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Quenched decay of correlations for slowly mixing systems

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QUENCHED DECAY OF CORRELATIONS FOR SLOWLY MIXING SYSTEMS

WAEL BAHSOUN[†], CHRISTOPHER BOSE[*], AND MARKS RUZIBOEV[‡]

ABSTRACT. We study random towers that are suitable to analyse the statistics of slowly mixing random systems. We obtain upper bounds on the rate of *quenched* correlation decay in a general setting. We apply our results to the random family of Liverani-Saussol-Vaienti maps with parameters in $[\alpha_0, \alpha_1] \subset (0, 1)$ chosen independently with respect to a distribution ν on $[\alpha_0, \alpha_1]$ and show that the *quenched decay of correlation* is governed by the fastest mixing map in the family. In particular, we prove that for every $\delta > 0$, for almost every $\omega \in [\alpha_0, \alpha_1]^{\mathbb{Z}}$, the upper bound $n^{1-\frac{1}{\alpha_0}+\delta}$ holds on the rate of decay of correlation for Hölder observables on the fibre over ω. For three different distributions ν on $[\alpha_0, \alpha_1]$ (discrete, uniform, quadratic), we also derive *sharp* asymptotics on the measure of return-time intervals for the quenched dynamics, ranging from $n^{-\frac{1}{\alpha_0}}$ to $(\log n)^{\frac{1}{\alpha_0}} \cdot n^{-\frac{1}{\alpha_0}}$ to $(\log n)^{\frac{2}{\alpha_0}} \cdot n^{-\frac{1}{\alpha_0}}$ respectively.

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1. INTRODUCTION

In this paper we study statistical properties of systems that evolve according to deterministic laws driven by a random process. Such systems are called random dynamical systems and they are often studied via analysis of a related deterministic system, the skew product map $T : \Omega \times X \to \Omega \times X$ given by:

$$T(\omega, x) := (\sigma\omega, f_\omega(x)),$$

where $\{f_\omega\}_{\omega\in\Omega}$ is a family of transformations that map X, the phase space, into itself, and σ is a measure preserving map on Ω, the noise space. The f_ω's are often referred to as the fibre maps and σ is called the base map or the driving system. The fibre maps are the deterministic components of the random system, while the base map invokes the required randomness, or time dependence, or parameter drift in the system.

Recently there has been a remarkable interest in studying statistical limit theorems for random dynamical systems [1, 4, 5, 9, 10, 12, 13, 15, 21]. Most of these results assume some knowledge about the rate of correlation decay of the random system under consideration. In this work, we develop random towers that are suitable to study *quenched*[1] correlation decay for slowly mixing random systems. We obtain a general result on the rate of *quenched* correlation decay. Moreover, we apply our results to answer the following questions: in what way does an individual map f_ω, or a group of f_ω's, dictate the rate of quenched[2] correlation decay of the random system? A second question is: how does the distribution on Ω (the measure preserved by σ) effect the quenched statistics of the system? We answer the above two questions in the framework of the Pomeau-Manneville family [23] using the version popularised by Liverani-Saussol-Vaienti [17]. Such systems have attracted the attention of both mathematicians and physicists (see [16] for a recent work in this area). In particular, for Liverani-Saussol-Vaienti (LSV) maps with parameters in $[\alpha_0, \alpha_1] \subset (0, 1)$ and base dynamics $([\alpha_0, \alpha_1]^{\mathbb{Z}}, \sigma, \nu^{\mathbb{Z}})$ we show

[1]Quenched results in random dynamical systems refer to pathwise results for almost every ω.

[2]In a simple model, yet important in the study of intermittent transition to turbulence [23], the first question was answered in [6] only for the annealed dynamics; i.e., for the dynamics averaged over Ω, and only for a specific distribution on Ω. Precisely [6] considered a system that has only two fibre maps and with the base system being a Bernoulli shift.

via a general random tower construction, that the *quenched decay of correlation* is governed by the fastest mixing map. Precisely, we prove that $n^{1-\frac{1}{\alpha_0}+\delta}$ is an upper bound on the rate of quenched decay of correlation, for all $\delta > 0$. To illustrate the role that $\delta > 0$ plays in the quenched decay rate, and to address the second question above, we also obtain *sharp* asymptotics on the position of return time intervals for the quenched dynamics in the Liverani-Saussol-Vaienti family that depend on the randomising distribution. In particular, we show how different distributions on $[\alpha_0, \alpha_1]$ (discrete, uniform, quadratic) change the *sharp* asymptotics on the position of return time intervals for the quenched dynamics from $n^{-\frac{1}{\alpha_0}}$ to $(\log n)^{\frac{1}{\alpha_0}} \cdot n^{-\frac{1}{\alpha_0}}$ to $(\log n)^{\frac{2}{\alpha_0}} \cdot n^{-\frac{1}{\alpha_0}}$.

In Section 2 we recall standard definitions and notation from random dynamical systems and present various natural notions of correlation decay in this setting. In Section 3 we build random towers for our system and detail the dynamical hypotheses that are in force throughout the paper. Our main general results are contained in Section 4 (Theorems 4.1 and 4.2) where we prove existence and correlation decay estimates respectively for the dynamics on the random towers. In Section 5 we present detailed computations applying our general results to the case of random LSV maps on the interval. Three different randomising distributions are investigated: discrete, uniform and quadratic. At the end of Section 5 we also compute exact asymptotics for the measure of the return sets on the base of the random towers. In Section 6 we prove Theorem 4.1. The expansion and distortion conditions and related estimates are the main tools used in this section. In the next section, Section 7, we introduce random stopping times and derive asymptotics on their distributions in preparation for a coupling argument. In Section 8 we obtain decay of correlation estimates (upper bounds) for observables on our random towers. Both future and past decay estimates are derived. We conclude with Section 9 where we present some technical results that are used repeatedly in the paper. Notation: We use $a \lesssim b$ if there exists universal constant C such that $a \leq Cb$; $\sim$, o, O will have their usual meaning.

2. RANDOM DYNAMICAL SYSTEMS

Let $(\mathbb{A}, \mathcal{F}, p)$ be a Borel probability space, let $\Omega = \mathbb{A}^{\mathbb{Z}}$ be equipped with the product measure $P := p^{\mathbb{Z}}$ and let $\sigma : \Omega \to \Omega$ denote the $P-$preserving two-sided shift map. Let $(X, \mathcal{B})$ be a measurable space. Suppose that $f_u : X \to X$ is a family of measurable maps defined for p-almost every $u \in \mathbb{A}$ such that the *skew product*

$$T : \Omega \times X \to \Omega \times X,\ T(\omega, x) = (\sigma\,\omega, f_{\omega_0}(x))$$

is measurable with respect to $\mathcal{F} \times \mathcal{B}$ were $\omega_k \in \mathbb{A}$ denotes the k-th coordinate of $\omega \in \Omega$. In order to simplify notation, we will normally write $f_\omega := f_{\omega_0}$ when there is no danger of confusion. So, for example, $f_{\sigma\omega} = f_{\omega_1}$. The resulting i.i.d. random map associated to the family $\{f_\omega\}$ can be viewed as follows: letting $X_\omega := \{\omega\} \times X$ denote the fiber over ω and $f^n_\omega = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_\omega : X_\omega \to X_{\sigma^n\omega}$ we have $T^n(\omega, x) = (\sigma^n\,\omega, f^n_\omega(x))$. We say that μ is a T-invariant measure if $\mu(T^{-1}A) = \mu(A)$ for any $A \in \mathcal{B} \times \mathcal{F}$. Assume

that $\{X_\omega\}_{\omega\in\Omega}$ forms a measurable partition[3] of $\Omega\times X$. We are interested in T-invariant probability measures, μ, such that $\pi_*\mu = P$, where π is the projection onto Ω. Then by Rokhlin's disintegration theorem (see [22] or [24]), for any such measure μ there exists a (essentially unique) system of probability measures μ_ω on X_ω such that for any $A\in\mathcal{F}\times\mathcal{B}$

$$\omega\mapsto\mu_\omega(A) \text{ is measurable and } \mu(A)=\int\mu_\omega(A)dP(\omega). \tag{2.1}$$

It is easy to check that μ is T-invariant if and only if $(f_\omega)_*\mu_\omega=\mu_{\sigma\omega}$ for P- a.e. ω, a property we naturally refer to as $f_\omega-$*equivariance* (or simply *equivarariance*, when the random map is understood) of the family $\{\mu_\omega\}$.

In this paper we study statistical properties of the equivariant family of measures $\{\mu_\omega\}$ for P-almost every $\omega\in\Omega$, when μ_ω is absolutely continuous with respect to Lebesgue measure m on X. More precisely we study *future* and *past quenched correlations*: given $\varphi,\psi: X\times\Omega\to\mathbb{R}$ define *future* and *past* fibre-wise correlations

$$\begin{aligned} Cor^{(f)}_{n,\omega}(\varphi,\psi)&=\int(\varphi_{\sigma^n\omega}\circ f^n_\omega)\psi_\omega d\mu_\omega-\int\varphi_{\sigma^n\omega}d\mu_{\sigma^n\omega}\int\psi_\omega d\mu_\omega,\\ Cor^{(p)}_{n,\omega}(\varphi,\psi)&=\int(\varphi_\omega\circ f^n_{\sigma^{-n}\omega})\psi_{\sigma^{-n}\omega}d\mu_{\sigma^{-n}\omega}-\int\varphi_\omega d\mu_\omega\int\psi_{\sigma^{-n}\omega}d\mu_{\sigma^{-n}\omega}. \end{aligned} \tag{2.2}$$

Definition 1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces on $\Omega\times X$ and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\rho_n=0$. We say that f_ω admits *quenched decay of correlations* at rate ρ_n if for P-almost all ω and for any $\varphi\in\mathcal{B}_1$ and $\psi\in\mathcal{B}_2$ there are constants C_ω and $C_{\varphi,\psi}$ such that

$$|Cor^{(f)}_{n,\omega}(\varphi,\psi)|\le C_\omega C_{\varphi,\psi}\rho_n,\quad |Cor^{(p)}_{n,\omega}(\varphi,\psi)|\le C_\omega C_{\varphi,\psi}\rho_n. \tag{2.3}$$

Remark 1. Note that if C_ω is P-integrable, then this implies the same rate for the integrated correlations; i.e., $\int_\Omega Cor^{(f)}_{n,\omega}(\varphi,\psi)dP\le\hat{C}_{\varphi,\psi}\rho_n$. The importance of knowing the rate of the integrated correlations is due to its relation to the annealed correlations of the skew product. Indeed, setting $\bar{\varphi}_\omega:=\int_X\varphi d\mu_\omega$ and $\bar{\psi}_\omega:=\int_X\psi d\mu_\omega$ we have

$$Cor_n(T,\varphi,\psi)=\int_\Omega Cor^{(f)}_{n,\omega}(\varphi,\psi)dP+Cor_n(\sigma,\bar{\varphi},\bar{\psi}).$$

3. Abstract tower setting

A main tool, in particular in the absence of spectral techniques, to study statistical properties of dynamical systems is the so called Young Tower [26, 27]. Young Towers have been used extensively to obtain rates of decay of correlations for nonuniformly hyperbolic systems (see for example [2, 3, 11, 19] and references therein). In this section we describe random towers which were first considered in [7] to study *quenched* statistical properties of i.i.d. unimodal maps. Later the work of [7] was extended in [8] to cover a wider class of i.i.d. unimodal maps. Building on ideas from [7, 8] we

[3]This is satisfied, for example, when $\mathbb{A}$ is Hausdorff so that $\{\omega\}$ is closed.

study random towers with slowly decaying tails. Let $\Lambda \subset X$ be a measurable set with $m(\Lambda) = 1$. Consider a family of maps $f_\omega : X \to X$, where f_ω depends only on the zeroth coordinate of ω. We say that f_ω admits a random tower on $\Lambda \subset X$ if for almost every $\omega \in \Omega$ there exists a countable partition $\{\Lambda_j(\omega)\}_j$ of Λ and a measurable return time function $R_\omega : \Lambda \to \mathbb{N}$ that is constant on each $\Lambda_j(\omega)$ such that $f_\omega^{R_\omega}(x) = f_{\sigma^{R_\omega(x)-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega(x) \in \Lambda$ for P-almost every $\omega \in \Omega$ and m -almost every $x \in \Lambda$. Given the above information we define a random tower for almost every ω as

$$\Delta_\omega = \left\{(x,\ell) \in \Lambda \times \mathbb{Z}_+ \mid x \in \cup_j \Lambda_j(\sigma^{-\ell}\omega),\ j, \ell \in \mathbb{Z}_+, 0 \le \ell \le R_{\sigma^{-\ell}\omega}(x) - 1\right\} \tag{3.1}$$

and random tower map $F_\omega : \Delta_\omega \to \Delta_{\sigma\omega}$ by

$$F_\omega(x,\ell) = \begin{cases} (x, \ell+1), & \text{if} \quad \ell + 1 < R_{\sigma^{-\ell}\omega}(x) \\ (f_{\sigma^{-\ell}\omega}^{\ell+1} x, 0), & \text{if} \quad \ell + 1 = R_{\sigma^{-\ell}\omega}(x). \end{cases} \tag{3.2}$$

Denote by $\Delta_{\omega,\ell} := \{(x,\ell) \in \Delta_\omega\}$ the ℓth level of the tower, which is a copy of $\{x \in \Lambda \mid R_{\sigma^{-\ell}\omega}(x) > \ell\}$; for instance $\Delta_{\omega,0} = \Lambda$ and $F_\omega^{R_\omega}|\Delta_{\omega,0} = f_\omega^{R_\omega}|\Lambda$. Let $\Delta = \{\Delta_\omega\}_{\omega\in\Omega}$. Then $F = \{F_\omega\}_{\omega\in\Omega}$ is a fibered system on Δ; see Figure 1 for a pictorial representation.

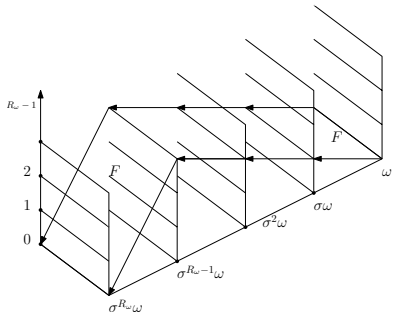


FIGURE 1. Dynamics on the tower

Notice that $\{\Lambda_j(\omega)\}_j$ induces a countable partition $\mathcal{P}_\omega$ on each Δ_ω:

$$\mathcal{P}_\omega := \left\{F_{\sigma^{-\ell}\omega}^\ell(\Lambda_j(\sigma^{-\ell}\omega)) \mid R_\omega|\Lambda_j(\sigma^{-\ell}\omega) \ge \ell + 1, \ell \in \mathbb{Z}_+\right\}.$$

For $(x,\ell) \in \Delta_\omega$, let $\hat{R}_\omega$ denote the first return time to the base of the tower $\Delta_{\sigma^{\hat{R}_\omega}\omega}$ i.e.

$$\hat{R}_\omega(x,\ell) = R_{\sigma^{-\ell}\omega}(x) - \ell. \tag{3.3}$$

The reference measure m and the σ-algebra on Λ naturally lifts to Δ_ω and by abuse of notation we call it m. The lifted σ-algebra will be denoted by $\mathcal{B}_\omega$. Let $R_\omega^1(x) = R_\omega(x)$ and for $n \ge 2$, $R_\omega^n(x) = R_{\sigma^{R_\omega^{n-1}}\omega}(F_\omega^{n-1}(x)) + R_\omega^{n-1}(x)$.Next we define the separation time $s : \Delta \times \Delta \to \mathbb{Z}_+ \cup \{\infty\}$ for almost every ω by setting $s(z_1, z_2) = 0$ if z_1 and z_2 lie in different towers Δ_ω and if $z_1, z_2 \in \Delta_\omega$ then

$$s(z_1,z_2) = \min\{n \ge 0 \mid (F_\omega^{R_\omega})^n(z_1) \text{ and } (F_\omega^{R_\omega})^n(z_2) \text{ lie in distinct elements of } \mathcal{P}_{\sigma^{R_\omega^n}\omega}\}.$$

Below we refer to Λ as the zeroth level of the tower. We assume that the random tower satisfies the following properties.

(P1) **Markov**: for each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega) : \Lambda_j(\omega) \to \Lambda$ is a bijection;

(P2) **Bounded distortion**: There are constants $D > 0$ and $0 < \gamma < 1$ such that for all ω and each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega)$ and its inverse are non-singular with respect to m with corresponding Jacobian $JF_\omega^{R_\omega}|\Lambda_j(\omega)$ which is positive and for each $x, y \in \Lambda_j(\omega)$ satisfies the following

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)} - 1\right| \leq D\gamma^{s(F_\omega^{R_\omega}(x,0),F_\omega^{R_\omega}(y,0))}; \tag{3.4}$$

(P3) **Weak expansion**: $\mathcal{P}_\omega$ is a generating partition for F_ω i.e. diameters of the partitions $\vee_{j=0}^n F_\omega^{-j}\mathcal{P}_{\sigma^j\omega}$ converge to zero as n tends to infinity;

(P4) **Return time asymptotics**: There are constants $C > 0$, $a > 1$, $b \geq 0$, $u > 0$, $v > 0$, a full measure subset $\Omega_1 \subset \Omega$ and a random variable $n_1 : \Omega_1 \to \mathbb{N}$ such that

$$\begin{cases} m\{x \in \Lambda \mid R_\omega(x) > n\} \leq C\frac{(\log n)^b}{n^a}, \text{ whenever } n \geq n_1(\omega), \\ P\{n_1(\omega) > n\} \leq Ce^{-un^v}; \end{cases} \tag{3.5}$$

(P5) **Aperiodicity**: There are $N \in \mathbb{N}$ and $\{t_i \in \mathbb{Z}_+ \mid i = 1, 2, ..., N\}$ such that g.c.d.$\{t_i\} = 1$ and $\epsilon_i > 0$ so that for almost every $\omega \in \Omega$ and $i = 1, 2, \dots N$ we have $m\{x \in \Lambda \mid R_\omega(x) = t_i\} > \epsilon_i$.

(P6) **Finiteness**: There exists an $M > 0$ such that $m(\Delta_\omega) \leq M$ for all $\omega \in \Omega$.

(P7) **Annealed return time asymptotics**: There are constants $C > 0$, $\hat{b} \geq 0$ and $a > 1$ such that $\int_\Omega m\{x \in \Lambda | \, R_\omega = n\}dP \leq C\frac{(\log n)^{\hat{b}}}{n^{a+1}}$.

3.1. **Tower projections.** For almost every $\omega \in \Omega$ and $(x, \ell) \in \Delta_\omega$ we define tower projections $\pi_\omega : \Delta_\omega \to X_\omega$ as $\pi_\omega(x, \ell) = f_{\sigma^{-\ell}\omega}^\ell(x)$. Then π_ω is a semi-conjugacy i.e. $\pi_{\sigma\omega} \circ F_\omega = f_\omega \circ \pi_\omega$. Indeed, for $(x, \ell) \in \Delta_\omega$ we have $f_\omega(\pi_\omega(x, \ell)) = f_\omega \circ f_{\sigma^{-\ell}\omega}^\ell(x)$. On the other hand, since $F(x, \ell) \in \Delta_{\sigma\omega}$

$$\pi_{\sigma\omega}(F_\omega(x, \ell)) = \begin{cases} \pi_{\sigma\omega}(x, \ell + 1) & \text{if} \quad R_{\sigma^{-\ell}\omega}(x) > \ell + 1 \\ \pi_{\sigma\omega}(f_{\sigma^{-\ell}\omega}^{\ell+1}(x), 0) & \text{otherwise} \end{cases} = f_{\sigma^{-\ell}\omega}^{\ell+1}(x).$$

Now, if ν_ω is an absolutely continuous family of F_ω-equivariant probability measures on Δ_ω, then $\mu_\omega := (\pi_\omega)_*\nu_\omega$ is a family of f_ω- equivariant probability measures on $\Omega \times X$. Since each f_ω is nonsingular, if $A \subset X$ is such that $m(A) = 0$ then $m(\pi_\omega^{-1}(A)) = 0$, which implies $\nu_\omega(\pi_\omega^{-1}(A)) = 0$, consequently $\mu_\omega(A) = 0$. Therefore each μ_ω is absolutely continuous.

4. Statement of main results

In this section we state general theorems concerning quenched correlation decay for slowly mixing systems. We start this section by introducing some function spaces on Δ, which are necessary to state the theorems. These spaces appeared in the present form in

[7]. Below we let constants $u > 0$, $v > 0$, $a > 1$, $b \geq 0$, $0 < \gamma < 1$ be as in (P2) and (P4) above and set

$$\mathcal{F}_\gamma^+ = \{\varphi_\omega : \Delta_\omega \to \mathbb{R} \mid \exists C_\varphi > 0, \forall I_\omega \in \mathcal{P}_\omega, \text{ either } \varphi_\omega|I_\omega \equiv 0 \text{ or } \varphi_\omega|I_\omega > 0 \text{ and } \left|\log \frac{\varphi_\omega(x)}{\varphi_\omega(y)}\right| \leq C_\varphi \gamma^{s(x,y)}, \forall x, y \in I_\omega\}.$$

Let $K_\omega : \Omega \to \mathbb{R}_+$ be a random variable with $\inf_\Omega K_\omega > 0$ and

$$P\{\omega \mid K_\omega > n\} \leq e^{-un^v}. \tag{4.1}$$

Define the space of random bounded functions as

$$\mathcal{L}_\infty^{K_\omega} = \{\varphi_\omega : \Delta_\omega \to \mathbb{R} \mid \exists C'_\varphi > 0, \sup_{x \in \Delta_\omega} |\varphi_\omega(x)| \leq C'_\varphi K_\omega\}$$

and a space of random Lipschitz functions

$$\mathcal{F}_\gamma^{K_\omega} = \{\varphi_\omega \in \mathcal{L}_\infty^{K_\omega} \mid \exists C_\varphi > 0, |\varphi_\omega(x) - \varphi_\omega(y)| \leq C_\varphi K_\omega \gamma^{s(x,y)}, \ \forall x, y \in \Delta_\omega\}.$$

Notice that the above spaces are defined for almost every $\omega \in \Omega$. By taking smallest possible values of C'_φ and C_φ the spaces $\mathcal{L}_\infty^{K_\omega}$ and $\mathcal{F}_\gamma^{K_\omega}$ equipped with the norms $\|\varphi\|_{\mathcal{L}_\infty} = C'_\varphi$ and $\|\varphi\|_{\mathcal{F}} = \max\{C_\varphi, C'_\varphi\}$ respectively are Banach spaces. Note that no regularity assumption is made in the ω variable, in particular, these functions need not be measurable in ω. Finally we let $\mathcal{B}$ be a σ-algebra on Δ defined as follows: $B \in \mathcal{B}$ if and only if for each ω the intersection $B_\omega = B \cap \Delta_\omega \in \mathcal{B}_\omega$. Let $\{\nu_\omega\}_{\omega \in \Omega}$ be a fibered equivariant family of measures i.e. $(F_\omega)_* \nu_\omega = \nu_{\sigma\omega}$. Let $\mu_\omega = (\pi_\omega)_* \nu_\omega$ and $\mu(A) = \int_\Omega \mu_\omega(A_\omega) dP(\omega)$. Then μ is T-invariant (for the skew product T on $\Omega \times X$). We say that ν is exact/mixing for F if μ is exact/mixing for the skew product T. We can formulate equivalent conditions as follows.

Definition 2.

(i) The fibered system $(F, \nu) = (F_\omega, \nu_\omega)_{\omega \in \Omega}$ is exact iff $\vee_{n=0}^\infty F^{-n}\mathcal{B}$ is trivial; i.e., for any $B \in \vee_{n=0}^\infty F^{-n}\mathcal{B}$, either for almost all ω, $\nu_\omega(B) = 0$ or for almost all ω, $\nu_\omega(B) = 1$.

(ii) The random skew product (F, ν) is mixing iff for all $\varphi, \psi \in L^2(\nu)$,

$$\lim_{n \to \infty} \left| \int_\Omega \int_{\Delta_\omega} \varphi_{\sigma^n\omega} \circ F_\omega^n \cdot \psi_\omega d\nu_\omega dP - \int_\Omega \int_{\Delta_\omega} \varphi_\omega d\nu_\omega dP \int_\Omega \int_{\Delta_\omega} \psi_\omega d\nu_\omega dP \right| = 0.$$

We note that in our situation exactness implies mixing ([7], section 4). The first result is the existence of absolutely continuous sample measures.

Theorem 4.1. For almost every $\omega \in \Omega$ there is a family $\nu_{\sigma^k\omega}$ which is equivariant and absolutely continuous with respect to m: $(F_{\sigma^k\omega})_* \nu_{\sigma^k\omega} = \nu_{\sigma^{k+1}\omega}$, with $\nu_\omega = h_\omega m$. Moreover, there exists K_ω satisfying (4.1) such that $h_\omega \in \mathcal{F}_\gamma^{K_\omega} \cap \mathcal{F}_\gamma^+$ for almost every $\omega \in \Omega$.

The next main result is about the decay of future and past quenched correlations.

Theorem 4.2. Let $\delta > 0$. Let K_ω be the function given in Theorem 4.1. There exits a full measure set $\Omega_0 \subset \Omega$ and a random variable C_ω on Ω_0 such that for every $\varphi \in \mathcal{L}_\infty^{K_\omega}$, $\psi \in \mathcal{F}_\gamma^{K_\omega}$ there exits a constant $C_{\varphi,\psi}$ such that for every $\omega \in \Omega_0$

(i) "Future" operational correlations :

$$\left| \int (\varphi_{\sigma^n\omega} \circ F_\omega^n)\psi_\omega dm - \int \varphi_{\sigma^n\omega} d\nu_{\sigma^n\omega} \int \psi_\omega dm \right| \leq C_\omega C_{\varphi,\psi} n^{1+\delta-a}.$$

(ii) "Past" operational correlations :

$$\left| \int (\varphi_\omega \circ F_{\sigma^{-n}\omega}^n)\psi_{\sigma^{-n}\omega} dm - \int \varphi_\omega d\nu_\omega \int \psi_{\sigma^{-n}\omega} dm \right| \leq C_\omega C_{\varphi,\psi} n^{1+\delta-a}.$$

Moreover, there exist constants $C > 0$, $u' > 0$, $v' \in (0,1)$ such that

$$P\{C_\omega > n\} \leq Ce^{-u'n^{v'}}.$$

Remark 2. A quenched correlation decay rate of the form $\frac{(\log n)^b}{n^{a-1}}$, which is analogous to what one expects in the deterministic setting, cannot be achieved since we want to get information on the integrability of the C_ω in Theorem 4.2. The shift of the Lipschitz constant K_ω, and hence the dependence of that constant on n, in equation (8.7) and the non-uniformity of the tail in (P4) are the main reasons for getting a rate at the order $\frac{1}{n^{a-1+\delta}}$, for any $\delta > 0$. See Footnote 7 for more details.

Theorem 4.3. Let $\{\nu_\omega\}$ be as in Theorem 4.1. The map $\omega \mapsto \nu_\omega$ is measurable in ω. The measure $\nu(\cdot) = \int_\Omega \nu_\omega(\cdot) dP$ is F-invariant, and the system (F, ν) is exact and hence mixing.

Proof. By Proposition 8.6, for almost every ω, we have

$$|(F_{\sigma^{-n}\omega}^n)_* m - \nu_\omega| = \int \left| \frac{d(F_{\sigma^{-n}\omega}^n)_* m}{dm} - h_\omega \right| dm \to 0.$$

Since $d(F_{\sigma^{-n}\omega}^n)_* m/dm$ is measurable in (ω, x), this implies that $(\omega, x) \mapsto h_\omega(x)$ is measurable. The integrability of h_ω with respect to P follows from that of K_ω. Thus, $\{\nu_\omega\}$ is a measurable family of measures and $\nu(\cdot) = \int_\Omega \nu_\omega(\cdot) dP$ is F-invariant. Using the same method as detailed in [7] we conclude that the system (F, ν) is exact and hence mixing. □

5. Applications to random LSV maps

In this section we illustrate our results with applications to the family of intermittent LSV maps as described in [17]. Let $0 < \alpha < 1$ and consider $f_\alpha : I \to I$ defined as

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & x \in \left[0, \frac{1}{2}\right], \\ 2x - 1, & x \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{5.1}$$

To define a random LSV map we fix two positive numbers $0 < \alpha_0 < \alpha_1 < 1$ and let ν be a probability measure on $[\alpha_0, \alpha_1]$. Set $\Omega = [\alpha_0, \alpha_1]^{\mathbb{Z}}$ and $P = \nu^{\mathbb{Z}}$. Then the shift map

$\sigma : \Omega \to \Omega$ preserves P. Let $\alpha : \Omega \to [\alpha_0, \alpha_1]$ be the projection to the zeroth coordinate and let $f_{\alpha(\omega)} = f_\omega$. We consider the skew product $T : \Omega \times I \to \Omega \times I$ defined by

$$T(\omega, x) = (\sigma(\omega), f_\omega(x)).$$

Compositions of f_ω are given by $f^n_\omega = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_{\sigma\omega} \circ f_\omega$.

For each ω we define a sequence of pre-images of $\frac{1}{2}$ as follows. Let $x_1(\omega) = \frac{1}{2}$, and

$$x_n(\omega) = \left(f_\omega|_{(0,1/2]}\right)^{-1} x_{n-1}(\sigma\omega) \text{ for } n \geq 2. \tag{5.2}$$

Further let

$$x'_0(\omega) = 1, x'_1(\omega) = \frac{3}{4}, \text{ and } x'_n(\omega) = \frac{x_n(\sigma\omega) + 1}{2} \text{ for } n \geq 2. \tag{5.3}$$

The sequences $\{x_n(\omega)\}$ and $\{x'_n(\omega)\}$ will allow us to define the random tower structure. First of all notice that from the definition of $x'_n(\omega)$ we have $f_\omega(x'_n(\omega)) = x_n(\sigma\omega)$, $f^2_\omega(x'_n(\omega)) = x_{n-1}(\sigma^2\omega), ..., f^n_\omega(x'_n(\omega)) = x_1(\sigma^n\omega) = \frac{1}{2}$. Let $\Lambda := \left(\frac{1}{2}, 1\right]$. The sequence $\{x'_n(\omega)\}_{n\geq 0}$ generates a partition $\mathcal{P}_\omega = \{(x'_n(\omega), x'_{n-1}(\omega)] \mid n \geq 0\}$ on each $\Lambda \times \{\omega\} = \Delta_{\omega,0}$. Define the return time $R_\omega : \Delta_{\omega,0} \to \mathbb{N}$ by setting

$$R_\omega|_{(x'_n(\omega), x'_{n-1}(\omega)]} = n. \tag{5.4}$$

A fibered system is obtained by defining a tower Δ_ω over each $\omega \in \Omega$ by

$$\Delta_\omega = \cup_{\ell=0}^{\infty} \cup_{i=\ell+1}^{\infty} (x'_i(\sigma^{-\ell}\omega), x'_{i-1}(\sigma^{-\ell}\omega)] \times \ell.$$

The fibered map $F : (\omega, \Delta_\omega) \to (\sigma\omega, \Delta_{\sigma\omega})$ from equation (3.1) can be expressed in this notation as follows: Let $(x, \ell) \in \Delta_\omega$ over ω, with $x \in (x'_i(\sigma^{-\ell}\omega), x'_{i-1}(\sigma^{-\ell}\omega)]$. There are two cases. If $i > \ell + 1$, then $F(x, \ell) = (x, \ell + 1) \in \Delta_{\sigma\omega}$ since $x \in (x'_i(\sigma^{-\ell}\omega), x'_{i-1}(\sigma^{-\ell}\omega)] = (x'_i(\sigma^{-\ell-1}\sigma\omega), x'_{i-1}(\sigma^{-\ell-1}\sigma\omega)]$. On the other hand, if $i = \ell + 1$ then $x \in (x'_{\ell+1}(\sigma^{-\ell}\omega), x'_\ell(\sigma^{-\ell}\omega)]$, where $R_{\sigma^{-\ell}\omega} \equiv \ell + 1$, so the interval is mapped bijectively to $(\frac{1}{2}, 1]$ by $f^{\ell+1}_{\sigma^{-\ell}\omega}$. Therefore in this case we have $F(x, \ell) = (f^{\ell+1}_{\sigma^{-\ell}\omega}(x), 0)$.

Proposition 5.1. *The fibered system $\{\Delta_\omega\}_{\omega\in\Omega}$ with fibered map F defined above satisfies properties (P1)-(P3) and (P5). In particular, the distortion condition (P3) is satisfied for any $\gamma \in [\frac{1}{2}, 1)$ and $D < \infty$.*

Proof. Since every map in the family f_α expands by at least a factor of 2 on return to the base interval $(\frac{1}{2}, 1]$, with full returns, we see that the Markov and weak expansion properties are satisfied. Furthermore, for each ω, $\{x \in \Lambda \mid R_\omega(x) = 1\} = (\frac{3}{4}, 1]$, which implies that the aperiodicity condition (P5) is satisfied. Since every f_α has negative Schwarzian derivative and this property is preserved under composition, we obtain the bounded distortion condition (P3) using the Koebe principle. See Lemmas 4.8 and 4.10 in [6] for computations related to Schwarzian derivatives and [20] for more details about the use of the Koebe principle. This completes the proof. □

It remains to establish appropriate estimates on the return time asymptotics as in (P4) and (P7) and the uniform bound in (P6). Observe, in view of the return-time formula (5.4), that

$$m\{R_\omega > \ell\} = \frac{1}{2} x_\ell(\sigma\omega). \tag{5.5}$$

We will estimate terms on the right hand side of this expression. Let $\underline{\alpha}_0$ (respectively $\underline{\alpha}_1$) denote the special sequences of all $\omega_k = \alpha_0$ (respectively, all $\omega_k = \alpha_1$). Following Lemma 4.4 in [6], we obtain coarse estimates on the location of the $x_n(\omega)$. Translated into our setting these estimates imply, for every $\ell, n \in \mathbb{N}$,

$$x_\ell(\underline{\alpha}_0) \leq x_\ell(\omega) \leq x_\ell(\underline{\alpha}_1). \tag{5.6}$$

It is well known (see [27] section 6, for example) that $x_\ell(\underline{\alpha}_0) \sim \frac{1}{2}\alpha_0^{-\frac{1}{\alpha_0}} \ell^{-\frac{1}{\alpha_0}}$ so if we define $c_\ell(\underline{\alpha}_0) := x_\ell(\underline{\alpha}_0)\ell^{\frac{1}{\alpha_0}}$ and write $x_\ell(\underline{\alpha}_0) = \frac{c_\ell(\underline{\alpha}_0)}{\ell^{\frac{1}{\alpha_0}}}$ then $\lim_\ell c_\ell(\underline{\alpha}_0) = \frac{1}{2}\alpha_0^{-\frac{1}{\alpha_0}} := c(\underline{\alpha}_0)$. We define $c_\ell(\underline{\alpha}_1)$ and $c(\underline{\alpha}_1)$ analogously using $x_\ell(\underline{\alpha}_1)$. Since, $m\{R_\omega > \ell\} = \frac{1}{2}x_\ell(\sigma\omega) \leq \frac{1}{2}x_\ell(\underline{\alpha}_1) \leq C\ell^{-\frac{1}{\alpha_1}}$ this implies $m(\Delta_\omega) \leq M$ for some $M > 0$ independent of ω. This proves (P6).

We now check that assumption (P4) is satisfied. Using definition (5.1) and the estimate $(1+x)^{-\alpha} \leq 1 - \alpha x + \frac{\alpha(1+\alpha)}{2}x^2$, valid for $\alpha > 0,\ x \geq 0$, and by substituting $x = [2x_n(\omega)]^{\alpha(\omega)}$ we obtain

$$\begin{aligned} \frac{1}{[x_n(\omega)]^{\alpha_0}} - \frac{1}{[x_{n-1}(\sigma\omega)]^{\alpha_0}} &\geq \alpha_0 2^{\alpha_0}[2x_n(\omega)]^{\alpha(\omega)-\alpha_0} \\ &\quad - \frac{\alpha_0(1+\alpha_0)}{2} 2^{\alpha_0}[2x_n(\omega)]^{2\alpha(\omega)-\alpha_0}. \end{aligned} \tag{5.7}$$

Iterating this one-step estimate along the sequence $x_k(\sigma^{\ell-k}\omega)$, $k = 1, 2, \ldots \ell$ we obtain

$$\begin{aligned} \frac{1}{[x_\ell(\omega)]^{\alpha_0}} \geq 2^{\alpha_0} + \alpha_0 2^{\alpha_0}\{&\sum_{k=2}^{\ell}[2x_k(\sigma^{\ell-k}\omega)]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\ &- \frac{1+\alpha_0}{2}\sum_{k=2}^{\ell}[2x_k(\sigma^{\ell-k}\omega)]^{2\alpha(\sigma^{\ell-k}\omega)-\alpha_0}\}. \end{aligned} \tag{5.8}$$

Combining equations (5.8) and (5.6) implies that, for any parameter $q \geq 0$

$$\begin{aligned}\frac{(\log \ell)^q}{\ell[x_\ell(\omega)]^{\alpha_0}} &\geq \frac{2^{\alpha_0}(\log \ell)^q}{\ell} + \alpha_0 2^{\alpha_0}\Bigg\{\frac{(\log \ell)^q}{\ell}\sum_{k=2}^{\ell}\left[\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\ &- \frac{1+\alpha_0}{2}\frac{(\log \ell)^q}{\ell-1}\sum_{k=2}^{\ell}\left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{2\alpha(\sigma^{\ell-k}\omega)-\alpha_0}\Bigg\} \\ &\geq \alpha_0 2^{\alpha_0}\frac{(\log \ell)^q}{\ell}\Bigg\{\sum_{k=2}^{\ell}\left[\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\ &- \frac{1+\alpha_0}{2}\left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{2\alpha(\sigma^{\ell-k}\omega)-\alpha_0}\Bigg\} = \frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell}A_k(\omega),\end{aligned} \tag{5.9}$$

where we have introduced notation $A_k(\omega)$ for the sequence of independent random variables $A_1 \equiv 0$ and for $k = 2, 3, \ldots \ell$

$$A_k(\omega) := \alpha_0 2^{\alpha_0}\left[\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} - \frac{1+\alpha_0}{2}\alpha_0 2^{\alpha_0}\left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{2\alpha(\sigma^{\ell-k}\omega)-\alpha_0}. \tag{5.10}$$

From now on we write $A_k := A_k(\omega)$. Note that $-\alpha_0 2^{\alpha_0} \leq A_k \leq \alpha_0 2^{\alpha_0}$ for every k.

Assumption (A1) (Asymptotics on expectations)
Assume[4] there are constants $q = q(\nu) \geq 0$ and a constant $c(\nu) > 0$ such that the following holds:

$$\frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell}E_\nu A_k \to c(\nu). \tag{5.11}$$

Fix any $0 < c < c(\nu)$. Pick N_1 so that for all $\ell > N_1$,

$$\frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell}E_\nu A_k \geq \frac{c + c(\nu)}{2}.$$

Note that, given expression (5.11), N_1 depends only on the choice of c, and in particular is independent of ω. Set $r_0 = \alpha_0 2^{\alpha_0}$.

Lemma 5.2. For each $t > 0$ we have

$$P\left\{\frac{(\log \ell)^q}{\ell}\left|\sum_{k=1}^{\ell}A_k - \sum_{k=1}^{\ell}E_\nu A_k\right| \geq t\right\} \leq \exp\left[-\frac{\ell t^2}{2r_0(\log \ell)^{2q}}\right]. \tag{5.12}$$

[4]Below in the examples, we will show that the assumption is satisfied for different types of distributions. In particular, we will show how different measures ν lead to different tail asymptotics.

Proof. We may apply the classical result of Hoeffding (see [14] Theorem 1, or [18] Lemma 1.2) to the sequence of independent random variables A_k, noting that instead of the bound $0 \leq A_k \leq 1$ we have $-r_0 \leq A_k \leq r_0$, accounting for the extra factor in the exponential. □

Next, we apply the previous lemma to obtain a large deviation estimate on the normalized sums of A_k. For each $\ell > N_1$:

(5.13)
$$\begin{aligned} P\left\{\frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell} A_k < c\right\} &= P\left\{\frac{(\log \ell)^q}{\ell}\{\sum_{k=1}^{\ell} A_k - \sum_{k=1}^{\ell} E_\nu A_k\} < c - \frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell} E_\nu A_k\right\} \\ &\leq P\left\{\frac{(\log \ell)^q}{\ell}\left|\sum_{k=1}^{\ell} A_k - \sum_{k=1}^{\ell} E_\nu A_k\right| > \frac{c(\nu)-c}{2}\right\} \\ &\leq \exp\left[-\frac{\ell[c(\nu)-c]^2}{8r_0(\log \ell)^{2q}}\right], \end{aligned}$$

where in the last line we have used equation (5.12). Now define

$$n_1(\omega) := \inf\{n > N_1 \mid \forall \ell > n,\ \frac{(\log \ell)^q}{\ell}\sum_{k=1}^{\ell} A_k \geq c\}.$$

If $\ell > n_1(\omega)$ then equation (5.9) implies that

$$\frac{(\log \ell)^q}{\ell[x_\ell(\omega)]^{\alpha_0}} \geq c \tag{5.14}$$

and hence

$$x_\ell(\omega) \leq \left[\frac{1}{c}\right]^{\frac{1}{\alpha_0}} \frac{[\log \ell]^{\frac{q}{\alpha_0}}}{\ell^{\frac{1}{\alpha_0}}} \leq 2c(\nu)^{-\frac{1}{\alpha_0}} \frac{[\log \ell]^{\frac{q}{\alpha_0}}}{\ell^{\frac{1}{\alpha_0}}}, \tag{5.15}$$

provided $\frac{c(\nu)}{2^{\alpha_0}} < c < c(\nu)$. We may take $b := \frac{q}{\alpha_0}$ and $a := \frac{1}{\alpha_0}$ and $C = c(\nu)^{-\frac{1}{\alpha_0}}$ in (P4). Note that the constant C is independent of n and ω. Finally, we estimate, for $n > N_1$ and fixed $0 < v < 1$

$$\begin{aligned} P\left\{n_1(\omega) > n\right\} &\leq \sum_{\ell > n} \exp\left[-\frac{\ell[c(\nu)-c]^2}{8r_0(\log \ell)^{2q}}\right] \\ &\leq \sum_{\ell > n} \exp\left[-u\ell^v\right] \leq C\exp\left[-un^v\right] \end{aligned} \tag{5.16}$$

for suitable constants $u > 0$ and $C < \infty$. We can remove the restriction $n > N_1$ by substitution of a larger constant C in the final expression

$$P\left\{n_1(\omega) > n\right\} \leq Ce^{-un^v},$$

completing the second condition in (P4). Finally we verify (P7).

First, we have

$$\begin{aligned}(P\times m)\{x\in\Lambda|\, R_\omega=n\} &= (P\times m)\{x\in\Lambda|\, R_\omega>n-1\}\\ &\quad -(P\times m)\{x\in\Lambda|\, R_\omega>n\}\\ &= \frac{1}{2}E_\nu[x_{n-1}(\omega)]-\frac{1}{2}E_\nu[x_n(\omega)]\\ &= \frac{1}{2}E_\nu[x_{n-1}(\sigma\omega)]-\frac{1}{2}E_\nu[x_n(\omega)]\\ &= \frac{1}{2}E_\nu[x_{n-1}(\sigma\omega)-x_n(\omega)],\end{aligned} \tag{5.17}$$

where we have used the fact that P is σ-invariant to write $E_\nu[x_{n-1}(\omega)]=E_\nu[x_{n-1}(\sigma\omega)]$. Now recall that $f_\omega(x_n(\omega))=x_{n-1}(\sigma\omega)$ and $f_\omega(x_n(\omega))=x_n(\omega)+2^{\alpha(\omega)}(x_n(\omega))^{\alpha(\omega)+1}$. Using this fact, (5.17), (5.15) and $0\le 2x_n(\omega)\le 1$, we obtain

$$\begin{aligned}&(P\times m)\{x\in\Lambda|\, R_\omega=n\}=\frac{1}{2}E_\nu[2^{\alpha(\omega)}(x_n(\omega))^{\alpha(\omega)+1}]\\ &\le \frac{1}{2}E_\nu[2^{\alpha_0}(x_n(\omega))^{\alpha_0+1}]=2^{\alpha_0-1}E_\nu[x_n(\omega)]^{\alpha_0+1}\\ &\le 2^{\alpha_0-1}\left(E_\nu[\mathbb{1}_{\{n_1(\omega)\le n\}}\cdot x_n(\omega)^{\alpha_0+1}]+E_\nu[\mathbb{1}_{\{n_1(\omega)>n\}}\cdot x_n(\omega)^{\alpha_0+1}]\right)\\ &= 2^{2\alpha_0}c(\nu)^{-1-\frac{1}{\alpha_0}}\frac{[\log n]^{\frac{q(\alpha_0+1)}{\alpha_0}}}{n^{\frac{1}{\alpha_0}+1}}+Ce^{-un^v}\le \hat{C}\frac{[\log n]^{\frac{q(\alpha_0+1)}{\alpha_0}}}{n^{\frac{1}{\alpha_0}+1}}.\end{aligned} \tag{5.18}$$

Choosing $\hat{b}=\frac{q(\alpha_0+1)}{\alpha_0}$ and $a=\frac{1}{\alpha_0}$ completes the verification of (P7).

Theorem 5.1. Let $0<\alpha_0<\alpha_1<1$ be fixed and $\Omega=[\alpha_0,\alpha_1]^{\mathbb{Z}}$ equipped with the product probability measure $P:=\nu^{\mathbb{Z}}$ and left shift σ. Let $f_\omega,\ \omega\in\Omega$ be a random family of LSV maps with respect to the measure $\nu^{\mathbb{Z}}$. Assume condition (A1) holds for the asymptotic expectations. Then there exists a family of absolutely continuous sample stationary measures μ_ω on $[0,1]$, for almost every $\omega\in\Omega$ (i.e. $f_{\omega *}\mu_\omega=\mu_{\sigma\omega}$). The map $\omega\to\mu_\omega$ is measurable, $\mu=\int_\Omega \mu_\omega dP$ is T-invariant and (T,μ) is exact and hence mixing.

Moreover, for every $\delta>0$ there exists a full measure subset $\Omega_0\subset\Omega$ and a random variable $C_\omega:\Omega_0\to\mathbb{R}_+$ so that for any $\varphi\in L^\infty[0,1]$, $\psi\in C^\eta[0,1]$, the class of $\eta-$ Hölder functions on $[0,1]$, we have

(i) "Future" correlations:

$$\left|\int(\varphi\circ f_\omega^n)\psi\, d\mu_\omega-\int\varphi d\mu_{\sigma^n\omega}\int\psi d\mu_\omega\right|\le C_\omega C_{\varphi,\psi}\frac{1}{n^{\frac{1}{\alpha_0}-\delta-1}}.$$

(ii) "Past" correlations:

$$\left|\int(\varphi\circ f_{\sigma^{-n}\omega}^n)\psi\, d\mu_{\sigma^{-n}\omega}-\int\varphi\, d\mu_\omega\int\psi d\mu_{\sigma^{-n}\omega}\right|\le C_\omega C_{\varphi,\psi}\frac{1}{n^{\frac{1}{\alpha_0}-\delta-1}}.$$

There exist constants $C > 0$, $u' > 0$ and $0 < v' < 1$ such that the random variable C_ω satisfies the following tail estimates: for all $n \in \mathbb{N}$

$$P\{C_\omega > n\} \leq Ce^{-u'n^{v'}}.$$

In particular, C_ω is integrable. Finally, every $\eta \in (0, 1]$ can be used by choosing[5] $\gamma \in [\frac{1}{2}, 1)$ so that $2^\eta\gamma \geq 1$.

Proof. Proposition 5.1 establishes conditions (P1) - (P3) and (P5). Since we are assuming (A1), condition (P4) follows from equations (5.5) and (5.15) with constants $b = \frac{q}{\alpha_0}$, $a = \frac{1}{\alpha_0}$, $C = c(\nu)^{-\frac{1}{\alpha_0}}$. The second condition in (P4) holds because of equation (5.16). (P7) is verified in (5.18).

Theorem 4.1 and Corollary 4.3 therefore apply and give existence and mixing of the sample stationary measures μ_ω. Finally we apply Theorem 4.2 to obtain the decay of correlations.

For $\varphi \in L^\infty([0, 1])$ and $\psi \in C^\eta([0, 1])$ define $\bar{\psi} = \psi \circ \pi_\omega, \bar{\varphi} = \varphi \circ \pi_\omega : \Delta_\omega \to \mathbb{R}$. Then we have $\int(\varphi \circ f_\omega^n)\psi d\mu_\omega = \int(\bar{\varphi} \circ F_\omega^n)\bar{\psi}h_\omega dm$. Now, to apply Theorem 4.2 it is sufficient to show $\bar{\psi}h_\omega \in \mathcal{F}_\gamma^{K_\omega}$ and $\bar{\varphi} \in \mathcal{L}_\infty^{K_\omega}$. The latter is obvious, since the projection π_ω is nonsingular and hence, for $\nu_\omega-$ a.e. (x, ℓ) we have $|\bar{\varphi}(x, \ell)| \leq \|\varphi\|_{L^\infty}$. For the former one we first note that since $|(f_\omega^{R_\omega})'| \geq 2$ we have $|x - y| \leq (\frac{1}{2})^{s(x,y)}$. Hence, for any $(x, \ell), (y, \ell) \in \Delta_\omega$ we have the inequality

$$|\bar{\psi}(x, \ell) - \bar{\psi}(y, \ell)| \leq \|\psi\|_\eta(\frac{1}{2})^{\eta \cdot s(x,y)} \leq \|\psi\|_\eta(\frac{1}{2^\eta\gamma})^{s(x,y)}\gamma^{s(x,y)}. \tag{5.19}$$

Now since $s((x, \ell), (y, \ell)) = s(x, y)$ the inequality (5.19) implies

$$\begin{aligned}|(\bar{\psi}h_\omega)(x, \ell) - (\bar{\psi}h_\omega)(y, \ell)| &\leq \|\psi\|_\infty\|h_\omega\|_{\mathcal{F}_\gamma^{K_\omega}}\gamma^{s((x,\ell),(y,\ell))} \\ &+ \|h_\omega\|_{\mathcal{L}_\infty}\|\psi\|_\eta\gamma^{s((x,\ell),(y,\ell))}.\end{aligned}$$

□

5.1. **Sharp asymptotics on the measure of return-time intervals.** Although we will not need lower bound estimates on the $x_n(\omega)$ to prove the main results in this paper, it is not difficult to identify conditions (see Assumption (A2) below) under which the upper bounds from the previous section are sharp. This condition will hold for all of the examples discussed in this paper.

Notice that from equation (5.15) and the summability derived in equation (5.16), an application of Borel-Cantelli yields, for almost every ω,

$$\liminf_\ell \frac{(\log \ell)^q}{\ell[x_\ell(\omega)]^{\alpha_0}} \geq c.$$

Keeping in mind that $0 < c < c(\nu)$ was arbitrary (and working through a sequence of choices c increasing to $c(\nu)$, applying Borel-Cantelli at each step) we obtain a set $\Omega_2 \subseteq \Omega$ of full $P-$measure such that for every $\omega \in \Omega_2$ we have

[5]Recall that γ is the regularity parameter in the distortion condition (P2).

$$\limsup_{\ell} \frac{\ell^{\frac{1}{\alpha_0}} x_\ell(\omega)}{[\log \ell]^{\frac{q}{\alpha_0}}} \leq \frac{1}{[c(\nu)]^{\frac{1}{\alpha_0}}}. \tag{5.20}$$

Now we concentrate on deriving lower bounds. Using definition (5.1) and the estimate $(1+x)^{-\alpha} \geq 1-\alpha x$, valid for $\alpha > 0,\ x \geq 0$, and by substituting $x = [2x_n(\omega)]^{\alpha(\omega)}$ we obtain

$$\frac{1}{[x_n(\omega)]^{\alpha_0}} - \frac{1}{[x_{n-1}(\sigma\omega)]^{\alpha_0}} \leq \alpha_0 2^{\alpha_0} [2x_n(\omega)]^{\alpha(\omega)-\alpha_0}. \tag{5.21}$$

Iterating (5.21) we obtain

$$\begin{aligned}
\frac{(\log \ell)^q}{\ell [x_\ell(\omega)]^{\alpha_0}} &\leq \frac{(\log \ell)^q}{\ell} 2^{\alpha_0} + \alpha_0 2^{\alpha_0} \frac{(\log \ell)^q}{\ell} \sum_{k=2}^{\ell} [2x_k(\sigma^{\ell-k}\omega)]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\
&= \frac{(\log \ell)^q}{\ell} 2^{\alpha_0} + \alpha_0 2^{\alpha_0} \frac{(\log \ell)^q}{\ell} \sum_{k=2}^{\lfloor\sqrt{\ell}\rfloor} [2x_k(\sigma^{\ell-k}\omega)]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\
&+ \alpha_0 2^{\alpha_0} \frac{(\log \ell)^q (\ell - \lfloor\sqrt{\ell}\rfloor)}{\ell \log(\ell - \lfloor\sqrt{\ell}\rfloor)^q} \cdot \frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)^q}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} [2x_k(\sigma^{\ell-k}\omega)]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \\
&= (I) + (II) + (III).
\end{aligned} \tag{5.22}$$

Clearly $(I) = o(1)$ and the same is true of (II) since

$$(II) \leq \alpha_0 2^{\alpha_0} \frac{(\log \ell)^q}{\lfloor\sqrt{\ell}\rfloor}.$$

In order to estimate (III) note that from equation (5.20), for all ℓ sufficiently large (depending on ω), for $\lfloor\sqrt{\ell}\rfloor + 1 \leq k \leq \ell$,

$$\alpha_0 2^{\alpha_0} [2x_k(\sigma^{\ell-k}\omega)]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} \leq \alpha_0 2^{\alpha_0} \left[3 \frac{(\log k)^{\frac{q}{\alpha_0}}}{[c(\nu)]^{\frac{1}{\alpha_0}} k^{\frac{1}{\alpha_0}}} \right]^{\alpha(\sigma^{\ell-k}\omega)-\alpha_0} =: A'_k(\omega).$$

From now on we write $A'_k := A'_k(\omega)$. In addition to Assumption (A1) we now assume[6] the following asymptotics on the $E_\nu(A'_k)$:

Assumption (A2) (Asymptotics on expectations revisited)

$$\frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)^q}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A'_k) \to c(\nu). \tag{5.23}$$

[6]We will see that for many examples, including the ones presented in the next section, Assumption (A2) will hold.

Another large deviations estimate as in the preceding section will give, for each $c(\nu) < c$ an integer $N_2 = N_2(c)$ so that for all $\ell > N_2$, $\frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)^q}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A'_k) \leq \frac{c+c(\nu)}{2}$ and

$$P\left\{\frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)^q}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} A'_k > c\right\} \leq \exp\left[-\frac{(\ell - \lfloor\sqrt{\ell}\rfloor)[c - c(\nu)]^2}{8r_0(\log(\ell - \lfloor\sqrt{\ell}\rfloor)^{2q}}\right].$$

Once again, application of Borel-Cantelli implies there exists a random variable $n_2(\omega)$, finite almost everywhere, such that for all $\ell > n_2(\omega)$

$$\frac{(\log \ell)^q}{\ell[x_\ell(\omega)]^{\alpha_0}} \leq c' \tag{5.24}$$

and for each $0 < v < 1$ there are constants $u > 0$ and $C < \infty$ so that

$$P\{n_2(\omega) > n\} \leq Ce^{-un^v}.$$

The factor $c' > c$ in equation (5.24) is necessary to account for the two $o(1)$ terms (I) and (II) in equation (5.22). Returning to equation (5.24), another sequence of Borel-Cantelli reductions over the parameter c' decreasing to $c(\nu)$ gives a full measure set $\Omega_2 \subseteq \Omega$ such that for every $\omega \in \Omega_2$

$$\liminf_\ell \frac{\ell^{\frac{1}{\alpha_0}} x_\ell(\omega)}{[\log \ell]^{\frac{q}{\alpha_0}}} \geq \frac{1}{[c(\nu)]^{\frac{1}{\alpha_0}}}. \tag{5.25}$$

We have therefore established the following (fibre-wise, or quenched) exact asymptotics

Proposition 5.3. *For random LSV maps as described in Theorem 5.1, assuming asymptotic growth conditions (5.11) and (5.23), we have the following exact asymptotics: There is a full measure subset $\Omega' \subseteq \Omega$ such that for all $\omega \in \Omega'$, for all $n = 0, 1, 2, \ldots$ we have*

$$x_\ell(\omega) \sim \left[\frac{(\log \ell)^q}{c(\nu)\ell}\right]^{\frac{1}{\alpha_0}}.$$

5.2. **Natural probability distributions on the parameter space.** In this subsection we verify assumptions (A1) and (A2) for some natural probability distributions ν on $[\alpha_0, \alpha_1]$.

5.2.1. *Example: Discrete distribution.* Here we assume ν is a discrete probability distribution; for concreteness, $\nu = p_1\delta_{\alpha_0} + p_2\delta_{\alpha_1}$ with $p_i > 0$ and $p_1 + p_2 = 1$.

Lemma 5.4. $\frac{1}{\ell}\sum_{k=1}^{\ell} E_\nu A_k \to \alpha_0 2^{\alpha_0} p_1$. Therefore, in condition (5.11) we can take $q := 0$ and $c(\nu) := \alpha_0 2^{\alpha_0} p_1 > 0$.

Proof. Write $A_k(\omega) = X_k(\omega) - Y_k(\omega)$ where

$$X_k(\omega) = \alpha_0 2^{\alpha_0} \left[\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right]^{\alpha(\sigma^{n-k}\omega)-\alpha_0}, \tag{5.26}$$

$$Y_k(\omega) = \frac{1+\alpha_0}{2}\alpha_0 2^{\alpha_0} \left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{2\alpha(\sigma^{n-k}\omega)-\alpha_0}.$$

Using σ-invariance of P, a direct calculation shows

$$E_\nu(X_k) = \alpha_0 2^{\alpha_0} \left\{ p_1 + p_2 \left[\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right]^{\alpha_1-\alpha_0} \right\},$$

while

$$E_\nu Y_k = \frac{1+\alpha_0}{2}\alpha_0 2^{\alpha_0} \left\{ p_1 \left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{\alpha_0} + p_2 \left[\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right]^{2\alpha_1-\alpha_0} \right\}.$$

Therefore,

$$\frac{1}{\ell-1}\sum_{k=2}^{\ell} E_\nu X_k = \alpha_0 2^{\alpha_0} p_1 + O(\ell^{1-\frac{\alpha_1}{\alpha_0}}),$$

whereas

$$\frac{1}{\ell-1}\sum_{k=2}^{\ell} E_\nu Y_k = O(\ell^{-\delta})$$

for $\delta = \min\{\alpha_0/\alpha_1, 2-\alpha_0/\alpha_1\} > 0$. It follows that

$$\frac{1}{\ell}\sum_{k=1}^{\ell} E_\nu A_k = \frac{\ell-1}{\ell}\alpha_0 2^{\alpha_0} p_1 + O(\ell^{-\zeta}),$$

where $\zeta := \min\{\delta, 1-\frac{\alpha_1}{\alpha_0}\} > 0$. Therefore Assumption (5.11) is satisfied by taking $q := 0$ and $c(\nu) := \alpha_0 2^{\alpha_0} p_1 > 0$. □

We now show the upper bound on the $x_\ell(\omega)$ obtained above is sharp.

Proposition 5.5 (Sharp asymptotics for the discrete probability distribution)**.** *For almost every* ω

$$x_\ell(\omega) \sim \left[\frac{1}{c(\nu)\ell}\right]^{\frac{1}{\alpha_0}},$$

where $c(\nu) = \alpha_0 2^{\alpha_0} p_1$.

Proof. We only need to verify Assumption (A2) and apply Proposition 5.3.

$$\frac{1}{\ell-\lfloor\sqrt{\ell}\rfloor}\sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A'_k) = \frac{\alpha_0 2^{\alpha_0}}{\ell-\lfloor\sqrt{\ell}\rfloor}\sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} p_1 + p_2\left[\frac{1}{c(\nu)k}\right]^{\frac{\alpha_1}{\alpha_0}-1}$$
$$= \alpha_0 2^{\alpha_0} p_1 + O\left(\ell^{\frac{\alpha_0-\alpha_1}{2\alpha_0}}\right).$$

Since $\alpha_0 2^{\alpha_0} p_1 = c(\nu)$ we have verified Assumption (A2). □

5.2.2. *Example: Uniform distribution.* Here we assume ν is the uniform probability distribution on $[\alpha_0, \alpha_1]$, that is, for $\alpha_0 < t < \alpha_1$,

$$\nu([\alpha_0, t]) = \frac{1}{\alpha_1 - \alpha_0}(t - \alpha_0).$$

We start with a lemma that will allow us to compute the appropriate expectations in condition (5.11).

Lemma 5.6. Let $c \geq 1$. Then, as $u \to \infty$

$$E_\nu \left[e^{-(c\alpha(\omega) - \alpha_0)u} \right] \sim \frac{1}{\alpha_1 - \alpha_0} \cdot \frac{1}{cu} \cdot e^{-(c-1)\alpha_0 u}. \tag{5.27}$$

Proof. We have

$$\begin{aligned} E_\nu \left[e^{-(c\alpha(\omega) - \alpha_0)u} \right] &= \frac{1}{\alpha_1 - \alpha_0} \int_{\alpha_0}^{\alpha_1} e^{-(cx - \alpha_0)u} dx \\ &= \frac{1}{\alpha_1 - \alpha_0} \cdot \frac{1}{cu} \left[e^{-(c-1)\alpha_0 u} - e^{-(c\alpha_1 - \alpha_0)u} \right] \\ &= \frac{1}{\alpha_1 - \alpha_0} \cdot \frac{1}{cu} e^{-(c-1)\alpha_0 u} \left[1 - e^{-c(\alpha_1 - \alpha_0)u} \right]. \end{aligned}$$

□

As in the previous section, we decompose $A_k(\omega) := X_k(\omega) - Y_k(\omega)$ according to equation (5.26).

Using σ invariance of P and Lemma 5.6 with $u = -\log\left(\frac{2c_k(\alpha_0)}{k^{\frac{1}{\alpha_0}}}\right)$, $c = 1$ and $u = -\log\left(\frac{2c_k(\alpha_1)}{k^{\frac{1}{\alpha_1}}}\right)$, $c = 2$, respectively, we obtain

$$\begin{aligned} E_\nu(X_k(\omega)) &\sim \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0} \frac{1}{\log k}, \\ E_\nu(Y_k(\omega)) &= o(\log k). \end{aligned} \tag{5.28}$$

It follows that $\log k \cdot E_\nu A_k \sim \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0}$. Now apply Lemma 9.2 of the Appendix to compute the asymptotics for the sum:

$$\frac{\log \ell}{\ell} \sum_{k=1}^{\ell} E_\nu A_k \to \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0}.$$

Therefore we can take $q = 1$ and $c(\nu) = \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0}$ in Assumption (A1)

Proposition 5.7 (Sharp asymptotics for the uniform probability distribution)**.** *For almost every* ω

$$x_\ell(\omega) \sim \left[\frac{\log \ell}{c(\nu)\ell}\right]^{\frac{1}{\alpha_0}},$$

where $c(\nu) = \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0}$.

Proof. We verify Assumption (A2) and apply Proposition 5.3.

$$\frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A'_k) = \alpha_0 2^{\alpha_0} \frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)}{\ell - \lfloor\sqrt{\ell}\rfloor} \times \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu\left(\left[3\frac{\log k}{c(\nu)k}\right]^{\frac{\alpha(\sigma^{n-k_\omega})-\alpha_0}{\alpha_0}}\right).$$

We can evaluate the individual expectations using Lemma 5.6 to obtain:

$$E_\nu\left(\left[3\frac{\log k}{c(\nu)k}\right]^{\frac{\alpha(\sigma^{n-k_\omega})-\alpha_0}{\alpha_0}}\right) \sim \frac{\alpha_0}{\alpha_1 - \alpha_0}\frac{1}{\log k}.$$

Now, two applications of Lemma 9.2 from the Appendix shows

$$\sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} \frac{1}{\log k} = \sum_{k=1}^{\ell} \frac{1}{\log k} - \sum_{k=1}^{\lfloor\sqrt{\ell}\rfloor} \frac{1}{\log k} \sim \frac{\ell - 2\lfloor\sqrt{\ell}\rfloor}{\log \ell}.$$

Applying this to the first estimate gives

$$\frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)}{\ell - \lfloor\sqrt{\ell}\rfloor} \sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A'_k) \sim \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0} \frac{\log(\ell - \lfloor\sqrt{\ell}\rfloor)}{\ell - \lfloor\sqrt{\ell}\rfloor} \frac{\ell - 2\lfloor\sqrt{\ell}\rfloor}{\log \ell} \to \frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0}.$$

Since $\frac{\alpha_0^2 2^{\alpha_0}}{\alpha_1 - \alpha_0} = c(\nu)$ we have verified Assumption (A2). □

5.2.3. *Example: Quadratic distribution.* Here we assume ν is the quadratic probability distribution on $[\alpha_0, \alpha_1]$, given by

$$\nu([\alpha_0, t]) = \frac{1}{(\alpha_1 - \alpha_0)^2}(t - \alpha_0)^2.$$

Again, we begin with a simple lemma that will allow us to estimate the expectations.

Lemma 5.8. Let $c \geq 1$. Then, as $u \to \infty$

$$E_\nu\left[e^{-(c\alpha(\omega)-\alpha_0)u}\right] \sim \frac{2}{(\alpha_1 - \alpha_0)^2} \cdot \frac{1}{(cu)^2} \cdot e^{-(c-1)\alpha_0 u}. \tag{5.29}$$

Proof. We have

$$\begin{aligned} E_\nu\left[e^{-(c\alpha(\omega)-\alpha_0)u}\right] &= \frac{2}{(\alpha_1-\alpha_0)^2}\int_{\alpha_0}^{\alpha_1} e^{-(cx-\alpha_0)u}(x-\alpha_0)dx \\ &= \frac{2}{(\alpha_1-\alpha_0)^2}\left\{\frac{-1}{cu}e^{-(c\alpha_1-\alpha_0)u}(\alpha_1-\alpha_0) - \frac{1}{(cu)^2}[e^{-(c\alpha_1-\alpha_0)u} - e^{-(c-1)\alpha_0 u}]\right\} \\ &= \frac{2}{(\alpha_1-\alpha_0)^2}\cdot\frac{1}{(cu)^2}e^{-(c-1)\alpha_0 u}\left\{1-e^{-c(\alpha_1-\alpha_0)u}[cu(\alpha_1-\alpha_0)+1]\right\}. \end{aligned}$$

□

Writing $A_k = X_k - Y_k$ as in equation (5.26), using invariance of P and Lemma 5.8 with we obtain

$$\begin{aligned} E_\nu(X_k(\omega)) &\sim 2\left(\frac{\alpha_0}{\alpha_1-\alpha_0}\right)^2\frac{1}{(\log k)^2}, \\ E_\nu(Y_k(\omega)) &\sim 2\left(\frac{\alpha_1}{\alpha_1-\alpha_0}\right)^2\frac{k^{-\frac{\alpha_0}{\alpha_1}}}{(\log k)^2} = o\left(\frac{1}{(\log k)^2}\right). \end{aligned} \tag{5.30}$$

It follows that $(\log k)^2\cdot E_\nu A_k \sim 2\left(\frac{\alpha_0}{\alpha_1-\alpha_0}\right)^2$ after which an application of Lemma 9.2 of the Appendix implies

$$\frac{(\log \ell)^2}{\ell}\sum_{k=1}^{\ell} E_\nu A_k \to 2\left(\frac{\alpha_0}{\alpha_1-\alpha_0}\right)^2.$$

We take $q=2$ and $c(\nu) = 2\left(\frac{\alpha_0}{\alpha_1-\alpha_0}\right)^2$ in the assumption (5.11). For this example, lower bounds can also be obtained by essentially following the steps in the previous example and using Lemma 5.8 in place of Lemma 5.6.

Proposition 5.9 (Sharp asymptotics for the quadratic probability distribution)**.** *For almost every* ω

$$x_\ell(\omega) \sim \left[\frac{(\log \ell)^2}{c(\nu)\ell}\right]^{\frac{1}{\alpha_0}},$$

where $c(\nu) = 2\left(\frac{\alpha_0}{\alpha_1-\alpha_0}\right)^2$.

Proof. We verify Assumption (A2) and apply Proposition 5.3. The key estimates are

$$E_\nu A_k' \sim \frac{2\alpha_0^2}{(\alpha_1-\alpha_0)^2}\frac{1}{(\log k)^2}$$

and

$$\frac{\log(\ell-\lfloor\sqrt{\ell}\rfloor)^2}{\ell-\lfloor\sqrt{\ell}\rfloor}\sum_{k=\lfloor\sqrt{\ell}\rfloor+1}^{\ell} E_\nu(A_k') \sim \frac{2\alpha_0^2}{(\alpha_1-\alpha_0)^2}\frac{(\log(\ell-\lfloor\sqrt{\ell}\rfloor))^2}{\ell-\lfloor\sqrt{\ell}\rfloor}\frac{(\ell-4\lfloor\sqrt{\ell}\rfloor)}{(\log \ell)^2} \sim c(\nu),$$

where we have again used Lemma 9.2 in the Appendix to estimate the sum and verify Assumption (A2) for this example. □

6. Proof of existence of absolutely continuous mixing sample measures

6.1. **Distortion estimates.** Here we prove consequences of bounded distortion which are key for many of the later computations. For any $n \geq 1$

$$\mathcal{P}_\omega^{(n)} = \vee_{i=0}^{n-1} F_\omega^{-i} \mathcal{P}_{\sigma^i\omega} \quad \text{and} \quad \mathcal{A}_\omega^{(n)} = \{A \in \mathcal{P}_\omega^{(n)} \mid F_\omega^n A = \Delta_{\sigma^n\omega,0}\}. \tag{6.1}$$

Lemma 6.1. For any $n \geq 1$, $A \in \mathcal{A}_\omega^{(n)}$ and $x, y \in A$ the following inequality holds

$$\left| \frac{JF_\omega^n(x)}{JF_\omega^n(y)} - 1 \right| \leq D,$$

for some $D > 0$.

Proof. The collection $\mathcal{A}_\omega^{(n)}$ is a partition of $F_\omega^{-n}\Delta_{\sigma^n\omega,0}$ and for any $x \in \Delta_{\sigma^n\omega,0}$ each $A \in \mathcal{A}_\omega^{(n)}$ contains a single element of $\{F_\omega^{-n}x\}$. For $x \in A$ let $j(x)$ be the number of visits of its orbit to $\Delta_{\sigma^k\omega,0}$ up to time n. Since the images of A before time n will remain in an element of $\mathcal{P}_{\sigma^k\omega}$, all the points in A have the same itineraries, up to time n and so $j(x)$ is constant on A. Therefore $JF_\omega^n(x) = J(F_\omega^{R_\omega})^j(\tilde{x})$, for the projection $\tilde{x}$ of x onto $\Delta_{\sigma^n\omega,0}$ (i.e. if $x = (z, \ell)$ then $\tilde{x} = (z, 0)$). Thus for any $x, y \in A$ from (3.4) we obtain

$$\left| \frac{JF_\omega^n(x)}{JF_\omega^n(y)} - 1 \right| = \left| \frac{J(F_\omega^{R_\omega})^j(\tilde{x})}{J(F_\omega^{R_\omega})^j(\tilde{y})} - 1 \right| \leq D. \tag{6.2}$$

□

Corollary 6.2. For any $A \in \mathcal{A}_\omega^{(n)}$, $F_\omega^n : A \to \Delta_{\sigma^n\omega,0}$ is a bijection and for each $y \in A$ we have

$$JF_\omega^n(y) \geq \frac{m(\Delta_{\sigma^n\omega,0})}{m(A)(1+D)}. \tag{6.3}$$

Proof. Lemma 6.1 implies $JF_\omega^n(x) \leq (1+D)JF_\omega^n(y)$. Integrating both sides of this inequality with respect to x over A gives

$$m(\Delta_{\sigma^n\omega,0}) = \int_A JF_\omega^n(x)dm \leq JF_\omega^n(y)(1+D)m(A),$$

which finishes the proof. □

Lemma 6.3.

(i) There exists a constant $M_0 \geq 1$ such that for all $n \in \mathbb{N}$ and $\omega \in \Omega$,

$$\frac{d(F_\omega^n)_* m}{dm} \leq M_0 m(\Delta_\omega) \leq M_0 M.$$

(ii) Let λ_ω be a family of absolutely continuous probability measures on $\{\Delta_\omega\}$ with $\frac{d\lambda_\omega}{dm} \in \mathcal{F}_\gamma^+$. For every $A \in \mathcal{A}_\omega^{(n)}$ let $\nu_{\sigma^n\omega} = (F_\omega^n)_*(\lambda_\omega|A)$ and $\varphi_\omega = \frac{d\nu_{\sigma^n\omega}}{dm}$. There exists $C_\lambda > 0$ such that for each $\omega \in \Omega$, for all $x, y \in \Delta_{\sigma^n\omega,0}$ we have

$$\left|\frac{\varphi_\omega(x)}{\varphi_\omega(y)} - 1\right| \leq D + [e^{C_\lambda} - 1](1 + D),$$

where D is as in (3.4).

Proof. To prove the item (i) we estimate the density $d(F_\omega^n)_* m/dm$ at an arbitrary point $x \in \Delta_{\sigma^n\omega}$ and consider three different cases according to the position of x. First of all, for any $x \in \Delta_{\sigma^n\omega,0}$, from Corollary 6.2 we have

$$\begin{aligned}\frac{d(F_\omega^n)_* m}{dm}(x) = \sum_{y \in F_\omega^{-n}x} \frac{1}{JF_\omega^n(y)} &\leq (D+1) \sum_{A \in \mathcal{A}_n} \frac{m(A)}{m(\Delta_{\sigma^n\omega,0})} \\ &\leq (D+1)\frac{m(\Delta_\omega)}{m(\Delta_{\sigma^n\omega,0})}.\end{aligned}$$

Since $m(\Delta_{\sigma^n\omega,0}) = m(\Lambda) = 1$ choosing $M_0 = D + 1$ finishes the proof for the case $x \in \Delta_{\sigma^n\omega,0}$.

For $x \in \Delta_{\sigma^n\omega,\ell}$ with $\ell \geq n$ we have $F_\omega^{-n}(x) = y \in \Delta_{\omega,\ell-n}$. Since $JF_\omega(y) = 1$ for any $y \in \Delta_\omega \setminus \Delta_{\omega,0}$,

$$\frac{d(F_\omega^n)_* m}{dm}(x) = \frac{1}{JF_\omega(y) \cdots JF_{\sigma^{n-1}\omega}(F_\omega^{n-1}y)} = 1.$$

Finally, let $x \in \Delta_{\sigma^n\omega,\ell}$, for $0 < \ell < n$. Then for any $y \in F_\omega^{-n}x$ the equality $F_\omega^{n-\ell}y = F_{\sigma^{n-\ell}\omega}^{-\ell}x \in \Delta_{\sigma^{n-\ell}\omega,0}$ holds. Hence, $JF_{\sigma^{n-\ell+j}\omega}(F_\omega^j y) = 1$ for all $j = 0, \ldots, \ell - 1$. Therefore, by the chain rule we obtain $JF_\omega^n(y) = JF_\omega^{n-\ell}(y)$. Hence the problem is reduced to the first case since

$$\frac{d(F_\omega^n)_* m}{dm}(x) = \sum_{y \in F_\omega^{-n}x} \frac{1}{JF_\omega^n y} = \sum_{y \in F_\omega^{\ell-n}(F_{\sigma^{n-\ell}\omega}^{-\ell}x)} \frac{1}{JF_\omega^{n-\ell}y} = \frac{d(F_\omega^{n-\ell})_* m}{dm}(F_{\sigma^{n-\ell}\omega}^{-\ell}(x)).$$

This finishes the proof of item (i).

To prove item (ii) we first note that $F_\omega^n : A \to \Delta_{\sigma^n\omega,0}$ is invertible. So for any $x \in \Delta_{\sigma^n\omega,0}$ there is a unique $x_0 \in A$ such that $F_\omega^n(x_0) = x$ and

$$\frac{d\nu_{\sigma^n\omega}}{dm}(x) = \frac{1}{JF_\omega^n(x_0)} \frac{d\lambda_\omega}{dm}(x_0).$$

Let $\varphi_\omega = \frac{d\nu_{\sigma^n\omega}}{dm}$ then for $x, y \in \Delta_{\sigma^n\omega,0}$, using Lemma 6.1 and the assumption on $\frac{d\lambda_\omega}{dm}$ we obtain

$$\left|\frac{\varphi_\omega(x)}{\varphi_\omega(y)} - 1\right| = \left|\frac{JF_\omega^n(y_0)}{JF_\omega^n(x_0)} \frac{\frac{d\lambda_\omega}{dm}(x_0)}{\frac{d\lambda_\omega}{dm}(y_0)} - 1\right| \leq D + [e^{C_\lambda \gamma^{s(x_0,y_0)}} - 1](1 + D).$$

□

Remark 3. It is important to note that the constant C_λ does not depend on ω. Moreover, if A and n are such that the orbits $F^j(x_0), F^j(y_0), j = 1, 2, \dots n$ see sufficiently many returns to the base, so that $C_\lambda \gamma^{s(x_0,y_0)} \leq \log 2$, then the upper bound in (ii) becomes $2D + 1$. The elements A for which this holds are independent of the starting measure λ.

6.2. **Proof of Theorem 4.1.**

Proof of Existence. Recall that $m(\Lambda) = 1$. Recall the definitions of $\mathcal{P}_\omega^{(j)}$ and $\mathcal{A}_\omega^{(j)}$ in (6.1), and for $A \in \mathcal{P}^{(j)}_{\sigma^{-j}\omega} | \Delta_{\sigma^{-j}\omega,0}$ let

$$\phi^\omega_{j,A} = \frac{d}{dm} \left(F^j_{\sigma^{-j}\omega}\right)_* (m_A),$$

where $m_A(B) = m(A \cap B)$. Clearly, $\phi^\omega_{j,A}$ is a density on Δ_ω, such that $\phi^\omega_{j,A} | \Delta_{\omega,\ell} \equiv 0$ for $\ell > j$. Below we consider two cases depending on A. First, notice that if $A \in \mathcal{A}^{(j)}_{\sigma^{-j}\omega}$ then $F^j_{\sigma^{-j}\omega} : A \to \Delta_{\omega,0}$ is a bijection. For $x, y \in \Delta_{\omega,0}$, let $x', y' \in A$ be such that $F^j_{\sigma^{-j}\omega}(x') = x$, and $F^j_{\sigma^{-j}\omega}(y') = y$. By the choice of A there exists i such that $F^j_{\sigma^{-j}\omega} = (F^{R_\omega}_{\sigma^{-j}\omega})^i$. The bounded distortion condition implies that

$$\left| \log \frac{\phi^\omega_{j,A}(y)}{\phi^\omega_{j,A}(x)} \right| = \left| \log \frac{JF^j_{\sigma^{-j}\omega}(x')}{JF^j_{\sigma^{-j}\omega}(y')} \right| \leq \sum_{\ell=0}^{i-1} D\gamma^{s(x,y)+(i-\ell)-1} \leq \frac{D}{1-\gamma} \gamma^{s(x,y)}. \tag{6.4}$$

Notice that the constant in equation (6.4) is independent of j, A and ω. Hence for all $x, y \in \Delta_\omega$ letting $D' = e^{\frac{D}{1-\gamma}}$ we have

$$\phi^\omega_{j,A}(y) \leq D' \phi^\omega_{j,A}(x).$$

Integrating both sides of the latter inequality over $\Delta_{\omega,0}$ with respect to x implies

$$\phi^\omega_{j,A}(y) \leq D' \frac{m(A)}{m(\Lambda)} = D' m(A). \tag{6.5}$$

On the other hand, if $A \in \mathcal{P}^{(j)}_{\sigma^{-j}\omega} | \Delta_{\sigma^{-j}\omega,0}$ such that $F^j_{\sigma^{-j}\omega}(A) \subset \Delta_{\omega,\ell}$ for $\ell > 0$ then $\phi^\omega_{j,A}(x) = \phi^{\sigma^{-\ell}\omega}_{j-\ell,A}(F^{-\ell}_{\sigma^{-\ell}\omega}(x))$. Hence we can apply (6.5). Futher define

$$\phi^\omega_n = \frac{d}{dm} \left(\frac{1}{n} \sum_{j=0}^{n-1} (F^j_{\sigma^{-j}\omega})_* (m | \Delta_{\sigma^{-j}\omega,0}) \right).$$

As above $\phi^\omega_n | \Delta_{\omega,\ell} \equiv 0$ for $\ell > j$. For $x \in \Delta_{\omega,\ell}$, $\ell \leq j$ we write ϕ_n as a convex combination of $\phi^\omega_{j,A}$ and obtain

$$\phi^\omega_n(x) \leq D'. \tag{6.6}$$

Notice that if $\phi_n^\omega(x) = 0$ and $\phi_n^\omega(y) \neq 0$ then $s(x,y) = 0$. Taking this into account for all $y \in \Delta_\omega$ such that $\phi_n^\omega(y) \neq 0$ we have

(6.7)

$$|\phi_n^\omega(x) - \phi_n^\omega(y)| \leq \frac{1}{n}\sum_{j=0}^{n-1} \sum_{A \in \mathcal{P}^{(j)}_{\sigma^{-j}\omega} | \Delta_{\sigma^{-j}\omega,0}} |\phi_{j,A}^\omega(y)| \left| \frac{\phi_{j,A}^\omega(x)}{\phi_{j,A}^\omega(y)} - 1 \right|$$

$$\leq D' \frac{(1-\gamma)(D'-1)}{D} \frac{1}{n}\sum_{j=0}^{n-1} \sum_{A \in \mathcal{P}^{(j)}_{\sigma^{-j}\omega} | \Delta_{\sigma^{-j}\omega,0}} \left| \log \frac{\phi_{j,A}^\omega(y)}{\phi_{j,A}^\omega(x)} \right| m(A) \leq D'(D'-1)\gamma^{s(x,y)},$$

where we have used equation (6.4) in the last step.

Hence $\phi_n^\omega \in \mathcal{F}_\gamma^+ \cap \mathcal{F}_\gamma^1$ (i.e. $K_\omega \equiv 1$). Since, $d(x,y) := \gamma^{s(x,y)}$ defines separable metric space structure on Δ_ω for each ω, by Arzela-Ascoli Theorem there exists a subsequence of $\phi_n^\omega \to \tilde{h}_\omega \in \mathcal{F}_\gamma^+ \cap \mathcal{F}_\gamma^1$. By a diagonal argument we construct along the sequence $\{\sigma^k\omega\}$, for almost every $\omega \in \Omega$, a convergent subsequence $\{\phi_{n_k}^{\sigma^\ell\omega} dx\}$ for every $\ell \in \mathbb{Z}$. The limiting measure is F_ω-equivariant i.e. $(F_{\sigma^k\omega})_* \nu_{\sigma^k\omega} = \nu_{\sigma^{k+1}\omega}$. Moreover, $h_\omega := \frac{d\nu_\omega}{dm} \in \mathcal{F}_\gamma^+ \cap \mathcal{F}_\gamma^K$ and $h_\omega \leq \frac{e^{-4C'}}{m(\Delta_\omega)}$ by construction. □

7. RANDOM COUPLING

7.1. **Estimates on the random recurrence times for the base.** For a single map, the recurrence time gives a key construction parameter for coupling arguments. In the setting of random maps, this recurrence time is ω dependent. Our first task is to obtain a suitable version of the recurrence time (see ℓ_0 below, and its use in the following Lemma 7.2). At this stage, it is useful for the reader to recall from section 3 our assumptions (P1)-P(7); in particular that $a > 1$ in (P4). Moreover, recall the regularity class of the equivariant densities defined by the random variable K_ω from Theorem 4.1.

Lemma 7.1. Let N and t_i be from the aperiodicity condition (P5). There is an $\ell_0 \in \mathbb{N}$ so that for every $\ell > \ell_0$ there are nonnegative integers c_i such that

$$\ell = \sum_{i=1}^{N} c_i t_i.$$

Proof. See Lemma A2 [25]. □

For $\ell \in \mathbb{N}$ define a random variable $V^\ell : \Omega \to \mathbb{R}$ by

$$V_\omega^\ell = m(\Delta_{\omega,0} \cap F_\omega^{-\ell}(\Delta_{\sigma^\ell\omega,0})).$$

Recall that every base $\Delta_{\omega,0} = \Lambda$.

Lemma 7.2. For each $\ell > \ell_0$ there is a constant $V(\ell) > 0$ so that for almost every $\omega \in \Omega$,

$$V_\omega^\ell \geq V(\ell).$$

Proof. The result follows from the aperiodicity condition (P5) and bounded distortion (P2). First, suppose $F_1 = F_\omega^{j_1} : \Lambda \to \Delta_{\sigma^{j_1}\omega}$ and $F_2 = F_{\sigma^{j_1}\omega}^{j_2} : \Lambda \to \Delta_{\sigma^{j_1+j_2}\omega}$, satisfying

$$\frac{m(F_i^{-1}\Lambda \cap \Lambda)}{m(\Lambda)} \geq \epsilon_i > 0,\ i = 1, 2.$$

Then, since F_i^{-1} are bijections when restricted to Λ, using bounded distortion, we get

$$\begin{aligned}\frac{m((F_2 \circ F_1)^{-1}\Lambda \cap \Lambda)}{m(\Lambda)} &\geq \frac{m(F_1^{-1}(F_2^{-1}\Lambda \cap \Lambda) \cap \Lambda)}{m(F_1^{-1}\Lambda \cap \Lambda)} \cdot \frac{m(F_1^{-1}\Lambda \cap \Lambda)}{m(\Lambda)} \\ &\geq \frac{1}{D}\frac{m(F_2^{-1}\Lambda \cap \Lambda)}{m(\Lambda)} \cdot \frac{m(F_1^{-1}\Lambda \cap \Lambda)}{m(\Lambda)} \\ &\geq \frac{1}{D}\epsilon_2 \cdot \epsilon_1.\end{aligned}$$

Now, for $\ell > \ell_0$, Lemma 7.1 implies F_ω^ℓ can be written as a composition of $F_{\omega_i}^{t_i}$ (with at most ℓ terms). Iterating the above estimate and using the lower bounds given in condition (P5) implies the existence of the required lower bound $V(\ell) > 0$. □

Remark 4. From the proof of the previous lemma, it is clear that one should not expect a lower bound on the $V(\ell)$, uniform over all ℓ.

7.2. **Random stopping times.** Let $\Delta = \{(\omega, x) | \omega \in \Omega, x \in \Delta_\omega\}$. Denote by $\Delta \otimes_\omega \Delta$ the relative product over Ω, that is $\Delta \otimes_\omega \Delta = \{(\omega, x, x') | \omega \in \Omega, x, x' \in \Delta_\omega\}$. These are measurable subsets of the appropriate product spaces ($\Omega \times \Lambda \times \mathbb{N}$ in the case of Δ, for example), and naturally carry the measures $P \times m$ and $\mathbb{P} := P \times m \times m$ respectively. We can lift the tower map F to a product action on $\Delta \otimes_\omega \Delta$ with the property $F_\omega \times F_\omega : \Delta_\omega \times \Delta_\omega \to \Delta_{\sigma\omega} \times \Delta_{\sigma\omega}$ by applying F in each of the x, x' coordinates.

With respect to this map, we define *auxiliary stopping times* $\tau_1^\omega < \tau_2^\omega < \ldots$ to the base as follows:

Let ℓ_0 be the constant given in Lemma 7.1. For $(\omega, x, x') \in \Delta \otimes_\omega \Delta$ set

$$\begin{aligned}\tau_1^\omega(x, x') &= \inf\{n \geq \ell_0 \mid F_\omega^n x \in \Delta_{\sigma^n\omega,0}\}; \\ \tau_2^\omega(x, x') &= \inf\{n \geq \tau_1^\omega(x, x') + \ell_0 \mid F_\omega^n x' \in \Delta_{\sigma^n\omega,0}\}; \\ \tau_3^\omega(x, x') &= \inf\{n \geq \tau_2^\omega(x, x') + \ell_0 \mid F_\omega^n x \in \Delta_{\sigma^n\omega,0}\}; \\ \tau_4^\omega(x, x') &= \inf\{n \geq \tau_3^\omega(x, x') + \ell_0 \mid F_\omega^n x' \in \Delta_{\sigma^n\omega,0}\};\end{aligned}$$

and so on, with the action alternating between x and x'. Notice that for odd i's the first (resp. for even i's the second) coordinate of $(F_\omega \times F_\omega)^{\tau_i^\omega}(x, x')$ makes a return to $\Delta_{\sigma^{\tau_i^\omega}\omega,0}$.

Let $i \geq 2$ be the smallest integer such that $(F_\omega \times F_\omega)^{\tau_i^\omega}(x, x') \in \Delta_{\sigma^{\tau_i^\omega}\omega,0} \times \Delta_{\sigma^{\tau_i^\omega}\omega,0}$. Then we define the *stopping time* T_ω by

$$T_\omega(x, x') := \tau_i^\omega(x, x').$$

Next define a sequence of partitions $\xi_1^\omega \prec \xi_2^\omega \prec \xi_3^\omega \prec ...$ of $\Delta_\omega \times \Delta_\omega$ so that τ_i^ω is constant on the elements of ξ_j^ω for all $i \leq j$, $i, j \in \mathbb{N}$. Given a partition $\mathcal{Q}$ of Δ_ω we write $\mathcal{Q}(x)$ to denote the element of $\mathcal{Q}$ containing x. With this convention, we let

$$\xi_1^\omega(x, x') = \left(\bigvee_{k=0}^{\tau_1^\omega - 1} F_\omega^{-k} \mathcal{P}_{\sigma^k \omega} \right)(x) \times \Delta_\omega.$$

Letting $\pi : \Delta_\omega \times \Delta_\omega \to \Delta_\omega$ be the projection to the first coordinate, we define

$$\xi_2^\omega(x, x') = \pi \xi_1^\omega(x, x') \times \left(\bigvee_{k=0}^{\tau_2^\omega - 1} F_\omega^{-k} \mathcal{P}_{\sigma^k \omega} \right)(x').$$

Let π' be the projection onto the second coordinate. We define ξ_3^ω by refining the partition on the first coordinate, and so on. If ξ_{2i}^ω is defined then we define ξ_{2i+1}^ω by refining each element of ξ_{2i}^ω in the first coordinate so that τ_{2i+1}^ω is constant on each element of ξ_{2i+1}^ω. Similarly ξ_{2i+2}^ω is defined by refining each element of ξ_{2i+1}^ω in the second coordinate so that τ_{2i+2}^ω is constant on each new partition element. Now we define a partition $\hat{\mathcal{P}}_\omega$ of $\Delta_\omega \times \Delta_\omega$ such that T_ω is constant on its element. For definiteness suppose that i is even and choose $\Gamma \in \xi_i^\omega$ such that $T_\omega|_\Gamma > \tau_{i-1}^\omega$. By construction $\Gamma = A \times B$ such that $F^{\tau_i^\omega}(B) = \Delta_{\sigma^{\tau_i^\omega}, 0}$ and $F^{\tau_i^\omega} A$ is spread around $\Delta_{\sigma^{\tau_i^\omega} \omega}$. We refine A into countably many pieces and choose those parts which are mapped onto the corresponding base at time τ_i^ω. Note that $\{T_\omega = \tau_i^\omega\}$ may not be measurable with respect to ξ_i^ω. However, since $\tau_{i+1}^\omega \geq \ell_0 + \tau_i^\omega$ and ξ_{i+1}^ω is defined by dividing A into pieces where τ_{i+1}^ω is constant, $\{T_\omega = \tau_i^\omega\}$ is measurable with respect to ξ_{i+1}^ω.

7.3. **Tail of the simultaneous return times.** In this section we estimate the tail of the simultaneous return time T_ω. We start this section with the lower bound on the measure of the set that made return at time τ_i^ω.

Lemma 7.3. Let λ_ω and λ'_ω be two probability measures on $\{\Delta_\omega\}$ with densities $\varphi_\omega, \varphi'_\omega \in \mathcal{F}_\gamma^+ \cap \mathcal{L}_\infty^{K_\omega}$. Let $\tilde{\lambda} = \lambda_\omega \times \lambda'_\omega$. For each ω, for each $i \geq 2$ and $\Gamma \in \xi_i^\omega$ such that $T_\omega|_\Gamma > \tau_{i-1}^\omega$ we have

$$\tilde{\lambda}\{T_\omega = \tau_i^\omega | \Gamma\} \geq C_{\tilde{\lambda}} V_{\sigma^{\tau_{i-1}^\omega} \omega}^{\tau_i^\omega - \tau_{i-1}^\omega}.$$

where $0 < C_{\tilde{\lambda}} < 1$. We can fix $C_{\tilde{\lambda}} = \frac{2D+1}{2(D+1)^2}$, independent of $\tilde{\lambda}$, for all i sufficiently large, i.e. $i \geq i_0(\tilde{\lambda})$.

Proof. In the proof we write λ for a measure on Δ_ω since the dependence on ω is clear from the context. For definiteness assume i is even. Then $\Gamma \in \xi_i^\omega$ has the property $F_\omega^{\tau_{i-1}^\omega}(\pi\Gamma) = \Delta_{\sigma^{\tau_{i-1}^\omega} \omega, 0}$ and $F_\omega^{\tau_i^\omega}(\pi'\Gamma) = \Delta_{\sigma^{\tau_i^\omega} \omega, 0}$. Together with the definition of T_ω this implies $\pi'\{T_\omega = \tau_i^\omega | \Gamma\} \cap \Gamma = \pi'\Gamma$. Therefore, letting $\nu_{\sigma^{\tau_{i-1}^\omega} \omega} := (F_\omega^{\tau_{i-1}^\omega})_*(\lambda | \pi\Gamma)$

we have

$$\tilde{\lambda}\{T_\omega = \tau_i^\omega|\Gamma\} = \frac{\lambda(\pi\{T_\omega = \tau_i^\omega\} \cap \pi\Gamma)}{\lambda(\pi\Gamma)} = \frac{\lambda(F_\omega^{-\tau_i^\omega}\Delta_{\sigma^{\tau_i^\omega}\omega,0} \cap F_\omega^{-\tau_{i-1}^\omega}\Delta_{\sigma^{\tau_{i-1}^\omega}\omega,0} \cap \pi\Gamma)}{\lambda(\pi\Gamma)}$$
$$= (F_\omega^{\tau_{i-1}^\omega})_*(\lambda|\pi\Gamma)(F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_{i-1}^\omega - \tau_i^\omega}\Delta_{\sigma^{\tau_i^\omega}\omega,0} \cap \Delta_{\sigma^{\tau_{i-1}^\omega}\omega,0}) = \nu_{\sigma^{\tau_{i-1}^\omega}\omega}(F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_{i-1}^\omega - \tau_i^\omega}\Delta_{\sigma^{\tau_i^\omega}\omega,0} \cap \Delta_{\sigma^{\tau_{i-1}^\omega}\omega,0}).$$

Finally, item (ii) of Lemma 6.3 applied to $\nu_{\sigma^{\tau_{i-1}^\omega}\omega}$ implies that

$$\tilde{\lambda}\{T_\omega = \tau_i^\omega|\Gamma\} \geq \frac{1}{1 + D + C_\lambda(D+1)} m(F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_{i-1}^\omega - \tau_i^\omega}\Delta_{\sigma^{\tau_i^\omega}\omega,0} \cap \Delta_{\sigma^{\tau_{i-1}^\omega}\omega,0}).$$

Now, the lemma holds with $C_{\tilde{\lambda}} = \min\{\frac{1}{(1+C_\lambda)(D+1)}, \frac{1}{(1+C_{\lambda'})(D+1)}\}$. In view of Remark 3 we can use $C_{\tilde{\lambda}} = \frac{2D+1}{2(D+1)^2}$ for all i sufficiently large. □

The next lemma estimates the distribution of τ_i^ω's on $\Delta_\omega \times \Delta_\omega$ by the measure of the tail of the random tower.

Lemma 7.4. Let $C_{\tilde{\lambda}}$ be as in Lemma 7.3. For each ω, for each i and $\Gamma \in \xi_i^\omega$

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega > \ell_0 + n|\Gamma\} \leq M_0 M C_{\tilde{\lambda}}^{-1} \cdot m\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} > n\}.$$

Proof. Suppose that i is even. Since τ_i^ω is constant on the elements of ξ_i^ω for every $\Gamma \in \xi_i^\omega$ we have

$$\pi\left(\{(x,x')|\hat{R}_\omega \circ F_\omega^{\tau_i^\omega + \ell_0}(x) > n\} \cap \Gamma\right) = \{x|\hat{R}_\omega \circ F_\omega^{\tau_i^\omega + \ell_0}(x) > n\} \cap \pi\Gamma.$$

Letting $\nu_{\sigma^{\tau_{i-1}^\omega}\omega} = (F_\omega^{\tau_{i-1}^\omega})_*(\lambda|\pi\Gamma)$, we have

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega - \ell_0 > n|\Gamma\} = \tilde{\lambda}\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_\omega^{\tau_i^\omega + \ell_0} > n|\Gamma\}$$
$$= \frac{\lambda(\pi\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_\omega^{\tau_i^\omega + \ell_0} > n\} \cap \pi\Gamma)}{\lambda(\pi\Gamma)} = (\lambda|\pi\Gamma)\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_\omega^{\tau_i^\omega + \ell_0} > n\}$$
$$= (F_\omega^{\tau_{i-1}^\omega})_*(\lambda|\pi\Gamma)\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_i^\omega - \tau_{i-1}^\omega + \ell_0} > n\} = \nu_{\sigma^{\tau_{i-1}^\omega}\omega}\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_i^\omega - \tau_{i-1}^\omega + \ell_0} > n\}.$$

Applying item (ii) of Lemma 6.3 we obtain

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega - \ell_0 > n|\Gamma\} \leq (1 + D + C_\lambda(1+D)) m\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} \circ F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_i^\omega - \tau_{i-1}^\omega + \ell_0} > n\}.$$

Finally, since the density of $(F_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_i^\omega - \tau_{i-1}^\omega + \ell_0})_* m$ is bounded above by the first item of Lemma 6.3 we have

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega > \ell_0 + n|\Gamma\} \leq M_0 M(1 + D + C_\lambda(1+D)) m\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} > n\}.$$

For i odd the calculation is analogous and we obtain for all i

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega > \ell_0 + n|\Gamma\} \leq M_0 M C_{\tilde{\lambda}}^{-1} m\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} > n\}.$$

□

Suppose we are given a sequence of positive integers $\ell_0 \leq \tau_1 < \tau_2 < \cdots < \tau_n < \ldots$ with $\tau_i - \tau_{i-1} \geq \ell_0$ for all $i \geq 2$, denoted $\vec{\tau}$, and a positive integer $q > 0$. Define associated subsets of $\Delta \otimes_\omega \Delta$

$$G_q(\vec{\tau}) = \{(\omega, x, x') | \tau_i^\omega(x, x') = \tau_i,\ i = 1, 2, \ldots q\}.$$

This is a *partition* into sets where a specified sequence of hitting times up to q is attained:

$$\cup_{\vec{\tau}} G_q(\vec{\tau}) = \cup_{\tau_1 < \tau_2 < \ldots \tau_q} G_q(\vec{\tau}) = \Delta \otimes_\omega \Delta.$$

For fixed ω denote $G_q^\omega(\vec{\tau}) = G_q(\vec{\tau}) \cap (\Delta_\omega \times \Delta_\omega)$, the cross section of $G_q(\vec{\tau})$ at ω. Let $G_q^\omega = \{(x, x') \in \Delta_\omega \times \Delta_\omega \mid \tau_j^\omega(x, x') = \tau_j, j = 1, \ldots, q\}$. The following lemma gives a useful estimate on the size of the elements $G_q(\vec{\tau})$.

Lemma 7.5. There exists a $C > 0$ such that for each fixed $\vec{\tau}$, $q > 0$

$$\mathbb{P}(G_q(\vec{\tau})) \leq C^q \mathbb{P}\{\tau_1^\omega(x, x') = \tau_1\} \prod_{j=2}^{q} \gamma(\tau_j - \tau_{j-1}).$$

where

$$\gamma(\tau_{i+1} - \tau_i) = \int_\Omega m\{x \in \Delta_\omega | \, \hat{R}_\omega = \tau_{i+1} - \tau_i - \ell_0\} dP(\omega).$$

Proof. Assume first that q is even and let G_q^ω be as above. Let $k_j = \tau_j - \tau_{j-1} - \ell_0$, $j = 1, \ldots, q$. We first show that

$$m \times m(G_q^\omega) \leq Dm \times m(G_{q-1}^\omega) m \times m\{\hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{q-2}}\omega}^{\tau_{q-1}-\tau_{q-2}+\ell_0} = k_q\}.$$

Indeed, for any $\Gamma_{q-1}^\omega \in \xi_{q-1}^\omega$ with $\tau_j^\omega|_{\Gamma_{q-1}^\omega} = \tau_j$ we have

$$\begin{aligned}
&\frac{m \times m(G_q^\omega \cap \Gamma_{q-1}^\omega)}{m \times m(\Gamma_{q-1}^\omega)} = \frac{m \times m(\{\tau_q^\omega = \tau_q\} \cap \Gamma_{q-1}^\omega)}{m \times m(\Gamma_{q-1}^\omega)} \\
&= \frac{m(\pi'\{\tau_q^\omega = \tau_q\} \cap \Gamma_{q-1}^\omega)}{m(\pi' \Gamma_{q-1}^\omega)} \leq D \frac{m(F_\omega^{\tau_{q-2}}(\pi'\{\tau_q^\omega = \tau_q\} \cap \Gamma_{q-1}^\omega))}{m(\Lambda)} \\
&\leq Dm(\Delta_{\sigma^{\tau_{q-2}}\omega, 0} \cap F_\omega^{\tau_{q-2}}\{\hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_\omega^{\tau_{q-1}+\ell_0} = k_q\}) \\
&= Dm\{\Delta_{\sigma^{\tau_{q-2}}\omega, 0} \cap \hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{q-2}}\omega}^{\tau_{q-1}-\tau_{q-2}+\ell_0} = k_q\} \\
&\leq Dm\{\hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{q-2}}\omega}^{\tau_{q-1}-\tau_{q-2}+\ell_0} = k_q\}.
\end{aligned}$$

Hence we have

$$\frac{m \times m(G_q^\omega \cap \Gamma_{q-1}^\omega)}{m \times m(\Gamma_{q-1}^\omega)} \leq Dm\{\hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{q-2}}\omega}^{\tau_{q-1}-\tau_{q-2}+\ell_0} = k_q\}.$$

Therefore,

$$\begin{aligned}
m \times m(G_q^\omega) &= \sum_{\Gamma_{q-1}^\omega \in \xi_{q-1}^\omega} m \times m(\Gamma_{q-1}^\omega) \frac{m \times m(G_q^\omega \cap \Gamma_{q-1}^\omega)}{m \times m(\Gamma_{q-1}^\omega)} \\
&\leq Dm \times m(G_{q-1}^\omega) m\{\hat{R}_{\sigma^{\tau_{q-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{q-2}}\omega}^{\tau_{q-1}-\tau_{q-2}+\ell_0} = k_q\}.
\end{aligned}$$

By induction, for any $q > 2$, we have

$$m \times m(G_q^\omega) \leq D^{q-2} m \times m(G_1^\omega) \prod_{j=2}^{q} m\{\hat{R}_{\sigma^{\tau_{j-1}+\ell_0}\omega} \circ F_{\sigma^{\tau_{j-2}}\omega}^{\tau_{j-1}-\tau_{j-2}+\ell_0} = k_j\}.$$

A similar argument applies to obtain the same formula when q is odd. Now by (i) of Lemma 6.3, we get

$$m \times m(G_q^\omega) \leq (DM_0M)^{q-2} m \times m(G_1^\omega) \prod_{j=2}^{q} m\{\hat{R}_{\sigma^{\tau_{j-1}+\ell_0}\omega} = k_j\}. \tag{7.1}$$

Notice that, $m\{\hat{R}_{\sigma^{\tau_{j-1}+\ell_0}\omega} = k_j\}$ depends only on $\omega_{\tau_{j-1}+\ell_0}, \ldots, \omega_{\tau_{j-1}+\ell_0+k_j-1}$ while $m\{\hat{R}_{\sigma^{\tau_j+\ell_0}\omega} = k_{j+1}\}$ depends only on $\omega_{\tau_{j+\ell_0}}, \ldots, \omega_{\tau_j+\ell_0+k_{j+1}-1}$. By definition of k_j, we have $\tau_{j-1}+\ell_0+k_j-1 = \tau_j - 1 < \tau_j + \ell_0$. Therefore, the product on the right hand side of (7.1) is formed of independent random variables. Moreover, observe that $m \times m(G_1^\omega)$ depends only on $\omega_{\ell_0}, \ldots \omega_{\tau_1-1}$ and $\tau_1 - 1 < \tau_1 + \ell_0$. Thus,

$$\int_\Omega m \times m(G_q^\omega) dP \leq (DM_0M)^{q-2} \int_\Omega m \times m(G_1^\omega) dP \prod_{j=2}^{q} \int_\Omega m\{\hat{R}_{\sigma^{\tau_{j-1}+\ell_0}\omega} = k_j\} dP.$$

Since σ is P invariant, taking $C^q := (DM_0M)^{q-2}$ gives the desired estimate. □

We now present two lemmas that will be invoked in the proof of Proposition 7.8 below.

Lemma 7.6. We have $\sum_{\tau_{j+1}-\tau_j=K}^{\infty} \gamma(\tau_{j+1} - \tau_j) \leq C(K) < \infty$. Moreover, $C(K) \to 0$ as $K \to \infty$.

Proof. Using assumption (P7) and Lemma 9.1 in the appendix, we have

$$\sum_{\tau_{j+1}-\tau_j=K}^{\infty} \gamma(\tau_{j+1} - \tau_j) \leq \sum_{\tau_{j+1}-\tau_j=K}^{\infty} C \frac{\log(\tau_{j+1} - \tau_j - \ell_0)^{\hat{b}}}{(\tau_{j+1} - \tau_j - \ell_0)^a} \leq \frac{C' \log(K)^{\hat{b}}}{(K - \ell_0)^{a-1}} := C(K). \tag{7.2}$$

Moreover; $C(K) \to 0$ as $K \to \infty$. □

Lemma 7.7. We have $\sum_{k \geq \ell_0}^{\infty} \mathbb{P}\{\tau_1^\omega(x, x') = k\} < \infty$.

Proof. Recall that $\tau_1^\omega(x,x') = \ell_0 + \hat{R}_{\sigma^{\ell_0}\omega} \circ F_\omega^{\ell_0}(x)$; i.e., $\tau_1^\omega(x,x')$ does not depend on x'. Therefore,

$$\sum_{k\geq \ell_0}^{\infty} \mathbb{P}\{\tau_1^\omega(x,x') = k\} = \sum_{k\geq 0}^{\infty} P \times m\{\hat{R}_{\sigma^{\ell_0}\omega} \circ F_\omega^{\ell_0}(x) = k\} \times m(\Delta_\omega)$$

$$\leq M \sum_{k\geq 0}^{\infty} P \times m\{\hat{R}_{\sigma^{\ell_0}\omega} \circ F_\omega^{\ell_0}(x) = k\} \leq M^2 M_0 \sum_{k\geq 0}^{\infty} P \times m\{\hat{R}_\omega = k\}$$

$$\leq M^2 M_0 C \sum_{k\geq 0} \frac{(\log k)^{\hat{b}}}{k^a} < \infty,$$

where we have used the first item of Lemma 6.3 and (P7). □

We can now present the main result of this section.

Proposition 7.8. *Let $\delta > 0$ be given. Let λ_ω and λ'_ω be two two families of probability measures on $\{\Delta_\omega\}$ with densities $\varphi, \varphi' \in \mathcal{F}_\gamma^+ \cap \mathcal{L}_\infty^{K_\omega}$. Let $\tilde{\lambda} := \lambda_\omega \times \lambda'_\omega$. Then there exists a constant $\hat{C}_{\tilde{\lambda}}$ and a subset $\Omega_5 \subset \Omega$ of full measure and a random variable n_5 on Ω_5 such that for any $n(\omega) > n_5(\omega)$ the following holds*

$$\tilde{\lambda}\{T_\omega > n\} \leq \hat{C}_{\tilde{\lambda}} \frac{(\log n)^b}{n^{a-1-\delta}}.$$

Moreover, there exist $C > 0$, $u' > 0, 0 < v' < 1$ such that for any n

$$P\{\omega \mid n_5(\omega) > n\} \leq Ce^{-u'n^{v'}}.$$

Proof. Let $c := \min\{\frac{\delta}{a+1}, 1/2\}$. For a.e. $\omega \in \Omega$ we have[7]

$$\begin{aligned} \tilde{\lambda}\{T_\omega > n\} &\leq \sum_{i<\lfloor n^c \rfloor} \tilde{\lambda}\{T_\omega > n; \tau_{i-1}^\omega \leq n < \tau_i^\omega\} + \tilde{\lambda}\{T_\omega > n; \tau_{\lfloor n^c \rfloor}^\omega \leq n\} \\ &\leq \sum_{i<\lfloor n^c \rfloor} \tilde{\lambda}\{\tau_{i-1}^\omega \leq n < \tau_i^\omega\} + \tilde{\lambda}\{T_\omega > n; \tau_{\lfloor n^c \rfloor}^\omega \leq n\} \\ &=: Y_1 + Y_2. \end{aligned} \tag{7.3}$$

We will show that the term Y_1 decays at the indicated log-polynomial rate (in n) while the term Y_2 decays as stretched exponential, which implies the result. First, for the term

[7] Notice that we have chosen $q = n^c$ to keep the proof and the estimates of Y_1, Y_2 and C_ω as simple as possible. One may try $q = (\log n)^d$ for sufficiently large d so that Y_2 decays faster than Y_1 and C_ω remains integrable and get a quenched decay rate of the form $\frac{(\log n)^{b+d'}}{n^{a-1}}$, for some $d' \geq d$, in Theorem 4.2. However, no matter how we choose $q = g(n)$, with $g(n) \to \infty$ as $n \to \infty$, a quenched correlation decay rate of the form $\frac{(\log n)^b}{n^{a-1}}$, which is analogous to what one expects in the deterministic setting, cannot be achieved since we want to get information on the integrability of the C_ω in Theorem 4.2. The shift of the Lipschitz constant K_ω, and hence the dependence of that constant on n, in equation (8.7) and the non-uniformity of the tail in (P4) are the main reasons for getting a rate at the order $\frac{1}{n^{a-1+\delta}}$, for any $\delta > 0$.

Y_1 we have:

$$\begin{aligned}\sum_{i<\lfloor n^c\rfloor}\tilde{\lambda}\{\tau_{i-1}^{\omega}\le n<\tau_i^{\omega}\} &= \sum_{i<\lfloor n^c\rfloor}\sum_{\Gamma\in\xi_{i-1}^{\omega}}\tilde{\lambda}\{\tau_{i-1}^{\omega}\le n<\tau_i^{\omega}\mid\Gamma\}\tilde{\lambda}(\Gamma)\\ &= \sum_{i<\lfloor n^c\rfloor}\sum_{\substack{\Gamma\in\xi_{i-1}^{\omega}\\ \tau_{i-1}^{\omega}|\Gamma\le n}}\tilde{\lambda}\{\tau_{i-1}^{\omega}\le n<\tau_i^{\omega}\mid\Gamma\}\tilde{\lambda}(\Gamma)\\ &\le \sum_{i<\lfloor n^c\rfloor}\sum_{\substack{\Gamma\in\xi_{i-1}^{\omega}\\ \tau_{i-1}^{\omega}|\Gamma\le n}}\sum_{j=1}^{i}\tilde{\lambda}\{\tau_j^{\omega}-\tau_{j-1}^{\omega}\ge\frac{n}{i}\mid\Gamma\}\tilde{\lambda}(\Gamma).\end{aligned}\tag{7.4}$$

For each term in the sum (7.4), using Lemma 7.4 we obtain,

$$\begin{aligned}\tilde{\lambda}\{\tau_j^{\omega}-\tau_{j-1}^{\omega}\ge\frac{n}{i}\mid\Gamma\}\tilde{\lambda}(\Gamma) &= \tilde{\lambda}\{\tau_j^{\omega}-\tau_{j-1}^{\omega}\ge(\frac{n}{i}-\ell_0)+\ell_0|\Gamma\}\tilde{\lambda}(\Gamma)\\ &\le MM_0C_{\tilde{\lambda}}^{-1}m\{\hat{R}_{\sigma^{\tau_{j-1}^{\omega}+\ell_0}\omega}>\frac{n}{i}-\ell_0\}\tilde{\lambda}(\Gamma)\\ &\le MM_0C_{\tilde{\lambda}}^{-1}\tilde{\lambda}(\Gamma)\sum_{k>\frac{n}{i}-\ell_0}m\{x\in\Lambda|R_{\sigma^{\tau_{j-1}^{\omega}+\ell_0-k}\omega}>k\}.\end{aligned}\tag{7.5}$$

For each $\omega\in\cap_{n\in\mathbb{Z}}\sigma^{-n}(\Omega_1)$, where Ω_1 is the full measure subset from condition (P4), we want to define a random variable $n_4(\omega)$ such that for any $n\ge n_4(\omega)$ we have $n_1(\sigma^{\tau_{j-1}^{\omega}+\ell_0-k}\omega)\le\lfloor n^{1-c}\rfloor$ for any $k\ge\frac{n}{i}-\ell_0$, $i=1,\dots,\lfloor n^{1-c}\rfloor$, so that we can apply the uniform decay rates from (P4). Below the constraint $\tau_{i-1}^{\omega}|\Gamma\le n$ is crucial.

$$n_4(\omega)=\inf\{m|\forall n>m,\forall N\in\{1,2,\dots n+\ell_0\},\forall k>\lfloor n^{1-c}\rfloor-\ell_0,n_1(\sigma^{N-k}\omega)<k\}.$$

We claim that n_4 has a stretched exponential tail.

$$\begin{aligned}P\{n_4(\omega)>m\} &\le \sum_{n>m}\sum_{N=1}^{n+\ell_0}\sum_{k\ge\lfloor n^{1-c}\rfloor-\ell_0}P\{n_1(\sigma^{N-k}\omega)>k\}\\ &= \sum_{n>m}\sum_{N=1}^{n+\ell_0}\sum_{k\ge\lfloor n^{1-c}\rfloor-\ell_0}P\{n_1(\omega)>k\}\\ &\le \sum_{n>m}(n+\ell_0)e^{-u'n^{v'}}\le e^{-u''m^{v''}},\end{aligned}$$

for an appropriate choice of $u''>0$, $0<v''<1$. Now, for $n>n_4$ using the fact that $\tau_{j-1}^{\omega}\le n$ and Lemma 9.1 we can further upper bound the sum in the equation (7.5) by

$$\sum_{k\ge\frac{n}{i}-\ell_0}m\{x\in\Lambda|R_{\sigma^{\tau_{j-1}^{\omega}+\ell_0-k}\omega}>k\}\le C\frac{[\log(\frac{n}{i}-\ell_0)]^b}{[\frac{n}{i}-\ell_0]^{a-1}}\le C'i^a\frac{[\log n]^b}{n^{a-1}}.\tag{7.6}$$

Now, inserting the estimate (7.6) back into equation (7.5) and substituting that result into (7.4) we obtain the final estimate on Y_1:

$$Y_1 \leq M^2 M_0 C_{\tilde{\lambda}}^{-1} K\hat{C} \sum_{i<\lfloor n^c \rfloor} i^a \frac{[\log n]^b}{n^{a-1}} \leq C' n^{c(a+1)} \frac{[\log n]^b}{n^{a-1}} \leq C' \frac{[\log n]^b}{n^{a-1-\delta}}.$$

Now we tackle the term Y_2 by decomposing $\Delta \otimes_\omega \Delta$ into two pieces. First for parameters $K > 0$ and $0 < \rho < 1$, and integer $q > 0$ define

$$B_q(K,\rho) = \{(\omega, x, x') | \#\{i | 2 \leq i \leq q, \tau_i^\omega - \tau_{i-1}^\omega > K\} > \rho q\}.$$

We are going to pick the parameters K, and $\rho \sim 1$ later, but the idea is that for points in $B_q(K,\rho)$ the first q return times have many (at least ρq) large (bigger than K) gaps. Our decomposition will be according to this $B_q(K,\rho)$ for $q = \lfloor n^c \rfloor$:

$$\begin{aligned} Y_2 &= \tilde{\lambda}\{T_\omega > n, \tau_q^\omega < n\} = \tilde{\lambda}(\{T_\omega > n\} \cap B_q(K,\rho)) \\ &+ \tilde{\lambda}(\{T_\omega > n\} \cap [B_q(K,\rho)]^c) \leq \tilde{\lambda}(B_q(K,\rho)) + \tilde{\lambda}(\{T_\omega > n\} \cap [B_q(K,\rho)]^c). \end{aligned} \tag{7.7}$$

In order to estimate the first term in this expression, fix a sequence of integers $2 \leq t_1 < t_2 < \cdots < t_s$ for $\rho q \leq s \leq q-1$ and define

$$B_q(K, \{t_i\}) = \{(\omega, x, x') | \tau_{t_i}^\omega - \tau_{t_i-1}^\omega > K, i = 1, 2, \dots s\}.$$

Then $B_q(K,\rho) = \cup_{s=\rho q}^{q-1} \cup_{t_1<\cdots<t_s} B_q(K, \{t_i\})$ and by Lemma 7.5 we can estimate measures of the terms on the right by

(7.8)

$$\begin{aligned} \mathbb{P}(B_q(K,\{t_i\})) &= \sum_{\tau_1<\tau_2<\dots\tau_q} \mathbb{P}(B_q(K,\{t_i\}) \cap G_q(\vec{\tau})) \\ &\leq \sum_{\substack{\tau_1<\tau_2<\dots\tau_q \\ \tau_{t_i}-\tau_{t_i-1}>K \\ i=1,2,\dots s}} \mathbb{P}(G_q(\vec{\tau})) \leq C^q \sum_{\tau_1} \mathbb{P}\{\tau_1^\omega(x,x') = \tau_1\} \sum_{\substack{\tau_2<\tau_3<\dots\tau_q \\ \tau_{t_i}-\tau_{t_i-1}>K \\ i=1,2,\dots s}} \prod_{j=2}^{q} \gamma(\tau_j - \tau_{j-1}) \\ &= C^q \sum_{\tau_1} \mathbb{P}\{\tau_1^\omega(x,x') = \tau_1\} \prod_{i=1}^{s} \sum_{\tau_{t_i}-\tau_{t_i-1}} \gamma(\tau_{t_i} - \tau_{t_i-1}) \prod_{j\neq t_i} \sum_{\tau_j-\tau_{j-1}} \gamma(\tau_j - \tau_{j-1}) \\ &\leq C^q \sum_{m\geq \ell_0} \mathbb{P}\{\tau_1^\omega(x,x') = m\} \left(\sum_{m>K} \gamma(m)\right)^s \left(\sum_{m\geq\ell_0} \gamma(m)\right)^{q-s}. \end{aligned}$$

Now applying Lemmas 7.7 and 7.6 we obtain, assuming $\rho > \frac{1}{2}$,

(7.9)

$$\mathbb{P}(B_q(K,\rho)) = \sum_{s=\rho q}^{q-1} \sum_{t_1<\cdots<t_s} \mathbb{P}(B_q(K,\{t_i\})) \leq \sum_{s=\rho q}^{q-1} [2\hat{C}[C(K)]^\rho]^q < [2e\hat{C}[C(K)]^\rho]^q.$$

We pick K large enough so that

$$2e\hat{C}[C(K)]^{\rho} := \kappa_1 < 1.$$

This shows $\mathbb{P}(B_q(K,\rho)) \leq \kappa_1^q$. Since we want estimates over individual fibres $\Delta_\omega \times \Delta_\omega$ we finally observe the above estimate shows $m \times m(B_q^\omega(K,\rho)) \leq \kappa_1^{q/2}$ except on a set of ω of measure at most $\kappa_1^{q/2}$. Once again, an application of Borel-Cantelli shows there is a full measure set $\Omega_5 \subseteq \Omega$ and $n_5(\omega) \geq n_4(\omega)$ with stretched exponential tails (there exists $u > 0$ and $0 < v < 1$ so that $P\{n_5 > n\} \leq e^{-un^v}$ and such that, for every $\omega \in \Omega_5$ and $n > n_5(\omega)$, $m \times m(B_q^\omega(K,\rho)) \leq \kappa_1^{q/2}$.

We now turn our attention to the complement of $B_q(K,\rho)$. Note that for each $\omega, \vec{\tau} = (\tau_1, \dots \tau_q)$ either $G_q^\omega(\vec{\tau}) \subseteq B_q^\omega(K,\rho)$ or $G_q^\omega(\vec{\tau}) \cap B_q^\omega(K,\rho) = \emptyset$. Let us call those $\vec{\tau}$ in the former class $\omega, q-$ good. The others we will call $\omega, q-$ bad. Therefore, for ω fixed

$$\begin{aligned}\tilde{\lambda}(\{T_\omega > n; \tau_q^\omega < n\} \cap [B_q^\omega(K,\rho)]^c) &\leq \tilde{\lambda}(\{T_\omega > \tau_q^\omega\} \cap [B_q^\omega(K,\rho)]^c) \\ &= \sum_{\vec{\tau}:\ \omega,q- \text{ bad}} \tilde{\lambda}(\{T_\omega > \tau_q^\omega\} \cap G_q^\omega(\vec{\tau})).\end{aligned} \tag{7.10}$$

We move now to estimate the individual terms in the sum over $\omega, q-$ bad terms. Note that each $G_q^\omega(\vec{\tau})$ is ξ_q^ω measurable. Therefore we can write

$$G_q^\omega(\vec{\tau}) = \cup_{\Gamma_q \in \xi_q^\omega,\, \Gamma_q \subseteq G_q^\omega(\vec{\tau})} \Gamma_q$$

as a disjoint union. Recall that $\{T_\omega > \tau_{q-1}\}$ is measurable with respect to ξ_q^ω. Therefore, for each Γ_q in the above decomposition, either $\Gamma_q \cap \{T_\omega > \tau_{q-1}\} = \Gamma_q$ or $\Gamma_q \cap \{T_\omega > \tau_{q-1}\} = \emptyset$. Call the former $\omega, q-$ good and the latter $\omega, q-$ bad. Finally, note that if Γ_q is $\omega, q-$ bad then $\Gamma_q \cap \{T_\omega > \tau_q\} = \emptyset$. Now we estimate:

$$\begin{aligned}&\tilde{\lambda}(\{T_\omega > \tau_q\} \cap G_q^\omega(\vec{\tau})) = \sum_{\Gamma_q:\ \omega,q- \text{ good}} \tilde{\lambda}(\{T_\omega > \tau_q\} \cap \Gamma_q) \\ &= \sum_{\Gamma_q:\ \omega,q- \text{good}} \tilde{\lambda}(\{T_\omega > \tau_q\} | \{T_\omega > \tau_{q-1}\} \cap \Gamma_q) \tilde{\lambda}(\{T_\omega > \tau_{q-1}\} \cap \Gamma_q) \\ &\leq \sum_{\Gamma_q:\ \omega,q- \text{good}} (1 - C_{\tilde{\lambda}} V_{\sigma^{\tau_{q-1}}\omega}^{\tau_q - \tau_{q-1}}) \tilde{\lambda}(\{T_\omega > \tau_{q-1}\} | \{T_\omega > \tau_{q-2}\} \cap \Gamma_q) \tilde{\lambda}(\{T_\omega > \tau_{q-2}\} \cap \Gamma_q) \\ &\dots \\ &\leq \sum_{\Gamma_q:\ \omega,q- \text{ good}} \Pi_{j=2}^q (1 - C_{\tilde{\lambda}} V_{\sigma^{\tau_{j-1}}\omega}^{\tau_j - \tau_{j-1}}) \tilde{\lambda}(\{T_\omega > \tau_1\} \cap \Gamma_q).\end{aligned}$$

Now, since each good Γ_q in the above sum is a subset of $G_q^\omega(\vec{\tau})$ that is $\omega, q-$ bad we know that $\#\{i | 2 \leq i \leq q, \tau_i - \tau_{i-1} \leq K\} > (1-\rho)q$. Therefore, in the above product, considering only those factors in the product, and keeping in mind the lower bound

given by Lemma 7.2 we get

$$\begin{aligned}\tilde{\lambda}(\{T_\omega > \tau_q\} \cap G_q^\omega(\vec{\tau})) &\leq (1 - C_{\tilde{\lambda}} V(K))^{(1-\rho)q} \sum_{\Gamma_q:\ \omega,q-\text{ good}} \tilde{\lambda}(\{T_\omega > \tau_1\} \cap \Gamma_q) \\ &\leq (1 - C_{\tilde{\lambda}} V(K))^{(1-\rho)q} \sum_{\Gamma_q:\ \omega,q-\text{ good}} \tilde{\lambda}(\Gamma_q).\end{aligned}$$

Finally, summing first over all the good Γ_q and then over all $G_q^\omega(\vec{\tau})$ for $\omega, q-$ bad $\vec{\tau}$ we obtain

$$\tilde{\lambda}(\{T_\omega > \tau_q\} \cap [B_q^\omega(K,\rho)]^c) \leq (1 - C_{\tilde{\lambda}} V(K))^{(1-\rho)q}.$$

Set $\kappa = \max\{\kappa_1, (1 - C_{\tilde{\lambda}} V(K))^{(1-\rho)}\} < 1$ and obtain

$$Y_2 \leq C'' \kappa^q = C'' \kappa^{\lfloor n^c \rfloor}$$

for all $n > n_5(\omega)$, giving the claimed stretched exponential decay. This completes the proof of the lemma. □

7.4. **Coupling.** Here we consider $\hat{F}_\omega = (F_\omega \times F_\omega)^{T_\omega}$ which is a mapping from $\hat{\Delta}_\omega = \Delta_\omega \times \Delta_\omega$ into $\hat{\Delta}_{\sigma^{T_\omega}\omega}$. Let $\hat{\xi}_1^\omega$ be the partition of $\hat{\Delta}_\omega$ on which T_ω is constant. Let $T_{1,\omega} < T_{2,\omega} \dots$ be stopping times on $\hat{\Delta}_\omega$ defined as

$$T_{1,\omega} = T_\omega, \quad T_{n,\omega} = T_{n-1,\omega} + T_{\sigma^{T_{n-1,\omega}}\omega} \circ \hat{F}_\omega^{n-1}.$$

For $u, z \in \hat{\Delta}_\omega$ we define a separation time $\hat{s}(u,z)$ associated with $\hat{F}_\omega$ as the smallest $n \geq 0$ such that $\hat{F}_\omega^n(u)$ and $\hat{F}_\omega^n(z)$ lie in distinct elements of $\hat{\xi}_1^{\sigma^{T_{n,\omega}}\omega}$ [8].

Let λ_ω and λ'_ω be two probability measures on $\{\Delta_\omega\}$ with densities $\varphi, \varphi' \in \mathcal{F}_\gamma^+ \cap \mathcal{L}_\infty^{K_\omega}$. Let $\tilde{\lambda} = \lambda_\omega \times \lambda'_\omega$ and $\Phi = d\tilde{\lambda}/d(m \times m)$, then $\Phi(x,x') = \varphi(x)\varphi'(x')$. The next lemma establishes the regularity of $\hat{F}_\omega$ and Φ.

Lemma 7.9. (1) For any $n > 0$, $u, z \in \hat{\Delta}_\omega$ with $\hat{s}(u,z) \geq n$

$$\left| \log \frac{J\hat{F}_\omega^n(u)}{J\hat{F}_\omega^n(z)} \right| \leq \hat{D}\gamma^{\hat{s}(\hat{F}_\omega^n u, \hat{F}_\omega^n z)},$$

where $\hat{D} \geq 2D$ is a constant.

(2) For any $n > 0$, $u, z \in \hat{\Delta}_\omega$

$$\left| \log \frac{\Phi(u)}{\Phi(z)} \right| \leq C_\Phi \gamma^{\hat{s}(u,z)},$$

where $C_\Phi = C_\varphi + C_{\varphi'}$.

[8] Notice that, for any $u = (x,x'), z = (y,y') \in \Delta_\omega$ if $\hat{s}(u,z) > n$ then $s(x,x') > n$ and $s(y,y') > n$.

Proof. Let $u = (x, x')$, $z = (y, y')$. For $n > 0$ choose k so that $\hat{F}^n_\omega(u) = (F_\omega \times F_\omega)^k(u)$. Then

$$\left|\log \frac{J\hat{F}^n_\omega(x,x')}{J\hat{F}^n_\omega(y,y')}\right| \le \left|\log \frac{JF^k_\omega(x)}{JF^k_\omega(y)}\right| + \left|\log \frac{JF^k_\omega(x')}{JF^k_\omega(y')}\right|$$
$$\le D\gamma^{s(F^k_\omega x, F^k_\omega y)} + D\gamma^{s(F^k_\omega x', F^k_\omega y')} \le \hat{D}\gamma^{\hat{s}(\hat{F}^n_\omega u, \hat{F}^n_\omega z)},$$

where we have used $\hat{s}(u,z) \le \min\{s(x,x'), s(y,y')\}$. Similarly for the second item we have

$$\left|\log \frac{\Phi(x,x')}{\Phi(y,y')}\right| \le \left|\log \frac{\varphi(x)}{\varphi(y)}\right| + \left|\log \frac{\varphi'(x')}{\varphi'(y')}\right| \le C_\Phi \gamma^{\hat{s}(u,z)}.$$

□

Let $\hat{\xi}^\omega_i$ be the partition of $\hat{\Delta}_\omega$ on which $T_{1,\omega}, \dots, T_{i,\omega}$ are constant. For $z \in \hat{\Delta}_\omega$ let $\hat{\xi}^\omega_i(z)$ be the element containing z. Given $\Phi(x,x') = \varphi(x)\varphi(x')$ let $i_1(\Phi)$ be such that $C_\Phi \gamma^{i_1} < \hat{D}$. For $i < i_1$ let $\hat{\Phi}_i \equiv \Phi$. For $i \ge i_1$, let

$$\hat{\Phi}_i(z) = \left[\frac{\hat{\Phi}_{i-1}(z)}{J\hat{F}^i_\omega(z)} - \varepsilon \min_{u \in \hat{\xi}^\omega_i(z)} \frac{\hat{\Phi}_{i-1}(u)}{J\hat{F}^i_\omega(u)}\right] J\hat{F}^i_\omega(z), \tag{7.11}$$

where ε is a small number that will be defined below. Since $(\hat{\Phi}_i - \hat{\Phi}_{i-1})/J\hat{F}^i_\omega$ is constant on every $\Gamma \in \hat{\xi}^\omega_i$, we have

$$\pi_*(\hat{F}^i_\omega)_*((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m)|\Gamma) = \pi'_*(\hat{F}^i_\omega)_*((\hat{\Phi}_{i-1} - \hat{\Phi}_i)(m \times m)|\Gamma).$$

Note that, $\hat{\Phi}_i$ is the density of the part of $\tilde{\lambda}$ which has not been matched up to time $T_{i,\omega}$.

Lemma 7.10. For all sufficiently small $\varepsilon > 0$ in (7.11), there exists $0 < \varepsilon_1 < 1$ independent of Φ such that for almost every ω and for all $i \ge i_1$

$$\hat{\Phi}_i \le (1 - \varepsilon_1)\hat{\Phi}_{i-1} \quad \text{on} \quad \hat{\Delta}_\omega.$$

We will introduce the following densities in order to prove Lemma 7.10. For $z \in \hat{\Delta}_\omega$ let

$$\tilde{\Psi}_{i_1-1}(z) = \frac{\Phi(z)}{J\hat{F}^{i_1-1}_\omega(z)}$$

and for $i \ge i_1$, let

$$\Psi_i(z) = \frac{\tilde{\Psi}_{i-1}(z)}{J\hat{F}_{\sigma^{T_{i-1},\omega}\omega}(\hat{F}^{i-1}_\omega(z))}, \quad \varepsilon_{i,z} = \varepsilon \cdot \min_{u \in \hat{\xi}_i(z)} \Psi_i(u), \quad \tilde{\Psi}_i(z) = \Psi_i(z) - \varepsilon_{i,z}.$$

Lemma 7.10 then follows from the following lemma:

Lemma 7.11. There exists $\hat{C}$ such that for all sufficiently small ε the following holds: for any $z \in \hat{\Delta}_\omega$ with $u \in \hat{\xi}^\omega_i(z)$ and $i \ge i_1$

$$\left|\log \frac{\tilde{\Psi}_i(u)}{\tilde{\Psi}_i(z)}\right| \le \hat{C}\gamma^{\hat{s}(\hat{F}^i_\omega u, \hat{F}^i_\omega z)}.$$

Proof. By definition of Ψ_i and item (1) Lemma 7.9 of we have

$$\left|\log \frac{\Psi_i(u)}{\Psi_i(z)}\right| \leq \left|\log \frac{\tilde{\Psi}_{i-1}(u)}{\tilde{\Psi}_{i-1}(z)}\right| + \hat{D}\gamma^{\hat{s}(\hat{F}^i_\omega z, \hat{F}^i_\omega u)}. \tag{7.12}$$

Since $\varepsilon_{i,z}$ is constant on $\hat{\xi}^\omega_i(z)$ we let $\varepsilon_i = \varepsilon_{i,z}$. We have

$$\begin{aligned}\left|\log \frac{\tilde{\Psi}_i(u)}{\tilde{\Psi}_i(z)} - \log \frac{\Psi_i(u)}{\Psi_i(z)}\right| &= \left|\log \frac{\Psi_i(u)-\varepsilon_i}{\Psi_i(u)}\frac{\Psi_i(z)}{\Psi_i(z)-\varepsilon_i}\right| \leq \left|\frac{\frac{\varepsilon_i}{\Psi_i(z)} - \frac{\varepsilon_i}{\Psi_i(u)}}{1-\frac{\varepsilon_i}{\Psi_i(z)}}\right| \\ &\leq \frac{\varepsilon_i}{\Psi_i(z)}\left|\frac{\Psi_i(u)}{\Psi_i(z)} - 1\right| \frac{1}{1-\varepsilon} \leq \frac{\varepsilon}{1-\varepsilon} C \left|\log \frac{\Psi_i(u)}{\Psi_i(z)}\right|.\end{aligned}$$

Notice that C in the latter inequality increases as $\frac{\Psi_i(u)}{\Psi_i(z)}$ increases. Allowing ε to depend on i, z, u and ω,[9] for a given $0 < \varepsilon' < \gamma^{-1} - 1$ we can choose ε small enough so that

$$\frac{\varepsilon}{1-\varepsilon} C < \varepsilon' \tag{7.13}$$

we obtain

$$\left|\log \frac{\tilde{\Psi}_i(u)}{\tilde{\Psi}_i(z)}\right| \leq (1+\varepsilon')\left|\log \frac{\Psi_i(u)}{\Psi_i(z)}\right|. \tag{7.14}$$

By (7.12) and (7.14) we obtain

$$\left|\log \frac{\tilde{\Psi}_i(u)}{\tilde{\Psi}_i(z)}\right| \leq (1+\varepsilon')\left(\left|\log \frac{\tilde{\Psi}_{i-1}(u)}{\tilde{\Psi}_{i-1}(z)}\right| + \hat{D}\gamma^{\hat{s}(\hat{F}^i_\omega u, \hat{F}^i_\omega z)}\right). \tag{7.15}$$

Moreover, for $i = i_1$ we have

$$\begin{aligned}\left|\log \frac{\tilde{\Psi}_{i_1}(u)}{\tilde{\Psi}_{i_1}(z)}\right| &\leq (1+\varepsilon')\left(\left|\log \frac{\Phi(u)}{\Phi(z)}\right| + \hat{D}\gamma^{\hat{s}(\hat{F}^i_\omega u, \hat{F}^i_\omega z)}\right) \\ &\leq (1+\varepsilon')(C_\Phi \gamma^{\hat{s}(u,z)} + \hat{D}\gamma^{\hat{s}(\hat{F}^{i_1}_\omega u, \hat{F}^{i_1}_\omega z)}) \\ &\leq (1+\varepsilon')2\hat{D}\gamma^{\hat{s}(\hat{F}^{i_1}_\omega u, \hat{F}^{i_1}_\omega z)}.\end{aligned}$$

Note that in the last inequality we have used $C_\Phi \gamma^{\hat{s}(u,z)} \leq \hat{D}\gamma^{\hat{s}(\hat{F}^{i_1}_\omega u, \hat{F}^{i_1}_\omega z)}$. Finally, using the relation $\hat{s}(\hat{F}^{i-j}_\omega u, \hat{F}^{i-j}_\omega z) = \hat{s}(\hat{F}^i_\omega u, \hat{F}^i_\omega z) + j$ we have

$$\left|\log \frac{\tilde{\Psi}_i(u)}{\tilde{\Psi}_i(z)}\right| \leq \hat{C}\gamma^{s(\hat{F}^i_\omega u, \hat{F}^i_\omega z))}, \tag{7.16}$$

where $\hat{C} = 2(1+\varepsilon')\hat{D}\sum_{j=0}^{\infty}[(1+\varepsilon')\gamma]^j$.

[9]Notice that when $i = i_1$ we can choose ε uniform for all u and z, allowing dependence only on $\hat{D}$ and C_Φ.

Now, we show by an inductive argument that ε in (7.13) can be chosen independent of i, u, z, ω. First of all notice that we can choose ε independent of u, z for $i = i_1$ because

$$\frac{\Psi_{i_1}(u)}{\Psi_{i_1}(z)} = \frac{\Phi(u)}{\Phi(z)} \frac{J\hat{F}_\omega^{i_1}(z)}{J\hat{F}_\omega^{i_1}(u)} \le (1+\hat{D})^2.$$

Let $j > i_1$ and suppose that ε is small enough so that (7.16) holds for all $i < j$ and $u \in \hat{\xi}_i^\omega(z)$. Then by (7.12) we have

$$\left|\log \frac{\Psi_i(u)}{\Psi_i(z)}\right| \le \hat{C} + \hat{D},$$

which implies that $\frac{\Psi_i(u)}{\Psi_i(z)} \in [e^{-(\hat{C}+\hat{D})}, e^{\hat{C}+\hat{D}}]$. Therefore C in (7.13) is bounded by $e^{\hat{C}+\hat{D}}$. Hence by choosing $\varepsilon < \varepsilon' e^{-(\hat{C}+\hat{D})}$ we conclude that the estimate in (7.15) holds for $i = j$. □

Lemma 7.12. Let $0 < \varepsilon_1 < 1$ be as in Lemma 7.10. For almost every ω and all $n \in \mathbb{N}$

$$|(F_\omega^n)_*(\lambda_\omega) - (F_\omega^n)_*(\lambda'_\omega)| \le 2\tilde{\lambda}\{T_{i_1,\omega} > n\} + 2\sum_{i=i_1}^{\infty} (1-\varepsilon_1)^{i-i_1+1} \tilde{\lambda}\{T_{i,\omega} \le n < T_{i+1,\omega}\},$$

where $\tilde{\lambda} = \lambda_\omega \times \lambda'_\omega$.

Proof. In Lemma 7.10 the estimates for the mass of $\tilde{\lambda}$ after the i^{th} iterate matching was given. Now we will relate that estimate to the iterates of F_ω. Define Φ_0, $\Phi_1, \dots$ as follows: for $z \in \Delta_\omega \times \Delta_\omega$ let

$$\Phi_n(z) = \hat{\Phi}_i(z) \quad \text{when} \quad T_{i,\omega}(z) \le n < T_{i+1,\omega}(z),$$

where $\hat{\Phi}_i(z)$ is as in (7.11). We first prove that $|(F_\omega^n)_*(\lambda) - (F_\omega^n)_*(\lambda')| \le 2\int \Phi_n d(m \times m)$. Below we use the notation $\Phi(m \times m)$ to denote a measure whose density with respect to $m \times m$ is Φ. First of all recall that $\Phi_0 = \Phi$ and write $\Phi = \Phi_n + \sum_{k=1}^n (\Phi_{k-1} - \Phi_k)$. We have

$$\begin{aligned} &|(F_\omega^n)_*(\lambda) - (F_\omega^n)_*(\lambda')| \\ &\quad = |\pi_*(F_\omega \times F_\omega)_*^n(\Phi(m \times m)) - \pi'_*(F_\omega \times F_\omega)_*^n(\Phi(m \times m))| \\ &\quad \le |\pi_*(F_\omega \times F_\omega)_*^n(\Phi_n(m \times m)) - \pi'_*(F_\omega \times F_\omega)_*^n(\Phi_n(m \times m))| \\ &\quad + \sum_{k=1}^n |(\pi_* - \pi'_*)(F_\omega \times F_\omega)_*^n((\Phi_{k-1} - \Phi_k)(m \times m))|. \end{aligned} \tag{7.17}$$

Since, for any $A \subset \Delta_{\sigma^n\omega}$ we have

$$\pi_*(F_\omega \times F_\omega)_*^n(\Phi_n(m \times m))(A) = \int_{F_\omega^{-n}(A) \times \Delta_\omega} \Phi_n d(m \times m),$$

the first term in the final sum in (7.17) is bounded by $2\int \Phi_n d(m \times m)$. Now, we claim that all other terms in (7.17) vanish. Let $A_k = \cup A_{k,i} \subset \hat{\Delta}_\omega$ be such that $A_{k,i} = \{z \in$

$\hat{\Delta}_\omega \mid k = T_{i,\omega}(z)\}$. By construction $A_{i,k}$ is a union of elements of $\hat{\xi}_i^\omega$ and $A_{k,i} \cap A_{k,j} = \varnothing$ for $i \neq j$ (because $T_{i,\omega} < T_{j,\omega}$ for $i < j$). For $\Gamma \in \hat{\xi}_i^\omega | A_{k,i}$ by definition of the Φ_i's we have $\Phi_{k-1} - \Phi_k = \hat{\Phi}_{i-1} - \hat{\Phi}_i$. On the other hand, $\Phi_{k-1} \equiv \Phi_k$ on $\hat{\Delta}_\omega \setminus A_k$. Hence for each k and for every $\Gamma \subset A_{k,i}$ we have

$$(F_{\sigma^k\omega}^{n-k})_* \pi_* (F_\omega \times F_\omega)_*^{T_{i,\omega}} (\Phi_k(m \times m)|\Gamma) = (F_{\sigma^k\omega}^{n-k})_* \pi'_* (F_\omega \times F_\omega)_*^{T_{i,\omega}} (\Phi_k(m \times m)|\Gamma)$$

which finishes the proof of claim.

It remains to estimate $\int \Phi_n d(m \times m)$. We have

$$\int \Phi_n d(m \times m) = \int_{\{T_{i_1,\omega} > n\}} \Phi_n d(m \times m) + \sum_{i=i_1}^{\infty} \int_{\{T_{i,\omega} \leq n < T_{i+1,\omega}\}} \Phi_n d(m \times m).$$

Note that $\Phi_n = \Phi$ on $\{T_{i_1,\omega} > n\}$. Hence we have

$$\int_{\{T_{i_1,\omega} > n\}} \Phi_n d(m \times m) = \int_{\{T_{i_1,\omega} > n\}} \Phi d(m \times m) = \tilde{\lambda}\{T_{i_1,\omega} > n\}.$$

Let n be such that $T_{i,\omega} \leq n < T_{i+1,\omega}$. By Lemma 7.10 we have $\Phi_n = \hat{\Phi}_i \leq (1 - \varepsilon_1)^{i-i_1+1} \Phi$. Hence

$$\begin{aligned} \int_{\{T_{i,\omega} \leq n < T_{i+1,\omega}\}} \Phi_n d(m \times m) &\leq \int_{\{T_{i,\omega} \leq n < T_{i+1,\omega}\}} (1 - \varepsilon_1)^{i-i_1+1} \Phi d(m \times m) \\ &= (1 - \varepsilon_1)^{i-i_1+1} \tilde{\lambda}\{T_{i,\omega} \leq n < T_{i+1,\omega}\}. \end{aligned}$$

□

8. DECAY OF CORRELATIONS

The main result of this section is the following proposition.

Proposition 8.1. *For every $\delta > 0$ there is a full measure set $\Omega_6 \subset \Omega$ and a random variable $n_6(\omega)$ such that for all probability measures $\lambda_\omega, \lambda'_\omega$ on $\{\Delta_\omega\}$ with $\frac{d\lambda_\omega}{dm}, \frac{d\lambda'_\omega}{dm} \in \mathcal{F}_\gamma^{K_\omega} \cap \mathcal{F}_\gamma^+$, there is C so that for any $n > n_6$, we have*

$$|(F_\omega^n)_*(\lambda_\omega) - (F_\omega^n)_*(\lambda'_\omega)| \leq C \frac{(\log n)^{b+a}}{n^{a-1-\delta}}.$$

Moreover, there exist $C' > 0$, such that

$$P\{n_6 > n\} \leq C' e^{-u' n^{v'}},$$

for some $u' > 0$ and $0 < v' < 1$.

Notice that the constant C in Proposition 8.1 depends on λ and λ' through the Lipschitz constants of their densities, which belong $\mathcal{F}_\gamma^{K_\omega} \cap \mathcal{F}_\gamma^+$ and hence C is independent of ω. Before proving the proposition, we prove the following auxiliary lemma.

Lemma 8.2. There exists $C_{\tilde{\lambda}}$ such that for all $i \geq 1$ and for any $\Gamma \in \hat{\xi}_i^\omega$

$$\tilde{\lambda}\{T_{i+1,\omega} - T_{i,\omega} > n|\Gamma\} \leq C_{\tilde{\lambda}}(m \times m)\{T_{\sigma^{T_{i,\omega}}\omega} > n\}.$$

Proof. By definition we have

$$\tilde{\lambda}\{T_{i+1,\omega} - T_{i,\omega} > n|\Gamma\} = \tilde{\lambda}\{T_{\sigma^{T_{i,\omega}}\omega} \circ \hat{F}^i_\omega > n\}.$$

Therefore, it remains to bound the density $d(\hat{F}^i_\omega)_*\tilde{\lambda}/d(m\times m)$. Let $\Gamma \in \hat{\xi}^\omega_i$. Any $z', u' \in \hat{\Delta}_{0,\sigma^{T_{i,\omega}}\omega}$ have unique pre-images $u, z \in \Gamma$. By definition we have

$$\begin{aligned}\left|\log\left(\frac{d(\hat{F}^i_\omega)_*\tilde{\lambda}}{d(m\times m)}(u')\Big/\frac{d(\hat{F}^i_\omega)_*\tilde{\lambda}}{d(m\times m)}(z')\right)\right| &= \left|\log\frac{J\hat{F}^i_\omega(z)}{J\hat{F}^i_\omega(u)} + \log\frac{\Phi(u)}{\Phi(z)}\right| \\ &\le \hat{D}\gamma^{\hat{s}(\hat{F}^i_\omega u, \hat{F}^i_\omega z)} + C_\Phi\gamma^{\hat{s}(u,z)} \le \hat{D} + C_\Phi =: \log C_{\tilde{\lambda}}.\end{aligned}$$

Since $\hat{D}$ is independent of Γ this implies $d(\hat{F}^i_\omega)_*\tilde{\lambda}/d(m\times m) < C_{\tilde{\lambda}}$. □

Proof of Proposition 8.1. Note that, by taking $T_{0,\omega} \equiv 0$ Lemma 7.12 implies

$$|(F^n_\omega)_*(\lambda) - (F^n_\omega)_*(\lambda')| \le 2(1-\varepsilon_1)^{1-i_1}\sum_{i=0}^{\infty}(1-\varepsilon_1)^i\tilde{\lambda}\{T_{i,\omega} \le n < T_{i+1,\omega}\}. \tag{8.1}$$

By choosing $A(n) \in \mathbb{N}$ so that $(1-\varepsilon_1)^{A(n)} \le n^{-2a}$, for any $i \ge A(n)$ we have

$$\sum_{i=A(n)}^{\infty}(1-\varepsilon_1)^i\tilde{\lambda}\{T_{i,\omega} \le n < T_{i+1,\omega}\} \le \sum_{i=A(n)}^{\infty}(1-\varepsilon_1)^i \le \frac{1}{\varepsilon_1 n^{2a}}. \tag{8.2}$$

Now we estimate $\tilde{\lambda}\{T_{i-1,\omega} \le n < T_{i,\omega}\}$ for $i \le A(n)$. Let $\tilde{m} = \frac{m\times m}{m(\Delta_\omega)^2}$. We proceed as in equation (7.4) in the estimate of Y_1. For every i we write

$$\begin{aligned}\tilde{\lambda}\{T_{i-1,\omega} \le n < T_{i,\omega}\} &\le \sum_{\substack{\Gamma\in\hat{\xi}^\omega_{i-1}\\ T_{i-1,\omega}|\Gamma\le n}}\sum_{j=0}^{i-1}\tilde{\lambda}\{T_{j+1,\omega} - T_{j,\omega} > \frac{n}{i} \mid \Gamma\}\tilde{\lambda}(\Gamma) \\ \text{(by Lemma 8.2)} \quad &\le C_{\tilde{\lambda}}\sum_{\substack{\Gamma\in\hat{\xi}^\omega_{i-1}\\ T_{i-1,\omega}|\Gamma\le n}}\tilde{\lambda}(\Gamma)\sum_{j=0}^{i-1}(m\times m)\{T_{\sigma^{T_{j,\omega}}\omega} > \frac{n}{i}\} \qquad (8.3) \\ &\le C_{\tilde{\lambda}}M^2\sum_{\substack{\Gamma\in\hat{\xi}^\omega_{i-1}\\ T_{i-1,\omega}|\Gamma\le n}}\tilde{\lambda}(\Gamma)\sum_{j=0}^{i-1}\tilde{m}\{T_{\sigma^{T_{j,\omega}}\omega} > \frac{n}{i}\}.\end{aligned}$$

Recall that there is a full measure set Ω_5 and a random variable n_5 which is finite on Ω_5, and $P\{n_5 > n\} \le Ce^{-un^v}$. Now define

$$n_6(\omega) = \inf\{n|\forall k \ge n, \forall N \in [1,k]\cap\mathbb{N}, n_5(\sigma^N\omega) \le k\}.$$

We now show that n_6 has a stretched exponential tail. Indeed,

$$\begin{aligned} P\{n_6(\omega) > n\} &\leq \sum_{k>n} \sum_{N=1}^{k} P\{n_5(\sigma^N\omega) > k\} \\ &= \sum_{k>n} \sum_{N=1}^{k} P\{n_5(\omega) > k\} \leq C' e^{-u' n^{v'}}, \end{aligned}$$

for an appropriate choice of $u' > 0$, $v' \in (0,1)$. Since $A(n) \sim \log n$, for any $\varepsilon > 0$ we have

$$P\{n_6(\omega) > \frac{n}{A(n)}\} \leq P\{n_6(\omega) > n^{1-\varepsilon}\} \lesssim e^{u' n^{(1-\varepsilon)v'}}.$$

Therefore, by Proposition 7.8 and the definition of $A(n)$, for any $\frac{n}{A(n)} > n_6$ we can estimate (8.3) as follows:

$$\begin{aligned} \tilde{\lambda}\{T_{i-1,\omega} \leq n < T_{i,\omega}\} &\leq C_{\tilde{\lambda}} M^2 C \sum_{j=0}^{i-1} \frac{(\log n)^b}{n^{a-1-\delta}} i^{a-1} = C_{\tilde{\lambda}} M^2 C \frac{(\log n)^b}{n^{a-1-\delta}} i^a \\ &\lesssim (\log n)^a \frac{(\log n)^b}{n^{a-1-\delta}} \lesssim \frac{(\log n)^{b+a}}{n^{a-1-\delta}}. \end{aligned} \tag{8.4}$$

Finally, using (8.4)

$$\sum_{i=1}^{A(n)} (1-\varepsilon_1)^i \tilde{\lambda}\{T_{i,\omega} \leq n < T_{i+1,\omega}\} \lesssim \frac{(\log n)^{b+a}}{n^{a-1-\delta}} \sum_{i=1}^{\infty} (1-\varepsilon_1)^i \lesssim \frac{(\log n)^{b+a}}{n^{a-1-\delta}}. \tag{8.5}$$

Thus, combining (8.2) and (8.5) finishes the proof. □

8.1. **Decay of future correlations (Proof of Theorem 4.2 item (i)).** Let $\psi \in \mathcal{F}_\gamma^{K_\omega}$ and $\varphi \in \mathcal{L}_\infty^{K_\omega}$. Also, let C_ψ, C'_ψ and C'_φ be the constants given in the definitions of $\mathcal{F}_\omega^{K_\omega}$ and $\mathcal{L}_\infty^{K_\omega}$ respectively. Let $\tilde{\psi} = A_\omega(\psi + (C'_\psi + 1)K_\omega + 1)$, where $A_\omega = (\int \psi dm + m(\Delta_\omega)[(C'_\psi+1)K_\omega + 1])^{-1}$. Then $\tilde{\psi} \in \mathcal{F}_\gamma^+ \cap \mathcal{F}_\gamma^{K_\omega}$, $\int \tilde{\psi} dm = 1$ and $|\tilde{\psi}(x)| \leq 2(C'_\psi + 1)$ for all $x \in \Delta_\omega$. The second assertion is obvious by the choice of A_ω. For the third one we use the inequality $A_\omega \leq m(\Delta_\omega)^{-1}(1+K_\omega)^{-1} \leq 1$. For the first claim we have

$$\left| \frac{\tilde{\psi}(x)}{\tilde{\psi}(y)} - 1 \right| \leq \frac{1}{K_\omega + 1} |\psi(x) - \psi(y)| \leq C_\psi \gamma^{s(x,y)}.$$

For the correlations we have the following relation

$$C_{n,\omega}^{(f)}(\varphi, \psi) = \frac{1}{A_\omega} C_{n,\omega}^{(f)}(\varphi, \tilde{\psi}) - m(\Delta_\omega)[K_\omega(C'_\psi + 1) + 1] C_{n,\omega}^{(f)}(\varphi, \frac{1}{m(\Delta_\omega)}). \tag{8.6}$$

Let λ be a probability with density $\frac{d\lambda}{dm} = \tilde{\psi}$. Then by Proposition 8.1 for every $n > n_6$ we have

$$\begin{aligned}\left|C^{(f)}_{n,\omega}(\varphi, \tilde{\psi})\right| &= \left|\int (\varphi_{\sigma^n\omega} \circ F^n_\omega)\tilde{\psi}_\omega dm - \int \varphi_{\sigma^n\omega} d\nu_{\sigma^n\omega} \int \tilde{\psi}_\omega dm\right| \\ &\leq \left|\int (\varphi_{\sigma^n\omega} \circ F^n_\omega) d\lambda_\omega - \int \varphi_{\sigma^n\omega} d\nu_{\sigma^n\omega}\right| \\ &\leq \sup_{x\in\Delta_{\sigma^n\omega}} \varphi_{\sigma^n\omega}(x) \cdot |(F^n_\omega)_*\lambda_\omega - (F^n_\omega)_*\nu_\omega| \\ &\leq C'_\varphi K_{\sigma^n\omega} C_{\lambda,\nu} \frac{(\log n)^{b+a}}{n^{a-1-\delta}}.\end{aligned} \tag{8.7}$$

Similarly, for the probability measure λ' with the constant density $m(\Delta_\omega)^{-1}$ we have

$$\left|C^{(f)}_{n,\omega}(\varphi, \frac{1}{m(\Delta_\omega)})\right| \leq C'_\varphi K_{\sigma^n\omega} C_{\lambda',\nu} \frac{(\log n)^{b+a}}{n^{a-1-\delta}}. \tag{8.8}$$

Define $C_{\lambda,\lambda'} := \max\{C_{\lambda,\nu}, C_{\lambda',\nu}\}$. Substituting (8.7) and (8.8) into (8.6), and using the inequality $A^{-1}_\omega \leq m(\Delta_\omega)[1 + (2C'_\psi + 1))K_\omega]$ we have

$$\left|C^{(f)}_{n,\omega}(\varphi, \psi)\right| \leq C_{\lambda,\lambda'} C'_\varphi m(\Delta_\omega)[2 + K_\omega(2 + 3C'_\psi)] K_{\sigma^n\omega} \frac{(\log n)^{b+a}}{n^{a-1-\delta}}. \tag{8.9}$$

Let $n_7(\omega) = \inf\{k \geq n_6(\omega) \mid \forall \ell > k,\ K_{\sigma^\ell\omega} \leq \ell^\delta\}$. Then

$$P\{n_7 > n\} \leq P\{n_6 > n\} + \sum_{k\geq n} P\{K_{\sigma^k\omega} > k^\delta\} \lesssim e^{-u'n^{v'}} + \sum_{k\geq n} e^{-uk^{v'}} \lesssim Ce^{-u'n^{v'}}.$$

Now, if $n > n_7$ then

$$\left|C^{(f)}_{n,\omega}(\varphi, \psi)\right| \leq C_{\lambda,\lambda'} C'_\varphi M[2 + K_\omega(2 + 3C'_\psi)] \frac{(\log n)^{b+a}}{n^{a-1-2\delta}}.$$

If $n \leq n_7$ then we let

$$C_\omega = n_7(\omega)^a K_\omega \sup_{n\leq n_7} K_{\sigma^n\omega}.$$

Hence, for all $n \in \mathbb{N}$ we have obtained

$$\left|\int (\varphi_{\sigma^n\omega} \circ F^n_\omega)\psi_\omega dm - \int \varphi_{\sigma^n\omega} d\nu_{\sigma^n\omega} \int \psi_\omega dm\right| \leq C_{\psi,\varphi} C_\omega \frac{(\log n)^{b+a}}{n^{a-1-2\delta}}.$$

It remains to show $P\{C_\omega > n\}$ has the desired decay rate. We write

$$P\{C_\omega > k\} \leq P\{K_\omega > k^{1/3}\} + P\{\sup_{n\leq n_7(\omega)} K_{\sigma^n\omega} > k^{1/3}\} + P\{n_7(\omega) > k^{1/(3a)}\}.$$

Notice that by the definition of n_7 we have

$$P\{\sup_{n\leq n_7(\omega)} K_{\sigma^n\omega} > k^{1/3}\} \leq P\{n_7(\omega) > k\} + \sum_{n=1}^{k} P\{K_{\sigma^n\omega} > k^{1/3}\}.$$

Hence, we have

$$P\{C_\omega > k\} \lesssim (k+1)e^{-uk^{v/3}} + e^{-u'k^{\delta v}} + e^{-u'k^{\delta v/(3a)}} \lesssim e^{-u'k^{v'/(3a)}}.$$

Then the conclusion of the theorem holds with u' and $v' := \frac{v'}{3a}$.

8.2. **Decay of past correlations.** To obtain decay of past correlations we need to prove the results of Sections 7 and 8.1 with the corresponding shift on ω. Below we use the notation $\omega = \sigma^{-n}\omega'$ for $\omega' \in \Omega$.

Lemma 8.3. Let $\lambda_{\omega'}$ and $\lambda'_{\omega'}$ be two probability measures on $\{\Delta_{\omega'}\}$ with densities $\varphi, \varphi' \in \mathcal{F}_\gamma^+ \cap \mathcal{L}_\infty^{K_{\omega'}}$. Let $\tilde{\lambda} = \lambda_{\omega'} \times \lambda'_{\omega'}$. For each $\omega' \in \Omega$ let $\omega = \sigma^{-n}\omega'$. Then for any $i \geq 2$ and $\Gamma \in \xi_i^\omega$, where such that $T_\omega|_\Gamma > \tau_{i-1}^\omega$ we have

$$\tilde{\lambda}\{T_\omega > \tau_i^\omega | \Gamma\} \geq 1 - C_{\tilde{\lambda}} V_{\sigma^{\tau_{i-1}^\omega}\omega}^{\tau_i^\omega - \tau_{i-1}^\omega}.$$

where $0 < C_{\tilde{\lambda}} < 1$. Dependence of $C_{\tilde{\lambda}}$ on $\tilde{\lambda}$ on can be removed if we only consider $i \geq i_0(\tilde{\lambda})$.

Lemma 8.4. Let $C_{\tilde{\lambda}}$ be as in Lemma 7.3. For each $\omega' \in \Omega$, let $\omega = \sigma^{-n}\omega'$. For every i and $\Gamma \in \xi_i^\omega$

$$\tilde{\lambda}\{\tau_{i+1}^\omega - \tau_i^\omega > \ell_0 + n | \Gamma\} \leq M_0 M C_{\tilde{\lambda}}^{-1} \cdot m\{\hat{R}_{\sigma^{\tau_i^\omega + \ell_0}\omega} > n\}.$$

Proposition 8.5. *Let $\delta > 0$ be given. Let $\lambda_{\omega'}$ and $\lambda'_{\omega'}$ be two probability measures on $\{\Delta_{\omega'}\}$ with densities $\varphi, \varphi' \in \mathcal{F}_\gamma^+ \cap \mathcal{F}_\infty^{K_{\omega'}}$. Then there exists a constant $\hat{C}_{\tilde{\lambda}}$ and a subset $\Omega_5 \subset \Omega$ full measure and a random variable $n_5(\omega')$ which is finite on Ω_5 such that for any $n > n_5$ letting $\omega = \sigma^{-n}\omega'$ we have*

$$\tilde{\lambda}\{T_\omega > n\} \leq \hat{C}_{\tilde{\lambda}} \frac{(\log n)^b}{n^{a-1-\delta}}.$$

Moreover, there exist $u' > 0, 0 < v < 1$ such that for any n

$$P\{\omega \mid n_5(\omega) > n\} \leq Ce^{-u'n^v}.$$

Proposition 8.6. *For every $\delta > 0$ there is a full measure set $\Omega_6 \subset \Omega$ and a random variable $n_6(\omega')$, which is finite on Ω_6 such that for all probability measures $\lambda_{\omega'}, \lambda'_{\omega'}$ on $\{\Delta_{\omega'}\}$ with $\frac{d\lambda_{\omega'}}{dm}, \frac{d\lambda'_{\omega'}}{dm} \in \mathcal{F}_\gamma^{K_{\omega'}} \cap \mathcal{F}_\gamma^+$, there is $C_{\lambda,\lambda'}$ so that for any $n > n_6(\omega')$ letting $\omega = \sigma^{-n}\omega'$ we have*

$$|(F_\omega^n)_*(\lambda_{\omega'}) - (F_\omega^n)_*(\lambda'_{\omega'})| \leq C_{\lambda,\lambda'} \frac{(\log n)^{b+a}}{n^{a-1-\delta}}.$$

Moreover, there exist $C > 0$, $u' > 0$ and $0 < v' < 1$ such that

$$P\{n_6 > n\} \leq Ce^{-u'n^{v'}}.$$

Using the above statements and following the same strategy as in the proof of future correlations we conclude decay of past correlations.

9. APPENDIX

9.1. **Sub-polynomial tail estimates.**

Lemma 9.1. Let $a > 1$ and $b > 0$. Then

$$\sum_{k>n} \frac{(\log k)^b}{k^a} \sim \frac{1}{a-1} \frac{(\log n)^b}{n^{a-1}}.$$

Proof. The proof is based on integration by parts. Since $\frac{(\log x)^b}{x^a}$ is monotonically decreasing on $(C, +\infty)$ for C big enough, we have

$$\sum_{k>n} \frac{(\log k)^b}{k^a} \leq \int_n^\infty \frac{(\log x)^b}{x^a}.$$

Let $K = [b] + 1$. Then first making change of variables $y = \log x$ and the integrating by parts K times we obtain

$$\int_n^\infty \frac{(\log x)^b}{x^a} = \frac{1}{a-1}(\log n)^b n^{1-a} + \sum_{i=2}^{K-1}(a-1)^{-i} \prod_{j=0}^{i-1}(b-j) + I_k(a,b), \tag{9.1}$$

where $I_k(a,b) = (a-1)^{-K} \prod_{j=0}^{K-1}(b-j) \int_{\log n}^\infty y^{b-K} e^{(1-a)y} dy$. Since $b - K < 0$ we conclude that $I_k(a) \leq (a-1)^{-K-1} \prod_{j=0}^{K-1}(a-j) n^{1-a}$, This shows that the dominant term in (9.1) is $\frac{1}{a-1}(\log n)^b n^{1-a}$. □

Lemma 9.2. Suppose $a > 0$ and $a_k \sim \frac{1}{(\log k)^a}$. Then $\sum_{k=2}^n a_k \sim \frac{n}{(\log n)^a}$.

Proof. A straightforward estimate, using the fact that $\sum_{k=2}^n \frac{1}{(\log k)^a} \to \infty$ shows that $\sum_{k=2}^n a_k \to \infty$ and $\sum_{k=2}^n a_k \sim \sum_{k=2}^n \frac{1}{(\log k)^a}$. We work with the latter sum. An elementary estimate shows

$$\int_2^{n+1} \frac{dx}{(\log x)^a} \leq \sum_{k=2}^n \frac{1}{(\log k)^a} \leq \frac{1}{(\log 2)^a} + \int_2^n \frac{dx}{(\log x)^a}.$$

Therefore $\sum_{k=2}^n \frac{1}{(\log k)^a} \sim \int_2^n \frac{dx}{(\log x)^a}$. We now estimate the integral.

$$\int_2^n \frac{dx}{(\log x)^a} = \frac{n}{(\log n)^a} - \frac{2}{(\log 2)^a} + a \int_2^n \frac{dx}{(\log x)^{a+1}}, \tag{9.2}$$

using integration by parts. The first term above is the claimed rate, and the second term is clearly $o\left(\frac{n}{(\log n)^a}\right)$. We will show the same is true for the third, integral term. We first upper-bound as follows:

$$a \frac{(\log n)^a}{n} \int_2^n \frac{dx}{(\log x)^{a+1}} \leq \frac{a}{n} \int_2^n \frac{dx}{\log x}.$$

Now simply estimate the right hand side by

$$\begin{aligned}\frac{a}{n}\int_2^n \frac{dx}{\log x} &= \frac{a}{n}\int_2^{\sqrt{n}} \frac{dx}{\log x} + \frac{a}{n}\int_{\sqrt{n}}^n \frac{dx}{\log x} \\ &\leq \frac{a}{\log 2}\frac{\sqrt{n-2}}{n} + \frac{a}{\log\sqrt{n}}\frac{n-\sqrt{n}}{n}.\end{aligned}$$

Since both terms are $o(1)$ in n we are done. □

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