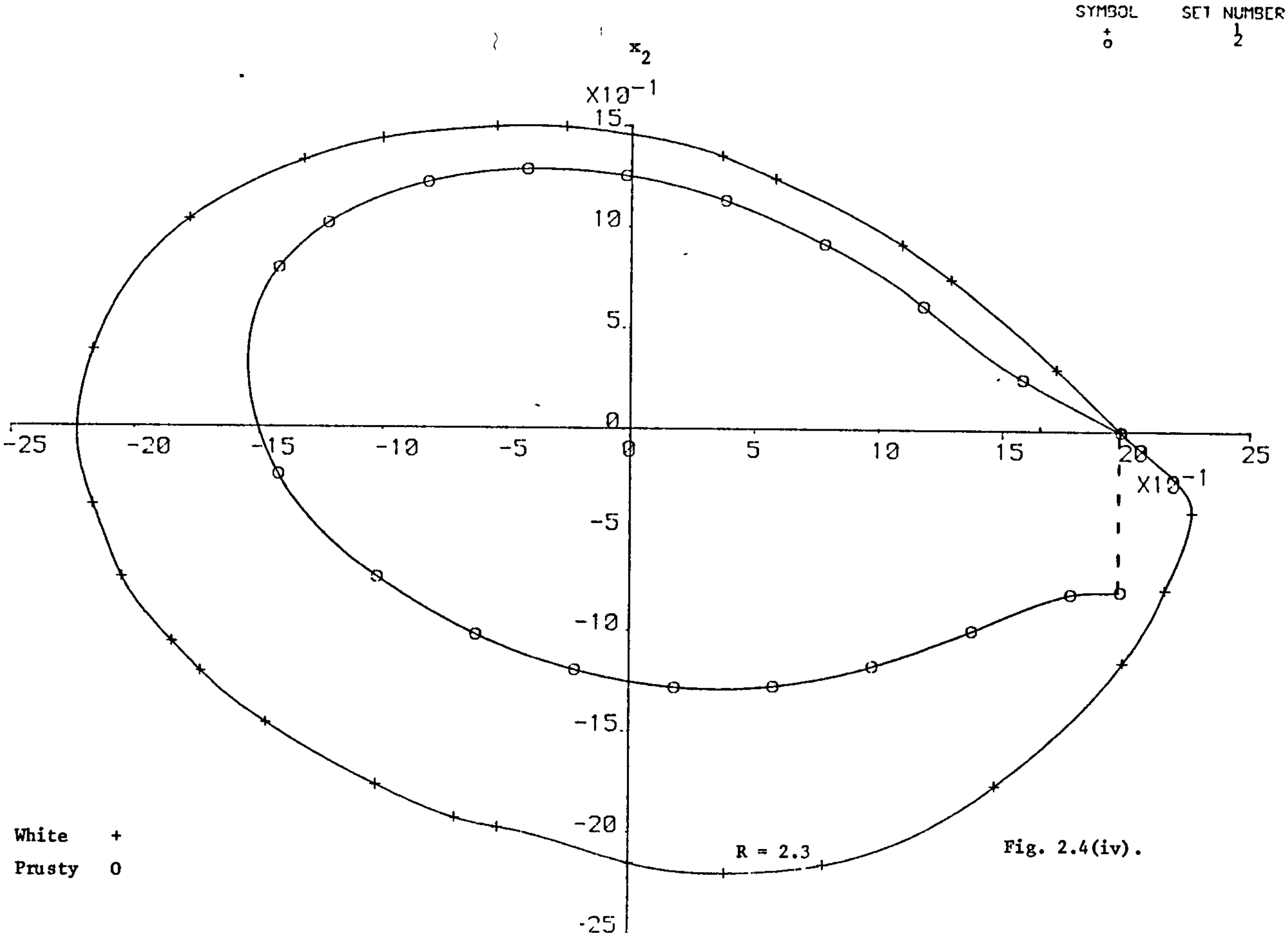
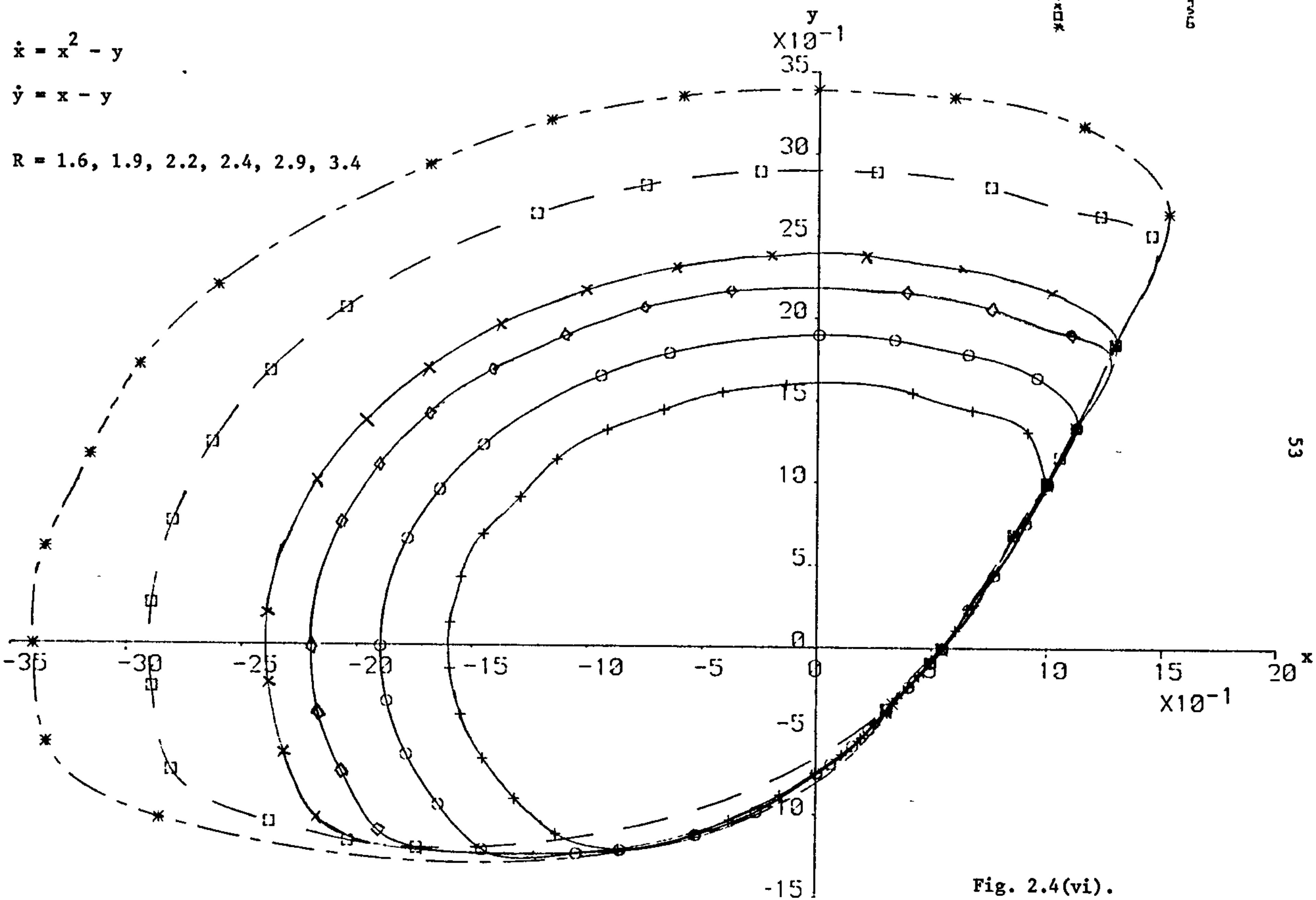

SYMBOL SET NUMBER
+ 1
o 2
x2
X10^-1
15
10
5
0
-5
-10
-15
-20
-25
x1
-25 -20 -15 -10 -5 0 5 10 15 20 25
X10^-1
White +
Prusty O
R = 2.3

Fig. 2.4(iv).


$\dot{x} = x^2 - y$
$\dot{y} = x - y$
R = 1.6, 1.9, 2.2, 2.4, 2.9, 3.4
y
X10^-1
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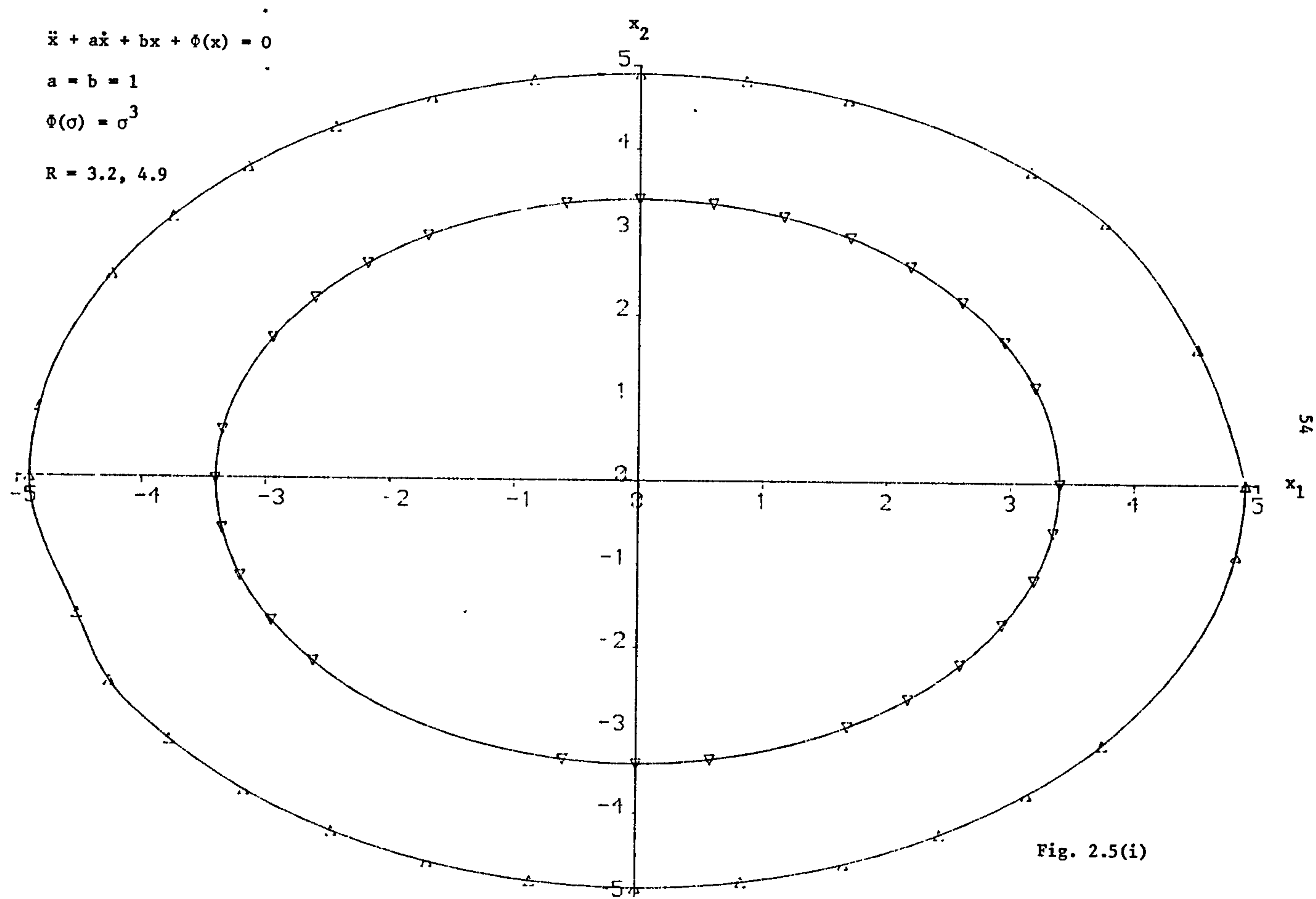
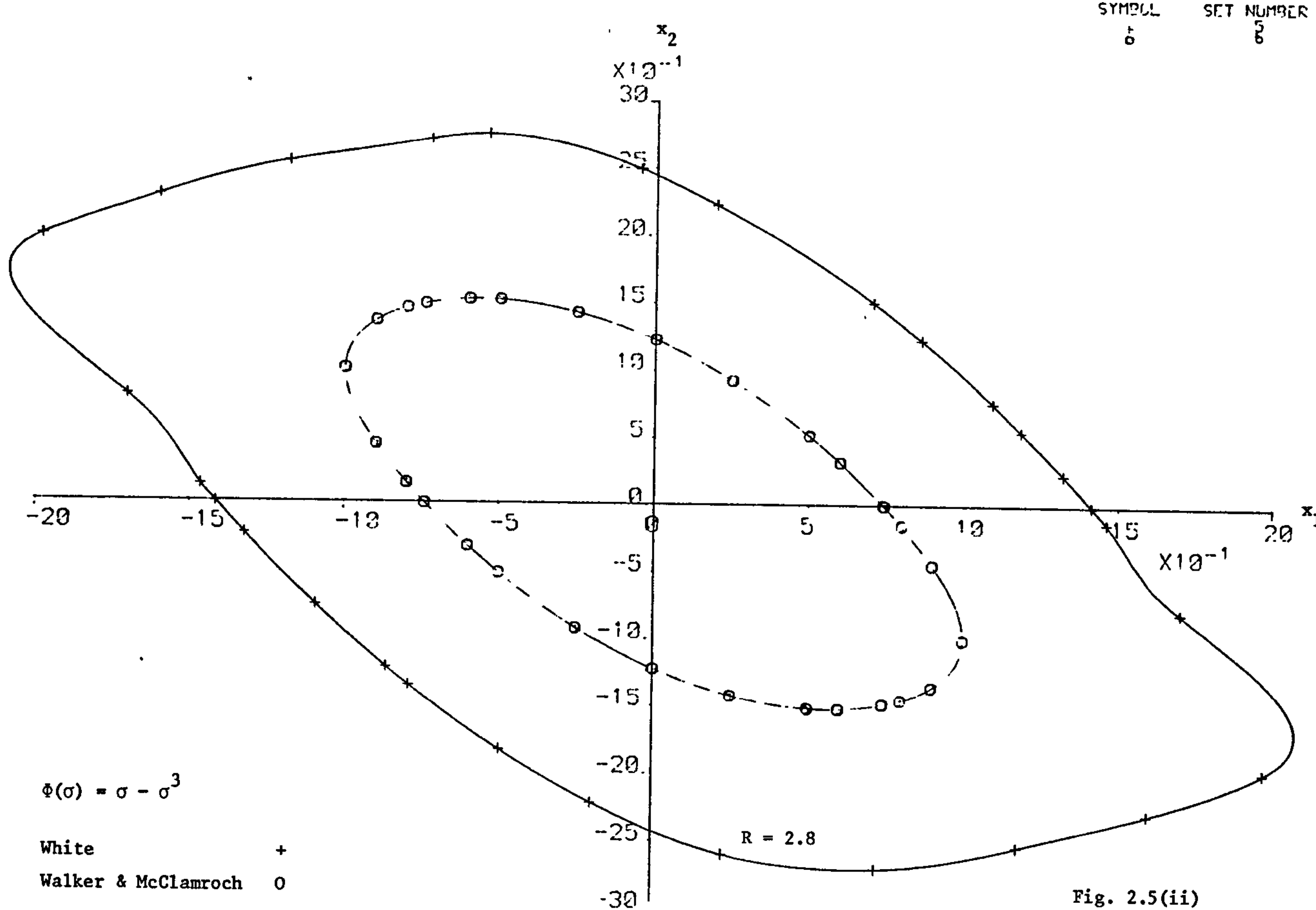
30
25
20
15
10
5.
0
-5
-10
-15
-35
-30
-25
-20
-15
-10
-5
0
5
10
15
20
x
X10^-1

Fig. 2.4(vi).



$\ddot{x} + a\dot{x} + bx + \Phi(x) = 0$
$a = b = 1$
$\Phi(\sigma) = \sigma^3$
R = 3.2, 4.9
x_2
x_1
5
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2
3
4
5

Fig. 2.5(i)



SYMBOL SET NUMBER
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x_2
X10^{-1}
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25
20
15
10
5
0
-5
-10
-15
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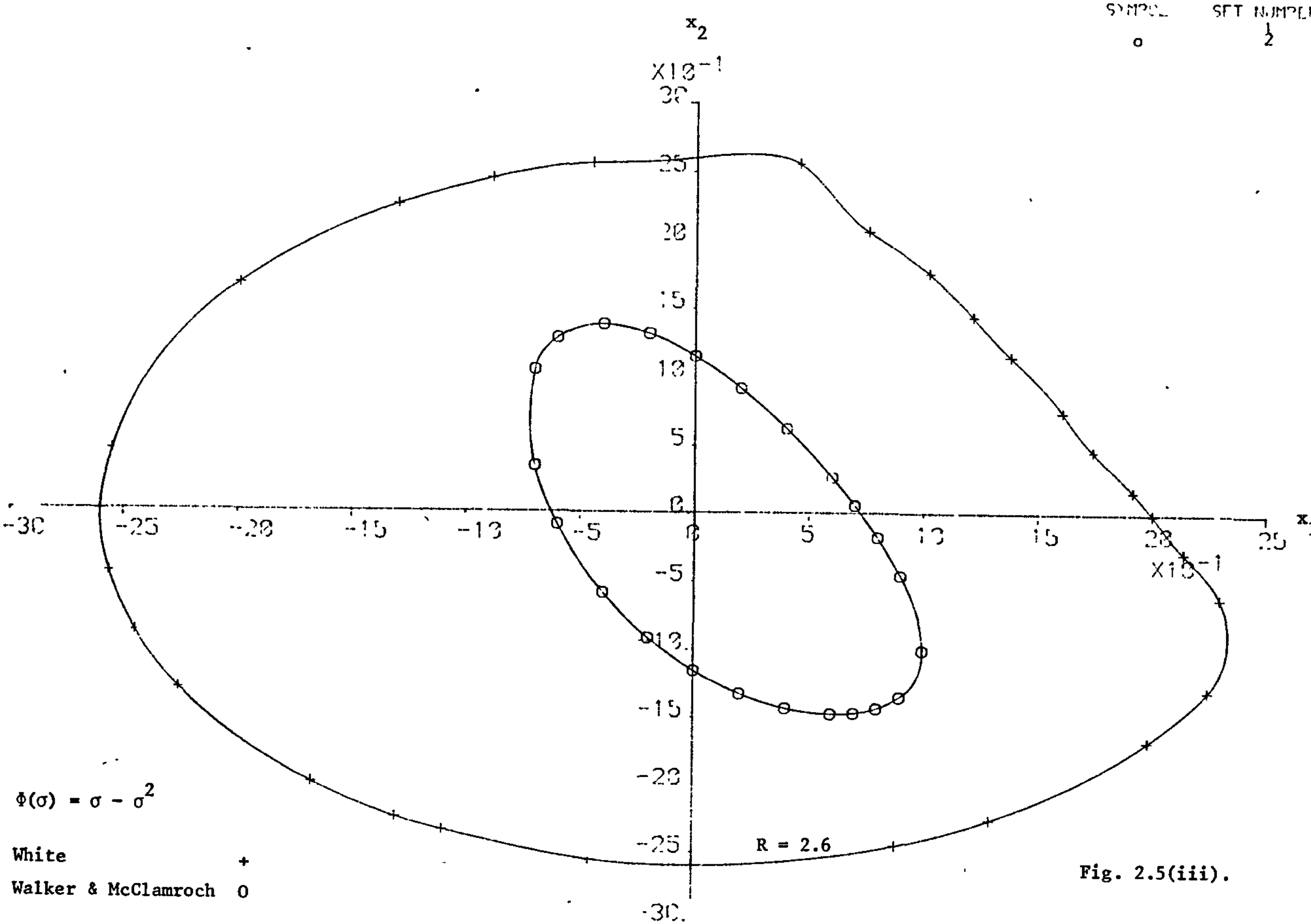
-25
-30
-20
-15
-10
-5
0
5
10
15
20
x_1
X10^{-1}
$\Phi(\sigma) = \sigma - \sigma^3$
White +
Walker & McClamroch O
R = 2.8

Fig. 2.5(ii)


SYMBOL SET NUMBER
o 1
2
x_2
X10^{-1}
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25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-30 -25 -20 -15 -10 -5 0 5 10 15 20 25
x_1
X10^{-1}
R = 2.6
$\Phi(\sigma) = \sigma - \sigma^2$
White +
Walker & McClamroch 0

Fig. 2.5(iii).

CHAPTER III

Scalar Time Varying Systems

3.1 INTRODUCTION.

The study and analysis of the asymptotic behaviour, stability and domain of attraction of nonautonomous systems has not received much attention compared to that of autonomous systems. Grujic [42] presents a refined analysis of the influence of initial data on dynamic behaviour and stability properties of nonstationary systems and establishes relationship from them. Many authors such as Kalman [41], Mandal [43], Newman [44] and Puri [45] study the time varying systems which are asymptotically stable in the large and hence the stability domain is the whole space.

After describing the method for determining the domain of attraction of autonomous system in Section 1.7 and applying it to second order power systems, we now apply the method to a scalar time varying system.

We know that for the second order autonomous system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{3.1.1}$$

the Zubov equation is

$$\frac{\partial V}{\partial x}(x,y)\ f(x,y) + \frac{\partial V}{\partial y}(x,y)\ g(x,y) = -\ \phi(x,y)(1-eV(x,y)) \tag{3.1.2}$$

which is of first order.

A scalar time varying system

$$\dot{x} = f(x,t) \tag{3.1.3}$$

will give a Zubov equation of order one of the form

$$\frac{\partial V}{\partial x}(x,t)\ f(x,t) + \frac{\partial V}{\partial t}(x,t) = -\ \phi(x,t)(1-eV(x,t)) \tag{3.1.4}$$

The function $V(x,t)$ according to Zubov $\boxed{1}$ will have the positive definiteness and decrescent properties in the stability region if $\phi(x,t)$ is positive definite and decrescent.

We have seen that the Zubov's equation (3.1.2) can be solved either analytically or in series form but in most cases analytical solution is not possible. It has been shown in various papers in the literature [2,3,21] that the truncated series solution gives only an approximate stability boundary and this boundary does not approach the true boundary monotonically with the increase of N, the partial sum number. We shall examine the Zubov equation (3.1.4) and solve it using a series solution technique.

3.2 SERIES SOLUTION.

Let a scalar time varying system (3.1.3) be written as

$$\dot{x} = \sum_{i=1}^{mf} a_i(t)\, x^i \tag{3.2.1}$$

or

$$\dot{x} = f_1 + f_2 + \dots + f_{mf} \tag{3.2.2}$$

where the f_i are functions of x and t and degree i in x.

Thus, we may write

$$f_r = a_r(t)\, x^r \,, \qquad r = 1,\dots,mf \;. \tag{3.2.3}$$

Let the function $\phi(x,t)$ in (3.1.4) be written as

$$\phi(x,t) = \sum_{i=2}^{m\phi} c_i(t)\, x^i \tag{3.2.4}$$

where $c_i(t)$ are assumed to be bounded functions,

or

$$\phi = \phi_2 + \phi_3 + \dots + \phi_{m\phi} \tag{3.2.5}$$

where the ϕ_i are functions of x and t and of degree i in x.

Thus

$$\phi_u = c_u(t)\ x^u , \qquad u = 2,\dots,m\phi \ . \tag{3.2.6}$$

We want to construct a Liapunov function V(x,t) of the form

$$V(x,t) = \sum_{i=2}^{mv} b_i(t)\ x^i \tag{3.2.7}$$

where $b_i(t)$ are bounded functions,

or

$$V = V_2 + V_3 + \dots + V_{mv} \tag{3.2.8}$$

where the V_i are functions of x and t and of degree i in x.

Thus

$$V_s = b_s(t)\ x^s, \qquad s = 2,\dots,mv \tag{3.2.9}$$

$$\frac{\partial V_s}{\partial x} = s b_s(t)\ x^{s-1}, \qquad s = 2,\dots,mv \tag{3.2.10}$$

$$\frac{\partial V_s}{\partial t} = \dot{b}_s(t)\ x^s , \qquad s = 2,\dots,mv \tag{3.2.11}$$

where $$\dot{b}_s(t) = \frac{db_s(t)}{dt} \ . \tag{3.2.12}$$

Equation (3.1.4) after substitution from (3.2.2), (3.2.5) and (3.2.8) becomes

$$\frac{\partial}{\partial x}(V_2 + \dots + V_{mv})(f_1 + \dots + f_{mf}) + \frac{\partial}{\partial t}(V_2 + \dots + V_{mv})$$

$$= -(\phi_2 + \dots + \phi_{m\phi})(1 - V_2 - \dots - V_{mv}) \tag{3.2.13}$$

$$\doteq -(\phi_2 + \dots + \phi_{m\phi}) \tag{3.2.14}$$

(3.2.13) is the regular equation, (3.2.14) is the modified equation. Equating terms of like degree in x, we have (for (3.2.14))

$$\frac{\partial V_2}{\partial x} f_1 + \frac{\partial V_2}{\partial t} = -\phi_2 \tag{3.2.15}$$

$$\frac{\partial V_3}{\partial x} f_1 + \frac{\partial V_3}{\partial t} = -\phi_3 - \frac{\partial V_2}{\partial x} f_2 \tag{3.2.16}$$

$$\frac{\partial V_4}{\partial x} f_1 + \frac{\partial V_4}{\partial t} = -\phi_4 - \frac{\partial V_2}{\partial x} f_3 - \frac{\partial V_3}{\partial x} f_2 \tag{3.2.17}$$

and so on.

Hence in general

$$\left.\begin{aligned} &\frac{\partial V_2}{\partial x} f_1 + \frac{\partial V_2}{\partial t} = -\phi_2 \\ &\frac{\partial V_n}{\partial x} f_1 + \frac{\partial V_n}{\partial t} = -\phi_n - \sum_{i=2}^{n-1} f_i \frac{\partial}{\partial x} (V_{n-i+1}) \end{aligned}\right\} \quad (3.2.18$$

for $n \geq 3$.

From the above equations we can find the coefficients $b_i(t)$ of the Liapunov functions when ϕ is known. Thus from (3.2.15) we can find $b_2(t)$ and a quadratic part V_2. Substitution of V_2 into (3.2.16) allows $b_3(t)$ to be determined and V_3 is obtained. Continuing the process, the highest degree term V_{mv} can be obtained. This will enable us to construct a Liapunov function by the series method and hence an exact domain of attraction can be determined if a closed form solution is derived from it.

Consider the scalar time varying system

$$\dot{x} = -x + 2e^{-t}x^2 \quad . \tag{3.2.19}$$

This system has the solution

$$x = \frac{1}{e^{-t} + \left(\frac{1}{x_o e^{t_o}} - e^{-2t_o}\right)e^t}$$

for a given x_o, t_o.

The solution approaches zero as $t \to +\infty$ and is infinite at

$$t = \frac{1}{2}\ln\frac{x_o e^{t_o}}{x_o e^{-t_o} - 1} \quad .$$

The domain of attraction of the equilibrium is

$$D_{t_o} : -\infty < x_o < e^{t_o} \quad .$$

Now we choose $\phi(x,t) = x^2$.

Using (3.2.9), (3.2.10), (3.2.11) and (3.2.18) we can find $b_2(t)$, $b_3(t), \ldots, b_{mv}(t)$ and hence V_2, $V_3, \ldots, V_{mv}$ can be obtained. The V-terms are

$$V_2 = \frac{x^2}{2}$$

$$V_3 = \frac{1}{2}e^{-t}x^3$$

$$V_4 = \frac{1}{2}e^{-2t}x^4$$

$$\vdots$$

$$V_n = \frac{1}{2}e^{-(n-2)t}x^n \quad .$$

The Liapunov function is therefore

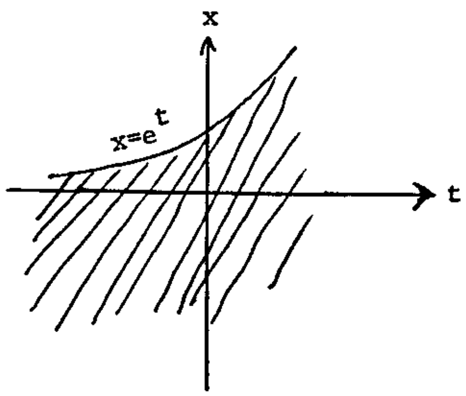
$$V = V_2 + V_3 + \ldots\ldots$$

$$= \frac{x^2}{2}\left[1 + e^{-t}x + e^{-2t}x^2 + e^{-3t}x^3 + \ldots\right]$$

$$= \frac{x^2}{2}\sum_{n=0}^{\infty}\left(e^{-t}x\right)^n = \frac{x^2}{2}\left(1 - xe^{-t}\right)^{-1} .$$

Thus we have a closed form solution of the Zubov equation and the domain of asymptotic stability is automatically found, that is $-\infty < x < e^t$.

From the series solution the even partial sums converge to the domain of asymptotic stability $-\infty < x < e^t$ and the odd partial sums converge to the region of convergence, $|x| < e^t$ as shown in Figures 3.2(i) and 3.2(ii) below. Hence this series procedure has also the usual problem of non-uniform convergence of the region of asymptotic stability to the domain of attraction that occurs in autonomous systems.



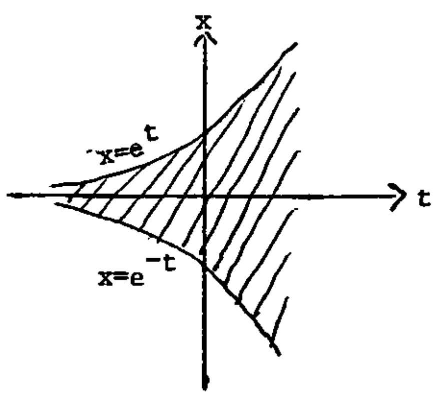


Figure 3.2(i) [even partial sums]. Figure 3.2(ii) [odd partial sums].

In order to illustrate further the occurrence of the non-uniform convergence, we consider both the even and odd partial sums of V. First we study the even partial sums by taking the second degree Liapunov function

$$V = V_2 = \frac{x^2}{2}$$

with $$\dot{V} = -x^2(1 - 2x\,e^{-t}) ,$$

$\dot{V}$ is negative definite for $-\infty < x < \frac{1}{2} e^t$.

Then the region of asymptotic stability $-\infty < x < \frac{1}{2} e^t$ lies inside the domain of attraction, $D : -\infty < x < e^t$.

Taking the partial sums as

$$V = V_2 + V_3 + V_4$$

gives $$\dot{V} = -x^2(1 - 4x^3\,e^{-3t}) .$$

The resulting RAS $-\infty < x < \frac{1}{\sqrt[3]{4}} e^t$ is bigger than $-\infty < x < \frac{1}{2} e^t$ but smaller than the domain of attraction D. As the even partial sums are increased the RAS grows bigger. The RAS boundary will converge to the actual boundary $x = e^t$ as n tends to infinity.

Next consider the odd partial sums by taking

$$V = V_2 + V_3 = \frac{x^2}{2}(1 + x\,e^{-t}) .$$

Now $$\dot{V} = -x^2(1 - 3x^2\,e^{-t})$$

and $\dot{V}$ is negative definite for $|x| < \frac{1}{\sqrt{3}} e^t$.

The RAS is therefore $|x| < \frac{1}{\sqrt{3}} e^t$ which lies inside $|x| < e^t$.

Increasing the odd partial sums to the fifth degree Liapunov function

$$V = V_2 + V_3 + V_4 + V_5 = \frac{x^2}{2}(1 + xe^{-t} + x^2e^{-2t} + x^3e^{-3t})$$

gives $\quad \dot{V} = -x^2(1 - 5x^4e^{-4t}) \quad .$

The RAS $|x| < \frac{1}{\sqrt[4]{5}} e^t$ is contained in $|x| < e^t$. As the odd partial sums are increased the RAS will converge to $|x| < e^t$. Hence from the above analysis the RAS is always contained in the domain of attraction for any partial sums.

3.3 DETERMINATION OF THE DOMAIN OF ATTRACTION OF SCALAR TIME VARYING SYSTEMS.

Consider the scalar time-varying system (3.1.3) and Zubov's equation arising from it,

$$\frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial t} = -\phi(x,t)(1 - eV) \tag{3.3.1}$$

where $\quad e = 0$ or 1 for the modified or regular form respectively.

The auxiliary equations of (3.3.1) are

$$\frac{dx}{f(x,t)} = dt = \frac{dV}{-\phi(x,t)(1-eV)} \quad . \tag{3.3.2}$$

Rearranging gives

$$\frac{dx}{dV} = -\frac{f(x,t)}{\phi(x,t)(1-eV)}$$

$$\frac{dt}{dV} = -\frac{1}{\phi(x,t)(1-eV)} \quad . \tag{3.3.4}$$

The rest of the algorithm will follow in the same manner as that discussed in Section 1.7.

In the second order autonomous system the domain of attraction is drawn with respect to the phase space, whereas the scalar time varying system the domain will be drawn with respect to the motion space. Since we are concerned, in this chapter, with determining the domain of attraction of time varying systems, the graph of x against t is appropriate. Therefore in the algorithm the initial point chosen will be (x_o, t_o) and will vary along the radial line from the origin. Hence it will be written as

$$\begin{aligned} x_o &= r_o \cos\theta \\ t_o &= r_o \sin\theta \end{aligned} \tag{3.3.6}$$

where θ is fixed and r_o is allowed to vary.

3.4 RELATIONSHIP BETWEEN SECOND ORDER TIME INVARIANT SYSTEM AND SCALAR TIME VARYING SYSTEM.

The Zubov's equations arising from the scalar time varying system and second order time invariant system are both of the same order, that is, for scalar time varying system we have a two-dimensional partial differential equation in x and t of order one while the second order time invariant system gives a partial differential equation in x and y of order one.

(3.1.2) and (3.1.4) can both be solved by the analytic method given in Sneddon [46]. The series method discussed by Hewit [5] and the series method described in Section 3.2 both have convergency problems. The V's, f's and ϕ's for the time invariant system are homogeneous in x and y whereas for the time varying system they are only homogeneous in x but not t.

For the time invariant system, the domain of attraction governed by the phase variables is considered and drawn in the phase space while in the scalar time varying system we get a domain in the motion space. In general, the Zubov equations for autonomous and time varying systems are of the same dimension only if the order of the time varying system is one less than that of autonomous system. In (3.1.2) the condition for the system to approach the stability boundary is

$$V \to \infty \quad \text{for} \quad (x,y) \in A \quad \text{and} \quad (x,y) \to (\bar{x},\bar{y}) \in \bar{A}|A$$

where A is the region of asymptotic stability, while in (3.1.4) the condition for the system to approach the boundary is

$$V \to \infty \quad \text{for} \quad (x,t) \in A \quad \text{and} \quad (x,t) \to (\bar{x},\bar{t}) \in \bar{A}|A.$$

Both systems require V and ϕ to be positive definite and decrescent. The initial condition for solving Zubov's equation in time varying system is $V(0,t) = 0$ and $V(0,0) = 0$ for autonomous system.

3.5 GRID METHOD.

Let us now compute the stability boundary of a scalar time varying system (3.1.3) by a finite difference method and compare the result with the analytical solution. Consider equation (3.1.3) with $f(0,t) = 0$. We want to solve equation (3.1.4) given positive definite ϕ and the initial condition

$$V(0,t) = 0 \tag{3.5.1}$$

The required stability condition is that V is positive definite in the neighbourhood of the origin and $V = \infty$ is the contour which will define the domain of attraction.

In order to solve (3.1.4) we set up a rectangular grid system

$$\left.\begin{aligned} x_m &= mh \;, & m &= -\infty \text{ to } \infty \\ t_n &= nk \;, & n &= -\infty \text{ to } \infty \end{aligned}\right\} . \tag{3.5.2}$$

Define

$$\left.\begin{aligned} V(mh,\ nk) &= V_m^n \\ \frac{\partial V}{\partial x}(mh,\ nk) &= \frac{V_m^n - V_{m-1}^n}{h} \\ \frac{\partial V}{\partial t}(mh,\ nk) &= \frac{V_m^n - V_m^{n-1}}{k} \end{aligned}\right\} . \tag{3.5.3}$$

V_m^n is the computed value of V at $x = mh$ and $t = nk$. Known functions f and ϕ at $x = mh$, $t = nk$ can be written as

$$\begin{aligned} f_m^n &= f(mh,\ nk) \\ \phi_m^n &= \phi(mh,\ nk) \end{aligned} \tag{3.5.4}$$

Substituting (3.5.3) and (3.5.4) into (3.1.4) gives

$$f_m^n \frac{(V_m^n - V_{m-1}^n)}{h} + \frac{(V_m^n - V_m^{n-1})}{k} = -\phi_m^n \ . \tag{3.5.5}$$

The initial condition (3.5.1) at the grid point becomes

$$V_o^n = 0 \qquad \text{for all } n \quad .$$

Rearranging (3.5.5) gives

$$V_m^n = \frac{\frac{k}{h} f_m^n V_{m-1}^n + V_m^{n-1} - k\phi_m^n}{\frac{k}{h} f_m^n + 1} \quad . \tag{3.5.6}$$

Putting $r = k/h$, (3.5.6) becomes

$$V_m^n = \frac{r f_m^n V_{m-1}^n + V_m^{n-1} - k\,\phi_m^n}{r f_m^n + 1} \tag{3.5.7}$$

Consider the example

$$\dot{x} = - x + 2x^2 e^{-t}$$

with $\qquad \phi(x,t) = x^2$.

Zubov's equation is

$$\frac{\partial V}{\partial x} (- x + 2x^2 e^{-t}) + \frac{\partial V}{\partial t} = - x^2$$

and its analytic solution is

$$V(x,t) = \frac{x^2}{2} (1 - x e^{-t})^{-1} \quad .$$

The exact stability boundary is $x = e^t$. Figure 3.5(i) shows the boundaries computed by the grid method with initial conditions $V_o^n = 0$ for all n and $V_m^o = \frac{m^2h^2}{2(1-mh)}$ chosen at V = 0.64 and V = 12 for h = k = 0.11 together with the boundaries obtained by the analytical method. The result indicates that for small V used, a fairly accurate curve is obtained when t is negative. As V increases the errors become larger. Figure 3.5(ii) shows the boundaries obtained by using $h = k = 0.11$ and $h = k = 0.05$ and the true boundaries. When h and k are small the result tends to converge nearer to the true stability boundary. This implies that the curve can be improved by reducing the values of the parameters h and k. The choice of the initial conditions also plays an important role in tracing the stability boundaries by grid methods. Only the initial condition $V(0,t) = 0$ is known but initial conditions on $V(x,0)$ are not available. There is no standard method for choosing these initial conditions. The choice that has been made for the above example is to take the lowest degree term in the series expansion of $V(x,t)$. Other initial conditions are also looked at to see the difference in the results. Figure 3.5(iii) shows the boundaries obtained by using different initial conditions such as

$$V(x,0) = \frac{x^2}{2(1-x)}\ , \ V(x,0) = \frac{x^4}{2(1-x)} \text{ and } V(x,0) = \frac{x^2}{2(1-x)^2}$$

$$\text{for } h = k = 0.11$$

which are taken at $V = 5$. The result shows some differences since there is no proper method for choosing these initial conditions.

From the results obtained (which are not extensive) we can conclude that the grid method has difficulty in producing an accurate stability boundary. The inability of the method to find the boundary points when solving Zubov's partial differential equation near the stability boundary is disappointing. This may be due to the fact that the analytic solution of V is not defined outside the domain of attraction. The information regarding the choice of initial condition will also affect the accuracy of the result.

3.6 APPLICATIONS.

i) General nonautonomous examples.

a) Consider a scalar time varying system

$$\dot{x} = - tx(1-x) \quad . \tag{3.6.1}$$

The analytical solution for given x_o, t_o is

$$x(t,x_o,t_o) = \cfrac{1}{\cfrac{1-x_o}{x_o\, e^{\frac{-(t^2-t_o^2)}{2}}} + 1} \quad . \tag{3.6.2}$$

The solution will tend to zero as $t \to \infty$ for $x_o < 1$. This implies that the origin is asymptotically stable. The solution will approach infinity as t tends to t_1 for $x_o > 1$, where t_1 is the finite escape time defined by

$$t_1^2 = t_o^2 + 2\, \ell n \frac{x_o}{x_o - 1} \quad .$$

The domain of attraction of the equilibrium is $- \infty < x < 1$ which is independent of t_o. A sketch of the trajectories of the system is shown in Figure 3.6(i).

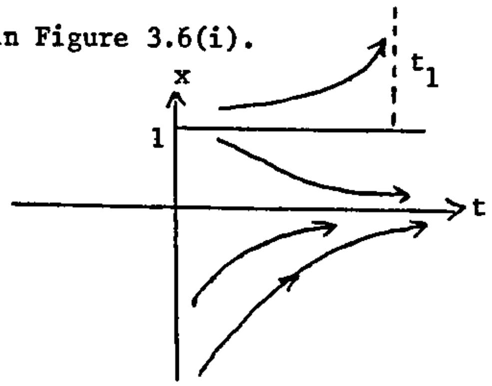


Fig. 3.6(i).

The Zubov's equation for system (3.6.1) is

$$\frac{\partial V}{\partial x}(-\,tx(1-x)) + \frac{\partial V}{\partial t} = -\,\phi(x,t) \quad .$$

Choose $\phi(x,t) = tx^2$, then the auxiliary equation becomes

$$\frac{dx}{-\,tx(1-x)} = dt = \frac{dV}{-tx^2} \quad .$$

Solving Zubov's equation analytically will give

$$V = -\,x - \ell n(1-x)$$

and the series solution is

$$V_m = \sum_{k=2}^{m} \frac{x^k}{k} \quad .$$

From the series solution we found that the odd partial sums give $|x| < 1$ as the region of convergence and even partial sums give the domain of attraction, $-\infty < x < 1$. The algorithm in Section 3.3 is applied to system (3.6.1) for different values of R and the result is shown in Figure 3.6(iii). Even for larger values of R we still have the region for which x is less than one.

b) Next we study a system which has the exact domain of attraction for every partial sum of the series solution.

The system is

$$\dot{x} = -\,tx(1-x^2) \quad . \tag{3.6.3}$$

The solution is

$$x(t,x_o,t_o) = \frac{1}{1 + \frac{1-x_o^2}{x_o^2}\exp(t^2-t_o^2)} \tag{3.6.4}$$

for arbitrary initial condition x_o, t_o with

$$\lim_{t\to\infty} x(t,x_o,t_o) = 0 \quad \text{if} \quad |x_o| < 1$$

and

$$\lim_{t\to t_1} x(t,x_o,t_o) = \infty \quad \text{if} \quad |x_o| > 1$$

where t_1 is the finite escape time given by

$$t_1^2 = t_o^2 + \frac{\ln x_o^2}{x_o^2-1} \quad .$$

The domain of attraction is $|x| < 1$ which is independent of t_o. The trajectories are represented in Figure 3.6(ii).

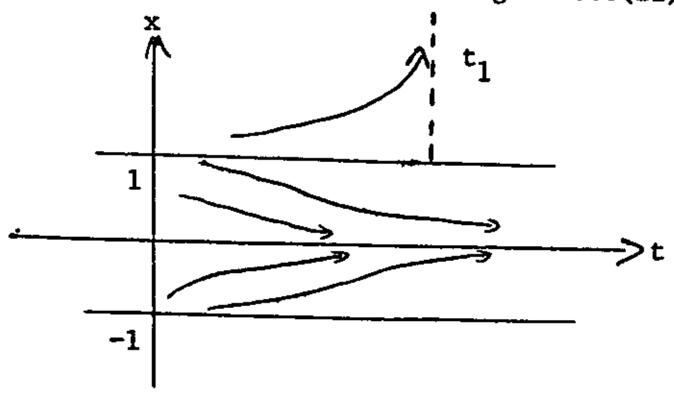


Fig. 3.6(ii).

Using $\phi(x,t) = 2tx^2$ and solving Zubov's equation gives the solution

$$V = -\ln(1-x^2) \quad .$$

The solution converges for $|x| < 1$. The series solution is

$$V_{2m} = \sum_{k=1}^{m} \frac{x^{2k}}{k}$$

and

$$\dot{V}_{2m} = - 2t(x^2 + \dots + x^{2m})(1-x^2) \quad .$$

We can show that every partial sum gives the exact domain of attraction, $-1 < x < 1$.

Figure 3.6(iv) shows the region of attraction of the equilibrium obtained by using R = 1.2 and 1.7. The regions drawn on these particular values of R are still included in the domain of attraction and are in the range $-1 < x < 1$.

c) Let us find the domain of attraction of the system

$$\dot{x} = - tx(1+x)(2-x) \quad . \tag{3.6.5}$$

The critical points for this system are 2, -1, 0. Zubov's equation is

$$\frac{\partial V}{\partial x}\Big[-tx(1+x)(2-x)\Big] + \frac{\partial V}{\partial t} = - \phi(x,t) \quad .$$

Choose $\phi(x,t) = 3tx^2$, then Zubov's solution becomes

$$V = - \ell n(1+x)(2-x)^2 \ .$$

$V = \infty$ when $x = -1$ or 2 and the domain of attraction is thus $-1 < x < 2$. Figure 3.6(v) shows the domain found by the numerical method described in Section 3.3. The region of convergence lies in the region $-1 < x < 2$. We can see that the result agrees with the analytical result.

ii) <u>Grujic's Examples [42]</u>

a) Grujic studies the equilibrium state $x = 0$ of the first order system

$$\dot{x} = -x + x^3(1 + \exp(t)) \tag{3.6.6}$$

and gives

$x^2 \leq \frac{1}{(1+\exp(t))}$ as an estimate

of the domain of attraction. He uses a Liapunov function $V = x^2$ and from the derivative he establishes the domain. The analytic solution is

$$x^2 = \frac{1}{1 + 2\exp(t) + \exp(2(t-t_o))\left[\frac{1}{x_o^2} - 1 - 2\exp(t_o)\right]}$$

for arbitrary x_o, t_o.

From the linear part of equation (3.6.6) it is clear that the system is asymptotically stable. This is true since the solution approaches zero as t tends to infinity.

Solving Zubov's equation by the series solution technique for $\phi = x^2$ gives

$$V_2 = \frac{x^2}{2}$$

$$V_4 = \frac{x^4}{4} + \frac{1}{3} e^t x^4$$

$$V_6 = \frac{x^6}{6} + \frac{7}{15} e^t x^6 + \frac{1}{3} e^{2t} x^6$$

$$V_8 = \frac{x^8}{8} + \frac{19}{35} e^t x^8 + \frac{4}{5} e^{2t} x^8 + \frac{2}{5} e^{3t} x^8$$

and so on, with the odd terms of V equal to zero.

Then

$$V = \frac{x^2}{2} + \left[\frac{1}{4} + \frac{1}{3}\,e^t\right]x^4 + \left[\frac{1}{6} + \frac{7}{15}\,e^t + \frac{1}{3}\,e^{2t}\right]x^6$$

$$+ \left[\frac{1}{8} + \frac{19}{35}\,e^t + \frac{4}{5}\,e^{2t} + \frac{2}{5}\,e^{3t}\right]x^8 + \ldots \quad .$$

In the series, the terms in t make it difficult to form a closed form solution. Perhaps, an appropriate positive definite ϕ will give a simple positive definite and decrescent V in closed form. Figure 3.6(vi) shows the stability regions of the equilibrium $x = 0$ obtained by the numerical algorithm. These are larger than that given by Grujic. From the tests performed, it is clearly seen that global asymptotic stability is indicated.

b) Consider a scalar time varying equation

$$\dot{x} = -\,x(1+2e^t) + x^3(1+e^t). \tag{3.6.7}$$

The equilibrium state $x = 0$ of equation (3.6.7) is asymptotically stable. Grujic shows that the domain of attraction of this system obtained by using the Liapunov function, $V = \frac{x^2}{2}$ is

$$x^2 < \frac{1 + 2e^t}{1 + e^t} \quad .$$

The regions of asymptotic stability obtained as a result of our study are shown in Figure 3.6(vii) and are found to be better than that of Grujic. Again the results indicate asymptotic stability in the large.

iii) Yoshizawa's Problems [47].

a) The time varying system

$$\dot{x} = -x + 2x^2 e^{-t} \tag{3.6.8}$$

possesses a convergent power series expansion about the origin on the right hand side and the equilibrium of the linear part of the system is asymptotically stable. So, according to Lehnigk [48] the nonlinear time varying system (3.6.8) is asymptotically stable. The solution is

$$x(t,x_o,t_o) = \frac{1}{e^{-t} + \left(\frac{1}{x_o e^{t_o}} - e^{-2t_o}\right) e^t}$$

for given x_o, t_o with

$$\lim_{t\to\infty} x(t,x_o,t_o) = \begin{cases} 0 & \text{if} \quad x_o < e^{t_o} \\ \infty & \text{if} \quad x_o = e^{t_o} \end{cases}$$

and

$$\lim_{t\to t_1} x(t,x_o,t_o) = \infty \qquad \text{if} \quad x_o > e^{t_o}$$

, where t_1 is the finite escape time defined by

$$t_1 = \frac{1}{2} \ln \frac{x_o e^{t_o}}{x_o e^{-t_o} - 1} \quad .$$

The solution will be near the origin for a sufficiently long period of time and will approach zero as $t \to \infty$. This implies that the system is eventually asymptotically stable and since the origin is the equilibrium state then from the theorem given in Appendix B the system is uniformly asymptotically stable. The domain of asymptotic stability is given by $-\infty < x_o < e^{t_o}$ which is dependent on t_o. Here we have a domain which depends on the initial data.

If we use $\phi(x,t) = x^2$ and solve Zubov's equation

$$\frac{\partial V}{\partial x}\left[-x + 2x^2 e^{-t}\right] + \frac{\partial V}{\partial t} = -x^2$$

by series solution we will get the series

$$V = \frac{x^2}{2}(1 + e^{-t}x + e^{-2t}x^2 + \dots)$$

$$= \frac{x^2}{2}\sum_{n=0}^{\infty}(e^{-t}x)^n$$

$$= \frac{x^2}{2}(1 - xe^{-t})^{-1}$$

$V = \infty$ when $x = e^t$ and the domain of attraction of the equilibrium is $-\infty < x < e^t$.

It is shown in Section 3.2 that the even partial sums of the series give the domain of attraction, $-\infty < x < e^t$ and the odd partial sums give the region of convergence $|x| < e^t$. The Zubov's equation can also be solved analytically by writing the auxiliary equation

$$\frac{dx}{-x(1-2x\,e^{-t})} = dt = \frac{dV}{-x^2} \quad .$$

Solving the first equation gives

$$x(t) = \frac{1}{e^{-t} + ce^{t}} \quad , \quad \text{where c is an arbitrary constant .}$$

Substituting x in $dt = \frac{dV}{-x^2}$ and eliminating the constant c, we will get the solution

$$V = \frac{x^2}{2(1-x\,e^{-t})}$$

and $x = e^{t}$ is the stability boundary.

The domain of attraction for various circles used is shown in Figure 3.6(viii) together with the true stability boundary. The algorithm computes the stability boundary which is almost identical to the true boundary within our test regions.

b) Next we study a system which has a stability region bounded by $|x| < e^{t}$.

The time varying system

$$\dot{x} = -x + 2x^3\, e^{-2t} \tag{3.6.9}$$

has a solution of the form

$$x^2(t,x_o,t_o) = \frac{1}{e^{-2t} + e^{2t}\left[\frac{1}{x_o^2\, e^{2t_o}} - e^{-4t_o}\right]}$$

for arbitrary x_o, t_o.

We also have

$$\lim_{t\to\infty} x(t,x_o,t_o) = \begin{cases} 0 & \text{if} \quad |x_o| < e^{t_o} \\ \infty & \text{if} \quad |x_o| = e^{t_o} \end{cases}$$

and

$$\lim_{t \to t_1} x(t,x_o,t_o) = \infty \qquad \text{if } |x_o| > e^{t_o}$$

with finite escape time

$$t_1 = \frac{1}{4} \ln \frac{x_o^2 e^{2t_o}}{x_o^2 e^{-2t_o} - 1} .$$

The domain of attraction of the equilibrium is $-e^{t_o} < x_o < e^{t_o}$ which is dependent on t_o.

The series solution obtained by using $\phi = x^4$ is

$$V = \frac{x^4}{4} (1 + e^{-2t} x^2 + e^{-4t} x^4 + \dots)$$

$$= \frac{x^4}{4(1-x^2 e^{-2t})}$$

$V = \infty$ gives $|x| = e^t$ as the stability boundary.

Figure 3.6(ix) shows the stability region obtained by the numerical method for circles of radii 0.7 and 0.9 together with the true stability boundary. The boundary plotted converges to the true stability boundary,

3.7 CONCLUSION.

An algorithm for the determination of the domain of attraction for a scalar time varying system is described and applied to many known problems of this nature. The series solution technique is presented and we encounter the similar problem of non-uniform convergence that has already occurred in autonomous system. A comparison between second order time invariant systems and scalar time varying systems is given. The condition of $\phi(x,t)$ in time varying system is being relaxed in some of our examples where a decrescent $\phi(x,t)$ is not being used. This requirement can also be found in [49], one of Zubov's papers. Examples of systems in which the domain of attraction depends on t_o or does not depend on t_o are also studied. The finite difference method used to solve the Zubov's partial differential equation arising from the scalar time varying system, does not produce an exact domain of attraction. The problem of choosing the right initial condition arises from this difference method.

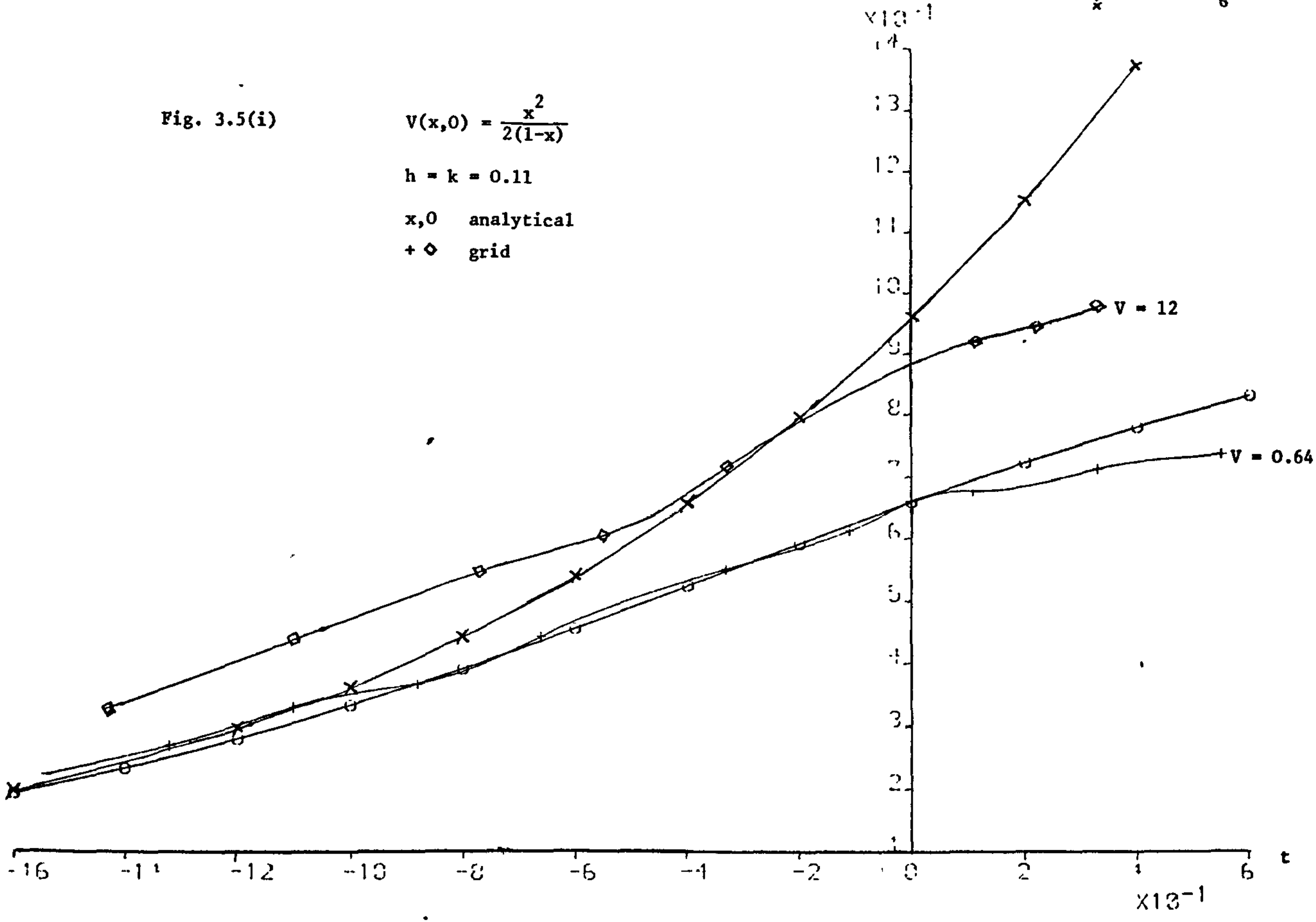


Fig. 3.5(i)

$$V(x,0) = \frac{x^2}{2(1-x)}$$

h = k = 0.11

x,0 analytical

+ ◇ grid

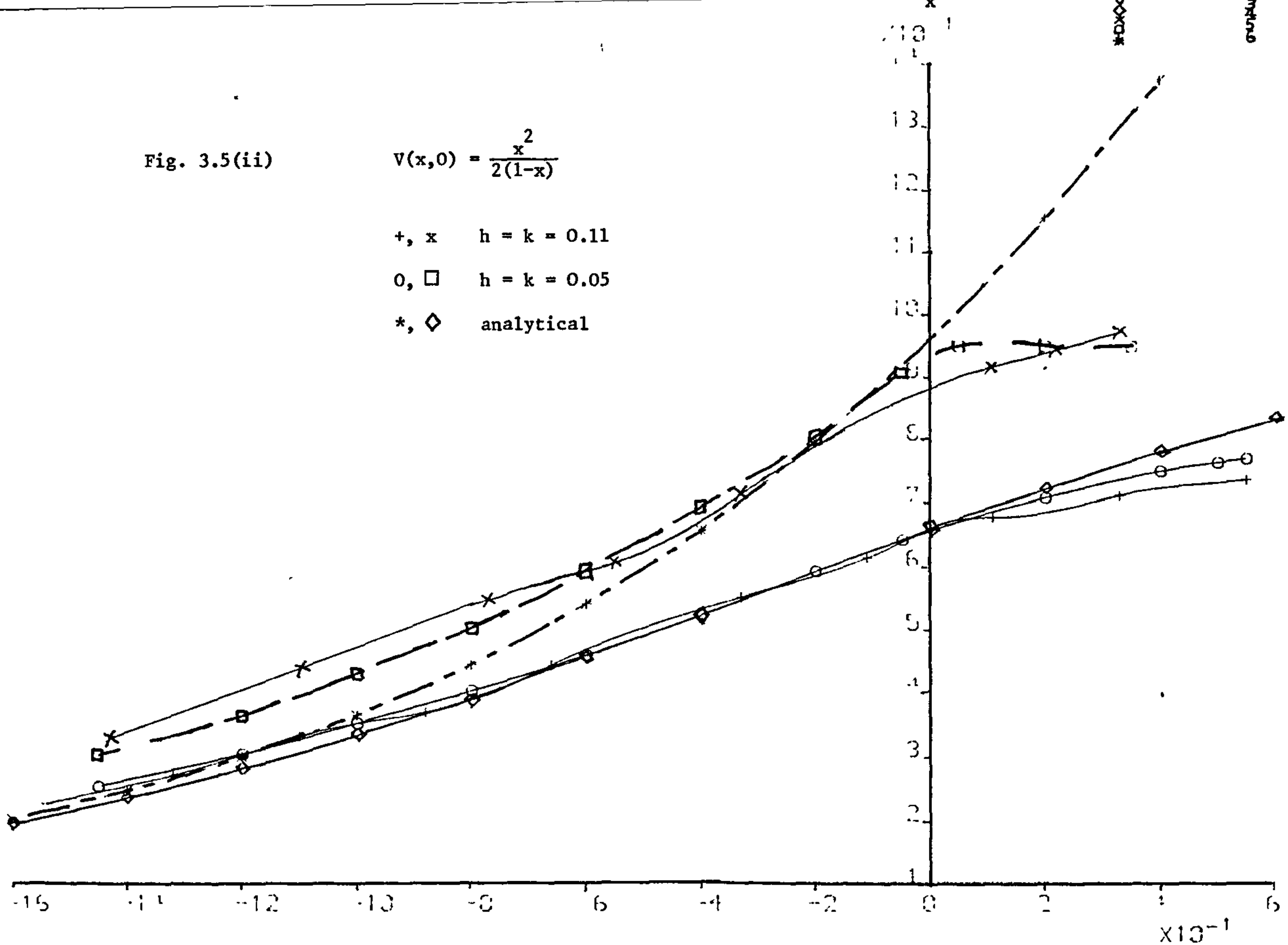


Fig. 3.5(ii) $V(x,0) = \frac{x^2}{2(1-x)}$

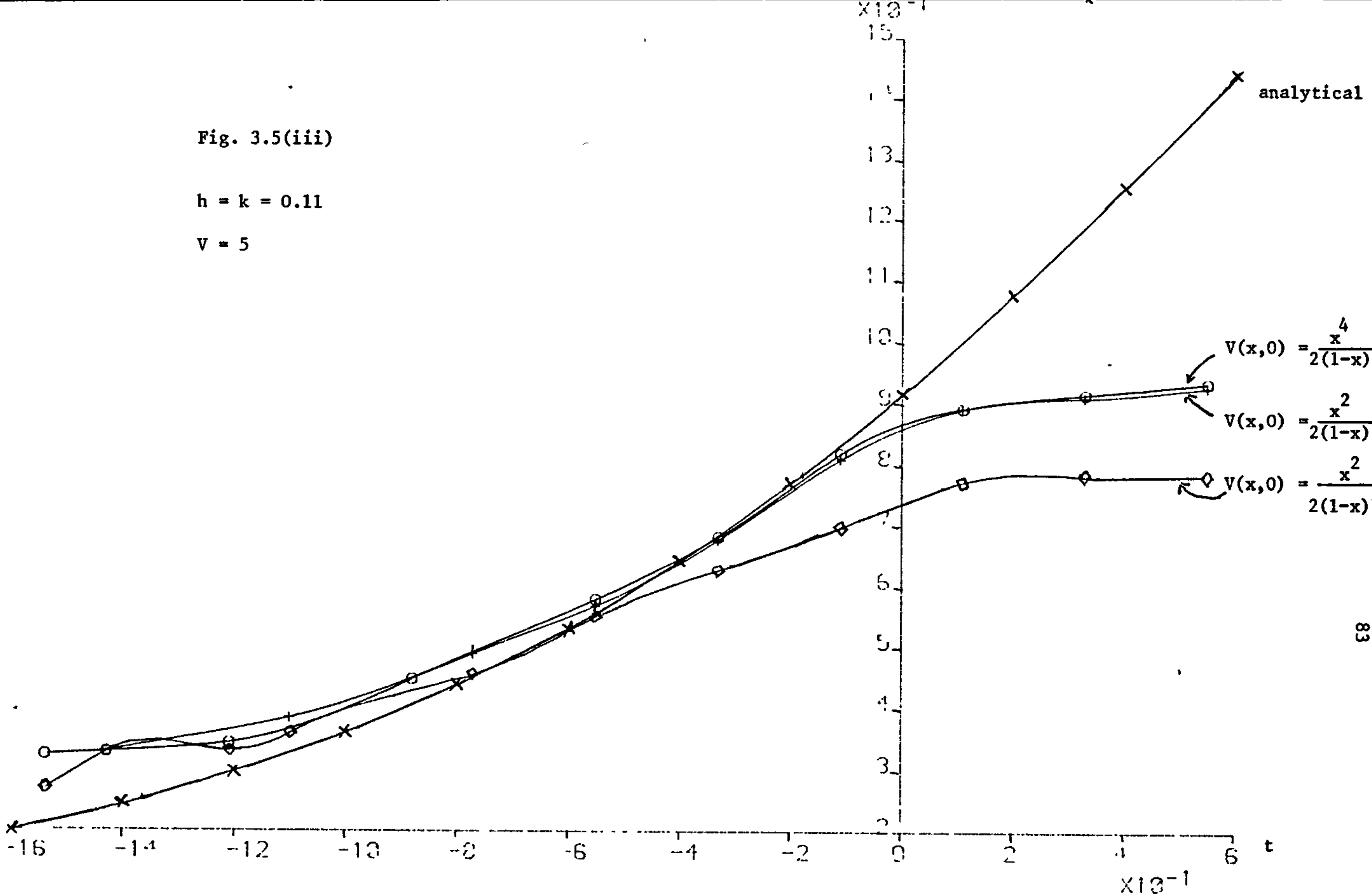
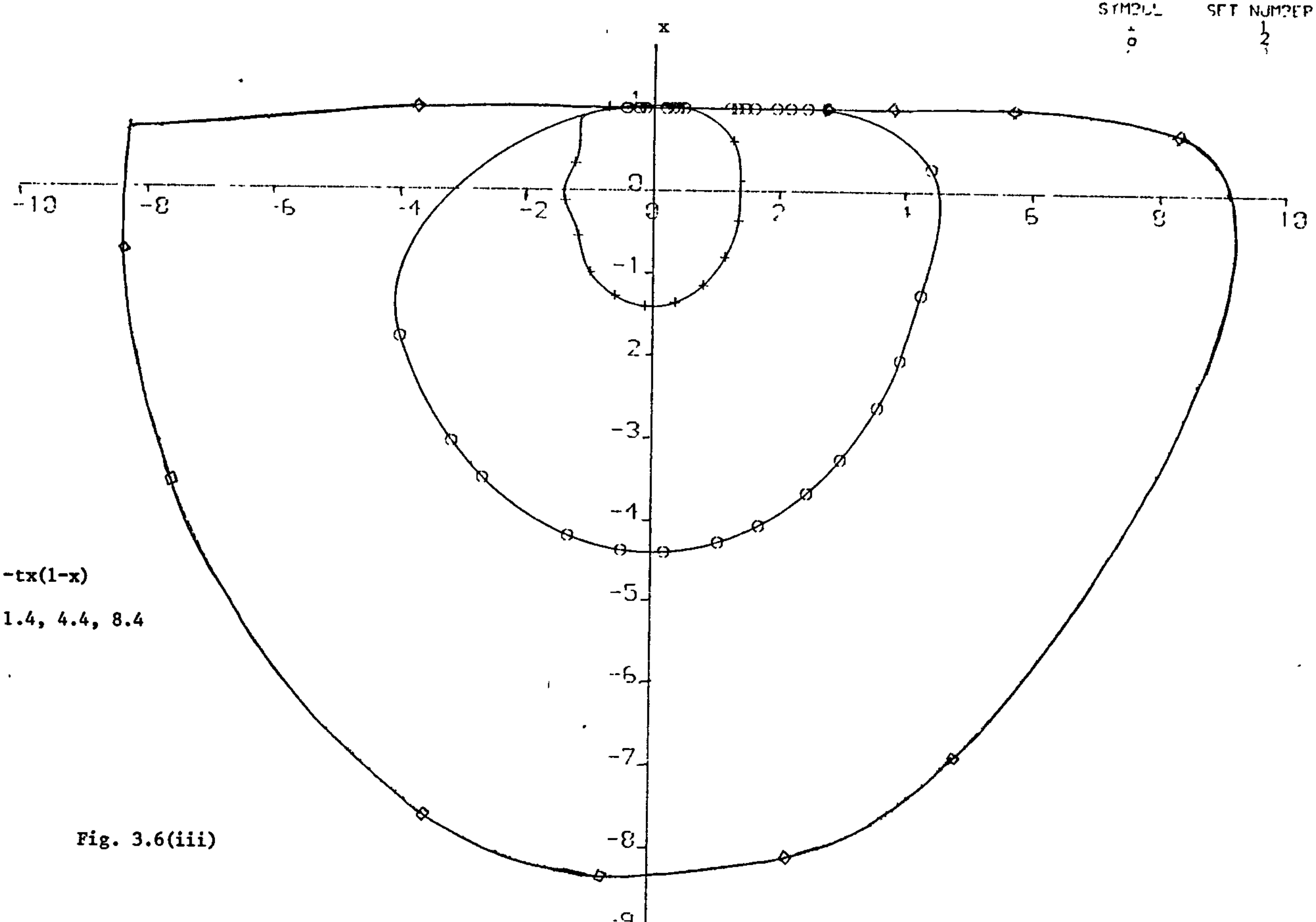


Fig. 3.5(iii)

$h = k = 0.11$

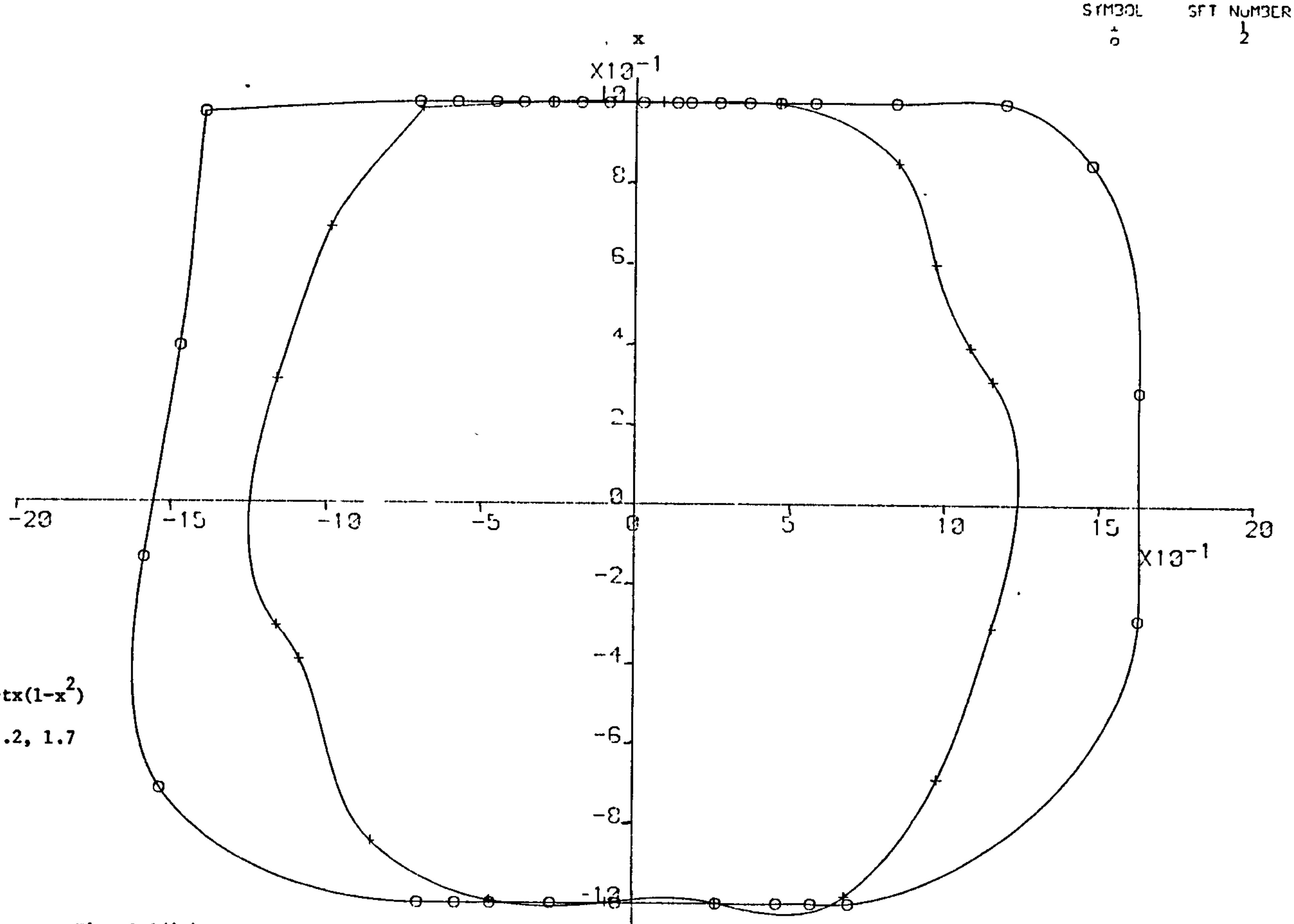
$V = 5$



$\dot{x} = -tx(1-x)$

R = 1.4, 4.4, 8.4

Fig. 3.6(iii)



$\dot{x} = -tx(1-x^2)$

$R = 1.2, 1.7$

Fig. 3.6(iv)

$\dot{x} = -tx(1+x)(2-x)$

$R = 1.2,\ 3.2$

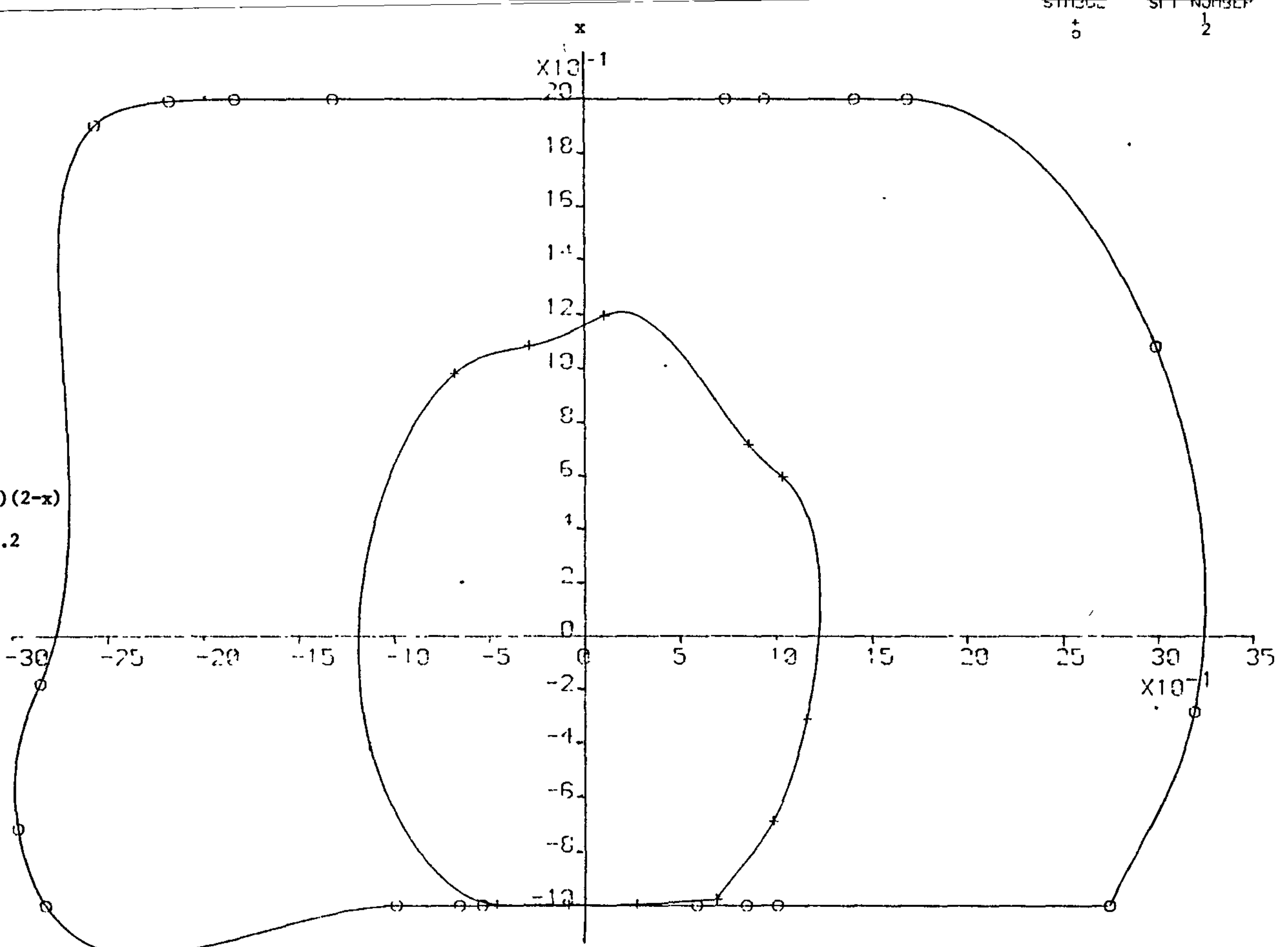


Fig. 3.6(v)

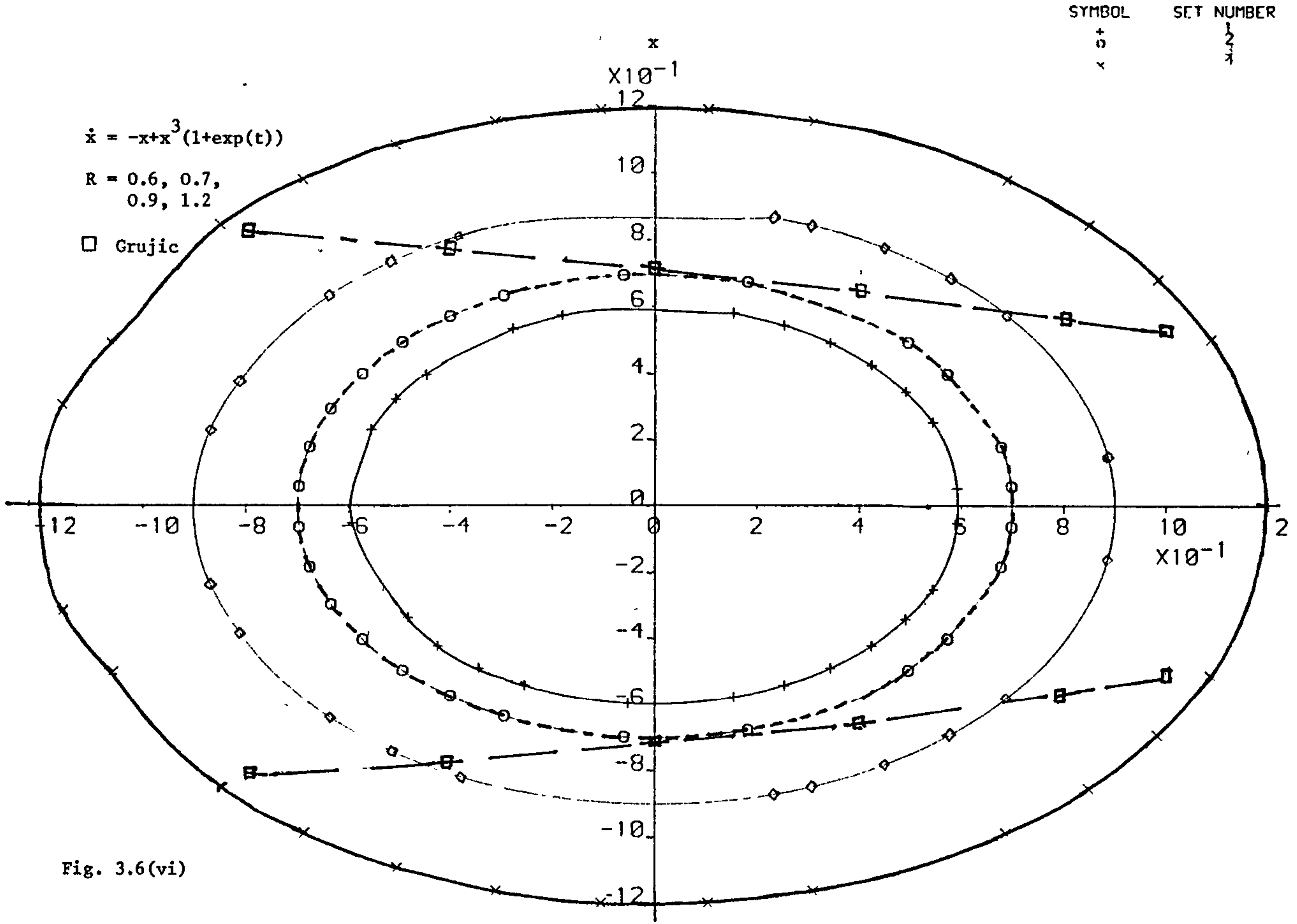
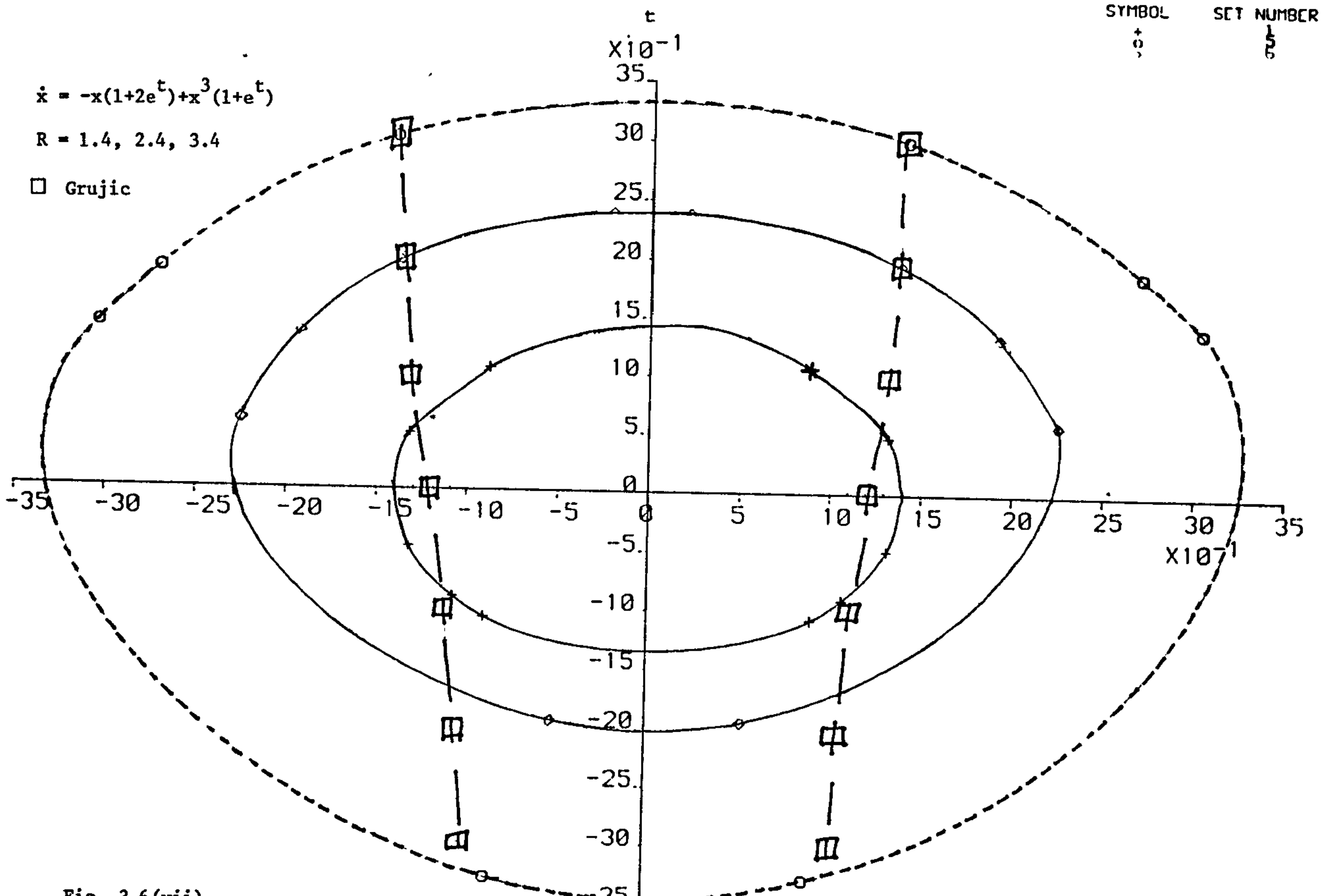

SYMBOL SET NUMBER
+ 1
o 2
x 3
x
X10^-1
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
t
-12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12
X10^-1
ẋ = -x+x^3(1+exp(t))
R = 0.6, 0.7,
0.9, 1.2
□ Grujic

Fig. 3.6(vi)



$\dot{x} = -x(1+2e^{t})+x^{3}(1+e^{t})$
R = 1.4, 2.4, 3.4
□ Grujic
SYMBOL
SET NUMBER
1
5
6
t
X10^-1
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
x
-35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35
X10^-1

Fig. 3.6(vii)

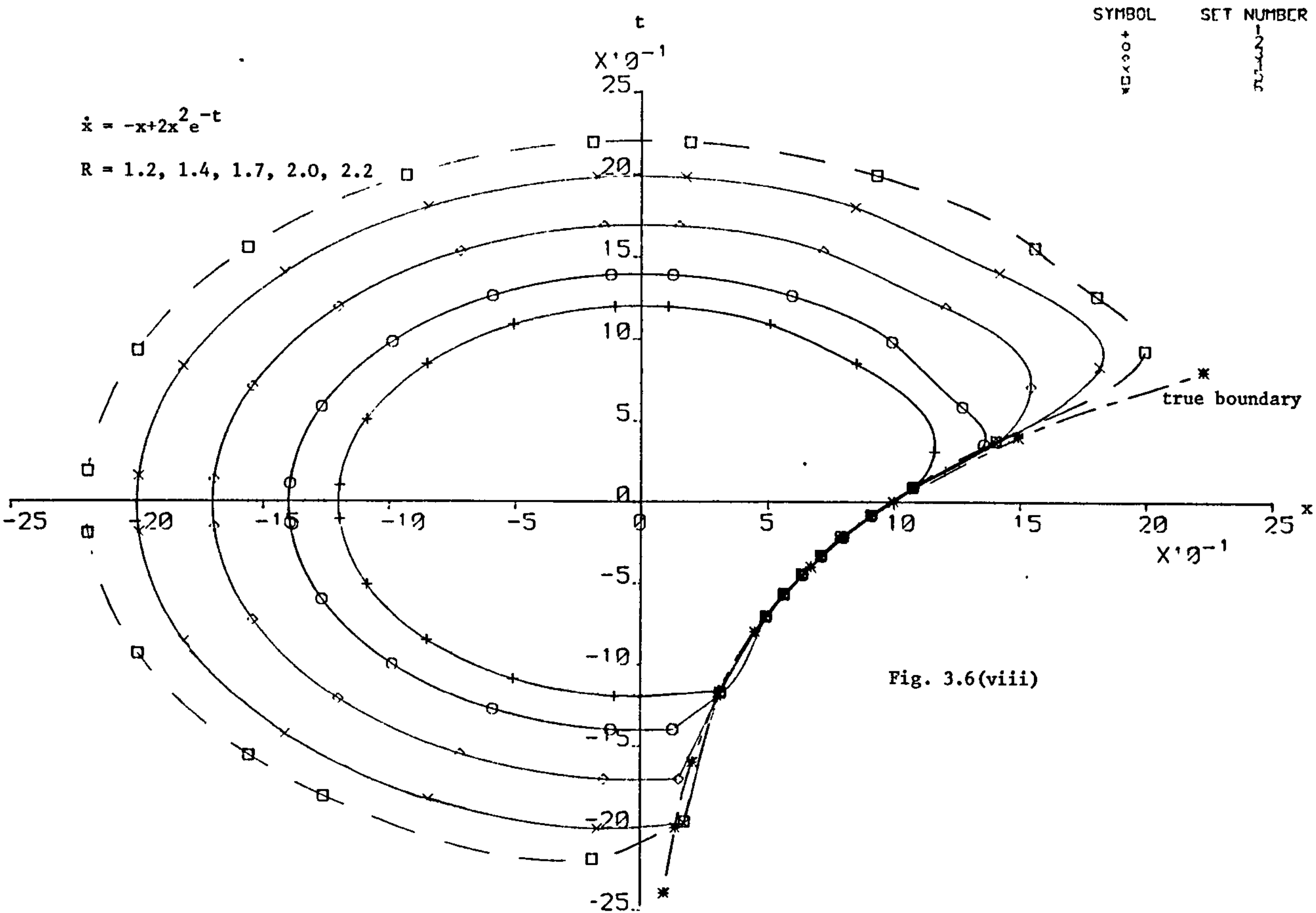
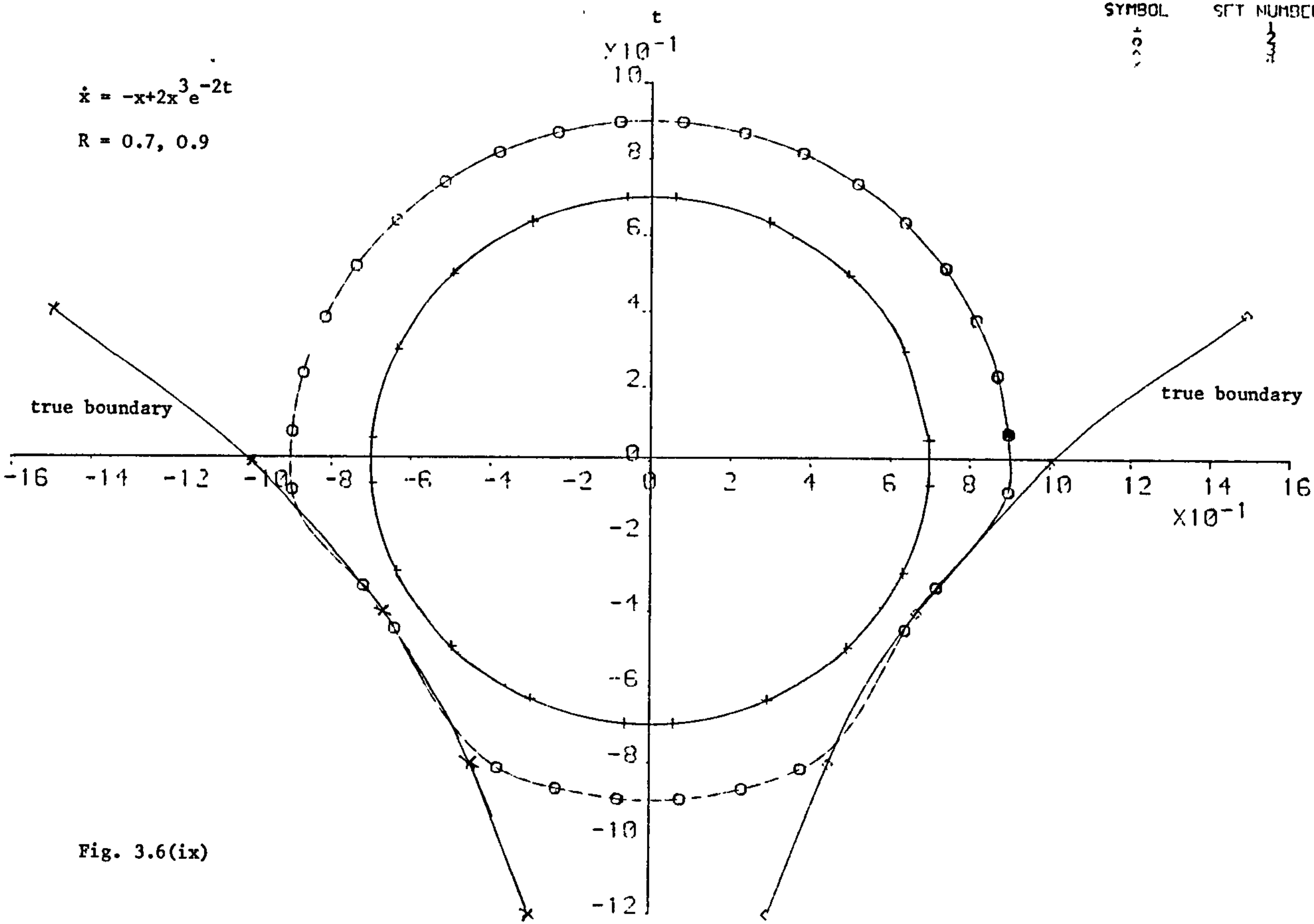

t
X'0^-1
SYMBOL
SET NUMBER
ẋ = -x+2x²e^-t
R = 1.2, 1.4, 1.7, 2.0, 2.2
true boundary
x
X'0^-1
-25
-20
-15
-10
-5
0
5
10
15
20
25

Fig. 3.6(viii)



SYMBOL
SET NUMBER
1
2
3
$\dot{x} = -x+2x^3e^{-2t}$
R = 0.7, 0.9
t
X10^-1
true boundary
true boundary
x
X10^-1
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
14
16
10
8
6
4
2
-2
-4
-6
-8
-10
-12

Fig. 3.6(ix)

CHAPTER IV

EXTENSION OF ZUBOV'S METHOD TO THIRD ORDER AUTONOMOUS SYSTEMS

4.1 INTRODUCTION.

In this chapter we will extend the numerical algorithm for determining the domain of attraction of second order nonlinear autonomous systems to systems of order three. The domain for third order autonomous systems will be spherical, ellipsoidal, or any regular or irregular solid form. We shall define our test region by the set, S_R given by

$$S_R = \left\{ \underline{x} : x_1^2 + x_2^2 + x_3^2 < R^2 \right\} \qquad (4.1.1)$$

where R is defined in (1.7.1).

In third order systems $\underline{\theta}$ is a fixed 2-dimensional vector given by

$$\underline{\theta}^T = (\theta_1, \theta_2) \qquad (4.1.2)$$

and the spherical polar coordinates are written as

$$x_1 = R \cos \theta_1, \quad x_2 = R \sin \theta_1 \cos \theta_2 \quad \text{and} \quad x_3 = R \sin \theta_1 \sin \theta_2.$$

By fixing one of the θ's and varying the other we will get a θ which can be applied in the algorithm for second order systems. The result will be presented in the form of cross sections of the stability surface for the three principal planes. However to vary $\underline{\theta}$ so that all the points of S_R have been covered and to compute the domain in solid form is quite complicated. An alternative is to obtain the result in the form of slices and build a solid shape from these slices by using the graphics package which is available in the Computer Centre of Loughborough University of Technology. Several examples are included and compared to show the improvement of the domains.

4.2 EXTENSION OF ALGORITHM TO THIRD ORDER SYSTEMS.

Consider a third order nonlinear system written in the form of three first order equations

$$\dot{x} = f(x,y,z)$$

$$\dot{y} = g(x,y,z) \tag{4.2.1}$$

$$\dot{z} = q(x,y,z)$$

with the origin assumed to be an equilibrium.

The Zubov equation is

$$\frac{\partial V}{\partial x} f(x,y,z) + \frac{\partial V}{\partial y} g(x,y,z) + \frac{\partial V}{\partial z} q(x,y,z) = - \phi(x,y,z)(1-eV) \tag{4.2.2}$$

where e = 0 or 1 for the modified or regular form respectively.

Transforming (4.2.2) to three ordinary differential equations will give

$$\frac{dx}{dV} = - \frac{f(x,y,z)}{\phi(x,y,z)(1-eV)}$$

$$\frac{dy}{dV} = - \frac{g(x,y,z)}{\phi(x,y,z)(1-eV)} \tag{4.2.3}$$

$$\frac{dz}{dV} = - \frac{q(x,y,z)}{\phi(x,z,y)(1-eV)} \quad .$$

Next we consider the variation of the initial point (x_o,y_o,z_o) along the radial line given by $(r_o, \underline{\theta})$ where $\underline{\theta}$ is a fixed two dimensional vector given by (4.1.2) and $(r_o,\underline{\theta})$ are given by the relations

$$x_o = r_o \cos \theta_1$$

$$y_o = r_o \sin \theta_1 \cos \theta_2 \qquad (4.2.4)$$

$$z_o = r_o \sin \theta_1 \sin \theta_2 \quad .$$

The function $W(r_o,\underline{\theta},h,p)$ in (1.7.14) can still be computed for fixed $\underline{\theta}$, h and p and by varying r_o the function $\hat{r}_o(\underline{\theta},h,p)$ defined by (1.7.15) can be obtained. The computation of $r^*(\underline{\theta},p)$ for given $\underline{\theta}$, p is identical to (1.7.16) but the procedure for varying scalar θ cannot be applied directly to vector $\underline{\theta}$. However by keeping either θ_1 or θ_2 fixed we can vary $\underline{\theta}$ and compute $r^*(\underline{\theta},p)$ and $J(\underline{\theta})$ for given $\underline{\theta}$. By doing so we are able to trace cross-sections of the stability surface for third order systems. For example, by fixing $\theta_2 = 0$ and varying θ_1 we can compute the stability boundaries in the principal plane, z = 0.

4.3 SYNCHRONOUS MACHINE WITH A VELOCITY GOVERNOR [30]

In Chapter two we have seen the application of White's method to power systems of order two. Various machines which take into account variable damping and saliency, constant damping without saliency and constant damping with saliency are studied and compared. Let us apply the algorithm in Section 4.2 to a third order system of a machine with constant damping and a velocity governor. The state equation for the system which includes the effect of a velocity governor and excludes the effect of flux decay can be taken directly from (2.3.6) as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - ax_2 - x_3 - b\left\{\sin(x_1+\delta_o) - \sin \delta_o\right\} \qquad (4.3.1)$$

$$\dot{x}_3 = - \gamma_1 x_3 + \gamma_2 x_2$$

where $a = 0.3$, $b = 1$, $\delta_o = 0.412$, $\gamma_1 = 0.1$ and $\gamma_2 = 0.002$. (4.3.1) has critical points at (0,0,0) and (2.318,0,0) with the origin a stable equilibrium point and the second singular point unstable. $\phi = x_1^2 + x_2^2 + x_3^2$ is used to compute the boundaries. Cross-sections of the stability surface for various principal planes are compared with the region obtained by Prabhakara et al. [30] as shown in Figure 4.3(i) - 4.3(iii). It is found that the application of the numerical method results in considerable improvement in the stability boundary estimates over the generalised Zubov method used by Prabhakara et al. The generalised Zubov method depends on the right choice of transformation of variables to solve the partial differential equation and defines the stability region as $V(x)$ = constant, where the constant is obtained from the relation constant = min.$(V:\dot{V} = 0)$. Also the general information for selecting suitable forms of transformation is not available and this will limit the use of the generalised Zubov method.

4.4 ZUBOV'S EXAMPLE.

Zubov shows that the system

$$\begin{aligned} \dot{x} &= - x + y + x(x^2+y^2) \\ \dot{y} &= - x - y + y(x^2+y^2) \end{aligned} \qquad (4.4.1)$$

has a limit cycle $x^2 + y^2 = 1$.

By introducing a third variable we generalize to a third order system

$$\dot{x} = -x + y + z + x(x^2 + y^2 + z^2)$$

$$\dot{y} = -x - y + z + y(x^2 + y^2 + z^2) \qquad (4.4.2)$$

$$\dot{z} = -x - y - z + z(x^2 + y^2 + z^2)$$

which has a limit surface

$$x^2 + y^2 + z^2 = 1 \quad .$$

This can be obtained analytically by solving Zubov's equation

$$\frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = -\phi$$

with $\qquad \phi = 2(x^2 + y^2 + z^2) \; .$

The V-function is found to be

$$V = -\ln(1 - x^2 - y^2 - z^2) \quad .$$

As $V \to \infty$, $x^2 + y^2 + z^2 \to 1$ and as $V \to 0$, $x^2 + y^2 + z^2$ will tend to zero. The limit surface is unstable since trajectories starting inside the unit sphere will tend to the origin as $t \to \infty$ and trajectories starting outside the unit sphere will move away from it. Next we apply the algorithm to this system. The domain for the various principal planes is found to be a unit circle as shown in Figure 4.4(i). Since the domain of the system is a unit sphere, the cross-sections of the stability boundaries in all the principal planes will be unit circles. Thus the algorithm gives the exact results in this example.

4.5 RESULT IN THE FORM OF SLICES.

It is useful to be able to find cross-sections of the stability boundaries at various lengths along the diameter of the solid sphere or cross-sections of the truncated sphere. For example, we wish to compute the stability boundaries at $y = 0.5$ or $y = k$ generally. In order to determine such cross-sections we have to introduce some transformation. A simple linear transformation in y and by fixing $\theta_2 = \frac{\pi}{2}$ will lead to the computation of the stability boundaries in the $x - z$ plane for various lengths along the y-axis. Similarly the boundaries on $x - y$ plane and $y - z$ plane for different heights on z-axis and x-axis respectively can be determined by transforming z or x and keeping one of the θ's fixed. Figure 4.5(i) shows the cross-sections of the stability boundaries of the slices of system (4.4.2) at levels $y = 0.5$, $y = 0.8$ and $y = 0$ in the $x - z$ plane. Other planes also give the same concentric circles which represent cross-sections of the stability boundaries of the truncated spheres. From the points obtained at all the three planes on different lengths of the axis, we can build a solid shape by utilizing the graphics package available from the Loughborough University of Technology Computer Centre. The result is a hemisphere shown in Figure 4.5(ii) viewed at four right angles.

Figure 4.5(iii) represents the cross-sections of the stability boundaries of system (4.3.1) at levels $x_3 = 1.2$, $x_3 = 0.5$ and $x_3 = 0$ in the $x_1 - x_2$ plane.

4.6 INGWERSON'S EXAMPLE.

A nonlinear third order system taken from Ingwerson [14] is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 \qquad (4.6.1)$$

$$\dot{x}_3 = -(x_1 + cx_2)^3 - bx_3$$

where $\quad b > 0, \quad c > 0 \quad \text{and} \quad bc - 1 > 0$. (4.6.2)

Ingwerson obtains the Liapunov function

$$V = \frac{bx_1^4}{4} + \frac{1}{4c}(x_1+cx_2)^4 - \frac{1}{4c}x_1^4 + \frac{b^2}{2}x_2^2 + bx_2x_3 + \frac{1}{2}x_3^2$$

$$= \frac{(bc-1)}{c}\frac{x_1^4}{4} + \frac{(x_1+cx_2)^4}{4c} + \frac{1}{2}(bx_2+x_3)^2$$

which is positive definite under condition (4.6.2).

$$\dot{V} = -(bc-1)(3x_1^2 + 3cx_1x_2 + c^2x_2^2)x_2^2$$

$$= -(bc-1)\left[(\frac{3}{2}x_1 + cx_2)^2 + \frac{3}{4}x_1^2\right]x_2^2$$

is negative semidefinite under the same condition as the positive definiteness of V. System (4.6.1) is therefore stable.* If b = 1.5 and c = 1, then

$$V = \frac{1}{8}x_1^4 + \frac{1}{4}(x_1+x_2)^4 + \frac{1}{2}(1.5x_2+x_3)^2$$

is positive definite and

* When 4.6.2 holds this system is in fact a.s.l.
(See D. G. Schultz, "Advances in Control Systems", 2, 1965, p. 32)

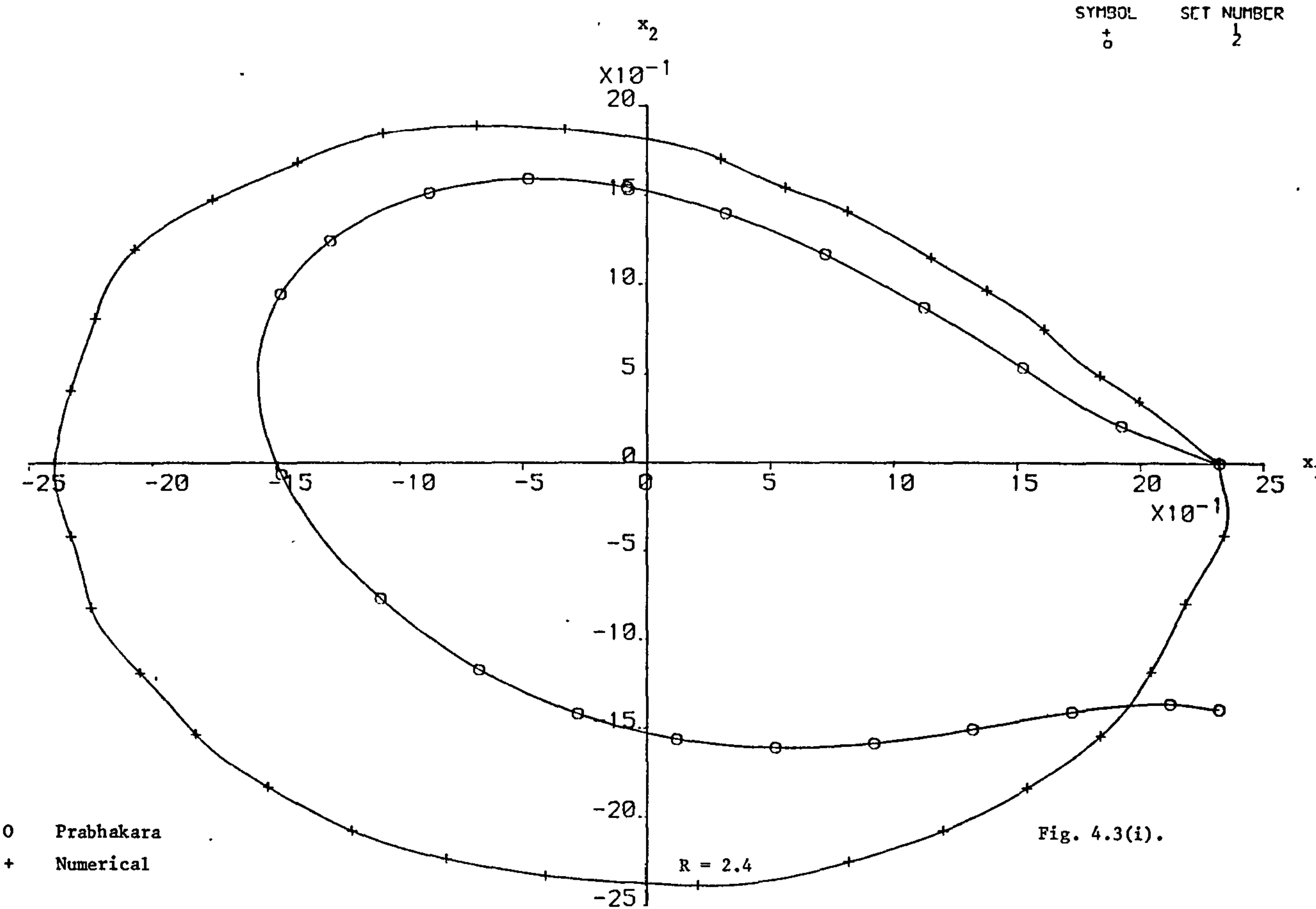
$$\dot{V} = -\frac{1}{2}\left[(1.5x_1+x_2)^2 + 0.75x_1^2\right]x_2^2$$

is negative semidefinite. Figures 4.6(i)-(iii) show the cross-section of the stability domain of (4.6.1) in the three principal planes by taking $\phi = x_1^2 + x_2^2 + x_3^2$, $b = 1.5$ and $c = 1$. Several tests have been made and larger circular regions are obtained in all three planes when the R's are increased.

So we conclude that this system is asymptotically stable in the large.

4.7 CONCLUSION.

The domain of attraction for third order nonlinear autonomous system has been investigated and the results are presented in the form of cross-sections in the three principal planes. The cross-sections of the stability boundaries at different intervals of the axis have been obtained. The main difficulty for computing the domains of third order system is to vary $\underline{\theta}$ so that all the points of S_R are covered in all directions and this will involve large computation time.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x_2
X10^{-1}
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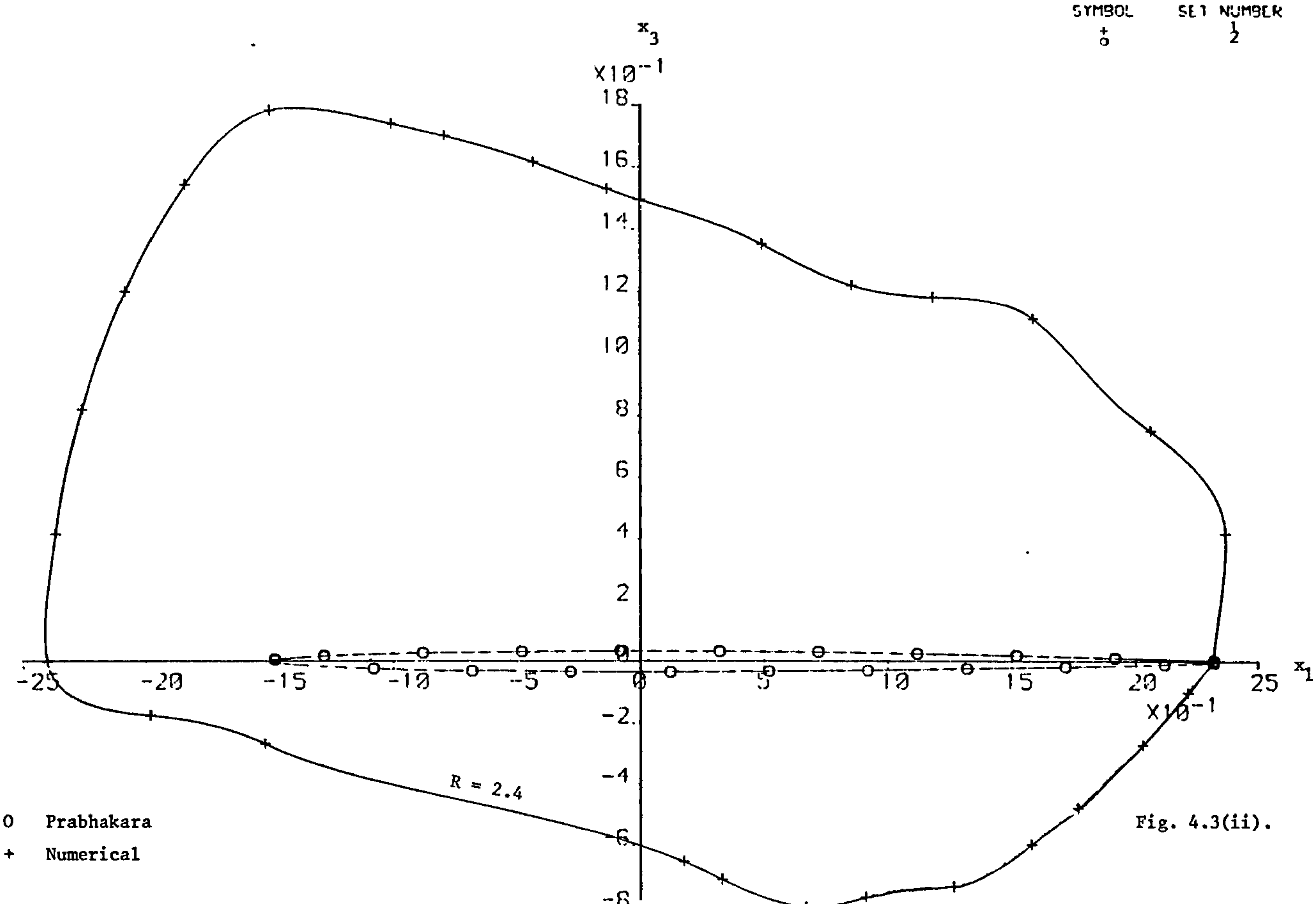
5
0
-5
-10
-15
-20
-25
-25 -20 -15 -10 -5 0 5 10 15 20 25
X10^{-1}
x_1
O Prabhakara
+ Numerical
R = 2.4

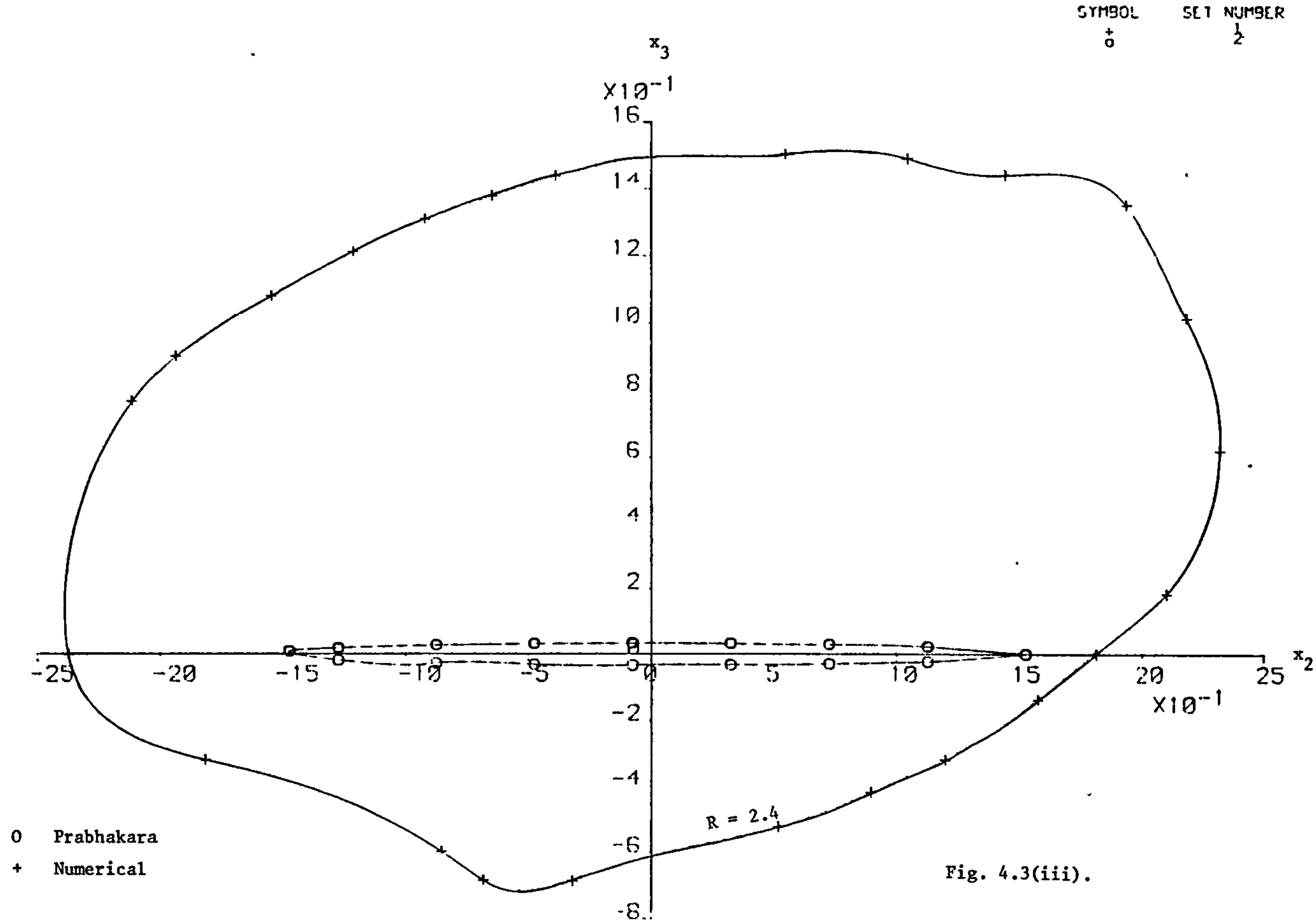
Fig. 4.3(i).



O Prabhakara

\+ Numerical

Fig. 4.3(ii).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x_3
X10^{-1}
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14
12
10
8
6
4
2
-2
-4
-6
-8
-25 -20 -15 -10 -5 0 5 10 15 20 25
x_2
X10^{-1}
R = 2.4
O Prabhakara
+ Numerical

Fig. 4.3(iii).

$$\dot{x} = -x+y+z+x(x^2+y^2+z^2)$$

$$\dot{y} = -x-y+z+y(x^2+y^2+z^2)$$

$$\dot{z} = -x-y-z+z(x^2+y^2+z^2)$$

$$\phi = 2(x^2+y^2+z^2)$$

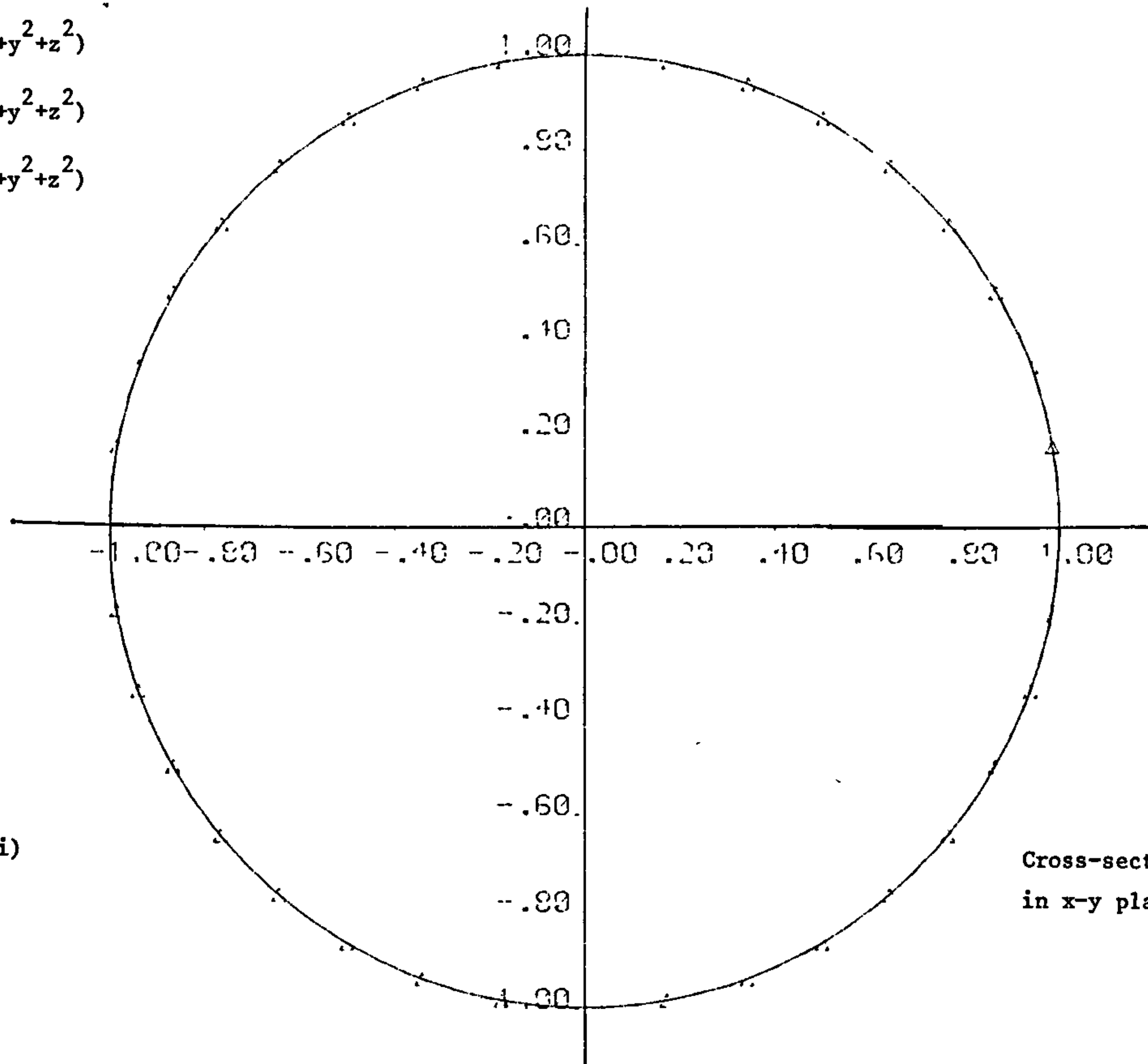


Fig. 4.4(i)

Cross-section of stability boundary in x-y plane, x-z plane and y-z plane.

$$\dot{x} = -x+y+z+x(x^2+y^2+z^2)$$

$$\dot{y} = -x-y+z+y(x^2+y^2+z^2)$$

$$\dot{z} = -x-y-z+z(x^2+y^2+z^2)$$

$$\phi = 2(x^2+y^2+z^2)$$

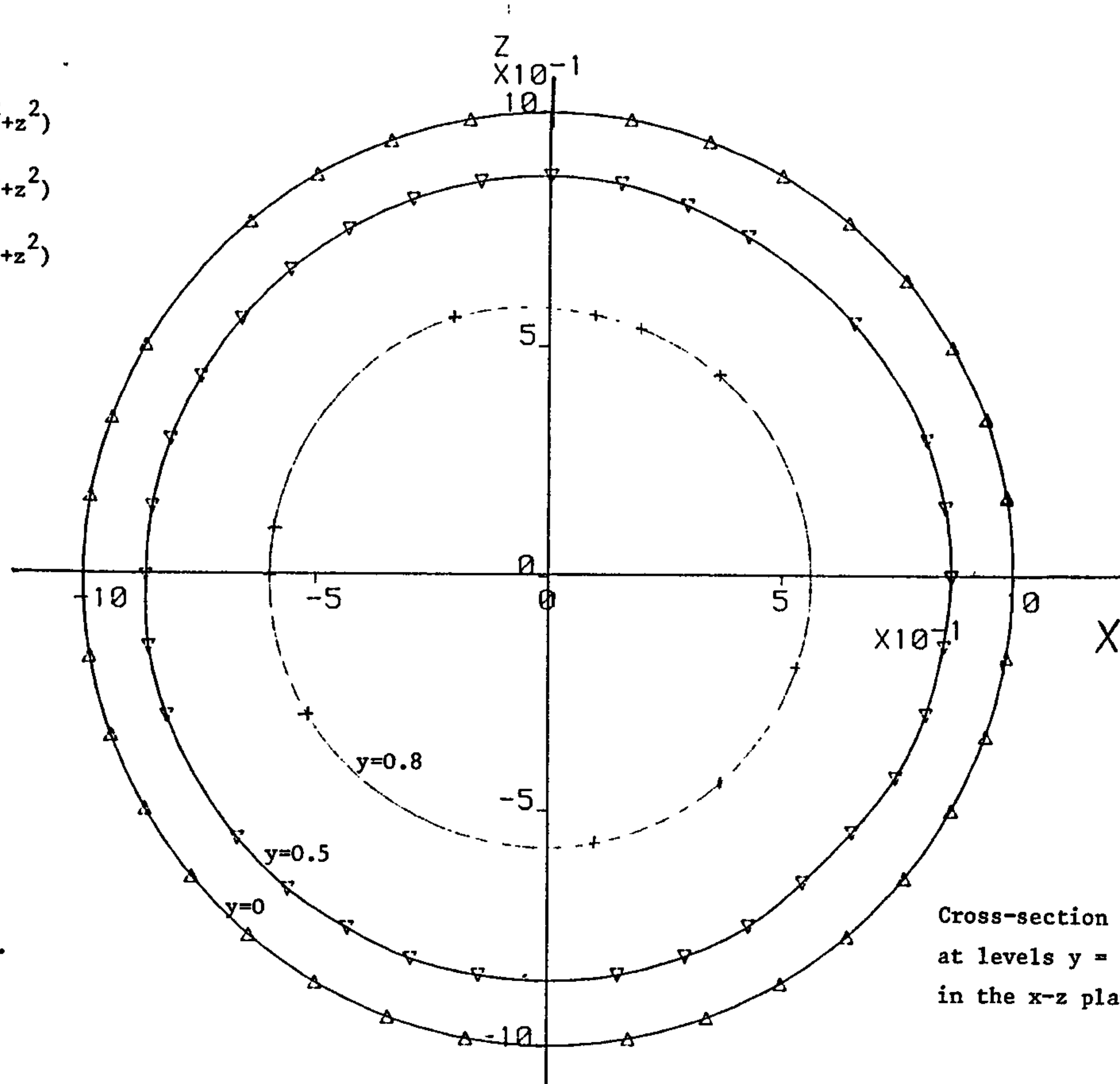


Fig. 4.5(i).

Cross-section of stability boundaries at levels y = 0.5, y = 0.8 and y = 0 in the x-z plane.

$$\dot{x} = -x+y+z+x(x^2+y^2+z^2)$$

$$\dot{y} = -x-y+z+y(x^2+y^2+z^2)$$

$$\dot{z} = -x-y-z+z(x^2+y^2+z^2)$$

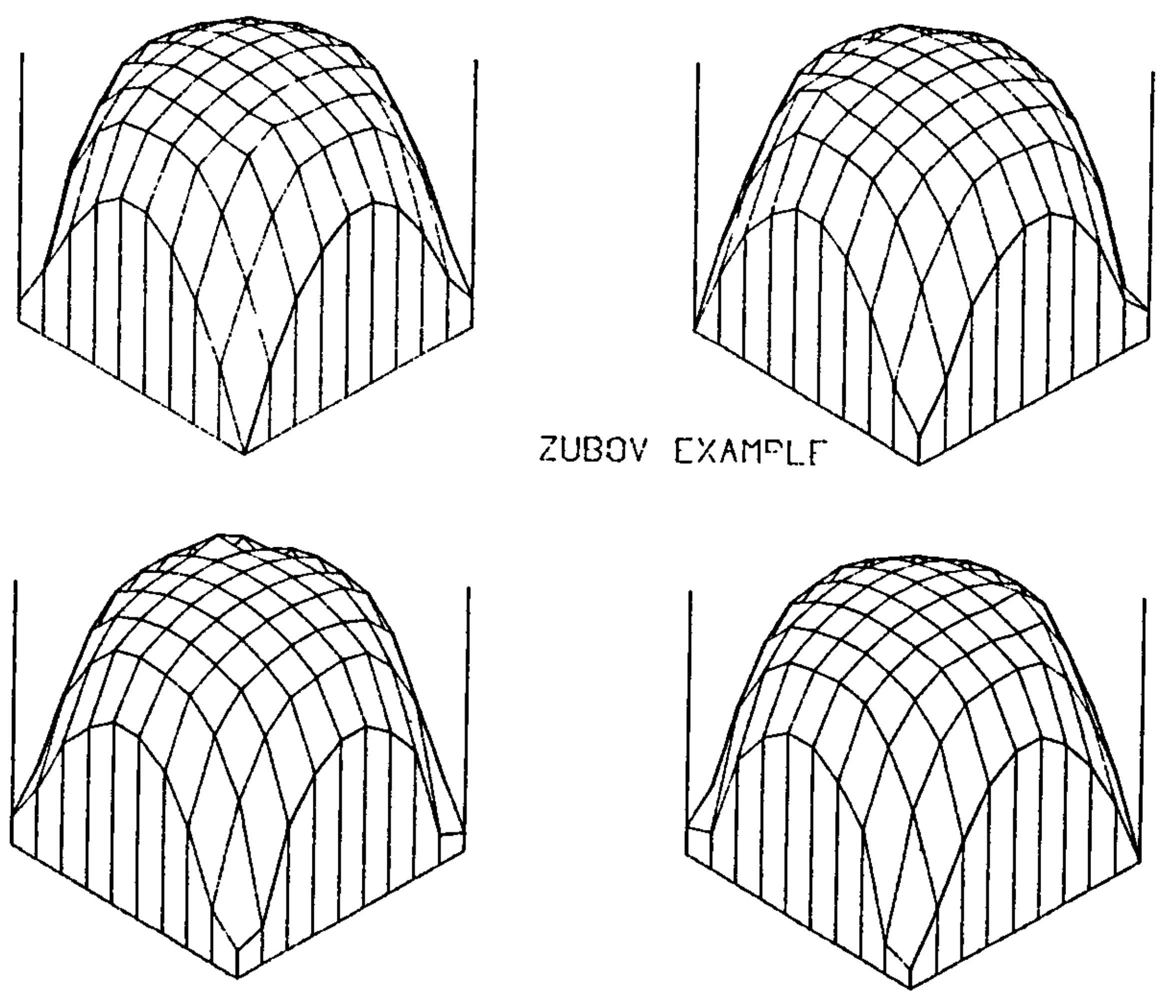


Fig. 4.5(ii).

Example of Prabhakara

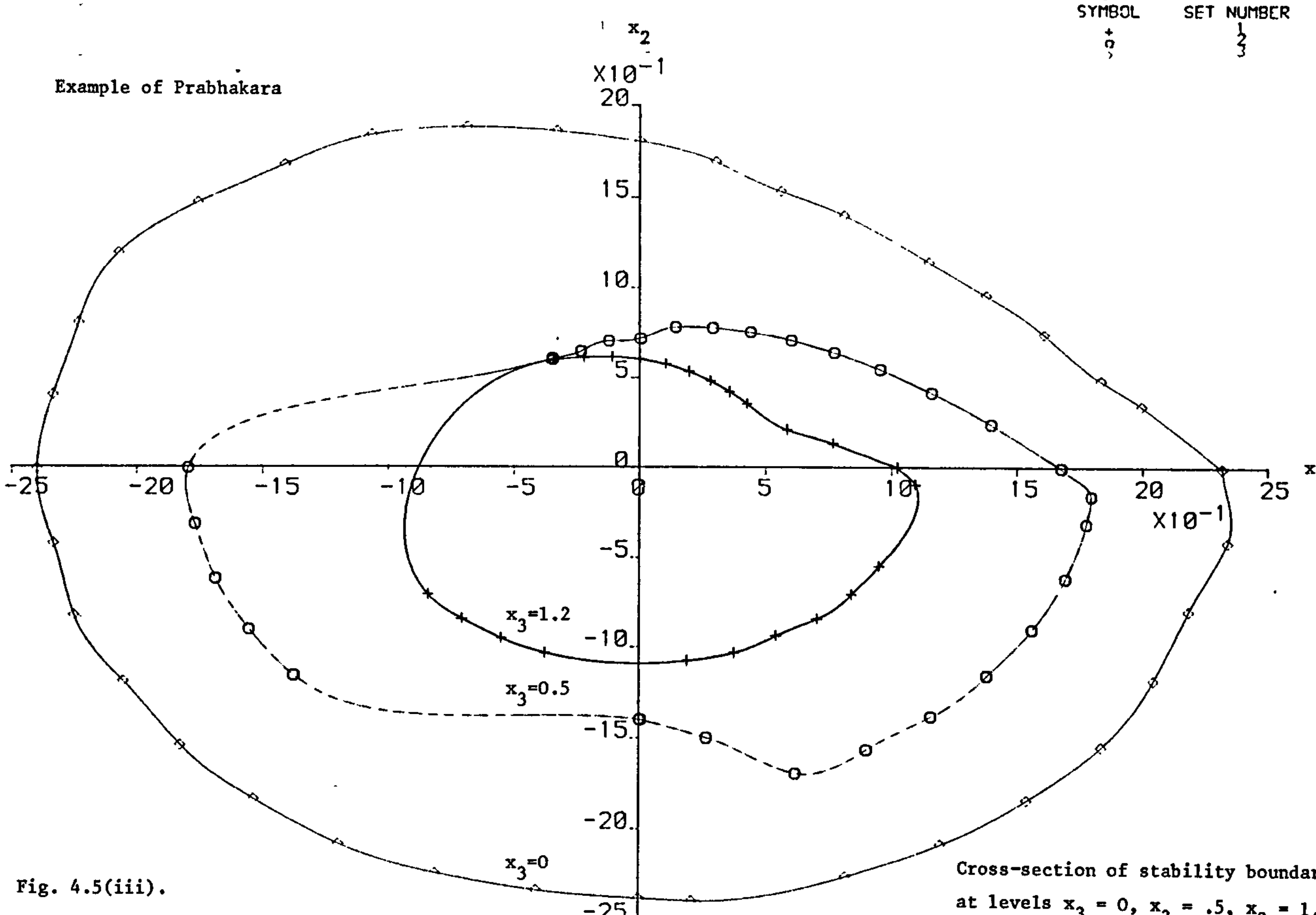


Fig. 4.5(iii).

Cross-section of stability boundaries at levels $x_3 = 0$, $x_3 = .5$, $x_3 = 1.2$ in the x_1-x_2 plane.

Ingwerson Example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -(x_1+cx_2)^3 - bx_3$$

$c = 1,\ b = 1.5$

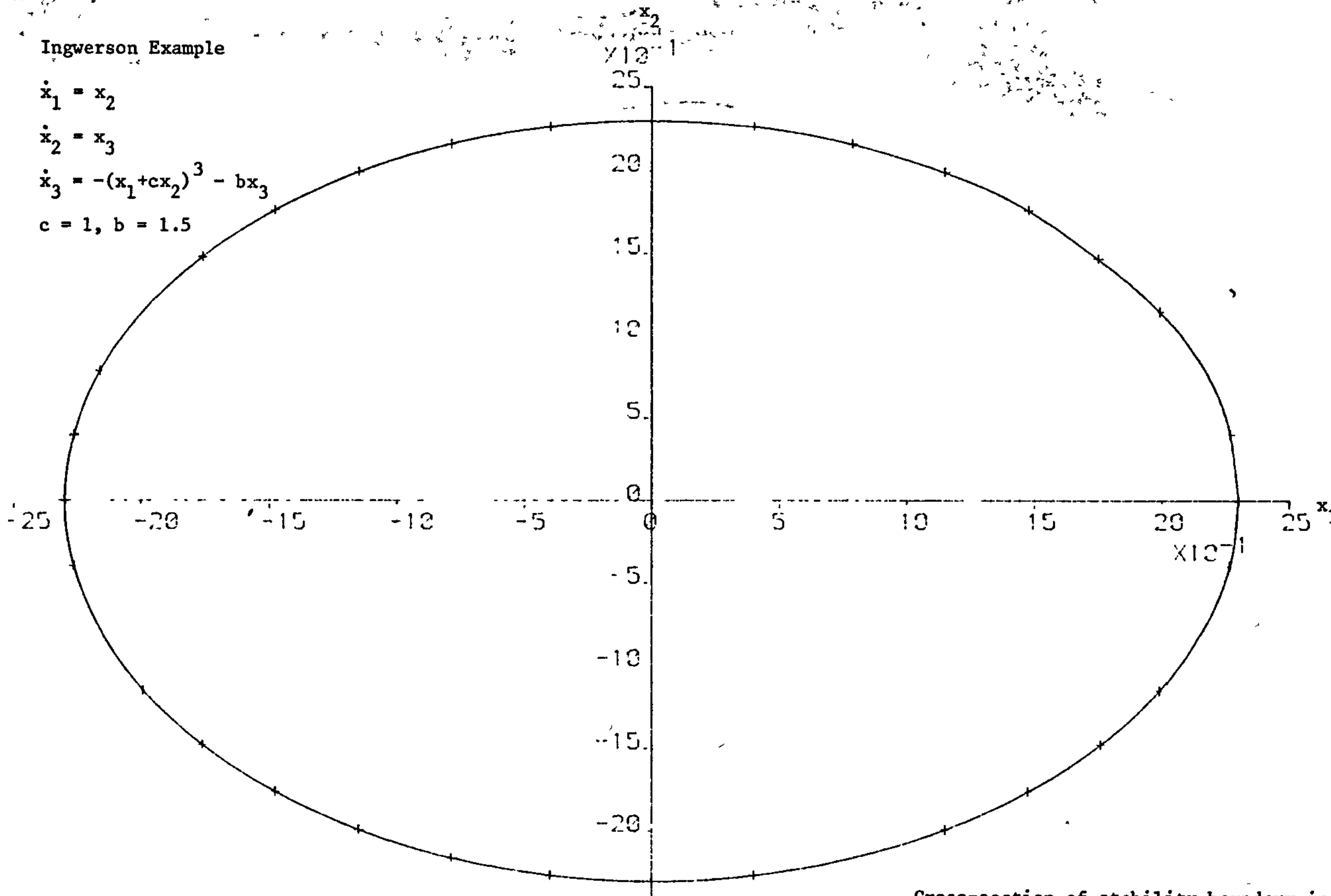


Fig. 4.6(i)

Cross-section of stability boundary in the $x_1 - x_2$ plane.

Ingwerson Example

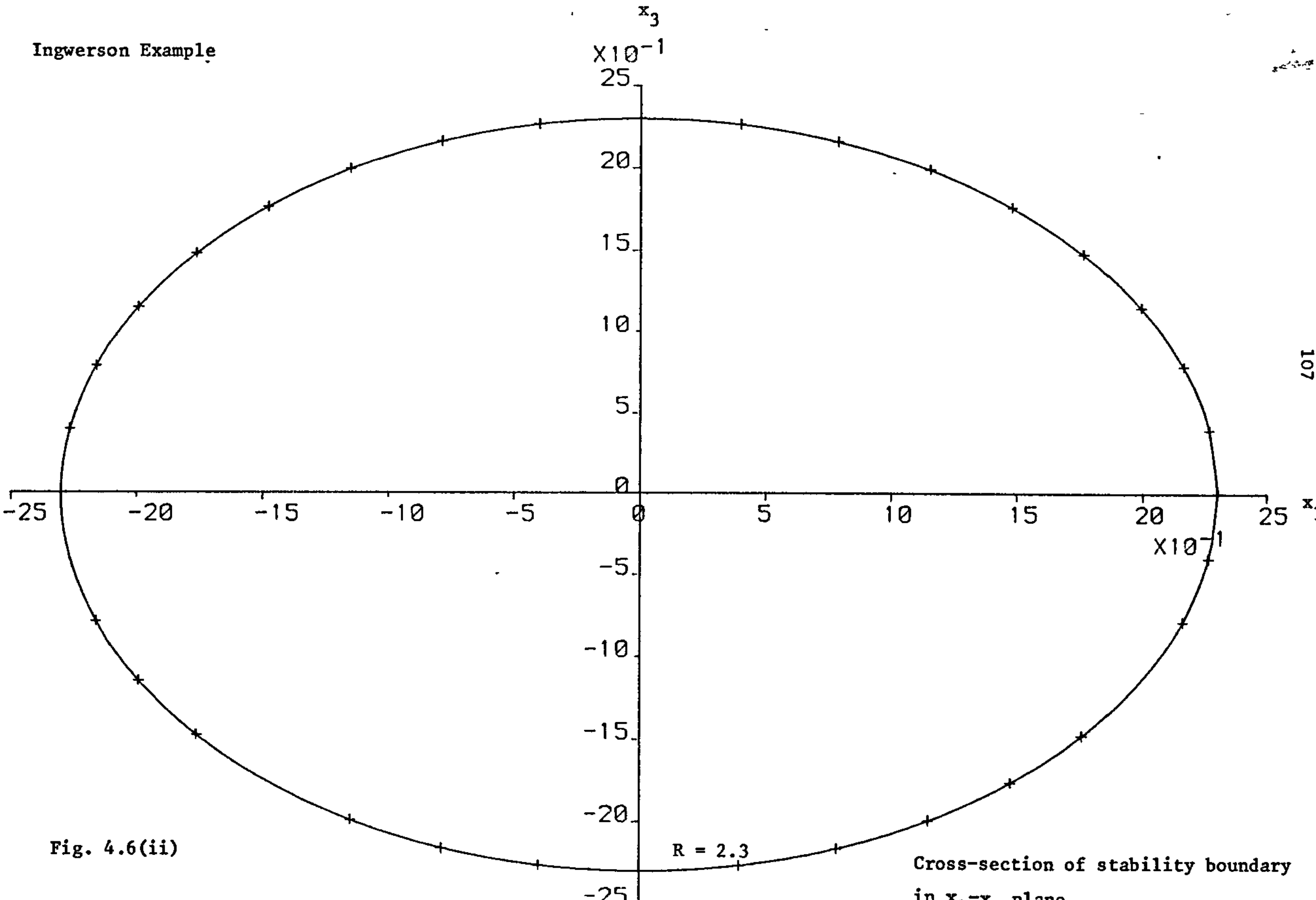


Fig. 4.6(ii)

Cross-section of stability boundary in x_1-x_3 plane.

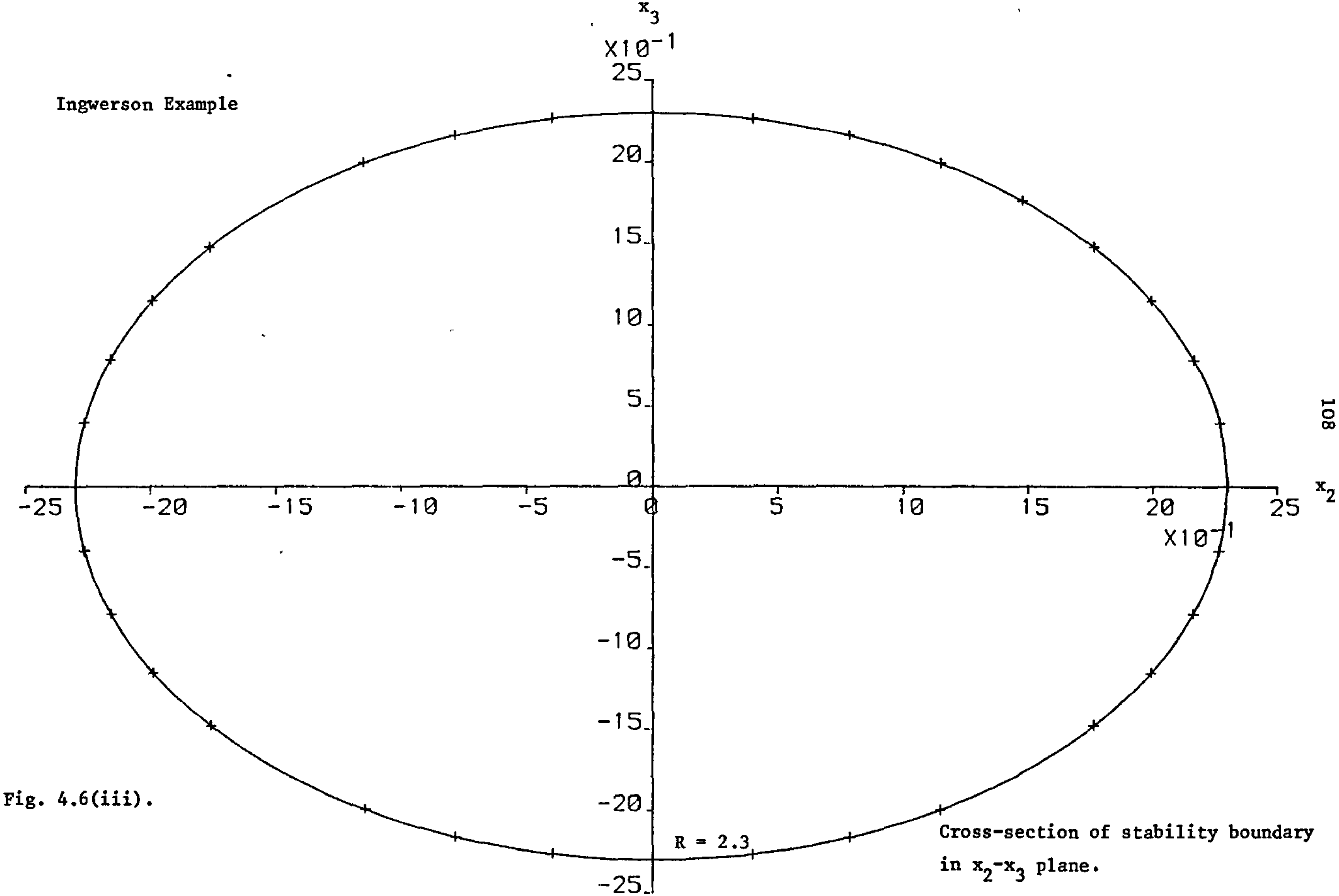


Fig. 4.6(iii).

Cross-section of stability boundary in x_2-x_3 plane.

Chapter V

Second Order Nonautonomous Systems

5.1 INTRODUCTION.

The task of establishing the stability conditions of nonlinear nonautonomous systems seems to pose a formidable problem to the analyst. Whereas almost complete success has been achieved for autonomous systems only limited success has been obtained for nonautonomous systems. Often, engineering problems can be reduced to, or may be approximated by, a second order differential equation with time varying coefficients and a method for determining its stability or stability domain would of course benefit the user.

The range of values of the parameters for which the system is asymptotically stable forms the stability domain of the parameters. We shall call this stability domain of the parameters the "stable parameter region". The region of the parameters for which the system is unstable is called an unstable parameter region. In most cases the system is asymptotically stable in the large for certain values of the parameters, but in some cases there are points in the parameter space for which the system is asymptotically stable in a specific region.

Robe and Jones [50] develop a numerical procedure for investigating the Liapunov stability of second order nonautonomous dynamical systems and present the stability results in terms of the stable parameter region. Michael [53] establishes a stability criterion for the damped Mathieu equation in which a parametric relation $|\varepsilon| < \omega^2$ is defined as the stable parameter region. Ramaran and Rao [51] obtain an improved stable parameter region given by

$$|\varepsilon| < \max\left[(\sqrt{\delta^2 + 4} - 2)^{\frac{1}{2}}\omega \quad , \quad \frac{\delta\omega^2}{\sqrt{\delta^2+4}}\right] .$$

This region is obtained from the use of a suitable Liapunov function. Similarly authors like Hahn [8], Loud [52] and many others obtain relationships between the parameters for which the system is asymptotically stable and present this relation in the parameter space.

In general this work deals with systems that are asymptotically stable in the large for parameters lying in the stable parameter region and there has been little research on the determination of the domain of attraction in the phase space of the equilibrium of nonautonomous systems. An algorithm for finding such domains will be described in the next section.

The use of the numerical algorithm for third order autonomous system on second order nonautonomous system will hopefully throw some light on the determination of domains of attraction in the phase space. Cross-sections of the stability boundaries in the x-y plane for different values of t for an example proposed by Lehnigk are examined. Tests have also been made on the Mathieu equation for various values of the parameters of the system and it is found that for a number of parameter values satisfying the relationship

$$(\omega-1)^2 - \varepsilon^2(\tfrac{1}{4} - \delta_1^2) > 0$$

the system is asymptotically stable in the large, but for parameters which do not satisfy the relationship the system is unstable.

5.2 DETERMINATION OF DOMAIN OF ATTRACTION OF SECOND ORDER NONLINEAR NONAUTONOMOUS SYSTEM.

Consider a second order nonlinear nonautonomous system

$$\begin{aligned} \dot{x} &= f(x,y,t) \\ \dot{y} &= g(x,y,t) \end{aligned} \tag{5.2.1}$$

with the origin assumed to be an equilibrium, i.e.

$$f(0,0,t) = g(0,0,t) = 0 \quad . \tag{5.2.2}$$

Zubov's equation is

$$\frac{\partial V}{\partial x} f(x,y,t) + \frac{\partial V}{\partial y} g(x,y,t) + \frac{\partial V}{\partial t} = - \phi(1-eV) \tag{5.2.3}$$

where e = 0 or 1 for the modified or regular form respectively and its auxiliary equations are

$$\frac{dx}{f(x,y,t)} = \frac{dy}{g(x,y,t)} = dt = - \frac{dV}{\phi(1-eV)} \quad . \tag{5.2.4}$$

Rearranging gives

$$\begin{aligned} \frac{dx}{dV} &= - \frac{f(x,y,t)}{\phi(1-eV)} \\ \frac{dy}{dV} &= - \frac{g(x,y,t)}{\phi(1-eV)} \\ \frac{dt}{dV} &= - \frac{1}{\phi(1-eV)} \quad . \end{aligned} \tag{5.2.5}$$

The form of (5.2.5) is similar to that of (4.2.3) and the algorithm described in Section 4.2 will be used to compute the domain of attraction. Again we encounter the vector $\underline{\theta}$ and we have to fix one component of θ to determine the cross-section of the stability boundary. Note that, to ensure positive definiteness and decrescency of V in the region of stability, a positive definite ϕ is chosen.

The connection between second order nonautonomous systems and third order autonomous ones is that the Zubov equation (5.2.3) is similar to (4.2.2) of the third order system. Also the V's, f's and ϕ's for autonomous system are homogeneous in x,y,z whereas for nonautonomous system they are only homogeneous in the phase variables x and y but not t. Hence the homogeneity of the phase variables is preserved in Zubov theory.

5.3 DAMPED MATHIEU EQUATION.

Consider a damped Mathieu equation

$$\ddot{x} + \delta\dot{x} + (\omega + \varepsilon\cos 2t)x = 0 \quad . \tag{5.3.1}$$

If we write

$$x(t) = e^{-\frac{1}{2}\delta t}\,\eta(t) \;, \tag{5.3.2}$$

then (5.3.1) transforms into

$$\ddot{\eta} + (\omega - \frac{1}{4}\,\delta^2 + \varepsilon\cos 2t)\eta = 0 \tag{5.3.3}$$

which is a form of the Mathieu equation

$$\ddot{x} + (\alpha + \varepsilon\cos 2t)x = 0 \tag{5.3.4}$$

with $\quad \alpha = \omega - \frac{1}{4}\delta^2$.

Jordan and Smith [54] show that a certain unstable region (which they called second unstable region) for (5.3.3) occurs near $\alpha = 1$ or $\omega = (1 + \frac{\delta^2}{4})$ and on the boundaries of this unstable region, periodic solutions of period 2π exist. Suppose the damping term, $\delta = \varepsilon\delta_1$ is small, then by letting

$$\omega = \omega(\varepsilon) = \omega_o + \varepsilon\omega_1 + \dots \tag{5.3.5}$$

and solutions of period 2π be

$$x(t) = x_o(t) + \varepsilon x_1(t) + \dots \tag{5.3.6}$$

and using perturbation method, the parametric stability boundary near $\omega = 1$ is found to be

$$\omega = 1 \pm \varepsilon\sqrt{\frac{1}{4} - \delta_1^2} \tag{5.3.7}$$

so long as $\delta_1^2 < \frac{1}{4}$. The set of parameter values for which the damped Mathieu equation (5.3.1) is asymptotically stable is therefore

$$(\omega-1)^2 - \varepsilon^2(\frac{1}{4} - \delta_1^2) > 0 \quad . \tag{5.3.8}$$

Taking $\quad \delta = 0.2$, $\varepsilon = 0.5$ and $\omega = 2.0$, (5.3.8) is satisfied and system (5.3.1) is stable. Figures 5.3(i)-(iii) show the cross-section of the stability boundaries of the above system drawn in the three principal planes using $\phi = x^2 + y^2$ and $R = 6.4$. Since this system is asymptotically stable in the large for the above parameter values full circles are obtained for increasing values of R.

When $\delta = 0.2$, $\omega = 1.0$ and $\varepsilon = 1.5$ are used we will get a list of unstable points from the computation where $J(\theta)$ of (1.7.17) are all zero. Hence the system is unstable for this set of parameter values. Moreover for these values of the parameters (5.3.8) is not satisfied.

5.4 LEHNIGK EXAMPLE.

Consider a system

$$\dot{\underline{x}} = A\underline{x} + g(\underline{x},t) \tag{5.4.1}$$

where the eigenvalues of the constant n×n matrix A have negative real parts and $g(x,t)$ is nonlinear with $g(0,t) = 0$.

The equilibrium of the equation of the first approximation is asymptotically stable. The function $g(x,t)$ is strictly nonlinear (i.e. $g(x,t)$ cannot be expanded into the form

$$g(\underline{x},t) = A'\underline{x} + g'(\underline{x},t),$$

where A' is not a zero matrix).

Lehnigk states that system (5.4.1) is asymptotically stable if the equation

$$\dot{\underline{x}} = A\,\underline{x} \tag{5.4.2}$$

of the first approximation is asymptotically stable and if for sufficiently small $\lambda > 0$

$$|g(\underline{x},t)| < \lambda\,|\underline{x}| \quad \text{for} \quad |\underline{x}| \neq 0 \text{ and } t \geqslant t_o > \tau,\ |x| \leqslant h_1 < h\ . \tag{5.4.3}$$

Let us take

$$A = \begin{bmatrix} -1 & -2 \\ 5 & -1 \end{bmatrix}$$

and

$$g(x,t) = \left[x \sin t \, , \, \frac{y}{t+4}\right] \; .$$

The eigenvalues of A have negative real parts; so the equilibrium of $\dot{\underline{x}} = A\underline{x}$ is asymptotically stable. The estimate of the domain of attraction given by Lehnigk is

$$x^2 + \frac{2y^2}{5} \leq 1 \quad . \tag{5.4.4}$$

In this example we use $\phi = x^2 + y^2$ and compute the cross-sections of the stability boundaries at t = 0, 1, 2, 3, 5, 6. Figure 5.4(i) shows these boundaries superimposed on each other in the x-y plane. The domains obtained are obviously greater than Lehnigk's domain defined by (5.4.4).

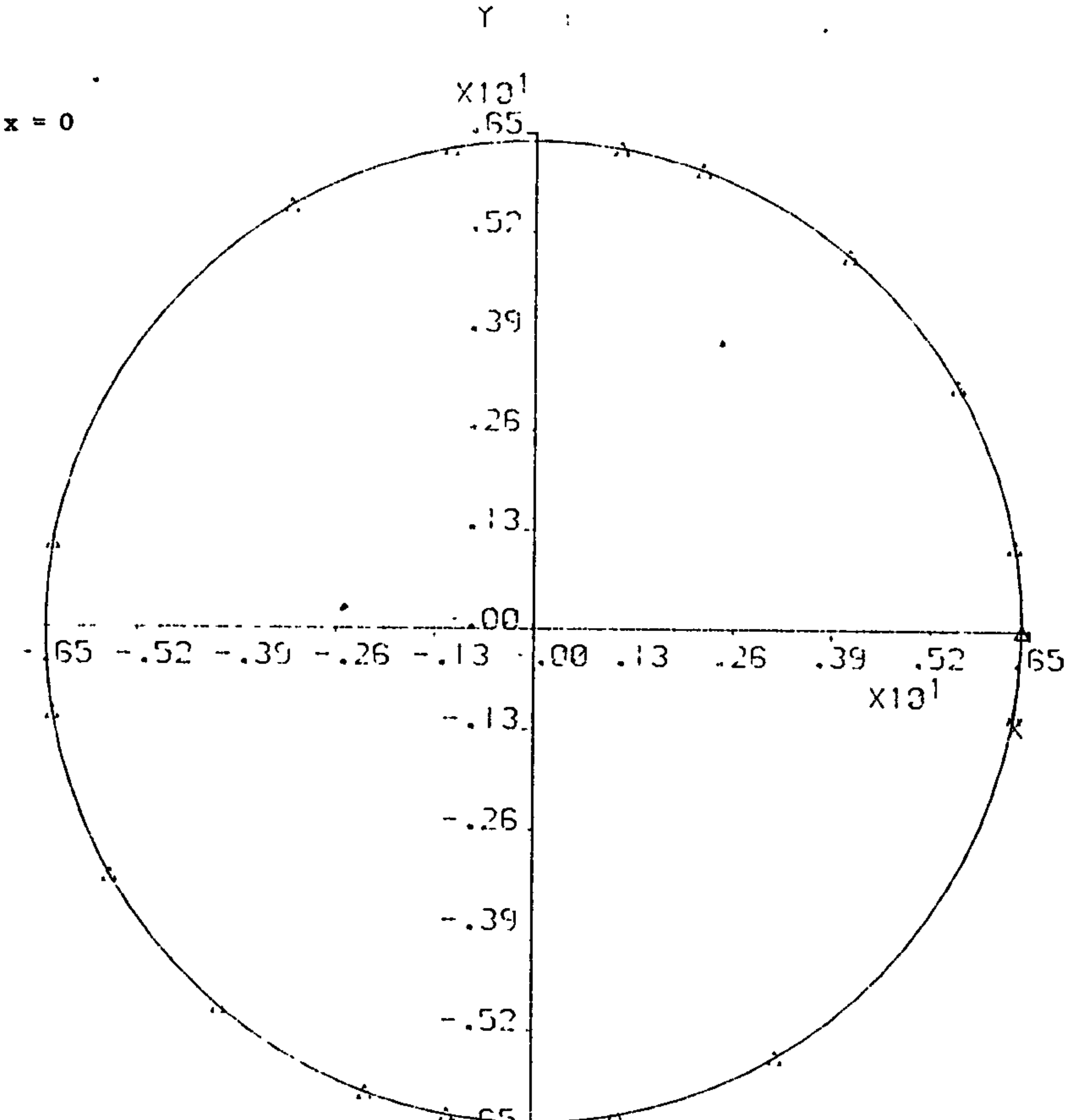
5.5 CONCLUSION.

The numerical algorithm for third order autonomous systems is used to determine the domains of attraction of second order nonautonomous systems. Domains of attraction of the nonautonomous system which are represented in terms of the phase variables can now be obtained. The Mathieu equation with damping term has been discussed and the results are presented in the form of cross-sections of the stability boundary. This system is stable only for a certain range of parameters represented by (5.3.8) calculated near the second unstable region. Estimates of the domain of attraction for Lehnigk's system have been obtained for different values of t as shown in Figures 5.4(i) and are found to be larger than that of Lehnigk. The domains vary for different values of t and this occurs because the second unstable critical point is not the same for different t.

$\ddot{x} = \delta\dot{x}+(\omega+\varepsilon\cos 2t)x = 0$

$\phi = x^2+y^2$

$R = 6.4$



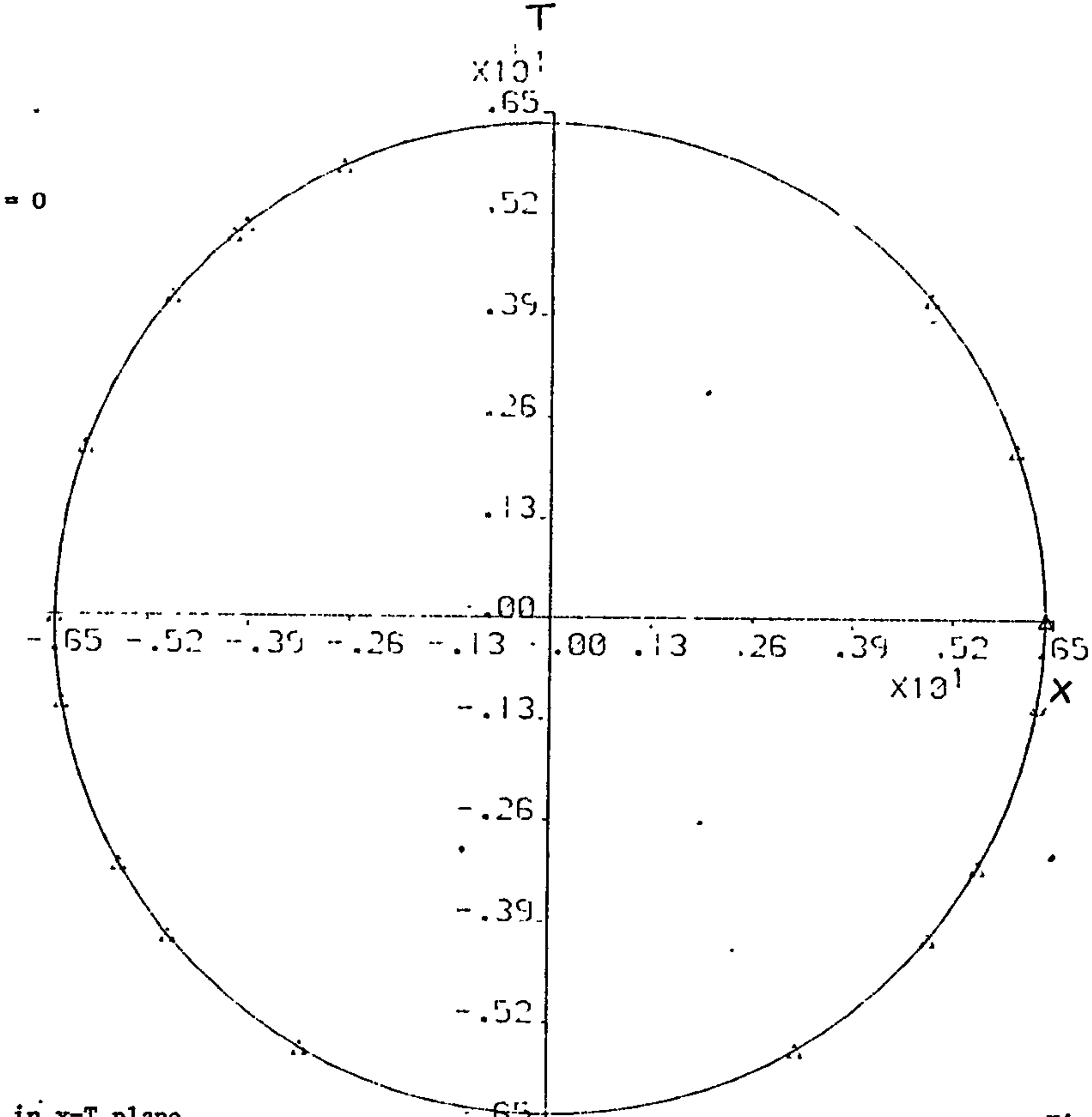
Stability region in x-y plane.

Fig. 5.3(i).

$\ddot{x} + \delta\dot{x} + (\omega + \varepsilon\cos 2t)x = 0$

$\phi = x^2 + y^2$

$R = 6.4$



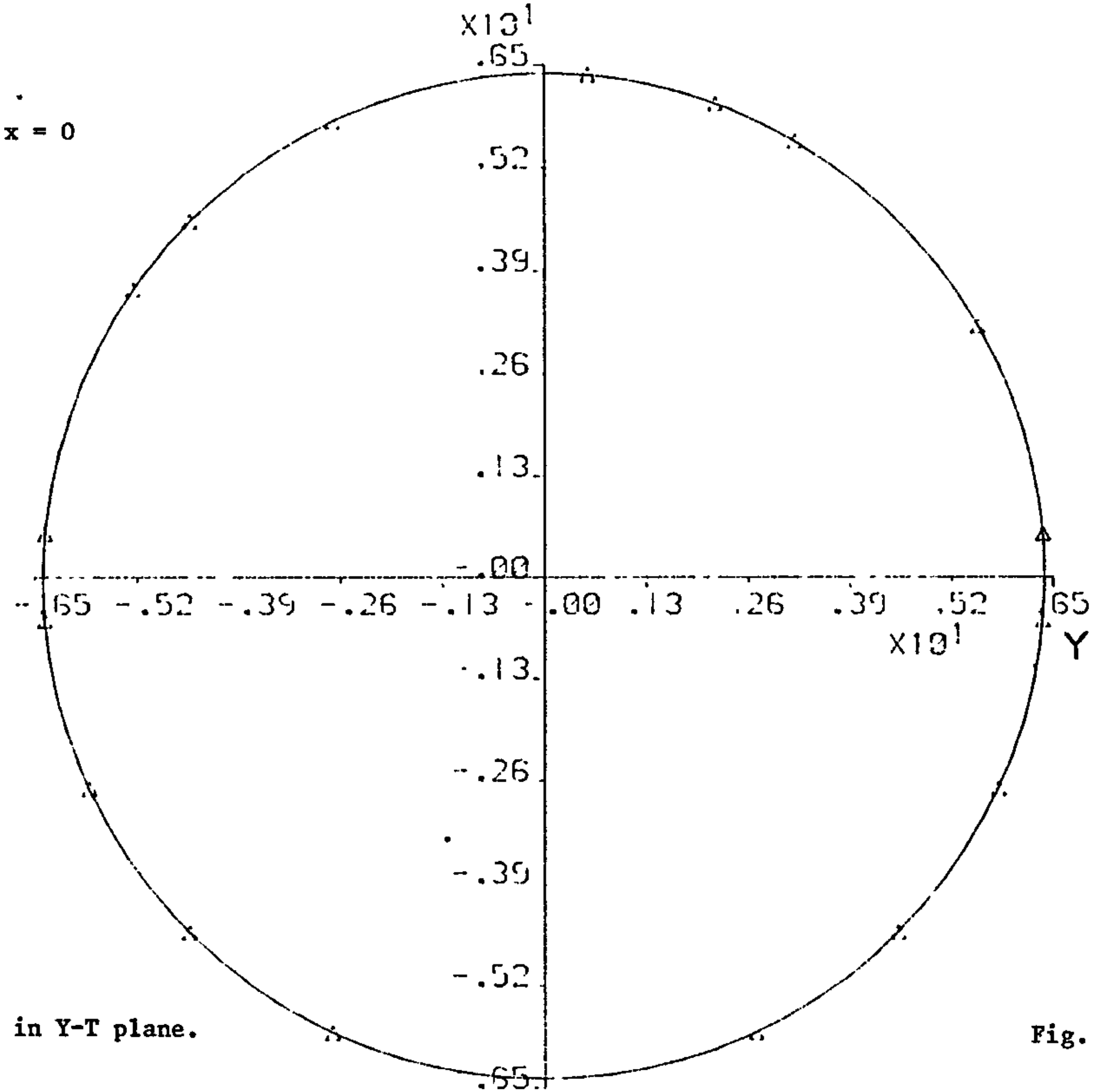
Stability region in x-T plane.

Fig. 5.3(ii).

$$\ddot{x} + \delta\dot{x} + (\omega + \varepsilon\cos 2t)x = 0$$

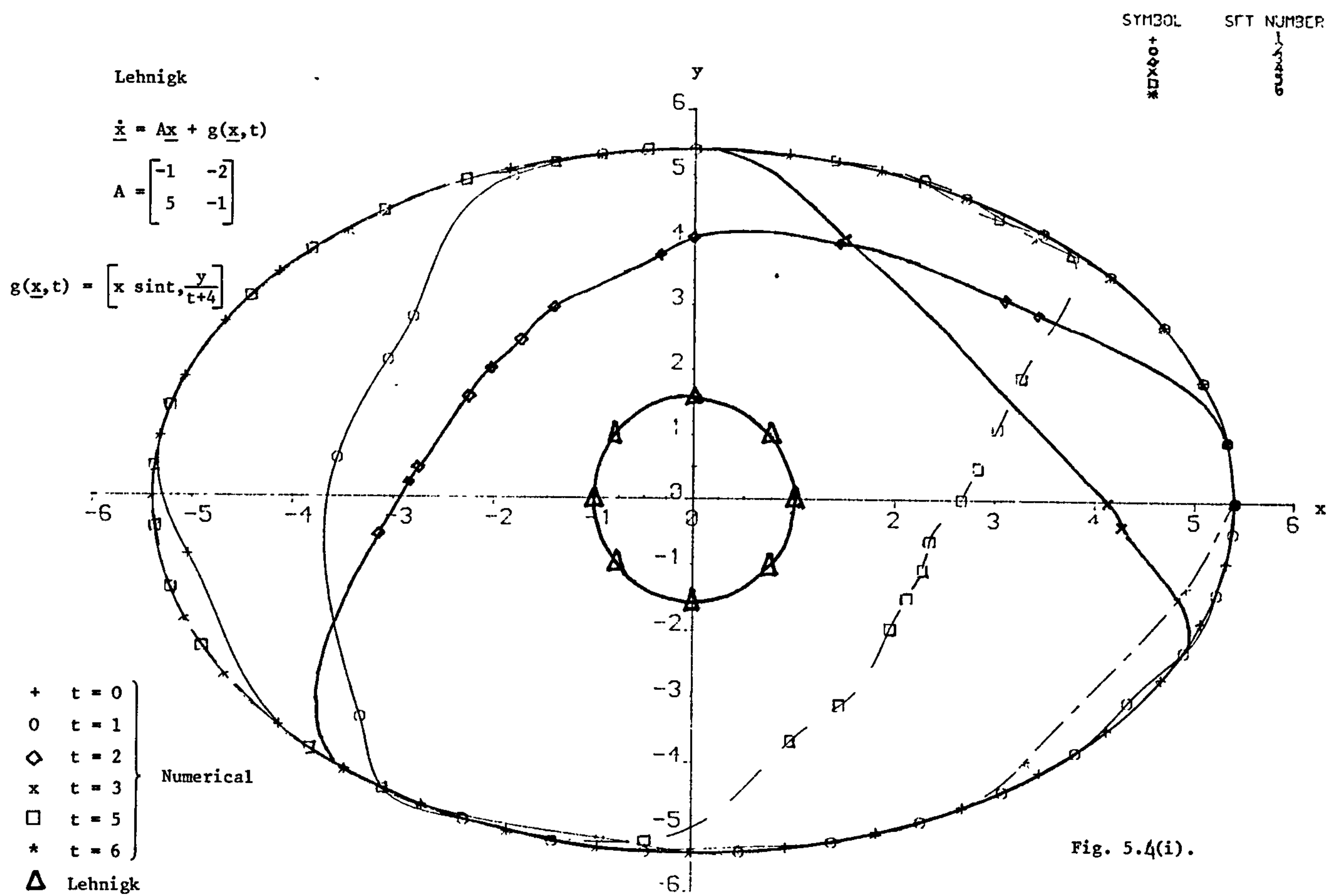
$$\phi = x^2 + y^2$$

$$R = 6.4$$



Stability region in Y-T plane.

Fig. 5.3(iii).



SYMBOL SET NUMBER
+ 1
o 2
◇ 3
x 4
□ 5
* 6
Lehnigk
$\dot{\underline{x}} = A\underline{x} + g(\underline{x},t)$
$A = \begin{bmatrix} -1 & -2 \\ 5 & -1 \end{bmatrix}$
$g(\underline{x},t) = \left[x \sin t, \frac{y}{t+4}\right]$
y
x
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
+ t = 0
0 t = 1
◇ t = 2
x t = 3
□ t = 5
* t = 6
Numerical
Δ Lehnigk

Fig. 5.4(i).

Chapter VI

Periodic Solutions

6.1 INTRODUCTION.

The present chapter is devoted to one of the more important problems encountered in the analysis and synthesis of nonlinear systems with self-oscillatory behaviour or periodic solution. The problem of stability of these solutions is a difficult one which has attracted attention for many years. The stability of a limit cycle is studied. The limit cycle may be stable, unstable or semi-stable depending on where the trajectories, emanating from the interior or exterior of the limit cycle, approach to as t tends to infinity. Methods for obtaining the periodic solution of nonlinear systems are described in the literature [13].

The theory on the region of attraction of periodic solutions of a system is required in the determination of such regions by a numerical method. Zubov [1] states a theorem for finding the domain of attraction of a periodic solution and gives an example in which the periodic solution is asymptotically stable in the large and the origin is an unstable singular point. Some definitions on asymptotic stability of the periodic solutions and a theorem on the domain of attraction of this solution are included since this theory will be used in the numerical method. In implementing this theory a periodic solution must be known a priori, so methods for approximating such solutions are given.

Finally we illustrate the theory by examples which have bounded domains of attraction and utilize circles and ellipses as our regions of approximation in the method described in Section 1.7 of Chapter one for the determination of domain of attraction.

6.2 DEFINITIONS AND THEOREM.

Consider a system of equations

$$\frac{dx_s}{dt} = F_s\ (x_1,\ldots,x_n), \quad s = 1,\ldots,n \tag{6.2.1}$$

whose right hand members are continuously differentiable in all their arguments. We shall assume that (6.2.1) has a periodic solution

$$x_s = \bar{x}_s(t) \tag{6.2.2}$$

with period T.

Definition 6.2.1.

The periodic solution (6.2.2) of system (6.2.1) is stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,t_o)$ such that $d(x_{1o},\ldots,x_{no}) < \delta$ implies $d[x_1(t),\ldots,x_n(t)] < \varepsilon$ for $t \in I$.

Here

$$d(x_1,\ldots,x_n) = \inf_{t\in[0,T]} \sqrt{\sum_{i=1}^{n} [x_i - \bar{x}_i(t)]^2} \tag{6.2.3}$$

Definition 6.2.2.

The periodic solution (6.2.2) of system (6.2.1) is asymptotically stable if

i) the periodic solution (6.2.2) is stable

ii) there exist a $\gamma > 0$ such that

$$d(x_{1o},\ldots,x_{no}) < \gamma \ \text{ implies } \ d[x_1(t),\ldots,x_n(t)] \to 0$$

as $t \to \infty$.

Definition 6.2.3.

The set A of all points $x_{1o},\ldots,x_{no}$ of the n-dimensional space is called the region of attraction of the periodic solution, if from $(x_{1o},\ldots,x_{no}) \in A$ it follows that

$$d[x_1(t),\ldots,x_n(t)] \to 0 \quad \text{as } t \to \infty \quad .$$

The definitions presented here coincide with those given in Chapter one where the origin is considered as the trivial solution. Instead of considering the stability of this trivial solution, the stability of periodic solution is being studied. This type of stability is sometimes known as orbital stability.

Definition 6.2.4.

A limit cycle is an isolated periodic solution of system (6.2.1), represented in the phase plane by an isolated closed path.

Definition 6.2.5.

A limit cycle is stable if all the trajectories (both interior and exterior) approach towards it as $t \to \infty$ and unstable if the trajectories move away from it.

At this stage it is necessary to state a theorem which leads to the investigation of the problem of asymptotic stability of a periodic solution of system (6.2.1).

Theorem 6.2.1. (Zubov).

In order for the region A of the space R^n, consisting of entire trajectories of system (6.2.1) and containing the set of all $\underline{x}$ satisfying

$d(x_1,\ldots,x_n) < \delta$ for sufficiently small δ, to be the region of attraction of the periodic solution (6.2.2) of system (6.2.1) it is necessary and sufficient that there exist two functions $V(x_1,\ldots,x_n)$ and $\phi(x_1,\ldots,x_n)$ satisfying the conditions

i) the function $V(x_1,\ldots,x_n)$ is defined and continuous in A, the function $\phi(x_1,\ldots,x_n)$ is defined and continuous in R^n.

ii) $V(x_1,\ldots,x_n)$ is positive definite in A and $\phi(x_1,\ldots,x_n)$ is positive definite in R^n, i.e. given any quantity $\gamma_2 > 0$ it is possible to find γ_1 and α_1 such that

$$V(x_1,\ldots,x_n) > \gamma_1 \quad \text{for} \quad d(x_1,\ldots,x_n) > \gamma_2$$

$$\text{and} \quad \phi(x_1,\ldots,x_n) > \alpha_1 \quad \text{for} \quad d(x_1,\ldots,x_n) > \gamma_2$$

iii) $V \to 0$ and $\phi \to 0$ as $d(x_1,\ldots,x_n) \to 0$.

iv) $\lim\limits_{x_s \to \tilde{x}_s} V(x_1,\ldots,x_n) = \infty$, where $\tilde{x}_s \in \bar{A} - A$ and is a point on the boundary of region A.

v) The total derivative of the function V, calculated by virtue of system (6.2.1), satisfies the relation

$$\frac{dV}{dt} = - \phi(x_1,\ldots,x_n).$$

The theorem stated above is equivalent to the theorem for autonomous system in which the region of asymptotic stability of the origin is considered. In Theorem 6.2.1 the region of asymptotic stability of a periodic solution is required. The V's and ϕ's also possess the positive definiteness and decrescency properties. The function $V \to \infty$ as one approaches a finite or infinite point of the boundary of A.

Let us apply the theorem to analyse an example given by Zubov

$$\dot{x} = x + y - x(x^2+y^2)$$

$$\dot{y} = -x + y - y(x^2+y^2) \quad .$$

We set $$\phi = \frac{2(1-x^2-y^2)^2(1+x^2+y^2)}{x^2+y^2}$$

then the partial differential equation corresponding to this system is

$$\frac{\partial V}{\partial x}\left[x + y - x(x^2+y^2)\right] + \frac{\partial V}{\partial y}\left[-x + y - y(x^2+y^2)\right] = -\frac{2(1-x^2-y^2)^2(1+x^2+y^2)}{x^2+y^2} \quad .$$

The solution of this equation is

$$V = \frac{\left[1 - (x^2+y^2)\right]^2}{x^2+y^2} \quad .$$

The function V is defined at all points of the phase space except at $x = y = 0$ and satisfies the conditions of Theorem 6.2.1. $V = 0$ for $x^2 + y^2 = 1$. The circle $x^2 + y^2 = 1$ is a periodic solution and its region of attraction is the whole space. The function V in the region of attraction apart from the periodic solution will take positive values. V will tend to infinity as one approaches a finite or infinite point of the boundary of the domain of attraction. Although this is a trivial example, it contains the characteristic behaviour of the integral curves of a nonlinear system in the presence of its periodic solution and also complies with the conditions of the above theorem. Here the region of attraction fills the entire phase space, while in the general case the region of attraction of a periodic solution in a nonlinear system may be located in a bounded portion of the space. Examples of this type of region of attraction will be included in the later section.

6.3 APPROXIMATIONS OF THE PERIODIC SOLUTIONS.

In order to apply the method of Section 1.7 usefully the periodic solution or approximate periodic solution of system (6.2.1) should be known a priori so that a positive definite and decrescent ϕ can be chosen and solving the Zubov's partial differential equation will guarantee the existence of positive definite and decrescent V. There are several ways of approximating the periodic solution of a system and one of the easiest methods is by harmonic balance. Take a general, second order equation

$$\ddot{x} + \varepsilon g(x,\dot{x}) + x = 0 \ . \tag{6.3.1}$$

Suppose the approximate periodic solution is $r \cos \omega t$ and that g has a Fourier Series

$$\begin{aligned} g(x,\dot{x}) &\simeq g(r \cos \omega t \, , \, - r\omega \sin \omega t\,) \\ &= A_1(r) \cos \omega t \, + B_1(r) \sin \omega t \, + \text{higher harmonics} \end{aligned} \tag{6.3.2}$$

with the constant term absent.

Then (6.3.1) becomes

$$(1-\omega^2)r\cos\omega t + \varepsilon A_1(r)\cos\omega t + \varepsilon B_1(r)\sin\omega t + \text{higher harmonics} = 0 \ . \tag{6.3.3}$$

This equation is true for all t if

$$(1-\omega^2)r + \varepsilon A_1(r) = 0, \quad B_1(r) = 0 \tag{6.3.4}$$

r and ω can thus be determined.

Consider the Van der Pol equation

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0 \ . \tag{6.3.5}$$

Assume an approximate solution $x = r\cos\omega t$.

Substituting in (6.3.5) gives

$$(1-\omega^2)r\cos\omega t = \varepsilon r\omega\left(\frac{r^2}{4} - 1\right)\sin\omega t + \frac{1}{4}\varepsilon r^3\omega\sin 3\omega t \ .$$

Equating the coefficients of $\cos\omega t$, $\sin\omega t$ gives

$$\omega^2 = 1 \quad \text{and} \quad r = 2 \ .$$

Hence the approximate solution is 2 cost and the approximate periodic solution in the phase plane is $x^2 + y^2 = 4$.

Another procedure for estimating such solutions is by the averaging method. We consider (6.3.1) with $g(0,0) = 0$ and $|\varepsilon| << 1$ and represent it by a system of equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\varepsilon g(x,y) - x \ . \end{aligned} \tag{6.3.6}$$

Suppose the solutions are nearly circles in the phase plane and $x = r\cos\theta$, $y = r\sin\theta$.

Differentiating $r^2 = x^2 + y^2$, $\theta = \tan^{-1} y/x$ and substituting in (6.3.6) gives

$$\dot{r} = -\varepsilon g(r\cos\theta, r\sin\theta)\sin\theta \tag{6.3.7}$$

and

$$\dot{\theta} = -1 - \frac{\varepsilon}{r} g(r\cos\theta, r\sin\theta)\cos\theta . \tag{6.3.8}$$

From (6.3.7) and (6.3.8)

$$\frac{dr}{d\theta} \simeq \varepsilon g(r\cos\theta, r\sin\theta)\sin\theta . \tag{6.3.9}$$

Since the motion is assumed to be periodic with period 2π and r returns to its original value over one period

$$\int_o^{2\pi} \frac{dr}{d\theta}\, d\theta = 0 = \varepsilon \int_o^{2\pi} g(r\cos\theta, r\sin\theta)\sin\theta\, d\theta . \tag{6.3.10}$$

Since $\frac{dr}{d\theta} = O(\varepsilon)$, then over one period

$$r(\theta) = r_o + O(\varepsilon) \tag{6.3.11}$$

where r_o is the value at the beginning of the period. The right hand side of (6.3.7) contains the function

$$g(r\cos\theta, r\sin\theta)\sin\theta$$

and can be represented as a Fourier-type series over a 'cycle', $0 \leq \theta \leq 2\pi$.

Therefore

$$g\left[r(\theta)\cos\theta, r(\theta)\sin\theta\right]\sin\theta = A_o\left[r(\theta)\right] + \sum_{n=1}^{\infty}\left\{A_n\left[r(\theta)\right]\cos n\theta + B_n\left[r(\theta)\right]\sin n\theta\right\} \tag{6.3.12}$$

where $A_n(r)$ and $B_n(r)$ are obtained by treating r as constant in the usual definitions of Fourier series coefficients:

$$A_o\left[r(\theta)\right] = \frac{1}{2\pi}\int_o^{2\pi} g\left[r(\theta)\cos u,\ r(\theta)\sin u\right]\sin u\ du$$

and for $n \geq 1$ (6.3.13)

$$A_n\left[r(\theta)\right] = \frac{1}{\pi}\int_o^{2\pi} g\left[r(\theta)\cos u,\ r(\theta)\sin u\right]\sin u \cos nu\ du$$

$$B_n\left[r(\theta)\right] = \frac{1}{\pi}\int_o^{2\pi} g\left[r(\theta)\cos u,\ r(\theta)\sin u\right]\sin u \sin nu\ du \quad .$$

Equation (6.3.9) becomes, for all θ,

$$\frac{dr}{d\theta} = \varepsilon A_o(r) + \varepsilon \sum_{n=1}^{\infty} \left[A_n(r)\cos n\theta + B_n(r)\sin n\theta\right]. \qquad (6.3.14)$$

Integrating over a cycle $\theta_o < \theta < \theta_o + 2\pi$ gives

$$r(\theta_o+2\pi) - r(\theta_o) = \varepsilon \int_{\theta_o}^{\theta_o+2\pi} A_o(r)d\theta + \varepsilon \sum_{n=1}^{\infty}\int_{\theta_o}^{\theta_o+2\pi} \left[A_n(r)\cos n\theta + B_n(r)\sin n\theta\right] d\theta \quad . \qquad (6.3.15)$$

Over such a cycle, r is constant to order ε, so the terms under the sumation sign disappears, and

$$r(\theta_o + 2\pi) - r(\theta_o) = 2\pi\varepsilon A_o(r). \qquad (6.3.16)$$

Therefore the simplified equation

$$\frac{dr}{d\theta} = \varepsilon A_o(r) \qquad (6.3.17)$$

at least gives the increments in r per cycle correctly to order ε.

The simplified equation corresponding to (6.3.7) is obtained by writing

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dr}{d\theta} + O(\varepsilon^2)$$

$$\simeq -\varepsilon A_o(r) \quad . \tag{6.3.18}$$

Similarly (6.3.8) becomes

$$\frac{d\theta}{dt} = -1 - \frac{\varepsilon}{r} m_o(r) \tag{6.3.19}$$

where

$$m_o(r) = \frac{1}{2\pi}\int_o^{2\pi} g\left[r \cos u,\ r \sin u\right]\cos u\ du$$

and r is treated as constant for the integration.

Let us find the approximate solution for the Van der Pol's equation (6.3.5) for small positive ε.

From (6.3.13)

$$A_o(r) = \frac{r}{2}\left(\frac{r^2}{4} - 1\right)$$

and the approximate equation (6.3.18) is

$$\frac{dr}{dt} = -\frac{\varepsilon r}{2}\left(\frac{r^2}{4} - 1\right)$$

and its solution when $r(0) = r_o$ is

$$r(t) = 2/\left\{1 - \left(1 - 4/r_o^2\right)e^{-\varepsilon t}\right\}^{\frac{1}{2}}$$

which tends to 2 as $t \to \infty$.

From (6.3.19), $\theta(t) = -t + \theta_o$.

Therefore the required approximate solutions are given by

$$x(t) = r(t)\cos\theta(t) = \frac{2\cos(t-\theta_o)}{\left\{1 - (1-4/r_o^2)e^{-\varepsilon t}\right\}^{\frac{1}{2}}}$$

As $t \to \infty$ the approximate periodic solution in the phase plane becomes $x^2 + y^2 = 4$.

In the above methods we have assumed that the system

$$\ddot{x} + \varepsilon g(x,\dot{x}) + x = 0$$

has at least one periodic solution and that its phase diagram contains either a limit cycle or a centre. This system is in a sense close to the linear equation $\ddot{x} + x = 0$ for small ε, and the solutions will be close to circles and thus the assumption of a closed path which is circular but with unknown radius is used to approximate the periodic solution. The application of the algorithm by using approximate periodic solutionsto determine the domain of attraction of a periodic solution of a system will be given in Section 6.5. In the next section we will use the exact periodic solution to determine such domains.

6.4 EXAMPLES AND RESULTS.

Let us now apply the numerical method described in Chapter 1 to problems which have periodic solutions.

i) Consider a system of equations

$$\begin{aligned} \dot{x} &= 2x + y - 3x(x^2+y^2) + x(x^2+y^2)^2 \\ \dot{y} &= 2y - x - 3y(x^2+y^2) + y(x^2+y^2)^2 \end{aligned} \qquad (6.4.1)$$

Let us solve (6.4.1) by using the substitution

$$r = x^2 + y^2$$

then
$$\dot{r} = 2r(1-r)(2-r) \quad . \tag{6.4.2}$$

Solving this first order differential equation gives

$$r = 1 \pm \frac{1}{\sqrt{1 + e^{4t}}}$$

r approaches one as t tends to infinity.

Therefore $r = 1$ or $x^2 + y^2 = 1$ is the stable limit cycle. Also r approaches zero or two as t tends to minus infinity. Furthermore

$$\dot{r} > 0 \qquad \text{for} \qquad r > 2$$

$$\dot{r} < 0 \qquad \text{for} \qquad 1 < r < 2$$

$$\dot{r} > 0 \qquad \text{for} \qquad 0 < r < 1 \quad .$$

From (6.4.2) we see that all trajectories inside the circle $x^2 + y^2 = 2$ approach the periodic solution $x^2 + y^2 = 1$ showing that this limit cycle is stable, that is, trajectories starting at $0 < r < 1$ and $1 < r < 2$ will spiral towards $x^2 + y^2 = 1$.

The closed path $x^2 + y^2 = 2$ is a periodic solution of (6.4.1) which is unstable.

Next choose $\phi = 4(1-r)^2$, and Zubov's equation becomes

$$\frac{dV}{dr} = - \frac{2(1-r)}{r(2-r)} \quad .$$

Solving the Zubov equation gives

$$V = - \ln r(2-r)$$

$V = \infty$ when $r = 2$ or 0. Hence the domain of attraction of the periodic solution $x^2 + y^2 = 1$ is $0 < r \leqslant 2$ or $x^2 + y^2 \leqslant 2$. We can also solve the Zubov partial differential equation corresponding to (6.4.1) by the characteristic method and obtain the same solution as above.

The auxiliary equations will then be

$$\frac{dx}{2x + y - 3x(x^2+y^2) + x(x^2+y^2)^2} = \frac{dy}{-x + 2y - 3y(x^2+y^2) + y(x^2+y^2)^2}$$

$$= \frac{dV}{-4(1-x^2-y^2)^2} .$$

Solving these equations give

$$\frac{xdx + ydy}{(x^2+y^2)(1-x^2-y^2)(2-x^2-y^2)} = \frac{dV}{-4(1-x^2-y^2)^2}$$

therefore $$V = -\ln(x^2+y^2)(2-x^2-y^2) .$$

Figure 6.4(i) shows the region of attraction of periodic solution of system (6.4.1) obtained by the numerical method using $\phi = 4(1-x^2-y^2)^2$. We see that for $R = 1.2$ the region is a circle of radius 1.2 and this is due to the fact that our circle of approximation is only $x^2 + y^2 = 1.2^2$, whereas the actual domain is $x^2 + y^2 \leqslant 2$. Also the domain is based on $S_R \cap D(f)$. However, for $R > \sqrt{2}$, we get a circle of radius $\sqrt{2}$ as shown in the figure.

We also apply the numerical method to system (6.4.2) derived from (6.4.1) using $\phi = 4(1-r)^2$. Equation (6.4.2) is a first order differential equation and its domain is presented in the motion space as shown in Figure 6.4(ii). The stability domain lies in the region $0 < r \leq 2$ and this is true since the stability domain of the periodic solution of system (6.4.1) is $x^2 + y^2 \leq 2$.

ii) Let us take another example

$$\begin{aligned} \dot{x} &= -y + x(1 - \sqrt{x^2+y^2})(2 - \sqrt{x^2+y^2}) \\ \dot{y} &= x + y(1 - \sqrt{x^2+y^2})(2 - \sqrt{x^2+y^2}) \quad . \end{aligned} \tag{6.4.3}$$

System (6.4.3) has periodic solutions $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Trajectories which lie on either side of the circle radius one but inside the circle of radius two will approach the circle $x^2 + y^2 = 1$ asymptotically. This circle is a stable limit cycle. Trajectories which lie interior or exterior to the circle of radius two will spiral away from the circle $x^2 + y^2 = 4$ and this circle is the unstable limit cycle. The bounded domain of attraction is $x^2 + y^2 \leq 4$. Figure 6.4(iii) shows the region of attraction of the periodic solution of system (6.4.3) obtained by using $\phi = 2(1 - \sqrt{x^2+y^2})^2$ with $R = 0.9$ and any $R > 2$. Notice that even for $R > 2$, we still get a domain defined by a circle radius two which is the actual domain.

By reversing the time of system (6.4.3) one will expect the origin to be a stable singular point and with $\phi = x^2 + y^2$ the stability boundary curve will be $x^2 + y^2 = 1$. This curve is in fact the stable limit cycle of system (6.4.3). The limit cycle can thus be approximated by taking the negative time of the system.

iii) Let us study a system of equations which has periodic solutions other than circular orbits. The system

$$\begin{aligned}
\dot{x} &= 2x + y - 3x(2x^2+y^2) + x(2x^2+y^2)^2 \\
\dot{y} &= -2x + 2y - 3y(2x^2+y^2) + y(2x^2+y^2)^2
\end{aligned} \tag{6.4.4}$$

has periodic solutions in the form of ellipses, viz.

$$\frac{x^2}{0.5} + y^2 = 1 \quad \text{and} \quad x^2 + \frac{y^2}{2} = 1.$$

The ellipse $\frac{x^2}{0.5} + y^2 = 1$ is the stable limit cycle and the other one is unstable.

One of the periodic solutions of system (6.4.4) has a finite domain of attraction and all trajectories within this domain will tend to the stable limit cycle as t tends to infinity. Figure 6.4(iv) shows the domain obtained by the numerical method where circles of radii 1.2, 1.3 and 1.6 are chosen as the region of approximation. $\phi = 4(1 - (2x^2+y^2))^2$ is used. Since the domain is elliptical in shape, it is therefore appropriate to estimate our region as ellipses rather than circles. So in polar coordinates $x = R\cos\theta$ and $y = (R+k)\sin\theta$ where k is constant. Use of ellipses will save computation time and the boundary is achieved much more quickly. Figure 6.4(v) shows the domain computed by using various sizes of ellipse. So depending on what shape the domain of attraction is, we can choose the forms of the region of approximation for computing the boundary. Note that if any trajectory begins on the stability boundary $2x^2 + y^2 = 2$ of a stable periodic solution, then it remains on it with increasing or decreasing time.

6.5 EXAMPLES ON APPROXIMATE PERIODIC SOLUTIONS.

In this section we shall try to find the domain of attraction of a periodic solution using an approximate periodic solution.

i) Consider the Van der Pol equation

$$\ddot{x} + \varepsilon(x^2-1)\,\dot{x} + x = 0 \tag{6.5.1}$$

with $\varepsilon > 0$.

This system has the origin as an unstable focal point and has a periodic solution (limit cycle) in the form of closed curve in the phase plane containing the origin. The phase trajectories emerging from the interior of the limit cycle approach the limit cycle and trajectories exterior to the limit cycle will also approach the limit cycle. Solutions of (6.5.1) will tend asymptotically to the periodic solution and hence this solution is asymptotically stable. By using the harmonic balance method the approximate periodic solution of (6.5.1) is $x^2 + y^2 = 4$. Using this solution as a guide for choosing ϕ, the relation $\phi = (4-x^2-y^2)^2$ is employed in the numerical method. Figure 6.5(i) shows the region of attraction of the periodic solution obtained by using the approximate periodic solution with $\varepsilon = 0.5$ and $R = 1.4$, 3.4 and 4.4. Since the domain of attraction of the periodic solution is the whole space it is obvious that from the characteristic of the method the estimate of the domain will be a larger circle if R is increased.

ii) Consider a periodic solution which has a bounded domain of attraction. Let us take system (6.4.3) which has periodic solutions $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Here although we approximate the solutions by the harmonic balance method we still get the exact periodic solutions. For the sake of utilising approximate solutions simply consider an approximation $x^2 + y^2 = 0.9^2$ which is near enough the periodic solution $x^2 + y^2 = 1$. Then using $\phi = 2(0.9 - \sqrt{x^2+y^2})^2$ in the algorithm we obtain the region of asymptotic stability of the periodic solution as shown in Figure 6.5(ii). The smaller circle is obtained when R = 1.6 is used while the bigger one is the domain of attraction of the periodic solution where values of R greater than two are used.

Next we take an approximate solution $1.1x^2+y^2 = 1$. With $\phi = 2(1-1.1x^2-y^2)^2$, the domain of attraction of the periodic solution obtained is shown in Figure 6.5(iii). Ellipses $\frac{x^2}{1.1^2} + \frac{y^2}{1.5^2} = 1$ and $\frac{x^2}{2.2^2} + \frac{y^2}{2.6^2} = 1$ are being used as the region of approximation. The former gives the inner ellipse while the latter gives the domain of attraction.

iii) Let us now consider system (6.4.4) where the periodic solutions are $2x^2+y^2 = 1$ and $x^2 + \frac{y^2}{2} = 1$. The solution $2x^2 + y^2 = 1$ is a stable limit cycle. Again the harmonic balance method will give the exact solution of the system. Consider an approximate solution $1.9x^2 + y^2 = 1$. Take $\phi = 4(1-1.9x^2-y^2)^2$. The domain of attraction of the periodic solution is shown in Figure 6.5(iv) where the above approximate solution is used. The regions of approximation used are ellipses $\frac{x^2}{1} + \frac{y^2}{1.4^2} = 1$ and $\frac{x^2}{1.2^2} + \frac{y^2}{1.6^2} = 1$. The stability boundary of the stable limit cycle is $x^2 + \frac{y^2}{2} = 1$.

From the examples given above, although an approximate solution is used we still obtain the exact stability boundary of the periodic solution. Thus without the knowledge of the actual periodic solution, the domain of attraction can be found by using the algorithm of Section 1.7. In many practical problems the exact periodic solutions are often difficult to obtain. Many attempts have been made to solve these problems through various approximation methods such as harmonic balance, averaging method and perturbation method. So knowing the approximate solution, the problem of determining the domain of attraction of a periodic solution of some systems can be solved as shown in our examples. These examinations will indicate that other engineering problems which have periodic solutions and bounded domains of attraction may also be solved without much difficulty.

6.5 CONCLUSION.

The domain of attraction of a periodic solution of an engineering problem or control system can now be estimated by this numerical technique. The limit cycle may be drawn by reversing the time of the system. We may also conclude whether the limit cycles are stable or unstable and determine the domain for the stable one. The test region may also be allowed to vary and this depends on the shape of the domain of the particular system. Although the examples treated are simple in nature, they contain the characteristic behaviour of the trajectories of nonlinear systems in the presence of periodic solutions and have a feature where the periodic solutions have bounded domains of attraction.

Often in many engineering systems the actual periodic solution is not always available, so finding the domain of attraction is a difficult problem. Clearly this difficulty is eased since the algorithm allows the use of approximate solution for determining the domains of attraction as discussed in Section 6.5. The definitions and theorems of stability of periodic solution are equivalent to the definitions and theorems of stability of the trivial solution, $x = 0$ and these make it possible to use the algorithm mentioned in Section 1.7.

$$\dot{x} = 2x+y-3x(x^2+y^2) + x(x^2+y^2)^2$$

$$\dot{y} = 2y-x-3y(x^2+y^2) + y(x^2+y^2)^2$$

$$\phi = 4(1-x^2-y^2)^2$$

∇ R = 1.2

Δ all R > $\sqrt{2}$

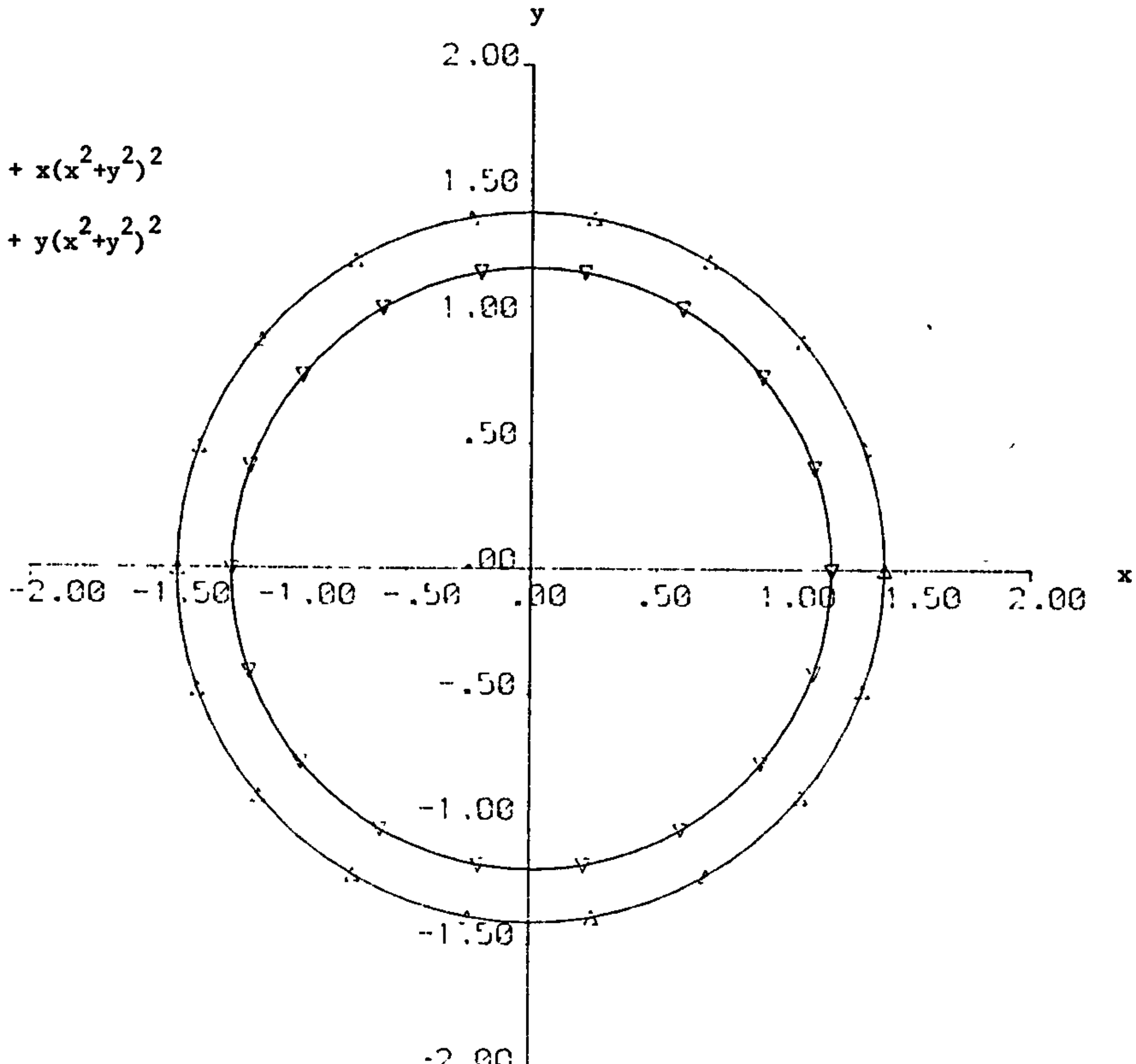
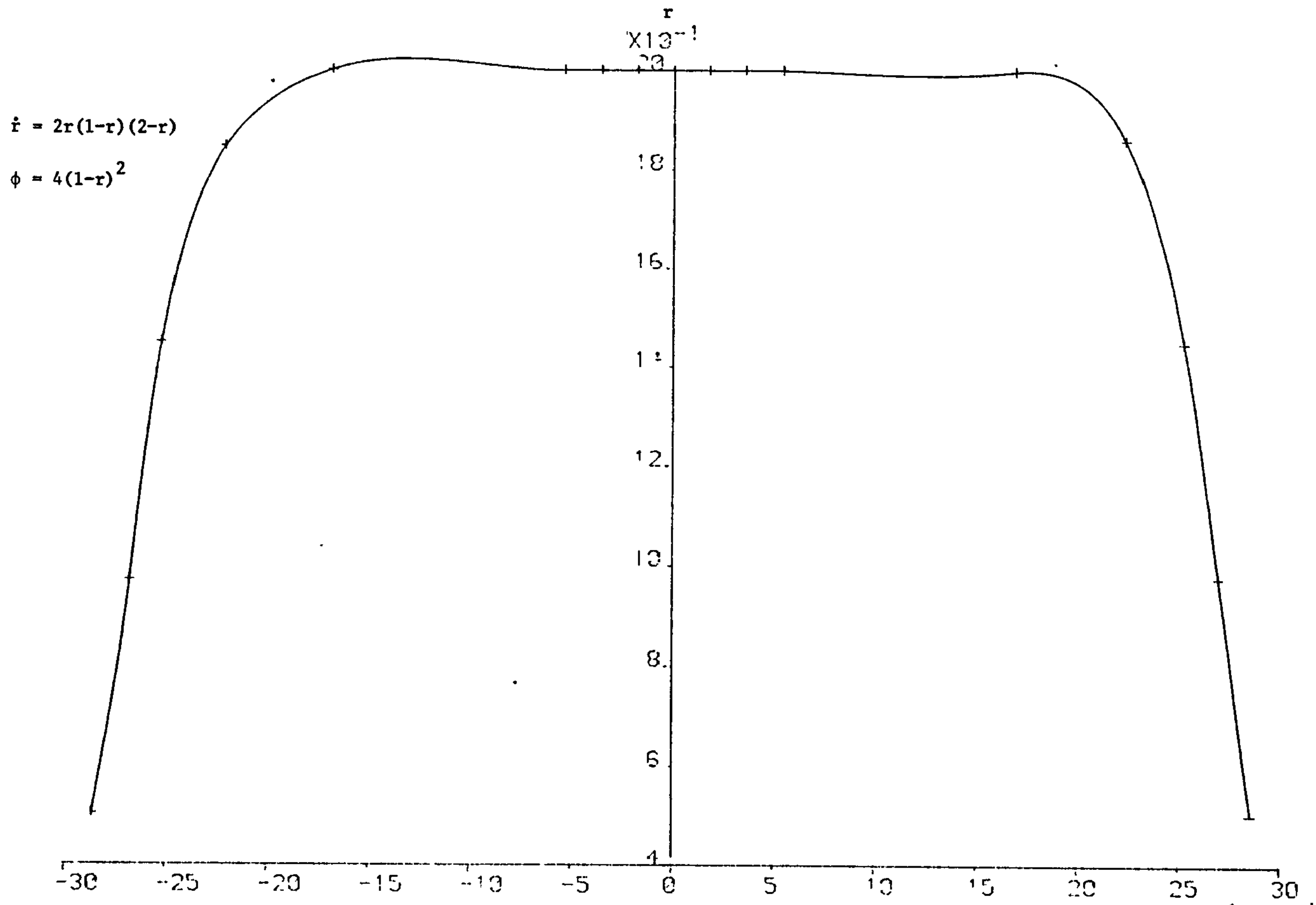


Fig. 6.4(i)



$\dot{r} = 2r(1-r)(2-r)$
$\phi = 4(1-r)^2$
r
X10^-1
20
18
16
14
12
10
8
6
4
-30
-25
-20
-15
-10
-5
0
5
10
15
20
25
30
X10^-1
t

Fig. 6.4(ii)

$\dot{x} = -y+x(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$

$\dot{y} = x+y(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$

$\phi = 2(1-\sqrt{x^2+y^2})^2$

∇ R = 0.9

Δ all R > 2

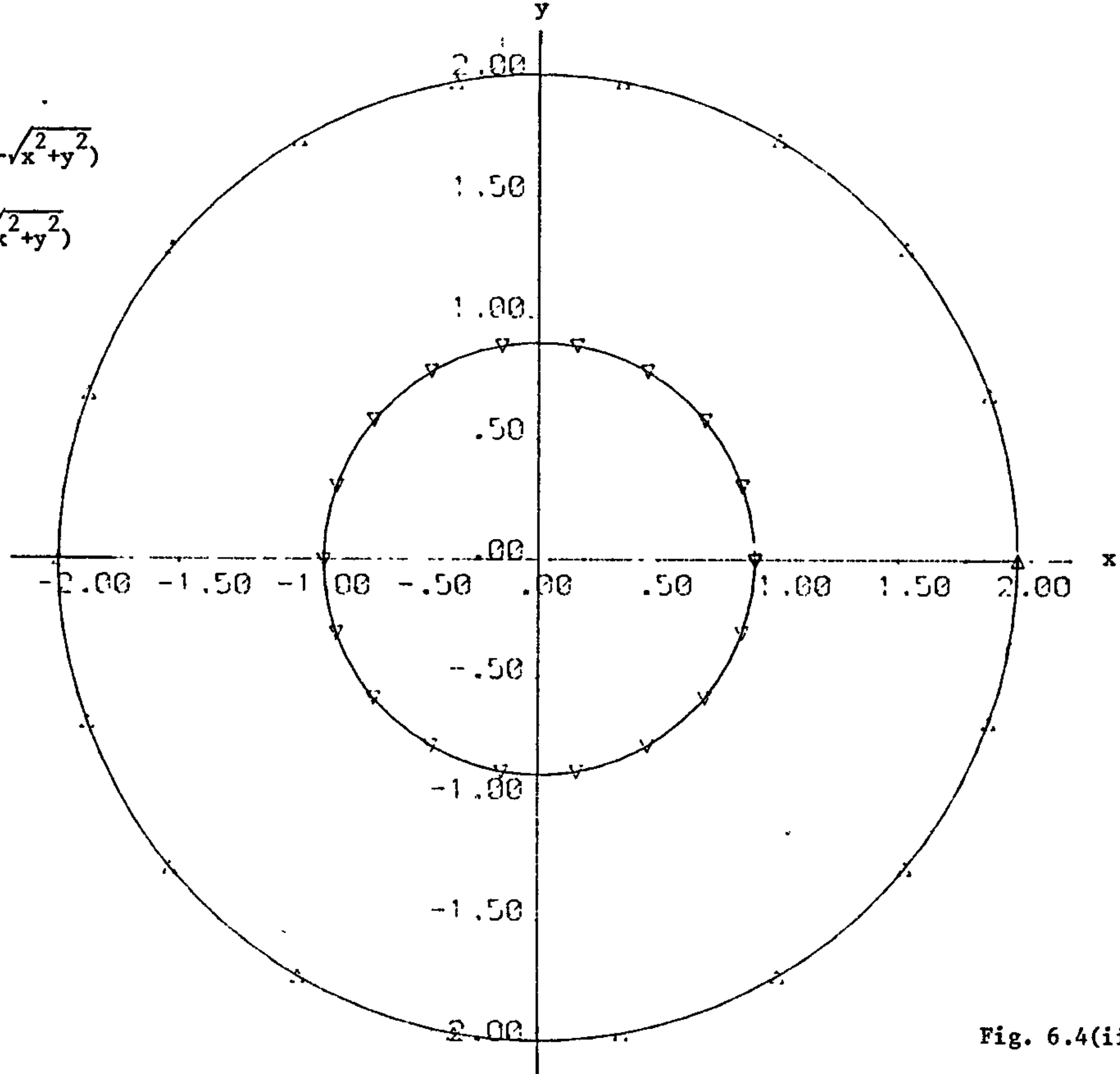


Fig. 6.4(iii)

$$\dot{x} = 2x+y-3x(2x^2+y^2) + x(2x^2+y^2)^2$$

$$\dot{y} = -2x+2y-3y(2x^2+y^2) + y(2x^2+y^2)^2$$

$$\phi = 4(1-(2x^2+y^2))^2$$

∇ R = 1.2

$+$ R = 1.3

Δ R = 1.6

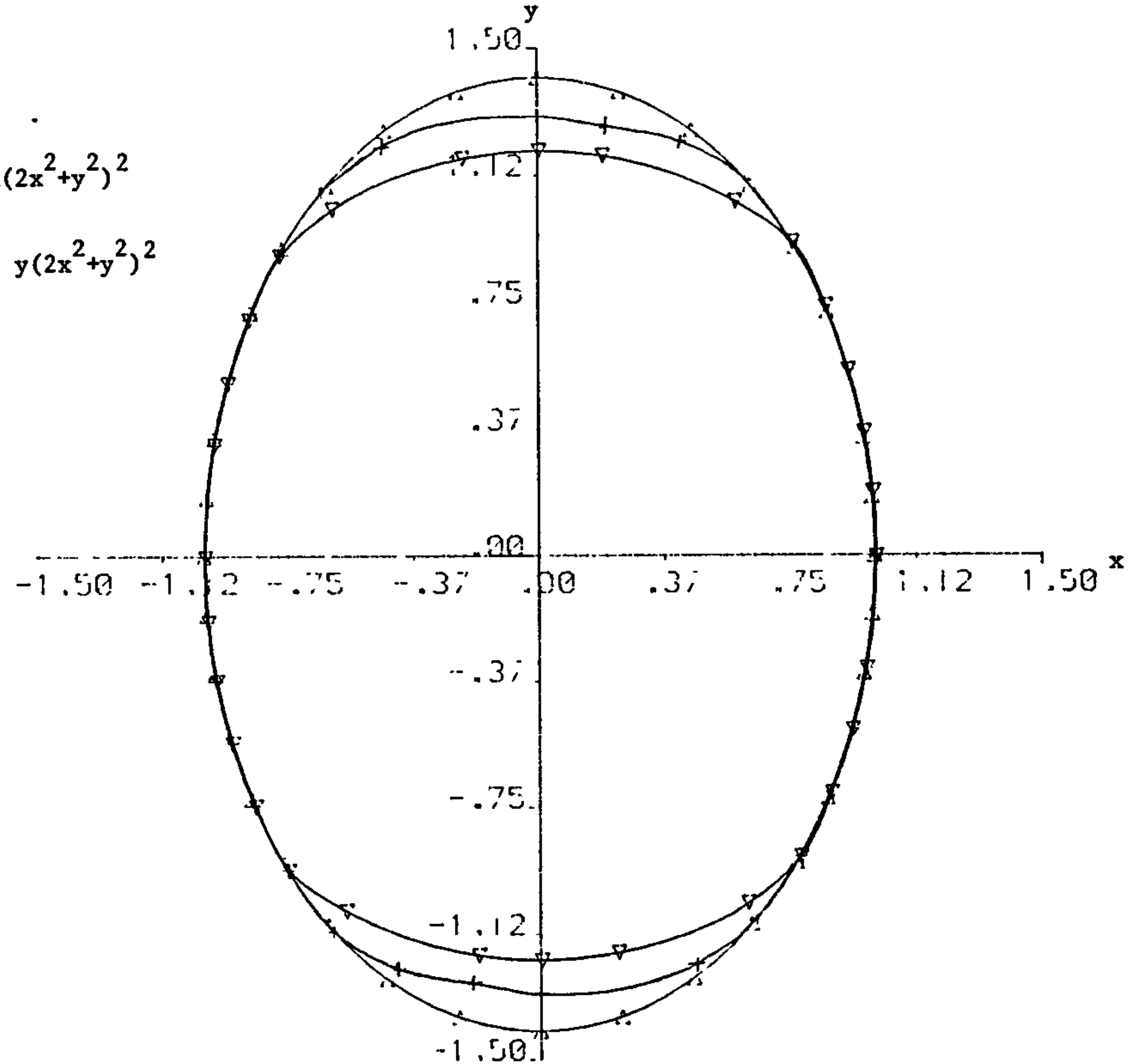


Fig. 6.4(iv)

$$\dot{x} = 2x+y-3x(2x^2+y^2) + x(2x^2+y^2)^2$$

$$\dot{y} = -2x+2y-3y(2x^2+y^2)+y(2x^2+y^2)^2$$

$$\phi = 4(1-(2x^2+y^2))^2$$

<u>Region of approximation</u>

$\nabla \quad \dfrac{x^2}{.8^2} + \dfrac{y^2}{1.2^2} = 1$

$+ \quad \dfrac{x^2}{1} + \dfrac{y^2}{1.4^2} = 1$

$\Delta \quad \dfrac{x^2}{1.2^2} + \dfrac{y^2}{1.6^2} = 1$

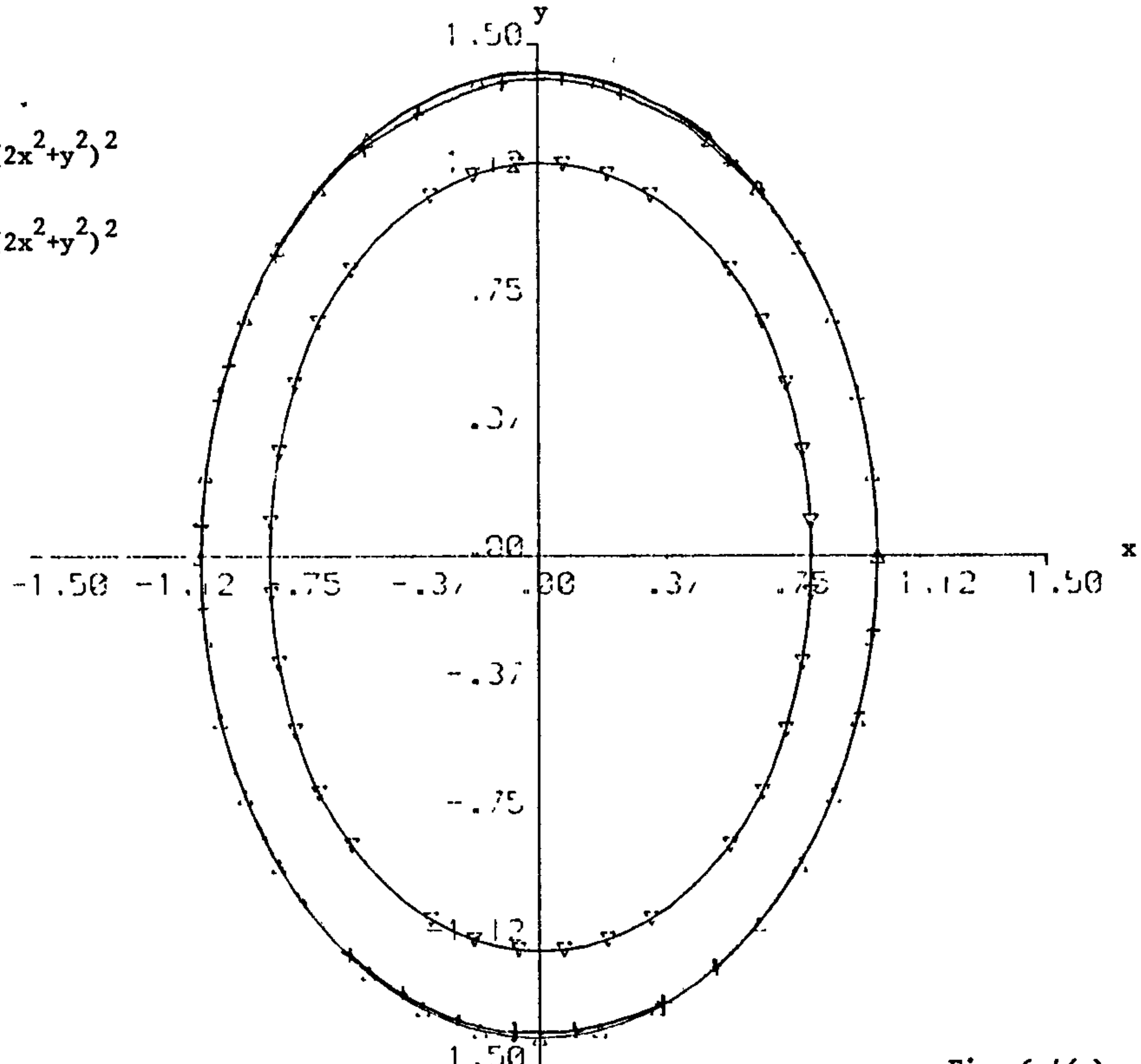


Fig. 6.4(v)

$\ddot{x} + \varepsilon(x^2-1)\,\dot{x} + x = 0$

$\phi = (4-x^2-y^2)^2$

$R = 1.4,\ 3.4,\ 4.4$

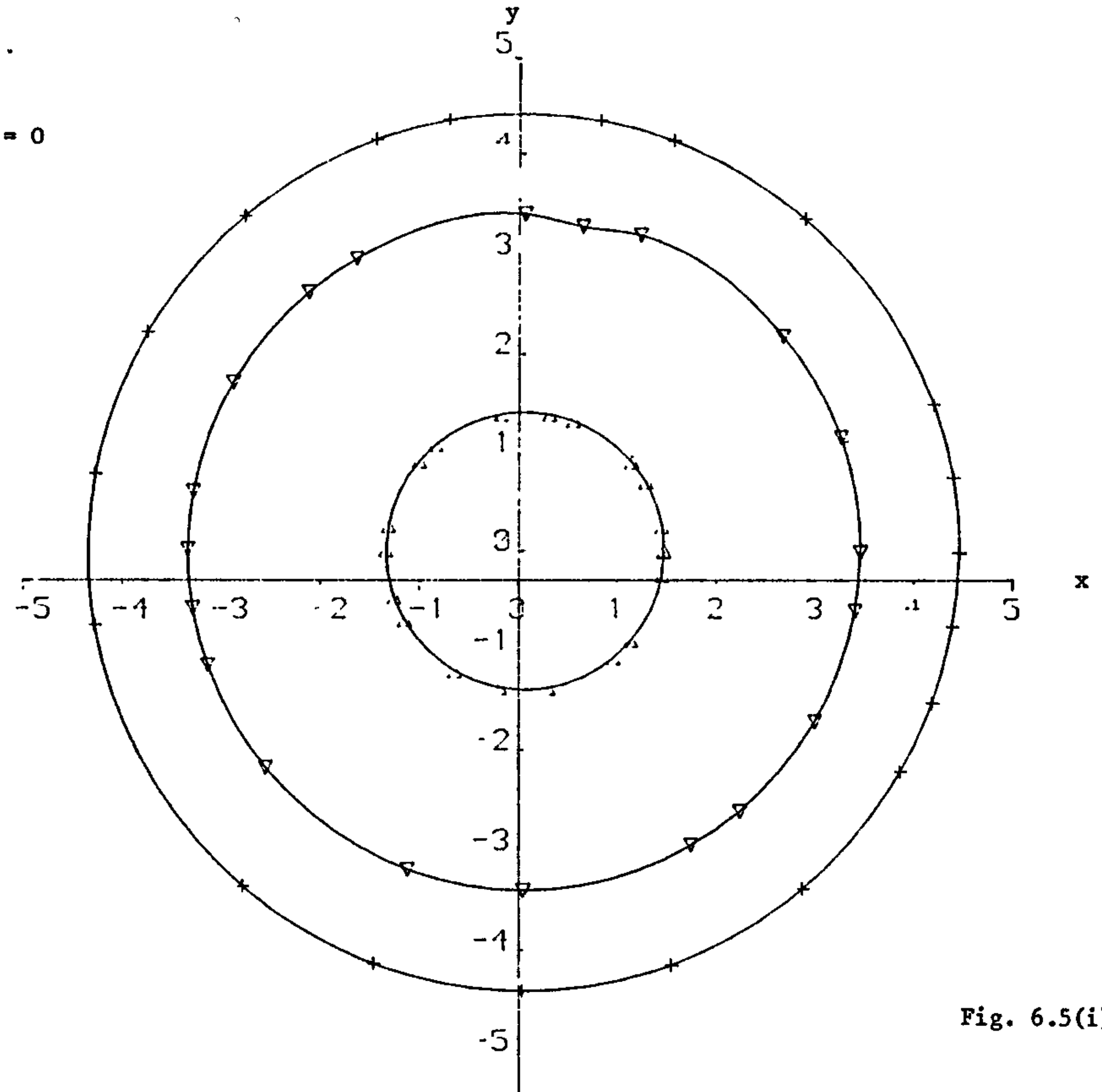


Fig. 6.5(i)

$$\dot{x} = -y + x(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$$

$$\dot{y} = x + y(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$$

$$\phi = 2(0.9 - \sqrt{x^2+y^2})^2$$

∇ R = 1.6

Δ all R > 2

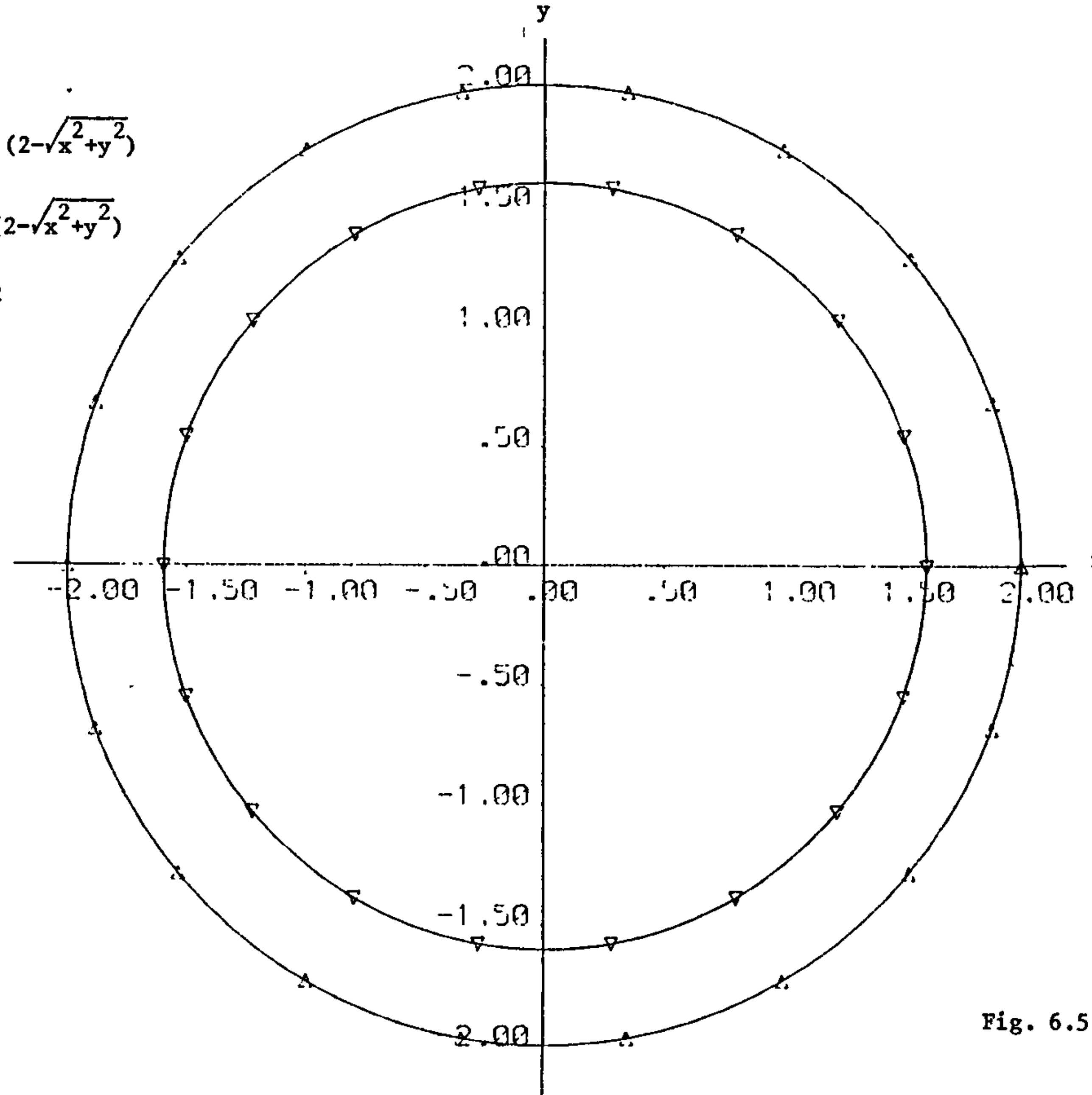


Fig. 6.5(ii)

$$\dot{x} = -y + x(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$$

$$\dot{y} = x + y(1-\sqrt{x^2+y^2})(2-\sqrt{x^2+y^2})$$

$$\phi = 2(1 - 1.1x^2-y^2)^2$$

<u>Region of approximations</u>

$\Delta \quad \dfrac{x^2}{1.1^2} + \dfrac{y^2}{1.5^2} = 1$

$\nabla \quad \dfrac{x^2}{2.2^2} + \dfrac{y^2}{2.6^2} = 1$

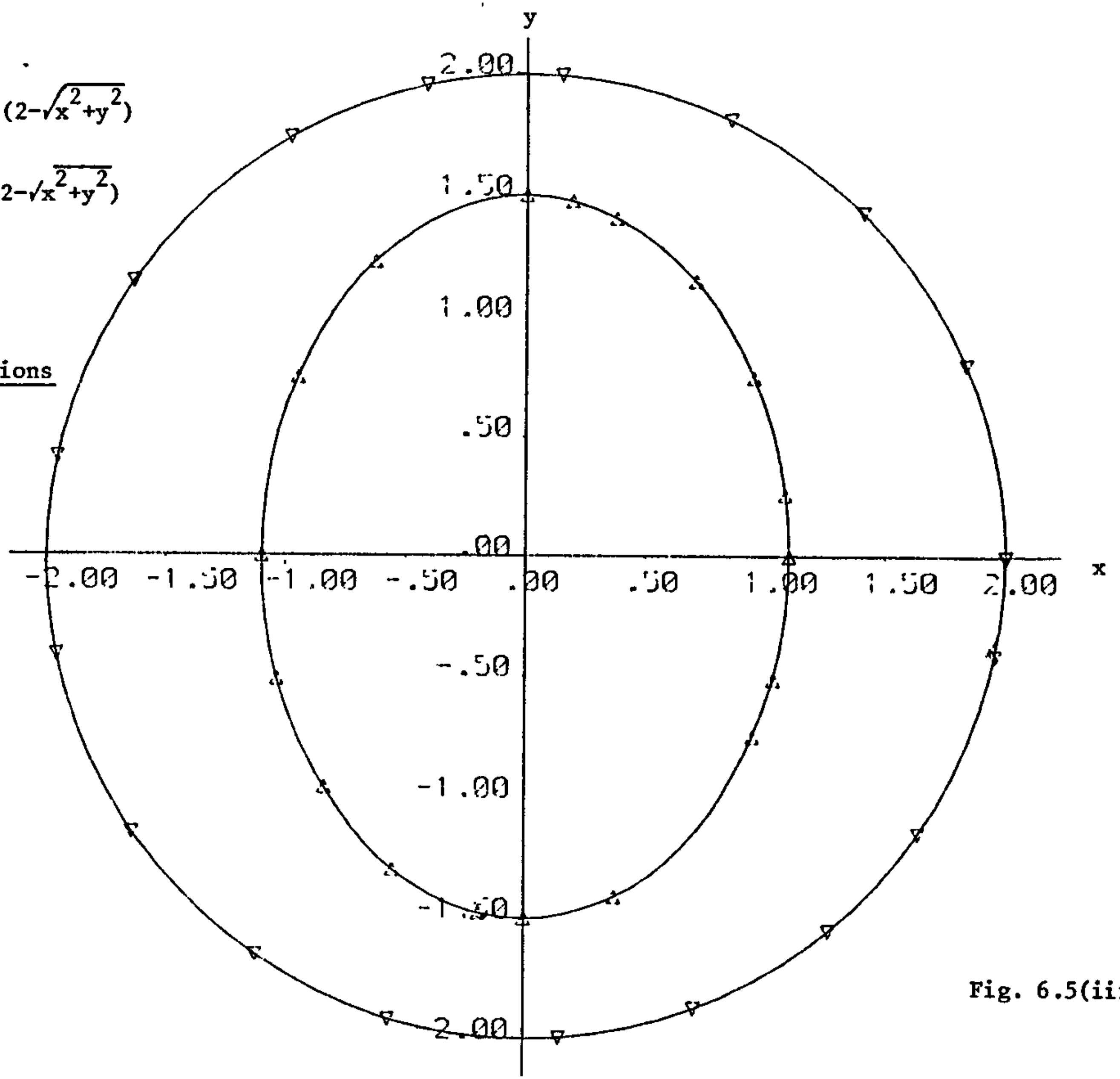


Fig. 6.5(iii)

$$\dot{x} = 2x + y - 3x(2x^2+y^2) + x(2x^2+y^2)^2$$

$$\dot{y} = -2x + 2y - 3y(2x^2+y^2) + y(2x^2+y^2)^2$$

$$\phi = 4(1 - 1.9x^2-y^2)^2$$

Region of approximation

$\Delta \quad x^2 + \dfrac{y^2}{1.4^2} = 1$

$\nabla \quad \dfrac{x^2}{1.2^2} + \dfrac{y^2}{1.6^2} = 1$

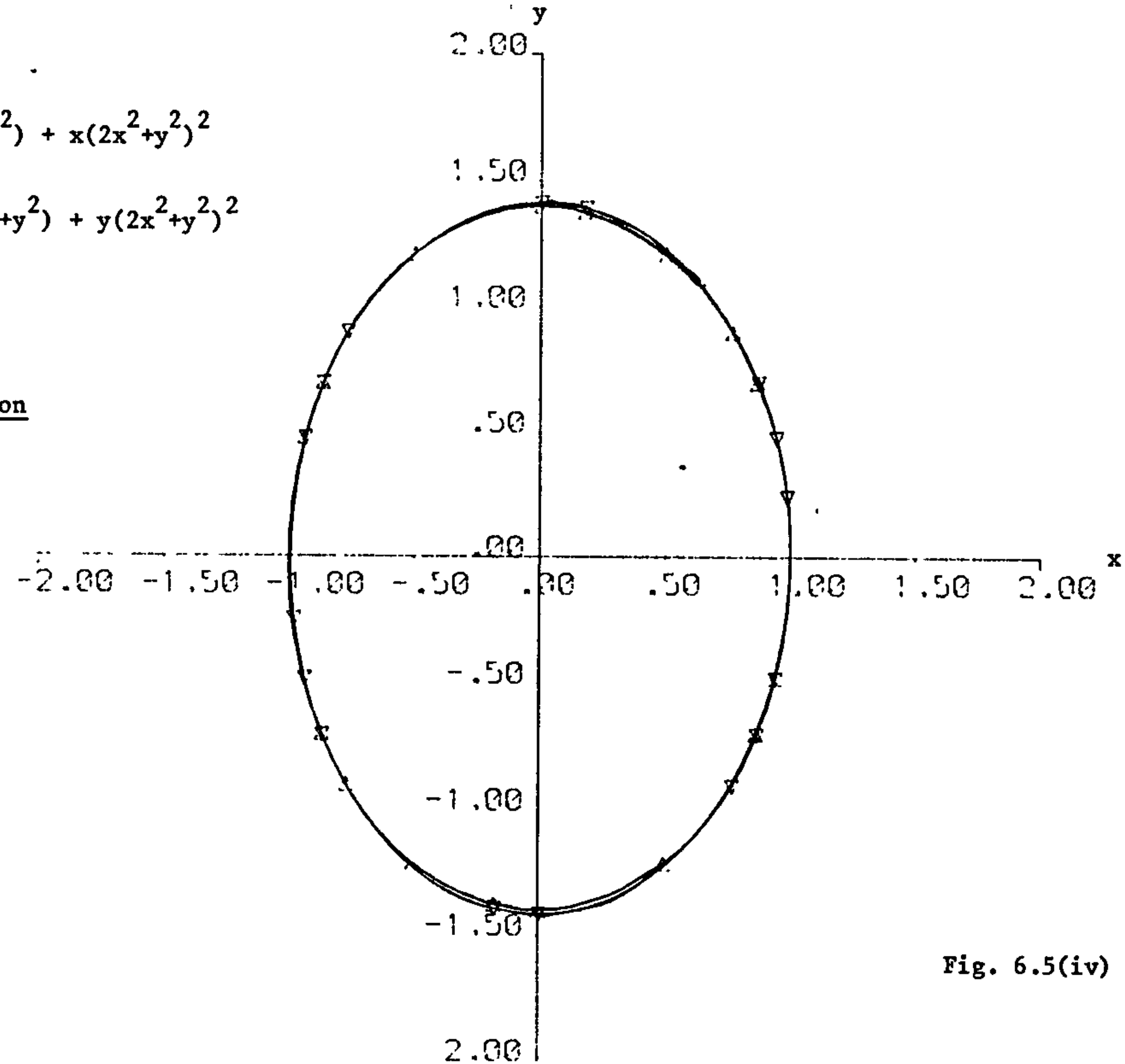


Fig. 6.5(iv)

CHAPTER VII

Global Optimization

7.1 INTRODUCTION.

Techniques for solving global optimization problems are still rather primitive and there is a considerable need for further research into this area. In many practical situations, we have optimization problems in which the objective function is not convex and possesses multiple minima. It is the identification of these minima that has stimulated researchers to devise techniques that may locate these minima automatically. In many techniques of global optimization we usually encounter with two main problems, viz:

i) locating the minima

ii) determining the domains of attraction of the minima.

The minimization algorithms [55,56,57] for locating a local minimum are well known and have been discussed extensively. Only a few attempts have been made towards solving the global minimization problem.

Trecanni et al. [58] propose an algorithm which utilizes the concept of region of attraction of the minimum by a Liapunov function approach. The objective of their method is to identify the saddle point which lies on the boundary of the region of attraction of a given minimum by successive approximations of the region of attraction. C.R. Corles [59] goes a step further by implementing an algorithm for global optimization in two dimensions based on the above approach in a different way. He locates a minimum and constructs a series of ellipses around it until a saddle point is located and continues to search for another minimum and then a further series of ellipses is constructed; repeating the process until all the minima have been identified. Branin [60] presents a trajectory method which is based on the integration of a series of differential equations whose trajectories lead to the stationary points or critical points of the objective function. However this technique does not determine whether all the stationary points have been found.

In this chapter, an attempt is made towards detecting global minima by determining the domains of attraction of the local minima. The method is more of an interactive graphical approach which locates a minimum by any local minimization method and computes the domain of attraction of this minimum by the method described in Section 1.7 of Chapter one. Then taking any point outside this domain, locate the second minimum and determine its domain; repeating the process until all the minima and domains have been identified. We also look at the possibility of obtaining all the minima and domains automatically by imposing certain criteria while determining the domain of a minimum. That is, if the distance of a point on the boundary of the domain from the minimum is less than R defined by (1.7.1), then the direction of this point from the minimum and magnitude of R are taken to find a point which acts as the starting point for another minimum. The result is illustrated by testing an example given in Storey [61]. We conclude by illustrating the graphical approach to functions with a single variable and with two variables.

7.2 FUNDAMENTALS OF GLOBAL OPTIMIZATION.

Definition 7.2.1.

A point $\bar{\underline{x}} \in S$ is the global minimum point of a function $f(\underline{x})$

on a set S, if $\underline{x} \in S$ implies $f(\underline{x}) \geq f(\overline{\underline{x}})$.

Definition 7.2.2.

A point $\underline{x}'$ is the local minimum point of a function $f(\underline{x})$ if there exists a neighbourhood M around $\underline{x}'$ such that if $\underline{x} \in M$ then $f(\underline{x}) \geq f(\underline{x}')$.

Definition 7.2.3.

The region of attraction of a minimum is the set of all neighbouring points which if used as initial points for a minimization algorithm will cause the algorithm to converge to that particular minimum.

Consider a differential equation

$$\dot{\underline{x}} = - \underline{g}(\underline{x}) \tag{7.2.1}$$

where $$\underline{g}(x) = \text{grad } f(\underline{x}) \quad . \tag{7.2.2}$$

The solutions of system (7.2.1) are the orthogonal trajectories of the contours of the function $f(\underline{x})$. In stability theory, the equilibrium points are obtained by equating the right hand side of (7.2.1) to zero, which is equivalent to finding the critical points of the function $f(\underline{x})$ in differential calculus. So the critical points of the function $f(\underline{x})$ are the equilibrium points of system (7.2.1) and all minima of $f(\underline{x})$ become asymptotically stable points, all maxima correspond to unstable points and saddle points remain saddle points. The region of attraction of a minimum of the function $f(\underline{x})$ is therefore the domain of attraction of the stable equilibrium point of system (7.2.1). The boundary of the domain of a minimum must pass through at least one saddle point and this saddle point also belongs to the boundary of the domain of another minimum if another minimum exists. Note that if the function has maxima then the domain of attraction is determined by reversing the time of system (7.2.1). Treccani et al. [58] use the local minimum as a starting point to identify a saddle point from which another minimum may be found if it exists. They consider the Liapunov function

$$V(\underline{x}) = \underline{x}^T H \underline{x} \tag{7.2.3}$$

where H is initially chosen as the Hessian of the function at this local minimum, so that its derivative corresponding to system (7.2.1),

$$\dot{V}(\underline{x}) = 2\underline{x}^T H \dot{\underline{x}} = - 2\underline{x}^T H \underline{g} \tag{7.2.4}$$

is negative whenever $\underline{x}^T H \underline{g} > 0$.

$\underline{x}^T H \underline{g}$ will be positive until either $g = 0$ (a stationary point is found) or g is tangential to the surface V = constant. Their algorithm is divided into four main steps.

<u>Step 1.</u>

A local minimum is first located by a local minimization algorithm and a linear transformation is performed so that this minimum is the new origin. The external point penalty function method is used to minimise

$$V_o = \underline{x}^T H \underline{x}$$

subject to the constraints

i) $\underline{x}^T \underline{x} - t \geqslant 0$

ii) $\dot{V}_o(\underline{x}) \geqslant 0$.

The first constraint removes the origin which would otherwise be the solution to the problem. This method determines the smallest value of $V_o(\underline{x})$ on which a point $\underline{x}_c$ exists with $\dot{V}_o(\underline{x}_c) = 0$. This point therefore limits the region in which $V_o(\underline{x})$ is a Liapunov function which gives information regarding asymptotic stability of the equilibrium point at the origin for system (7.2.1) represented in the transformed variables. The external point penalty function method does not guarantee that a minimum is found with $\dot{V}_o(\underline{x}) = 0$. So a sequence of minimization is performed with an additional constraint

$$B - V_o(\underline{x}) - e \geq 0$$

where B is the value of $V_o(\underline{x})$ found in the previous search, until an acceptable solution is obtained.

Step 2.

The second step is to identify the largest level set of $f(\underline{x})$ contained in $V_o(\underline{x}_c)$. The problem is therefore

$$\text{minimise} \quad f(\underline{x})$$
$$\text{such that} \quad V_o(\underline{x}) = V_o(\underline{x}_c) \quad .$$

Let the solution be $\underline{x} = \underline{z}_o$. If $\underline{g}(\underline{z}_o) = 0$ then the saddle point has been identified.

Step 3.

The region of attraction is extended by constructing another quadratic function

$$V_i(\underline{x}) = \underline{x}^T H_i \underline{x} \quad .$$

H_i is defined in such a way that one of the axes of the ellipsoid defined above lies on a straight line joining the origin to the point $\underline{z}_{i-1}$ found in the previous step. The minimization performed is

Minimize $V_i(\underline{x})$

subject to the constraints

i) $\dot{V}_i(\underline{x}) \geq 0$

ii) $\underline{x}^T \underline{z}_{i-1} \geq 0$, this restricts the minimization to the half of the quadratic surface on the same side as $\underline{z}_{i-1}$

iii) a constraint that restricts the minimization to points external to the surfaces $V_j(\underline{x}_c)$, $j = 1,\ldots,i-1$

iv) a constraint that prevents the plane formed by ii) cutting the earlier surfaces.

Step 4.

Repeat step 2 by considering the surface formed by the boundaries of $V_j(\underline{x}_c)$, $j = 1,\ldots,i$. If $\underline{z}_i$ is the solution and $g(\underline{z}_i) = 0$ then the required saddle point, $\underline{z}_i$ has been identified, otherwise return to step 3.

The following theoretical results are produced in relation to their algorithm.

Theorem 7.2.1.

The local minimization outlined in step 1 will converge after a finite number of steps.

Theorem 7.2.2.

The level sets identified in steps 2 and 4 are contained in the region of asymptotic stability of the minimum.

Theorem 7.2.3.

If the sequence z_i is finite it terminates at a saddle point. If it is infinite and not convergent, then there does not exist any saddle point and if some conditions on the sequence of matrices H_i are maintained then the limit point of the sequence will be a saddle point.

C.R. Corles, however, has shown that for certain functions it is theoretically impossible to construct a sequence H_i which satisfies the conditions on the eigenvalues of H_i. The detailed description of the method of Corles can be obtained in his paper [59].

7.3 INTERACTIVE GRAPHICAL AID TECHNIQUE.

This technique attempts to solve global optimization problems by initially locating a local minimum from an arbitrary point and then computing its domain of attraction. A point outside the computed domain is taken as the starting point for searching for the next minimum and domain and continues until all the minima and domains have been identified. We shall consider the differential equation defined by equations (7.2.1) and (7.2.2). The global optimization problem is solved through the following steps.

i) An initial estimate is taken as a starting point for searching for the local minimum of $f(\underline{x})$ by any minimization algorithm. Newton's method which is given in Appendix C is used in our examples. This may not be the best method for locating a local minimum. In this graphical aid technique, however, we do not require a critical analysis of the minimization method. So the use of the method of Newton is sufficient for our purpose but other minimization methods can also be employed.

ii) Perform a linear transformation so that the minimum is the new origin of the transformed equation of (7.2.1). Compute the domain of attraction of the new origin by the numerical method described in Section 1.7. For the purpose of our drawings transfer the origin back to the minimum. Points on the boundary of the domain of attraction are also transferred to the old axis. Trace the boundary of the domain of this minimum.

iii) Take any point outside the computed domain, search for the next minimum, if it exists. Follow step (ii).

iv) Repeat step (iii) until all the minima and domains of attraction have been identified.

This approach is easy to apply since the set of differential equations can be formed from any given function. Hence we are able to solve the global optimization problem by computing the domain of attraction of the stable equilibrium point of the differential equation which is in fact the domain of attraction of the minimum of the function.

Since our interest is only to locate the minima and their domains of attraction, no attempt is made towards analysing the rate of convergence to its local minimum as this has been discussed in many optimization references.

7.4 FUNCTIONS OF TWO VARIABLES.

Let us first implement the method for two dimensional problems.

i) Consider the function

$$f(x,y) = x^3 + \frac{y^2}{6} - xy \quad . \tag{7.4.1}$$

This function has a minimum at (1,3) and a saddle point at (0,0). The contours of this function are shown in Figure 7.4(i). The set of differential equations formed by using (7.2.1) is

$$\begin{aligned} \dot{x} &= - 3x^2 + y \\ \dot{y} &= x - \frac{y}{3} \quad . \end{aligned} \tag{7.4.2}$$

Figure 7.4(ii) shows the domain of attraction of the stable critical point of (7.4.2) which is the domain of the minimum of (7.4.1). Notice that the saddle point (0,0) lies on the stability boundary.

ii) Consider the problem given in Storey [61] which has two minima and one saddle point. The function

$$f(x,y) = (x^2+y^2-1)^2 + (x+y-1)^2 \tag{7.4.3}$$

has minima at (1,0) and (0,1) and a saddle point at $\left[\left(\frac{1}{4}\right)^{1/3}, \left(\frac{1}{4}\right)^{1/3}\right]$. The contours of (7.4.3) are shown in Figure 7.4(iii).

The differential system is

$$\begin{aligned} \dot{x} &= - 4x(x^2+y^2-1) - 2(x+y-1) \\ \dot{y} &= - 4y(x^2+y^2-1) - 2(x+y-1) \quad . \end{aligned} \tag{7.4.4}$$

The domains of attraction of the minima are shown in

Figure 7.4(iv) where $\phi(x,y) = 2(x^2+y^2)$ and $R = 1.0$ are used. Points starting in region A will converge to (0,1) and points in the region B will converge to (1,0) in the minimization procedure. We do not expect the minimization algorithms to converge to the exact solution, but to converge to some small neighbourhood of the minimum. A test has been made by taking the minima at (0.9, 0.0) and (0.1, 1.0) and the domain is shown in Figure 7.4(v). Although the true minimum is not used, the domain of a point in the small neighbourhood of the minimum can still be computed. The region of stability of the minima is found to be symmetric about the line $x = y$. The typical effect of using different values of R is shown in Figure 7.4(ix).

iii) Consider the function

$$f(x,y) = 2(x-y)^2 - x^4 - y^4 \tag{7.4.5}$$

which has maxima at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ and a saddle point at (0,0). The contours of (7.4.5) are shown in Figure 7.4(vi). The differential system is formed by taking (7.2.1) with time reversed so that the equilibrium points of the system are asymptotically stable, so

$$\begin{aligned} \dot{x} &= 4(x-y) - 4x^3 \\ \dot{y} &= -4(x-y) - 4y^3 \ . \end{aligned} \tag{7.4.6}$$

Clearly the critical points of (7.4.6) are $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ and (0,0). Taking $\phi = 4(x^2+y^2)$ and $R = 2.9$ the domains of the maxima are plotted as shown in Figure 7.4(vii). The line $x = y$ forms the boundary of the two domains of attraction of the maxima.

iv) Next we study an example of a function which has four minima. The function

$$f(x,y) = (9x^2+4y^2+12y-16x-16xy-1)^2 + (x^2+2y-2x-2xy+1)^2 \qquad (7.4.7)$$

has roots at (-1,-1), (1,-1), (1,2) and (3,1).

The differential equations arising from (7.2.1) are

$$\dot{x} = -4\Big[(9x-8y-8)(9x^2+4y^2+12y-16x-16xy-1)$$
$$+ (x-y-1)(x^2+2y-2x-2xy+1)\Big]$$

$$\dot{y} = -4\Big[(4y-8x+6)(9x^2+4y^2+12y-16x-16xy-1)$$
$$+ (1-x)(x^2+2y-2x-2xy+1)\Big] \quad . \qquad (7.4.8)$$

Figure 7.4(viii) shows the domains of attraction of all the four roots by using $\phi = 4(x^2+y^2)$. For functions with four minima, the computation of the domains is not accurate. It is very difficult to compute the boundaries which pass through the saddle points of the function since the trajectories are quite complicated. The figure shows roughly on which line the saddle points are located. Since this is a graphical aid technqiue, points well outside the computed domain of a known minimum may be taken as an initial guess for locating another minimum. The contours may become narrower as they approach the saddle points.

7.5 FUNCTIONS OF A SINGLE VARIABLE.

Let us now apply the technique to a one-dimensional optimization problem.

i) The function

$$f(x) = - 2x^2 + 2x^3 - \frac{x^4}{2} \tag{7.5.1}$$

has a minimum at x = 1 and maxima at x = 0, 2.

The differential equation

$$\dot{x} = 2x(1-x)(2-x) \tag{7.5.2}$$

has a stable equilibrium point at x = 1.

The lines x = 0 and x = 2 form the boundary of the domain of attraction of the minimum, x = 1 as shown in Figure 7.5(i). Different sizes of ellipses are used as our regions of approximations. The maxima of (7.5.1) become unstable roots for system (7.5.2). Any points inside the domain of attraction of the minimum will converge to x = 1 in the minimization algorithm.

ii) Consider the function

$$f(x) = - \frac{x^2}{2} + \frac{x^4}{4} \tag{7.5.3}$$

with minima at x = 1 and x = - 1.

The scalar equation is

$$\dot{x} = x(1-x^2) \quad . \tag{7.5.4}$$

Figure 7.5(ii) shows the domains of attraction of x = 1 and x = - 1 obtained by using the graphical approach. Any point starting above the positive x-axis will locate the minimum x = 1 and likewise points below the x-axis will converge to x = - 1 when used in any minimization algorithm.

iii) Take the function

$$f(x) = -2x^2 + \frac{5x^4}{4} - \frac{x^5}{5} \qquad (7.5.5)$$

with minima at $x = 1$ and $x = -1$ and maxima at $x = 0, 2, -2$.

Forming a scalar equation gives

$$\dot{x} = x(1-x^2)(4-x^2) \quad . \qquad (7.5.6)$$

The origin of (7.5.6) is an unstable equilibrium point. The domains of attraction of the minima are shown in Figure 7.5(iii). Any initial point taken in the interval $0 < x < 2$ will converge to $x = 1$ and points in the interval $-2 < x < 0$ will converge to $x = -1$ when used in the graphical aid technique. The domain of attraction of the maxima $x = 0, 2, -2$ obtained by reversing the time of (7.5.6) is given in Figure 7.5(iv). If the function is maximized in the region $-1 < x < 1$ then $x = 0$ will be located. If maximization starts at $x > 1$ the maximum $x = 2$ is located while for $x < -1$ the maximum $x = -2$ can be identified.

iv) Consider a polynomial function which has a minimum and maximum close to each other. Such a function is

$$f(x) = -\frac{x^5}{5} + \frac{9x^4}{4} - \frac{29.08}{3}x^3 + 19.76x^2 - 18.82x \qquad (7.5.7)$$

and the minima are at $x = 1.107$ and $x = 2.905$ and the maxima at $x = 1.888$ and $x = 3.1$. Figure 7.5(v) shows the graph of this function.

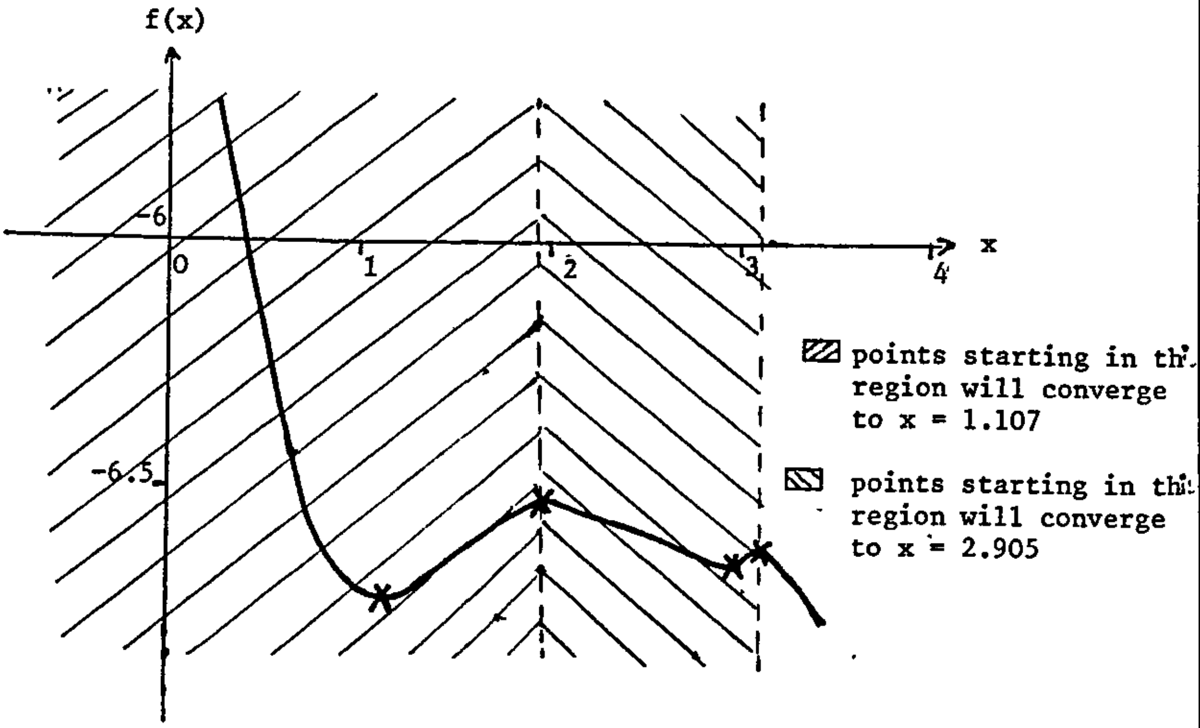


Figure 7.5(v)

Forming the differential equation gives

$$\dot{x} = x^4 - 9x^3 + 29.08x^2 - 39.52x + 18.82 \quad . \qquad (7.5.8)$$

The minima of function (7.5.7) are the stable points of system (7.5.8) while the maxima are the unstable points. The roots 2.905 and 3.1 are quite close to each other. The domain of attraction of $x = 1.107$ is $-\infty < x < 1.888$ and the domain of attraction of $x = 2.905$ is $1.888 < x < 3.1$. Figure 7.5(vi) shows the domains of attraction of $x = 1.107$ and $x = 2.905$. Any point starting below the line $x = 1.888$ will find the minimum $x = 1.107$ and points in the region $1.888 < x < 3.1$ will converge to $x = 2.905$ when a minimization is performed. Although the roots and values of the function at these roots are close to each other the domains of attraction of the minima can still be determined. The closeness of these roots does not affect the determination of the domains of attraction.

v) Next we study a function which has transcendental terms and has k^{th} order continuous derivatives. McCormick [62] considers the function

$$f(x) = x \sin x - \exp(-x) \tag{7.5.9}$$

in finding the global minimum in the interval $[0, 2\pi]$ by using the method of constant signed higher derivatives. In this interval there is only one internal minimum which is located at $x = 4.91177$. Let us now take a bigger interval where there exists multiple minima. For the purpose of applying the technique of Section 7.3 we take the interval $[0, 5\pi]$ where minima occur at $x = 4.91177$ and $x = 11.0855$ and maxima occur at $x = 2.074$, 7.978 and 14.207. The graph of this function is given in Figure 7.5(vii).

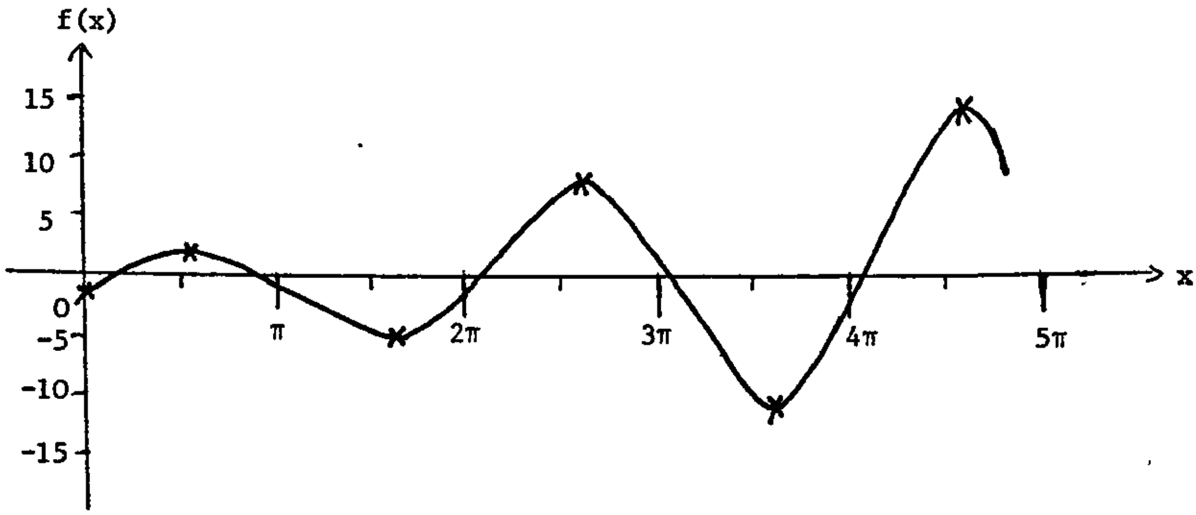


Figure 7.5(vii)

The differential equation formed by using (7.2.1) and (7.2.2) is

$$\dot{x} = - \sin x - x \cos x - \exp(-x) . \qquad (7.5.10)$$

The stable points of system (7.5.10) are therefore $x = 4.91177$ and $x = 11.0855$. The maxima are the unstable ones and will form the boundaries of the domains of attraction of the minima. Figure 7.5(viii) shows the domains of attraction of the minima $x = 4.91177$ and $x = 11.0855$. With the technique of Section 7.3 points starting in the region $2.074 < x < 7.978$ will locate the minimum $x = 4.91177$ while points starting in the region $7.978 < x < 14.207$ will find the minimum $x = 11.0855$. Similarly using this technique we can also locate all the minima and compute the domains of attraction of the minima of this function if a bigger interval is considered. For this example, Figure 7.5(viii) indicates the accuracy of the technique, the exact boundaries of the domains of attraction of the minima being obtained to 3 dec. places. This technique certainly has the advantage over McCormick's method since all the minima and domains can be obtained. McCormick only considers the function in the interval $[0, 2\pi]$ where only one internal minimum exists and no attempt is made towards locating the minima and determining their domains of attraction in a bigger interval.

7.6 DISCUSSION OF GRAPHICAL AID TECHNIQUE.

The interactive graphical aid technique locates the minima by any local minimization algorithm such as Newton (see Appendix C) and computes the domains of attraction of these minima by the method of Section 1.7. In this technique results of stability theory have been borrowed to solve global optimization problems. It is called a graphical aid technique since it only considers points outside the computed domains of attraction of the minima for locating other minima and computing their domains of attraction.

Sections 7.5 and 7.4 illustrate the use of the technique on one-dimensional and two-dimensional optimization problems respectively. In the two-dimensional problem the technique successfully locates the minima and plots their domains of attraction. An exact stability boundary, x = y is obtained for system (7.4.4) where the minima of function (7.4.3) are (1,0) and 0,1) and the stability regions are symmetric about the line x = y. Figure 7.4(iv) also indicates that the saddle point $\left[\left(\frac{1}{4}\right)^{1/3}, \left(\frac{1}{4}\right)^{1/3}\right]$ lies on the stability boundary of the two minima. In problem (iv) of Section 7.4 the function contains four minima and the problem of locating these minima is a difficult one. By using this technique, although the domains of attraction obtained are not very accurate all the four minima can still be located. This is because points well outside the computed domains may be taken as an initial guess for locating another minimum. In a single variable function all the minima and domains of attraction can be easily found by this technique and this is shown by the examples considered in Section 7.5. Thus the problem of global optimization which is admittedly difficult can be solved by the interactive graphical aid technique.

7.7 POSSIBLE AUTOMATIC METHOD.

The problem of identifying minima and determining their domains of attraction automatically is still a difficult problem to solve. One factor is that the time involved in locating the minima and computing the domains may exceed the limit of our computer usage time. Complicated functions with many minima will certainly need a lot of time to determine the minima and domains. In this section we try to devise an automatic algorithm which relies on the technique proposed in Section 7.3. In the graphical aid technique, an arbitrary point outside the computed domain is chosen as an initial guess for locating the minimum. In order to implement the method automatically, the direction of a point on the stability boundary whose distance from the minimum is less than the given R is selected from the computation of the domain of attraction of the minimum. This direction is used to find the starting point (which lies outside the computed domain) for the location of the next minimum. We summarize the algorithm through the following steps.

i) Given an arbitrary point, find a local minimum by any minimization algorithm.

ii) Transfer the minimum to the origin. Compute the domain of attraction of the minimum and also pick the direction of a point on the stability boundary whose distance from the minimum is less than the value of R defined by (1.7.1).

iii) The direction, say ϕ, picked from (ii) is used to find the starting point (which lies outside the computed domain and is defined by $(R\cos\phi\,,\ R\sin\phi)$) for the location of the next minimum. Locate the minimum. Repeat step (ii).

iv) Repeat step (iii) until all the minima and domains of attraction have been identified.

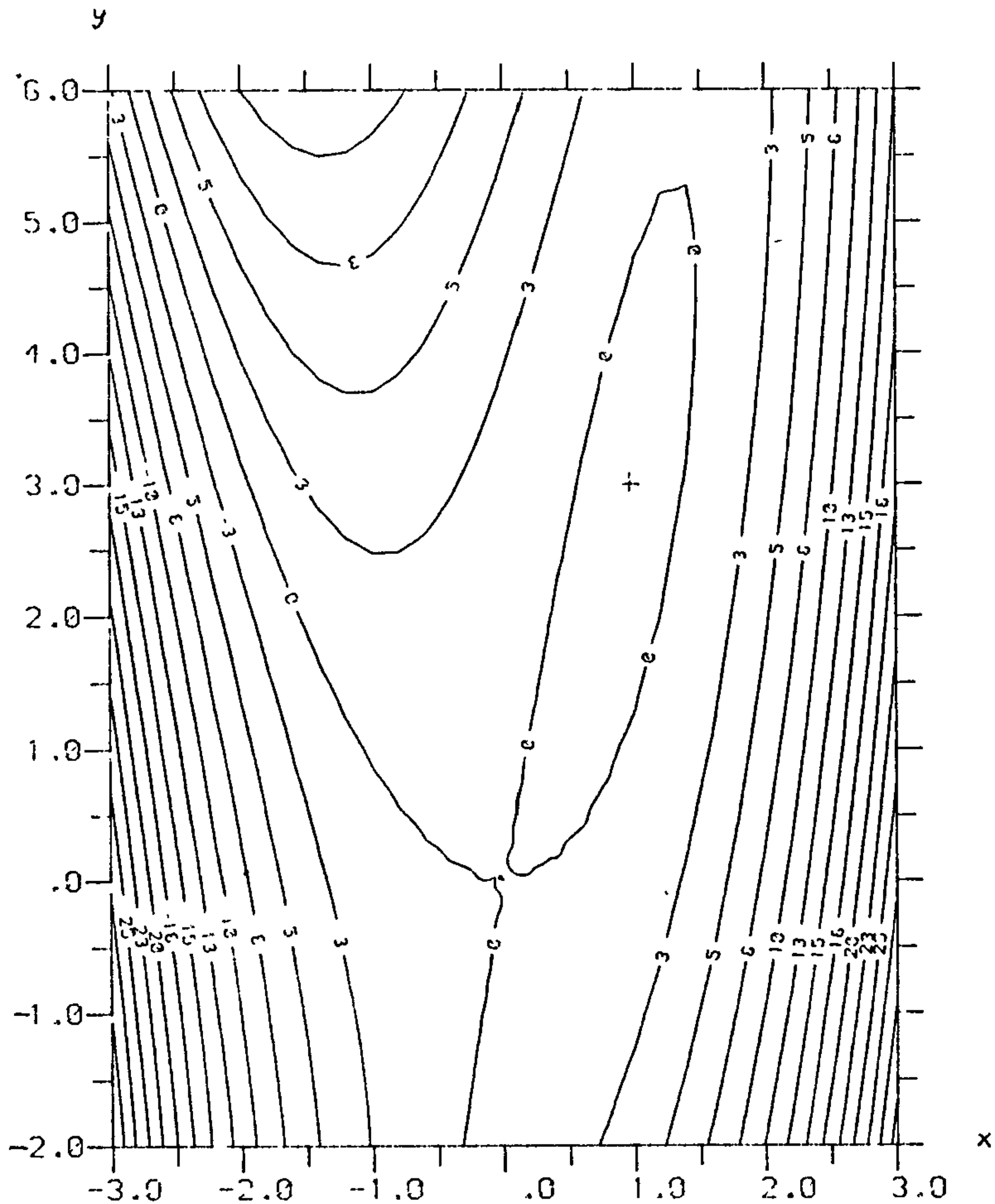
The method is applied to the problem of Section 7.4.ii with $R = 1.0$ and the result is as shown in Figure 7.4(iv). The arbitrary initial point is (0.3, −0.1) and the minimum located is (1, 0). The angle picked is 160^{o}. So the starting point for locating the next minimum is ($\cos 160^{o}$, $\sin 160^{o}$) which lies outside the computed domain. Although we are able to locate the minima and compute the domains automatically, the computation time is very large. So complicated functions with many minima may require a considerable amount of time.

7.8 CONCLUSION.

The idea of combining stability theory and optimization theory throws some light on solving the global optimization problem. The graphical aid technique is another space covering technique which successfully locates the minima and computes their domains. For functions with two minima or maxima the stability boundaries traced are accurate as seen in examples (ii) and (iii) of Section 7.4. But functions with four minima the boundaries are difficult to compute especially the boundaries which pass through the saddle points and the domains are not drawn accurately. The method also solves the one-dimensional optimization problem with multiple minima. In this one-dimensional optimization problem, an example from a recent article of McCormick [62] is studied in a larger interval and the exact boundaries of the domains of attraction of the minima are obtained as shown in Figure 7.5(viii).

The automatic method looked promising in that it is able to locate the minima and to determine their domains of attraction automatically as seen from the application of the method to the problem of Section 7.4.ii which has two minima. However the computation time is very large. So for functions with many minima, the computing time involved may exceed the limit of our computer usage time.

The graphical aid technique concerns only in solving the global optimization problem by trying to locate all the minima as well as determining their domains of attraction. No critical analysis regarding the efficiency and accuracy of this technique is given and this would be a worthwhile topic of further research.



Contours of

$$f(x,y) = x^3 + \frac{y^2}{6} - xy$$

Fig. 7.4(i)

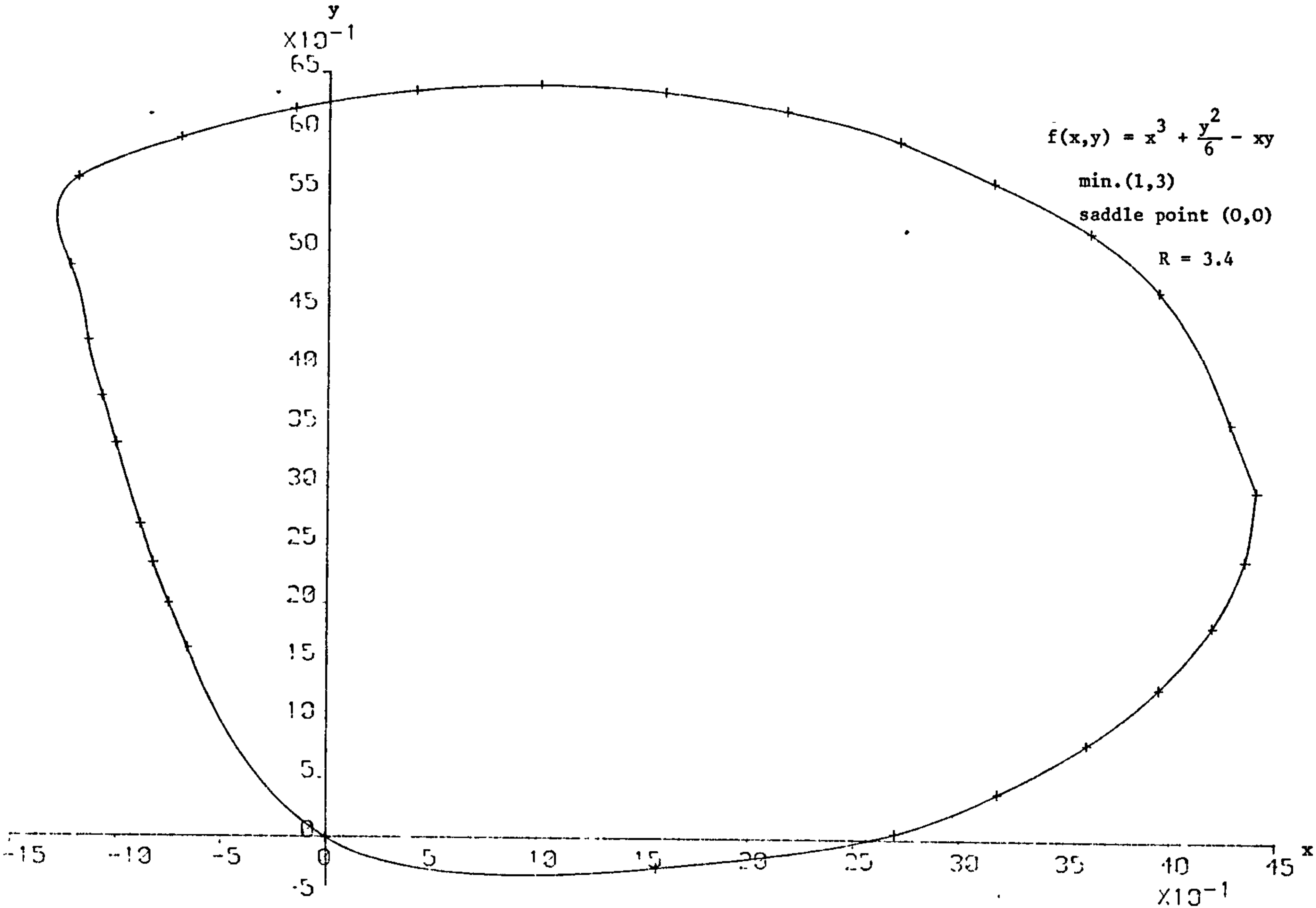
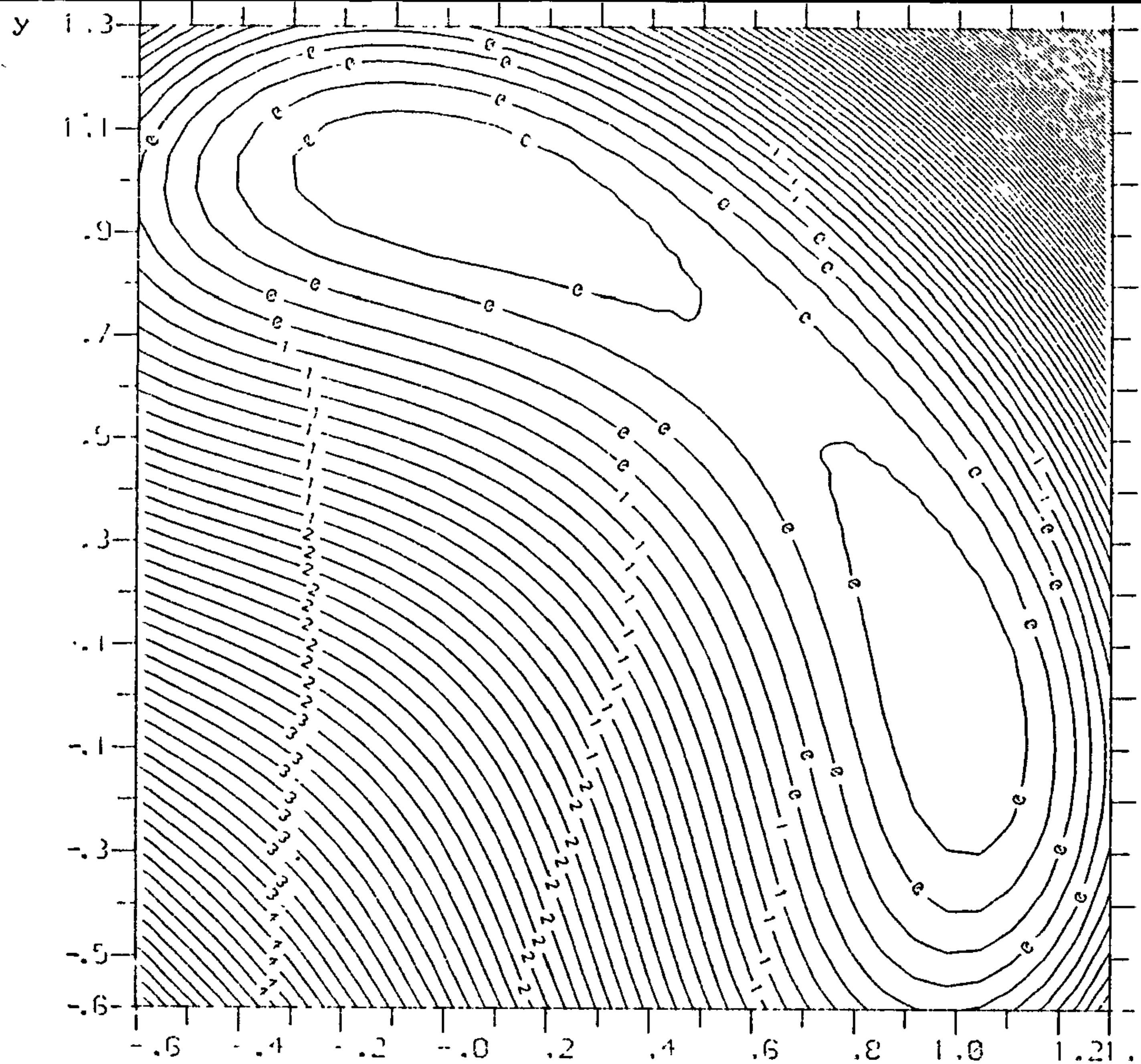


Fig. 7.4(ii)



Contours of

$$f(x,y) = (x^2+y^2-1)^2 + (x+y-1)^2$$

Fig. 7.4(iii)

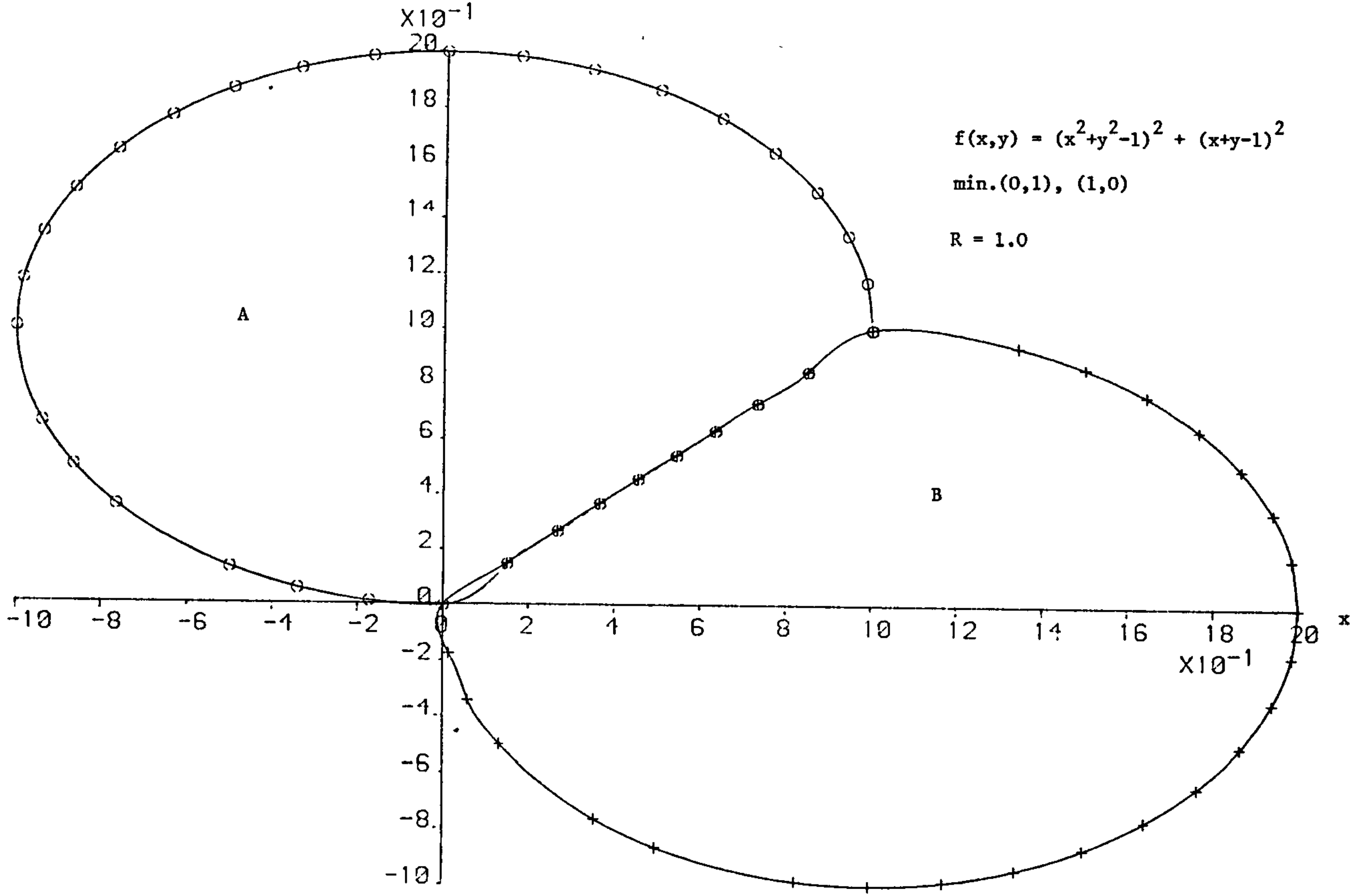

X10^-1
$f(x,y) = (x^2+y^2-1)^2 + (x+y-1)^2$
min.(0,1), (1,0)
R = 1.0
A
B
x
X10^-1
-10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20
20 18 16 14 12 10 8 6 4 2 0 -2 -4 -6 -8 -10

Fig. 7.4(iv)

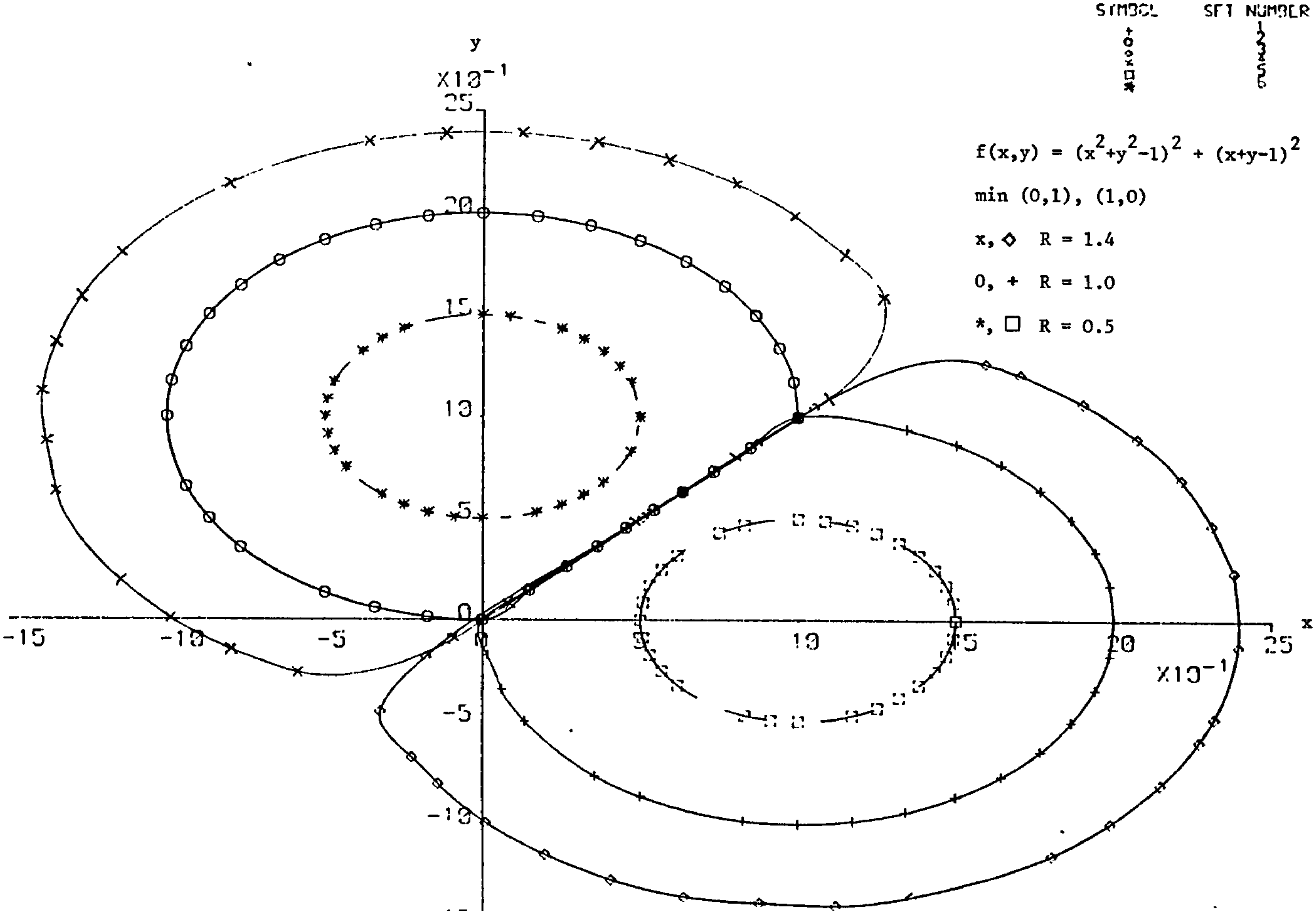

SYMBOL SET NUMBER
y
X10^-1
25
20
15
10
5
0
-5
-10
-15
x
-15 -10 -5 0 5 10 15 20 25
X10^-1
f(x,y) = (x^2+y^2-1)^2 + (x+y-1)^2
min (0,1), (1,0)
x, ◇ R = 1.4
0, + R = 1.0
*, □ R = 0.5

Fig. 7.4(ix)

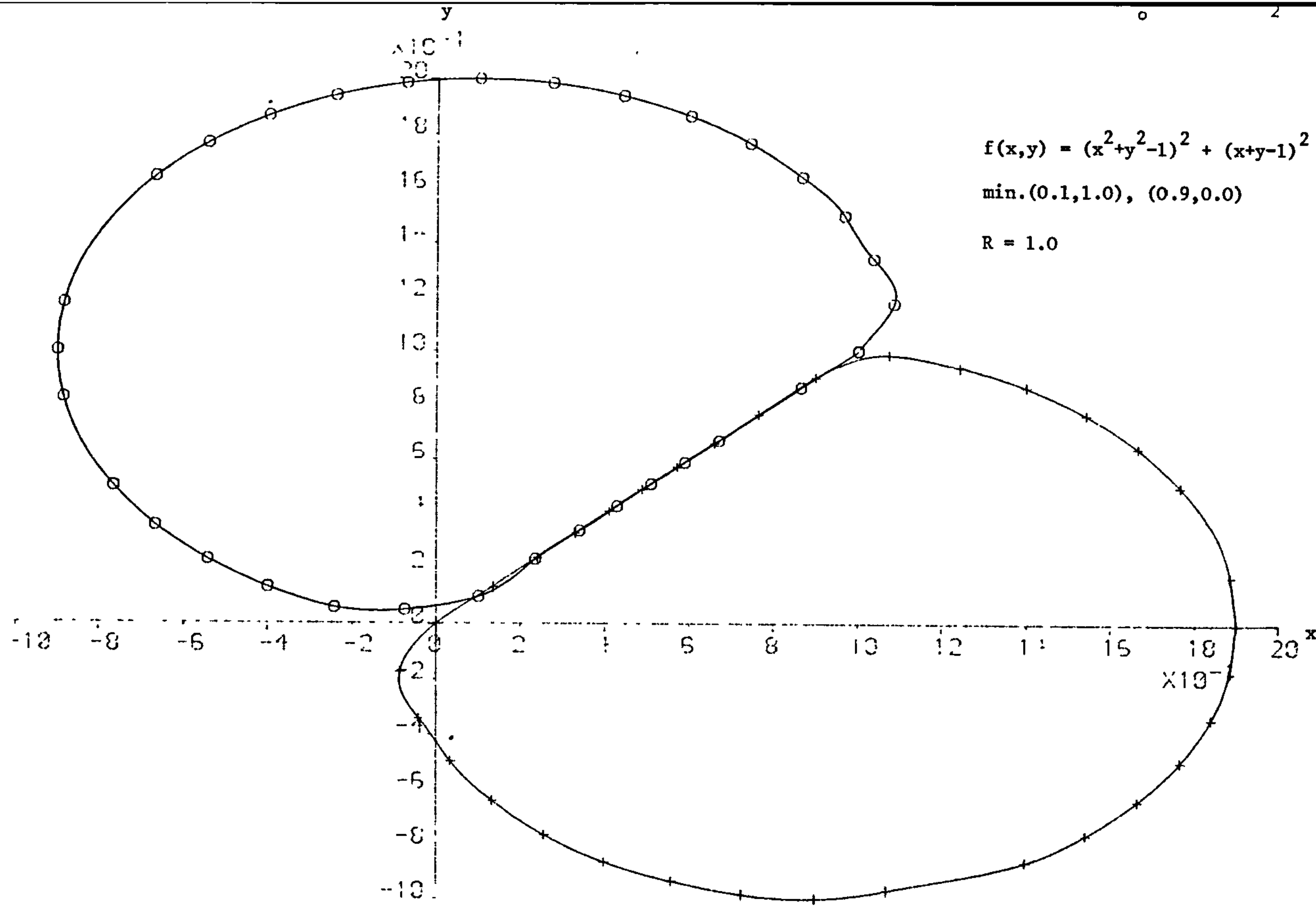
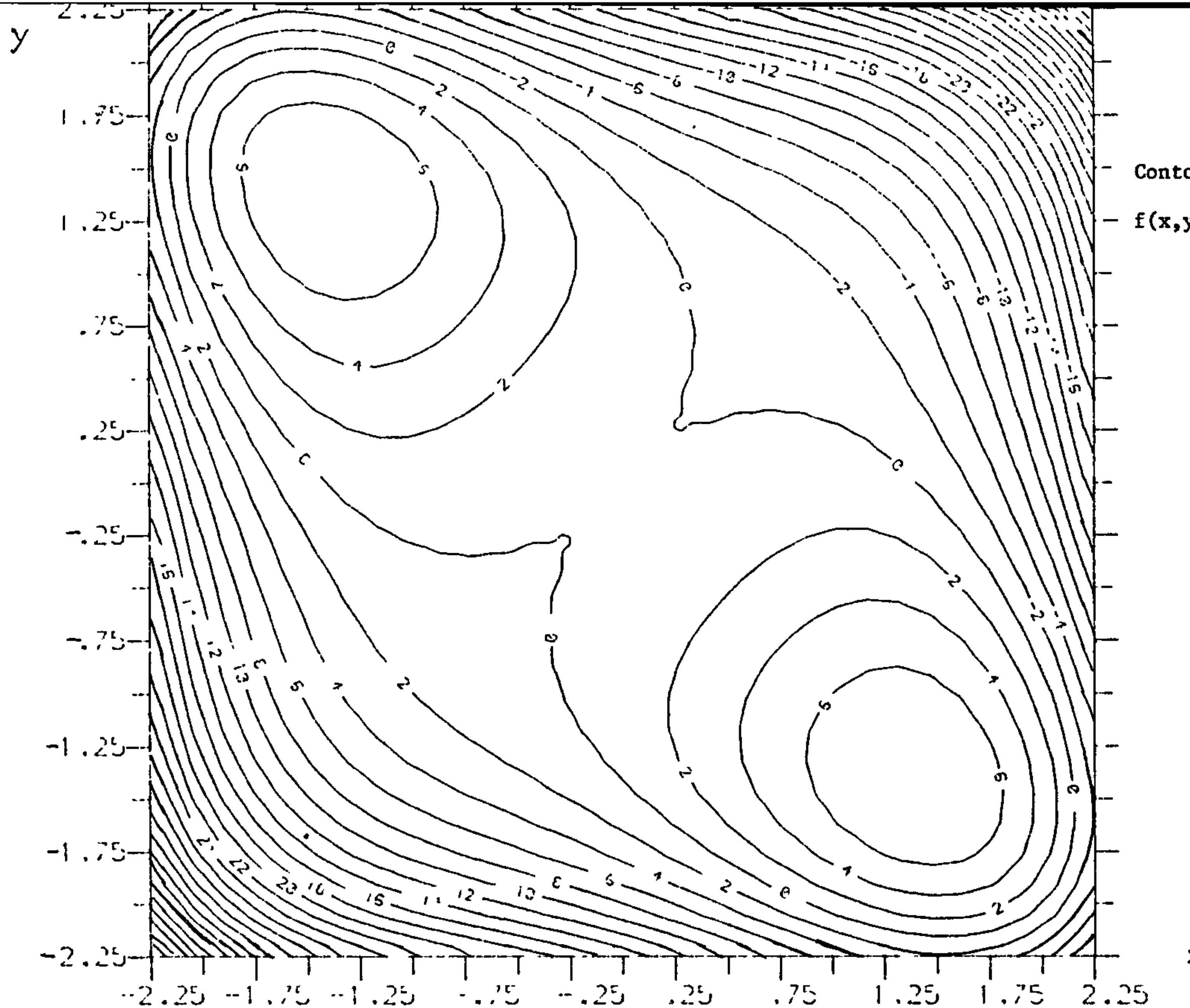


Fig. 7.4(v)



Contours of

$f(x,y) = 2(x-y)^2-x^4-y^4$

Fig. 7.4(vi)

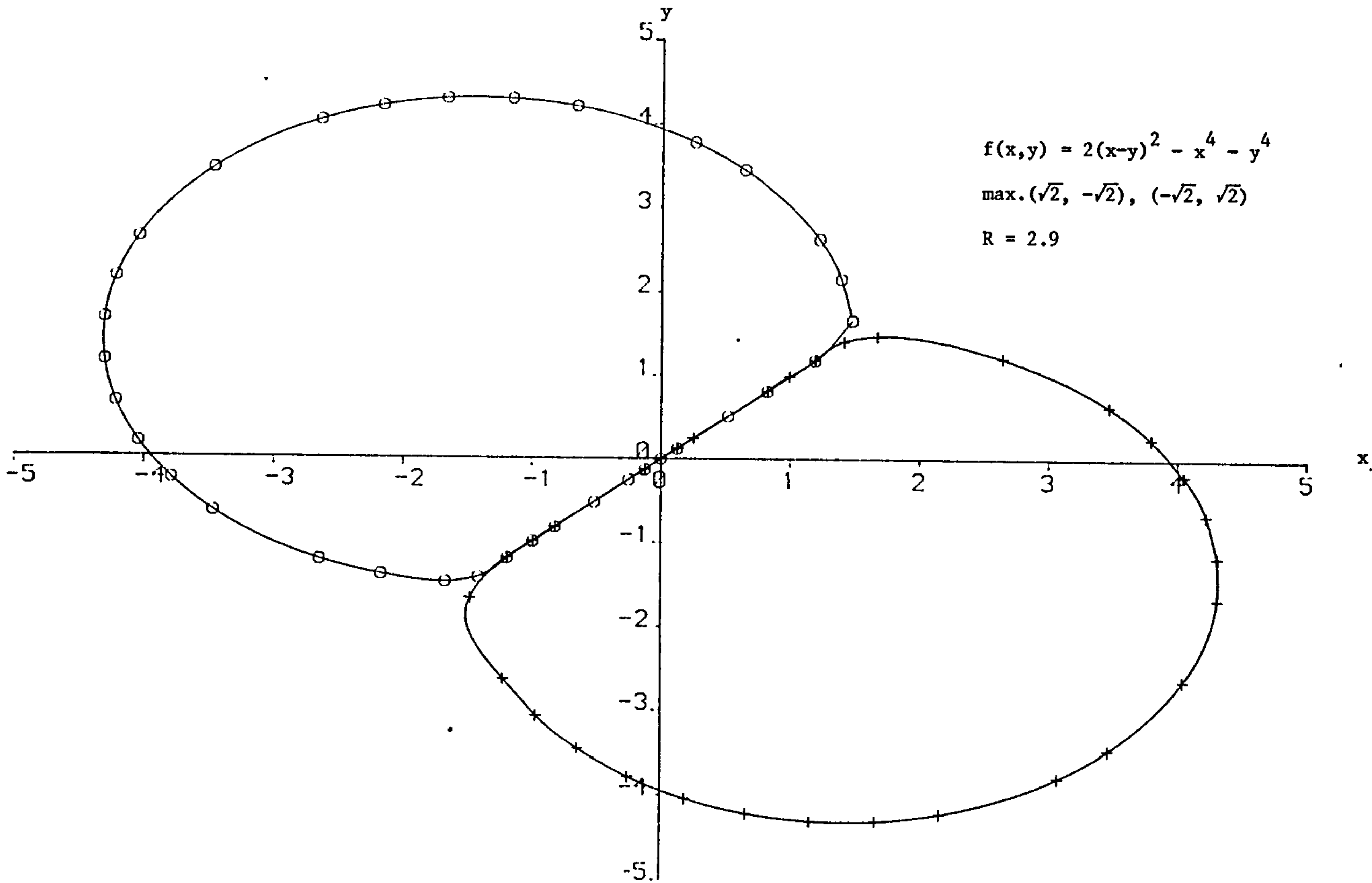


Fig. 7.4(vii)

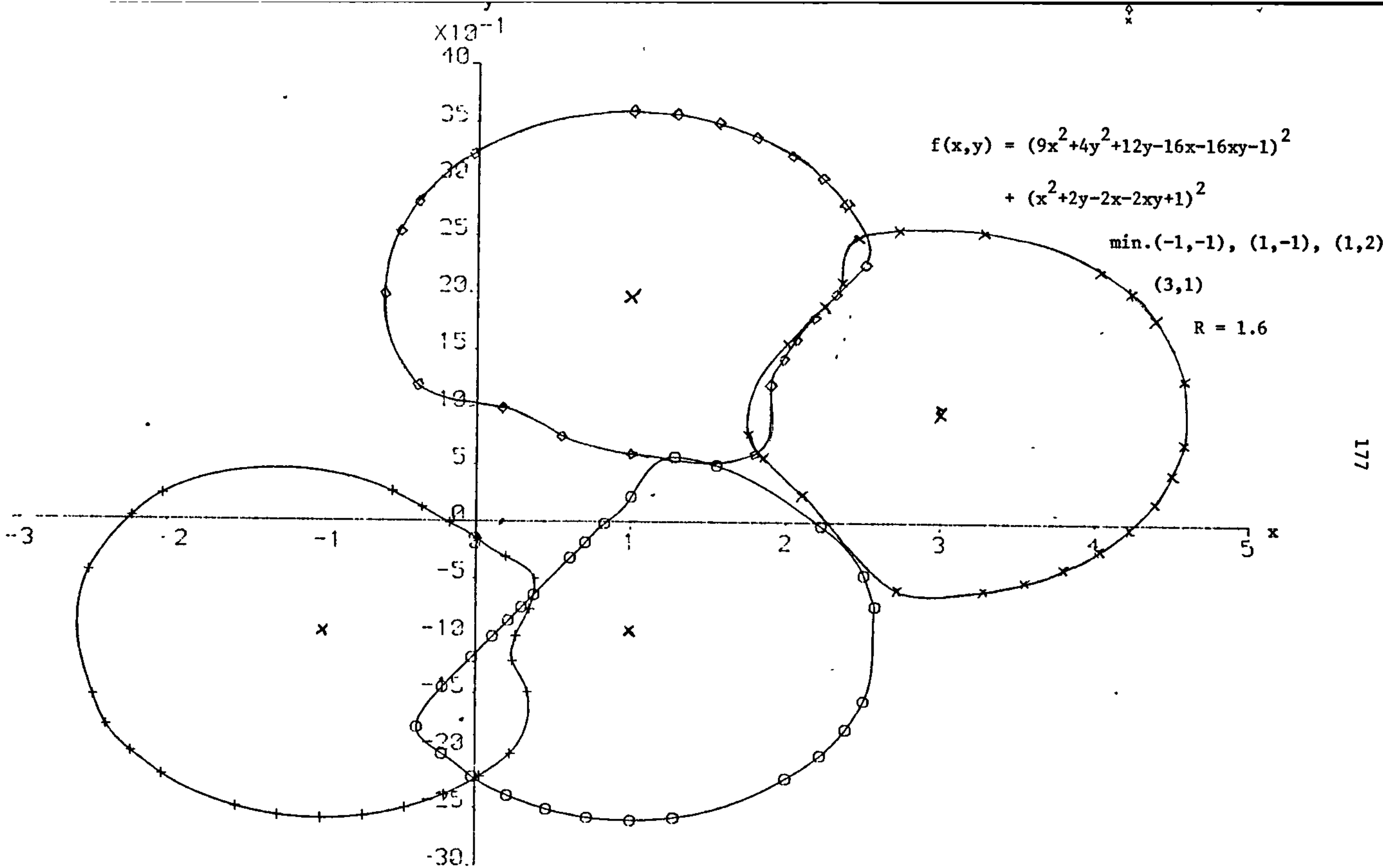

X10^-1
y
x
$f(x,y) = (9x^2+4y^2+12y-16x-16xy-1)^2$
$+ (x^2+2y-2x-2xy+1)^2$
min.(-1,-1), (1,-1), (1,2),
(3,1)
R = 1.6

Fig. 7.4(viii)

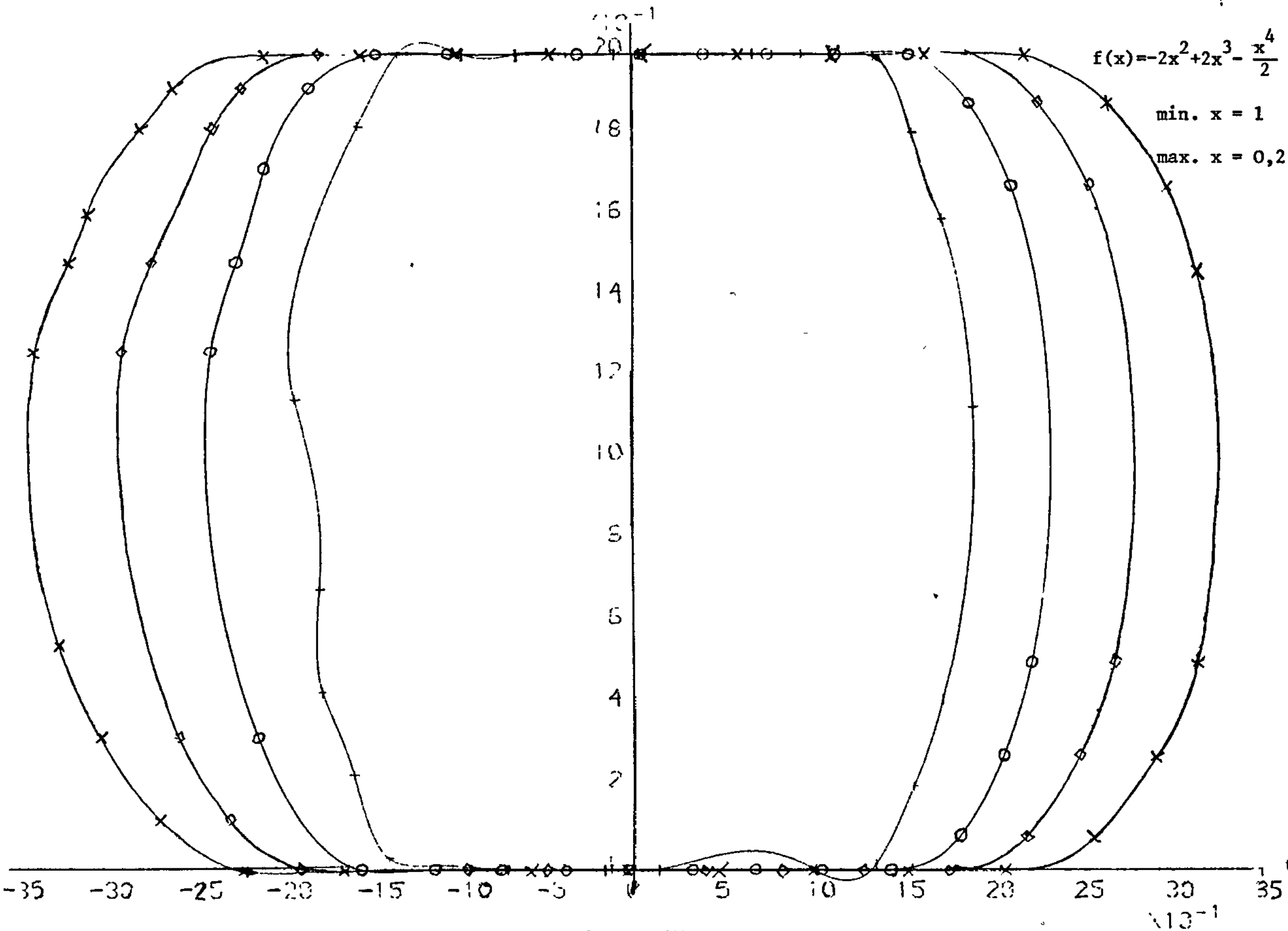


Fig. 7.5(i)

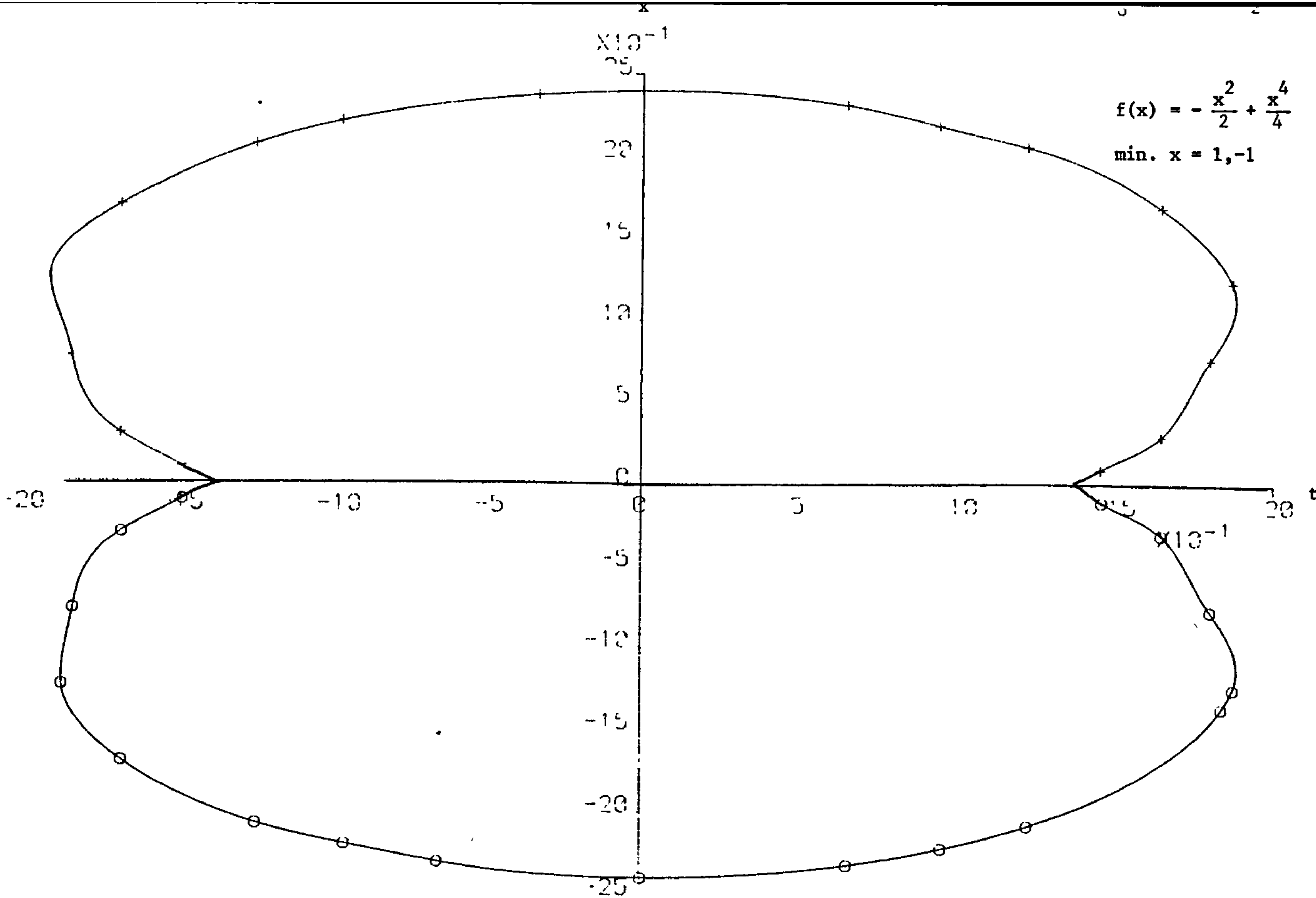

x
X10⁻¹
25
20
15
10
5
0
-5
-10
-15
-20
-25
-20
-15
-10
-5
0
5
10
15
20
t
X10⁻¹
f(x) = - x^2/2 + x^4/4
min. x = 1,-1

Fig. 7.5(ii)

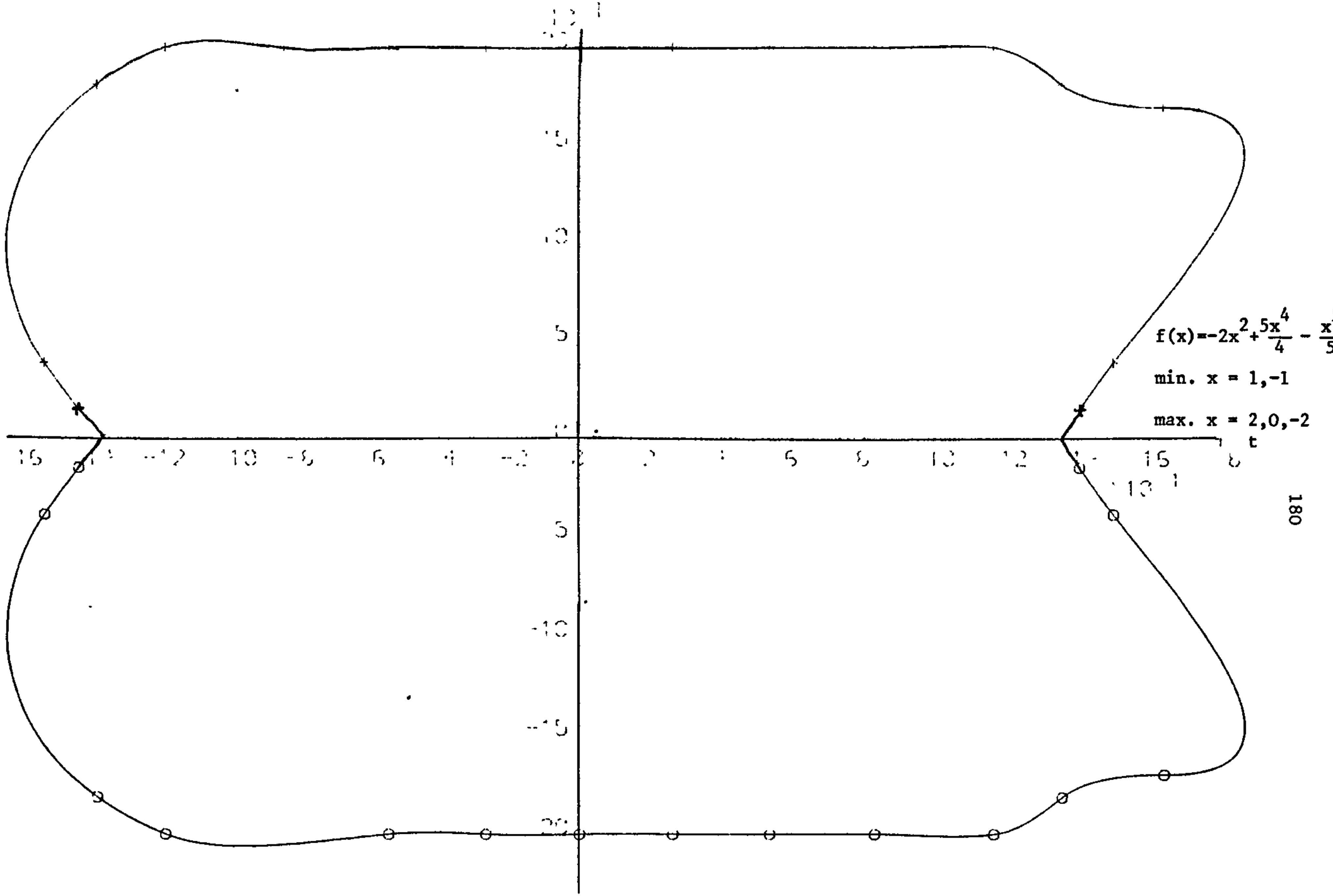
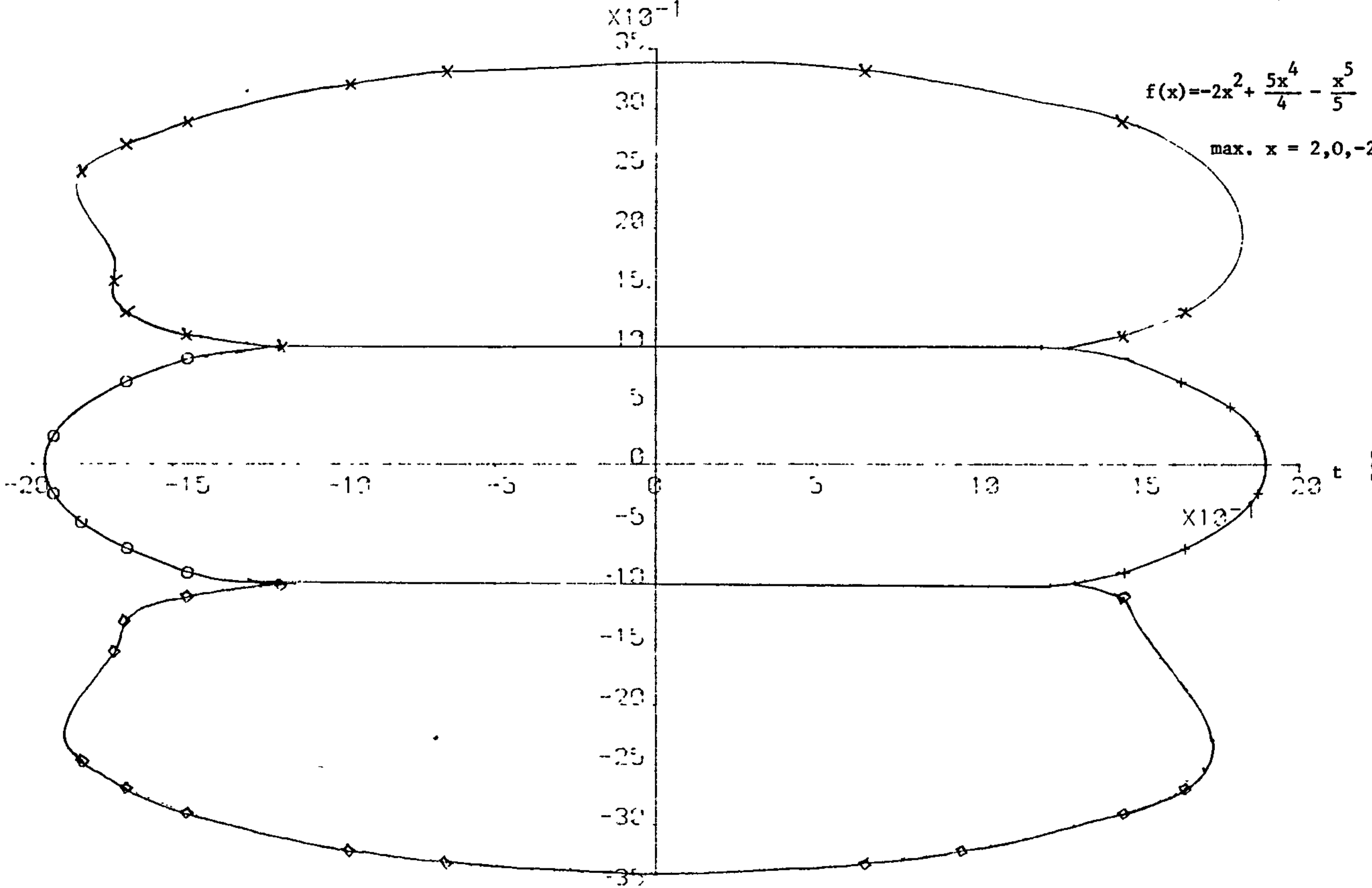
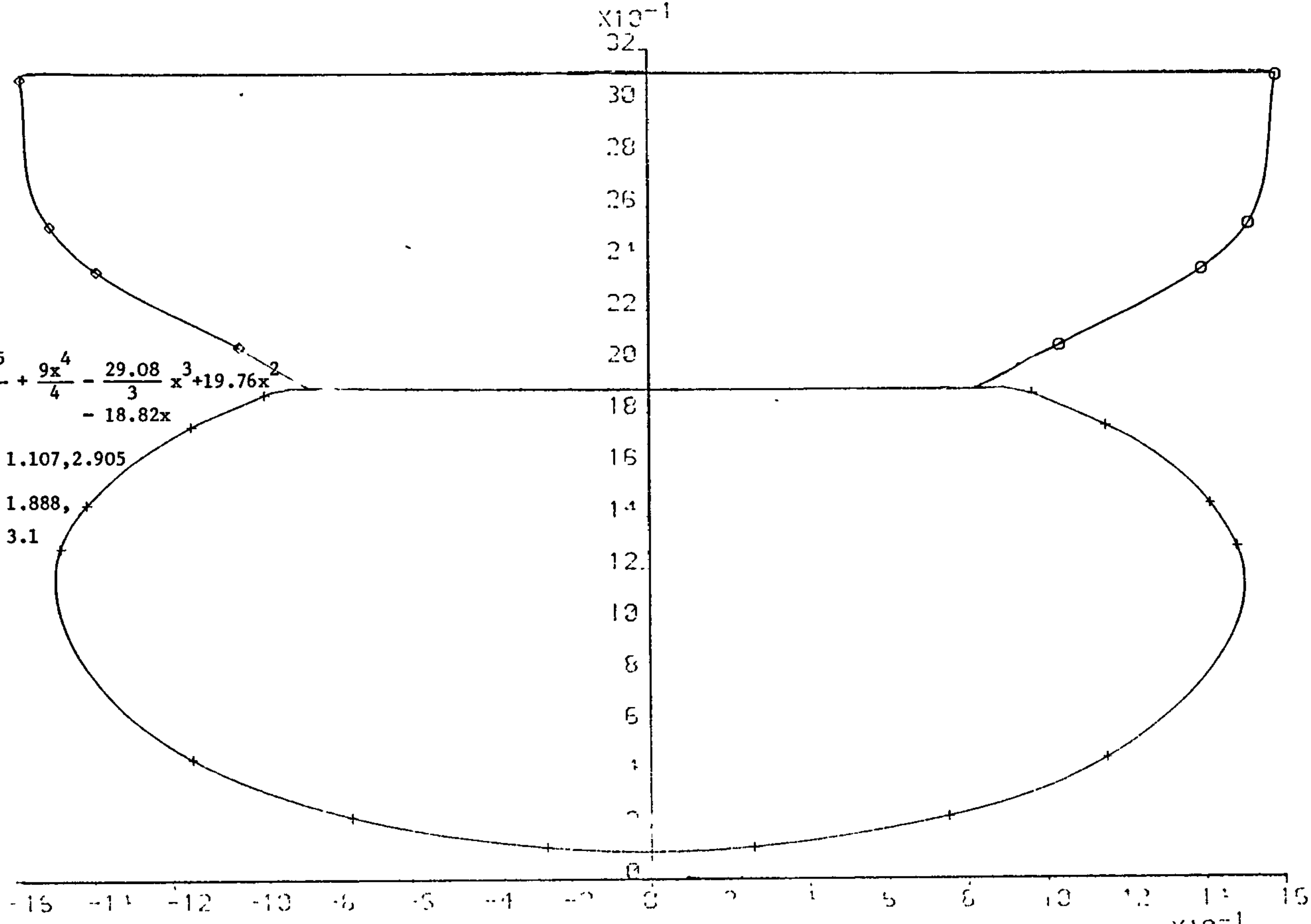


Fig. 7.5(iii)



$f(x) = -2x^2 + \frac{5x^4}{4} - \frac{x^5}{5}$
max. x = 2,0,-2
X10^-1
X10^-1
t
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
-20
-15
-10
-5
0
5
10
15
20

Fig. 7.5(iv)



$$f(x) = -\frac{x^5}{5} + \frac{9x^4}{4} - \frac{29.08}{3}x^3 + 19.76x^2 - 18.82x$$

min, x = 1.107, 2.905

max. x = 1.888, 3.1

Fig. 7.5(vi)

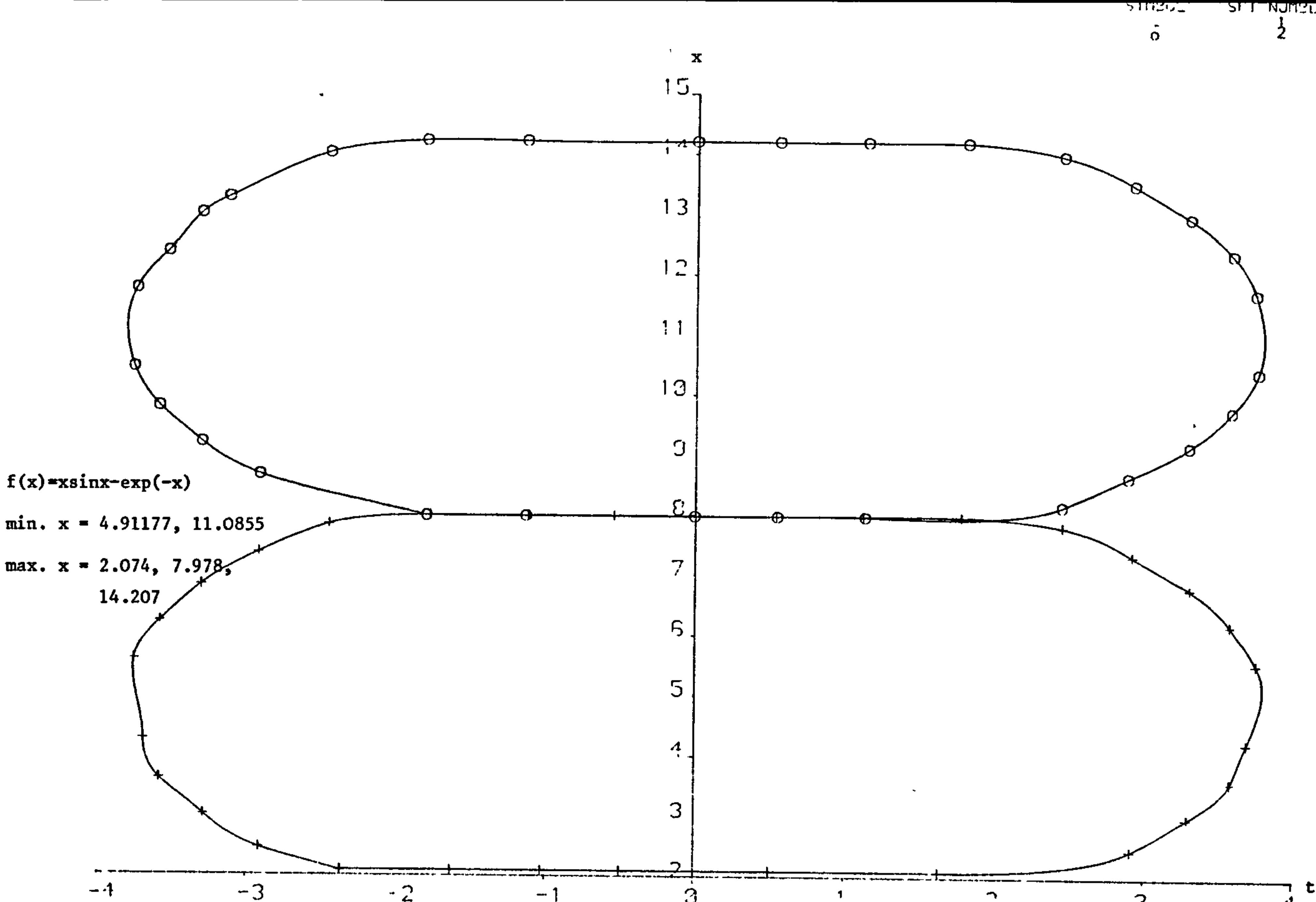

SYMBOL SET NUMBER
o 1
+ 2
x
t
f(x)=xsinx-exp(-x)
min. x = 4.91177, 11.0855
max. x = 2.074, 7.978,
14.207
15
14
13
12
11
10
9
8
7
6
5
4
3
2
-4
-3
-2
-1
0
1
2
3
4

Fig. 7.5(viii)

CHAPTER VIII

SUMMARY AND CONCLUSIONS

The work of this thesis is mainly concentrated with applying the method of Zubov to some practical problems and extending it to higher order systems, time varying systems and periodic systems and also solving global optimization problems. Several comments have been included throughout the preceding chapters and in this chapter it is appropriate to summarize and draw some conclusions regarding the work and make suggestions for further research.

Zubov's theory plays an important role in generating a Liapunov function and estimating the domain of attraction of a stable equilibrium of the given system. The fact that $V = 1$ or ∞ which defines the exact stability boundary, gives some incentives in trying to establish the complete domain of attraction. However, solving Zubov's equation is not an easy matter. Some authors have attempted to solve the Zubov's equation by the classical methods for solving first order partial differential equation and some have tackled the problem through series and Lie series construction procedures. The series solution however has a convergency problem. Also in most problems it is extremely difficult to obtain a closed form solution of Zubov's equation.

Miyagi and Taniguchi solve Zubov's equation by the classical Lagrange Charpit method. The method determines an arbitrary non negative function $\phi(x)$ which allows the Liapunov function to be determined. In the process of solving the partial differential equation the authors have assumed many arbitrary functions so that a Liapunov function can be formed easily. However Zubov's theory is not applied fully in determining the domain of attraction, so the exact stability region cannot be achieved. An attempt has been made to apply the Lagrange Charpit method to a scalar time varying system in the example of section 1.6(i) and the actual domain of attraction $(-\infty, 1)$ is obtained for all $t \geqslant 0$ when the condition $V = \infty$ is imposed. Here again many arbitrary functions have been assumed. Further research is required as to what criteria should be used to select these arbitrary functions.

An obvious way of solving Zubov's equation is by numerical methods. The result of using finite difference methods is not encouraging because of inaccuracies as the stability boundary is approached. Also the choice of the initial condition $V(\underline{x}, 0)$ will affect the accuracy of plotting the stability boundary.

White has developed a numerical method which determines the domain of attraction of a stable equilibrium of a system. This method overcomes the problem of nonuniform convergence and finds the domain of attraction accurately. Zubov's equation is transformed into a pair of first order ordinary differential equations in the case of second order autonomous system. The method initiates near the boundary of the domain of asymptotic stability and computes trajectories which either tend to the origin or away from it, depending on where the computation is initiated. The approach of using V to compute $\underline{x}(V)$ is different from the Texter algorithm where $\underline{x}(t)$ is computed. In White's method, a circle defined by (1.7.1) is used as the region of approximation but other shapes could also be assumed. For systems which have elliptical regions of asymptotic stability the use of an ellipse as the region of approximation has an advantage over the circular approximation and allows the boundary points to be computed more quickly. In most numerical algorithms the starting point for computing the stability boundary is taken near the boundary but a method which determines the domain of attraction by initiating near the origin would form a worthwhile area of research.

In Chapter two we have applied White's method to some practical problems like power systems and control systems and the results are encouraging because considerable improvement in the estimates of the domain of attraction has been achieved. In the Luré problem the sector condition has been relaxed so that the system is asymptotically stable but not in the large. The region of attraction obtained clearly shows the superiority of the method.

The method of White has been extended to scalar time varying system and the results are presented in motion space. Accurate domains of attraction have been obtained. Here the Zubov's equation in some of the examples requires a positive definite and decrescent ϕ and in some a positive definite ϕ only. There is still an uncertainity concerning the nature of ϕ. For example, in section 3.7(i) only a positive definite ϕ is used, but still the Zubov's equation can be solved analytically giving a closed form solution and the exact domain of attraction can be inferred.

The series solution for scalar time varying systems also suffers from the problem of nonuniform convergence. In the Yoshizawa example we have shown that the even partial sums of the series converge to the domain of attraction $-\infty < x < e^t$ while the odd partial sums converge to the region of convergence $|x| < e^t$. In the series construction procedure, the V's, f's and ϕ's for second order time invariant systems are homogeneous in x and y but for the scalar time varying systems they are only homogeneous in x. The construction of the series solution for a second order time varying system by following the same line of argument as the scalar time varying system forms another possible topic of research. In the series, the bounded functions of time will be determined by solving the linear equations formed by equating the coefficients of the phase variables.

In the extension of the algorithm to third order autonomous systems the problem of varying the vector $\underline{\theta}^T = (\theta_1, \theta_2)$ to cover all directions of the state space is encountered. One way of resolving this problem is to compute the cross-section of the stability boundaries in all the principal planes. So to vary $\underline{\theta}$, one of its components is fixed so that the variation of scalar θ discussed in White's algorithm can be applied. The domain of attraction of a third order autonomous system is in general represented by a volume in three dimensions. The cross-sections of the stability boundaries at different levels of the three axes are computed and from these boundary points a solid figure is drawn by using the graphics package which is available from the Loughborough University of Technology Computer Centre. A method of varying vector $\underline{\theta}$ so that all dimensions are covered will certainly ease the problem of determining the domain of attraction of third order autonomous systems. The domains of attraction of second order time varying systems represented in the phase space are obtained by using the algorithm for third order autonomous systems.

The extension of the method to periodic systems requires the knowledge of the periodic solution being studied. The only difference in the method is that instead of studying the trivial solution $\underline{x} = \underline{0}$, we consider the periodic solution. So the form of ϕ will be different from the former case but the positive definiteness and decrescency of ϕ are still preserved. In many periodic systems it is often difficult to obtain the exact periodic solution. The advantage of the method is that knowledge of approximate periodic solutions is sufficient for the determination of domains of attraction.

It is known that the solution of global optimization problems is a difficult task. The use of stability theory gives us the possibility of determining the domains of attraction of minima of a function using the differential equations formed by taking the negative gradient of the function. The domains of attraction of the stable critical points (which are the minima of the function) of this system are obtained by White's method. In the one-dimensional minimization problems, the results obtained are impressive. In the two-dimensional problems the computed domains of attraction of the function with two minima are accurate but this is not so for functions with four minima. This is because the contours may become narrower as they approach saddle points. The result tells roughly the location of the saddle points. Nevertheless, all the four minima of the function can be located since points well outside the computed domain of attraction of a known minimum can be considered as starting points for identifying another minimum. Since we are only interested in identifying the minima and determining their domains of attraction, the critical analysis of the graphical aid technique is not given and there is certainly a scope for further research in this area. Extension of the method to a three-dimensional minimization problem forms another possible research topic.

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APPENDIX A

Notations used for the power systems.

M	- inertia constant
δ	- load angle
δ_o	- steady state load angle
D, a, a_i	- damping coefficients
P_{mi}	- mechanical power input
P_{mio}	- steady state mechanical power input
P_e	- electrical power output
E'_q	- voltage proportional to field flux linkage
E'_{qo}	- steady state voltage proportional to field flux linkage
E_{ex}	- exciter voltage
v	- infinite busbar voltage
x_d, x_q	- direct-axis and quadrature-axis synchronous reactances
x'_d, x'_q	- direct-axis and quadrature-axis transient reactances
x''_d, x''_q	- direct-axis and quadrature-axis subtransient reactances
x_e	- external reactances
T'_{do}	- open circuit transient time constant
T'_d	- short circuit transient time constant
T''_{do}, T''_{qo}	- open circuit subtransient time constants
T_e	- equivalent time constant of the governor system
b	- amplitude of the fundamental of power angle curve
c	- amplitude of the second harmonic of power angle curve
η_1	- $1/T'_d$

η_2 — $\dfrac{(x_d - x_d')v}{(x_e + x_d')T_{do}'}$

γ_1 — $1/T_e$

γ_2 — k/T_e

k — governor amplification factor

V — Liapunov function

Appendix B

Definitions and Theorems of Eventual Stability

Definition 1:

The origin of a system $\dot{x} = F(t,x)$ is said to be eventually stable if, for a given $\varepsilon > 0$, there exist numbers δ and T such that

$$\|x_o\| < \delta \text{ implies } \|x(t,x_o,t_o)\| < \varepsilon \text{ for all } t \geq t_o > T .$$

Definition 2:

The origin of the system $\dot{x} = F(t,x)$ is said to be eventually asymptotically stable if

i) it is eventually stable and

ii) there exists an $h > 0$ and a T_o such that

$$\|x_o\| < h \text{ and } t_o \geq T_o \text{ imply } x(t,x_o,t_o) \to 0 \text{ as } t \to \infty .$$

Theorem 1:

Eventual stability of the origin is equivalent to the following: given $\varepsilon > 0$ there exist numbers δ and T such that

$$\|x(t_1,x_o,t_o)\| < \delta \text{ for some } t_1 \geq T \text{ implies } \|x(t,x_o,t_o)\| < \varepsilon \text{ for all } t \geq t_1 .$$

Theorem 2:

Eventual asymptotic stability of the origin is equivalent to:

i) eventual stability of the origin

ii) there exist an $h > 0$ and a T_o such that

$\|x(t_1,x_o,t_o)\| < h$ for some $t_1 \geq T_o$ implies

$x(t,x_o,t_o) \to 0$ as $t \to \infty$.

APPENDIX C

Newton's Method.

The general iteration for this technique is

$$\underline{x}^{(n+1)} = \underline{x}^{(n)} - t^{(n)}\; G^{-1}\;(\underline{x}^{(n)})\;\underline{\nabla}f(\underline{x}^{(n)})$$

where $G^{-1}(\underline{x}^{(n)})$ is the inverse of the Hessian at the point $\underline{x}^{(n)}$, $\nabla f(\underline{x}^{(n)})$ is the gradient of the function at the point $\underline{x}^{(n)}$ and $t^{(n)}$ is a scalar parameter obtained so that $f(\underline{x}^{(n+1)})$ is minimized.

The initial point $\underline{x}^{(o)}$ is taken to be a suitable estimate of the minimum.