Mathematical investigations

MATHEMATICAL INVESTIGATIONS

BY

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ABSTRACT

Since the publication of Mathematics Counts in 1982 there has been a growing interest in investigational work in the mathematics classroom. There have been many books published specifically on investigational work and the related topic of problem solving. Class texts have been published claiming to follow the suggestions of Mathematics Counts including investigational work.

The new examination at 16, the General Certificate of Secondary Education appears to be moving towards containing work of an investigational nature.

In the first chapter the nature of investigational work is examined. Distinctions are drawn between problem solving and investigational work. A list of characteristics of investigational work is considered with a view to clarifying exactly what constitutes investigational work in mathematics.

In the second chapter the role of investigational work is considered both in the curriculum as a whole and more specifically in the mathematics curriculum. Particular attention is paid to the aims and objectives of mathematics education as set out in Mathematics From 5 to 16.

The third chapter considers how investigational work can be introduced into the secondary school both in the short term and over a greater period of time.

The next chapter examines how an investigational approach is used in a recently published mathematics scheme, SMP 11 - 16.

In chapter five the various roles that the micro-computer can play in investigational work is examined by considering a number of computer programs.

Finally the difficulties in assessment presented by investigational work are compared with methods of assessment currently in practice. Several forms of assessment are suggested for investigational work undertaken in timed examinations and also as coursework within the school.

DECLARATION

The work in this dissertation is entirely my own work.

ROBERT JOHN BLACK.

ACKNOWLEDGEMENTS

I wish to record my thanks to my supervisor, Dr. R.P. Knott, for his help, advice and encouragement with this dissertation.

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CONTENTS

CHAPTER 1

A DEFINITION OF INVESTIGATIONAL WORK.

A DEFINITION OF INVESTIGATIONAL WORK

Before the role of investigational work in the mathematics curriculum can be evaluated and before its implementation can be planned it is necessary to consider exactly what constitutes investigational work in mathematics. There is no better place to start than at what many regard as the current source of inspiration for investigation work, namely paragraph 243 of Mathematics Counts.

> Mathematics teaching at all levels should include opportunities for : -
>
> - exposition by the teacher;
> - discussion between teacher and pupils and between pupils themselves;
> - appropriate practical work;
> - consolidation and practice of fundamental skills and routines;
> - problem solving, including the application of mathematics to everyday situations;
> - investigation work.
>
> In setting out this list we are aware that we are not saying anything which has not already been said many times and over many years. The list which we have given has appeared, by implication if not explicitly, in official reports, DES publications, HMI discussion papers and the journals and publications of the professional mathematical associations. Yet we are aware that although there are some classrooms in which teaching includes, as a matter of course, all the elements which we have listed, there are still many in which the mathematics teaching does not include even a majority of these elements.
>
> (Cockcroft 1982)

There are two points that arise from paragraph 243. First it follows on from a discussion of the nature of mathematical understanding.

The report draws the conclusion that in mathematics teaching an analysis of topics reveals five components. There are facts and skills, conceptual structures and general strategies and appreciation. These are to be implemented by a variety of styles for the teaching of mathematics listed in paragraph 243. Thus investigational work is one of several styles for the teaching of mathematics. This is important to appreciate as investigational work may take many different forms depending on which of the five components of mathematics, facts, skills, conceptual structures, general strategies or appreciation, that it is applied to.

Second, investigational work has been advocated by many agencies and practised in many classrooms over the years. The implication is that there should be many sources that give examples of investigational work and that it should be possible to detail exactly what its nature is.

The authors of the Cockcroft Report felt that it was necessary to consider more fully each of the six styles of teaching listed in paragraph 234. Exposition, discussion, practical work and practice do not directly concern us here. Problem solving and investigational work bear closer consideration. [One common conception of investigational work, the authors note, is that of a project or piece of coursework. This may be undertaken individually or as a member of a group and would consist of an extensive piece of work taking quite a long time to complete.] While admitting this interpretation as one form which mathematical investigations can take they argue that it is not the only form. [As the most fundamental level they see mathematical investigations arising from questions asked by pupils about current work. Key questions here are "what would happen if ....?" and "could we have done the same thing with ....?".][Sometimes these questions may be resolved after a few moments of discussion with an individual pupil for group of pupils. Sometimes the individual or group may require some time to attempt to find the answer to the question themselves. On other occasions the individual or group may return at a later date to the question.]

The essential role of the teacher then is to provide an environment where pupils can be encouraged to ask these sorts of questions and to think in this sort of way. The teacher must above all provide time for pupils not only to follow interesting lines of thought but also, perhaps as importantly, to follow false trails.

A further quotation sums up the author's views.

> The idea of investigation is fundamental both to the study of mathematics itself and also to an understanding of the ways in which mathematics can be used to extend knowledge and to solve problems in very many fields.
>
> (Cockcroft 1982)

Investigational work then can be seen from two view points. One is as a style of teaching that enables pupils to gain a greater understanding of mathematical facts, skills and conceptual structures. The other is to put their mathematical facts, skills and knowledge of conceptual structures to use in doing some mathematics, by solving problems.

Problem solving then is closely connected with investigational work. In the Cockcroft Report the distinction is drawn by giving the name 'problem-solving' to the ability to apply mathematics to a variety of situations.

This definition does not clearly differentiate between the two. Perhaps this is because while it is possible to give examples of 'problem solving' that are clearly not 'investigational work' and vice versa there is a good deal of common ground between the two.

In Teaching Styles - a response to Cockcroft 243 produced as a discussion book by the ATM their analysis of paragraph 243 brackets problem solving and investigational work together. They felt that the most helpful classification was to attach the word 'convergent' to problem solving and 'divergent' to investigational work. They did not however wish to separate the two.

However it was agreed that some elements of investigation were implicit in problem solving and some of problem solving in investigation.

Barbara Jaworski in Investigation in Mathematics Teaching clarifies the position still further.

> A question which teachers sometimes ask is "what is the difference between problem solving and investigation?". One answer to this might be that problem solving usually involves solutions or answers. Investigation often does not and might only throw up further questions. However, to imply that solving a problem may not involve investigation would not be correct. Probably it is truer to say that problem solving may involve investigation, but that investigation is much more than just problem solving.
>
> (B. Jaworski 1984)

The common ground between problem solving and investigational work is exemplified in Teaching Problem Solving by Randall Charles and Frank Lester. They list six different types of mathematical problems.

<u>Drill Exercise</u>

1. $\begin{array}{r} 346 \\ \underline{\times 28} \end{array}$

<u>Simple Translation Problem</u>

2. Jenny has 7 tropical fish in her aquarium. Tommy has 4 tropical fish in his aquarium. How many more fish does Jenny have than Tommy?

<u>Complex Translation Problem</u>

3. Ping-Pong balls come in packs of 3. A carton holds 24 packs. Mr. Collins, the owner of a sporting goods store, ordered 1800 Ping-Pong balls. How many cartons did Mr. Collins order?

Process Problem

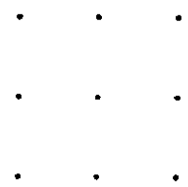
4. A chess club held a tournament for its 15 members. If every member played one game against each other member, how many games were played?

Applied Problem

5. How much paper of all kinds does your school use in a month?

Puzzle Problem

6. Draw 4 straight line segments to pass through all 9 dots in the figure. Each segment must be connected to an endpoint of at least one other line segment.



(R. Charles and F. Lester 1982)

Using the criteria of the Cockcroft Report for definitions of problem solving and investigational work these six types would be interpreted differently. Type 1 would be assigned to consolidation and practice while types 2,3 and 5 would be assigned to problem solving.

Types 4 and 6 are intrinsically different from the others. It is these two that warrant closer examination.

First the Puzzle Problem. Charles and Lester suggest that this type of problem can usually only be solved by a lucky guess or by thinking of an unusual way of looking at the problem. Often they involve a 'trick'. For this reason many people find them frustrating. The 'trick' in this problem is to realise that the straight lines can extend beyond the square of dots.

Second the Process Problem. Charles and Lester suggest a variety of approaches that pupils may adopt to solve this problem. One approach is to use the strategy of simplification. This involves finding how many games would be played if the club had only 2 members, 3 members, 4 members and so on. Pupils then try to identify a pattern and extend the pattern to 15 members.

Another approach is to use diagrams. Two alternative diagrams are suggested.

1)

	Amy	Ben	Cathy	Dave	Opel
Amy		X	X	X	X
Ben			X	X	X
Cathy				X	X
Dave					X
Opel					

(An X represents a game is played)

2)

1-2, 1-3, 1-4, ..., 1-15
2-3, 2-4, ..., 2-15
3-4, ..., 3-15
4-15
⋮
14-15.

The authors claim that the Process Problem is richer than Problem types 1, 2 and 3 as there are several different ways in which to proceed towards a solution. Also, as there is no obvious calculation that will yield up the solution the problem solver needs to think carefully about the problem, to play around with some ideas and to try and come up with a sensible plan of action. The problem solver has to make a conjecture about what to do, test the conjecture, play hunches, perhaps guess and otherwise use a wide variety of thinking processes.

Thus, the sort of problems that involves, amongst other strategies, planning, guessing, estimating, forming conjectures, testing conjectures and looking for patterns is called, by the authors, a process problem.

This is an apt name, for the type of problem is being defined not by what it is about but by the processes or strategies needed to solve it.

It is interesting to compare this view of problem solving with the view of mathematical investigations expounded by David Burghes in his article Mathematical Investigations which appeared in Teaching Mathematics and its Applications Vol. 3 No 2 1984. As with Charles and Lester, Burghes divides mathematical investigations into different types.

> Classification of mathematical investigations.
> We give four categories of investigation. Any classification will have overlaps, often depending on how the investigation is used in the classroom. The four suggested areas of mathematical investigations are : -
>
> A: Eureka investigations
> B: Escalator investigations
> C: Decision problems
> D: Real problems
>
> (D. Burghes 1984)

An examination of these four types shows a remarkable similarity with the analysis of Charles and Lester. I will consider these in reverse order.

Real Problems

In this category, we are dealing with problems of direct relevance. In the form in which they are posed they have little to do with mathematics and mathematical analysis may or may not help to solve them.

> Indeed, often the most difficult part of the investigation is to translate the real problem into a mathematical one, i.e. the formulation of a mathematical model.
>
> (D. Burghes 1984)

Burghes gives several examples, including where to place a rugby ball to maximise the chance of a successful goal-kick, a problem of a shot putter and the relationship of angle of put and speed of put to range and a problem of stock control.

The similarity to Charles and Lesters Applied Problem is quite clear. It can be argued that the Applied Problem/Real Problem should be grouped with the Process Problem as both include strategies of simplification, forming a conjecture and testing a conjecture. I would argue however that they are very different. In for example the case of the shot putter in the Real Problem the simplifications may be to ignore air resistance. This simplification changes the nature of the problem. It uses a strategy of mathematical modelling of saying that an easier or simpler problem to solve has a solution that approximates to the actual problem under consideration.

In the chess club problem the simplification is of a different kind. By solving easier or simpler problems with 2 or 3 or 4 members the nature of the problem is not changed. It is made easier in terms of the calculations but the problem remains essentially the same.

When I come to attempt to draw a distinction between problem solving and investigational work it is this difference which will exclude this type of problem from the general heading of investigational work.

· Decision Problems

These are problems which are related to real problems (i.e. problems originating outside mathematics), but which either have already been turned into a mathematical problem or it is at least clear what the mathematical problem involves.

There is no element of critical judgement in deciding what factors should be included in the problems or what weighting should be given to the factors.

(D. Burghes 1984)

Burghes gives several examples, amongst them the travelling salesman problem. In this problem there are a number of towns with known distances between them. The problem is to find the route which starts and finishes at a given town, passes through all the towns and which has the minimum total journey length.

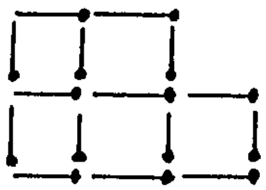
There is said to be no judgement required as the problem can be solved, perhaps at great length, by considering every possible route and its associated distance. The 'Decision Problem' is in fact a 'Real Problem' after the simplification has taken place. The judgement occurs in the 'Real Problem' as to what factors to leave out, and what weight to give to those factors included.
As this type of problem is included in a 'Real Problem' I would not therefore include this in the classification of investigational work.

Eureka investigation

This type of investigation is characterised by having a 'breakthrough' to solve the problem. Usually little useful progress can be made until the breakthrough has been made - the solution then becomes obvious.

(D. Burghes 1984)

Burghes gives several examples, one of which is a traditional matchstick problem. The object is to remove three matches so that you are left with only three squares.



Now certainly many pupils would stare at this problem, perhaps trying one or two possibilities but generally waiting for inspiration. The procedure fits Burghes definition well. However it could be likened to the decision problem in as much as one could systematically remove all possible combinations of three matches until left with three squares. Such an approach would be tedious and could be short circuited by an inspiration or judgement or whatever one would care to call it.

A further example that is given satisfies the definition more exactly. The object here is to prove that the sequence

$$1!, \quad 1! + 2!, \quad 1! + 2! + 3!, \quad \ldots$$

has one and only one square number in it (excluding 1).
Being an infinite sequence the Decision Problem method is not a feasible method of solution. Some 'trick' or 'breakthrough' or 'insight' is required. This matchstick problem will be entitled an Eureka Problem rather than Eureka Investigation as it is similar to the Decision Problem except that a 'breakthrough' can lead to a quick solution.

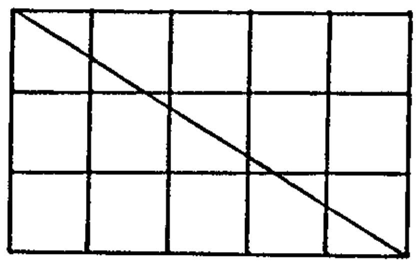
It is the fourth type of investigation that brings us closer to what is generally accepted as an investigation.

> Escalator investigation
>
> These investigations are characterised by being able to achieve some success throughout the investigation. In other words, success is not dependent on a vital breakthrough.
>
> (D. Burghes 1984)

Once again the investigations included under this heading seem to lack a common unity. One example he gives is the four "fours" problem. Four fours and the usual mathematical signs and symbols are used to make the integers from 1 to 20. This seems to be a collection of Eureka Problems. Each number is independent of the others or nearly so. It is true that having obtained 8 as $4 \times 4 - 4 - 4$ one can permutate the last two signs to obtain $16 = 4 \times 4 - 4 + 4$ and $24 = 4 \times 4 + 4 + 4$

Compare this with another example he gives called 'cross-overs'.

> The diagonal drawn on a 3 x 5 grid makes 6 cross-overs with grid lines.
>
> Find the number of diagonal cross-overs for the following grids :-
>
> (i) 4 x 7, 6 x 11, 7 x 9, 5 x 2
>
> (ii) 4 x 6, 3 x 6, 5 x 15, 3 x 9, 6 x 15
>
> Find a formula for the number of cross-overs for a grid of size a x b when
>
> (i) a,b are co-prime
>
> (ii) a,b are not co-prime and c = h.c.f.(a,b)



(D. Burghes 1984)

The difference here is that a formula or generalisation is required. This is taking the investigation a definite step beyond the mere collection of data or results. The results need to be scrutinised, a pattern spotted and a conclusion arrived at.

The Cross-overs investigation involves considerably more strategies or stages in solution than the other investigations listed. Suppose the investigation was posed in its barest form : -

> Find a formula for the number of cross-overs for a grid of size a x b.

The investigation might be tackled in the following way.

1) Several rectangular grids could be drawn, the diagonals drawn in and the number of cross-overs counted;

2) The results could be recorded in the form 3 x 5 : 6 cross-overs;

3) A conclusion would be drawn from the results.

The strategies used in the solution of this investigation could be listed as follows : -

1) Use helpful diagrams
2) Tabulate the results
3) Spot a pattern
4) Make a generalised statement.

We begin to see then the common ground of mathematical investigations being defined in terms of the strategies or procedures used to arrive at a solution of the mathematical investigation.

In a further article entitled Investigations in Teaching Mathematics and its Applications Vol 3 No 3 1984 Paul Ernest examined a mathematical investigation in detail noting the strategies or stages in its solutions which he lists as the features of an investigation. It is this detailed analysis of a particular mathematical investigation that gives us a clearer idea of what constitutes mathematical investigations in general.

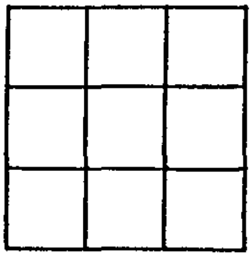
We will consider a mathematical investigation posed to an experienced practitioner. The question

"How many squares are there on a chess board?"

provides the starting point for the investigation. Unlike the cross-over investigation referred to above no hints or clues are given as to where to start.

Notice too that this starting point is presented in the form of a particular problem. Charles and Lester could classify this as a Process Problem while Burghes would classify it as an Escalater Investigation. As the solution of the chess board problem is examined we will see the similarities between the two views of the chess board problem, one as a piece of investigational work and one as a piece of problem solving.

Let us consider the chess board problem. There are many squares of many different sizes ranging from the obvious 1 x 1 through 2 x 2, 3 x 3, 4 x 4, 5 x 5, 6 x 6, 7 x 7 up to the single 8 x 8 square. Obviously there are 64 1 x 1 squares, slightly less 2 x 2 squares and so on down to a single 8 x 8 square. Rather than count them all at this stage we could simplify the problem. This gives us a way into the problem. We take a simpler, miniature of the problem and consider a 3 x 3 square.



Counting the different size squares we can tabulate our results.

Size	1 x 1	2 x 2	3 x 3
Number of squares	9	4	1

So we have a total of 1 + 4 + 9 = 14 squares in all.

Consideration of a 4 x 4 square board gives these results.

Size	1 x 1	2 x 2	3 x 3	4 x 4
Number of squares	16	9	4	1

giving a total of 1 + 4 + 9 + 16 = 30 squares.

Considering the trivial cases of a 1 x 1 board and a 2 x 2 board which give respectively

Size	1 x 1
Number of squares	1
Total of	1 square

Size	1 x 1	2 x 2
Number of squares	4	1
Total of 1 + 4 =	5 squares	

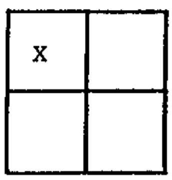
We are in a position to pause and look for a pattern from which we can make a prediction. Putting the results for the four boards together we obtain

Board Size	Number of squares
1 x 1	1
2 x 2	1 + 4 = 5
3 x 3	1 + 4 + 9 = 14
4 x 4	1 + 4 + 9 + 16 = 30

Having spotted the pattern as the sum of the square numbers we are now in a position to make a hypothesis. This is that for a 5 x 5 board the total number of squares will be 1 + 4 + 9 + 16 + 25 = 55.
We now check our hypothesis by counting the squares and find that, in this case at least, it is correct. We can predict, with some confidence, that for an 8 x 8 board there will be 1+4+9+16+25+36+49+64 = 204 squares in all.

At the stage we have now reached, the particular problem is complete as the solution has been found. We ought really to prove or at least justify or explain the correctness of the solution.

We take a 2 x 2 square and mark a cross in the top left hand corner square



This 2 x 2 square can be placed at the top of the chess board. The cross can only occupy 7 of the squares of the chess board in the top row. As the 2 x 2 square is moved into different positions on the 8 x 8 chess board we obtain a pattern of X's as shown

X	X	X	X	X	X	X	
X	X	X	X	X	X	X	
X	X	X	X	X	X	X	
X	X	X	X	X	X	X	
X	X	X	X	X	X	X	
X	X	X	X	X	X	X	
X	X	X	X	X	X	X	

We can conclude therefore that whatever the size of board there will be one less this number, squared, 2 x 2 squares.

A similar argument gives the number of 3 x 3 squares on a board as $(n - 2)^2$ if the board is $n \times n$. By continuing this argument we arrive at the generalisation for an $n \times n$ board that the number of squares is $1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2 + n^2$.

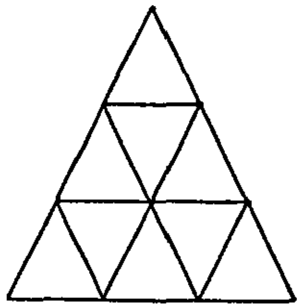
Not only have we proved the result for the chess board but we have generalised the result to cover any $n \times n$ board.

Although this particular investigation is complete it can provide the starting point for a number of similar investigations. The investigation has the potential to be extended. Possible extensions are : -

1) What about the number of squares on a $m \times n$ rectangular board?
2) What about the number of rectangles on an $n \times n$ square board?

3) What about the number of rectangles on a $m \times n$ rectangular board?

4) What about the number of triangles on grids like the one below?



5) What about the number of cubes in an $n \times n \times n$ cube?

Ernest lists the features that often occur in an investigation and could certainly occur in the investigation above as these :

The initial problem or situation.
An exploration of the situation. Looking for a way in.
The accumulation of facts from the exploration.
The tabulation or ordering of the facts, in the hope that a pattern might stand out.
Conjectured patterns coming out of facts.
Testing the conjectures against new data.
Possibly a disproof of the conjecture by an example that does not fit.
Modification of the conjectures so that it fits the new data, or generation of a new conjecture.
Confirmation of the conjecture by new data.
Attempts to justify (or even prove) the conjecture.
Generalisation of the conjecture.
New situations, or problems suggested as extensions of the original situation.

(Ernest 1984)

In this examination of what constitutes investigational work we have seen the overlap between the views of Charles and Lester with what they consider as problem solving and the views of Burghes and Ernest with what they consider as mathematical investigations. Certainly there are types of problem solving that we can detach from inclusion in mathematical investigations. These include problems such as the shot putting problem mentioned above. This is more of a mathematical modelling problem than a mathematical investigation. However the chess club problem classified by Charles and Lester as a Process Problem certainly comes under the umbrella of mathematical investigations.

Ernest makes the point well :

> Of course the difference between problem solving and investigatory approaches is just a matter of degree or even terminology.
>
> (Ernest 1984)

Ernest however goes on to clarify the difference as follows.

> The student may solve the problem, but the investigation does not go beyond problem solving until the student generalises the problem and tries to solve that: or until the student thinks up related questions or goes beyond the starting task in some other way.
>
> (Ernest 1984)

As we saw with the squares on a chess board problem this particular Process Problem is identical to a mathematical investigation up to the stage where the problem is generalised and solved or where it is used as generating further starting points.

I would therefore describe a mathematical investigation as a particular part of problem solving characterised by the features listed previously by Ernest.

In examining investigational work we have so far considered mathematical investigations and the types of problem solving that are a subset of mathematical investigations.

The mathematical investigations have been of the form where a pupil brings his or her collection of facts, skills and conceptual structures to a particular problem or investigation and uses the strategies learnt to solve the problem or complete the investigation. The emphasis has been on application of already acquired expertise rather than on furthering this expertise. True the pupil may improve or polish the expertise already gained but this is a by product rather than the reason for doing the mathematical investigation.

We can view mathematical investigations then as a culmination of or raison d'être for doing mathematics. From a particular starting point the pupil has the freedom both to choose the particular way he or she develops or defines the investigation and the methods of solution that will be used. The pupil is essentially the controller.

In the same way that there is a merging between problem solving and investigational work so there is a merging between pupils controlling the development and methods of solution of an investigation, and being directed by a teacher to use the same strategies to solve a problem where the teacher has a specific goal in mind. There is a style of teaching or learning that involves a pupil in the use of some of the features of mathematical investigations to learn certain facts in mathematics, to acquire certain skills and increase the pupils knowledge of certain concepts. Ernest names this approach 'Guided Discovery'. He distinguishes between 'guided discovery', 'problem solving' and an 'investigatory approach' as follows :

Method	Teachers Role	Students Role
1. Guided discovery	Poses problem, or chooses situation with goal in mind. Guides student towards solution or goal.	Follows guidance
2. Problem solving	Poses Problem. Leaves method of solution open.	Finds own way to solve problem.
3. Investigatory approach	Chooses starting situation (or approves students choice).	Defines own problems within situation. Attempts to solve them.

(Ernest 1984)

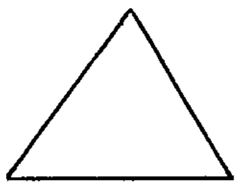
In Mathematics from 5 to 16 HMI consider the aims of mathematics teaching in schools. The aims are to be achieved through certain specific objectives. HMI consider five main categories of these objectives :

A. Facts
B Skills
C Conceptual Structures
D General Strategies
E Personal Qualities

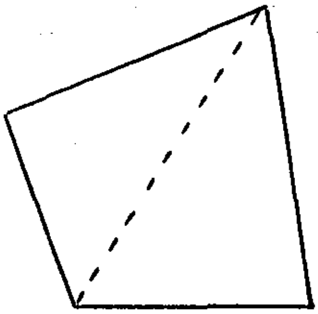
We will examine the role of investigational work in implementing a mathematics curriculum in more detail in Chapter 2. For the moment we will see how an investigatory approach can be used to aid the teaching of the first three objectives. As we will see later the fourth objective, General Strategies has been covered by investigations. The fifth objective, Personal Qualities, will also be examined in the next chapter.

First the acquisition of facts. It is possible, in some instances, to so direct an investigation that a pupil may arrive at the facts to be acquired without being directly told them. It will of course be necessary to remove the freedom of choice available in an investigation in order that prescribed avenues of thought are followed. This can often be done by explicitly telling the pupil what diagrams are helpful, what simplification to make, how to build up the diagrams and finally how to tabulate the information gained.

Perhaps one of the most common examples of this is the derivation of the formula for the sum of the interior angles of a polygon. The self discovery of the formula may proceed as follows.

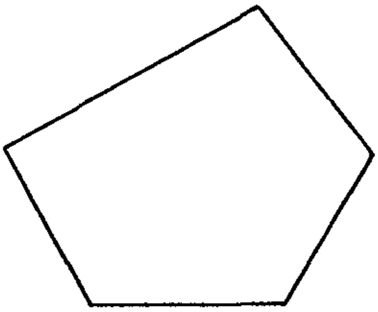


What do the angles of a triangle add up to?



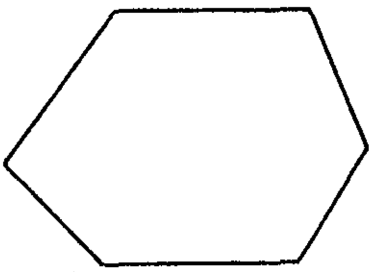
A

The quadrilateral has one diagonal drawn in from vertex A.
It divides the quadrilateral into two triangles.
What is the sum of the angles of these two triangles?
What is the sum of the interior angles of the quadrilateral?



A

Draw in all the diagonals of the pentagon through the vertex A.
How many triangles is the pentagon divided into?
What is the sum of the angles of these triangles?
What is the sum of the interior angles of the pentagon?



A

Draw in all the diagonals of the hexagon through the vertex A. Use the triangles to find the sum of the interior angles of the hexagon.

Repeat for a seven sided polygon.
Copy and complete the following table.

Number of sides	Sum of interior angles
3	
4	
5	
6	
7	

Can you see a pattern in the second column?
Predict what the sum of the interior angles will be for an octagon.

Draw an octagon to check.

What will the sum of the interior angles be for a 9, 10, 11, 12, 13 sided polygon?

Write down the formula for the sum of the interior angles of an n sided polygon.

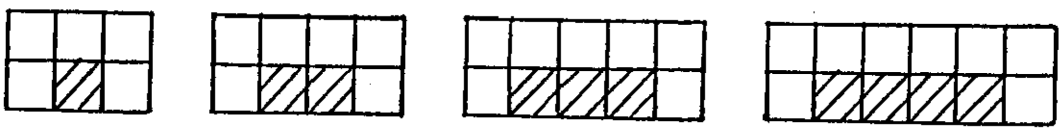
In this example the pupils is lead through some of the stages or strategies of a typical mathematical investigation. In this case these are

1) use a helpful diagram;
2) tabulate the results being systematic;
3) spot a pattern;
4) form a hypothesis and test it;
5) make a generalisation.

Second the acquisition of skills. The Cockcroft Report defines skills as follows :

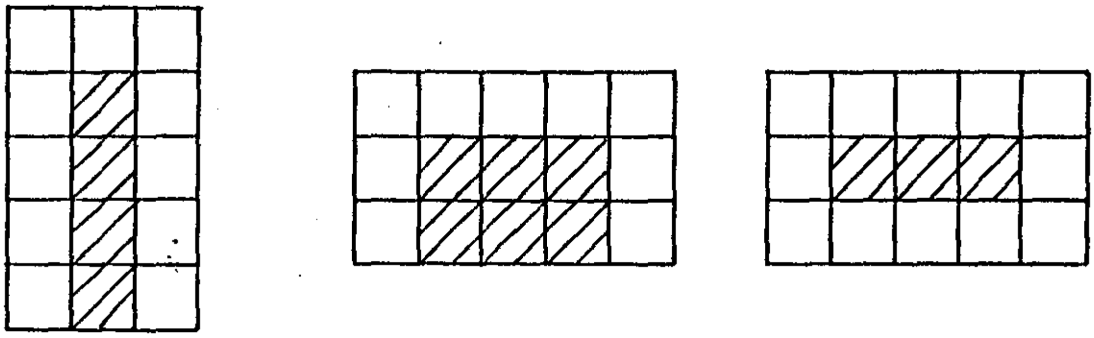
> Skills include not only the use of the number facts and the standard computational procedures of arithmetic and algebra, but also of any well established procedures which it is possible to carry out by the use of routine.
>
> (Cockcroft 1982)

In the SMP 11 -16 mathematics scheme algebra is introduced through the booklet Discovering Rules. Use is made of small red and white square tiles. Pupils are required to construct a border of white tiles around a row of red tiles in the following fashion.



Pupils are then instructed to fill in a table with columns for red tiles and white tiles. The pupils are then lead to obtain the number of red tiles given the number of red tiles as a formula $w = r + 4$.

The same procedures are carried out for various different enclosures examples of which are shown below.



The strategies involved here are

1) use a helpful diagram
2) tabulate the results
3) spot a pattern
4) make a generalisation.

By these procedures pupils acquire the algebraic skill of forming algebraic relations from particular problems.

Third, conceptual structures. Conceptual structures are described as follows in the Cockcroft Report :

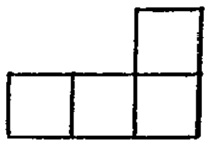
> Conceptual structures are richly interconnecting bodies of knowledge, including the routines required for the exercise of skills. It is these which make up the substance of mathematical knowledge stored in the long term memory. They underpin the performance of a memory failure or to adapt a procedure to a new situation.
>
> (para 240) (Cockcroft 1982)

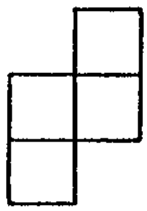
The development of a concept is markedly different from the development of a skill. A skill is usually mastered by repetition without much, if any variety. On the other hand a concept is developed by providing a wide variety of experiences in which the concept is present in some form.

In developing the concept of area it is necessary not only that the pupil remembers certain formulae, such as the area of a rectangle and the area of a triangle, but also that the pupil should have such a grasp of the concept of area that should a new, but only slightly different, situation arise the pupil will be able to adapt a procedure to cope with it.

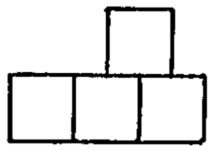
A pupil may think that the area may be connected only with squares and rectangles. This may be so if the only experience has been in square counting with squares and rectangles drawn on a centimetre square grid. Further experience for the pupil can be given by setting the following investigation.

Investigate how many different shapes there are whose area is 4 square centimetres.
Nearly all will start with shapes made up of whole squares such as these

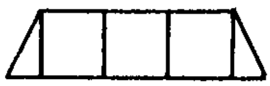




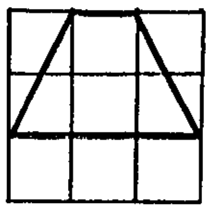
or



Someone, perhaps the teacher, may suggest



Further shapes such as



could arise.

This investigational approach helps pupils appreciate that the area of shape is not restricted to squares and rectangles even if the unit of area measurement is square centimetres. It also offers pupils some hint of a procedure to adopt when asked to find the area of shapes other than squares and rectangles, namely that of dividing up into separate squares, rectangles and right angled triangles.

We can interpret investigational work as being a major part of mathematical education. It embraces an attitude to, or an approach to, or a way of thinking about mathematics. It has a part to play not only in the application of mathematics through what are usually termed 'investigations' or 'problem solving activities' but also in the process of acquiring facts and skills and in the development of conceptual structures by children.

CHAPTER 2

THE ROLE OF INVESTIGATIONAL WORK IN THE MATHEMATICS CURRICULUM.

THE ROLE OF INVESTIGATIONAL WORK IN THE MATHEMATICS CURRICULUM

In the previous chapter we saw how investigational work in mathematics could be considered from two view-points. On the one hand it could be viewed as a problem solving activity where the pupils accumulated mathematical facts and skills together with their appreciation of the conceptual structures could be brought to bear on particular problems or applied to particular starting points. This activity was characterised by a set of features, amongst them simplifying, tabulating results, looking for patterns, hypothesising and testing and generalising. Pupils undertaking this type of investigation to a large extent chose their own lines of development. A group of pupils undertaking an investigation may well arrive at several different but equally valid conclusions. This sort of investigation can be termed divergent.

On the other hand investigational work may be viewed as one of the styles of teaching that enables pupils to acquire facts and skills and to develop their understanding of conceptual structures. While still employing many of the features of a divergent investigation pupils are directed as to which features to use and in what fashion to use them so that a specific objective can be achieved. This sort of investigational work can be termed convergent.

What I intend to examine in this chapter is the role of both of these strands of investigational work in the mathematics curriculum.

There are two perspectives of the mathematics curriculum. One is as certain agencies would wish it to be. Such an agency may at one extreme be the mathematics department of a school. Its views would be enshrined in the departments aims and objectives. At the other extreme the agency may be central government expressing what it considers are appropriate aims and objectives through such documents as Mathematics From 5 to 16. The other perspective is as certain agencies view what is actually going on in mathematical education in the classroom. Such an agency may once again be central government as the monitoring of standards by the Assessment of Performance Unit.

Sometimes these two perspectives are contained in the same document. In Better Schools the Secretary of State for Education and Science presents both the governments view of 'The Present Situation'(Chapter 1) and also the governments views on what the curriculum should be in 'The Primary and Secondary Curriculum' (Chapter 2).

Likewise Mathematics Counts reports specifically on what was the current practice in schools at the time the report was being compiled and also indicates how it feels the mathematics curriculum should be developed.

Important too are the views of examination boards as these influence the mathematics curriculum in secondary schools and hence often reflect the mathematics curriculum of those pupils entered for an external examination. They do not necessarily reflect all aspects of the curriculum as previously the majority of external examinations have only assessed the content part of the mathematics curriculum in terms of facts and skills and to a limited extent the ability to solve problems.

The development of the GCSE however, requiring as it does, to meet the needs of government dictated national criteria will perhaps give another, more representative view of the curriculum. The place of investigational work within the examination system will be examined in greater detail in Chapter 6.

I will examine first the view of the curriculum in general as put forward in Better Schools and the Curriculum from 5 to 16. I will then examine the mathematics curriculum as put forward in Mathematics From 5 to 16.

Better Schools is wide ranging in its coverage of the educational system in England and Wales. Some parts refer to the curriculum in general, some specifically to mathematics. I will pick out those instances where comment is directly relevant to investigational work.

In the first chapter, entitled The Present Situation, the document outlines the progress and improvements made in education over the last thirty years.

In primary schools one such improvement is that now teachers

> teach English and Mathematics in a way which transcends simple skills.
>
> (para 5)

and in secondary schools there is a curriculum which

> widens mathematics beyond computation and increases its depth.
>
> (para 6)

Thus Better Schools sees the mathematics curriculum extending beyond simple skills and computation, beyond then what is generally referred to as 'basic skills'.
Further

> In a majority of schools over-concentration on the practice of basic skills in literacy and numeracy unrelated to a context in which they are needed means that those skills are insufficiently extended and applied.
>
> (para 18)

Investigational work and particularly investigations closely related to problem solving offer a suitable context in which numeracy skills as well as other skills can be practiced in a relevant way.

We have seen that the investigational approach to learning, what Ernest in Chapter 1 referred to as the 'Guided Discovery' approach, requires pupils to take an active part in their learning process. Pupils have to carry out a number of activities, carefully selected by the teacher, so as to arrive at a particular fact or so as to acquire certain skills. This aspect of the investigational approach fits in well with the view expressed in paragraph 15.

> At their best, the schools in England and Wales grapple with their tasks with a strong sense of purpose, ... by motivating them towards active, well directed enquiry rather than passive learning.
>
> (para 15)

The investigational approach to teaching contains within it many other approaches or styles of teaching which enhance the learning process for the child. Certainly Better Schools advocates a widening of horizons in terms of approaches and styles available to teachers.

> Curricular guidelines exist for English and Mathematics .... often do not make explicit the different approaches to learning, ... which are needed as children progress through the school.

The example of the squares on a chess board problem in Chapter 1 illustrated the many different features or characteristics of an investigation. These features enable pupils to partake of a wide range of activities that they would not otherwise have to in a more conventional teaching style.

In the interpretation of the problem or investigation, in deciding what approach to take, in the exploration of the situation, in deciding what facts to tabulate and how to tabulate them, in deciding what conjectures to make and how to test them and in the generalisation of the conjecture there is much opportunity for oral work. This oral work will generally take the form of discussion between pupils as they explore together the mathematical situations that arise. It can also involve discussion between pupils and the teacher as the teacher asks pertinent questions, suggests avenues to follow and generally guides the pupils through the investigation. Also as pupils tackle the investigation they will constantly be making their own decisions about how to proceed in the investigation. They will be assuming some responsibility for the pace, direction and method that the investigation requires. Once again this meets a criticism raised in Better Schools.

> In about half of all classes much work in classrooms is so closely directed by the teacher that there is little opportunity either for oral discussion or for posing and solving practical problems. Pupils are given insufficient responsibility for pursuing their own enquiries and deciding how to tackle their work.
>
> (para 19)

As individual children or groups of children work on an investigation there arises, quite naturally, the opportunity to match work to the pupils needs and capabilities. The criticism often voiced is that in many classrooms the pace of the lesson is often geared to the middle level of ability. This means that the more able pupils are insufficiently stretched and therefore spend a great deal of their time practising skills that they have already mastered. On the other hand the less able are tackling work that pushes them beyond what they are mathematically ready for.

The problem arises when selecting skill and concept objectives for a group, which even if setted, contains a wide range of mathematical ability and is at different stages of mathematical maturity.

The selection of a suitable investigation for a group of children together with judicious counselling by the teacher enables pupils of all abilities to be stretched to their fullest capabilities. This is by no means an easy task. A balance needs to be struck between direction which can become

> often excessive direction by the teacher of pupils' work
>
> (para 25)

and providing opportunities for pupils to learn for themselves through discussion, to develop their ideas through a process of struggling with the problem in hand. Too frequently teachers intervene to make life easy for pupils and consequently have to intervene again and again as pupils fail to build up the necessary confidence to progress on their own.

Better Schools encompasses all aspects of the educational scene. The role of investigational work in the curriculum can be examined in more detail in specific curriculum documents published by the DES. Curriculum from 5 to 16 seeks to stimulate discussion about the whole curriculum of pupils in compulsory education. This document then gives a context in which to set the mathematics curriculum.

It is pertinent to remember that a school's curriculum is more than just a collection of subjects to be taught. It encompasses all those activities designed or encouraged within the schools organisational framework to promote the personal, social and physical development of children. It will include not only the formal programme of lessons, usually hung on a subject framework, but also the 'informal' programme of extra curricular activities such as the general 'ethos' of the school and the way in which the school is organised and managed. Thus the mathematics curriculum stretches beyond facts, skills and concepts to include attitudes as well.

Curriculum From 5 to 16 lists five educational aims, three of which have a direct bearing on the mathematics curriculum. These are :

1) to help pupils to develop lively,enquiring minds, the ability to question and argue rationally and to apply themselves to tasks and physical skills;

2) to help pupils acquire knowledge and skills relevant to adult life and employment in a fast changing world;

3) to help pupils to use language and number effectively.

The use of language is of concern in mathematics. In Aspects of Secondary Education (HMSO 1979) concern was expressed that the curricula in many schools, was dominated by writing mainly of the kind requiring notes or summaries. This was particularly true in years 4 and 5 where the impending external examinations greatly influenced the curriculum. While mathematics does not often involve a large amount of writing what little that does take place is often of a formal note taking kind. A consequence of this is that talk tends to be squeezed out of lessons. Talk is a vital element in enabling children to come to grips with new ideas and to internalise their understanding of these ideas. Children need to be able to develop reasoned argument, involving the putting forward of ideas, and their discussion and refinement. Investigational work offers an opportunity for the development of the 'talk' aspect of language.

It is often difficult to provide a situation where constructive talk can take place. If groups of 4 or 5 children are embarking upon an investigation together the opportunity for talk arises naturally. They discuss their interpretation of the problem, how to set about making progress, they float ideas and generally use the whole range of verbal communication to help them in their work.

While talk is taking place within the group some are using the complementary activity of listening. This too is a vital element for without talk and listening there is no communication.

Talking/listening is one of the two axis of communication. Talking is the first step in internalising thoughts and of being certain in ones own mind of ones understanding. Reading/writing is the other axis. Verbalising ones ideas is a relatively coarse method. If one is not clear the listener can interrupt and ask for clarification or further explanation. The talker has immediate feedback and is able to adjust his or her output.

There is no such luxury with the written word. It is not usually read in the presence of the writer. The written word requires a greater precision and therefore a greater clarification of thoughts in the writers mind. This aspect is provided when children 'write up' their investigational work.

The first aim listed above is one that all teachers would hopefully aspire to. A prerequisite of a lively, enquiring mind is the opportunity for enquiry to take place. Enquiry in mathematics entails children following their own trains of thoughts and ideas without being too constrained by conditions imposed by the teacher. This is not to suggest that children have 'a blank sheet of paper' and pursue what ever they fancy. Some starting point is necessary and some direction essential to facilitate enquiry. Investigational work provides the opportunity for children to develop enquiring minds. Within broad guidelines children can take various lines of enquiry.

The multiplicity of lines of enquiry emanating from one starting point is well illustrated by an investigation entitled 'Magic Squares'.

Children are asked to place the numbers 3,3,3,5,5,5,7,7,7 in a 3 by 3 square so that each row, column and diagonal adds up to 15.

The following four solutions are arrived at.

7	3	5
3	5	7
5	7	3

5	3	7
7	5	3
3	7	5

5	7	3
3	5	7
7	3	5

3	7	5
7	5	3
5	3	7

From here children can pursue many different ideas amongst them :

1) what other possible combinations of numbers in the form a,a,a,b,b,b,c,c,c, would give a total of 15?

2) what combination gives other totals like 12 and 18 and how are the number of combinations related to the total?

3) can the idea be extended to 4 by 4 squares with numbers of the form a,a,a,a,b,b,b,b,c,c,c,c,d,d,d,d ?

4) can all the solutions be obtained from one solution by rotations and reflections?

5) when the four solutions are put together they make a 6 x 6 magic square with certain symmetric properties. What are they and how do they change if you change the order of the squares?

6) what about 3 by 3 magic squares with the integers 1 to 9?

7) what about 3 by 3 squares where no row, column or diagonal adds up to the same total?

The ability to question and argue rationally can take two forms. The first when the child questions and argues with another as happens when two or more children are involved with an investigation as a group. The second when the child questions and argues with him or her self when attempting to come to some conclusions about a piece of work. Investigational work offers a format in which questioning and arguing rationally can take place.

The second listed aim concerned with helping pupils to acquire knowledge and skills relevant to adult life and employment in a fast changing world is one that requires further amplification. In the past the mathematics taught in schools has tended to be of a kind more concerned with acquiring sophisticated techniques rather than with applying these techniques to a wide range of problems. Partly this has been because what was seen as being relevant to the world of work was thought of as being either constant or at worst changing slowly. The skills and knowledge would be learnt in the classroom and then used in the workplace.

As technology has developed it has become more obvious that skills and knowledge acquired in the classroom may well be inappropriate for the workplace in 10 or 15 years time. What is more important is an expertise in applying a limited amount of knowledge and skills in a creative way to a wide range of problems. This requires the opportunity to develop strategies and processes of enquiry and problem solving. Investigational work is predominantly involved with this emphasis.

In highlighting the role of investigational work within the curriculum as a whole we should not forget that investigational work can take on many different forms of organisation and that it forms one of several different varieties of learning and teaching experiences available to the child. Curriculum From 5 to 16 points out that no single style of teaching is suitable for all purposes. Sometimes it will be appropriate for the class to work individually, sometimes in pairs, sometimes in groups of 4 or 5 and sometimes as a whole class.

Curriculum From 5 to 16 then considers in detail the areas of learning and experience. These include aesthetic and creative, human and social, linguistic and literary, mathematical, moral, physical, scientific, spiritual and technological. The role of mathematical investigational work in these areas will be examined in more detail when I consider Mathematics From 5 to 16.

Curriculum From 5 to 16 considers a second perspective to areas of learning and experience namely that of 'elements of learning'. The four elements of learning are found and need to be developed through the nine areas of learning and experience. These elements are :

knowledge
concepts
skills
attitudes

While they will be examined in detail in Mathematics From 5 to 16 it is worth remembering that they do form a part of a general approach to the curriculum as a whole.

There is more knowledge that is potentially useful and appropriate to children in school than can possibly be assimilated in the time available. Much of mathematics syllabuses are overladen with factual content built up over a long period of time. Schools will therefore need to be selective in deciding what is to be taught.

As well as being understandable to children, and worth knowing, the content taught should form an integral part of the curriculum and be such as to achieve the aims agreed upon. Mathematical investigations shift the focus from content in isolation to content in use, in a particular problem. Knowledge is then seen not as an accumulation of unrelated facts that may in the future be of some use but as a more limited collection of items that are of great use in solving particular problems.

An over emphasis on collecting many items of knowledge can have the consequence that children are unable to relate one item to another. The forming of concepts involves the making of generalisations from a wide range of different experiences embodying the same themes. Investigational work enables children to be presented with a variety of situations from which, eventually, concepts may be extracted. It is this way of approaching learning, the presenting of experiences enabling concepts to be extracted rather than the telling of the concepts and the presenting of situation where they arise that is most effective in enabling children to develop a good understanding of concepts.

Skills form a vital ingredient of any curriculum. They are the capacity or competence to perform a task. Although a skill may be specific, as for example in the addition of two numbers, it is applicable in a wide variety of contexts. The usefulness of a skill is that it enables a particular activity to be completed successfully. It is therefore important that the acquisition of skills, achieved by practice, should go hand in hand with their application. Skills are therefore best acquired in situations that are in themselves relevant and worthwhile and seen as such by children.

Although the skill referred to as the addition of two numbers is a mathematical skill there are more general skills that run across areas of learning and experience and which have many opportunities for development in investigational work.

Communication is probably the most obvious skill that permeates nearly all activity in schools.

Communication is often thought of as the ability to speak and to listen, to write and to read. It involves too the ability to communicate using diagrams, graphical representation, charts, tables and symbols. Communication is fundamental to mathematics.

> We believe that all these perceptions of the usefulness of mathematics arise from the fact that mathematics provides a means of communication which is powerful, concise and unambiguous.
>
> (Cockcroft 1982)

Further skills include that of study skills, of problem solving skills, of personal and social skills and of attitudes.

Study skills involves being able to recognise relationships or patterns within information or data, the ability to select and extract relevant information from a variety of sources and having collected information to interpret it and to be able to draw conclusions from it.

Problem solving includes the processes of being able to pose pertinent questions, to be able to put forward hypotheses and to devise ways of testing these hypotheses, to be able to reach a conclusion and to be able to make predictions on the basis of these conclusions.

Personal and social skills essentially involve working with other people. They require a consideration of other people's views, of being able to contribute to a group, to co-operate and where appropriate to take a lead within the group.

Attitudes are the manifestation of personal qualities and values. They involve such attributes as independence of thought, co-operation and persistence and many more that teachers would hope are engendered in children during their time at school. Often in the usual mathematics curriculum provided by schools there is little opportunity for independence of thought or for co-operation between children.

Frequently children are presented with work that takes them down a very narrow, prescribed course of action. They have very little chance to pursue ideas that they have generated themselves.

Children, too, tend to work individually. The answers to a problem must be their answers, arrived at on their own without help from others with the exception perhaps of the teacher. Rarely do children have to work together in a small group of 3 or 4 to produce a joint piece of work. The skills associated with co-operating, namely, listening, talking, discussing, being tolerant of others, arriving at a consensus, sharing out tasks etc., are not given the opportunity to develop. All of these skills form part of and are developed through investigational work.

Having examined the role of investigational work in mathematics in the curriculum as a whole I will now examine its role in mathematics.

As we might well expect our examination of the mathematics curriculum will mirror very closely the curriculum in general. Just as general curriculum aims and objectives cut across subject boundaries so specific mathematical aims and objectives will cut across areas within mathematics. While it is possible to take the aims and objectives as expoused by Mathematics From 5 to 16 and match them with investigational work in mathematics, we need to bear in mind that there are other elements of the mathematics curriculum that have contributions to make. Mathematics From 5 to 16 lists 10 aims of mathematics teaching. Some of these aims have been covered previously and will be only lightly touched on.

1. <u>Mathematics as an essential element of communication</u>

   Mathematics offers a way of interpreting the world we live in. It provides a system for describing, for hypothesising and verifying, and for predicting. It provides a means of working with our ideas and observations and above all for interpreting and communicating them to other people. The Mathematics curriculum must be able to provide for all of these aspects of communication.

However proficient a child may be in performing some operation, such as the multiplication of two numbers, there is something seriously wrong if the child cannot both give an example of when the operation might be used and whether or not the answer is a reasonable one.

All too often mathematics education has been involved with the learning of operations out of a context in which they are meaningful. Children have therefore gained little insight into when the operation might be useful and to the reasonableness of an answer.

In investigational work mathematics is put in a context. By being given a problem but not the information about how to solve the problem children develop the skill of interpreting the problem and the subsequent information gained. They develop the powers of analysis and of communication of information and ideas as they follow the investigation through. Sometimes this communication may be at a very simple level in the form of diagrams or a table of results. Some form of description or generalisation may be attempted in the form of written words. At the highest level of sophistication it may be in the form of algebraic symbols.

As the investigational work has produced some ideas from the child rather than the performing of a particular skill there arises naturally a need for the child to communicate these ideas to others. Investigational work provides an opportunity for, in fact requires, communication to take place.

2. Mathematics as a powerful tool

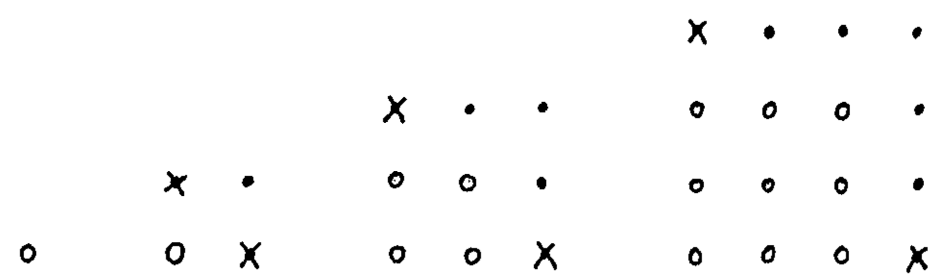
The nature of mathematics, its structure and symbols offer a powerful tool in the solution of many problems. For most of the time, for the majority of people, it is the final result, the solution of the problem that is of paramount importance rather than the means used to obtain it. Unfortunately mathematics teaching in schools has tended to concentrate on the acquisition of skills in isolation, devoid of a context that gives justice to the power of mathematics to solve a wide range of problems.

Investigational work, particularly that closely related to problem solving, has a different approach. The object is to arrive at some solution to a problem or to discover something about a particular topic. The emphasis is on the result and its consequences rather than the particular skills or techniques that have been used. Thus starting with a particular problem and achieving a solution gives some meaning to the skills and techniques used.

3. <u>Appreciation of relationships within mathematics</u>

Mathematics in schools is sometimes taught as a seemingly arbitary collection of items with little or no relationship between the items. This detracts from the pupils understanding of mathematics. Relationships between items enables children to develop a better understanding of mathematics as relationships give a different view on a topic. In investigational work children are encouraged to use helpful diagrams and to tabulate results. This, in a large number of cases, gives an immediate link between a pattern developing pictorially and the same pattern developing numerically. Thus the two views of the same idea support and reinforce each other.

This can be seen in the investigation involving the sum of consecutive odd numbers starting from 1.



By translating the last group of dots added to the previous square and adding 2 more (indicated by crosses) another square is formed. As the two added crosses give the next consecutive odd number it can be seen that a square number is obtained each time.

4. <u>Awareness of the fascination of mathematics</u>

For many children there will be some moment, or hopefully several moments, during their mathematical education when they will experience the fascination of mathematics. This fascination may take many different forms and be experienced at any level. It may come from the appreciation of pattern in multiplication tables by children in Junior School. It may come from the aesthetic appeal to order that the integers from 1 to 9 can be used to make a magic square. Above all the spark of fascination comes from the child. It cannot often be given to a child by the teacher pointing out something that fascinates the teacher. It can often come from the teacher providing an environment or a context which offers opportunities for this spark to take place. Mathematical investigations have many starting points which are rich in possible generators of fascination.

5. <u>Imagination, initiative and flexibility of mind in mathematics</u>

The state of the art of mathematics today is the result of numerous individuals over the centuries exercising imagination, initiative and flexibility of mind when tackling various problems. Progress is made and the store of knowledge increased by such a process. This process is essentially a creative activity. Many children in schools do not experience this creative aspect of mathematics. They experience at second hand and from a distance other peoples achivements. It is rather as if art at school consisted of children viewing and discussing works of art rather than children actually doing painting or whatever. In painting each childs picture will be unique and whatever its artist merit will none the less be a creative work. In mathematics it is more difficult, at times even impossible, for children to provide an original, unique piece of mathematics. However if the discovery is new for the child then it is as creative for that child as any original painting or poem.

Investigational work often entails giving a child a starting point which enables the child to exercise his or her creative ability.

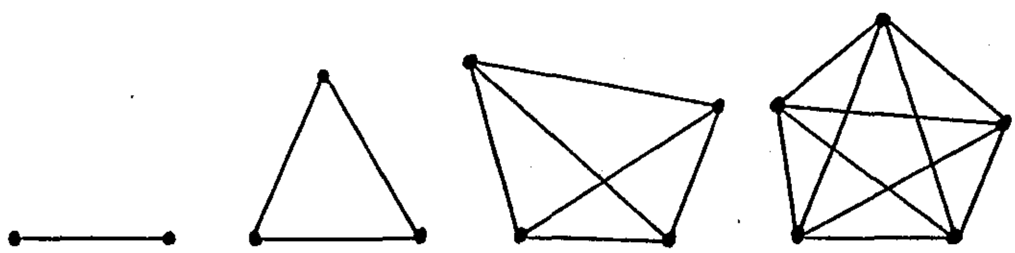
If the child can arrive at a formula or a conclusion that is a step on from what he or she already knew or had been told then, however trivial that conclusion may be in the context of mathematical knowledge, it is a genuine act of creativity.

6. <u>Working in a systematic way</u>

It is generally accepted that careful and accurate work is necessary when tackling any mathematical task. This is often thought of only in terms of precision in the execution of number operations in algebraic manipulations and constructions in geometry. Working in a systematic way goes beyond this. In the previous aim the impression may have been given that creativity 'comes in a flash', that there is a sudden moment of insight when the previously hidden pattern in the mathematics is revealed. This may be how the creative process works. It is however usually preceded by careful and systematic gathering and organisation of information. This can be clearly seen in the handshake investigation.

> Investigate the number of handshakes that take place when everyone in a group shakes hands with everyone else just once.

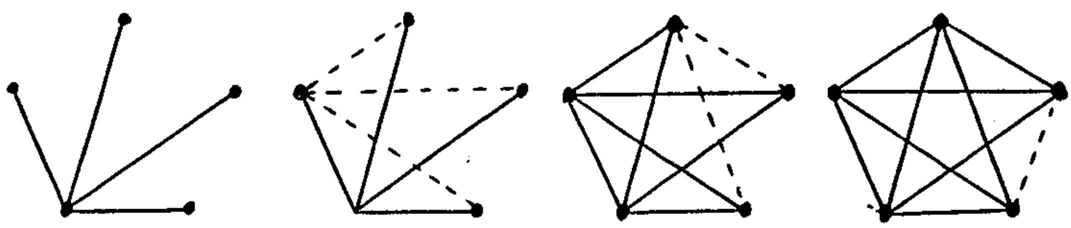
If the investigator precedes in a systematic way it is possible to build up a table of results from which a pattern can be spotted and a rule extracted.



number of people in the group	number of handshakes
2	1
3	3
4	6
5	10

It is instructive to note the influence that being systematic has on the solution of this particular problem. First the desire for a table of values imposes upon the initial mathematics a strategy of starting with the simplest possible case, namely 2 people in the group and increasing the number of people in the group by 1 each time.

Second after having tabulated the results and seen the pattern of triangular numbers the systematic approach offers a way of obtaining an algebraic formulae for any size group. By being systematical in joining the dots, representing the people in the group, a further pattern emerges.



So for 5 people in the group the number of handshakes is

$$4 + 3 + 2 + 1$$

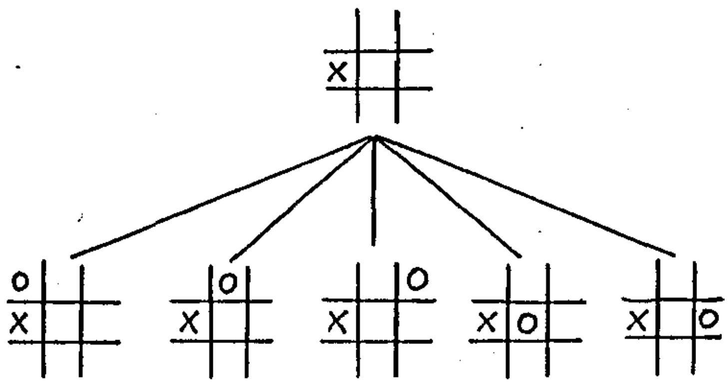
Thus for n people in the group the number of handshakes is

$$(n - 1) + \ldots + 5 + 4 + 3 + 2 + 1$$

The method of adding the terms of an arithmetic progression by namely writing the numbers in reverse order underneath and adding gives the formulae $\frac{1}{2} n (n + 1)$

Being systematic clarifies the thought processes. As well as the strategies mentioned already it involves a process of appraisal and review as various approaches are tried. This type of systemisation proves particularly useful in investigation where many alternatives present themselves.

Consider an investigation of the game of noughts and crosses. In searching for a successful strategy it is important to ensure that no possible series of moves is omitted. A systematic 'tree' approach ensures that all possible moves are covered.



7. <u>Working independently</u>

Working independently does not just refer to the organisation of children in the classroom. It also refers to the way in which a child approaches the task upon which they are working. It is connected with the fifth aim of using ones imagination and initiative. Working independently involves children in being able to cope with slightly unfamiliar situations without always needing the teacher to direct them. It involves children being able, to a certain degree, to take responsibility for the direction in which their learning is going. Decisions as to what information to use, or even to seek, what techniques to use, what methods to employ and probably most important of all the asking of questions .. why? ... what if I change this? ... is there another way? ... is there a better way? ... are all indicators of an independence of thinking and approach that investigational work encourages and, to a large extent, cannot function without.

8. Working co-operatively

Working co-operatively may seem an antithesis of working independently but it is not. Working independently involves a frame of mind in doing mathematics as indicated above. Working co-operatively is an organisational aspect of mathematics. In fact children can work independently and co-operatively at the same time.

Working co-operatively involves two or more children exchanging ideas, listening to anothers point of view, perhaps allocating different parts of a large investigation to individual members of the group. Suppose a group is investigating the tessellations of pentominoes. Individual members of the group may take two or three of the pentominoes and produce the tessellations for these.

Working co-operatively is more than this though. In the example above each individual is working separately at an identical activity. Sometimes children will need to work together, while doing two different tasks. Two children may be investigating the distance different prisms roll down a slope. One will be releasing the prism and the other measuring.

Investigation work in mathematics is a complex activity. There are many tasks within the main activity that can be shared and hence require co-operation.

9. In depth study in mathematics

Mathematics is often taught in a very compartmentalised way. Lessons are often self contained packages of knowledge where a skill is learnt and practiced before the next skill is learnt and practiced. It is not uncommon for a text book to be organised so that children complete a chapter a week or at most a fortnight. There is little opportunity to study a topic in depth.

Investigational work is different in nature to this. Investigations are usually open ended or divergent. A good investigation 'asks' as many questions as are answered. There will be many avenues all of which may be followed for a short way or a few which may be followed for a considerable distance.

Under the heading 'Working in a systematic way' the game of noughts and crosses was considered as an example of an investigation. There are of course a limited number of possible moves in this game and the investigation of noughts and crosses may soon be completed. Other, relatively 'simple' positional games such as De Bonos L game and Four in a Line, may be investigated in a similar fashion. Thus the strategies of a number of positional games may be investigated in depth.

10. <u>Pupils' confidence in their mathematical abilities</u>

Mathematics is a difficult subject both to learn and to teach. For children to make progress they need a delicate balance of challenge and struggle on the one hand and success and achievement on the other. This requires a holding back by teachers so that activities can be provided that enable children to use the knowledge and skills that they have mastered in a relevant way that leads to success rather than pushing children on to new content for which they are not ready or at which they might experience repeated failure.

Investigational work provides a format in which children can gain confidence in their mathematical abilities. Provided that the teacher exercises judgement in selecting a suitable starting point that is a rich enough investigation then a child will be doing mathematics at a level at which he or she chooses and can cope with. Once again judgement needs to be exercised by the teacher as to when to push and challenge the child. The child will only choose skills and strategies that he or she is confident in using. Success is therefore likely and confidence increased.

As can be seen for an examination of the suggested aims of mathematics in Mathematics From 5 to 16 there is a close match between these stated aims and the approaches used and manner in which investigational work takes place.

In Mathematics From 5 to 16 these aims are followed through into more specific objectives. These form five main categories : -

A. Facts
B. Skills
C. Conceptual Structures
D. General Strategies
E. Personal Qualities

(HMSO 1985)

These five categories are subdivided into a further twenty four objectives. Many of the objectives listed under Facts and Skills will be present whenever mathematics is taking place and will not therefore be alluded to specifically here. However a number of the objectives are particularly covered in investigational work and are worth highlighting.

B. Skills

Objective 6

Sensible use of a calculator.

Previous to the availability of the calculator calculations were done either mentally, or with pencil and paper or using logarithmic tables or by using a slide rule. Often an exercise of rather horrendous calculation would be presented to be completed using one particular method. These calculations were usually developed to a level beyond which the child would encounter them in a problem.

In examinations questions were often prefaced with the phrases 'using logarithms' or 'without using logarithms'. Children, then, rarely had the opportunity to develop a discerning and sensible approach as when to use the various arithmetical methods available to them.

The calculator, in terms of an aid to calculation, a natural successor to logarithms and slide rules has presented different problems to its predecessors.

Where as logarithms and slide rules have needed the acquisition of quite complicated skills, the calculator is easy to use. It is therefore available to many more children at a much earlier age. The problem then arises that its indiscriminant use may prevent children gaining an understanding of the conceptual structure underlying arithmetic. To avoid this some teachers may therefore relegate the calculators use to a check on pencil and paper calculations rather than involving it in the mainstream of mathematical learning.

Any activity where the calculator plays a relevant and real role in mathematics rather than a contrived role is to be welcomed. In mathematical investigations the calculator plays a subservient role to the main mathematical thrust. It is used where appropriate and not merely to obtain 'the' answer. It is used to aid the justification of a conclusion that would otherwise be difficult to arrive at rather than to obtain the result itself.

An example of this is the investigation where children are asked to find the largest number obtainable by multiplying a three digit number by a two digit number where individual digits are 1,2,3,4 and 5.

Initially the calculator will be used to obtain the answer to such products as 543 x 21. As the investigation continues products such as 435 x 21 will be eliminated without the use of the calculator as children see that the answer will be less than the former product. As the calculator is a tool to be used to aid the investigation children are not under pressure to use it all the time. They then learn to discriminate and to make sensible use of a calculator.

Objective 8

Ability to communicate mathematics

When children are working at mathematics there are two main functions of communications.

One is in the form of discussion, of talking and listening between children as they formulate their ideas and experiment with them. This is a 'working' communication that is essential when a group of children is working on an investigation and is frequently lacking when children are only practicing skills. The other form of communication is what might be termed 'presentation' communication. At the end of the investigation children will need to describe and explain their results and discoveries to others. This will usually be in the form of a written description illustrated with diagrams, charts, graphs and perhaps models. It could also be in the form of an oral presentation with a child describing to the class, with the help of an overhead projector, or slides and diagrams the results of the investigations.

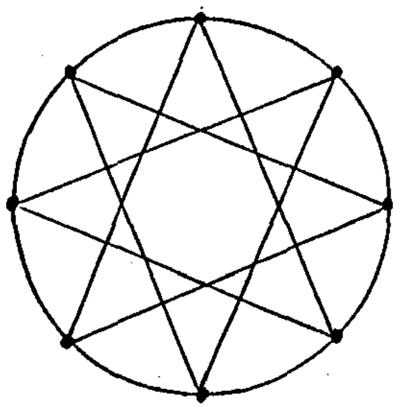
Objective 9

The use of micro-computers in mathematical activities.

(C) as a tool for pupils to use in doing mathematical tasks.

As with calculators, computers open up a wide range of investigational work that was hither to unavailable in an easily attainable form to children. The computer can be used in two distinct ways in investigational work. The first is in a very specific way to solve a particular investigation. Circle is a piece of computer software on the SMILE disc THE FIRST THIRTY.

The program gives you the option of choosing the number of points on the circle and the size of a jump. The diagram below shows the diagram obtained with eight points on the circle and a jump of three as points on the circle are joined.



This program allows an investigation of when stars, as opposed to convex polygons are formed to be carried out more easily. The second way computers can be used to aid investigational work is in by the software providing a medium in which the child can operate. Papert describes this as a micro world in Mindstorms. The classic example of a piece of software that provides a micro world in which a child can explore and discover things about mathematics in LOGO.

Both these aspects of using computers in investigational work will be considered in detail in Chapter 5.

C. Conceptual Structures

Objective 11

The relationships between concepts.

One aspect of being 'good at mathematics' is the ability to recognise the relationships between one concept and another. Some would argue that this is at the heart of mathematical understanding. In Able Children Denton and Postlethwaite construct a checklist of mathematical aptitudes. Two items are : -

(1) Quick to see when a method of solution that has been applied to problems in one topic will solve problems of a similar nature in another.

(2) When attempting to solve new sorts of problems displays a strong tendency to interpret and solve the problem in :

a) a spatial/geometric way or

b) an analytic/logical way

(Denton 1985)

Both of these exhibit the ability to link apparently disjoint areas of mathematics. Mathematical investigations offer the opportunity for different areas of mathematics to be linked together. In the investigation Magic Squares referred to in this chapter there is a clear example of this.

Children arrive at the different possible solutions listed below of placing the numbers 3,3,3,5,5,5,7,7,7 in a 3 by 3 square so that each row, column and diagonal adds up to 15.

7	3	5
3	5	7
5	7	3

5	3	7
7	5	3
3	7	5

5	7	3
3	5	7
7	3	5

3	7	5
7	5	3
5	3	7

In checking that all solutions have been obtained or in generating solutions to other 3 by 3 magic squares it can be useful to recognise the geometric connection between one solution and the other three solutions in terms of rotations and reflections. By performing all eight isometric transformations of the square it is possible to quickly check that all possible arrangements of the numbers have been included.

Skemp in The Psychology of Learning Mathematics writes of schemas. Skemp sees a schema as the linking or inter relating of concepts to form new structures. This is a very powerful component in the learning of mathematics as it enables concepts that have already been assimilated to be combined to make a more complex structure. The new structure enables us to gain a greater understanding of what we are considering.

Objective 12

Selecting appropriate data.

In much of mathematics teaching children are presented with sufficient information to solve a particular problem.

They are seldom given too much information and required to make a selection from the information. They are rarely given insufficient information and required to choose which further information is necessary to solve the problem.

In investigation work both cases exist at the same time. By being given a starting point children are not given sufficient information to solve the problem. However they are free to generate information by their own investigating. Of the information generated some will be superfluous to the solving of the problem and will have to be rejected.

Objective 13

Using mathematics in context.

Divergent mathematical investigations involve using various mathematical skills to solve a particular problem. Almost by definition they require the using of these skills in a context rather than in isolation.

It is in the section General Strategies that investigational work can be of the greatest benefit. The headings for the objectives in this section are : -

Objective 15	Ability to estimate
Objective 16	Ability to approximate
Objective 17	Trial and error methods
Objective 18	Simplifying difficult tasks
Objective 19	Looking for pattern
Objective 20	Reasoning
Objective 21	Making and testing hypotheses
Objective 22	Proving and disproving

Rather than deal with these objectives in detail here I shall develop these further in Chapter 3 dealing with the introduction of a course of investigational work in the classroom.

E. Personal Qualities

Objective 23

Good work habits.

The good work habits necessary for pupils to reach the highest standards in mathematics are these described in the aims, of being :

imagination, creative flexible;
systematic;
independent in thought and action;
co-operative;
persistent.

Opportunities for these qualities to be developed are provided within investigational work.

Objective 24

A positive attitude to mathematics.

One of the major stumbling-blocks to achievement in mathematics is a negative attitude. This is exemplified in feelings of anxiety, helplessness and fear when attempting even a relatively straightforward piece of mathematics.

Children need to experience success, children need to feel that they are working at something meaningful and relevant if they are to develop a positive attitude to mathematics.

I believe that much of this development can be achieved through investigation work.

From the detailed examination of what should constitute the aims and objectives of a mathematics course it is clear that investigational work is able to play a substantial part in achieving these aims and objectives.

CHAPTER 3

INTRODUCING INVESTIGATIONAL WORK.

INTRODUCING INVESTIGATIONAL WORK

We saw in Chapter 1 that investigational work could be considered from two view points. One was the 'Problem Solving' or 'Investigatory Approach' view point. This was where a problem or starting point was given to the child. The child had then to attempt to solve the problem or investigate the starting point using his or her accumulated store of facts, skills and conceptual knowledge together with a collection of strategies.

The other, the 'Guided Discovery Approach' viewpoint was to use the same collection of strategies to arrive at the acquisition of certain specified facts, skills and concepts. Rather than being told the required fact or being told how to perform the required skill and then being given exercises to consolidate this skill, another approach is used. Children are lead through a number of situations where they can arrive at the particular fact or acquire the particular skill.

In this chapter I shall examine how children can be introduced to the 'Investigatory Approach'.

I shall consider two different ways in which investigational work can be introduced to children.

1) As a module of work for children to complete over a short period of time.

2) As an on going activity built into the main course.

As a module of work for children to complete over a short period of time.

Problems with Patterns and Numbers is a module produced jointly by the Shell Centre for Mathematical Education and the Joint Matriculation Board. The objective of the module is to prepare children in the fourth or fifth year of an 11 - 16 or 11 - 18 secondary school for certain questions in an O level examination.

The examination questions anticipated are of a markedly different kind from those usually found on O-level examination papers.

> This Module introduces a somewhat new type of examination question in which the mathematical processes involved, especially the choice and explanation of strategies and discussion of results are as important as the answers obtained.
>
> (JMB Shell Centre for Mathematical Education, 1984)

A specimen examination question is

Reverses

Here is a row of numbers : 2, 5, 1, 4, 3.
They are to be put in ascending order by a sequence of moves which reverse chosen blocks of numbers, always starting at the beginning of the row.

Example

2, 5, 1, 4, 3	reversing the first 4 numbers gives	4, 1, 5, 2, 3
4, 1, 5, 2, 3	reversing the first 3 numbers gives	5, 1, 4, 2, 3
5, 1, 4, 2, 3	reversing all 5 numbers gives	3, 2, 4, 1, 5

. . . . .
. . . . .
. . . . .
. . . . .
1, 2, 3, 4, 5

(i) Find a sequence of moves to put the following rows of numbers in ascending order.

a) 2, 3, 1

b) 4, 2, 3, 1

c) 7, 2, 6, 5, 4, 3, 1

(ii) Find some rules for the moves which will put any row of numbers in ascending order.

(JMB, Shell Centre for Mathematical Education, 1984)

Such a question tests far more than the retention of facts and skills and their application to a familiar problem. The questions set here are of a more varied, more open and less standardised kind than is normally encountered on an examination paper. The emphasis is less on content and more on the specific strategies that can be applied to solving mathematical problems.

The Module lists the following strategies :

1) try some simple cases
2) find a helpful diagram
3) organise systematically
4) make a table
5) spot patterns
6) find a general rule
7) explain why the rule works
8) check regularly

(JMB, Shell Centre for Mathematical Education, 1984)

If such strategies have not been developed in the classroom already then it will be necessary for different teaching styles from normal to be employed. These will include children working more independently both alone and within a group. Working within a group, including as one of a pair, will involve more discussion than normally takes place.

The role of the teacher will change from that of someone who is giving detailed explanation to someone that gives advice on the strategies to be used and offers general encouragement.

The Module provides material than enables teachers to develop the appropriate styles of teaching. The Module contains :

1) A book containing suitable investigations with suggestions about how they can be introduced and specimen examination questions, sample pupil scripts and marking schemes.

2) A pack of mastercopies for photocopying containing pupil and teacher material from the above book.

3) A disk of five computer programs involving investigations.

4) Booklets containing detailed notes on two of the computer programs.

5) A video showing investigational work being undertaken in the classroom.

6) A booklet to accompany the video.

I will examine in detail the 'pencil and paper' classroom investigational material presented in the book mentioned in section 1).

The classroom material is presented in four units. Each of the first three units constitutes roughly one week's work. The units introduce children to investigational work, gradually providing children with less guidance as progress is made through the units. The fourth unit is a collection of investigations to supplement those in the first three units.

Unit A

Unit A forms the basic core of Module. It presents the child and teacher with detailed instructions as how to proceed with the investigations. Before this however there are three Introductory Problems. These problems are different in nature from the questions that would normally be found in a text book. The purpose of these questions is to give children an introduction to the sort of work that the Module entails. They will be able to see that investigational work seldom involves going straight to the answer and requires different techniques to arrive at the answer or answers.

It is suggested that all children should complete the first problem, make considerable progress on the second but little progress on the third.

The thinking behind this is that children should be able to appreciate some of the differences in this type of work by successfully completing the first problem but also appreciate that to be successful on more difficult problems they will need to develop certain strategies and skills.

The first problem, that it is anticipated will be completed by all children, is called Target.

1. Target

On your calculator you are only allowed to use the keys

$\boxed{3}$ $\boxed{4}$ $\boxed{x}$ $\boxed{-}$ $\boxed{=}$

You can press them as often as you like.
You are asked to find a sequence of key presses that produce a given number in the display. For example, 6 can be produced by

$$3 \times 4 - 3 - 3 =$$

a) Find a way of producing each of the numbers from 1 to 10. You must "clear" your calculator before each new sequence.

b) Find a second way of producing the number 10. Give reasons why one way might be preferred to the other.

(JMB, Shell Centre for Mathematical Education, 1984)

This problem is a good starting problem as it is easy to make initial progress. However as further work is done on the problem the virtues of making a table and organising systematically becomes apparent.

The second problem involves two discs with a 7 on one and a 10 on the other. Children are required to find what numbers are on the back of the discs if the two discs can give totals of 11, 12, 16 and 17.

The third problem involves finding how many matches, home and away there are in a league of 30 teams.

As it is recommended that no more than one hour is allowed for these problems and that no help is to be given by the teacher it is unlikely that much progress would be made by the majority of children in the third problem.

It is the third problem that provides the first detailed consideration of the strategies necessary in investigational work.

Children are lead through the problem being told exactly which strategy to employ and how to use the strategy. Finally a check list of key strategies is given as a summary and to be used on two further investigations.

These are given in Appendix A.

The main emphasis while tackling the Tournament Problem is on organising the problem. The necessity to bring the problem down to a manageable size by 'trying some simple cases' is the first strategy encountered. This will be in contrast to the childs own original attempt at the problem during the Introductory Problems stage where simplication probably did not take place.

The way in which 'finding a helpful diagram', in this case a grid, helps in 'organising the information systematically' is brought out too.

Finally the strategies of 'making a table' so that 'a pattern can be spotted' and 'the pattern used' to find the particular solution to the problem are dealt with.

The last problem on the sheet A1, the Money Problem, forms a link with the next part of the unit.

The Tournament Problem concentrated on the strategies involved once a particular line of approach had been decided upon.

The second detailed consideration is of the different approaches that can be employed to tackle a particular problem.

These are given in Appendix B

Once again the instructions on the sheet A2 are very detailed and prescriptive. This method is however very useful at this stage. It enables several different techniques, drawing a graph, systematic counting, finding a rule and finding a pattern to be considered while tackling one problem.

Finally the unit concludes with Solving a Whole Problem. The problem is given but unlike the previous two sections no detailed help or structure for solving the problem is given. However a list of strategies is given (appendix A p.2) which implies a structure if the strategies are used in the order suggested.

The second unit tackles a number of problems similar in difficulty to those in the first unit. Children are given less guidance than in the first section the only written help being in the form of an appropriate Pupil's Checklist consisting of recommended strategies to be employed. These checklists are slightly different for each problem and form a structure to the way in which a problem can be tackled without too much chance of false trails being followed.

These are given in Appendix C

The third unit consists of problems, with some explanation, for children to tackle. There is no pupil's checklist available automatically for each problem for each child to consult. It is expected that by this stage children will have assimilated the necessary techniques, skills and strategies to make a good attempt on these problems unaided.

It is anticipated that on some occasions some children will need some help.

This is provided through the teacher who has a strategy checklist appropriate to each problem.

These are given in Appendix D

The teacher is also encouraged to help the children not by telling them what to do and when to do it but by using a series of graded questions.

It is suggested that questions can be of two types, one to be used freely and one when the first has not been successful, to be used sparingly.

> Use freely any hints that make children think about the way they are tackling the problem :
>
> "What have you tried?"
> "Well, what do you think?"
> "What are you trying to do?"
> "Why are we doing this?"
> "What will we do when we get this result?"
>
> Use sparingly, particularly later on, hints about which strategies they should use :
>
> "What have you found out so far?"
> "Have you seen anything that is like this in any way?"
> "How can we organise this?"
> "Let's draw up a table of results".
> "Can you see any pattern?"
> "Have you tried some simple cases?"
> "What examples should we choose?"
> "How can we start?"
> "Have you checked if that works?"
>
> (JMB, Shell Centre for Mathematical Education, 1984)

Throughout the book there are solutions to the problems and for some of the problems examples of childrens attempts at the problems.

There are also suggestions as to how the material may be used with the class.

In considering this Module there are points that need to be borne in mind. First the Module is intended for children preparing for an O-level examination. Second the Module is intended for children in the fourth or fifth year of secondary school. Both of these factors greatly influence the construction and content of the Module.

The shortness of time set aside for the Module, roughly three weeks for the introduction of materials in the classroom, means that children must be fairly rapidly brought to a position where they have acquired all the strategies necessary to attempt this type of investigational work. Little opportunity is available for children to work towards their own development of the strategies necessary to the successful completion of investigational work. These strategies are presented, already refined, very early in the course.

The nature of the questions to be set in the examination determine what type of investigational work is to be encountered. The examples given in the Module tend to be fairly narrow, well defined problems that do not allow the child much choice as to how the problem may be developed or to select how the problem is to be interpreted.

The Module does provide valuable help to the teacher to develop investigational work. It is very thorough in its coverage of this particular type of investigation and gives good guidance to the teacher to introduce investigational work to children.

However the necessity to reach a particular level of competence in such a short period of time means that a tight schedule has to be adhered to. There appears to be little flexibility in the material to cater for the different rates of progress of different children. If the Module is to be completed within the time limit children, by necessity, must move on to the next section without perhaps having either fully experienced or grasped the work.

The main reasons for the draw backs mentioned is the lack of time at this juncture of the course. This can be largely overcome by the introduction of investigational work at an earlier stage in the course. This should enable children to spend more time in developing the required strategies, of seeing when they are appropriate and being given time to struggle with an investigation, perhaps fail and later return to it.

It is the opportunity to pursue an investigation in depth over a lengthy period of time that is not available within the Module but which can be accommodated within a longer term course of investigational work.

<u>As an on going activity built into the main course</u>. .

The Module considered which was written in support of the JMB O-level examination can be contrasted with the Investigations and Stretchers Booklet produced to supplement the SMP 11 - 16 scheme.

The SMP 11 - 16 scheme will be considered in more detail in Chapter 4 where the investigational approach as a vehicle for teaching facts and skills will be examined. As with the JMB examination the SMP 11 - 16 pilot examinations contain a question requiring an investigational approach. Whereas the JMB question should take 20 minutes the SMP questions take longer. An investigational question from the SMP 11 - 16 pilot specimen paper is as follows : -

<u>SMP 11 - 16 "Mock" coursework task : TREES</u>

<u>Code</u> : Mock LMU

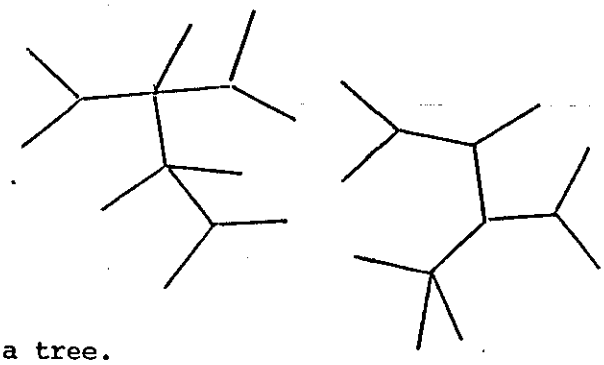
<u>Category</u> : S (in school,under supervision)

Remember :

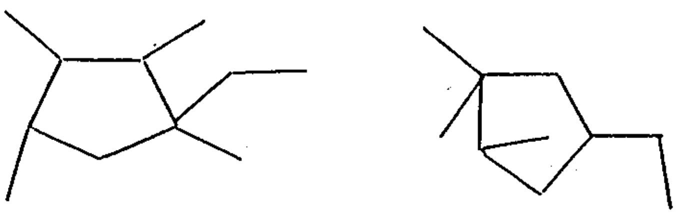
(1) Describe how you tackle this investigation, so that someone else can follow what you did.

(2) Answers by themselves are not enough. You must show the working that leads to them.

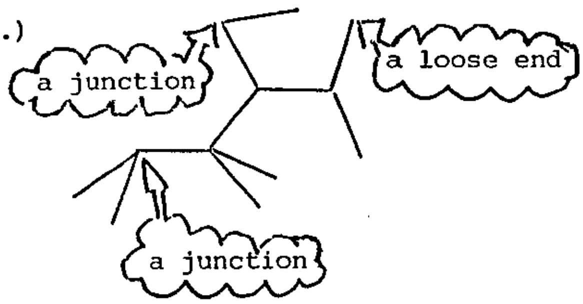
These shapes are made with matchsticks. They are called 'trees'.



There are no closed loops in a tree. So these, for example, are not trees.



This tree has 7 loose ends and 5 junctions. (A junction is where two or more matchsticks meet.)



Find out if there is any rule connecting the number of loose ends, the number of junctions and the number of matches in a tree.

If you find a rule, write down clearly what it is.

As can be seen both JMB and SMP questions are similar in type, although the JMB question because of the shorter period of time allowed is more directed.

The approach adopted by the SMP 11 - 16 course is far less structured as to how and when the strategies used in investigational work are to to introduced.

The philosophy behind this is summed up in the booklet Investigations and Stretchers.

> Investigational work, on the other hand, shifts the emphasis towards pupils developing, at a low level, mathematical thought processes and strategies themselves : generalisation from particular examples, the invention of shorthand notation, testing hypotheses, looking at and extending simple solutions, making simple deductions - these can all arise from mathematical activities which require very little specific guidance from the teacher.
>
> (SMP 1985)

This view, diametrically opposite to that implied by the JMB, Shell Centre Module, is modified later in the booklet where it is suggested that you cannot 'teach' investigations but

> there are approaches which will emerge as proving useful and can be discussed at an appropriate stage with the class. These include tabulating data systematically, investigating first a simpler case, keeping one variable constant, and stopping - once you have had some experience - to ask where you have met a similar problem before.
>
> (SMP 1985)

There is a great deal of agreement between the two in what the strategies are, less if any agreement as to how to pursue a course of investigational work.

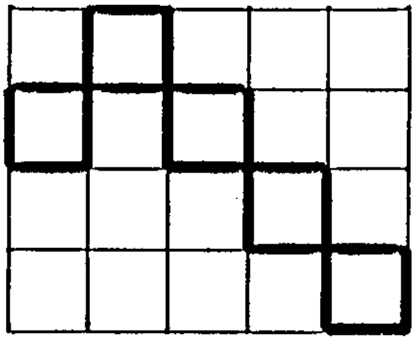
The SMP 11 -16 scheme leaves the decisions as to when to emphasise or even suggest appropriate strategies to the teacher. This is not unreasonable as one or more members of the class may arrive at an appropriate strategy in the course of an investigation. This strategy can be discussed and its effectiveness compared with other approaches. In this way a class collection of strategies can be built up over a period of time.

Investigations are presented as a collection of starting points with little in the way of comment about specific investigations and some comments applying to investigations in general.

Examples of starting points and specific comments

1) Open chains

An open chain is formed by 5 (or 6,7,8, etc) grid squares which touch only at corners. What are the largest and smallest possible areas of the containing rectangle?



Comment.

Finding the rules for the largest and smallest areas is a nice example of simple generalisation.

In an 'open chain', no square touches more than two other squares.

If this condition is relaxed, we get investigation 11, which can be thought of as an extension of this one.

2) Card shuffle

A pack containing an even number of cards is split into two equal piles and shuffled so that cards interleave alternately. (The top card of the pack stays on top and the bottom card on the bottom). If this shuffling is continued, will the pack return to its original order? If so, after how many shuffles?

Comment.

This illustrates a useful strategy - trying simple cases first, for example a pack of six cards, or even four. Pupils will need to invent a method or notation for recording the effects of the shuffles.

(SMP, 1985)

The general comments include the following :

1) in response to suggestions or questions reply "try it and see ..." ;

2) allow pupils to follow blind alleys;

3) keep emphasising the need for accuracy and checking;

4) do not allow the recording and the writing up of investigations to become a chore;

5) allow occasions to pull the finding of an investigation together.

These two approaches represent two extremes. The JMB Shell Centre Module is very structured and directional allowing for very little initiative to be shown by the child following the course. The SMP booklet contains short starting points and very sparse comments in support.

An ideal course lies somewhere between these two viewpoints. It is necessary for the teacher who, like the children in the class, is probably unaware initially of all the strategies and techniques available to be made aware of them. It is equally true that the accumulation of these strategies is a slow process and that children need time to effectively acquire them.

The teacher must be aware of what the overall aims of investigational work are and what elements are contained in the particular investigation under consideration. The teacher needs to use his or her judgement as to when to introduce investigations containing certain strategies, when to emphasise certain points and when to curtail the investigation.

As children within a class develop their awareness at different rates a heavily structured course is probably not the best type to embark upon.

There needs to be a structure but both the timing of and the amount of material between the introduction of different points needs to be flexible and in the hands of the teacher.

A teacher may embark upon compiling a course of investigational work as follows. First the teacher needs to draw up a list of strategies used in, attributes associated with or features of investigational work. Several such lists have already been referred to in the preceding chapters.

Ernest listed features found in investigational work as

The intitial problem or situation
An exploration of the situation
The accumulation of facts
The tabulation or ordering of facts
The conjecturing of patterns
The testing of conjectures
The disproof of a conjecture by counter example
The modification of the conjecture
The confirmation of a conjecture
The attempts to justify a conjecture
The generalisation of a conjecture
The suggestion of new situations.

(Ernest, 1984)

A further list from Starting Mathematical Investigations published by the Northamptonshire Mathematics Resource Centre contains similar elements.

interpreting a problem
selecting relevant information
simplifying the problem
tabulating information
drawing a diagram or graph
spotting a pattern
systematically covering all possibilities
using trial and error

finding a rule
finding an algebraic relationship
hypothesising and testing
looking for counter examples
generalising
explaining and proving
extending the problem
formulating new problems

(Northamptonshire Mathematics Resource Centre 1985)

Second the teacher will need to collect a variety of investigations that illustrate the strategies, attributes and features listed. There are now many sources of starting points for investigations available to choose from. Among them are :

Points of departure 1 - The Association of Teachers of Mathematics
Points of departure 2 - The Association of Teachers of Mathematics
Mathematical Activities - Brian Bolt
More Mathematical Activities - Brian Bolt
Sources of Mathematical Discovery - Lorraine Mottershead.

It is important that a careful selection is made from the collection of investigations when constructing the course of investigational work. An appropriate variety of investigations is necessary if all the attributes listed are to be covered.

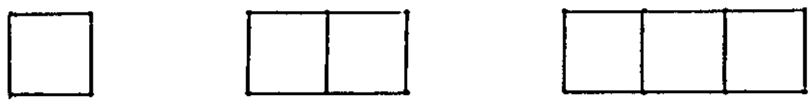
It is also necessary to devise some sort of grading system for the investigations as the difficulty in making progress within an investigation varies considerably from investigation to investigation. What usually makes one investigation harder than another is the difficulty in spotting a pattern and subsequently in making an algebraic generalisation. Investigations can therefore be classified in difficulty by considering the table of results.

Examples of the types of classification that might be made are given below. The 'numbers' refer to the data in the results column of the investigation.

1. <u>Numbers increase by a constant amount</u>

Investigation : Square snakes

Snakes can be built up from squares like this :



Snakes must be in a straight line.
Each square has sides of length 1
Investigate the perimeter of the snakes.

The following table of results is obtained.

Number of squares	Perimeter
1	4
2	6
3	8
4	10

The pattern in the sequence of numbers in the perimater column is very easy for children to spot. It is also fairly easy for a connection to be spotted between corresponding numbers in the 'number of squares' column and the 'perimeter' column.

2. <u>Numbers increase by a variable amount</u>

Investigation : Handshakes

Investigate how many handshakes take place when everyone in a group shakes hands with everyone else.

The following table of results is obtained

Number of people	Number of handshakes
2	1
3	3
4	6
5	10
6	15

The pattern in the sequence of numbers in the 'number of handshakes' column is still relatively easy for children to spot. However the connection between the 'number of people' column and the 'number of handshakes' column is not.

3. Numbers have an 'obvious' relationship to another variable

Investigation : Windows

Take an ordinary 100 number squares.
Draw squares on the grid to enclose 4, 9, 16 etc numbers.
Investigate what happens when you perform the following on the numbers in the corners of the squares :

(top right corner number x bottom left corner number) - (top left corner number x bottom right corner number)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The following table of results is obtained.

Size of square	Answer
2 x 2	10
3 x 3	40
4 x 4	90
5 x 5	160

The pattern in the sequence of numbers in the 'answer' column may be seen either as a variable difference (30, 50, 70, 90, ... etc) or as being directly related to the corresponding entry in the 'size of square' column in the form of size of square less one, squared, times ten.

Generally children will try and analyse their results by looking for a pattern down a column of numbers rather than looking for a relationship between two columns of numbers. If this is the case then the order of difficulty of these three examples is correct. However, if children are looking for a relationship between two columns of numbers, then Windows is easier than Handshakes.

It is not always possible to grade investigations absolutely. None the less it is important to be aware of the degrees of complexity that can arise.

4. Numbers are a combination of several relationships

Investigation : Squares on a chessboard.

Investigate how many squares of different sizes there are on different sized square grids.

The following table of results is obtained.

Size of square	Number of squares
1	1
2	5
3	14
4	30

The pattern of the sequence in the 'number of squares' column is difficult for children to spot unless they are able to make the connection between the intermediate results obtained from a systematic approach and the final results.

$$1 = 1$$
$$5 = 1 + 4$$
$$14 = 1 + 4 + 9$$
$$30 = 1 + 4 + 9 + 16$$

The four classifications of investigations considered have only involved a 'results' column depending on one variable, the number of people in the Handshakes investigation or the size of the squares in the Windows investigation for example.

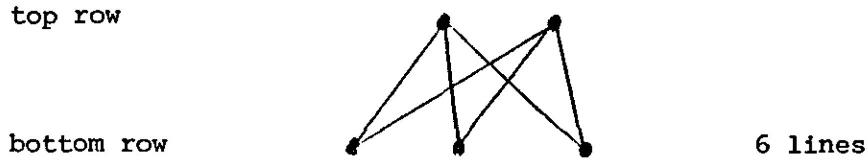
Some investigations involve two variables and are consequently of greater difficulty.

5. <u>Numbers are directly related in a unique simple way to the two variables</u>

Investigation : Lines joining two rows of dots.

Investigate the number of lines joining the dots in one row to all the dots in another row.

Example :



The following table of results can be obtained.

Number of dots in top row	Number of dots in bottom row	Number of lines
1	2	2
2	2	4
2	3	6
3	3	9
4	3	12

The relationship

number of lines = (numbers of dots in top row) x (number of dots in bottom row) is relatively easy to spot.

In this investigation it is of no advantage to be systematical in choosing the number of dots in the top and bottom rows. In the following investigation the relationship is more complex and hence a systematic approach is necessary.

6. <u>Numbers are directly related in a unique but not obvious way to the two variables</u>

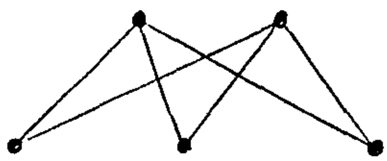
Investigation : Intersection of lines joining two rows of dots.

Investigate the number of intersections of lines joining the dots in one row to all the dots in another row.

Example :

top row

bottom row



3 intersections

The following table of results can be ontained

Number of dots in top row	Number of dots in bottom row	Number of intersections
1	2	0
2	2	1
2	3	3
3	3	9
4	3	18

It is by no means obvious what the relationship is in this case.
It is helpful here to hold one of the variables constant and
to make several tables with this variable taking different values.
This is a useful strategy applicable to certain types of investigations.

dots in top row	dots in bottom row	intersection
2	2	1
2	3	3
2	4	6
2	5	10

dots in top row	dots in bottom row	intersection
3	2	3
3	3	9
3	4	18
3	5	30

dots in top row	dots in bottom row	intersection
4	2	6
4	3	18
4	4	36
4	5	60

It can now be seen that the 'intersection' column is the sequence of the triangle numbers multiplied by a factor dependent on the number of dots in the top row.

A further table can be obtained.

dots in top row	multiplying factor
2	1
3	3
4	6
5	10

If t is the number of dots in the top row and b the number of dots in the bottom row then

$$\text{number of intersections} = \frac{t(t - 1)\ b(b - 1)}{4}$$

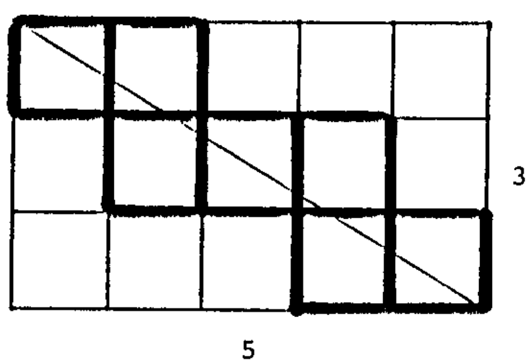
To be able to arrive at such a formula, whether expressed algebraically or in words requires considerable experience and expertise in investigational work.

7. <u>Numbers are directly related to two variables but not in a unique way</u>

Sometimes the relationship is not of the explicit kind of the last two examples. The relationship can vary depending on the relationship between the two variables themselves.

Investigation : The diagonal of a rectangle.

The rectangle has sides of integer lengths. It is divided into unit squares. Investigate how many squares a diagonal passes through.



passes through 7 squares.

The following table of results can be obtained.

length of rectangle	breadth of rectangle	number of squares
5	3	7
5	4	8
4	3	6
4	2	4
6	4	8
3	2	4
9	6	12

These results can be further tabulated as follows :

length and breadth have no factors in common

length of rectangle	breadth of rectangle	number of squares
5	3	7
5	4	8
4	3	6
3	2	4

number of squares = length + breadth - 1 length and breadth have a factor of 2 in common

length of rectangle	breadth of rectangle	number of squares
4	2	4
6	4	8
10	8	16
8	6	12

number of squares = length + breadth - 2

and so on.

The various rules can be combined as

number of squares = length + breath - highest common factor of the length and breadth.

Such a generalisation as this is a stage on from an immediate generalisation. There appears to be several 'different' generalisations depending on the relationship between the length and the breadth of the rectangle. These then have to be combined into one generalisation taking into account the number to be subtracted.

The purpose of this classification is not to determine an absolute hierarchy of investigations with regard to difficulty. In the investigations considered it is for example arguable whether example 7 is harder than example 6. The purpose is more to appreciate that there are many different types of generalisations to be made and to group them roughly according to their difficulty.

Finally the teacher will need to match the strategies that are considered essential to investigational work with the investigations themselves and to devise a plan for introducing the investigations. The first can best be done by drawing up a grid with the characteristics of investigations in general against particular investigations. Such a grid analysis

	Square Snakes	Handshakes	
Being systematic	✓	✓	
Trial and error			

is a way of ensuring that all the characteristics of investigational work are covered within the course. A selection of investigations can then be made that are appropriate to the mathematical development of the children for whom they are intended.

Investigational work may be introduced in several ways. At different stages of the course different introductions will be appropriate.

Verbally as a group or class activity

This is probably one of the most effective ways of introducing investigational work. The teacher can respond to the childrens work, highlighting those points which he or she sees as being valuable, encouraging and questioning.

It is however, also a way full of pitfalls. It is all too easy for a teacher to over direct, to push children down a particular path of exploration and to thrust results in front of them rather than allow the children to make their own, albeit, slow and faltering progress.

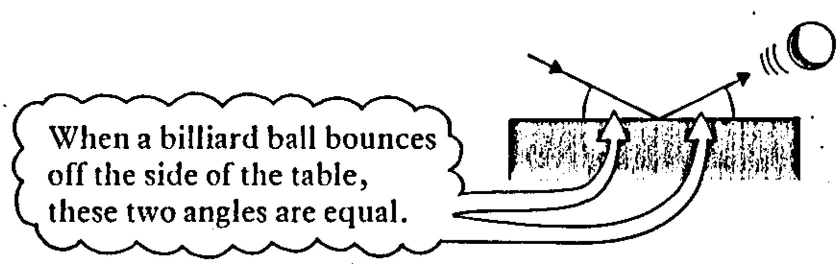
The strategies that the teacher needs to develop to avoid these pitfalls apply equally to investigations introduced in other ways and will be considered at the end of this chapter.

In a written, directed format

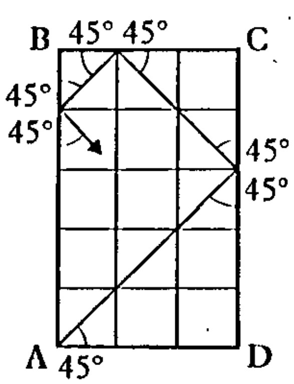
The advantage of presenting an investigation in a written form is that the teachers influence in responding to questions is removed, at least for some of the time. The child can make what he or she will of the investigation.

The disadvantage is that a written worksheet that enables a child to carry out an investigation must of necessity, in the early stages of investigational work at least, be directed. Particular lines of development are followed irrespective of whether they are how the child would have proceeded.
An example of this is Billiards from the SMP 11 -16 Stretchers.

Billiards



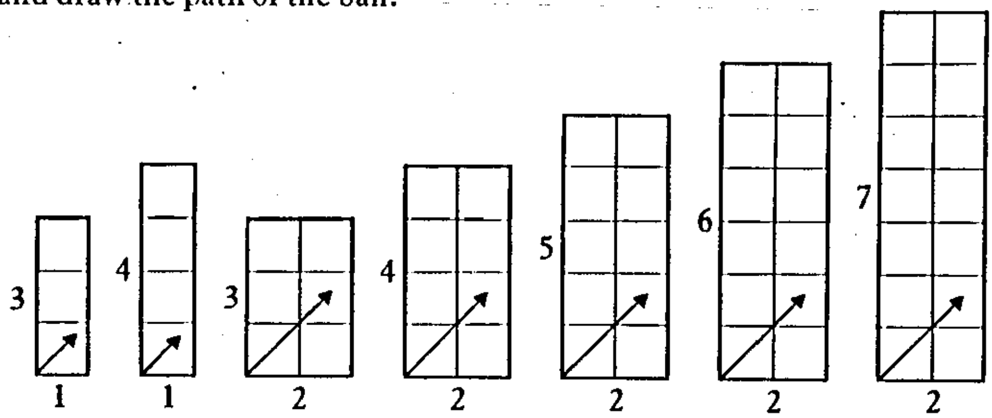
1



If a ball is hit from corner A at an angle of 45° to the edge of the table, all its bounces will be at 45°.

Eventually it will reach a second corner and stop. Which one?

2 Copy each of these 'billiard tables' onto squared paper and draw the path of the ball.



3 Look at the 3 by 1 and 4 by 1 tables in question 2.
At which corner will the ball finish on a 5 by 1 table
top left or top right?
What about a 6 by 1 table?
Write about any pattern you notice.

4 Where will the ball finish on a 50 by 1 table?
Where on a 77 by 1 table?

5 Look carefully at your drawings of
the 3 by 2, 4 by 2, 5 by 2, 6 by 2 and 7 by 2 tables.

Where will the ball finish – top left, top right,
bottom left or bottom right – on tables which are
8 by 2
9 by 2
and 10 by 2?
Try to answer without drawing any more tables.
Write about any pattern you notice.

6 Where will the ball finish on tables which are
18 by 2
36 by 2
43 by 2
and 25 by 2?

7 Investigate for yourself what happens on tables whose length and width are both odd.

8 Investigate what happens on tables whose width is 3.

(SMP, 1985)

In a written but open format

In this form the investigation is explained together with a number of possible avenues of development that may be explored. The main difference from the previous method of introduction is that children are given little help as to how to organise their investigation and no hints as to what they might be expected to be able to find out.

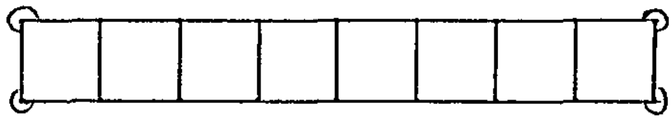
The Billiard Table Investigation

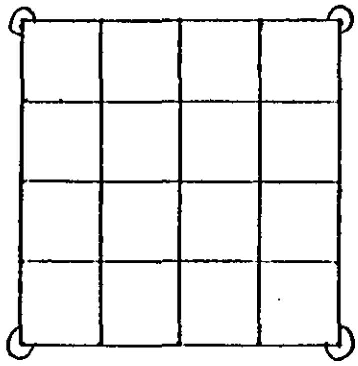
Below are a few questions to start you off but remember that an important part of an investigation is asking your own questions.

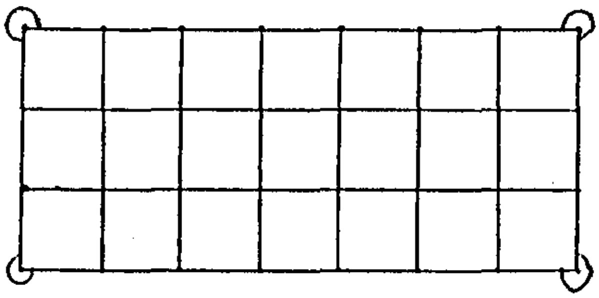
What would happen if the billiard table was of a different size, say 2 x 6, 5 x 10, 4 x 8 etc.

On some tables the ball will travel over every square.
Which tables?
How many times does the ball hit the side of the table?
Which pocket does the ball fall into?

Use the drawings below on separate paper to investigate billiard tables. What answers do you think you will get?
What questions will you ask?







(Investigator 1984)

As a starting point

Eventually it is hoped that when children have gained sufficient experience and expertise in investigational work they will be able to pursue their own lines of inquiry without too much assistance from the teacher. That is not to say that the teacher will not be there to offer suggestions as to what to do next or to offer encouragement. Starting points for this stage are essentially terse as follows.

> Investigate what happens when a billiard ball is struck from a corner of different sized billiard tables.
> These billiard tables only have a pocket at each corner.

As a visual starting point

During the Autumn term of 1985 the BBC broadcast a series of 10 minute television programmes entitled Mathematical Investigations. These programmes introduced a number of starting points for investigational work. In some ways they were similar to the written introductions to investigations in as much as the development of the investigation was pre-determined. However, the technical facilities available to television enabled the investigations to be presented in a very different way.

A typical example of one of these 10 minute programmes was Shuffles. This introduced two linked starting points for investigational work. These were the shuffling of cards and permutations in bell ringing.

The medium of television considerably enhanced the introduction of permutations in bell ringing. The programme consisted of shots of bell-ringers in a church and the recording of the sound of the rounds that they were being rung. Subsequently these recordings were the background to a visual description of how the rounds operated. The overall effect was an introduction that could not have been achieved by any other means in the classroom.

In considering the ways that investigations could be introduced the pitfalls involved in verbal communication with children were mentioned. The teacher needs to be well prepared as how to handle this aspect of investigational work.

An effective strategy that teachers need to develop is that of avoiding telling children the correct answer or providing a ready made explanation for the children.

> When pupils do ask their own questions, how easy it is to jump in with a 'well explained answer'. Mary Boole, in the 18th century, coined the phrase 'teacher lust' to describe teachers who are too ready to give their own explanations, rather than encouraging pupils to think and explain for themselves.
>
> (Jaworski 1985)

In these circumstances the teacher needs to employ a series of 'deflecting questions'. In reply to questions from the children such as

"Is this right?" or

"Do we have to do it like this?"

are needed 'deflecting questions' such as

"What do you think about it?"

"How would you do it then?"

This is all part of helping children to develop independence in their learning. It is not easy for a teacher to stand back and allow children to struggle and sometimes to fail yet it is necessary if children are to become independent in their mathematical thought.

In conclusion it is worth bearing in mind some suggestions made by Marion Bird in Investigator 2.

> I should like to end by posing some suggestions of factors which might all have a part in restraining pupils still further from starting to carry out their own inquiries :
>
> 1. Squashing any attempts at symbolising or making up terms to refer to things by immediately telling them that they should be using particular symbols or terms which we see as the official ones.
>
> 2. Telling pupils conventional rules, but omitting to tell them that they are conventions.
>
> 3. Telling pupils exactly how to record their findings. ("Set your results in a table like this ..." etc).
>
> 4. Insisting that they always work on their own.
>
> 5. Insisting that the whole class always finishes one topic at the same time and starts a new one together.
>
> 6. Concentrating on answers only.
>
> (Bird, 1984)

CHAPTER 4

INVESTIGATIONAL WORK AS AN APPROACH FOR LEARNING FACTS, SKILLS AND CONCEPTS.

INVESTIGATIONAL WORK AS AN APPROACH FOR LEARNING FACTS, SKILLS AND CONCEPTS

In this chapter I shall examine how the Guided Discovery Approach viewpoint of investigational work can be used to enable a child to acquire facts, skills and concepts. In contrast to the vast majority of commercial mathematics schemes which are organised in an exposition and consolidation and practice way, the SMP 11 - 16 scheme makes use of the Guided Discovery Approach.

The SMP 11 -16 scheme is a mathematics scheme that covers the age range of children in compulsory education in secondary schools. It encompasses all abilities except for the lowest 15% of the ability range although in practice many schools only exclude the bottom 5% from the scheme.

During the first two years of the course all pupils follow an individualised learning scheme based on a series of just over a hundred booklets.

In the third year of the course children follow one of three courses based on text books. These courses match the proposed G.C.S.E. examinations in terms of the difficulty of work and the estimated percentage of children electing to enter the G.C.S.E. examinations.

The Yellow Course is aimed at those children intending to enter the Higher examination.

The Blue Course is aimed at children intending to enter either the Intermediate or Lower examination. In the fourth year of the course there is an off shoot of the Blue Course called the Red Course. The Red Course is aimed at those children intending to enter the Intermediate examination while the Blue Course continues and is aimed at those children intending to enter the Lower examination.

The Green Course is aimed at those children for whom the G.C.S.E. is considered inappropriate.

The part of the course which covers the first two years is organised in the following way.

The booklets are arranged in four levels, Level 1, Level 2, Level 3 and Level 4. Each level is divided into two sections (a) and (b). Further, with the exception of Level 1, all levels have a section (e), comprising of extension booklets. The levels form a hierarchical structure. In broad terms the more able child will complete all or nearly all booklets, the average child will complete the core course of Levels 1 to 4 but without the extension booklets and the less able child will complete Levels 1 and 2.

Booklets are grouped into five topics, Number, Space, Algebra, Graphs and Statistics.

Each booklet has up to 18 A5 pages and is written so as to be completed by a child largely on his or her own although the child may need to ask the teacher for help at some point. In some booklets the child is directed to check with the teacher at specific crucial points in the booklet.

I shall first examine how the investigational approach is used to enable children to acquire certain specific algebraic facts and skills in the first two years of the course.

Two main threads are developed in the part of the course dealing with algebra.

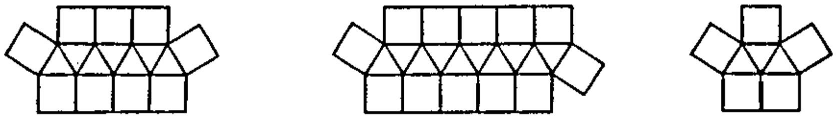
1) The ability to construct a generalised formula using letters from a problem given either a) pictorially or
b) in words.

2) Having been given an algebraic formula be able to use and interpret the formula.

These are best illustrated by questions taken from the booklets.

1(a) Formulas : extension 1 Level 3(e)

F4



Write the formula connecting t (the number of triangles) and s (the number of squares).

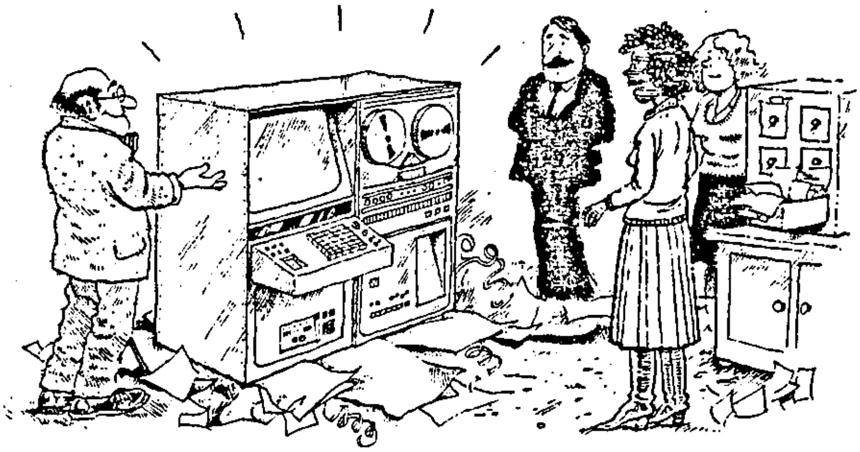
1(b) Formulas : extension 2 Level 3(e)

The Nutties lost a lot of money on their Leeds concert. They decide to make a comeback by producing a record.

E4 It takes 3 hours to set up the recording studio and 2 hours for each track they record. Let t stand for the number of tracks, and h stand for the total number of hours. Write down the formula for h in terms of t.

2 Formulas and Graphs Level 4(e)

G Selling computers



The company employs three other salespeople.
Each one sells a different type of computer, and each has a different pay formula.

Joe Williams	$P = 50n + 9000$
Devon Kingston	$P = 500n + 3000$
Shailesh Patel	$P = 200n + 6000$

G11 What does each formula say, in words?

G12 One of the three salesmen sells large computers, one medium size and one small.

Say which you think is which, and give your reasons.

G13 Draw the graphs of the three formulas on one pair of axes.

G14 How many computers does each salesman have to sell to earn more than £15 000?

G15 How many does each have to sell to earn over £20 000?

G16 The company recovers and starts to do well again.

(a) Joe Williams gets a pay rise of £2000 a year.
Write down his new pay formula.

(b) Devon Kingston gets an extra £50 for each computer he sells.
Write down his new pay formula.

(c) Shailesh Patel gets a rise of £1000 a year and an extra £25 for each computer he sells.
Write down his new pay formula.

(SMP 1983)

In doing investigational work the following strategies are often employed.

Start with simple cases
Use helpful diagrams
Be systematic
Tabulate results
Spot a pattern
Make a hypothesis
Test the hypothesis
Make a generalisation
Justify or prove

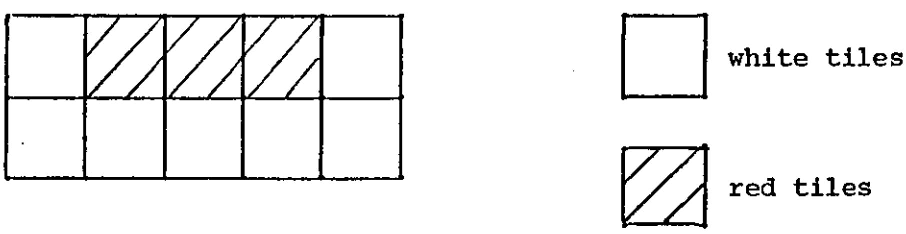
I shall examine how these strategies are used to develop the child's skills in solving the examples given previously.

Algebra is first introduced in Level 2. It will be met by children of O Level ability towards the end of their first year at secondary school and by nearly all children by the end of their second year.

In the booklet Discovering Rules Level 2(b) the strategies listed previously are all used, some explicitly and some implicitly. The objective of this booklet is that children can be given a problem expressed essentially in a pictorial form and be able to generalise the formula for the problem, the formula being written in the 'machine chain' form.

The first problem is to derive a 'machine chain' formula for a doorstep pattern of tiles which enables the number of white tiles needed to be calculated given the number of red tiles.

Typical doorstep patterns are of this form.



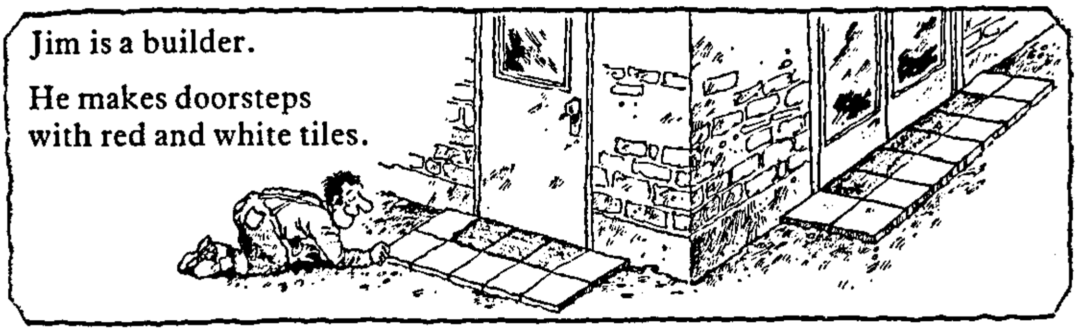
Start with simple cases.

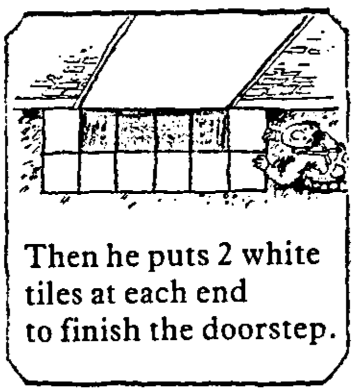
Two doorstep patterns have to be constructed using red and white tiles. The number of red tiles used for these two together with the two examples given in the booklet, one by way of an explanation of how to build the pattern and one as an illustration, are 5, 8, 4 and 3 respectively.

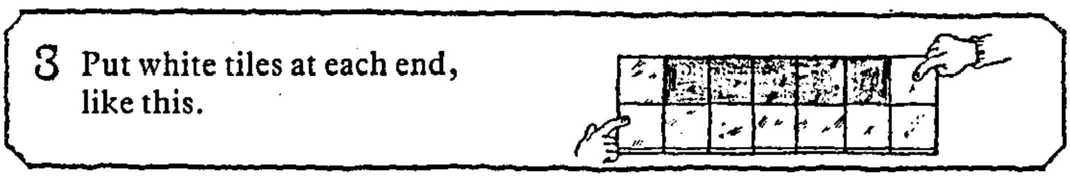
Use helpful diagrams.

The text uses a 'strip cartoon' approach in certain places. Clear diagrams illustrating the results of the patterns are used.

A Doorsteps







A1 How many white tiles did you use in your doorstep?

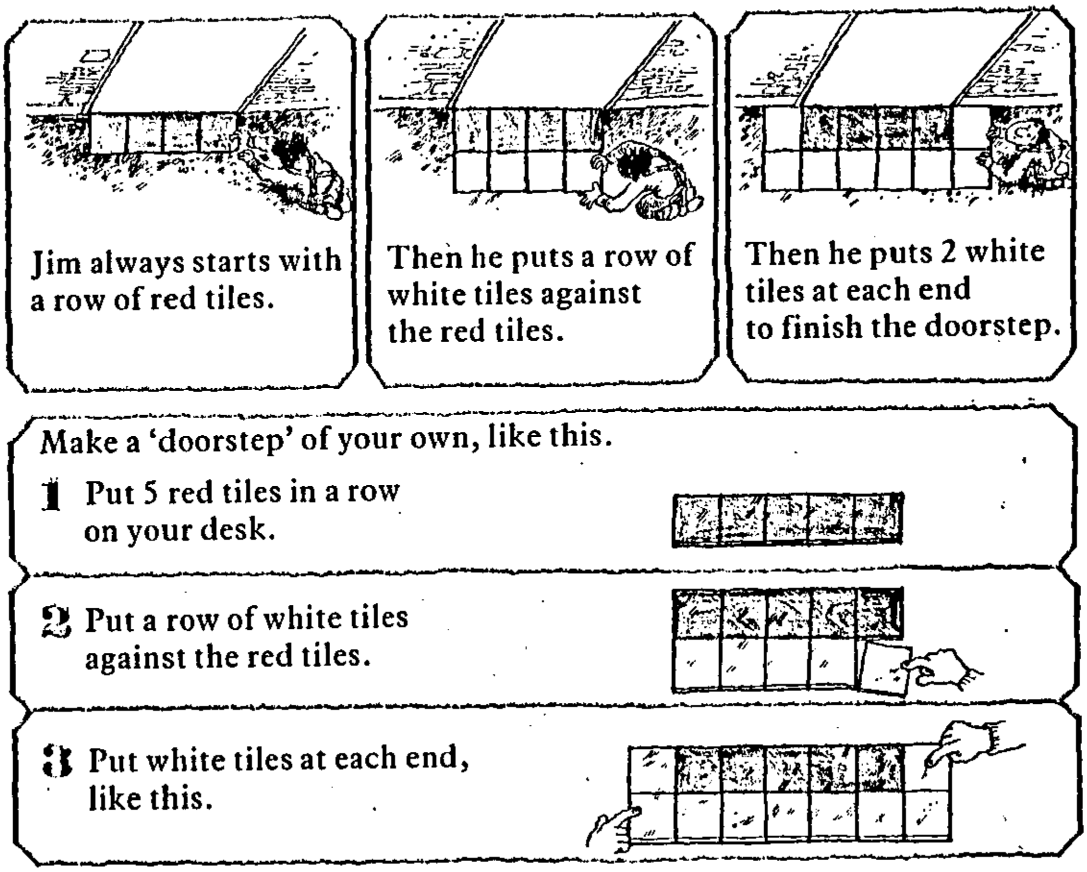
(SMP 1983)

Be systematic.

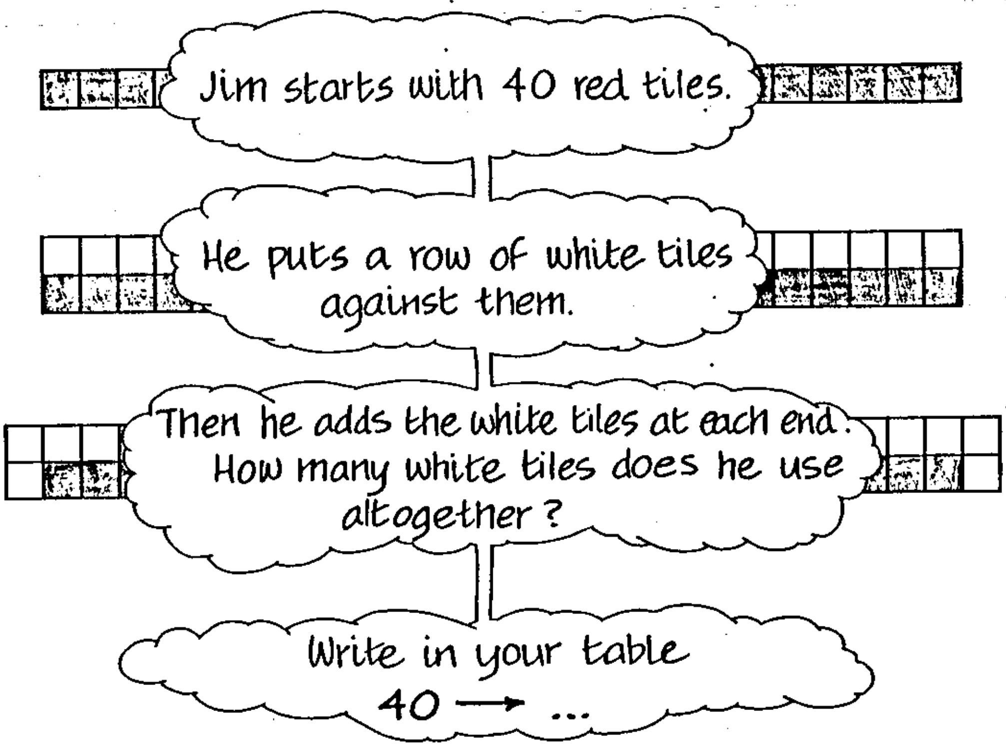
There are two aspects of being systematic. One is in following the strategies listed above in the order given. This is implied in the booklet by presenting the strategies in this order.

The other is in being systematic in the making of the tile pattern so that a method can be generalised and a formula written down.

The booklet uses both text and diagrams to suggest a systematic approach.



(SMP 1983)



(SMP 1983)

Both of these follow the same method of

placing the red tiles

placing a parallel row of white tiles of equal length

placing a further 4 white tiles, 2 at each end.

Tabulate results.

Children are required to tabulate their results in a specific fashion.

Number of red tiles	Number of white tiles
5 →	9

(SMP 1983)

Spot a pattern

Make a hypothesis

Test the hypothesis.

These three strategies are implied by the asking of two further questions that are to be answered without the use of tiles. Children have to find how many white tiles are needed if 40 and 100 red tiles are used.

It is anticipated that children will spot the pattern that the number of white tiles is 4 more than the number of red, that they will put this forward as a hypothesis and test it by checking the hypothesis with their results already tabulated and perhaps on a further pattern.

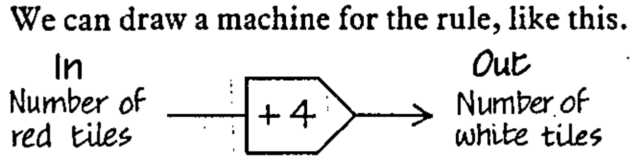
This process will not necessarily be as clear cut in childrens minds as indicated here. The spotting of the pattern of 4 more will merge with or come after the making of a hypothesis. To some the hypothesis will be so 'obvious' as to not require testing.

Make a generalisation.

Children are required to write down the rule that they have used both in words

> When you know the number of red tiles
> you can work out the number of white tiles.
> Write down the rule you use.

and in the 'machine chain' form.



(SMP 1983)

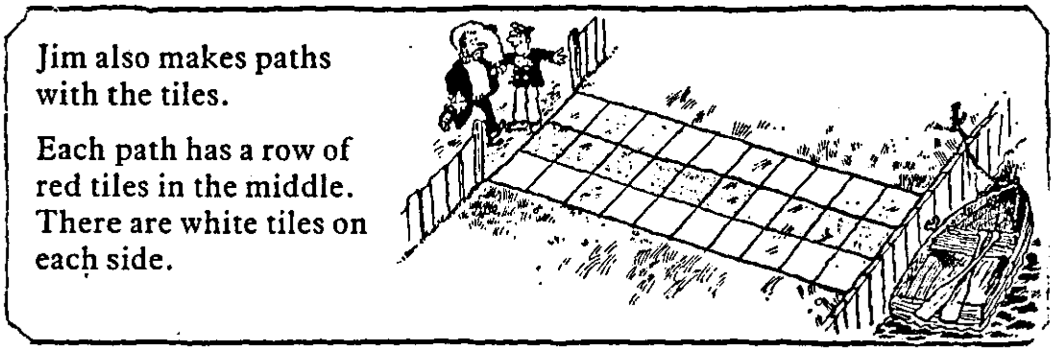
Justify or prove.

The booklet does not require the child to justify or prove their formula although the construction of a generalised formula in this case is closely linked with a justification of its validity.

The booklet continues with a series of problems using the same strategies as for the doorstep problem.

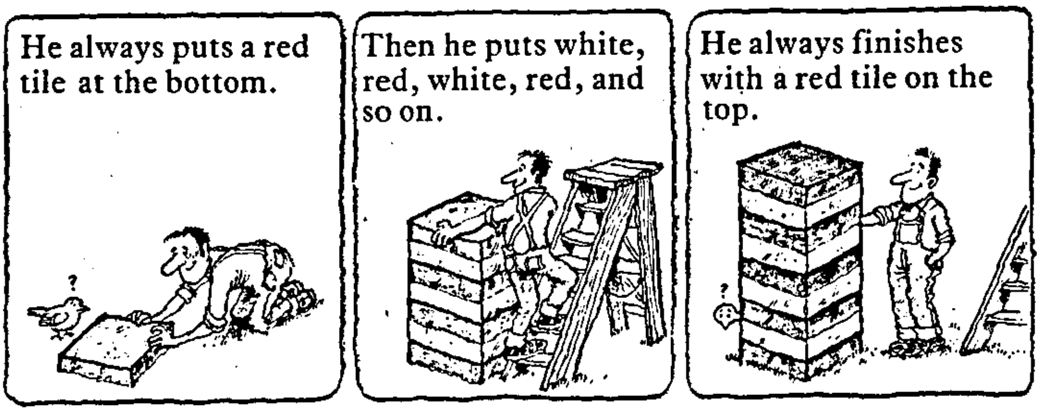
These are :

Paths



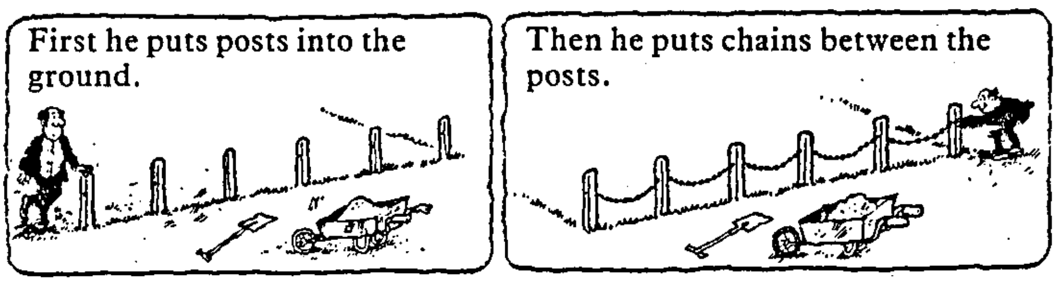
Pillars

Jim uses the tiles to make pillars.



Chain fences

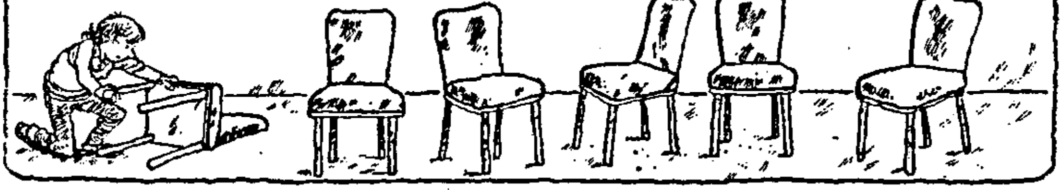
Sam makes fences with posts and chains.



(SMP 1983)

Chairs

Connie makes chairs like these.

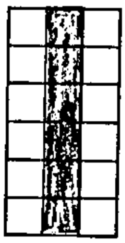


Towers

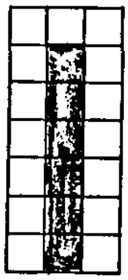
This is how to make a 'tower'.

Start with a column of red tiles.

Put a column of white tiles against each side.



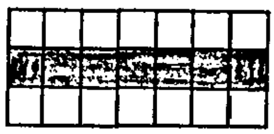
Put 3 white tiles across the top to finish the tower.



Borders

Start with a row of red tiles.

Put white tiles above and below to make a 'sandwich' like this.



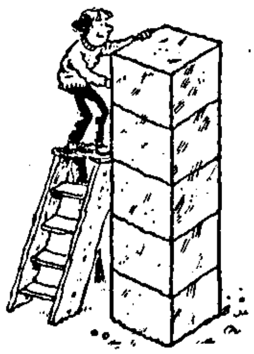
Put white tiles at each end to make a border.

(SMP 1983)

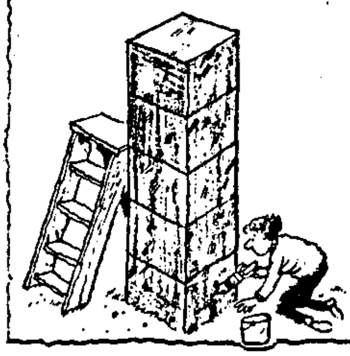
Painting cubes 1

You may need some cubes to help you do this section.

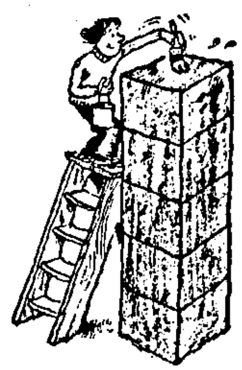
Sue builds pillars with cubes.
Here she has made a pillar with 5 cubes.



Now she has to paint the pillar.
She starts by painting the squares on the sides of the pillar.



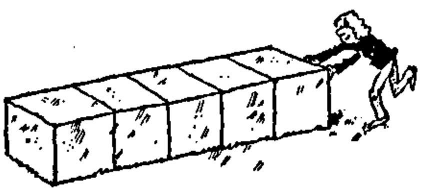
Then she paints the square on the top.



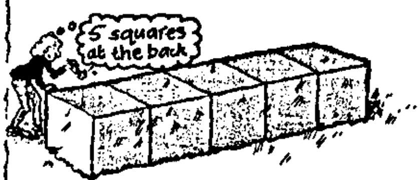
Painting cubes 2

Sue builds walls with cubes.
All her walls are 1 cube high.

Here she has used 5 cubes.

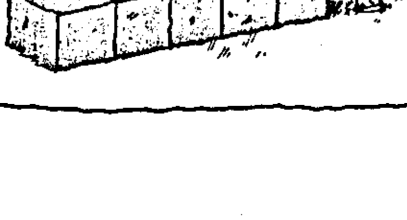


Now she has to paint the wall.
She paints the squares at the front, on top and at the back.



(She does **not** paint underneath the wall!)

Then she paints the squares at each end.



I7 How many squares has Sue painted altogether?

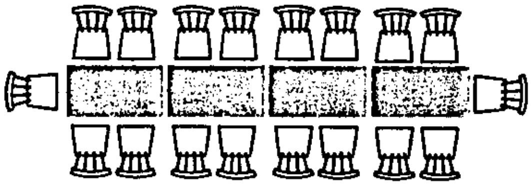
(SMP 1983)

Discovering rules : extension   Level 2(e) continues in the same way.

The pictures used continue to be of a diagrammatic form rather than of a pictorially representative form as is used in subsequent booklets.

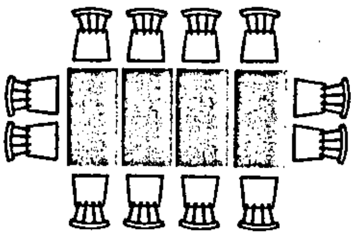
Seats and tables

A café owner has rectangular tables. He arranges them in rows like this



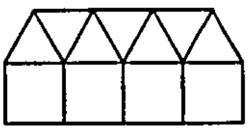
Arrangement A

or like this.

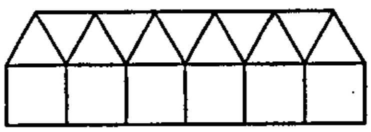


Arrangement B

Squares and triangles

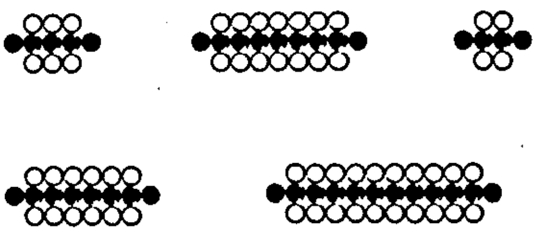


4 squares, 7 triangles



6 squares, 11 triangles

Spots



(SMP 1983)

Generalisation still takes place in the 'machine chain' format although the machine chains are either longer or require higher order operations.

Tile arrangements

In $\qquad$ Out

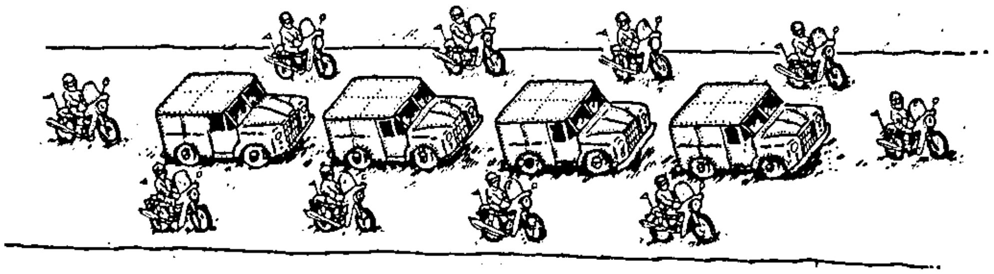
$r \longrightarrow \boxed{-3} \longrightarrow \boxed{\times 2} \longrightarrow w$

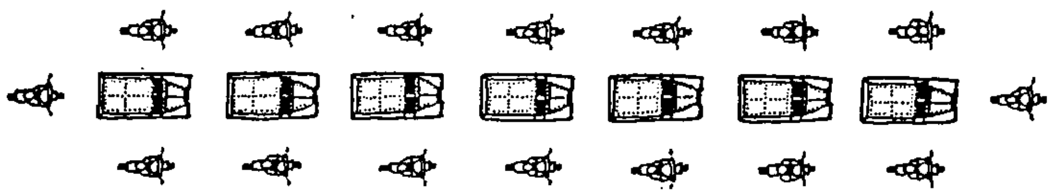
Layers

$r \longrightarrow \boxed{\div 2} \longrightarrow \boxed{+1} \longrightarrow \boxed{\times 3} \longrightarrow w$

In Formulas - Level 3(a) pictures of a pictorially representative form are used first alongside those of a diagrammatic form as in

Car and motorbikes

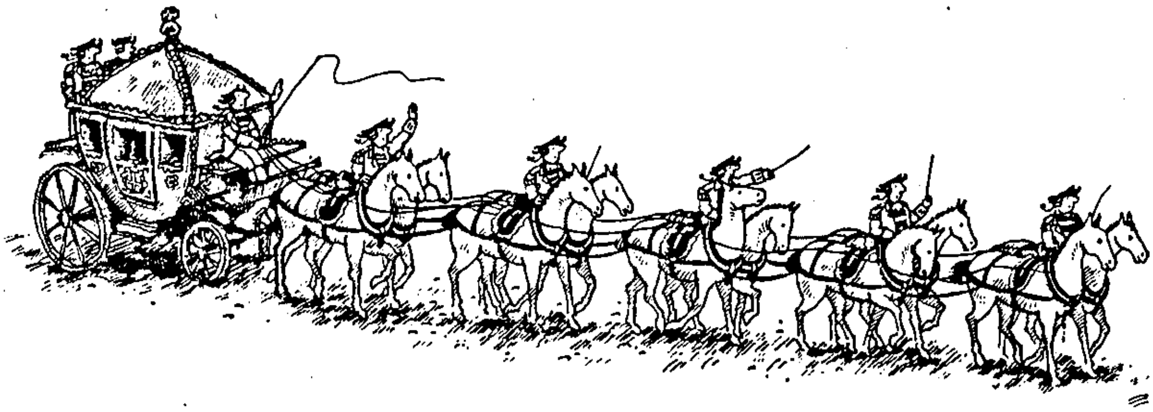




(SMP 1983)

and then only in a pictorially representative form as in

Horses and servants



(SMP 1983)

The uses of strategies becomes less explicit and less structured although children are required to start with simple cases and tabulate results directly in certain questions.

Parallel to these booklets in another part of the algebra course various forms of algebraic notations have been introduced. Algebraic formulas are introduced alongside 'machine chain' formulas.

This is the 'machine chain' for question D6.

$$c \longrightarrow \boxed{\times 2} \longrightarrow \boxed{+2} \longrightarrow m$$

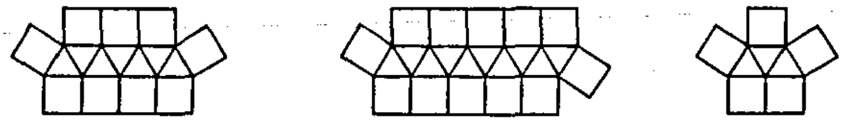
When we write the formula we use brackets, like this

$(c \times 2) + 2 = m$

(SMP 1983)

There is some substitution in formulas.

In Formulas : extension 1 Level 3(e) the strategies developed in the proceeding booklets are used to solve problems without the child being led through the strategies in great detail. This booklet finishes with the child being given three diagrams and being asked to find a generalised formula.



Write the formula connecting t (the number of triangles) and s (the number of squares).

(SMP 1983)

It should be pointed out that while these booklets are being followed sequentially they are not tackled consecutively. Other booklets specifically on algebraic topics will also be tackled. These algebraic booklets will be interspersed with booklets on other topics.

Formulas : extension 2 Level 3(e) is concerned with generalising a formula where the information is not given in a pictorial form but in words.

As this follows on from problems where the information is given in pictorial form the work is presented in a more condensed form. Children are still expected to follow the following strategies:

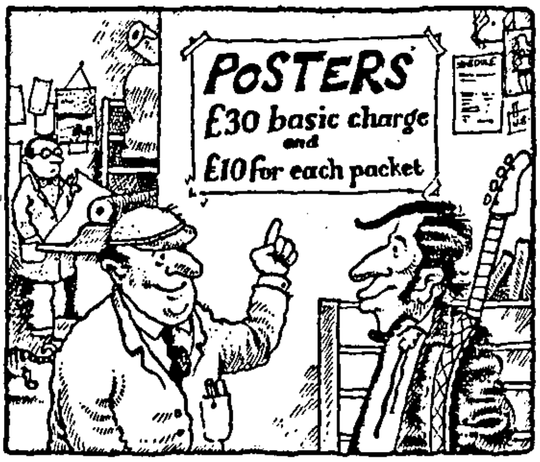
1. Start with simple cases
2. Be systematic
3. Tabulate results
4. Spot a pattern
5. Make a hypothesis
6. Test the hypothesis
7. Make a generalisation
8. Justify or prove.

Notice that in these type of problems the strategy 'Use a helpful diagram' is not employed.

Strategies 1 and 3 are combined in completing a table of results.
Strategies 4,5,6 and 7 are combined in deriving a 'machine chain' formula.
Strategies 2 and 8 are used in carrying out the other strategies.

A Printing posters

Brian and the Nutties are a new rock band.
They want to have some posters printed,
so Brian goes to find out how much it will cost.



The printer tells Brian that he must first pay a basic charge of £30. This is to set up the printing machine.

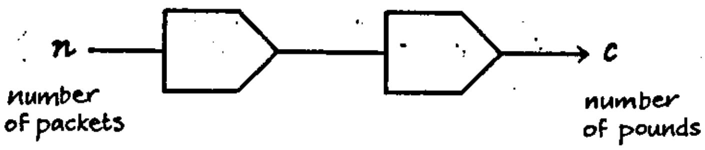
Then he has to pay an extra £10 for each packet of posters printed.

A1 How much does it cost altogether to have 4 packets of posters printed?

A2 Copy this table and complete it.

Number of packets	Cost in pounds
1 →	→
4 →	→ 70
10 →	→
3 →	→

A3 Draw the machine chain to show how you work out the cost.



(SMP 1983)

Towards the end of this booklet child are led to derive a formula from the information in words without going directly through the intermediate stages although they will probably be drawn on in constructing the formula.

E6 Brian and the Nutties offer to play at children's parties. They charge £2 for Brian and £1 for each Nutty who goes with him.

Let n be the number of Nutties who go with Brian.
Let c be the number of pounds charged.
Write down the formula for c in terms of n.

Form: ext 2

(SMP 1983)

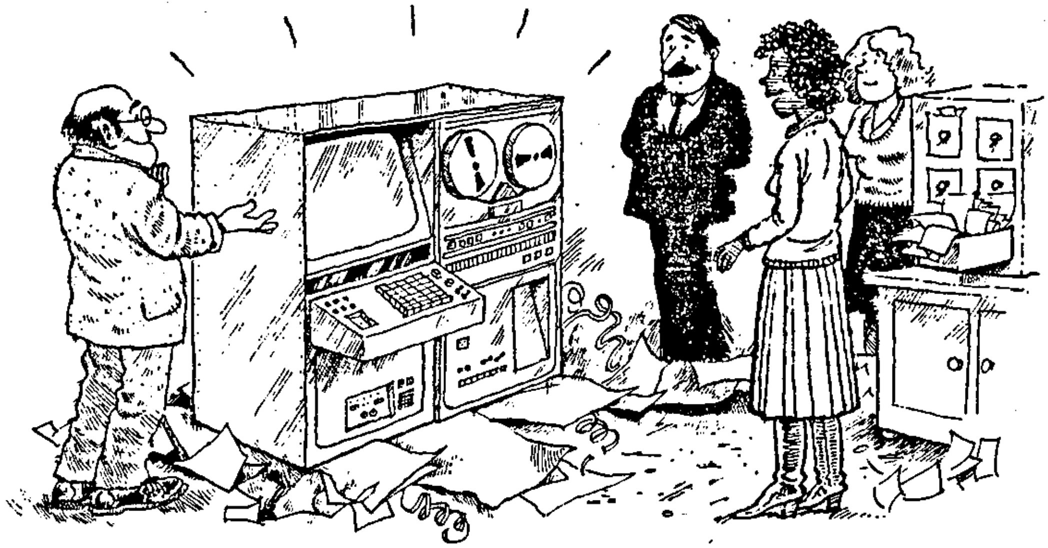
This topic is completed with two further booklets.

Using formulas Level 4(a) covers practice in subtituting in algebraic formulae like $y = 4x + 7$ and $z = \frac{x}{y} + w$ etc.

Formulas and graphs Level 4(e) draws together many of the skills acquired in previous booklets on algebra and requires children to apply them to graphs and to interpret formulas generally.

In one of the final exercises of the booklet children are given information about a problem in words and required to generalise a formula. This involves drawing on skills gained in previous booklets following an investigational approach.

G Selling computers



Angela Williams works for a computer manufacturer.
Her job is to sell one type of computer. She is paid
£10 000 a year plus £100 for every computer she sells.

G1 Let P stand for Angela's total pay in pounds.
Let n stand for the number of computers she sells.

Copy and complete this formula for her pay. $P = \ldots n + 10\,000$

(SMP 1983)

A second formula, $P = 150n + 5000$, is given.
The same values of n are substituted into both formulas and graphs of the formulas drawn.

Subsequently three different formulas are given and children are required to interpret them.

The company employs three other salespeople.
Each one sells a different type of computer, and each has a different pay formula.

Joe Williams	$P = 50n + 9000$
Devon Kingston	$P = 500n + 3000$
Shailesh Patel	$P = 200n + 6000$

G11 What does each formula say, in words?

G12 One of the three salesmen sells large computers, one medium size and one small.

Say which you think is which, and give your reasons.

(SMP 1983)

Although I have concentrated on a particular topic in algebra the approach outlined above is followed for many of the areas in the other topics of Number, Space, Graphs and Statistics.

CHAPTER 5

THE ROLE OF THE MICRO COMPUTER IN INVESTIGATIONAL WORK.

THE ROLE OF THE MICRO COMPUTER IN INVESTIGATIONAL WORK

The great strength of computers in investigational work is that they perform long and tedious calculations quickly and accurately. They thus enable children to accumulate information from which they can start to make hypotheses. These hypothese can then easily be tested by the computer.

As well as calculations computers can also produce diagrams that would otherwise be difficult to produce by hand. These can once again form the basis from which hypotheses can be made.

As information can so easily be gained from an appropriate computer program it does not matter how often information is required. This encourages the strategy of 'trial and error' so often a precursor of forming a hypothesis.

In this chapter I shall examine how computer programs can be used in different ways to aid investigational work in mathematics. I shall consider first four different types of computer programs that fulfil different roles. These types are : -

1) A program that aids a specific investigation.

2) A program that aids a limited number of investigations closely related to a parent investigation.

3) A program that aids a limited but broad area of investigational work.

4) A program that aids a wide range of investigations.

I shall then consider how small programs, written in BASIC and up to 20 lines long, can be used in investigational work.

Finally I shall consider how some computer programs may form part of a large investigation.

A program that aids a specific investigation

Investigation

a) There are two jugs. One has a capacity of 12 units the other a capacity of 5 units.
Either jug may be filled.
Either jug may be emptied.
The contents of one jug may be poured into the other.
There is a limitless supply of liquid.

Show how capacities from 1 unit to 12 units of capacity can be obtained.

b) The 12 unit jug is retained and a second jug of a different capacity is used.

Investigate what capacities the second jug must have to be able to obtain all capacities from 1 unit to 12 units.

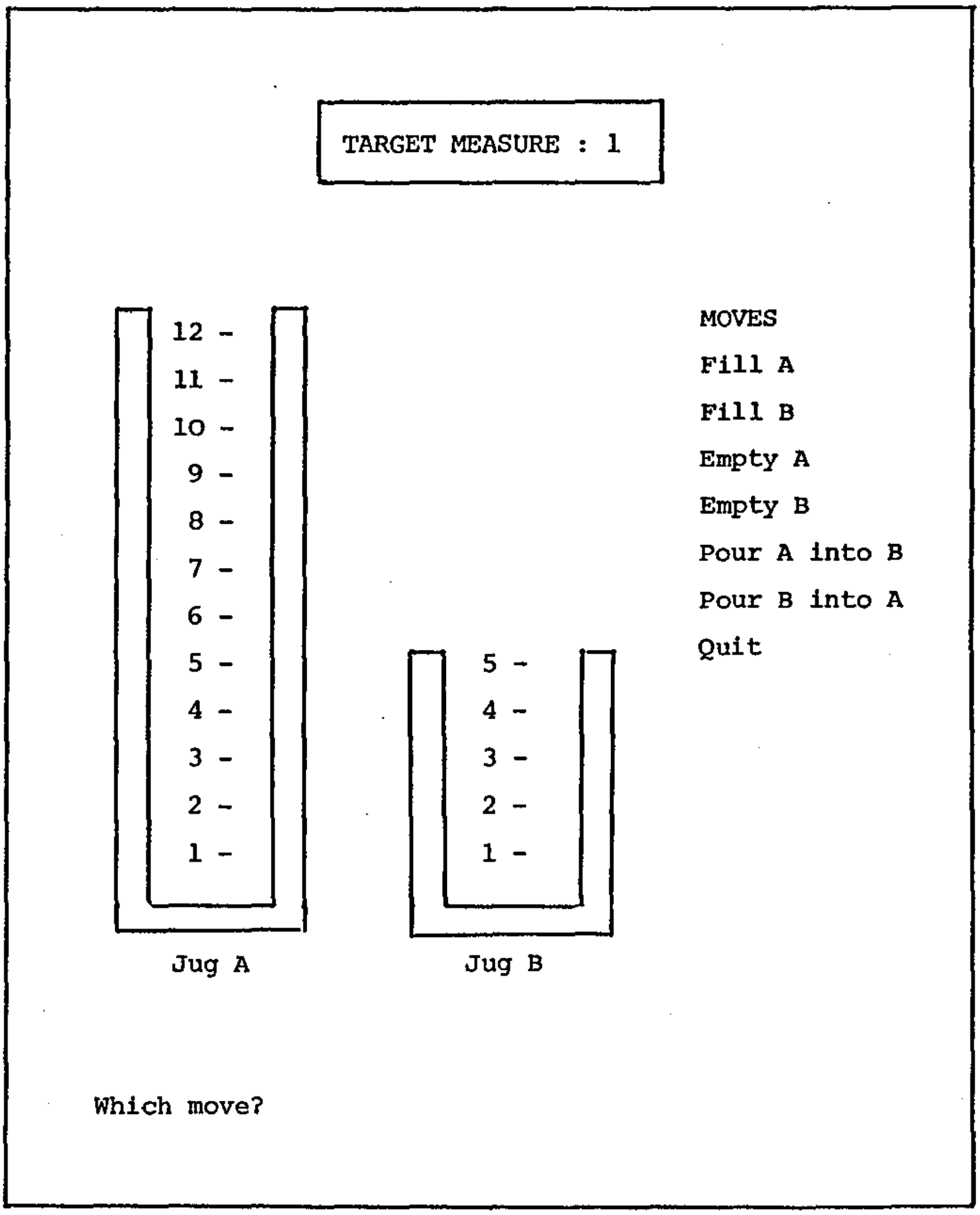
This investigation presents considerable difficulties for children. It is difficult for children to devise an accurate means of recording their results as they try to obtain a particular capacity. This can lead not only to frustration but to children losing their way in the investigation obtaining incorrect results.

A program that goes a long way in overcoming these difficulties is Jug from Microsmile, The First Thirty Programs.

This program enables the operations of filling, emptying and pouring from one jug to another to be performed easily and accurately and the results of these operations to be clearly seen.

The program allows the capacity of the two jugs and the target measure to be chosen.

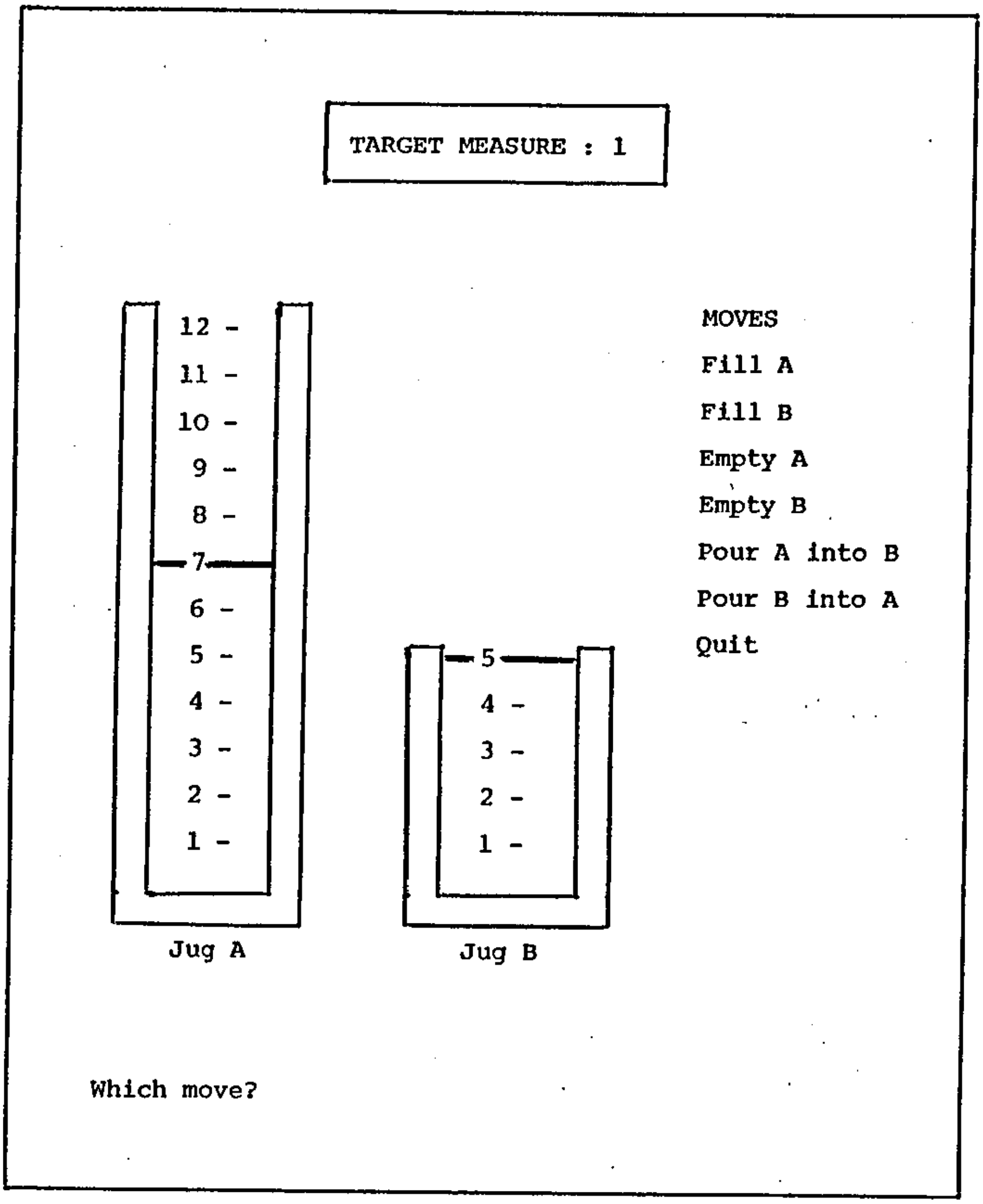
A typical screen display at the start of the investigation would be



After the moves   Fill A

Pour A into B

The screen would show



Each operation requires only two entries.

To Fill A : typing F produces the word Fill on the screen
then typing A produces the words Fill A on the screen

and completes the operation.

To Pour A into B : typing P produces the word Pour on the screen
then typing A produces the words Pour A into B on the screen

and completes the operation.

A record of the operations used is kept by the program.
While facilitating the execution of the filling, emptying and pouring operation this program does not directly give results. The selection and sequencing of the operations still remains with the investigator. The child still has to determine how to obtain a capacity of say 1 unit.

This program can be contrasted with another program entitled Polygon from the disk Games, Activities and Investigations for the Primary Classroom by Anita Straker. This piece of software supports the following investigation.

Investigation

Investigate how the number of diagonals of a polygon are related to the number of its sides.

The initial screen display is

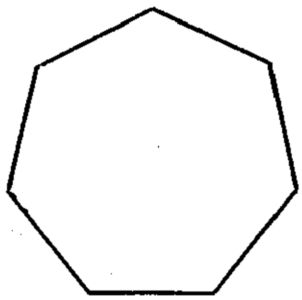
Find the number of diagonals of any polygon
Press 1, 2, 3 or 4 to : -
1. Try a new polygon
2. See all the results so far
3. See if you have guessed the rule
or 4. To stop.

If the first option is selected the extra line

How many sides for the polygon?

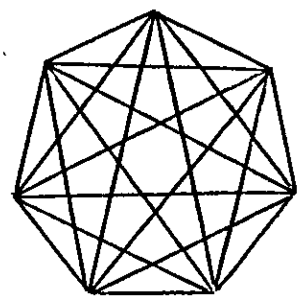
In response to typing 7 the screen shows

A 7 -gon



Guess the number of diagonals

After either a correct or incorrect guess the computer draws in the diagonal and tells you how many there are



There are 14 diagonals

The difference between Jugs and Polygons is that Polygons enables the child to obtain the number of diagonals of any polygon without working out what the number is. An incorrect guess elicits the response 'Watch' and proceeds to draw in the diagonals and then gives the number.

A piece of software such as Polygons needs to be used with forethought and care. If the program is used from the start of the investigation some of the 'process' benefits of investigational work are lost. Every answer will be correct. An important part of the investigatory process is to obtain a set of results and to be almost able to see a pattern in them. Checking the misfit result and correcting it, if it is wrong, is a vital part of the investigation.

This investigation should really be started with pencil and paper. There are four reasons for this.

1) The question can arise as to whether the polygons need necessarily be regular (as in the program) or whether they can be irregular or concave without effecting the results.

2) The actual method the child uses to draw in the diagonals may be helpful later in making a generalisation about the number of diagonals for an n sided polygon.
If the child is being systematic in drawing in the diagonals he or she may start at one vertex and move around the vertices in order.
For a 4 sided polygon one draws 1 and 1 diagonals
For a 5 sided polygon one draws 2 and 2 and 1 diagonals
For a 6 sided polygon one draws 3 and 3 and 2 and 1 diagonals
For a 7 sided polygon one draws 4 and 4 and 3 and 2 and 1 diagonals

This would not necessarily happen when the computer draws in the lines as either the child may 'switch off' while waiting for the results or may not have chosen to draw in the diagonals in the same way as the computer.

3) The difficulty of drawing polygons and counting the number of diagonals becomes increasingly difficult as the number of sides increases. The child therefore sees a good reason for trying to arrive at a rule or generalisation.

4) This difficulty imposes a limit on the number of pieces of data that the child has to work with. To increase the number the child, by necessity, must include simple cases (the triangle, the square) that may otherwise have been omitted.

It is after the initial stage where the child has drawn several polygons, tabulated the results and hopefully spotted a pattern and made a hypothesis that the computer can be used.

The computers main use is that it can check a hypothesis about how many diagonals a polygon with a large number of sides has. Such a polygon would be difficult to draw accurately. A subsidiary use of the computer would be to check the results already obtained.

Both of these programs aid specific investigations. It would be difficult to use them for other investigations. If for example Polygon was used to investigate the number of regions that a polygon is divided into by its diagonals two problems would arise.

1) For a polygon with a reasonable number of sides the regions would still have to be counted by the child. On paper they could be ticked off as counted, on the screen they could not.

2) For a polygon with a large number of sides some of the regions would be 'lost' because of the low resolution of the screen. This is not so important with the diagonals as the computer calculates and gives the number of diagonals.

<u>A program that aids a limited number of investigations closely related to a parent investigation</u>

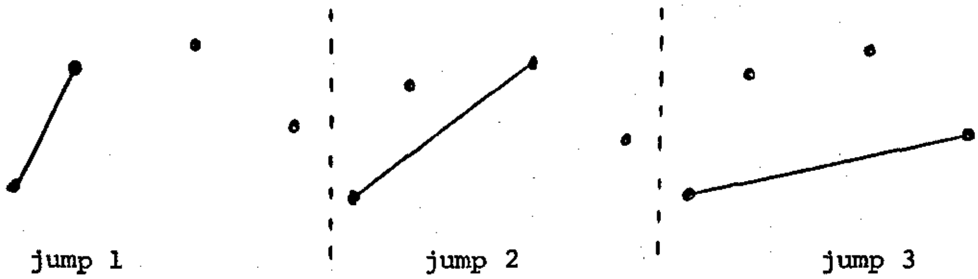
Sometimes a mathematical situation may give rise to a number of different but related investigations.

Such a mathematical situation is presented by the computer program Circle from Microsmile The First Thirty Programs.

The program allows you to choose the number of points around the circle and to decide the size of each jump.

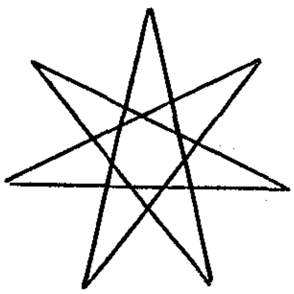
A jump of 1 joins two consecutive points
A jump of 2 misses a point

and so on



jump 1   jump 2   jump 3

The screen display for 7 points on the circle with a jump of 3 is



Points : 7
Jump size : 3

Lines : 7
Revolutions: 3

Do you want another?

With the diagrams drawn and the information provided there are a number of sub-investigations that may be followed. All are essentially involved with the relationship between the number of points on the circle and the number of jumps.
Some of these are : -

1) Investigate the relationship between the number of points on the circle and the jump size when the shapes generated are triangles, squares, pentagons, etc.

2) Investigate the relationship between the number of points on the circle and the jump size when only half the points on the circle are used as vertices of the shape.

3) Investigate the relationship between the number of points on the circle and the jump size when stars are generated.

It is interesting to consider the ways in which this program can be used with children as the basis of a piece of investigational work. At one extreme it can be used when children are undertaking investigational work for the first time. They will need some guidance as to the strategies involved. These strategies will be introduced by the use of appropriate questions. If the work is presented in a written form the instructions and questions may take the following form. The strategies to be introduced are on the right.

<u>An Investigation using Circle : Triangles</u>

Start with 6 points on the circle. Find the shapes that you get with jump sizes 1 to 6

Start with a simple case

Be systematic

Record your results in the table below.

Points on circle 6	
Jump Size	Shape
1	
2	
3	
4	
5	
6	

Tabulate the results

Draw up similar tables for 7,8,9,10,11, and 12 points on a circle and use the program to fill them in.

For what numbers of points on a circle have you got a triangle?	Spotting patterns
Draw up a table for 15 points on a circle and use the program to fill it in.	Making a hypothesis and testing it
Write down which number of points on a circle will give rise to triangles	Generalise

At the other extreme children may be given a brief 'one liner' as follows.

Investigate how to produce triangles using the circle program.

A program that aids a limited but broad area of investigational work

The computer programs considered in this chapter so far, Jugs, Polygon and Circle, have involved specific investigations. There are a number of programs that while being about a specific topic in mathematics are such that they are open to many different investigational approaches. Such a program is Tilekit.

Tilekit is written by Derek Ball and published by the Association of Teachers of Mathematics on the disk Some More Lessons in Mathematics with a Micro computer.

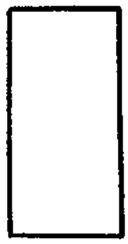
The program allows the user to create tessellations of certain shapes easily and quickly on the screen. The shapes available are :

Big Triangle	Small Triangle
Big Square	Small Square
Big Hexagon	Small Hexagon
Big Octagon	Small Octagon
Big Dodecagon	Small Dodecagon
Rhombus 1	Oblong 1
Rhombus 2	Oblong 2
Parallelogram 1	Parallelogram 3
Parallelogram 2	Parallelogram 4

The various rhombuses, oblongs (rectangles) and parallelograms give different transformations of an original figure as shown.

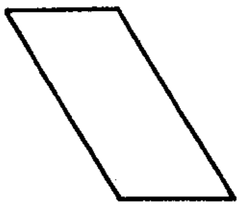
rhombus 1

rhombus 2

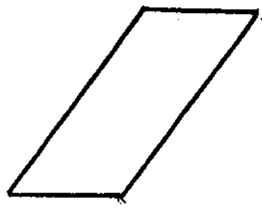


oblong 1

oblong 2



parallelogram 1



parallelogram 2

parallelogram 3

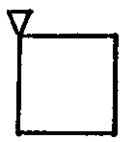
parallelogram 4

These shapes may be produced on the screen by two key strokes.

For example	BT	produces a Big Triangle
	P1	parallelogram 1

At the beginning of a sequence the pointer is at the centre of the screen. Shapes are drawn clockwise with the pointer ending up outside the shape pointing along the last side from the starting point.

Before



After Big Square

The commands used are similar to those of 'turtle' geometry programs except that the commands of Tilekit are simpler and more limited.

The Move commands are :

Move Round	Moves the pointer to the beginning of next side of the shape.
Move Forward	Moves the pointer in the direction it is pointing for a distance equal to the length of the side of the small shapes.
Move Back	As for Move Forward but in the opposite direction.
Move Hidden	All subsequent moves are hidden.
Move Draw	All subsequent moves are displayed.
Turn Round	Rotates the pointer through 180°.
Turn Slightly	Rotates the pointer through 15°.
Wipe	Rubs out the last shape drawn.
New picture	Clears the screen and re-centres the pointer.

All these commands are executed by two key strokes except for Wipe (W) and New pictures (N).

There are limited programming facilities

Create	Allows a program to be written.
Loop	Allows a series of commands to be executed up to 9 times.
End Loop	Ends the loop.
End Program	Ends the program.
Go	Runs the created program.
Display	Displays the created program.

As this program has a large number of commands and a large number of different shapes that can be drawn on the screen the possibilities of the programs use are consequently numerous too.

Some suggestions are :

1. Use just one instruction repeated (e.g. Big Hexagon). This will produce several copies of the same shape around a point. How many of the shapes fit around a point? Are there any 'rules'? Can you predict how many dodecagons will fit around a point. What are the rules?

2. Which shapes can be made to cover the screen? Which pairs of shapes fit together to cover the screen? Discuss this with a group. Try using just squares and triangles. There are boring ways of doing this and interesting ways. Look for repeating units of pattern in the tessellation.

3. Can the different sizes of parallelogram be used together to form other shapes? Similarly for the rhombus and oblong.

4. Which shapes will fit inside others?

5. Work with a group to build a symmetrical pattern on the screen using several different shapes. Much of the value of this activity lies in the work required to overcome the difficulty of describing where you want to put the shape next. There can be rules restricting, say, hand waving and pointing at the screen if this seems appropriate.

(A.T.M. 1985)

It is interesting to note that while suggestion 1 is a restricted investigation, similar in many ways to Jugs or Polygon, the other suggestions are more open ended. This is not surprising as the broadness of the program allows many different avenues of investigation to be followed.

This raises the issue of when to use particular computer programs in the mathematical development of the child. When children are beginning to embark upon investigational work it is important that they are aware of and use the strategies of investigational work.

These strategies will be taught through the structure of the investigation presented usually by direct instructions and direct questions. These strategies will also be highlighted orally by the teacher during the investigation.

When these strategies are familiar children will require less direction in their use. It will only be necessary to give a starting point and a few general directions.

There needs therefore to be careful planning on the part of the teacher as to the development of an investigational based course and the appropriate choice of software to support it.

A program such as Tilekit is perhaps more appropriate when children are familiar with the investigational approach as it is so rich in possible avenues of exploration.

A program that aids a wide range of investigations

The program Tilekit, considered in the previous section, enabled eight different shapes, with some restrictions (in the case of some only regular polygons) and some variations (in some cases size in others transformations of the original shape). By removing these restrictions and allowing any shapes to be created a still wider range of investigational work is opened up.

There are several 'turtle' geometry programs in existence. I shall consider the program Arrow produced by Oxfordshire County Council.

Arrow is a computer language which enables lines to be drawn on the screen. One of the most powerful features of Arrow is the ability to invent new commands from a combination of the existing commands and to treat these new commands in exactly the same way as the old ones.

Instructions to the computer are subdivided into primatives and commands.

Primatives

FORWARD	Moves the pointer forward a specified distance.
BACKWARD	Moves the pointer backward a specified distance.
LEFT	Changes the direction a specified number of degrees to the left.
RIGHT	Changes the direction a specified number of degrees to the right.
PEN OFF	Moves the pointer without drawing.
PEN ON	Moves the pointer with drawing.
SCALE	Enables the scale to be changed either horizontally, vertically or both.
COLOUR	Changes the drawing colour to one of 8 specified colours.
INK	Changes the drawing colour to one of 3 specified colours.
CENTRE	Moves the pointer to the centre of the screen.
CLEAN	Cleans the screen but leaves the pointer in its previous position.
FRESH	Cleans the screen and centres the pointer.
IF ... THEN ... IF ... THEN ... ELSE	Gives a choice of action equal less than less than or equal to greater than greater than or equal to not equal to.
END	Automatically at the end of a procedure but can be used as IF SIDE 20 THEN END
REPEAT	Allows a set of instructions to be repeated a specific number of times.
LIMIT	Signifies the end of a set of instructions to be repeated.
MAKE ... BECOME	Changes the value of a variable.

Commands

BUILD Allows a procedure to be built.

There are also a number of commands that facilitate editing of procedures, change the speed of drawing and control loading, saving and printing. These increase the scope of the investigations that can be undetaken. In comparison with Tilekit there are not many commands. The main differences are :

1. the ability to draw any shape rather than a limited selection of shapes;

2. the ability to define and name a shape by using a procedure and to use the shape subsequently by calling it by name;

3. the ability to use variables;

These extra facilities broaden the range of investigations that can be undertaken.

The investigation suggested in the section on programs that aid a limited number of investigations closely related to a parent investigation involved the program Circle. The investigation about stars can now be extended as follows :

1. Investigate the relationship between the angles of the points of the star and the number of points.

2. Investigate stars whose arms alternate in length from 20 to 40 units.

3. Investigate stars where the angles of the points alternate in size.

The investigations suggested in the section on programs that aid a limited but broad area of investigational work involved the program Tilekit. The investigation about what shapes cover the screen can be extended to include irregular polygons.

The use of small programs in investigational work

Perhaps the ultimate use of a computer is when the investigator writes or adapts programs to aid an investigation. In many ways this is an extension of what was seen in part 4 of the previous section where a program that aids a wide range of investigations was considered.

In many ways Arrow is a computer language related to other computer languages such as LOGO or BASIC. The main difference is in the number of commands available and the scope of the set of commands. As the number and scope of the commands are increased so then are the possible applications.

The use of small programs, written in say BASIC by the investigator to aid investigations is not very widespread. This is understandable as the more sophisticated the language the more difficult it becomes for the child to understand and to program in the language.

However if children have had some experience of programming within a program such as Tilekit or Arrow the transition to a language such as BASIC can be made smoother. It may also be necessary to restrict the commands used within the small programs.

Small programs may be defined as programs using up to 20 lines of instructions.

Although short these programs can use the 'number cruncher' ability of the computer to carry out investigations that would otherwise be beyond most children because of the length of time necessary to perform tedious calculations.

This use of small programs is well put by T.J. Fletcher in Micro computers and Mathematics in Schools.

> By this we mean such things as the properties of prime numbers, of sequences such as the triangular numbers, squares and cubes, and properties of divisibility.
>
> (T.J. Fletcher 1983)

An example of a small program is the following program concerning the factors of numbers.

```
10    INPUT  N
20    FOR  F = 1  TO  N/2
30    IF INT(N/F)  = N/F THEN PRINT F
40    NEXT F
50    PRINT N
```

(Mathematics Association 1985)

This program calculates and prints the factors of a number,including 1 and the number itself.

Such a program can form the basis of a number of investigations. Some of these are :

1) Investigation of those numbers with only two factors (i.e. prime numbers).

2) Investigation of those numbers with an odd number of factors.

3) Investigation of those numbers with only three factors.

4) Investigation whether you can always find a number with a given number of factors.

An advantage of small programs such as the one above is that they can be adapted and modified to aid further investigations.

```
10    INPUT  N
15    LET  S = 0
20    FOR  F = 1  TO  N/2
30    IF  INT(N/F)  = N/F THEN LET  S = S + F
40    NEXT  F
50    IF  S = N  THEN PRINT N
```

This program tests whether the factors of a number, excluding the number itself, add up to the origianl number. It can be used to find perfect numbers.

As it stands the program will only test individual numbers to see where they are perfect numbers but further adaptation allows any numbers up to M to be tested.

```
10  INPUT M
20  FOR N = 1 TO M
30  LET S = 0
40  FOR F = 1 TO N/2
50  IF INT(N/F) = N/F THEN LET S = S + F
60  NEXT F
70  IF S = N THEN PRINT N
80  NEXT N
```

(Mathematics Association 1985)

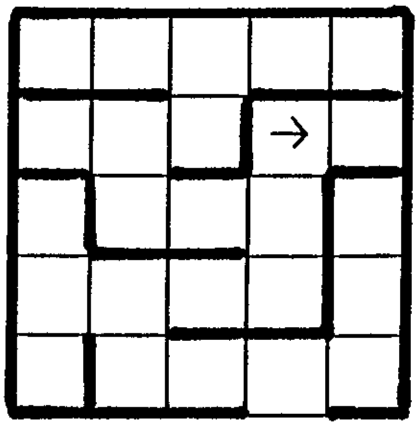
In these examples the small program may be viewed as a tool in aiding the investigation. If the program is given to the child this may well be the case. However when the child either writes the program or adapts a given program then this becomes very much part and parcel of the investigational approach. The writing of the program will probably involve hypothesising as to the commands to be used, testing the program to see if it works with simple, known cases and, if necessary, further adjustment.

The third use of a computer is as a support resource in an investigation not relying totally on the computer.

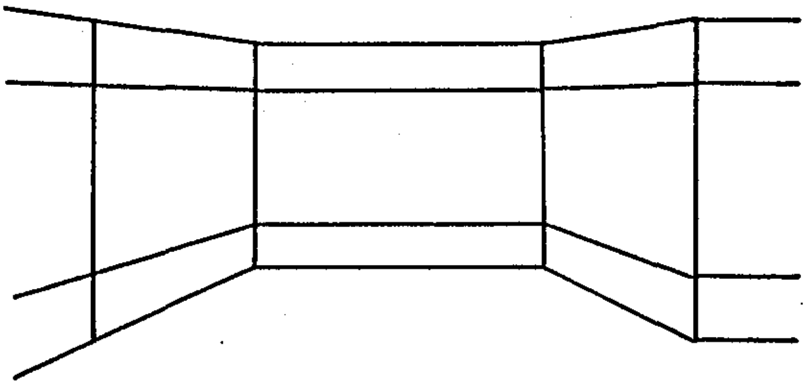
The starting point of an investigation might be 'mazes'.

There are many activities that may be undertaken that do not involve the use of a computer. These involve the design and construction of different mazes and strategies for navigating a maze.

One particular type of maze is that which is based on a square grid. It has the same entrance and exit. The problem is how to escape from such a maze when one is placed in it.

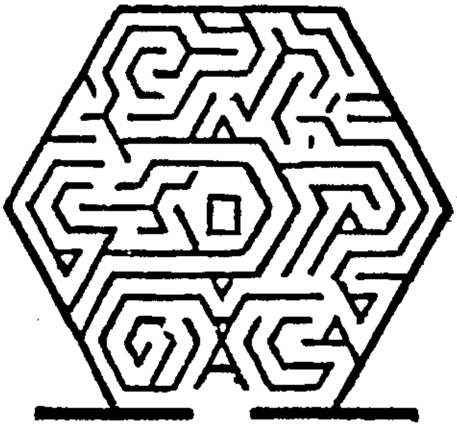


The only information you have is the view of the maze from where you are standing. The view from the position shown is

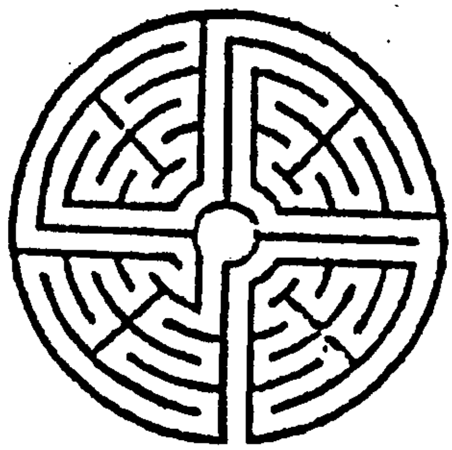


A program that supports this investigation is 3D Maze from Microsmile The Next 17. Views such as the one above are shown on the screen. You may move forward or turn left or right. After each of these the screen shows the appropriate view.

The program is useful in testing hypotheses as how to navigate a maze of this particular type and in forming an algorithm for solving other types of mazes such as the following.



Arley Hall, Cheshire



From The British Museum

As can be seen from the previous considerations computers have an important role in investigational work. They support all aspects of investigational work, sometimes as the major tool for the investigation, sometimes as one of several resources.

CHAPTER 6

ASSESSING INVESTIGATIONAL WORK.

ASSESSING INVESTIGATIONAL WORK

In Chapter 1 a distinction was made between what was described as the 'Guided Discovery Approach' and what was described as the 'Problem Solving Approach' or 'Investigatory Approach' in investigational work.

In the 'Guided Discovery Approach' the objective was for the child to learn certain facts and to acquire certain skills and concepts. Although the approach was different the facts, skills and concepts remained very much the same as with other styles of teaching. There is no reason then for the methods of assessment that were previously used to be as effective here in assessing the 'Guided Discovery Approach' of investigation work.

The 'Investigatory Approach' involves the use of various strategies in the solving of certain problems or in undertaking certain investigations. As the method of working is in some ways very different from conventional work in mathematics different forms of assessment need to be devised.

I want to concern myself here with the assessment of investigational work in the examination context. This falls into two categories. First those questions set under examination conditions to be completed in a limited period of time, for example 20 minutes. Second those questions which are not set under examination conditions and which do not have a fixed time limit. Such questions would come under the heading of school based course work assessment.

Limited time examination questions

It has always been general practice in traditional examinations to award marks in longer questions where candidates do not obtain the correct final answer. The principle behind this has been that candidates should receive credit for using the correct method but making a mistake, usually of the arithmetic kind.

Similar methods of assessment can be used in investigation questions of the type under consideration. In this case marks will be awarded for the various strategies used in the investigation.

> ..., mark schemes will be designed to give credit for :
>
> (i) showing an understanding of the problem,
> (ii) organising information systematically,
> (iii) describing and explaining the methods used and the results obtained,
> (iv) formulating a generalisation or rule, in words or algebraically.

(J.M.B. Shell Centre for Mathematical Education 1984)

In short questions, say to be completed in 20 minutes, it is possible to be fairly objective concerning the awarding of marks within the question. This is because the question will by necessity have to be very directed. Here is a typical question to be completed in 20 minutes.

Reverses.

Here is a row of numbers : 2, 5, 1, 4, 3.
They are put into ascending order by a sequence of moves which reverse chosen blocks of numbers, *always starting at the beginning of the row.*

Example :

2, 5, 1, 4, 3	reversing the first 4 numbers gives	4, 1, 5, 2, 3
4, 1, 5, 2, 3	reversing the first 3 numbers gives	5, 1, 4, 2, 3
5, 1, 4, 2, 3	reversing the 5 numbers gives	3, 2, 4, 1, 5
. . . . .		
. . . . .		
. . . . .		
. . . . .		
1, 2, 3, 4, 5		

(i) Find a sequence of moves to put the following rows of numbers in ascending order.

(a) 2, 3, 1

(b) 4, 2, 3, 1

(c) 7, 2, 6, 5, 4, 3, 1

(ii) Find some rules for the moves which will put any row of numbers in ascending order.

(J.M.B. Shell Centre for Mathematical Education 1985)

This question very much leads children through a sequence of activities each with specific answers. It is unlikely that there will be many, if any, solutions essentially different from the one catered for in the mark scheme.

Reverses ... Marking Scheme.

(i) Showing an understanding by dealing successfully with simple cases.

(a) 2 marks for 231 $\rightarrow$ 321 $\rightarrow$ 123 (or any correct solution)

(b) 2 marks for 4231 $\rightarrow$ 1324 $\rightarrow$ 3124 $\rightarrow$ 2134 $\rightarrow$ 1234 (or any correct solution)

Showing a systematic attack in the extension to a more difficult case.

(c) 3 marks for 7265431 $\rightarrow$ 1345627 $\rightarrow$ 6543127 $\rightarrow$ 2134567 $\rightarrow$ 1234567 (or any correct solution)

Part marks : Give 2 marks if the solution is basically correct but contains an error (e.g. in transcribing numbers).

Give 1 mark if the solution is incomplete but there is an intention to collect numbers ...567 or ...321 at the right hand end.

(ii) Formulating some general rules; describing and explaining them.

1 mark for saying that the highest (or lowest) number is to be brought to the right hand end.

1 mark for saying how this is done.

1 mark for completing the description of the method.

(J.M.B. Shell Centre for Mathematical Education 1985)

There is nothing in this marking scheme that is intrinsically different from conventional marking schemes in terms of having to make subjective judgements about the childrens work.

If the length of time allowed for the examination question is longer than the 20 minutes allowed for Reverses then it is possible for little or no help to be given to the child in the investigation. This would appear then to make the construction of a marking scheme and hence the assessment of a childs work relative to another child very difficult.
The pilot examination for the SMP 11 - 16 consists of two parts, written papers and a school based element. The school based element has two sections.
Section 1 takes the form of 'tasks' carried out during the two years prior to the written examination.
Some of the tasks are to be done in school, under supervision.
The other tasks are meant to be done out of school in the candidates own time or as part of the homework allocation.
Section 11 takes the form of orally administered tests.

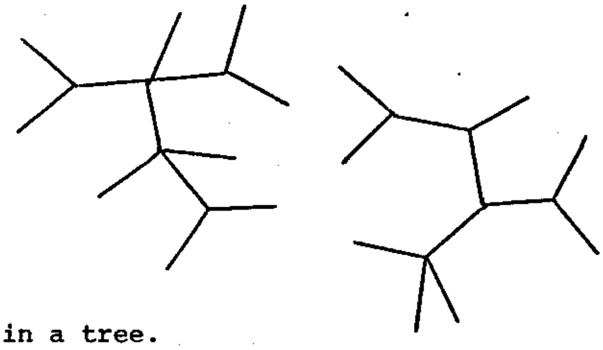
The following question is from an SMP 11 - 16 'Mock' coursework task on Investigations.

'Mock' coursework task : TREES

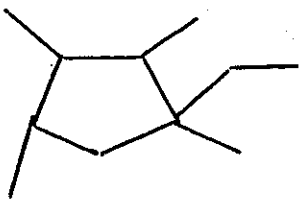
Remember :

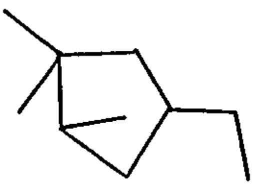
1) Describe how you tackle this investigation, so that someone else can follow what you did.
2) Answers by themselves are not enough. You must show the working that leads to them

These shapes are made
with matchsticks.
They are called 'trees'

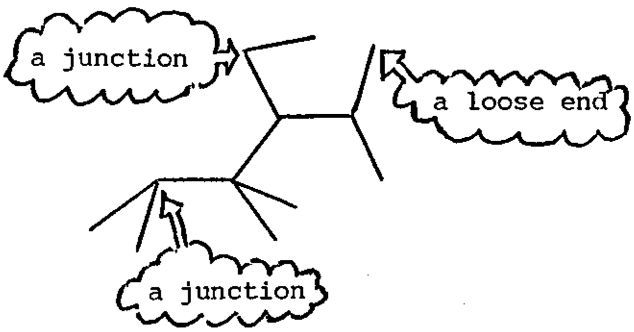


There are no closed loops in a tree.
So these, for example, are not trees.





This tree has 7 loose
ends and 5 junctions.
(A junction is where
two or more matches
meet).



Find out if there is any rule connecting the number of loose ends, the number of junctions and the number of matches in a tree.
If you find a rule, write down clearly what it is.

(SMP, 1985)

Although this is entitled a coursework task it is to be done under supervision, that is examination conditions, within a given period of time and therefore differs from what is normally understood by course work.

The question does not indicate how to proceed with the investigation only requiring the candidate to explain clearly how the investigation was set out. The main problem then is that candidates will tackle the investigation in different ways making it very difficult to lay down a precise mark scheme.

There will however be some strategies that candidates will need to employ to be successful with this investigation. These are :

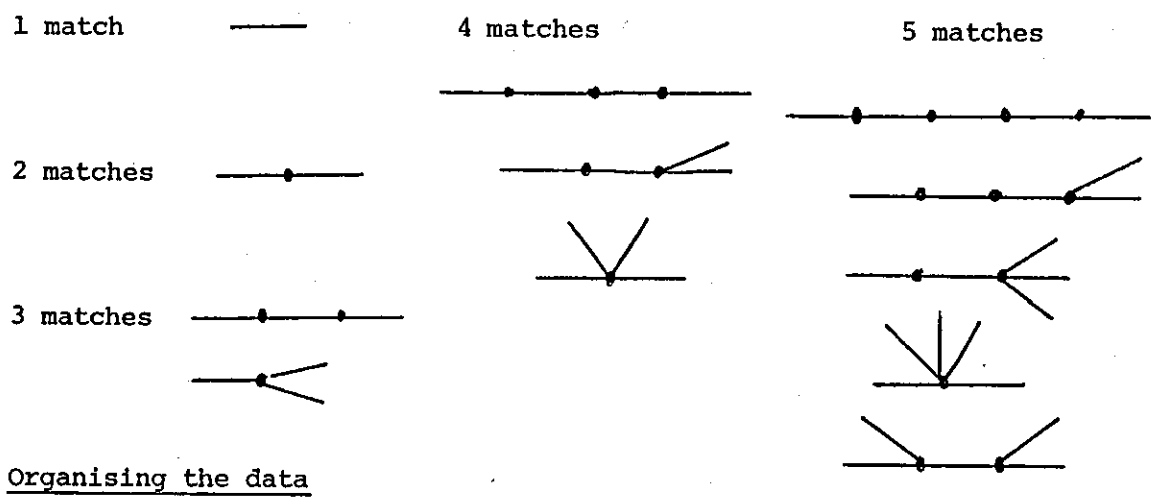
1) work in a systematic way
2) make use of helpful diagrams
3) start with simple cases
4) tabulate results
5) spot patterns
6) make a hypothesis
7) test the hypothesis
8) make a generalisation

Some of these strategies will merge together and so a mark scheme must make allowances for this. It must also be assumed that candidates are familiar with carrying out investigations and in presenting their results in a written form.

Mark Scheme for TREES

Generating data

Starting with simple cases	trees with either 1 or 2 matches	1
Being systematic	attempting to draw all trees with 3, 4 .. matches	1
Using a helpful diagram	actually drawing the trees	1
Successfully obtaining possible trees	3 matches	1
	4 matches all correct 1 or 2 scores 1	2
	5 matches 4 or 5 correct 1, 2 or 3 scores 1	2
	Section total	8



Organising the data

Use of a table of results	2
in a systematic fashion	2
Section total	4

Matches	Loose ends	Junctions
1	2	0
2	2	1
3	3	1
3	2	2
4	2	3
4	3	2
4	4	1
5	2	4
5	3	3
5	4	2
5	4	2
5	4	2
5	5	1

Interpreting the data

Statement of rule

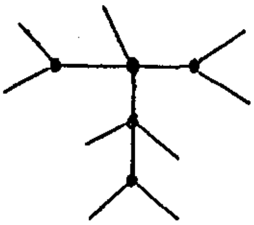
Loose ends plus junctions is one more than the number of matches

or equivalent 4

correct but not clear 3

correct but different statement 2

Testing of rule on a more complicated tree



Matches = 13
Loose ends = 9
Junctions = 5

Loose ends + junction = 14 2
Matches = 13

Section total 6

Total for Question 18

Both of these mark schemes have certain elements in common. They refer to a particular investigation which is of a narrow, limited variety. There is no likelihood of candidates embarking upon work other than that which is covered in the mark scheme.

School based course work

School based coursework presents problems for examination boards. The main problem is how to compare two pieces of work that are perhaps quite widely different although on the same topic. The East Midlands Regional Examination Board (EMREB) has for a number of years offered a Certificate of Secondary Education (CSE) which contains a course work element.

The course work element comprises 40% of the total mark for the examination. Candidates submit 15 topics of which a least 10 must be taken from one of the syllabuses of the option paper that the course work replaces. If schools wish to include topics outside the syllabus they must submit their whole course work scheme to the Board for verification.

The course work topics may involve open ended situations or be highly structured. They must however show evidence that the following objectives have been applied :

1. investigate, explain and, if possible, solve a problem;
2. collect and interpret data;
3. generalise and develop a mathematical situation;
4. recognise the appropriate method to tackle a problem; including possible use of computer;
5. hypothesise and then test the hypothesis;
6. use books of reference;
7. show initiative and create or invent valid methods new to them;
8. adapt and apply mathematics in unfamiliar situations.

(EMREB, 1984)

The mark scheme for each topic is as follows :

A.	Correctness	Maximum Mark 10
	This includes :	accuracy of calculation, method, appropriateness of method.
B.	Content	Maximum Mark 10
	This includes :	understanding of mathematical concepts, particularly those related to Section 3 of the syllabus; relevance of material to the topic; clarity of mathematical argument; amount of research; degree of difficulty of work done.
C.	Independence	Maximum Mark 5
		This includes evidence of the candidate's ability to work without constant guidance of the teacher, to use reference books appropriately, to extend his/her investigation beyond the work done in class.

D.	Completeness	Maximum Mark 5 This includes evidence that the particular project has shape and form, that in terms of the candidate's mathematical knowledge it has been brought to a satisfactory conclusion.
E.	Presentation	Maximum Mark 5 This includes evidence of the candidate's ability to carry out the work in a logical sequence and to present it carefully and neatly.
F.	Evidence of Retention	Maximum Mark 5 At the completion of each topic the candidate is required to show evidence that he/she has understood and is able to recall the essential mathematical concepts included in that topic.

(EMREB 1984)

The marking scheme was devised for a wide range of possible topics and not specifically for investigational work although the list of objectives suggest that EMREB anticipated that an investigatory approach would be adopted when completing the course work topics.

It is worth bearing in mind the CSE coursework mark scheme when considering how to devise a mark scheme specifically for investigational work.

In his article Investigations in Teaching Mathematics and its Applications Paul Ernest suggests the following marking framework.

1. Creative ability (30 marks)
   How much of the work appears to show originality of approach or handling? Give credit for originality or boldness of conjectures irrespective of the success or outcome. Are generalisations generated? Is the situation extended?

2. Systematic thought (30 marks)
   Is there a systematic generation and presentation of data?
   Is there careful verification (or disproof) of conjectures?
   Are there attempts to explain or even prove conjectures?
   Are appropriate diagrams and labels used systematically to present the data?

3. Thoroughness (10 marks)
   Is the investiation thoroughly pursued?

4. Skills (10 marks)
   Are various mathematical skills and background knowledge displayed?

5. Clarity (10 marks)
   Is the presentation clearly written?

6. Attractiveness (10 marks)
   Is the presentation made attractive by means of diagrams, models, colours, layout etc?

7. Further questions (20 marks)
   Does the investigation end with further questions and directions to pursue?

(Ernest, 1984)

While this marking framework is specifically designed for assessing a candidates piece of course work there are still a number of difficulties to overcome. These problems are essentially those faced by English teachers when they mark an essay. One concern is the balance between technical competence, spelling and grammar in the case of an essay, arithmetic and algebraic correctness in the case of mathematics and what might be termed the creative content.

This is perhaps easier to resolve in an essay where incorrect spelling or grammar can be separated from the creative content.

In mathematics incorrect arithmetic or algebra will often nullify the creative or insight part of the investigation.

Ultimately there has to be some subjective judgement as to the quality of creative thought or insight displayed by the child in the piece of course work. This can never really be removed. It has to be accepted that, as in assessing essays in English or a piece of work in Art, that two different people will assess the merits of the course work differently.

This presents a major difficulty when a large number of pieces of course work from a large number of different schools need to be moderated. The scheme outlined above would be difficult to moderate partly due to the wide range of marks available to the assessor and partly due to a lack of concensus, through inexperience, as to what consituted good, bad and indifferent creative ability.

Ernest puts forward a second assessment procedure based on that of the Mathematical Association devised for assessing projects but applicable to investigations. It has two features that are different from the marking framework just considered. First the range of marks are much narrower. The advantage of this is that the full range is more likely to be used being more easy to define than the range of 30 marks considered previously. Second, there is an allowance, in the form of mark deductions, for help to be given to the candidate, always a difficult point to accommodate with course work.

Source of marks	Range of Marks	Deductions
1. Data collection (subject of analysis)	0 to 4	0 to 2
2. Skills acquisition (means of analysis)	0 to 4	0
3. Analysis (mathematical development)	0 to 4	0 to 2
4. Presentation (writing up the whole report)	0 to 4	0 to 2
5. Originality and difficulty of the project	0 to 4	0
	Total Marks	20

1. Data collection. This can literally mean the generation and listing of data, the results of the initial exploratory investigation of the situation.

2. Skills acquisition. For investigations this might be termed conjecture generation processes. We expect to see conjecture patterns arising from the tabulated data or results of the investigation so far.

3. Analysis. This section is relevant to attempts to confirm, falsify, justify or even prove the conjectures: also to generalise the initial situation to give new problems.

4. Presentation. The writing up of a clear account of the investigation, the communication of processes and findings is an essential feature of an investigation.

5. Originality and difficulty. Different starting situations may vary on difficulty. However more important from the point of view of assessment is the originality of the work of the student, the creativity of the work, how far it has developed.

(Ernest 1984)

With this assessment there is a drop in the creative or originality and difficulty marks from 25% to 20%, This is still a reasonable percentage when it is considered that the group of candidates being considered in the General Certificate of Secondary Education (GCSE) will be assessed within three separate levels of examination.

This assessment procedure is moving towards the type of assessment procedure used for Computer Studies projects in GCE examinations. In these assessment procedures there are a large number of categories each receiving a small number of marks. Such a system could be adopted for assessing investigational work.

Assessment Procedure

Category	Maximum Marks	
Difficulty of original project	5	
Use of appropriate diagrams	2	
Systematic approach to collecting data	4	
Completeness of collecting data	2	
Correctness of collecting data	2	15
Appropriate recording of data (tables, graphs etc)	2	
Systematic approach to tabulation	2	4
Spotting of patterns	5	5
Forming an hypothesis	3	
Test the hypothesis	2	
Conclusion (reforming hypothesis etc)	2	7
Degree of generalisation (words /alegraic, special cases etc)	5	
Justification or proof	5	10
Possible extensions of investigations	4	4
Documentation	5	5
		50

Such a marking scheme has certain advantages. By being fairly categorical about what marks are awarded for it is possible to have a degree of consistency between different markers and different scripts.

The subjective nature of the first marking framework considered is to a large extent removed by alloting a small range of marks to many categories. Markers are much more likely to use the full range of marks available when the range is 0 to 4 than when it is 0 to 30.

The disadvantage of such a marking scheme is that it has been devised with a particular type of investigation in mind. If the investigation is not of this type some of the categories may become redundant. This can be overcome by ignoring the redundant categories, giving a mark out of less than 50 and then scaling the mark to be out of 50.

The assessing of investigations carried out for course work presents many difficulties.If it can be accepted that the marking of these investigations will not be as exact as that of conventional written examinations then I would maintain that by using a working scheme similar to the last one considered assessment does not present an insuperable problem.

CHAPTER 7

CONCLUSIONS.

CONCLUSIONS

This dissertation has looked at investigational work in mathematics from the point of view of a mathematics teacher who wishes to use investigation work in his or her classroom. The approach has been essentially a utilitarian one. Attention has been paid to the practical aspects of identifying what constitutes investigational work in mathematics, of how investigational work fits into the mathematics curriculum, of how investigational work can be introduced in the classroom and finally of how investigational work can be assessed.

This dissertation can be looked upon as a guide to implementing investigational work in mathematics rather than providing theoretical reasons for undetaking this kind of work.

A prerequisite of undertaking investigational work in mathematics is to be clear as to exactly what constitutes investigational work. The first chapter shows that it is possible to define investigational work in such a way that its components can be easily identified. Investigational work is seen as a collection of strategies that are usually applied in a particular sequence. These strategies define a process of doing mathematics, of applying mathematics to particular situations rather than the straight acquisition of facts and skills.

The identification of these strategies or characteristics such as being systematical, tabulating results, starting with simple cases, spotting patterns, making and testing hypotheses and extending the original problem is crucial. It enables judgements to be made as to whether undertaking investigational work is worthwhile and if it is it enables ways of 'teaching' investigational work to be developed.

In the second chapter it was seen how investigational work fulfilled the vast majority of the aims and objectives of the mathematics curriculum. The need to do this should not be underestimated.

Teachers in general feel, in the past and to an even greater extent in the present, that they have been subject to curriculum initiatives that have not always been carefully thought through.

These initiatives have caused the teacher in the classroom to make many changes, to learn new styles of teaching and to acquire new skills. This is not lightly undertaken without a conviction that the new initiatives are useful and hence are going to become part and parcel of classroom practice.

The close relationship between current establishment thinking as to the structure of the mathematics curriculum as exemplified by Mathematics From 5 to 16 and the nature of investigational work in mathematics provides such reassurance.

Having decided what constitutes investigational work in mathematics and that it is deserving of a prominent place in the mathematics curriculum the question arises as to how to implement it in the classroom. The next three chapters showed how this could be achieved.

The third chapter showed that it was feasible to devise a course involving investigational work virtually at any point in the secondary school but most successfully at the earliest possible time. All strategies inherent in investigational work could be covered in a carefully planned graduated course. The nature of such a course was to offer an opportunity for children to put into practice the facts and skills that they had acquired.

The next chapter showed how investigational work could be also harnessed in a more directed way to enable children to acquire facts and skills in an effective way. Although only an example from algebra was considered the approach was applicable to a wide range of topics. It was thus demonstrated that the investigational approach could be applied both to the acquisition of facts and skills and to their application.

The role of the computer in investigational work was considered in the next chapter.

There have been some misgivings about how successfully computers could be used in the mathematics class and whether in fact they brought anything worthwhile to the mathematics curriculum. This chapter showed that they can provide a very powerful mathematical situation where investigational work can be undertaken.

Finally the crucial topic of assessment was considered. A stumbling-block to much innovative curriculum development in mathematics has been that it has not been possible to assess the work undertaken by children. Consequently it has not been felt justifiable to undertake work if it cannot be discovered whether the objectives of the work have been achieved.

The final chapter reviewed a number of schemes for assessing investigational work drawing on previous examples of traditional course work assessment. An assessment scheme suitable for investigational work was suggested.

It can thus be seen that investigational work has a vitally important role to play within the mathematics curriculum and that it is perfectly feasible to devise a course of action, to implement it and to assess it.

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COMPUTER SOFTWARE.

COMPUTER SOFTWARE

Arrow	Oxfordshire County Council
Circle	The First Thirty Programs S.M.I.L.E.
Jugs	The First Thirty Programs S.M.I.L.E.
Tilekit	Some More Lessons in Mathematics with a Micro computer A.T.M.
Polygon	Activities and Investigations for the Primary Classroom Capital Media
3D Maze	Microsmile The Next 17

APPENDICES.

A1 ORGANISING PROBLEMS

> **The Tournament**
>
> A tournament is being arranged. 22 teams have entered. The competition will be on a league basis, where every team will play all the other teams twice—once at home and once away. The organiser wants to know how many matches will be involved.

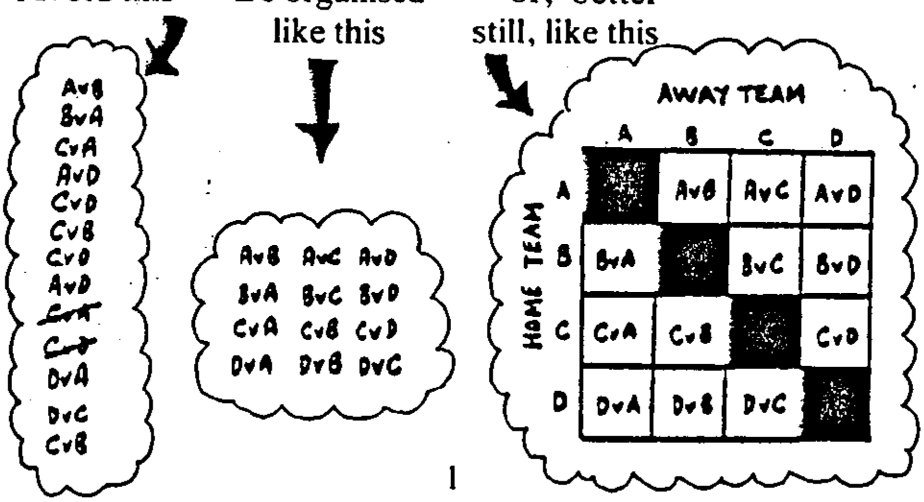
Often problems like this are too hard to solve immediately. If you get stuck with a problem, it often helps if you first **try some simple cases.**

So, suppose we have only 4 teams instead of 22.

Next, if you can **find a helpful diagram,** (table, chart or similar), it will help you to **organise the information systematically.**

For example,

Avoid this | Be organised like this | or, better still, like this



By now, you should be able to see that our 4 teams require 12 matches.

* How many matches will 6 teams require?
  How many matches will 7 teams require?
  Invent and do more questions like these.

* **Make a table** of your results. This is another key strategy . . .

Number of Teams	4	6	7			
Number of Matches	12					

* Try to **spot patterns** in your table.
  Write down what they are.
  (If you can't do this, check through your working, reorganise your information, or produce more examples.)

* Now try to **use your patterns** to solve the original problem with 22 teams.

* Try to **find a general rule** which tells you the number of matches needed for *any* number of teams. Write down your rule in words and, if you can, by a formula.

* **Check** that your rule always works.

* **Explain** why your rule works.

Now use the key strategies . . .

Try some simple cases

Find a helpful diagram

Organise systematically

Make a table

Spot patterns

Use the patterns

Find a general rule

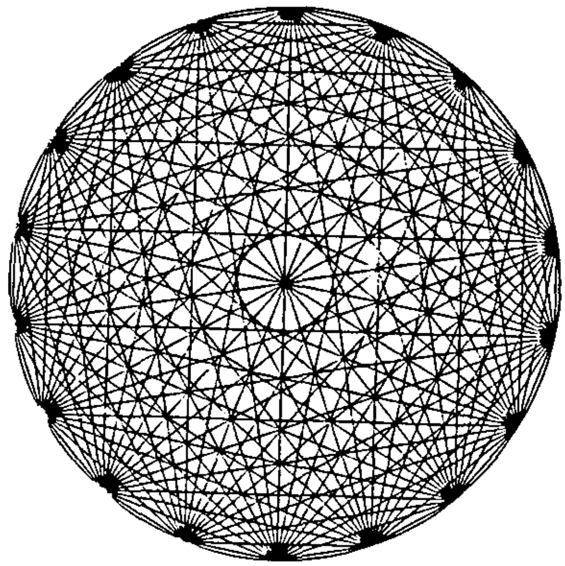
Explain why it works

Check regularly

to solve the problems on the next page . . .

11 (46)

Mystic Rose



This diagram has been made by connecting all 18 points on the circle to each other with straight lines. Every point is connected to every other point . . .

How many straight lines are there?

Money

Suppose you have the following 7 coins in your pocket . . .

1p, 2p, 5p, 10p, 20p, 50p, £1

How many different sums of money can you make?

4

A2 TRYING DIFFERENT APPROACHES

In the last lesson, you looked at the following problem . . .

Money

Suppose you have the following 7 coins in your pocket . . .

1p, 2p, 5p, 10p, 20p, 50p, £1.

How many different sums of money can you make?

We will now look at different ways of solving this problem and compare their advantages and disadvantages.

Method 1 "Method of 'differences' "

* Continue the table below for a few more terms:

Number of coins used	1	2	3	
Number of sums that can be made	1	3	7	

2 4

* Explain where the numbers 1, 3 and 7 in the table come from.
  (Do these numbers depend on the particular coins that are chosen?)
* Try to see a pattern in the differences between successive numbers in the table.
  How will this pattern continue?
  Are you sure? Try to explain it.
* Solve the problem with the 7 coins using this method.

1

Method 2 "Systematic Counting"

							sum made
						1	1p
					2		2p
					2	1	3p
				5			5p
				5		1	6p
				5	2		7p
				5	2	1	8p
			10				10p
			10			1	11p
			10		2		12p
			10		2	1	13p
			10	5			15p
			10	5		1	16p
			10	5	2		17p
			10	5	2	1	18p
		20					20p
		20				1	21p
£1	50	20	10	5		1	£1·86
£1	50	20	10	5	2		£1·87
£1	50	20	10	5	2	1	£1·88

14 (50)

2

This diagram shows a systematic attempt to list all the possible sums of money that can be made. (There is insufficient room to reproduce it all!)

* Try to see a quick way of counting the number of different sums.
* Solve the problem with 7 coins using this method.

Method 3 "Finding a Rule"

* Try to find a rule which links the number of coins with the number of sums of money directly.

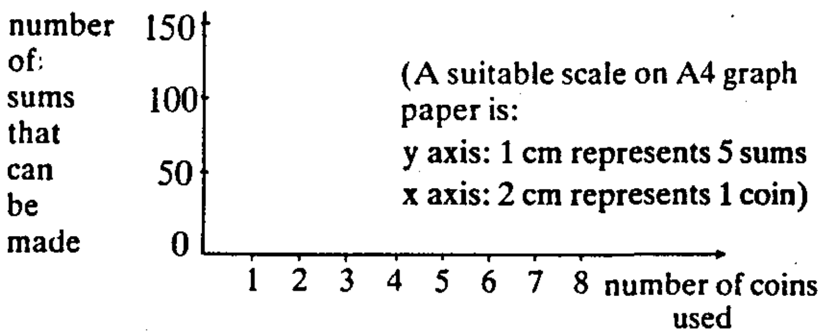
Number of coins used	Rule ?	Number of sums that can be made
1	$\xrightarrow{?}$	1
2	$\xrightarrow{?}$	3
3	$\xrightarrow{?}$	7
4	$\xrightarrow{?}$	. . .
7	$\xrightarrow{?}$	. . .

* Try to express your rule in words. Check that your rule always works.

  If you can, express your rule as a formula.
* Solve the problem for 7 coins using your rule.

Method 4 "Drawing a Graph"

* Draw a graph to show the relationship between the number of coins and the number of sums that can be made.



(A suitable scale on A4 graph paper is:
y axis: 1 cm represents 5 sums
x axis: 2 cm represents 1 coin)

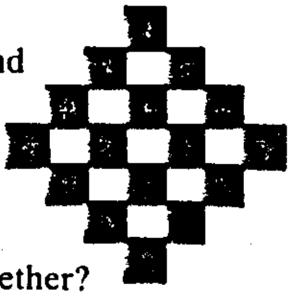
* Try to use your graph to solve the problem for 7 coins.

13 (50)

* **What are the advantages and disadvantages of these 4 methods?**
* **Can you invent any other methods?**

Try to solve the following problem using four different methods.
Which method do you prefer for this problem? Why?

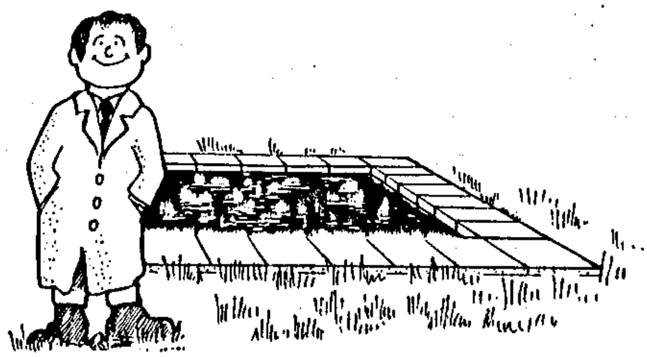
Town Hall Tiles
This pattern is made up of black and white tiles. It is 7 tiles across.



In the Town Hall there is a pattern like this which is 149 tiles across.
How many tiles will it contain altogether?

APPENDIX C.

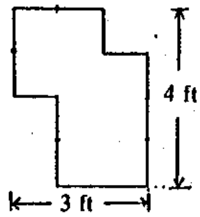
POND BORDERS



Joe works in a garden centre that sells square ponds and paving slabs to surround them. The paving slabs used are all 1 foot square.

The customers tell Joe the dimensions of the pond, and Joe has to work out how many paving slabs they need.

* How many slabs are needed in order to surround a pond 115 feet by 115 feet?
* Find a rule that Joe can use to work out the correct number of slabs for any square pond.
* Suppose the garden centre now decides to stock rectangular ponds. Try to find a rule now.
* Some customers want Joe to supply slabs to surround irregular ponds like the one below:—



(This pond needs 18 slabs. Check that you agree).

Try to find a rule for finding the number of slabs needed when you are given the overall dimensions (in this case 3 feet by 4 feet).

Explain why your rule works.

17 (72)

POND BORDERS . . . PUPIL'S CHECKLIST

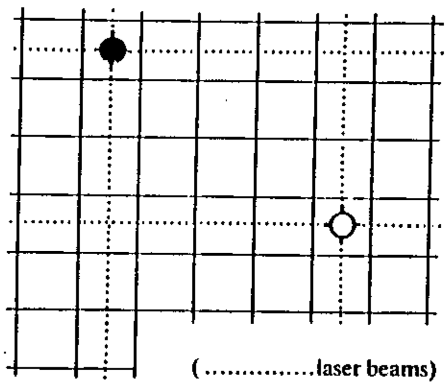
Try some simple cases	• Try finding the number of slabs needed for some small ponds.
Be systematic	• Don't just try ponds at random!
Make a table	• This should show the number of slabs needed for different ponds. (It may need to be a two-way table for rectangular and irregular ponds).
Spot patterns	• Write down any patterns you find in your table. (Can you explain *why* they occur?) • Use these patterns to extend the table. • Check that you were right.
Find a rule	• Either use your patterns, or look at a picture of the situation to find a rule that applies to any size pond.
Check your rule	• Test your rule on small and large ponds. • Explain *why* your rule always works.

18 (73)

APPENDIX D.

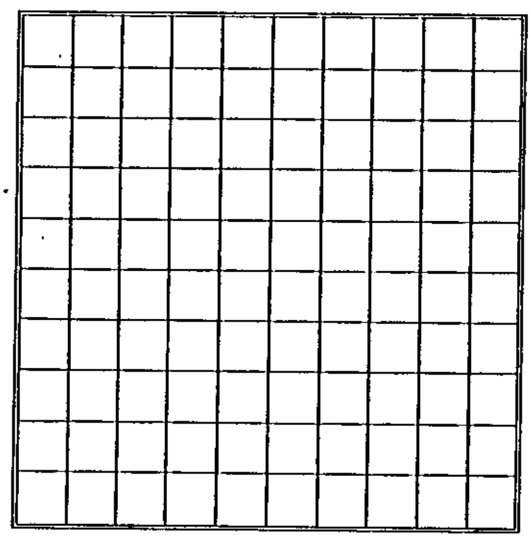
LASER-WARS

○ and ● represent two tanks armed with laser beams that annihilate anything which lies to the North, South, East or West of them. They move alternately. At each move a tank can move any distance North, South, East or West but cannot move across or into the path of the opponent's laser beam. A player loses when he is unable to move on his turn.



(..............laser beams)

- Play the game on the board below, using two objects to represent the tanks. Try to find a winning strategy, which works *wherever* the tanks are placed to start with.



- Now try to change the game in some way . . .

25 (96)

LASER-WARS . . . STRATEGY CHECKLIST

Try some simple cases	* Try playing on a smaller board. Try playing just the end of the game.
Be systematic	* Try starting from different positions, in a systematic way.
Spot patterns	* Find positions from which you can *always* win or from which you *must* lose. * Are there other positions from which you can always reach winning positions? * Look for symmetry in the game.
Find a rule	* Write down a description of "how to win this game". Explain why you are sure it works. * Extend your rule so that it applies to large boards.
Check your rule	* Try to beat someone who is playing according to your rule. * Can you convince them that it always works?
Change the game	* Limit the number of squares that can be moved by the vehicles. * Change the direction in which the lasers fire. * Change the playing area. (E.g.: use a triangular grid). * Try 3 players. (One vehicle each)