# The Spectrum of Non-Local Discrete Schrödinger Operators with a $\delta$-Potential

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**Abstract**

The behaviour of the spectral edges (embedded eigenvalues and resonances) is discussed at the two ends of the continuous spectrum of non-local discrete Schrödinger operators with a $\delta$-potential. These operators arise by replacing the discrete Laplacian by a strictly increasing $C^1$-function of the discrete Laplacian. The dependence of the results on this function and the lattice dimension are explicitly derived. It is found that while in the case of the discrete Schrödinger operator these behaviours are the same no matter which end of the continuous spectrum is considered, an asymmetry occurs for the non-local cases. A classification with respect to the spectral edge behaviour is also offered.

# 1 Introduction

## 1.1 Non-local discrete Schrödinger operators

The spectrum of discrete Schrödinger operators has been widely studied for both combinatorial Laplacians and quantum graphs; for some recent summaries see [C97, G01, BCFK06, EKKST08, BK12, P12, KS13] and the references therein. Specifically, eigenvalue behaviours of discrete Schrödinger operators on $l^2(\mathbb{Z}^d)$ are discussed in e.g. [EKW10, BS12, HSSS12]. However, for discrete non-local (such as fractional) Schrödinger operators only few results are known.

In this paper we define generalized discrete Schrödinger operators which include discrete fractional Schrödinger operators and others whose counterparts on $L^2(\mathbb{R}^d)$

are currently much studied [BB00, BB09, LM12, K13]. In [HIL12] we have introduced a class of generalized Schrödinger operators whose kinetic term is given by so called Bernstein functions of the Laplacian. These operators are non-local and via a Feynman-Kac representation generate subordinate Brownian motion killed at a rate given by the potential. Their discrete counterparts studied in this paper also have a probabilistic interpretation in that they generate continuous time random walks with jumps on $\mathbb{Z}^d$.

In the present paper we consider a class of Schrödinger operators obtained as a strictly increasing $C^1$-function of the discrete Laplacian and a $\delta$-potential. This includes, in particular, Bernstein functions (see below) of the discrete Laplacian. In the presence of a $\delta$-potential the above probabilistic picture then describes free motion with a "bump" which can be interpreted as an impurity in space. Our aim here is to investigate the spectrum of such operators, specifically, embedded eigenvalues and resonances at the edges of the continuous spectrum.

Let $d \geq 1$ and $L$ be the standard discrete Laplacian on $l^2(\mathbb{Z}^d)$ defined by

$$L\psi(x) = \frac{1}{2d} \sum_{|x=y|=1} (\psi(y) - \psi(x)).$$

Also, let $V(x) = v\delta_{x,0}$ be $\delta$-potential with mass $v$ concentrated on $x = 0$, i.e., $V\psi(x) = 0$ for $x \neq 0$ and $V\psi(x) = v\psi(x)$. Then the operator

$$h = L + v\delta_{x,0}, \quad v \in \mathbb{R} \tag{1.1}$$

is the discrete Schrödinger operator with $\delta$-potential. In order to define a non-local version of $h$, we use Fourier transform on $l^2(\mathbb{Z}^d)$. Let $\mathbb{T}^d = [-\pi, \pi]^d$ be the $d$-dimensional torus, and set

$$\mathscr{H} = L^2(\mathbb{T}^d). \tag{1.2}$$

The Fourier transform $\mathscr{F} : l^2(\mathbb{Z}^d) \to \mathscr{H}$ is then defined by $\mathscr{F}\psi(\theta) = \sum_{n\in\mathbb{Z}^d} \psi(n)e^{-in\cdot\theta}$ for $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$. Then the discrete Laplacian $L$ transforms as

$$\mathscr{F}L\mathscr{F}^{-1} = \frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1),$$

i.e., the right hand side above is a multiplication operator on $\mathscr{H}$. In this paper we use a non-local discrete Laplacian $\Psi(L)$ defined for a suitable function $\Psi$ by applying Fourier transform.

**Definition 1.1 (Non-local discrete Laplace and Schrödinger operators)** For a given $\Psi \in C^1((0,\infty))$ such that $\Psi'(x) > 0$, $x \in (0,\infty)$, we define the *non-local discrete Laplacian* $\Psi(L)$ by

$$\Psi(L) = \mathscr{F}^{-1}\Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\mathscr{F}.$$

Also, we call

$$h = \Psi(L) + v\delta_{x,0}, \quad v \in \mathbb{R}, \tag{1.3}$$

*non-local discrete Schrödinger operator with $\delta$-potential.*

An example of such a function is $\Psi(u) = u^{\alpha/2}$, $0 < \alpha < 2$, which describes a discrete Laplacian of fractional order $\alpha/2$. Other specific choices will be given in Example 2.9 below.

Under Fourier transform (1.3) is mapped into

$$H_v = \mathscr{F} h \mathscr{F}^{-1} = \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right) + v(\Omega, \cdot)_{\mathscr{H}}\Omega, \tag{1.4}$$

where $\Omega = (2\pi)^{-d/2}\mathbb{1} \in \mathscr{H}$.

Since $\sigma(L) = [0, 2]$ and $\Psi$ is strictly increasing, it is immediate that $\sigma(\Psi(L)) = \Psi([0, 2]) = [\Psi(0), \Psi(2)]$. In what follows we consider the spectrum of $H_v$ instead of $h$. Note that the map $\Phi \mapsto v(\Omega, \Phi)\Omega$ is a rank-one operator, and thus the continuous spectrum of the rank-one perturbation $H_v$ of $L$ is $[\Psi(0), \Psi(2)]$, for every $v \in \mathbb{R}$.

## 1.2 $\Psi(*)$-resonances and $\Psi(*)$-modes

As it will be seen below, for a sufficiently large value of $-v > 0$ there exists an eigenvalue $E_-(v)$ of $H_v$ strictly smaller than $\Psi(0)$. Suppose that $E_-(v) \uparrow \Psi(0)$ as $v \uparrow v_0$ with some $v_0 \neq 0$. If $\Psi(0)$ is an eigenvalue of $H_{v_0}$, we call the eigenvector associated with $\Psi(0)$ a $\Psi(0)$-*mode*. If $\Psi(0)$ is not an eigenvalue of $H_{v_0}$, we call it a $\Psi(0)$-*resonance*. Similarly, for a sufficiently large $v > 0$ it will be seen that there exists an eigenvalue $E_+(v)$ strictly larger than $\Psi(2)$. Suppose that $E_+(v) \downarrow \Psi(2)$ as $v \downarrow v_2$ with some $v_2 \neq 0$. If $\Psi(2)$ is an eigenvalue of $H_{v_2}$, we call the eigenvector associated with $\Psi(2)$ a $\Psi(2)$-*mode*, and a $\Psi(2)$-*resonance* whenever $\Psi(2)$ is not an eigenvalue of $H_{v_2}$.

For the discrete Schrödinger operator $L + V$ these modes and resonances were studied in e.g. [HSSS12], in particular, their dependence on the dimension $d$. For $d = 1, 2$, there is no 0-mode, 2-mode, 0-resonance or 2-resonance, for $d = 3, 4$ there are 0 and 2-resonances, and for $d \geq 5$ there are 0 and 2-modes. This shows that the eigenvalue behaviour at both edges (0 and 2) is the same (see Table 1). As it will be seen below, for the case of the fractional Laplacian we have the remarkable fact that the edge behaviours are in general different at the two sides (see Table 2). Note that $\sigma(\sqrt{L}) = [0, \sqrt{2}]$.

| | 2-mode | 2-resonance | 0-mode | 0-resonance |
|---|---|---|---|---|
| $d = 1$ | no | no | no | no |
| $d = 2$ | no | no | no | no |
| $d = 3$ | no | yes | no | yes |
| $d = 4$ | no | yes | no | yes |
| $d \geq 5$ | yes | no | yes | no |

Table 1: Modes and resonances of $L + V$

| | $\sqrt{2}$-mode | $\sqrt{2}$-resonance | 0-mode | 0-resonance |
|---|---|---|---|---|
| $d = 1$ | no | no | no | no |
| $d = 2$ | no | no | no | yes |
| $d = 3$ | no | yes | yes | no |
| $d = 4$ | no | yes | yes | no |
| $d \geq 5$ | yes | no | yes | no |

Table 2: Modes and resonances of $\sqrt{L} + V$

# 2 Eigenvalues

## 2.1 A criterion for determining the eigenvalues

Consider the eigenvalue equation

$$H_v \Phi = E\Phi$$

or, equivalently,

$$E\Phi - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\Phi = v(\Omega, \Phi)\Omega. \tag{2.1}$$

The following result gives an integral test to spot the eigenvalues of $H_v$.

**Lemma 2.1** *$E$ is an eigenvalue of $H_v$ for a given $v$ if and only if*

$$\int_{\mathbb{T}^d} \frac{1}{\left(E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\right)^2} d\theta < \infty \tag{2.2}$$

*and*

$$\int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta \neq 0. \tag{2.3}$$

*Furthermore, if $E$ is an eigenvalue of $H_v$, then the coupling constant $v$ satisfies*

$$v = (2\pi)^d \left( \int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)} d\theta \right)^{-1}. \tag{2.4}$$

Proof. To show the necessity part, suppose that $E$ is an eigenvalue and $\Phi$ an associated eigenvector. Assuming $(\Omega, \Phi) = 0$, we have $(\Omega, H_v\Phi) = 0$ and thus $(H_v\Omega, \Phi) = (\mathbb{1}, \Phi) + v\Phi(0) = v\Phi(0) = 0$. Hence $H_v\Phi(x) = L\Phi(x) = E\Phi(x)$, for all $x \in \mathbb{Z}^d$. Since $L$ has no point spectrum, $\Phi = E\Phi$ is a contradiction. This gives

$$(\Omega, \Phi) \neq 0 \quad \text{and} \quad \Phi = \frac{(\Omega, \Phi)}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)} \in \mathscr{H}.$$

Thus (2.2) follows, and $(\Omega, \Phi) \neq 0$ implies (2.3).

For the sufficiency part, suppose now that (2.2) and (2.3) hold. Define the $L^2(\mathbb{Z}^d)$-function

$$\Phi = \frac{cf}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)}$$

with a chosen $c$. It is straightforward to see that $\Phi$ satisfies $H_v\Phi = E\Phi$ whenever for $v$

$$v(2\pi)^{-d} \int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)} d\theta = 1 \tag{2.5}$$

holds. By (2.3) it follows that there exists $c$ such that (2.5) is satisfied, hence $E$ is an eigenvalue of $H_v$. □

In order to investigate $\Psi(*)$-resonances and $\Psi(*)$-modes we use Lemma 2.1 and estimate the two integrals

$$I(E) = \int_{\mathbb{T}^d} \frac{1}{|E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)|^2} d\theta, \tag{2.6}$$

$$J(E) = \int_{\mathbb{T}^d} \frac{1}{E - \Psi\left(\frac{1}{d}\sum_{j=1}^d (\cos\theta_j + 1)\right)} d\theta \tag{2.7}$$

at the two ends $E = \Psi(*)$ of the interval $[\Psi(0), \Psi(2)]$.

## 2.2 The location of eigenvalues

**Lemma 2.2** *Let $E \in \mathbb{R} \setminus [\Psi(0), \Psi(2)]$. Then there exists $v \neq 0$ such that $E$ is an eigenvalue of $H_v$.*

PROOF. In this case it is easily seen that $I(E) < \infty$ and $J(E) \neq 0$. Then $E$ is an eigenvalue and $v$ is given by (2.4). □

**Lemma 2.3** *We have $\sigma(H_v) \cap (\Psi(0), \Psi(2)) = \emptyset$, for every $v \neq 0$.*

PROOF. Due to monotonicity of $\Psi$, there is a unique $x \in (0, 2)$ such that $\Psi(E) = \Psi(x)$. Thus

$$\left| E - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)\right| \leq C\left|\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1 - x)\right|$$

with some $C > 0$. Hence

$$I(E) \geq \int_{\mathbb{T}^d} \frac{1}{|C\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1 - x)|^2} d\theta.$$

It is directly seen that the right hand side diverges, and thus the lemma follows. □

Next consider the cases $E = \Psi(2)$ and $E = \Psi(0)$. For a systematic discussion of the eigenvalue behaviour of $H_v$ we introduce the following concept.

**Definition 2.4** We say that $\Psi$ is of $(a, b)$-*type* whenever

$$\lim_{x\to 0+} \frac{\Psi(x) - \Psi(0)}{x^a} \neq 0, \tag{2.8}$$

$$\lim_{x\to 0} \frac{\Psi(2) - \Psi(2 - x)}{x^b} \neq 0. \tag{2.9}$$

**Lemma 2.5** *Let $\Psi$ be of $(a, b)$-type. Then we have the following behaviour.*

*(1) $J(E) \neq 0$ for both $E = \Psi(0)$ and $E = \Psi(2)$.*

*(2) For $E = \Psi(2)$ we have that $I(E) < \infty$ if and only if $d \geq 1+4a$, and $J(E) < \infty$ if and only if $d \geq 1 + 2a$.*

*(3) For $E = \Psi(0)$ we have that $I(E) < \infty$ if and only if $d \geq 1+4b$, and $J(E) < \infty$ if and only if $d \geq 1 + 2b$.*

PROOF. Since $\Psi$ is strictly increasing, the first statement follows directly.

Let $\Psi$ be of $(a, b)$-type. Then we have at $\theta \approx (0, \ldots, 0)$,

$$\Psi(2) - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right) \approx \left(\frac{1}{2d}\sum_{j=1}^{n}\theta_j^2\right)^a$$

and at $\theta \approx (\pi, \dots, \pi)$,

$$\Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right) - \Psi(0) \approx \left(\frac{1}{2d}\sum_{j=1}^{n}(\theta_j - \pi)^2\right)^b.$$

Hence

$$I(\Psi(2)) \approx \int_{\mathbb{T}^d} \frac{1}{\left(\sum_{j=1}^{n}\theta_j^2\right)^{2a}} d\theta \approx \int_0^1 \frac{r^{d-1}}{r^{4a}} dr,$$

and similarly

$$J(\Psi(2)) \approx \int_{\mathbb{T}^d} \frac{1}{(\sum_{j=1}^{n}\theta_j^2)^b} d\theta \approx \int_0^1 \frac{r^{d-1}}{r^{2b}} dr.$$

Thus the lemma follows for $E = \Psi(2)$. For the case of $E = \Psi(0)$ the proof is similar. □

From these lemmas we can derive the spectral edge behaviour of $H_v$. The next theorem is the main result in this paper.

**Theorem 2.6** *Assume that $\Psi$ is of $(a, b)$-type. Let*

$$v_2 = (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{1}{\Psi(2) - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta\right)^{-1} > 0, \tag{2.10}$$

$$v_0 = (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{1}{\Psi(0) - \Psi\left(\frac{1}{d}\sum_{j=1}^{d}(\cos\theta_j + 1)\right)} d\theta\right)^{-1} < 0. \tag{2.11}$$

*The spectral edge behaviour of $H_v$ is as follows.*

*(1) Suppose that $v > 0$. Then the following cases occur:*

- *(i) Let $d < 1 + 2b$. Then for all $v > 0$ there exists an eigenvalue $E > \Psi(2)$.*
- *(ii) Let $1 + 2b \le d < 1 + 4b$. Then for $v > v_2$ there exists an eigenvalue $E > \Psi(2)$, while for $v \le v_2$ there is no eigenvalue.*
- *(iii) Let $1 + 4b \le d$. Then for $v > v_2$ there exists an eigenvalue $E > \Psi(2)$, for $v = v_2$ the value $E = \Psi(2)$ is an eigenvalue, while $v < v_2$ there is no eigenvalue.*

*(2) Suppose that $v < 0$. Then the following cases occur:*

- *(i) Let $d < 1 + 2a$. Then for all $v < 0$ there exists an eigenvalue $E < \Psi(0)$.*

*(ii)* *Let* $1 + 2a \leq d < 1 + 4a$. *Then for* $v < v_0$ *there exists an eigenvalue* $E < \Psi(0)$, *while for* $v \geq v_0$ *there is no eigenvalue.*

*(iii)* *Let* $1 + 4a \leq d$. *Then for* $v < v_0$ *there exists an eigenvalue* $E < \Psi(0)$, *for* $v = v_0$ *the value* $E = \Psi(0)$ *is an eigenvalue, while for* $v > v_0$ *there is no eigenvalue.*

PROOF. Consider the case $v > 0$ and let $d < 1 + 2b$. Then for all $E > \Psi(2)$ we have $I(E) < \infty$ and $J(E) \neq 0$. Thus there exists $v$ such that $E$ is an eigenvalue of $H_v$.

Let $1 + 2b \leq d < 1 + 4b$. Then for all $E > \Psi(2)$ we have that $I(E) < \infty$ and $J(E) \neq 0$. Thus $E$ is an eigenvalue of $H_v$. Since $J(E) < \infty$, it follows that $E \downarrow \Psi(2)$ as $v \downarrow v_2 > 0$. However, $E = \Psi(2)$ is not an eigenvalue since $I(E) = \infty$.

Let $d \geq 1 + 4b$. Then for all $E > \Psi(2)$ we have $I(E) < \infty$ and $J(E) \neq 0$. Thus $E$ is an eigenvalue of $H_v$. Since $J(E) < \infty$, we obtain $E \downarrow \Psi(2)$ as $v \downarrow v_2 > 0$. Since $I(E) < \infty$, we have that $E = \Psi(2)$ is also an eigenvalue. The cases for $v < 0$ can be dealt with similarly.

□

**Remark 2.7** Note that in general $-v_0 \neq v_2$.

**Remark 2.8** From the above it is seen that the spectral edge behaviour of $H_v$ depends on the dimension $d$ as well as on the parameters $a$ and $b$, and the result is different according to which edge is considered. For a summary see the tables below.

| $v > 0$ | $d < 1 + 2b$ | $1 + 2b \leq d < 1 + 4b$ | $1 + 4b \leq d$ |
|---|---|---|---|
| $\Psi(2)$-mode | no | no | yes |
| $\Psi(2)$-resonance | no | yes | no |

| $v < 0$ | $d < 1 + 2a$ | $1 + 2a \leq d < 1 + 4a$ | $1 + 4a \leq d$ |
|---|---|---|---|
| $\Psi(0)$-mode | no | no | yes |
| $\Psi(0)$-resonance | no | yes | no |

Table 3: $\Psi(0)$ and $\Psi(2)$-modes and resonances

It is worthwhile to see the implications more closely for some specific choices of function $\Psi$.

**Example 2.9**

(1) *Discrete Schrödinger operator:* Let $\Psi(u) = u$. Then $\Psi$ is of $(1, 1)$-type and $H_v = L + V$. See Table 1.

(2) *Fractional discrete Schrödinger operator*: Let $\Psi(u) = u^{\alpha/2}$ for $0 < \alpha < 2$. Then $\Psi$ is of $(\alpha/2, 1)$-type and $H_v = L^{\alpha/2} + V$.

(3) *Relativistic fractional discrete Schrödinger operator*: Let $\Psi(u) = (u+m^{2/\alpha})^{\alpha/2} - m$ for $0 < \alpha < 2$ and $m > 0$. Then $\Psi$ is of $(1, 1)$-type.

(4) *Discrete jump-diffusion operator*: Let $\Psi(u) = u + bu^{\alpha/2}$ with $0 < \alpha < 2$. Then $\Psi$ is of $(\alpha/2, 1)$-type.

(5) *Rotationally symmetric geometric discrete $\alpha$-stable operator*: Let $\Psi(u) = \log(1 + u^{\alpha/2})$ for $0 < \alpha < 2$. Then $\Psi$ is of $(\alpha/2, 1)$-type.

(6) *Higher order discrete Laplacian*: Let $\Psi(u) = u^{\beta}$ for $\beta > 1$. Then $\Psi$ is of $(\beta, 1)$-type.

(7) *Bernstein functions of the discrete Laplacian*: Let $\Psi$ be a Bernstein function with vanishing right limits, i.e., $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ which can be represented in the form $\Psi(u) = bu + \int_0^\infty (1 - e^{-uy})\nu(dy)$, where $b \geq 0$ and $\nu$ is a Lévy measure with mass on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge y)\nu(dy) < \infty$. Then it follows that $\Psi'(2) = b + \int_0^\infty ye^{-2y}\nu(dy) \neq 0$. Furthermore, since $\Psi$ is concave, we have $a = \alpha/2$ with some $\alpha \geq 2$. Hence $\Psi$ is of $(\alpha/2, 1)$-type with some $0 \leq \alpha \leq 2$. Note that the first five examples above are specific cases of Bernstein functions.

# 3 A classification of spectral edge behaviour

The functions $\Psi$ of the discrete Laplacian can be classified according to the behaviour of the eigenvalues at the two ends of the interval $[\Psi(0), \Psi(2)]$.

**Definition 3.1** We call $\Psi$ *normal type* if $\Psi$ is $(1, 1)$-type, and *fractional type* if $\Psi$ is $(\alpha/2, 1)$-type with $0 < \alpha < 2$.

The two types show qualitatively different behaviour and we discuss them separately.

## 3.1 Normal type

Let $\Psi$ be of normal type. In this case the spectral edge behaviour is the same as that of the discrete Schrödinger operator $L + V$. The following result has been obtained in [HSSS12].

**Proposition 3.2** *Let $\Psi$ be normal type. We have the following cases.*

*(1) Let $d = 1$ or 2. For every $v > 0$ there exists an eigenvalue $E > \Psi(2)$, and for every $v < 0$ an eigenvalue $E < \Psi(0)$.*

*(2) Let $d = 3$ or $4$. If $v > 0$, then there exists $v_2 > 0$ such that for all $v > v_2$ an eigenvalue $E > \Psi(2)$ exists, and for $v \leq v_2$ no eigenvalue exists. If $v < 0$, then there exists $v_0 < 0$ such that for all $v < v_0$ an eigenvalue $E < \Psi(0)$ exists, and for $v < v_0$ no eigenvalue exists.*

*(3) Let $d \geq 5$. If $v > 0$, then there exists $v_2 > 0$ such that for all $v > v_2$ an eigenvalue $E > \Psi(2)$ exists, for $v = v_2$ the value $E = \Psi(2)$ is an eigenvalue, and for $v < v_2$ no eigenvalue exists. If $v < 0$, then there exists $v_0 < 0$ such that for all $v < v_0$ an eigenvalue $E < \Psi(0)$ exists, for $v = v_0$ the value $E = \Psi(0)$ is an eigenvalue, and for $v > v_0$ no eigenvalue exists.*

Thus the spectral edge behaviour for positive and negative $v$ is qualitatively the same, and the details only depend on the dimension $d$.

## 3.2 Fractional type

In the fractional type case we have the following spectral edge behaviour.

**Theorem 3.3** *Let $\Psi$ be of fractional type. The following cases occur.*

*(1) If $v > 0$, then the spectral edge behaviour is the same as for normal type $\Psi$ with $v > 0$.*

*(2) If $v < 0$, then we have the following cases:*

*(i) Let $d < 1 + \alpha$. Then for every $v < 0$ there exists an eigenvalue $E < 0$.*

*(ii) Let $1 + \alpha \leq d < 1 + 2\alpha$. There exists $v_0 < 0$ such that for all $v < v_0$ an eigenvalue $E < 0$ exists, while for $v \leq v_0$ no eigenvalue exists.*

*(iii) Let $d \geq 1 + 2\alpha$. There exists $v_0 < 0$ such that for all $v < v_0$ an eigenvalue $E < 0$ exists, for $v = v_0$ the value $E = 0$ is an eigenvalue, and for $v > v_0$ no eigenvalue exists.*

In the fractional case it is seen that the edge behaviour for positive and negative $v$ are in general different from each other, in contrast with the normal type case.

### 3.2.1 The case of $\alpha = 1$

For $\alpha = 1$ the spectral edge behaviour of $H_v = \sqrt{L} + V$ is displayed for dimensions $d = 1, ..., 4$ and $d \geq 5$ in Table 2. For dimensions $d = 2, 3, 4$ the edge behaviours at 0 and $\sqrt{2}$ are again different. We have displayed the specific situations in Figures 1-4 below, where $\oplus$ denotes a resonance, $\bullet$ an eigenvalue, and $\times$ denotes a value which is not an eigenvalue.

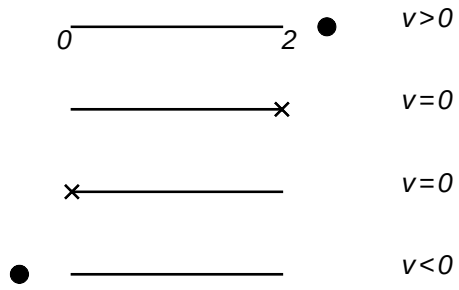


Figure 1: $\alpha = 1$ and $d = 1$

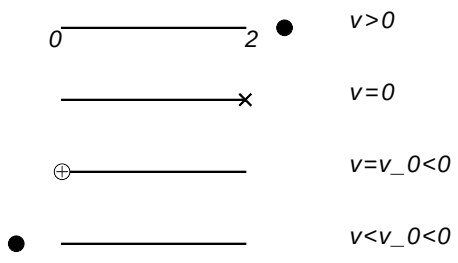


Figure 2: $\alpha = 1$ and $d = 2$

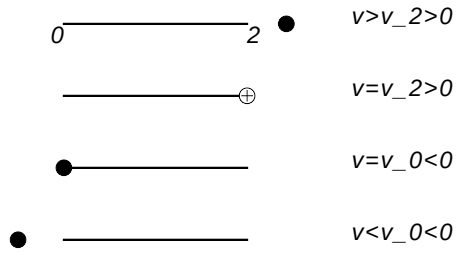


Figure 3: $\alpha = 1$ and $d = 3, 4$

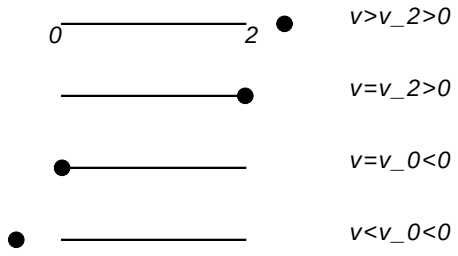


Figure 4: $\alpha = 1$ and $d \geq 5$

### 3.2.2 Massless and massive cases

Consider the Bernstein function $\Psi(u) = \sqrt{u + m^2} - m$ with $m \geq 0$. This allows to define the relativistic discrete Schrödinger operator $\sqrt{L + m^2} - m + V$. Then it follows that $\Psi(u)$ is $(1, 1)$-type for $m > 0$, and $(1/2, 1)$-type for $m = 0$. In particular, the edge behaviours of $\sqrt{L} + V$ and $\sqrt{L + m^2} - m + V$ are different. More generally, consider the Bernstein function $\Psi(u) = (u + m^{2/\alpha})^{\alpha/2} - m$, with $0 < \alpha < 2$ and $m \geq 0$. This defines the relativistic rotationally symmetric $\alpha$-stable operator $(L + m^{2/\alpha})^{\alpha/2} - m$. We conclude that $\Psi(u)$ is of $(1, 1)$-type for $m > 0$ but of $(\alpha/2, 1)$-type for $m = 0$. Thus the edge behaviours of $(L + m^{2/\alpha})^{\alpha/2} - m + V$ and $(L)^{\alpha/2} + V$ are different.

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