This item was submitted to Loughborough's Research Repository by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Stability analysis, singularities and topology of integrable systems

PLEASE CITE THE PUBLISHED VERSION

VERSION

AM (Accepted Manuscript)

PUBLISHER STATEMENT

This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

Bolsinov, Alexey V.. 2019. “Stability Analysis, Singularities and Topology of Integrable Systems”. figshare. https://hdl.handle.net/2134/23043.

Stability Analysis, Singularities and Topology of Integrable Systems

Alexey Bolsinov
Loughborough University
and
Moscow State University

GDIS 2016, Russia, Izhevsk
Geometry, Dynamics and Integrable Systems

Contributors and References

- Bolsinov, A. V.; Oshemkov, A. A. *Singularities of integrable Hamiltonian systems*. Topological methods in the theory of integrable systems, 1–67, Cambridge Sci. Publ., Cambridge, 2006.
- Bolsinov, A. V.; Oshemkov, A. A. *Bi-Hamiltonian structures and singularities of integrable systems*. Regul. Chaotic Dyn. 14 (2009), no. 4–5, 431–454.
- Bolsinov, A. V.; Borisov, A. V.; Mamaev, I. S. *Topology and stability of integrable systems*. Russian Math. Surveys 65 (2010), no. 2, 259–317.
- Bolsinov A. V.; Izosimov A. M. *Singularities of bi-Hamiltonian systems*, Comm. Math. Phys. 331 (2014) no. 2, 507–543.
- Izosimov A. M. *Stability in bihamiltonian systems and multidimensional rigid body*, J. Geom. and Phys. 62 (2012) 2414-2423.

Plan

Integrable Hamiltonian ODEs: Singularities and bifurcations

- Integrable systems, Lagrangian fibration and its singularities
- Why singularities are important?
- Non-degenerate singularities

Compatible Poisson brackets and bi-Hamiltonian systems

- Poisson structure: symplectic leaves, Casimir functions and singular set
- Compatible Poisson structures and the family of commuting Casimirs

From singularities of Poisson brackets to singularities of Lagrangian fibrations

- Zero order theory: Where are the singularities?
- First order theory: Linearisation of Poisson pencils and a criterion of non-degeneracy
- Example: stability of stationary rotations of an n-dim rigid body

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \ldots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \ldots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Momentum mapping $\Phi = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$

Singular Lagrangian fibration on M whose generic fibers L_a (i.e., connected components of $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$) are Liouville tori with quasi-periodic dynamics

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \ldots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Momentum mapping $\Phi = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$

Singular Lagrangian fibration on M whose generic fibers L_a (i.e., connected components of $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$) are Liouville tori with quasi-periodic dynamics

Set of critical points $S = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$

Integrable systems and their singularities

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \ldots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Momentum mapping $\Phi = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$

Singular Lagrangian fibration on M whose generic fibers L_a (i.e., connected components of $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$) are Liouville tori with quasi-periodic dynamics

Set of critical points $S = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$

General problem: Describe S and its properties

Integrable systems and their singularities

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \dots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Momentum mapping $\Phi = (f_1, \dots, f_n) : M^{2n} \to \mathbb{R}^n$

Singular Lagrangian fibration on M whose generic fibers L_a (i.e., connected components of $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$) are Liouville tori with quasi-periodic dynamics

Set of critical points $S = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$

General problem: Describe S and its properties

SINGULARITIES ARE IMPORTANT!

Integrable systems and their singularities

Symplectic manifold (M^{2n}, ω)

Hamiltonian system $\dot{x} = X_H(x) = \omega^{-1}(dH(x))$

Liouville integrability: there exist $f_1, \ldots, f_n : M^{2n} \to \mathbb{R}$ which:

- first integrals of $X_H(x)$;
- commute;
- independent almost everywhere.

Momentum mapping $\Phi = (f_1, \ldots, f_n) : M^{2n} \to \mathbb{R}^n$

Singular Lagrangian fibration on M whose generic fibers L_a (i.e., connected components of $\Phi^{-1}(a)$, $a \in \mathbb{R}^n$) are Liouville tori with quasi-periodic dynamics

Set of critical points $S = \{x \in M \mid \operatorname{rank}(df_1(x), \cdots, df_n(x)) < n\}$

General problem: Describe S and its properties

SINGULARITIES ARE IMPORTANT! Why?

Theorem (A. Borisov, I. Mamaev, AB (2010))

Two degrees of freedom

Let $\gamma(t)$ be a stable periodic solution. Then γ is singular, i.e., belongs to the singular set S (unless the system is resonant). Moreover, in the real analytic case $\gamma(t)$ is stable if and only if $\gamma(t)$ coincides with the common level of the integrals H and f:

$$\{\gamma(t), t \in \mathbb{R}\} = \{H(x) = H(x_0), f(x) = f(x_0)\}, \quad x_0 = \gamma(t_0)$$

Theorem

Let $P \in M^{2n}$ be an equilibrium point of a non-resonant integrable system. If P is stable then P is a critical point of Φ and, moreover, $\operatorname{rank} \Phi(P) = 0$, i.e., P is a common equilibrium point for all the integrals $f_1, \ldots, f_n$.

Theorem (A. Borisov, I. Mamaev, AB (2010))

Two degrees of freedom

Let $\gamma(t)$ be a stable periodic solution. Then γ is singular, i.e., belongs to the singular set S (unless the system is resonant). Moreover, in the real analytic case $\gamma(t)$ is stable if and only if $\gamma(t)$ coincides with the common level of the integrals H and f:

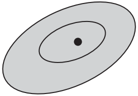
$$\{\gamma(t), t \in \mathbb{R}\} = \{H(x) = H(x_0), f(x) = f(x_0)\}, \quad x_0 = \gamma(t_0)$$

Theorem

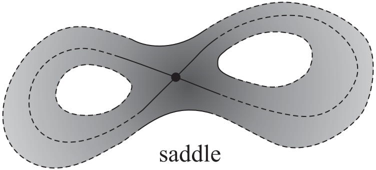
Let $P \in M^{2n}$ be an equilibrium point of a non-resonant integrable system. If P is stable then P is a critical point of Φ and, moreover, $\operatorname{rank} \Phi(P) = 0$, i.e., P is a common equilibrium point for all the integrals $f_1, \dots, f_n$.

Strange conclusion: for stability analysis of integrable systems, we do not need to consider the Hamiltonian equations, the only important thing is the momentum mapping and its singularities (or, equivalently, the corresponding singular Lagrangian fibration).

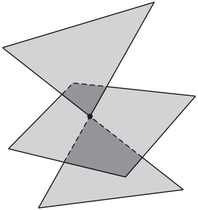
Non-degenerate singularities



center
(elliptic)

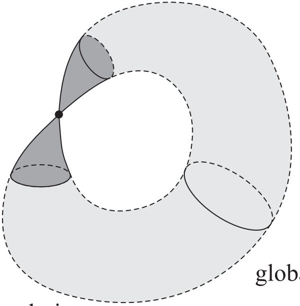


saddle
(hyperbolic)



local

$\simeq$



global

focus type singularity

Non-degenerate singularities

Non-degenerate singularities

Definition

Let $x_0 \in M^{2n}$ be a singular point of rank zero, i.e., $df_i(x_0) = 0$, $i = 1, \dots, n$. It is called non-degenerate, if the quadratic parts $d^2 f_1(x_0), \dots, d^2 f_n(x_0)$ are linearly independent and for a generic linear combination $f = \sum a_i f_i$, the roots of $\det(\omega^{-1} d^2 f(x_0) - t \cdot \mathrm{Id}) = 0$ are all distinct.

Definition

Let $x_0 \in M^{2n}$ be a singular point of rank zero, i.e., $df_i(x_0) = 0$, $i = 1, \ldots, n$. It is called non-degenerate, if the quadratic parts $d^2 f_1(x_0), \ldots, d^2 f_n(x_0)$ are linearly independent and for a generic linear combination $f = \sum a_i f_i$, the roots of $\det(\omega^{-1} d^2 f(x_0) - t \cdot \mathrm{Id}) = 0$ are all distinct.

Theorem (Vey, Eliasson: neighbourhood of a singular point)

The algebraic type of a non-degenerate singularity is its complete topological, smooth and even symplectic invariant.

Topological interpretation: every non-degenerate singularity can be represented as the product of the simplest singularities, i.e., elliptic, hyperbolic and focus.

Definition

Let $x_0 \in M^{2n}$ be a singular point of rank zero, i.e., $df_i(x_0) = 0$, $i = 1, \dots, n$. It is called non-degenerate, if the quadratic parts $d^2 f_1(x_0), \dots, d^2 f_n(x_0)$ are linearly independent and for a generic linear combination $f = \sum a_i f_i$, the roots of $\det(\omega^{-1} d^2 f(x_0) - t \cdot \mathrm{Id}) = 0$ are all distinct.

Theorem (Vey, Eliasson: neighbourhood of a singular point)

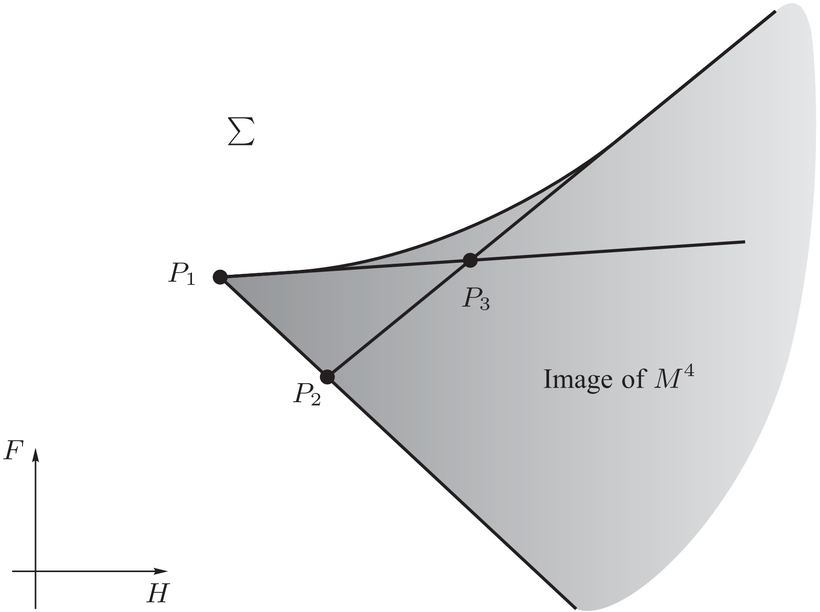
The algebraic type of a non-degenerate singularity is its complete topological, smooth and even symplectic invariant.

Topological interpretation: every non-degenerate singularity can be represented as the product of the simplest singularities, i.e., elliptic, hyperbolic and focus.

Theorem (Nguyen Tien Zung: neighbourhood of a singular fiber)

Every non-degenerate singularity (under some natural assumptions) can be topologically represented as an almost direct product type of elementary singularities.

Example: Neumann system on S^2



Definition

A Poisson bracket $\{\ ,\ \}$ on M is a bilinear operation on $C^\infty(M)$ defined by:

$$f, g \mapsto \{f, g\} = \sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

and satisfying the Jacobi identity. Here $A = (A^{ij})$ is a skew-symmetric $(2, 0)$-tensor field called a Poisson structure.

Important!

We do not assume that A is non-degenerate (in the sense $\det A \neq 0$). We do not assume that A is or constant rank either.

- Casimir functions $f \in C^\infty(M)$ such that

$$\{f, g\}_A = 0 \quad \text{for any } g \in C^\infty(M).$$

- M is foliated into symplectic leaves
- Singular set of A:

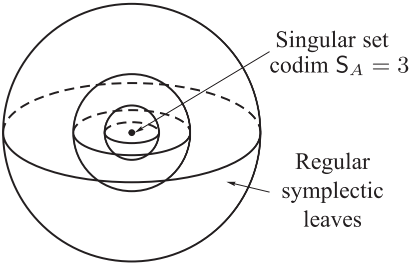
$$S_A = \{x \in M \mid \operatorname{rank} A(x) < \operatorname{rank} A = \max_{x \in M} \operatorname{rank} A(x)\}.$$

$so(3)$–bracket: $\quad A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$

Casimir function: $\quad F = x^2 + y^2 + z^2$

Symplectic leaves are spheres centered at the origin + one singular leaf $\{0\}$

Singular set is $S_A = \{\text{rank}\, A < 2\} = \{0\}, \quad \text{codim}\, S_A = 3$



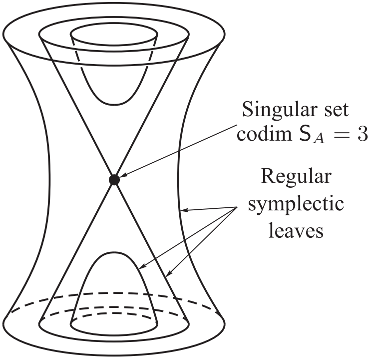
Example 2

$sl(2,\mathbb{R})$–bracket: $\quad A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix}$

Casimir function: $\quad F = x^2 + yz$

Symplectic leaves: hyperboloids, two halves of the cone + one singular leaf $\{0\}$

Singular set is $S_A = \{\text{rank}\, A < 2\} = \{0\}$, $\quad \text{codim}\, S_A = 3$



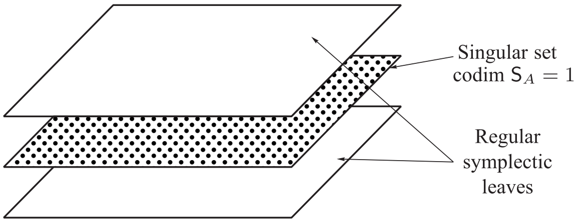
Example 3

Heisenberg–Lie bracket: $A = \begin{pmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Casimir function: $F = z$

Symplectic leaves: planes $\{z = \mathrm{const} \neq 0\}$ $+$ points on $\{z = 0\}$

Singular set is $S_A = \{\mathrm{rank}\, A < 2\} = \{z = 0\}$, $\quad$ codim $S_A = 1$



Definition

Two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

Let M be a manifold endowed with a linear family $\Pi = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \Pi$ are degenerate so that each of them possesses non-trivial Casimir functions.

Definition

Two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

Let M be a manifold endowed with a linear family $\Pi = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \Pi$ are degenerate so that each of them possesses non-trivial Casimir functions.

Proposition

Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \Pi$, then
1) the family of functions

$$\mathcal{F}_\Pi = \{\text{all Casimir functions of all generic } A_\mu\}$$

consists of its first integrals;
2) these integrals commute (w.r.t. every $A_\lambda \in \Pi$)

Definition

Two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

Let M be a manifold endowed with a linear family $\Pi = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \Pi$ are degenerate so that each of them possesses non-trivial Casimir functions.

Proposition

Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \Pi$, then
1) the family of functions

$$\mathcal{F}_\Pi = \{\text{all Casimir functions of all generic } A_\mu\}$$

consists of its first integrals;
2) these integrals commute (w.r.t. every $A_\lambda \in \Pi$)

Natural questions to discuss: PROPERTIES of $\mathcal{F}_\Pi$

Definition

Two Poisson structures A and B are compatible if $\mu A + \lambda B$ is again a Poisson structure.

Let M be a manifold endowed with a linear family $\Pi = \{A_\lambda = A + \lambda B\}$ of compatible Poisson brackets. Assume that all $A_\lambda \in \Pi$ are degenerate so that each of them possesses non-trivial Casimir functions.

Proposition

Let $\dot{x} = v(x)$ be a dynamical system which is Hamiltonian w.r.t. each generic $A_\mu \in \Pi$, then
1) the family of functions

$$\mathcal{F}_\Pi = \{\text{all Casimir functions of all generic } A_\mu\}$$

consists of its first integrals;
2) these integrals commute (w.r.t. every $A_\lambda \in \Pi$)

Natural questions to discuss: PROPERTIES of $\mathcal{F}_\Pi$

- Completeness
- Set of critical points
- Equilibrium points
- Non-degeneracy conditions, types
- Stability

General scheme

$\boxed{\text{Family of compatible Poisson brackets}}$

$\downarrow$

$\boxed{\text{Commuting integrals}}$

$\downarrow$

$\boxed{\text{Lagrangian fibration}}$

$\downarrow$

$\boxed{\text{Its singularities}}$

$\downarrow$

$\boxed{\text{Dynamical properties and stability analysis}}$

Zero-order theory

Zero-order theory

Theorem

$\mathcal{F}_\Pi$ is complete iff Π is of Kronecker type, i.e. at a generic point $x \in M$, the rank of $A(x) + \lambda B(x)$ is the same for all $\lambda \in \bar{\mathbb{C}}$.

Theorem
$\mathcal{F}_\Pi$ is complete iff Π is of Kronecker type, i.e. at a generic point $x \in M$, the rank of $A(x) + \lambda B(x)$ is the same for all $\lambda \in \bar{\mathbb{C}}$.

Let $\Pi = \{A + \lambda B\}$ be of Kronecker type, so that $\mathcal{F}_\Pi$ is complete and defines the structure of a Lagrabgian fibration on M.
Consider the singular set for this fibration

$$S_\Pi = \{x \in M \mid \dim d\mathcal{F}_\Pi \text{ is not maximal}\}$$

and singular sets for each Poisson structure

$$S_\lambda = \{x \in M \mid \operatorname{rank} A_\lambda(x) < \operatorname{rank} \Pi\}.$$

Theorem
$\mathcal{F}_\Pi$ is complete iff Π is of Kronecker type, i.e. at a generic point $x \in M$, the rank of $A(x) + \lambda B(x)$ is the same for all $\lambda \in \bar{\mathbb{C}}$.

Let $\Pi = \{A + \lambda B\}$ be of Kronecker type, so that $\mathcal{F}_\Pi$ is complete and defines the structure of a Lagrabgian fibration on M.
Consider the singular set for this fibration

$$S_\Pi = \{x \in M \mid \text{ dim } d\mathcal{F}_\Pi \text{ is not maximal}\}$$

and singular sets for each Poisson structure

$$S_\lambda = \{x \in M \mid \text{ rank } A_\lambda(x) < \text{rank } \Pi\}.$$

Theorem

$$S_\Pi = \bigcup_{\lambda \in \bar{\mathbb{C}}} S_\lambda$$

Theorem

$\mathcal{F}_\Pi$ is complete iff Π is of Kronecker type, i.e. at a generic point $x \in M$, the rank of $A(x) + \lambda B(x)$ is the same for all $\lambda \in \bar{\mathbb{C}}$.

Let $\Pi = \{A + \lambda B\}$ be of Kronecker type, so that $\mathcal{F}_\Pi$ is complete and defines the structure of a Lagrabgian fibration on M.
Consider the singular set for this fibration

$$S_\Pi = \{x \in M \mid \dim d\mathcal{F}_\Pi \text{ is not maximal}\}$$

and singular sets for each Poisson structure

$$S_\lambda = \{x \in M \mid \operatorname{rank} A_\lambda(x) < \operatorname{rank} \Pi\}.$$

Theorem

$$S_\Pi = \bigcup_{\lambda \in \bar{\mathbb{C}}} S_\lambda$$

Theorem

$x \in M$ is a common equilibrium point for $\mathcal{F}_\Pi$ if and only if the kernels of all generic brackets at this point coincide: $\operatorname{Ker} A_\lambda(x) = \operatorname{Ker} A_\mu(x)$, for all $A_\lambda(x)$ and $A_\mu(x)$ generic.

Linearisation of a Poisson structure

Linearisation of a Poisson structure

According to the splitting theorem (A.Weinstein), locally each Poisson structure A splits into direct product of a non-degenerate Poisson structure $A_{\rm sympl}$ and the transversal structure $A_{\rm transv}$ which vanishes at the given point:

$$A = A_{\rm sympl} \times A_{\rm transv}$$

The transversal Poisson structure $A_{\rm transv}$ is well defined and we can consider its linearisation just by taking the linear terms in the Taylor expansion

$$A_{\rm transv}(x) = \sum c_{ij}^k x_k + \dots$$

According to the splitting theorem (A.Weinstein), locally each Poisson structure A splits into direct product of a non-degenerate Poisson structure A_{sympl} and the transversal structure A_{transv} which vanishes at the given point:

$$A = A_{\mathrm{sympl}} \times A_{\mathrm{transv}}$$

The transversal Poisson structure A_{transv} is well defined and we can consider its linearisation just by taking the linear terms in the Taylor expansion

$$A_{\mathrm{transv}}(x) = \sum c_{ij}^{k} x_k + \dots$$

Definition

From the algebraic viewpoint, the linearisation of A at a point $x \in M$ is a Lie algebra $\mathfrak{g}_A$ defined on $\mathrm{Ker}\, A(x)$ as follows. Let $\xi, \eta \in \mathrm{Ker}\, A(x)$ and f, g be smooth functions such that $df(x) = \xi$, $dg(x) = \eta$. Then, by definition,

$$[\xi, \eta] = d\{f, g\}(x) \in \mathrm{Ker}\, A(x)$$

Linearisation of a Poisson pencil

Linearisation of a Poisson pencil

$\Pi = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$. Let us take $x \in M$, fix $\lambda \neq \infty$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$ we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, the lineraisation of A_λ at the point x,
- the restriction of B onto $\operatorname{Ker} A_\lambda$.

Linearisation of a Poisson pencil

$\Pi = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$. Let us take $x \in M$, fix $\lambda \neq \infty$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$ we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, the lineraisation of A_λ at the point x,
- the restriction of B onto $\operatorname{Ker} A_\lambda$.

We can think of them as two Poisson structures on $\mathfrak{g}_\lambda^*$:

- the first on is linear, i.e., the standard Lie-Poisson structure related to $\mathfrak{g}_\lambda$,
- the second one is constant $B|_{\mathfrak{g}_\lambda}$.

$\Pi = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$.
Let us take $x \in M$, fix $\lambda \neq \infty$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$ we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, the lineraisation of A_λ at the point x,
- the restriction of B onto $\operatorname{Ker} A_\lambda$.

We can think of them as two Poisson structures on $\mathfrak{g}_\lambda^*$:

- the first on is linear, i.e., the standard Lie-Poisson structure related to $\mathfrak{g}_\lambda$,
- the second one is constant $B|_{\mathfrak{g}_\lambda}$.

Proposition

These two Poisson structures are compatible, i.e. generate, a Poisson pencil $\mathrm{d}_\lambda \Pi(x)$.

Linearisation of a Poisson pencil

$\Pi = \{A_\lambda = A + \lambda B\}$ is a pencil of compatible Poisson brackets and $x \in M$. Let us take $x \in M$, fix $\lambda \neq \infty$ and consider the kernel $\operatorname{Ker} A_\lambda(x)$.

On $\operatorname{Ker} A_\lambda$ we can introduce two natural structures:

- the Lie algebra $\mathfrak{g}_\lambda = \mathfrak{g}_{A_\lambda}$, the lineraisation of A_λ at the point x,
- the restriction of B onto $\operatorname{Ker} A_\lambda$.

We can think of them as two Poisson structures on $\mathfrak{g}_\lambda^*$:

- the first on is linear, i.e., the standard Lie-Poisson structure related to $\mathfrak{g}_\lambda$,
- the second one is constant $B|_{\mathfrak{g}_\lambda}$.

Proposition

These two Poisson structures are compatible, i.e. generate, a Poisson pencil $\mathrm{d}_\lambda \Pi(x)$.

Definition

This Poisson pencil $\mathrm{d}_\lambda \Pi(x)$ is called the λ-linearisation of the pencil Π at $x \in M$.

Linear pencils

Consider two compatible Poisson brackets on a vector space V:

$$\text{linear } A + \text{constant } B.$$

What are "compatibility conditions" for this kind of brackets?

Linear pencils

Consider two compatible Poisson brackets on a vector space V:

linear A + constant B.

What are "compatibility conditions" for this kind of brackets?

Standard situation is "shift of argument" (Manakov, Mischenko, Fomenko) :

The brackets $\{f, g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$, $\{f, g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$ are compatible for each $a = (a_i) \in V \simeq \mathfrak{g}^*$.

Linear pencils

Consider two compatible Poisson brackets on a vector space V:

linear A + constant B.

What are "compatibility conditions" for this kind of brackets?

Standard situation is "shift of argument" (Manakov, Mischenko, Fomenko) :

The brackets $\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$, $\{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ are compatible for each $a = (a_i) \in V \simeq \mathfrak{g}^*$.

Situation can be different:

For $\{f,g\}_A(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$ there may exist constant compatible brackets

$$\{f,g\}_B(x) = \sum B_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

which are not of the above type. The compatibility condition can be written as

$$B([\xi,\eta],\zeta) + B([\eta,\zeta],\xi) + B([\zeta,\xi],\eta) = 0.$$

Non-degenerate linear pencils

Thus, a linear pencil is defined as a pair $(\mathfrak{g}, B)$ where

- $\mathfrak{g}$ is a finite-dimensional Lie algebra
- $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric form compatible with $\mathfrak{g}$ (2-cocycle).

Notation: $\Pi^{\mathfrak{g}, B}$

Thus, a linear pencil is defined as a pair $(\mathfrak{g}, B)$ where

- $\mathfrak{g}$ is a finite-dimensional Lie algebra
- $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric form compatible with $\mathfrak{g}$ (2-cocycle).

Notation: $\Pi^{\mathfrak{g},B}$

For this special kind of Poisson pencils $\Pi = \Pi^{\mathfrak{g},B}$ we can construct the family of commuting functions $\mathcal{F}_\Pi$ and ask the question about the structures of singular points. We will say that Π is complete, if $\mathcal{F}_\Pi$ is complete.

Non-degenerate linear pencils

Thus, a linear pencil is defined as a pair $(\mathfrak{g}, B)$ where

- $\mathfrak{g}$ is a finite-dimensional Lie algebra
- $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric form compatible with $\mathfrak{g}$ (2-cocycle).

Notation: $\Pi^{\mathfrak{g},B}$

For this special kind of Poisson pencils $\Pi = \Pi^{\mathfrak{g},B}$ we can construct the family of commuting functions $\mathcal{F}_\Pi$ and ask the question about the structures of singular points. We will say that Π is complete, if $\mathcal{F}_\Pi$ is complete.

Definition

We say that a complete linear pencil $\Pi^{\mathfrak{g},B}$ is *non-degenerate*, if $0 \in \mathfrak{g}^*$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$.

Thus, a linear pencil is defined as a pair $(\mathfrak{g}, B)$ where

- $\mathfrak{g}$ is a finite-dimensional Lie algebra
- $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric form compatible with $\mathfrak{g}$ (2-cocycle).

Notation: $\Pi^{\mathfrak{g},B}$

For this special kind of Poisson pencils $\Pi = \Pi^{\mathfrak{g},B}$ we can construct the family of commuting functions $\mathcal{F}_\Pi$ and ask the question about the structures of singular points. We will say that Π is complete, if $\mathcal{F}_\Pi$ is complete.

Definition

We say that a complete linear pencil $\Pi^{\mathfrak{g},B}$ is *non-degenerate*, if $0 \in \mathfrak{g}^*$ is a non-degenerate singular point for the family $\mathcal{F}_\Pi$.

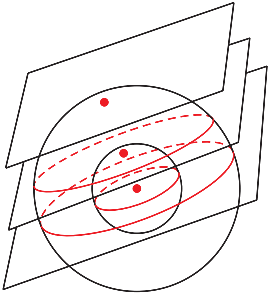
Problem. Describe all pairs $(\mathfrak{g}, B)$ such that the pencil $\Pi^{\mathfrak{g},B}$ is non-degenerate.

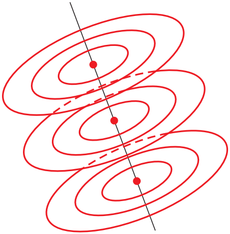
Example

If $\mathfrak{g} \simeq so(3)$ and B is arbitrary, then $\Pi^{\mathfrak{g},B}$ is non-degenerate.

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$$

Casimir functions: $\quad F_1 = x^2 + y^2 + z^2, \quad F_2 = ax + by + cz$





Examples: semisimple case $sl(2,\mathbb{R})$

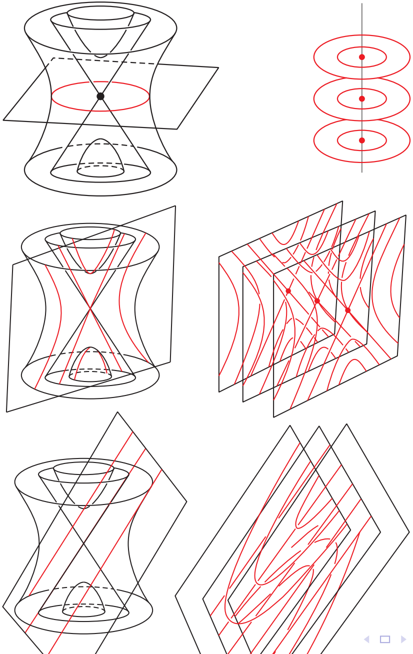
$\mathfrak{g} \simeq sl(2,\mathbb{R})$ and the constant bracket B is defined by an element $\xi \in sl(2,\mathbb{R}) \simeq sl(2,\mathbb{R})^*$:

$$\xi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & y & -z \\ -y & 0 & 2x \\ z & -2x & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & c & -b \\ -c & 0 & 2a \\ b & -2a & 0 \end{pmatrix}$$

Casimir functions: $\quad F_1 = x^2 + yz, \quad F_2 = 2ax + by + cz$

Examples: semisimple case $sl(2,\mathbb{R})$



Classification of non-degenerate pencils

Theorem (A. Izosimov)

Theorem (A. Izosimov)

Complex case. *A linear pencil* $\Pi^{\mathfrak{g},B}$ *is non-degenerate iff*

$$\mathfrak{g} \simeq \left(\bigoplus so(3,\mathbb{C})\right) \oplus \left(\left(\bigoplus \mathfrak{D}\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{C}\right)$$

where $\mathfrak{D}$ *is the diamond Lie algebra,* $\mathfrak{h}_0$ *is a commutative ideal which belongs to the center of* $(\bigoplus \mathfrak{D})$*, and* $\operatorname{Ker} B$ *is a Cartan subalgebra of* $\mathfrak{g}$*.*

Theorem (A. Izosimov)

Complex case. *A linear pencil $\Pi^{\mathfrak{g},B}$ is non-degenerate iff*

$$\mathfrak{g} \simeq \left(\bigoplus so(3,\mathbb{C})\right) \oplus \left(\left(\bigoplus \mathfrak{D}\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{C}\right)$$

where $\mathfrak{D}$ is the diamond Lie algebra, $\mathfrak{h}_0$ is a commutative ideal which belongs to the center of $(\bigoplus \mathfrak{D})$, and Ker B *is a Cartan subalgebra of $\mathfrak{g}$.*

Real case. *A linear pencil $\Pi^{\mathfrak{g},B}$ is non-degenerate iff*

$$\mathfrak{g} \simeq \left(\bigoplus so(3,\mathbb{R})\right) \oplus \left(\bigoplus sl(2,\mathbb{R})\right) \oplus \left(\bigoplus so(3,\mathbb{C})\right) \oplus$$
$$\left(\left(\left(\bigoplus \mathfrak{g}_{ell}\right) \oplus \left(\bigoplus \mathfrak{g}_{hyper}\right) \oplus \left(\bigoplus \mathfrak{g}_{foc}\right)\right)/\mathfrak{h}_0\right) \oplus \left(\bigoplus \mathbb{R}\right)$$

- *$\mathfrak{g}_{ell}$ and $\mathfrak{g}_{hyp}$ are the non-trivial central extensions of $e(2)$ and $e(1,1)$ (equivalently, they are real forms of $\mathfrak{D}$),*
- *$\mathfrak{e}_{foc} = \mathfrak{D}$ treated as real Lie algebra,*
- *$\mathfrak{h}_0$ is a commutative ideal which belongs to the center.*

and Ker B *is a Cartan subalgebra of $\mathfrak{g}$. The type of the singularity is naturally defined by the "number" of elliptic, hyperbolic and focus components in the above decomposition.*

Let $\Pi = \{A + \lambda B\}$ be an arbitrary pencil of compatible Poisson brackets. We consider the family of commuting Casimirs $\mathcal{F}_\Pi$ and a singular point $x \in S_\Pi$. This means that at this point for some $\lambda_i \in \bar{\mathbb{C}}$, the rank of $A(x) + \lambda_i B(x)$ drops. The set of such λ_i's is called the spectrum of the pencil Π at $x \in M$. Notation: $\Lambda(x)$.

For each $\lambda_i \in \Lambda(x)$ we can consider the λ_i-linearisation.

General criterion

Let $\Pi = \{A + \lambda B\}$ be an arbitrary pencil of compatible Poisson brackets. We consider the family of commuting Casimirs $\mathcal{F}_\Pi$ and a singular point $x \in S_\Pi$. This means that at this point for some $\lambda_i \in \bar{\mathbb{C}}$, the rank of $A(x) + \lambda_i B(x)$ drops. The set of such λ_i's is called the spectrum of the pencil Π at $x \in M$. Notation: $\Lambda(x)$.

For each $\lambda_i \in \Lambda(x)$ we can consider the λ_i-linearisation.

Is x non-degenerate?

Let $\Pi = \{A + \lambda B\}$ be an arbitrary pencil of compatible Poisson brackets. We consider the family of commuting Casimirs $\mathcal{F}_\Pi$ and a singular point $x \in S_\Pi$. This means that at this point for some $\lambda_i \in \bar{\mathbb{C}}$, the rank of $A(x) + \lambda_i B(x)$ drops. The set of such λ_i's is called the spectrum of the pencil Π at $x \in M$. Notation: $\Lambda(x)$.

For each $\lambda_i \in \Lambda(x)$ we can consider the λ_i-linearisation.

Is x non-degenerate?

Theorem

The singular point x is non-degenerate if and only if for every $\lambda_i \in \Lambda(x)$,

1. *the λ_i-linearisation of Π at x is non-degenerate;*
2. *the corank of the λ_i-linearisation equals to* corank Π.

The Williamson type of x is just the "sum" of the types of all λ_i-linearisations.

Stability of stationary rotations for n-dimensional rigid body

Consider a stationary rotation of an $2n$-dim rigid body.

Consider a stationary rotation of an $2n$-dim rigid body.
Let $\mathbb{R}^{2n} = \bigoplus_{i=1}^{2} V_i$ be the decomposition of $\mathbb{R}^{2n}$ into 2-dimensional subspaces spanned by the eigenvectors of the inertia tensor J, ω_i be the angular velocity of rotation in the plane V_i and J_i and J_i' the eigenvalues of J (principal moments of inertia) that correspond to V_i. Consider the function

$$y = f_i(x) = \frac{(x - J_i^2)(x - {J_i'}^2)}{\omega_i^2 (J_i + J_i')^2},$$

which defines a parabola on the (x, y)-plane. The collection of all these parabolas is called the *parabolic diagram* $\mathcal{P}$ of the stationary rotation.

Consider a stationary rotation of an $2n$-dim rigid body.
Let $\mathbb{R}^{2n} = \bigoplus_{i=1}^{2} V_i$ be the decomposition of $\mathbb{R}^{2n}$ into 2-dimensional subspaces spanned by the eigenvectors of the inertia tensor J, ω_i be the angular velocity of rotation in the plane V_i and J_i and J'_i the eigenvalues of J (principal moments of inertia) that correspond to V_i. Consider the function

$$y = f_i(x) = \frac{(x - J_i^2)(x - {J'_i}^2)}{\omega_i^2 (J_i + J'_i)^2},$$

which defines a parabola on the (x, y)-plane. The collection of all these parabolas is called the *parabolic diagram* $\mathcal{P}$ of the stationary rotation.

Theorem (A.Izosimov)

1. *The equilibrium point (stationary rotation) $X \in \mathrm{so}(n)$ is non-degenerate if and only if the parabolic diagram $\mathcal{P}$ is generic.*
2. *If $\mathcal{P}$ is generic, all intersections are real and located in the upper half plane, then the equilibrium is stable.*
3. *If there is either a complex intersection or an intersection point in the lower half plane, then the equilibrium point is unstable.*

Many thanks for your attention

Below are some additional slides which might be helpful

What is the diamond Lie algebra $\mathfrak{D}$?

$\mathfrak{D}$ is a four dimensional Lie algebra generated by e, f, t, h with the following relations

$$[t, e] = f, \quad [t, f] = -e \quad \text{and} \quad [e, f] = h, \quad [h, \mathfrak{D}] = 0. \tag{1}$$

In other words, $\mathfrak{D}$ (as a complex Lie algebra) is the non-trivial central extension of $e(2, \mathbb{C})$.
Matrix representation:

$$\alpha e + \beta f + \theta t + \gamma h \quad \mapsto \quad \begin{pmatrix} 0 & \alpha & \beta & 2\gamma \\ 0 & 0 & -\theta & \beta \\ 0 & \theta & 0 & -\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Casimir functions: $\quad F_1 = f^2 + e^2 + 2th, \quad F_2 = h.$

The complex diamond Lie algebra $\mathfrak{D}$ has 2 different real forms

- $\mathfrak{g}_{ell}$ defined by (1) and
- $\mathfrak{g}_{hyp}$ defined by $[t, e] = e$, $[t, f] = -f$, and $[e, f] = h$.

Fundamental example: Lie-Poisson structure

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{g}^*$ its dual space.
The Lie-Poisson bracket on $\mathfrak{g}^*$ is defined by:

$$\{f,g\}(x) = \langle x, [df(x), dg(x)] \rangle = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where $f,g \in C^\infty(\mathfrak{g}^*)$, $df(x), dg(x) \in \mathfrak{g} \simeq (\mathfrak{g}^*)^*$ and $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

In Cartesian coordinates, the Poisson structure A is linear:

$$A_{ij} = \sum_k c_{ij}^k x_k$$

and this property is characteristic for all Lie-Poisson brackets.

$$\begin{aligned} \text{corank}\, A &= \operatorname{ind} \mathfrak{g} \\ \text{Symplectic leaves} &= \text{coadjoint orbits} \\ \text{Casimir functions} &= \text{coadjoint invariants} \end{aligned}$$

Theorem

Let A and B be two skew-symmetric bilinear forms. Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:

$$A \mapsto \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \qquad B \mapsto \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}$$

where the pairs of the corresponding blocks A_i and B_i can be of the following three types (see next slide)

Types of blocks

Types of blocks

	A	B
Jordan block ($\lambda \in \mathbb{R}$)	$\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$

Types of blocks

	A	B
Jordan block ($\lambda \in \mathbb{R}$)	$\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$
Jordan block ($\lambda = \infty$)	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$

	A	B
Jordan block ($\lambda \in \mathbb{R}$)	$\begin{pmatrix} & J(\lambda) \\ -J^\top(\lambda) & \end{pmatrix}$	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$
Jordan block ($\lambda = \infty$)	$\begin{pmatrix} & Id \\ -Id & \end{pmatrix}$	$\begin{pmatrix} & J(0) \\ -J^\top(0) & \end{pmatrix}$
Kronecker block	$\left(\begin{array}{c\|c} & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & \end{array}\right)$	$\left(\begin{array}{c\|c} & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & \end{array}\right)$

On the dual space $\mathfrak{g}^*$ of an arbitrary Lie algebra $\mathfrak{g}$ there are two natural compatible Poisson brackets:

$$\{f,g\}(x) = \sum c_{ij}^k x_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} \quad \text{and} \quad \{f,g\}_a(x) = \sum c_{ij}^k a_k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

where $a = (a_i) \in \mathfrak{g}^*$ is a fixed element.

Proposition

For each $\lambda \in \mathbb{R}$, the bracket $\{\ ,\ \}_\lambda = \{\ ,\ \} + \lambda\{\ ,\ \}_a$ is isomorphic to $\{\ ,\ \}$ (by means of translation $x \to x + \lambda a$). In particular,

- the Casimir functions of $\{\ ,\ \}_\lambda$ are of the form $f(x+\lambda a)$, where f is a coadjoint invariant of $\mathfrak{g}$;
- the singular set of $\{\ ,\ \}_\lambda$ is $\mathrm{Sing} + \lambda a$, where Sing is the set of singular coadjoint orbits of $\mathfrak{g}$;
- the kernel of $\{\ ,\ \}_\lambda$ at the point $x \in \mathfrak{g}^*$ is the ad^*-stationary subalgebra of $x + \lambda a$, i.e., $\mathrm{ann}(x+\lambda a) = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi(x+\lambda a) = 0\}$.

Mischenko-Fomenko: The family of functions $\mathcal{F}_a = \{f(x+\lambda a) \mid \lambda \in \mathbb{R},\ f \text{ is a Casimir of } \mathfrak{g}\}$ is in bi-involution.

What about non-singular fibers?

Simplest case:

What about non-singular fibers?

Simplest case: $\dim M = 2$, $f_1 = H$ and
$L_a = \{H(x) = a\}$ is compact and regular. Then topologically, L_a is just a circle.

What about non-singular fibers?

Simplest case: dim $M = 2$, $f_1 = H$ and
$L_a = \{H(x) = a\}$ is compact and regular. Then topologically, L_a is just a circle. Moreover, its neighborhood $U(L_a)$ is fiberwise symplectomorfic to:

$$M^2_{\mathrm{reg}} = D^1 \times S^1, \quad \text{where } D^1 \text{ is an interval}, \quad \omega = ds \wedge d\phi$$

and the Hamiltonian of the system is $F = F(s)$, i.e., fibers are circles $S^1 \times \{s\}$, $s \in D^1$.

What about non-singular fibers?

Simplest case: $\dim M = 2$, $f_1 = H$ and
$L_a = \{H(x) = a\}$ is compact and regular. Then topologically, L_a is just a circle. Moreover, its neighborhood $U(L_a)$ is fiberwise symplectomorfic to:

$$M^2_{\mathrm{reg}} = D^1 \times S^1, \quad \text{where } D^1 \text{ is an interval}, \quad \omega = ds \wedge d\phi$$

and the Hamiltonian of the system is $F = F(s)$, i.e., fibers are circles $S^1 \times \{s\}$, $s \in D^1$.

General case: $\dim M = 2n$.

Theorem (Arnold–Liouville)

1) Let L_a be regular, compact and connected. Then L_a is an n dimensional torus and the dynamics of X_H on L_a is quasi-periodic.
2) There exists a neighborhood $U(L_a)$ which is fiberwise symplectomorphic to the canonical model

$$M^{2n}_{\mathrm{reg}} = \underbrace{M^2_{\mathrm{reg}} \times \cdots \times M^2_{\mathrm{reg}}}_{n \text{ times}}$$