

# Resolution of a shock in hyperbolic systems modified by weak dispersion

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## Abstract

We present a way to deal with dispersion-dominated “shock-type” transition in the absence of completely integrable structure for the systems that one may characterize as strictly hyperbolic regularized by a small amount of dispersion. The analysis is performed by assuming that, the dispersive shock transition between two different constant states can be modelled by an expansion fan solution of the associated modulation (Whitham) system for the short-wavelength nonlinear oscillations in the transition region (the so-called Gurevich – Pitaevskii problem). We consider as single-wave so bi-directional systems. The main mathematical assumption is that of hyperbolicity of the Whitham system for the solutions of our interest. By using general properties of the Whitham averaging for a certain class of nonlinear dispersive systems and specific features of the Cauchy data prescription on characteristics we derive a set of transition conditions for the dispersive shock, actually by-passing full integration of the modulation equations. Along with model KdV and mKdV examples, we consider a non-integrable system describing fully nonlinear ion-acoustic waves in collisionless plasma. In all cases our transition conditions are in complete agreement with previous analytical and numerical results.

Modern theory of dispersive shocks is based on the analysis of the Whitham averaged equations describing modulations of nonlinear short-wavelength oscillations in the transition region between two smooth regimes. If the wave dynamics is governed by one of the completely integrable equations, exact solutions in terms of the Riemann invariants are also available for the corresponding Whitham system providing full asymptotic description of such a transition. For nonintegrable systems describing many physically important cases of nonlinear dispersive wave propagation such modulation solutions are not readily (if at all) available. In this paper we show that, by using some general properties of the Whitham equations connected with their “averaged” origin one is able to obtain a set of transition conditions representing the “dispersive” analog of the

traditional shock conditions of classical dissipative gas dynamics. The developed method does not make use of the Riemann invariants for the modulation equations and can be applied to nonintegrable conservative systems. In particular, the obtained conditions allow for determination of the lead solitary wave amplitude in terms of a jump for hydrodynamic variables across the dispersive shock.

## 1 Introduction

It is well known that the resolution of breaking singularities in dispersive media occurs through generation of short-wavelength nonlinear oscillations. The wave-like transition between two smooth or constant hydrodynamic states is generally called a dispersive shock (or an undular bore, especially in the context of water waves). The main observable feature of the dispersive shock is formation of solitary waves in the vicinity of one of its edges. At the opposite edge, the wave structure degenerates into linear wavepacket. The phenomenon of the dispersive shock formation is quite ubiquitous and its physical contexts range from gravity water waves and bubbly fluid dynamics to space plasma physics, fibre optics and Bose-Einstein condensates.

One should be clear from the very beginning that purely dispersive resolution of a shock is a physical idealization and some amount of dissipation is inevitably present in real continuous media so we distinguish between two types of dispersive shocks depending on the actual role of dissipation relative to that of nonlinearity and dispersion in the wave development and evolution. There is some terminological confusion in the literature where the terms “dispersive shock”, “undular bore” and “collisionless shock” are used as for the dispersion-dominated waves so for the waves where dispersion and dissipation are in balance. In both cases, dispersion plays the decisive role in the formation of local oscillatory structure but the global properties of the oscillations zone in the two cases are drastically different.

The weakly dissipative undular bores (we use here this term to distinguish from our main subject – conservative dispersive shocks) despite their oscillatory structure exhibit global properties characteristic for classical, turbulent bores or shock waves: they have steady (though oscillatory in space) profile and constant effective width such that the speed of the shock propagation and the transition conditions can be derived within the frame of the classical theory of hyperbolic conservation laws (see [1] for instance). The qualitative theory of such steady undular bores has been first developed by Benjamin and Lighthill [2] in the context of shallow-water waves and by Sagdeev [3] for rarefied plasma flows. The quantitative description of the weakly dissipative undular bores has been made in [4], [5], [6] on the basis of the unidirectional KdV-Burgers equation and in [7] using the integrable version of the bi-directional Boussinesq equations modified by small viscous term.

The developed in [2] – [7] theory of steady undular bores, however, is valid only for the fully established regime when nonlinearity, dispersion and dissipation are in balance. Contrastingly, in the *dispersion-dominated* case, the traditional analysis of the mass, momentum and energy balance across the undular bore transition can not be applied (at least directly). The reason for that is that the boundaries of the dissipationless undular bore (a dispersive shock) diverge with time, i.e instead of a single shock speed defined by the jump conditions one has now two different speeds  $s^+ > s^-$  determining motion of the transition region

boundaries. These speeds, however, can not be found without the analysis of the nonlinear oscillatory structure of the transition region since in the dissipationless case dispersion not only dramatically modifies the fine structure of the shock transition but also, along with nonlinearity, determines its location. The “conservative” dynamics of dispersive shocks is of a considerable interest on its own and also, in many cases, can be viewed as an unsteady intermediate asymptotic in a general setting when the small dissipation is taken into account. Physical examples of such expanding dispersive shocks include atmospheric undular bores (morning glory) [8], optical shocks in the long-distance optical communication systems [9] and collisionless shocks in Bose-Einstein condensates [10].

In a weakly nonlinear case, when the original conservative system can be approximated by one of the exactly integrable equations, the study of the dispersive shock phenomenon have stimulated discovery of a whole new class of mathematical problems which can be broadly described as the singular semi-classical limits in integrable systems. In the Lax-Levermore-Venakides theory [11], [12] (see also [13]) developed originally for the KdV equation, and more recently extended to the defocusing NLS [14] and focusing mKdV [15] equations the evolution of the dispersive shock is modelled by the zero dispersion limit of the exact multi-soliton (multi-gap) solution of the original dispersive wave equation. The main characteristic feature of the zero-dispersion limit is co-existence of smooth and rapidly oscillating regions in the solution after the breaktime. In the smooth regions the zero-dispersion limit exists in a strong sense and is given by the classical solution of the dispersionless equation whereas in the oscillating regions this limit exists in a weak, averaged sense and is governed by a certain system of quasi-linear equations *different from the dispersionless limit*. This system turned out to coincide with the modulation equations obtained in 1960’s by Whitham [16] by averaging nonlinear single-phase wavepackets and later generalised by Flaschka, Forest and McLaughlin [17] to a multiphase case. The subsequent development of the Lax-Levermore theory [18], [13] has shown that the local waveform in the oscillating regions is indeed described by the multiphase solutions. The weak limits in the Lax-Levermore problem then can be regarded as the result of the Whitham averaging over these solutions.

A direct formulation of the dispersive shock problem in terms of the Whitham equations had been proposed (although without rigorous justification) much earlier by Gurevich and Pitaevskii (GP) [19] who supplied the Whitham system for the KdV equation with natural matching conditions at the boundaries separating smooth and oscillating regions and solved the problem analytically for the initial data in the form of a step. The discovery by Tsarev of the generalised hodograph method [20], and Krichever’s algebro-geometric construction [21] allowed for obtaining of many new exact global solutions to the Whitham equations (see for instance [22], [23] and references therein). Availability of such solutions heavily relies on the integrability of the Whitham systems considered and, in particular, on the existence of the Riemann invariant representation.

Thus, either way, integrability seems to be an essential part of the analytic theory of dispersive shocks. However, in many physically relevant situations, integrable systems although providing valuable insight into the qualitative properties of nonlinear dispersive wave propagation fail to yield satisfactory quantitative agreement with experimental data (see for instance discussion in [24] in the context of finite-amplitude internal shallow water waves). This explains the growing interest in the derivation of relatively simple non-integrable models which, while providing a more accurate description of physical effects, still seem to be amenable to analytic treatment. In the context of the dispersive shock theory this interest

is supported by a strong numerical evidence that such features of zero-dispersion limits of integrable systems as formation of oscillatory zones and weak convergence are true for the systems that are not completely integrable but are structurally similar to integrable ones (see for instance [13] [25], [26]). On the other hand, the Whitham method used in the direct Gurevich-Pitaevskii description of dispersive shocks does not require integrability from the governing system. What is needed is just the existence of the periodic travelling wave solutions and availability of sufficient number of conservation laws. So, in the absence of rigorous general approach it seems a natural idea to *assume* the single-phase Whitham description for a “non-integrable” dispersive shock and explore analytic consequences of such an assumption. Hyperbolicity of the Whitham equations (which also has to be assumed, for instance, on the grounds of numerical evidence of modulational stability) would be an essential part of such a problem formulation. Of course, such a heuristic approach requires validation. This can be made by (i) testing its results on integrable systems where exact solutions are available and (ii) by comparison with available numerical results for non-integrable systems.

In this paper, we develop the outlined approach for the the decay of an initial step problem for a broad class of systems that one may characterise as strictly hyperbolic modified by weak dispersion. It is clear that in the step decomposition problem the solution of the (quasilinear) Whitham equations must depend on  $x/t$  alone, which implies principal availability of several integrals of motion for their self-similar reductions. It is also clear that existence of the similarity solutions does not rely on complete integrability or existence of the Riemann invariants for the Whitham equations ( this, unfortunately, does not guarantee that such a solution would be available analytically in the absence of the Riemann invariants). However, the Whitham equations being obtained by averaging over periodic family have a number of special properties distinguishing them from the general class of hyperbolic quasilinear systems. In particular, they allow for exact reductions to the original dispersionless equations in the zero-amplitude (linear) and zero-wavelength (soliton) limits. This general property together with the self-similarity imposes a number of restrictions on possible values of the modulation parameters at the linear and soliton edges of the dispersive shock defined by natural boundary conditions. We derive these restrictions by finding a set of integrals along the edge characteristics of the Whitham expansion fan. These integrals, of course, are equivalent to the basic integrals for the similarity solution evaluated at the edge points. A surprising fact is that, the edge parameters are in a simple and universal way expressed in terms of the linear dispersion relation of the system and one of its nonlinear “dispersionless” characteristic velocities.

The obtained set of restrictions can be viewed as a “dispersive” replacement for the classical shock conditions and includes for a general bi-directional case

- (i) a condition for admissible jumps for hydrodynamic variables across the dispersive shock (which *does not* coincide with the classical jump conditions);
  - (ii) the speeds of the dispersive shock edges as functions of a given jump satisfying (i);
- and, since the parameters in (ii) turn out to be determined not uniquely,
- (iii) the set of inequalities selecting the unique valid set of parameters (ii) and providing consistency of the whole construction. These inequalities represent an analog of entropy conditions known in classical gas dynamics.

Thus, the obtained conditions allow one to derive main quantitative characteristics of the dispersive shock transition bypassing the integration of the Whitham system. In particular, they allow for obtaining the amplitude of the leading (or trailing depending on the sign of

the dispersion in the system) solitary wave in the dispersive shock, the major parameter in observational/experimental data. For instance, the conditions readily yield the well-known result of the original Gurevich-Pitaevskii paper [19] which reads that the amplitude of the lead solitary wave in the KdV dispersive shock is  $2\Delta$  where  $\Delta$  is the value of the initial step.

We apply the obtained general dispersive shock conditions to two integrable (KdV and defocusing mKdV) equations and one non-integrable system describing fully nonlinear ion-acoustic waves in collisionless plasma. In all cases our description is in complete agreement with previous analytic/numerical results. We also discuss the accuracy and some restrictions of the developed approach.

## 2 Gurevich-Pitaevskii problem for the KdV equation

We start with an exposition of the theory of Gurevich and Pitaevskii (GP) [19] formulated originally for the KdV equation and later generalised to other integrable equations by different authors (see [27] for a detailed introduction and many useful references). Although all formulas in this section are known very well, it is instructive for the purposes of this paper to have them handy as they will allow us later to draw parallels with “non-integrable” theory.

We take the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0. \quad (1)$$

Let the initial perturbation  $u(x, 0) = u_0(x)$  have the form of a large-scale ( $\Delta x \gg 1$ ) monotonically decreasing function with a single inflection point. As a special important case, one considers a smooth step

$$u_0(-\infty) = u^-, \quad u_0(+\infty) = u^+, \quad u'_0(x) < 0, \quad (2)$$

so that the characteristic width of the transition region, say  $l$  is finite. Of course, without loss of generality the initial step for the KdV equation can be normalized to unity but for our purposes it is convenient to retain arbitrary values for  $u^-, u^+$ ; we only assume  $u^- > u^+$ .

The qualitative picture of the KdV evolution of a smooth monotonic profile is as follows. During the initial stage of evolution,  $|u_{xxx}| \ll |uu_x|$  and one can neglect the dispersive term in the KdV (1). The evolution at this stage is approximately described by the dispersionless (classical) limit of the KdV equation:

$$u \approx \beta(x, t) : \quad \beta_t + \beta\beta_x = 0, \quad \beta(x, 0) = u_0(x). \quad (3)$$

The evolution (3) leads to a gradient catastrophe :  $t \rightarrow t_b : \quad \beta_x \rightarrow -\infty$ . For  $t > t_b$  (without loss of generality we put  $t_b = 0$ ), the dispersive term  $u_{xxx}$  should be taken into account in the vicinity of the breaking point and, as a result, the regularization of the singularity happens through the generation of small-scale nonlinear oscillations confined to a finite, albeit expanding, space region. This oscillatory structure represents a dispersive analog of a shock wave. Near the leading edge of such a *dispersive shock* (or an *undular bore* in a different terminology) the oscillations are close to successive solitary waves and in the vicinity of the trailing edge they are nearly linear (see Fig. 1). This structure has been recovered in a rigorous theory using the complete integrability of the KdV equation (see for instance [28]) but in the Gurevich-Pitaevskii approach it is an assumption based, say,

on the results of numerical simulations. This approach is consistent with the aims of this paper which develops a way to deal with generally non-integrable systems, where the IST formalism is not available in principle.

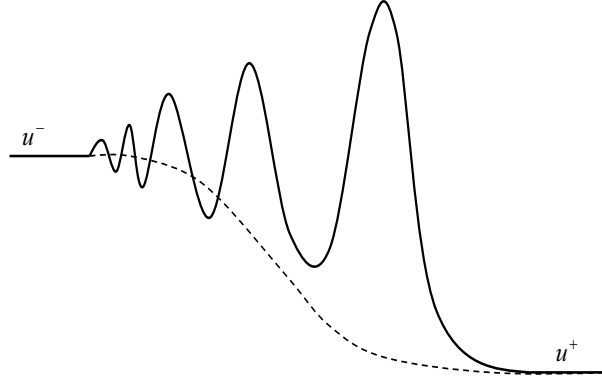


Figure 1: Oscillatory structure of the dispersive shock evolving from the initial step (dashed line).

We shall model the local waveform of the dispersive shock by the single-phase periodic solution of the KdV equation travelling with constant velocity  $c$ :  $u(x, t) = u(\theta)$ ,  $\theta = x - ct$ . This solution is specified by the ordinary differential equation

$$(u_\theta)^2 = -G(u), \quad u(\theta + 2\pi/k) = u(\theta), \quad (4)$$

where

$$G(u) = \frac{1}{3}(u - u_1)(u - u_2)(u - u_3), \quad (5)$$

$u_3 \geq u_2 \geq u_1$  being constants of integration. The phase velocity  $c$  and the wavenumber  $k$  are expressed in terms of the roots  $u_j$  as

$$c = \frac{1}{3}(u_1 + u_2 + u_3), \quad k = \pi \left( \int_{u_2}^{u_3} \frac{du}{\sqrt{-G(u)}} \right)^{-1} = \frac{\pi}{2\sqrt{3}} \frac{(u_3 - u_1)^{1/2}}{K(m)}, \quad (6)$$

where  $K(m)$  is the complete elliptic integral of the first kind. The modulus  $m$  and the amplitude  $a$  of the travelling wave are expressed in terms of  $\{u_j\}$  as

$$m = \frac{u_3 - u_2}{u_3 - u_1} \quad a = u_3 - u_2. \quad (7)$$

Eq. (4) is integrated in terms of Jacobian elliptic function  $\text{cn}(\xi; m)$  to give, up to an arbitrary phase shift,

$$u(x, t) = u_2 + a \text{cn}^2 \left( \sqrt{\frac{u_3 - u_1}{3}}(x - ct); m \right). \quad (8)$$

When  $m \rightarrow 0$  ( $u_2 \rightarrow u_3$ ) the solution (8) turns into the vanishing amplitude harmonic wave

$$u(x, t) \approx u_3 - a \sin^2[k_0(x - c_0 t)], \quad a = u_3 - u_2 \ll 1, \quad (9)$$

where  $k_0 = k(u_1, u_3, u_3)$ ,  $c_0 = c(u_1, u_3, u_3)$ . The relationship between  $c_0$  and  $k_0$  is obtained from Eqs. (6) considered in the limit  $u_2 \rightarrow u_3$ :

$$c_0 = u_3 - k_0^2 \quad (10)$$

and agrees with the KdV linear dispersion relation for small-amplitude waves propagating on the background  $u = u_3$ .

When  $m \rightarrow 1$  ( $u_2 \rightarrow u_1$ ), the cnoidal wave (8) turns into a soliton

$$u_s(x, t) = u_1 + a_s \operatorname{sech}^2[\sqrt{a_s/3}(x - c_s t)], \quad (11)$$

which speed  $c_s = c(u_1, u_1, u_3)$  is connected with the amplitude  $a_s$  by

$$c_s = u_1 + a_s/3. \quad (12)$$

In Eqs. (11), (12),  $u_1$  plays the role of a background which can be put equal to zero for the uniform solutions (as well as  $u_3$  in (9), (10)). However, this is not the case if one is interested in modulated travelling waves where the local integrals  $u_j$  become slow functions of  $x$  and  $t$  and the background term can not be eliminated by the passage to the moving reference frame. This simple fact will play an important role in what follows.

We now consider slowly modulated cnoidal waves by letting the constants of integration  $u_j$  be functions of  $x$  and  $t$  on a large spatio-temporal scale  $\Delta x \gg 1$ ,  $\Delta t \gg 1$ . Then the evolution equations for  $u_j(x, t)$  can be obtained by averaging any three KdV conservation laws  $\partial_t P_j + \partial_x Q_j = 0$  over the period of the travelling wave (8) [16] and are referred to as the modulation, or Whitham, equations. The averaging is made according to (4) as

$$\overline{F}(u_1, u_2, u_3) = \frac{k}{2\pi} \int_0^{2\pi/k} F(u(\theta; u_1, u_2, u_3)) d\theta = \frac{k}{\pi} \int_{u_2}^{u_3} \frac{F(u)}{\sqrt{-G(u)}} du. \quad (13)$$

In particular, the mean is calculated as

$$\bar{u} = u_1 + 2(u_3 - u_1)E(m)/K(m), \quad (14)$$

where  $E(m)$  is the complete elliptic integral of the second kind.

As a result, the KdV modulation system is obtained in a conservative form

$$\frac{\partial}{\partial t} \overline{P}_j(u_1, u_2, u_3) + \frac{\partial}{\partial x} \overline{Q}_j(u_1, u_2, u_3) = 0, \quad j = 1, 2, 3, \quad (15)$$

where  $\overline{P}_j, \overline{Q}_j$  are expressed in terms of the complete elliptic integrals.

One of the modulation equations can be replaced by the so-called wave number conservation law, which represents a consistency condition in the formal perturbation procedure equivalent to the Whitham averaging (see e.g. [29]),

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad (16)$$

where  $\omega = kc$  is the frequency. Of course, Eq. (16) can be obtained as a consequence of the three modulation equations (15) [16].

It has been discovered in [16] that, upon introducing symmetric combinations

$$\beta_1 = \frac{u_1 + u_2}{2}, \quad \beta_2 = \frac{u_1 + u_3}{2}, \quad \beta_3 = \frac{u_2 + u_3}{2} \quad (17)$$

the system (15) reduces to its diagonal (Riemann) form

$$\frac{\partial \beta_j}{\partial t} + V_j(\beta_1, \beta_2, \beta_3) \frac{\partial \beta_j}{\partial x} = 0, \quad j = 1, 2, 3. \quad (18)$$

where the characteristic velocities  $V_3 \geq V_2 \geq V_1$  are certain combinations of complete elliptic integrals of the first and the second kind. They can be conveniently represented in a “potential” form [30], [31]

$$V_j = \frac{\partial(kc)}{\partial \beta_j} / \frac{\partial k}{\partial \beta_j} \quad (19)$$

following from the consistency of wave number conservation law (16) with the Riemann system (18). Here  $k$ ,  $m$  and  $c$  are expressed in terms of the Riemann invariants as

$$k = \frac{\pi}{\sqrt{6}} \frac{(\beta_3 - \beta_1)^{1/2}}{K(m)}, \quad m = \frac{\beta_2 - \beta_1}{\beta_3 - \beta_1}, \quad c = \frac{1}{3}(\beta_1 + \beta_2 + \beta_3). \quad (20)$$

Of course, the existence of the Riemann invariants for the quasi-linear system of the third order (15) is a nontrivial fact and this feature is connected with the preservation of the KdV integrability under the averaging. A regular way of obtaining the KdV-Whitham system in the Riemann form has been developed by Flaschka, Forest and McLaughlin [17] using the methods of finite-gap integration. For the single-phase case of our interest an elementary general approach not requiring algebraic geometry can be found in the monograph [27]. The explicit expressions for  $V_j$  in terms of the complete elliptic integrals of the first and the second kind can be found elsewhere (see for instance [32], [29]). We will present here only their limiting properties.

In the harmonic limit

$$m = 0 : \quad V_3 = \beta_3, \quad V_2 = V_1 = 2\beta_1 - \beta_3 \quad (21)$$

so that the Whitham system reduces to

$$\beta_2 = \beta_1, \quad \frac{\partial \beta_3}{\partial t} + \beta_3 \frac{\partial \beta_3}{\partial x} = 0, \quad \frac{\partial \beta_1}{\partial t} + (2\beta_1 - \beta_3) \frac{\partial \beta_1}{\partial x} = 0. \quad (22)$$

In the soliton limit

$$m = 1 : \quad V_2 = V_1 = \frac{1}{3}(\beta_1 + 2\beta_3), \quad V_3 = \beta_3 \quad (23)$$

and the Whitham system reduces to

$$\beta_2 = \beta_3, \quad \frac{\partial \beta_1}{\partial t} + \beta_1 \frac{\partial \beta_1}{\partial x} = 0, \quad \frac{\partial \beta_3}{\partial t} + \frac{1}{3}(\beta_1 + 2\beta_3) \frac{\partial \beta_3}{\partial x} = 0. \quad (24)$$

Thus, the Whitham system admits nontrivial *exact* reductions to the dispersionless limit via singular limiting transitions  $\beta_2 \rightarrow \beta_1$  (linear limit) and  $\beta_2 \rightarrow \beta_3$  (soliton limit). In both



limits one of the Whitham equations converts into the Hopf equation, while the remaining two merge into one for the Riemann invariant along a double characteristics. Such a special structure of the Whitham equations makes it possible to formulate the following natural matching problem for (18) [19].

Let the upper  $(x, t)$  half-plane be split into three domains :  $\{(x, t > 0) : (-\infty, x^-(t)) \cup [x^-(t), x^+(t)] \cup (x^+(t), +\infty)\}$  (see Fig. 2), in which the solution is governed by different equations: outside the interval  $[x^-(t), x^+(t)]$  it is governed by the dispersionless limit (3) while within the interval  $[x^-(t), x^+(t)]$  the dynamics is described by the Whitham equations (18) so that the following matching conditions must be satisfied:

$$\begin{aligned} x = x^-(t) : \quad & \beta_2 = \beta_1, \quad \beta_3 = \beta \\ x = x^+(t) : \quad & \beta_2 = \beta_3, \quad \beta_1 = \beta \end{aligned} \quad (25)$$

where  $\beta(x, t)$  is the solution of the Hopf equation (3) and the (free) boundaries  $x^\pm(t)$  are unknown at the onset.

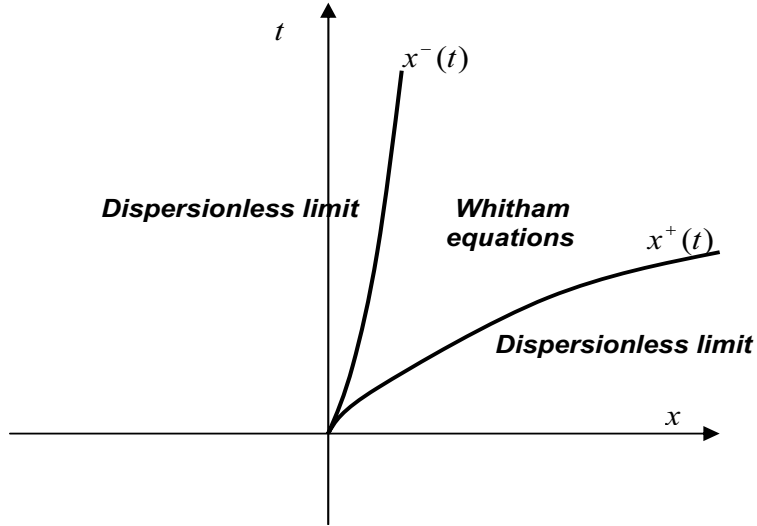


Figure 2: Splitting of the  $(x, t)$  - plane in the Gurevich – Pitaevskii problem

**Remark.** One can notice that formulated in this way, the GP problem contains an implicit assumption about the spatial structure of the dispersive shock, namely, it identifies the leading edge with the soliton and the trailing edge with the linear wave from the very beginning. Of course, for the KdV equation this wave pattern can be inferred from previous numerical simulations or from simple physical reasoning that owing to the negative sign of the dispersion in the KdV equation, the longer waves must propagate with greater speed. However, from mathematical point of view, such an assumption must be confirmed or rejected by the actual solution so, strictly speaking, at this point we actually don't know if  $x^+ > x^-$ .

For the initial data with a single breaking point, the conditions (25) uniquely define the global solution  $\beta_j(x, t)$  of the Whitham equations (18) (see [29]). It follows from Eq. (25) and the limiting properties of the Whitham velocities (21), (23) that the free boundaries  $x^\pm(t)$  are determined by the double eigenvalues of the Whitham system for  $m = 0$  ( $x = x^-(t)$ ) and  $m = 1$  ( $x = x^+(t)$ ) and are found from the ordinary differential equations

$$dx^-/dt = V_2(\beta_1, \beta_1, \beta_3)|_{x=x^-} = V_1(\beta_1, \beta_1, \beta_3)|_{x=x^-} \equiv V^-(x^-, t), \quad (26)$$

$$dx^+/dt = V_2(\beta_1, \beta_3, \beta_3)|_{x=x^+} = V_3(\beta_1, \beta_3, \beta_3)|_{x=x^+} \equiv V^+(x^+, t) \quad (27)$$

where  $\{\beta_j = \beta_j(x, t)\}$  is the solution of the GP problem. We note that the curves  $x^\pm(t)$  are sometimes referred to as the phase transition boundaries since they separate the regions of the zero-phase (external smooth flow) and the single phase (dispersive shock) solutions.

The local integrability of the Riemann system (18) is based on certain relationships between the characteristic velocities and has been established by Tsarev [20] who proposed a generalisation of the classical hodograph method applicable to the hydrodynamic-type systems with the number of field variables exceeding two. There is, however, a special important case when the generalised hodograph transform degenerates and the Tsarev integrability scheme becomes redundant. This is the case of the decay of an initial step (2), which implies that the modulation variables are the functions of  $s = x/t$  alone for  $t \gg l$ , where  $l$  is the characteristic width of the (smooth) initial step. In this case, the GP problem has the asymptotic solution in the form of the centred simple wave in which all but one Riemann invariants are constant:

$$\beta_1 = u^+, \quad \beta_3 = u^-, \quad V_2(u^-, \beta_2, u^+) = s, \quad (28)$$

or explicitly,

$$\frac{1}{3}(u^- + 2u^+ + m\Delta) - \frac{2}{3} \frac{(1-m)m\Delta}{E(m)/K(m) - (1-m)} = \frac{x}{t}, \quad (29)$$

where  $\Delta = u^- - u^+$  is the magnitude of the initial jump. Since the solution (29) represents a characteristic fan it never breaks for  $t > 0$  and therefore is global. The self-similar boundaries  $s^-$  and  $s^+$  of the dispersive shock are found from the solution (29) by putting in it  $m = 0$  and  $m = 1$  respectively:

$$s^- = s(0) = u^+ - \Delta, \quad s^+ = s(1) = u^+ + \frac{2}{3}\Delta, \quad (30)$$

One can see that  $s^+ > s^-$  so the assumed wave pattern with the leading soliton and trailing linear wavepacket was indeed correct. Now the amplitude of the lead soliton is simply

$$a^+ = 2(\beta_3 - \beta_1) = 2\Delta, \quad (31)$$

whereas the value of the wavenumber of the trailing wave packet follows from Eq. (20a) evaluated for  $\beta_2 = \beta_1$  which yields

$$k^- = \sqrt{\frac{2}{3}(\beta_3 - \beta_1)} = \sqrt{\frac{2}{3}\Delta}. \quad (32)$$

We see that the simple form (28) of the analytic solution to the GP problem is possible owing to the availability of the Riemann invariant form (18) which admits  $\beta_j = \text{constant}$  as an exact solution. The formulas (30) – (32) are obtained as consequences of this global solution. So, since the existence of the Riemann invariants is due to complete integrability of the original KdV equation one may conclude that the possibility of obtaining the “global” formulas (30)–(32) relies on the integrability as well. This is not in fact so. In the next section, we will show that these formulas can be drawn directly from the Whitham system in “physical” variables using some very general properties of the Whitham systems subjected to natural boundary (matching) conditions of the GP type. Actually, Eqs. (31), (32) will be obtained without making use of the Riemann invariant form for the Whitham equations, and bypassing their explicit integration.

### 3 “Non-integrable” reformulation of the Gurevich - Pitaevskii problem

#### 3.1 Formulation

Let us reformulate the GP problem in terms of the Whitham system in its original, non-diagonal form (15). Along with the set of natural “mathematical” variables  $u_1, u_2, u_3$  we will use an equivalent set of “physical” variables  $\bar{u}, k, a$  which are connected with  $u_1, u_2, u_3$  by means of Eqs. (14), (6), (7).

From here on we are not going to exploit subtle algebraic properties underlying the integrability of (15). Instead, we will use some general properties of the system (15) connected with its “averaged” origin and distinguishing it from a general class of hyperbolic quasi-linear systems of the third order. In all relevant cases we will indicate if the obtained relationships could be extended to a more general context.

We start with the KdV modulation system in conservative form (see [16] for instance)

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\bar{u}^2}{2} \right) = 0, \quad \frac{\partial}{\partial t} \left( \frac{\bar{u}^2}{2} \right) + \frac{\partial}{\partial x} \left( \frac{\bar{u}^3}{3} - \frac{\bar{u}_\theta^2}{2} + u u_{\theta\theta} \right) = 0, \quad \frac{\partial k}{\partial t} + \frac{\partial \omega(\bar{u}, k, a)}{\partial x} = 0. \quad (33)$$

It is not difficult to show that the averaging (13) over the period of the travelling wave specified by Eqs. (4), (5) implies the following general relationships for the averaged variables in the harmonic ( $m = 0$ ) and soliton ( $m = 1$ ) limits

$$\overline{F(u)}|_{u_2=u_3} = F(u_3); \quad \overline{F(u)}|_{u_2=u_1} = F(u_1); \quad (34)$$

In particular, this implies that  $\bar{u}(u_1, u_3, u_3) = u_3$  and  $\bar{u}(u_1, u_1, u_3) = u_1$  (this can also be directly seen from the explicit expression (14)). Then, since  $u_2 = u_3$  is equivalent to  $a = 0$  and  $u_2 = u_1$  to  $k = 0$ , the relationships (34) assume in the physical variables the form

$$\overline{F(u)}|_{a=0} = F(\bar{u}); \quad \overline{F(u)}|_{k=0} = F(\bar{u}); \quad (35)$$

The relationships (35) immediately imply that in both harmonic and soliton limits,  $\overline{u^2} = \bar{u}^2$  hence the first modulation equation (33) turns into the Hopf equation for the mean value  $\bar{u}_t + \bar{u}\bar{u}_x = 0$ . It can also be readily shown using (4), (13) that the same is true for the second modulation equation (33) as well. Therefore:

i) the modulation system (33) admits *exact* reduction to a lower order system both for  $a = 0$  and  $k = 0$ . The limiting transitions  $a \rightarrow 0$  and  $k \rightarrow 0$  are, therefore, singular;

ii) the limiting transitions  $a \rightarrow 0$  and  $k \rightarrow 0$  provide two possible ways of passage to the dispersionless limit *in the modulation equations*.

We note that the same conclusions have been inferred from the Riemann equations (22), (24).

Of course, the described limiting behaviour is quite expected for a modulation system since in both afore-mentioned limits the oscillations do not contribute into the averaging hence the averaged (semi-classical) system must agree with non-oscillatory classical limit of the original equation. It is also clear that this fact is not unique to the KdV equation and must hold for other dispersive-hydrodynamics systems supporting the periodic travelling waves with the quadratic behaviour of the potential curve  $G(u)$  (5) near its extrema. One

can see, in particular, that this (generic for weakly dispersive systems) behaviour of the potential curve guarantees exponential decay for the soliton solutions so that they do not contribute into the averaging.

Now the GP natural boundary conditions (25) can be reformulated in terms of the matching of the mean flow in the oscillatory region with the smooth external flow

$$\begin{aligned} x = x^-(t) : \quad & a = 0, \quad \bar{u} = \beta, \\ x = x^+(t) : \quad & k = 0, \quad \bar{u} = \beta, \end{aligned} \quad (36)$$

where  $\beta(x, t)$  is defined by the classical limit (3). Using explicit expression for  $\bar{u}$  (14) and the relationships (17) one can easily see that the conditions (36) indeed equivalent to (25). For the particular case of the decay of an initial discontinuity (2), the conditions (36) assume the form

$$\begin{aligned} x = s^-t : \quad & a = 0, \quad \bar{u} = u^-, \\ x = s^+t : \quad & k = 0, \quad \bar{u} = u^+. \end{aligned} \quad (37)$$

We note that when passing from (36) to (37) we have made an implicit assumption about the structure of the dispersive shock by implying that  $s^+ > s^-$ .

The boundaries  $x^\pm(t)$  of the modulation solution are defined by Eqs. (26), (27) in terms of the characteristic velocities of the modulation systems where either  $m = 0$  (trailing edge) or  $m = 1$  (leading edge). This definition, of course, does not rely on the existence of the Riemann invariants and must hold in general case. This can be explained in the following way. It is clear that in order to provide continuous matching of the solutions of two (consistent) quasilinear hyperbolic systems of different order (the Whitham system and the Hopf equation in our case) the matching lines must *necessarily* be the multiple characteristics for the system with a higher order. Thus, the natural matching conditions (36) must be supplemented with the definition of the boundaries  $x^\pm(t)$  in terms of the double eigenvalues of the modulation system (see [33] for a detailed description of the characteristics behaviour in the GP problem).

We represent the Whitham system (15) in a generic quasi-linear form

$$\mathbf{y}_t + A(\mathbf{y})\mathbf{y}_x = 0, \quad (38)$$

where  $\mathbf{y} = (\bar{u}, k, a)^T$  and  $A(\mathbf{y})$  is the coefficient matrix. As is well known (see for instance [34], [35]) the quasilinear system (38) can be represented in a characteristic form

$$\sum_{j=1}^3 b_j^{(m)} \frac{d_m y^j}{dt} = 0, \quad \frac{d_m}{dt} \equiv \frac{\partial}{\partial t} + V_m \frac{\partial}{\partial x}, \quad m = 1, 2, 3, \quad (39)$$

where the characteristic velocities  $V_j(\mathbf{y})$  are the eigenvalues of the matrix  $A(\mathbf{y})$ , and  $\mathbf{b}^{(m)}(\mathbf{y})$  is its left eigenvector corresponding to  $m$ -th eigenvalue:  $\mathbf{b}^{(m)} A = V_m \mathbf{b}^{(m)}$ . We assume that the characteristic velocities  $V_1(\mathbf{y}) \leq V_2(\mathbf{y}) \leq V_3(\mathbf{y})$  are real for the solutions of our interest so the system (38) is hyperbolic (of course, for the KdV equation, the hyperbolicity of the Whitham system is a proven fact [36] but in our, “non-integrable” approach it is an assumption).

It can be readily seen using an equivalent characteristic representation (39) in “mathematical” coordinates,  $\mathbf{y} \mapsto \mathbf{y}^* = (u_1, u_2, u_3)$ , that the fact that the modulation system (38) admits exact reductions for  $u_2 = u_3$  ( $a = 0$ ) and  $u_2 = u_1$  ( $k = 0$ ) implies that the multiple

roots of the potential curve  $G(u)$  necessarily correspond to multiple eigenvalues of the averaged system. This, again, is a consequence of the properties of the Whitham averaging (13). It follows from the characteristic velocities ordering that the double eigenvalues can be either  $V_2 = V_3$  or  $V_2 = V_1$ . The remaining single eigenvalue in both cases is  $\bar{u}$  since the Hopf equation is an exact reduction of the Whitham system in both limits. If the explicit expressions for the characteristic velocities are known, the correspondence between the double roots of the travelling wave potential function  $G(u)$  and the double eigenvalues of the averaged system is established directly (see (21), (23)). Alternatively, one can perform an asymptotic analysis of the modulation system for  $m \ll 1$  ( $a \ll 1$ ) and  $(1 - m) \ll 1$  ( $k \ll 1$ ) and establish the sought correspondence. Either way, even without additional analysis it is clear that, due to the accepted ordering of  $V_j$ 's, the double eigenvalues should be identified with the values of the middle characteristic velocity  $V_2$  in the harmonic and the soliton limits. So we define the boundaries  $x^\pm(t)$  of the KdV modulation solution by the characteristic equations (cf. (26), (27))

$$dx^-/dt = V_2(\bar{u}, k, 0)|_{x=x^-}, \quad dx^+/dt = V_2(\bar{u}, 0, a)|_{x=x^+}. \quad (40)$$

For the decay of an initial discontinuity Eqs. (40), in view of (37), assume the form

$$s^- = V_2(u^-, k^-, 0), \quad s^+ = V_2(u^+, 0, a^+) \quad (41)$$

where the values  $a^+$  and  $k^-$  are to be found from the solution of the GP problem.

In the subsequent sections we will find simple effective expressions for double characteristics of the Whitham system using physical variables  $\bar{u}, k, a$  and bypassing the full eigenvalue analysis. Moreover, we will show that the edge parameters  $a^+, k^-$  and hence  $s^+, s^-$  can be found from a certain set of conditions *not involving the global integration of the Whitham equations*. As a result, we will present a way to “fit” the dispersive shock into the solution of the dispersionless limit equations without the detailed analysis of its internal oscillatory structure in the same manner as the traditional shock is embedded in the solution of the inviscid equations of ideal gas. As will follow from the construction, the availability of these conditions does not depend on the existence of the Riemann invariants for the Whitham system.

### 3.2 Trailing edge

We start with the determination of the trailing edge speed  $s^-$  for the problem of the decay of an initial discontinuity (2) in the KdV equation using the reformulation of the GP problem in physical variables made in the previous subsection. The trailing edge is defined for the solution of the GP problem by the characteristic where  $a = 0$ . Our analysis below will be based on the special features of the Cauchy data prescription on characteristics (see e.g. [35]).

It follows from the characteristic form (39) that the differentials of the modulation variables along the  $i$ -th characteristic are not independent but related by means of the expression

$$b_1^{(i)}(\bar{u}, k, a)d_i\bar{u} + b_2^{(i)}(\bar{u}, k, a)d_ik + b_3^{(i)}(\bar{u}, k, a)d_ia = 0. \quad (42)$$

Let  $a = 0$  on this characteristic (which necessarily implies that it is a double characteristic – see explanation in the previous section). Since  $a = 0$  must be an exact solution of the

modulation system, the ordinary differential equation

$$b_1^{(i)}(\bar{u}, k, 0)d_i\bar{u} + b_2^{(i)}(\bar{u}, k, 0)d_ik = 0 \quad (43)$$

necessarily represents a characteristic equation for the reduction as  $a = 0$  of the full modulation system. This equation can always be integrated using an integrating factor to give the relationship between admissible values of  $\bar{u}$  and  $k$  *on the characteristic*, i.e:

$$\Phi_i(\bar{u}, k) = C_0, \quad (44)$$

$C_0$  being constant of integration. The point is that this relationship is *local*, so it does not depend on the specific solution under study but is determined (up to a constant) only by the coefficients of the Whitham system evaluated for  $a = 0$ . Therefore, the integral (44) can be derived, bypassing the technically involved route via full characteristic form (42), by a direct substitution of the ansatz  $k = k(\bar{u})$  into the reduction of the Whitham system for  $a = 0$ . We also remark that the value  $C_0$  represents, in fact, a Riemann invariant for this  $2 \times 2$  reduction. Of course, the existence of this Riemann invariant does not depend on the existence of the Riemann invariants for the full Whitham system (15).

It follows from the relationship (35a) and Eqs. (6), (14) considered for  $u_2 = u_3$ , that

$$\lim_{a \rightarrow 0} \bar{u}^2 = \bar{u}^2, \quad \lim_{a \rightarrow 0} \omega = \lim_{u_2 \rightarrow u_3} ck = k\bar{u} - k^3 \equiv \omega_0(\bar{u}, k). \quad (45)$$

Obviously, the second expression in Eq. (45) represents the KdV linear dispersion relation, where the linearization is made about the slowly varying mean background  $\bar{u}(x, t)$ . Now we can immediately obtain the reduction of the Whitham equations (33)

$$a = 0, \quad \bar{u}_t + \bar{u}\bar{u}_x = 0, \quad k_t + (\omega_0(\bar{u}, k))_x = 0. \quad (46)$$

The equations (46) form a *closed* system. We emphasize that the system (46) is not an asymptotic system, it is an *exact reduction* of the full modulation system. Of course, the reduction (46) is equivalent to the limiting diagonal system (22) (which is readily verified by a direct calculation) but here it was derived directly from the original modulation equations (33) without using the Riemann invariant form. Actually, the limit (46) has a clear meaning of the formal modulation system for a zero-amplitude wave packet propagating on a slowly varying background flow  $\bar{u}$  and can be postulated on the physical level of reasoning directly, even without turning to the Whitham equations (33) and the formal derivation of the linear dispersion relation (45) for modulated waves via nonlinear travelling wave solution. Now, looking for the integral  $k(\bar{u})$  of Eqs. (46) we arrive at the ordinary differential equation (which is an equivalent of Eq. (43)),

$$\frac{dk}{d\bar{u}} = \frac{\partial\omega_0/\partial\bar{u}}{\bar{u} - \partial\omega_0/\partial k}, \quad (47)$$

which, upon substituting the linear dispersion relation from (45) is readily integrated to give

$$k = \sqrt{\frac{2}{3}(\bar{u} + C_1)}. \quad (48)$$

Thus Eq. (48) is a relationship between values of  $k$  and  $\bar{u}$  *on the characteristic* of the Whitham system on which  $a = 0$  (there is no relation to any particular global solution yet).

Now we apply the relationship (48) to the GP problem (37), where the characteristic where  $a = 0$  is associated with the trailing edge  $x = s^-t$ . First, we find the constant  $C_1$  from the second boundary condition (37) which, in the space of the field variables prescribes  $k = 0$  when  $\bar{u} = u^+$  (this condition does not contain the amplitude  $a$  so it must hold everywhere in the plane  $k = 0$  of the three-dimensional space with the coordinates  $\{\bar{u}, k, a\}$ , including the line  $(\bar{u}, 0, 0)$  as well). Thus  $C_1 = -u^+$ . Then, putting  $\bar{u} = u^-$  in (48) we obtain the value of the wavenumber at the trailing edge of the dispersive shock  $x = s^-t$ ,

$$k^- = \sqrt{2\Delta/3}, \quad (49)$$

where  $\Delta = u^- - u^+$  is the jump across the dispersive shock. This agrees with the value of  $k^-$  given by Eq. (32) obtained as a consequence of the full modulation solution (61).

Now we find the self-similar co-ordinate (the speed) of the trailing edge  $s^-$  which, according to (40), is calculated as the double characteristic velocity of the Whitham system for the solution of the GP problem. It readily follows from the limiting system (46), that its characteristic velocities are

$$\bar{u}, \quad \frac{\partial \omega_0}{\partial k}(\bar{u}, k) = \bar{u} - 3k^2. \quad (50)$$

The latter, of course, is the group velocity of the linear wave packet propagating on the varying mean flow background  $\bar{u}(x, t)$ , which is perfectly reasonable from physical point of view. Now we need to identify one of the velocities (50) with the double eigenvalue  $V_2(\bar{u}, k, 0)$  in (41a) to evaluate the trailing edge speed. It is clear from physical reasoning that this must be the linear group velocity. Later we will present additional formal conditions eliminating any possible ambiguity in the identification of the double eigenvalue. Of course, for the KdV case it is not difficult to show using explicit formulas (14), (6) that expressions (50) are equivalent to those obtained from the Riemann form of the Whitham equations in the harmonic limit (see Eq. (21)) and our identification  $V_2(\bar{u}, k, 0) = \partial \omega_0 / \partial k$  is indeed correct. Thus, we get from Eq. (41a) for the trailing edge

$$s^- = \frac{\partial \omega_0}{\partial k}(u^-, k^-). \quad (51)$$

In this form the equation of the trailing edge can be interpreted as a kinematic boundary condition for linear modulated waves and can be postulated as a part of the problem formulation.

Setting (50), (49) into (51) we obtain for the trailing edge

$$s^- = u^- - 3(k^-)^2 = u^+ - \Delta. \quad (52)$$

This value, again, agrees with the full solution of the GP problem (see Eq. (30)).

### 3.3 Leading edge

In principle, one could proceed with the leading edge in the same fashion as we did with the trailing edge i.e. by indirect integration of the full characteristic 3-form along a characteristic where  $k = 0$ , where this form degenerates into

$$b_1^{(i)}(\bar{u}, 0, a)d_i\bar{u} + b_2^{(i)}(\bar{u}, 0, a)d_ia = 0. \quad (53)$$

That would imply finding the integral  $a(\bar{u})$  for the exact zero-wavenumber reduction of the modulation system. Such a reduction, however, must include full modulation equation for the soliton amplitude, which can be obtained without too much trouble for the KdV equation but is not readily available for more complicated systems, especially if the solitary wave solution can not be found explicitly. One should remark that the “obvious” universal soliton amplitude equation  $a_t + c_s(a)a_x = 0$  (see [34], Sec. 16.6) is correct only if the background flow  $\bar{u}$  is constant. The full amplitude equation for solitary waves (see [37], [38] for instance) takes into account variations in all modulation parameters and inevitably contains the term proportional to  $\bar{u}_x$  which is vital for deriving the characteristic equation (53). As a result, unlike the wave number conservation law, the *full* soliton amplitude equation does not have an explicit universal form and a straightforward application of the method proposed in Section 3.2 would be essentially equivalent to the consideration of the general characteristic equation (42) in the soliton limit. For “real” non-integrable systems that could be a very difficult technical task. This complication, however, can be bypassed by introducing a conjugate wavenumber defined for the KdV as

$$\tilde{k}(u_1, u_2, u_3) = \pi \left( \int_{u_1}^{u_2} \frac{du}{\sqrt{G(u)}} \right)^{-1} = \frac{\pi}{2\sqrt{3}} \frac{(u_3 - u_1)^{1/2}}{K(m')}, \quad m' = 1 - m \quad (54)$$

– instead of the amplitude  $a$  and the ratio  $\Lambda = k/\tilde{k}$  instead of the original wavenumber  $k$  (6). The new set of modulation variables we are going to use is  $(\bar{u}, \Lambda, \tilde{k})$ . It is readily seen that the soliton limit  $k = 0$  ( $m = 1$ ) corresponds to  $\Lambda = 0$ ,  $\tilde{k} = \sqrt{a_s/3} = \mathcal{O}(1)$  in new variables. On the other hand, in the harmonic limit  $a = 0$  ( $m = 0$ ) one has  $\tilde{k} = 0$ . So  $\tilde{k}$  indeed plays the role analogous to the amplitude. For further convenience, we rewrite the matching conditions (37) using new variables

$$\begin{aligned} x = s^-t : \quad & \tilde{k} = 0, \quad \bar{u} = u^-, \\ x = s^+t : \quad & \Lambda = 0, \quad \bar{u} = u^+. \end{aligned} \quad (55)$$

Now, as was shown in Section 3.1, the two first averaged conservation laws (33) in the soliton limit reduce to the Hopf equation:

$$\Lambda = 0 : \quad \bar{u}_t + \bar{u}\bar{u}_x = 0. \quad (56)$$

To obtain the equation for  $\tilde{k}$  in the soliton limit we set the new variables in the wave number conservation law in (33) to obtain:

$$\tilde{k}\Lambda_t + \tilde{\omega}\Lambda_x + \Lambda(\tilde{k}_t + \tilde{\omega}_x) = 0, \quad (57)$$

where  $\tilde{\omega} = \tilde{\omega}(\bar{u}, \Lambda, \tilde{k}) = \tilde{k}c$  is the conjugate frequency.

Now, using the arguments identical to those in the previous subsection we infer that if  $\Lambda = 0$  on some characteristic then the values of  $\tilde{k}$  and  $\bar{u}$  on this characteristic must be connected by a local relationship  $\tilde{k}(\bar{u})$  which must not depend on the particular solution. We consider equation (57) in the small vicinity of the mentioned characteristic where  $\Lambda \ll 1$ . Then, using  $\tilde{k} = \tilde{k}(\bar{u})$  and Eq. (56) for the leading order, we obtain an asymptotic equation

$$\frac{\partial \Lambda}{\partial t} + \frac{\tilde{\omega}_s}{\tilde{k}} \frac{\partial \Lambda}{\partial x} + \frac{\Lambda}{\tilde{k}} \frac{\partial \bar{u}}{\partial x} \left\{ \frac{d\tilde{k}}{d\bar{u}} \left( \frac{\partial \tilde{\omega}_s}{\partial \tilde{k}} - \bar{u} \right) + \frac{\partial \tilde{\omega}_s}{\partial \bar{u}} \right\} = \mathcal{O} \left( \Lambda \frac{\partial \Lambda}{\partial x} \right), \quad (58)$$



where the reduction  $\tilde{\omega}_s(\bar{u}, \tilde{k}) = \tilde{\omega}(\bar{u}, 0, \tilde{k})$  can be called a *soliton dispersion relation*. Since the sought characteristic integral  $\tilde{k}(\bar{u})$  must not depend on the way  $\Lambda$  tends to zero, the expression in the brackets (which does not depend on  $\Lambda$ ) must be identically zero and Eq. (58) splits into

$$\frac{d\tilde{k}}{d\bar{u}} = \frac{\partial \tilde{\omega}_s / \partial \bar{u}}{\bar{u} - \partial \tilde{\omega}_s / \partial \tilde{k}} \quad (59)$$

and

$$\frac{\partial \Lambda}{\partial t} + \frac{\tilde{\omega}_s}{\tilde{k}} \frac{\partial \Lambda}{\partial x} = \mathcal{O} \left( \Lambda \frac{\partial \Lambda}{\partial x} \right). \quad (60)$$

Eqs. (59), (60) have been obtained in [39] using somewhat more particular arguments. One can not help noticing that the equations (47) and (59) for the characteristic integrals  $k(\bar{u})$  (linear characteristic) and  $\tilde{k}(\bar{u})$  (soliton characteristic) are identical in terms of the corresponding dispersion relations  $\omega_0(\bar{u}, k)$  and  $\tilde{\omega}_s(\bar{u}, \tilde{k})$ . The latter, however, is yet to be found.

It follows from (60), (56) that in the soliton limit the characteristic velocities of the Whitham system are

$$\frac{\tilde{\omega}_s}{\tilde{k}}, \quad \bar{u}. \quad (61)$$

One can see from the definition of  $\tilde{\omega}_s$  that the characteristic velocity  $\tilde{\omega}_s/\tilde{k}$  coincides with the soliton speed  $c_s(\bar{u}, \tilde{k}) = c(u_1, u_1, u_3)$ , which is, of course, expected. We identify  $c_s$  with the double characteristic velocity  $V_2 = V_3$  of the full Whitham system in the soliton limit, then the classical speed  $\bar{u}$  would correspond to  $V_1$ . The conditions eliminating possible ambiguity in the characteristic velocity identification for the reduced Whitham system will be presented later. Now, similarly to the trailing edge case, the definition of the leading edge in Eq. (41) can be represented as a kinematic boundary condition for the lead soliton (cf. (51) )

$$s^+ = c_s(u^+, \tilde{k}^+), \quad (62)$$

where  $\tilde{k}^+ = \tilde{k}(u^+)$ . One can see that in this form the definition of the leading edge can be adopted as a part of the problem formulation.

At this point, one may take advantage of the explicit expressions (14), (6) for  $\bar{u}(u_1, u_2, u_3)$ ,  $\tilde{k}(u_1, u_2, u_3)$  and  $c(u_1, u_2, u_3)$  to find  $\tilde{\omega}_s(\bar{u}, \tilde{k})$  and solve Eq. (59) using the matching conditions (55). Such an attractive route, however, might not be readily (if at all) available for actual non-integrable systems, where the physical modulation parameters often can not be expressed in a simple way in terms of the roots of the potential curve. So we proceed with some apparently "roundabout" way, which would be more universally applicable to other systems.

We observe that  $\tilde{k}$  and  $\tilde{\omega} = \tilde{k}c$  defined by (54) and (6a) as functions of the roots  $u_1, u_2, u_3$  can be viewed as the wavenumber and the frequency in the *conjugate travelling wave* associated with the same set of roots  $u_j$  as in Eq. (4) but inverted potential curve (now oscillations occur between the roots  $u_1$  and  $u_2$ ):

$$(\tilde{u}_{\tilde{\theta}})^2 = G(\tilde{u}), \quad \tilde{u}(\tilde{\theta} + 2\pi/\tilde{k}) = \tilde{u}(\tilde{\theta}). \quad (63)$$

where  $\tilde{u}$  is new field variable and  $\tilde{\theta} = \tilde{x} - \tilde{c}\tilde{t}$  is a new travelling phase associated with "old" phase velocity  $c = (u_1 + u_2 + u_3)/3$ . Eq. (63) can be obtained from (4) by the change of variables  $u \mapsto \tilde{u}$ ,  $x \mapsto i\tilde{x}$ ,  $t \mapsto i\tilde{t}$ , which corresponds to a mere change of the dispersion sign in the KdV equation (1),

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} - \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0. \quad (64)$$

As a matter of fact, in our particular KdV case the functions  $u(\theta)$  and  $i\tilde{u}(-i\tilde{\theta})$  defined for the same set of roots  $u_j$  represent the same analytic (elliptic) function in the complex  $x$ -plane with the periods  $2\pi/k$  and  $2\pi i/\tilde{k}$  along the real and the imaginary axes. Generally, of course, this is not the case, so we are not going to take advantage of the analytic properties of  $u(\theta)$  here. Instead, we observe that the soliton limit for the original travelling wave equation (4) ( $u_2 \rightarrow u_1$ ) corresponds to the harmonic limit for the conjugate equation (63). Therefore, the relation between  $\tilde{\omega}_s$  and  $\tilde{k}$  can be obtained as a *linear dispersion relation* for the conjugate KdV equation (64), i.e.

$$\tilde{\omega}_s = \tilde{k} < \tilde{u} > + \tilde{k}^3, \quad (65)$$

where the brackets  $< \dots >$  denote the averaging over the conjugate family (63) (cf. (13))

$$< F > (u_1, u_2, u_3) = \frac{\tilde{k}}{\pi} \int_{u_1}^{u_2} \frac{F(\tilde{u})}{\sqrt{G(\tilde{u})}} d\tilde{u}. \quad (66)$$

It is not difficult to show by a direct calculation that  $< F > (u_1, u_1, u_3) = F(u_1)$ . Then, it follows from Eq. (34) that for  $u_2 = u_1$  ( $k = 0$ ) we have

$$< F(u) > |_{k=0} = \overline{F(u)}|_{k=0} \quad (67)$$

and, in particular,  $< u > = \bar{u}$ . Again, similarly to Eqs. (34), (35) this property is due to the quadratic behaviour of the potential curve  $G(u)$  (5) in the vicinity of the double root corresponding to the soliton limit and is not confined to the KdV example alone.

It follows from Eq. (67) that  $\tilde{\omega}_s(< u >, \tilde{k}) = \tilde{\omega}_s(\bar{u}, \tilde{k})$  and therefore, the soliton dispersion relation can be obtained from the original linear dispersion relation  $\omega_0(\bar{u}, k)$  by the formal change

$$k \mapsto i\tilde{k}, \quad \omega_0 \mapsto i\tilde{\omega}_0 \quad (68)$$

In other words, the soliton dispersion relation is found as

$$\tilde{\omega}_s(\bar{u}, \tilde{k}) = -i\omega_0(\bar{u}, i\tilde{k}) = \tilde{k}\bar{u} + \tilde{k}^3. \quad (69)$$

Of course, in view of  $\tilde{k}(u_1, u_1, u_3) = \sqrt{a_s/3}$  and  $\bar{u}(u_1, u_1, u_3) = u_1$ , Eq. (69) is equivalent to the classic relationship (11) between the speed and the amplitude of the KdV soliton but here it was obtained in terms of the linear dispersion relation so the outlined procedure can be easily generalised to more complicated systems, where the explicit analysis of the travelling wave solution is not as readily available as in the KdV case.

The possibility of expressing the soliton speed in terms of the linear dispersion relation might look somewhat surprising but can be explained by the following simple argument (see [40]): the value of  $u$  in the soliton tail, which moves with the same speed as the rest of the soliton, is very small, so one can, in principle, infer this speed from the linear theory.

**Remark.** We emphasize that the obtained relationships between the original and the conjugate averaged variables essentially represent algebraic identities between integrals of the form (13) and (66) associated with a given Riemann surface  $\mu^2 = G(u)$  and do not imply any connection between the spatio-temporal modulation dynamics for the original and the conjugate equations (1) and (64).

Now setting Eq. (69) into Eq. (59) we obtain after elementary integration

$$\tilde{k} = \sqrt{\frac{2}{3}(C_2 - \bar{u})}, \quad (70)$$

where  $C_1$  is constant of integration. We emphasize that Eq. (70) (as well as Eq. (48) in the linear case) is a general relationship between values of  $\tilde{k}$  and  $\bar{u}$  on the characteristic of the Whitham system where  $\Lambda = 0$  and is not tied to any particular global solution.

Now we apply the relationship (70) to the GP problem (55), where the characteristic on which  $\Lambda = 0$  is associated with the leading edge  $x = s^+t$ . First, we find the constant  $C_2$  from the first boundary condition (55) which prescribes  $\tilde{k} = 0$  when  $\bar{u} = u^-$  in the three-dimensional space with the coordinates  $\{\bar{u}, \Lambda, \tilde{k}\}$ . This condition does not contain  $\Lambda$  so it must hold on the line  $\{\bar{u}, 0, 0\}$  as well and, therefore, can be applied to (70). Thus  $C_2 = u^-$ . Then, putting  $\bar{u} = u^+$  in (70) we obtain the value of the conjugate wavenumber at the trailing edge of the dispersive shock  $s = s^+$ :

$$\tilde{k}^+ = \sqrt{2\Delta/3}. \quad (71)$$

We observe that, for the KdV equation  $\tilde{k}^+ = k^-$  (see (49)). Of course, this is not a general relationship although one can expect significant symmetry between the expressions for  $k^-$  and  $\tilde{k}^+$  in other dispersion-hydrodynamic systems.

The speed of the leading edge (62) is now calculated with the aid of the conjugate dispersion relation (69) as

$$s^+ = \frac{\tilde{\omega}_s(u^+, \tilde{k}^+)}{\tilde{k}^+} = u^+ + \frac{2}{3}\Delta, \quad (72)$$

which agrees with the global solution of the full Whitham system (see Eq. (30)). The KdV soliton amplitude is connected with its velocity by the relation (12), which in our case assumes the form  $s^+ = u^+ + a^+/3$ . Hence the lead soliton amplitude is  $a^+ = 3(s^+ - u^+) = 2\Delta$ , which, again, agrees with the full modulation solution.

### 3.4 “Entropy” conditions.

One can see that the valid solution to the GP problem with  $u^- > u^+$  must satisfy the inequalities

$$s^- < u^-, \quad s^+ > u^+, \quad s^+ > s^-, \quad (73)$$

These inequalities ensure that the “external”, Hopf characteristics  $x = u^-t$  and  $x = u^+t$  starting from the  $x$ -axis on either side of the dispersive shock region intersect its edges when continued in the direction of increasing  $t$  thus transferring the initial data *into* the dispersive shock zone  $s^-t < x < s^+t$  (see Fig. 3). The inequalities (73) represent a dispersive-hydrodynamic analog of the classical gas- dynamic *entropy conditions* [1]. As a matter of fact, these inequalities are redundant when the explicit solution of the GP problem is available (one can see that conditions (73) are indeed satisfied by the boundaries of the solution (28)). However, in the absence of the full modulation solution, and without physical assumptions about the spatial structure of the dispersive shock there is an ambiguity in the determination of the edges using the construction proposed in Sections 3.2, 3.3. This ambiguity has been pointed out when we made an identification of the leading and trailing edges with the soliton and the harmonic wave in the natural boundary conditions (37) and when we identified linear group velocity and the soliton velocity with the double eigenvalues of the full Whitham system in respective limits. One may speculate about the possible analog of the entropy for

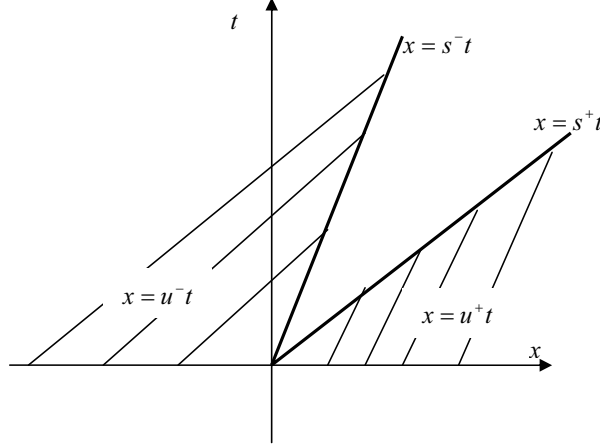


Figure 3: Classical (Hopf) characteristics transfer initial data into the Whitham zone

dispersive hydrodynamics. An attractive candidate is given by the integral

$$S \propto \int_{-\infty}^x k dx' \geq 0. \quad (74)$$

For the KdV dispersive shock the *positive* function  $k(x, t)$  is supported on the interval  $s^- t < x < s^+ t$  and one can see that the integral (85) increases when crossing the dispersive shock.

### 3.5 Geometric interpretation

We now place the obtained relationships in a more general context. The quasilinear hyperbolic system of the third order (38) admits the centred expansion fan solution

$$F_1(\bar{u}, k, a) = I_1, \quad F_2(\bar{u}, k, a) = I_2, \quad V_k(\bar{u}, k, a) = x/t, \quad (75)$$

$I_1, I_2$  being constants and  $V_k$  is one of the characteristic velocities so that the solution satisfied the GP matching conditions (37) (we know that for the KdV equation this is  $V_2$  – see Eq. (28)). The two first expressions in (75) define, for given  $I_1, I_2$ , two integral surfaces in the space of the field variables  $\bar{u}, k, a$  with  $k > 0, a > 0$  (physical restrictions). Their intersection yields the solution curve  $\{\Gamma(s; I_1, I_2) : \bar{u} = \bar{u}(s), k = k(s), a = a(s)\}$ , where the parameter  $s = x/t$  changes on the interval  $[s^-, s^+]$  so that  $a(s^-) = 0$  and  $k(s^+) = 0$ .

We now briefly outline how one proceeds with this geometric construction. First we observe that, the endpoints of the solution curve  $\Gamma(s)$  lie on the lines of intersection of two integral surfaces parametrised by  $I_1$  and  $I_2$  with the coordinate planes  $a = 0$  and  $k = 0$ . It follows from (75) that there are four these lines:  $F_{1,2}(\bar{u}, k, 0) = I_{1,2}$  and  $F_{1,2}(\bar{u}, 0, a) = I_{1,2}$ . Since the Whitham system admits exact  $2 \times 2$  hyperbolic reductions for  $a = 0$  and  $k = 0$ , the equations of the intersection lines can be obtained directly, as the characteristic integrals (Riemann invariants) of the reduced systems, i.e as solutions in the form  $k(\bar{u})$  and  $a(\bar{u})$  respectively. The constants  $I_{1,2}$  are then expressed in terms of the initial parameters  $u^-, u^+$  by applying the boundary conditions (37) to the characteristic integrals as it was done in Sections 3.2, 3.3. Then, using the kinematic conditions (51), (62), one evaluates the speeds

of the trailing and the leading edges and only after that, the valid pair of the endpoints is selected by the “entropy” inequalities (73).

Essentially, this general construction has been realised in Sections 3.2, 3.3 by using the arguments from the characteristics theory and introducing a different (conjugate) basis of the field variables when considering the leading edge.

## 4 Dispersive shocks in simple-wave led dispersive equations

It is clear that the presented construction is not restricted by the KdV equation alone, and can be naturally generalised to the “KdV-like” nonlinear dispersive systems possessing the basic properties necessary for the Whitham averaging. We represent the governing equation in the form

$$u_t + V(u)u_x + K_3[u] = 0, \quad (76)$$

where  $V(u)$  is a real function and  $K_3$  is a real differential operator of the third order with respect to spatial or mixed derivatives such that the equation (76) has real linear dispersion relation  $\omega = \omega_0(k)$ . We assume the following general properties for the equation (76):

(i) it admits the hyperbolic classical (dispersionless) limit obtained formally by introducing the stretched independent variables  $x' = \epsilon x$ ,  $t' = \epsilon t$  and tending  $\epsilon \rightarrow 0$  while assuming finiteness of the derivatives with respect to  $x'$  and  $t'$ ,

$$u_{t'} + V(u)u_{x'} = 0. \quad (77)$$

In terms of the linear dispersion relation this property implies  $\omega_0 \sim k$  for  $k \ll 1$  and is associated with weakly dispersive waves.

(ii) it possesses at least two conservation laws;

(iii) it supports periodic travelling waves parametrised by three independent integrals of motion such that the travelling wave solution allows for a harmonic (zero-amplitude) and a solitary wave (zero wavenumber) limits. We will assume the following KdV-like behaviour for the “potential” function  $G(u)$  in the ordinary differential equation  $(u_\theta)^2 = -G(u)$  specifying the travelling wave solution (cf. Eq. (5)): a) The function  $G(u)$  has at least three real zeros  $u_3 \geq u_2 \geq u_1$  such that the oscillations occur between  $u_2$  and  $u_3$  (the latter is assumed just for sake of definiteness); b) In both nearly linear  $(u_3 - u_2)/(u_3 - u_1) \ll 1$  and nearly soliton  $(u_2 - u_1)/(u_3 - u_1) \ll 1$  configurations the generic (quadratic) asymptotic behaviour is assumed:

$$G(u) = (u - u_1)(u - u_2)(u_3 - u)G^*(u), \quad (78)$$

where

$$G^*(u) = \mathcal{O}(1) \quad \text{for} \quad \left| \frac{u - u_2}{u_3 - u_1} \right| \ll 1 \quad (79)$$

so that the limiting transitions (34), (67) for the mean values can be easily shown to take place;

(iv) the corresponding Whitham system of the third order (two averaged “hydrodynamic” conservation equations plus the wave number conservation law) is hyperbolic for the solutions under study.

The equations of the form (76) possessing the mentioned properties (i) - (iv) may be characterised as *simple-wave led weakly dispersive*. We note that all assumptions except (iv) can usually be explicitly verified for a specific system. We partially address the hyperbolicity issue (iv) in the end of this section.

We consider initial data in the form of an arbitrary step

$$u_0(x) = u^- \quad \text{for } x < 0; \quad u^+ \quad \text{for } x > 0, \quad . \quad (80)$$

and assume that the dispersive shock transition for  $t > 0$  can be modelled by the centred expansion fan solution to the Whitham equations. Now one can see that all arguments used in the derivation of the dispersive shock conditions for the KdV equation in a “non-integrable” reformulation hold for the case being considered: one just needs to replace the Hopf “dispersionless” speed with  $V(u)$  and the KdV linear dispersion relation with the dispersion relation corresponding to (76). So we present only the final relationships without making any assumptions about mutual position of the soliton and harmonic edges.

First we introduce the linear dispersion relation for Eq. (76) by considering an infinitesimal perturbation of a mean level  $\bar{u}$

$$u \approx \bar{u} + u_1 e^{i(kx - \omega t)}, \quad u_1 \ll 1, \quad (81)$$

which yields  $\omega = \omega_0(\bar{u}, k)$  to leading order. Let the dispersive shock transition be confined to an interval  $s^- t \leq x \leq s^+ t$ . We introduce two sets of parameters  $\{k^-, s^-; \tilde{k}^+, s^+\}_1$  and  $\{\tilde{k}^-, s^-; k^+, s^+\}_2$ . The set  $\{k^-, s^-; \tilde{k}^+, s^+\}_1$  is associated with the negative dispersion wave pattern when the soliton appears at the leading edge of the dispersive shock. We define this set in the following way. First we find two functions  $k(\bar{u})$  and  $\tilde{k}(\bar{u})$  from the ordinary differential equations

$$\frac{dk}{d\bar{u}} = \frac{\partial \omega_0 / \partial \bar{u}}{V(\bar{u}) - \partial \omega_0 / \partial k}, \quad k(u^+) = 0, \quad (82)$$

$$\frac{d\tilde{k}}{d\bar{u}} = \frac{\partial \tilde{\omega}_s / \partial \bar{u}}{V(\bar{u}) - \partial \tilde{\omega}_s / \partial \tilde{k}}, \quad \tilde{k}(u^-) = 0. \quad (83)$$

Then the values of the wavenumber at the trailing edge  $k^-$  and the lead soliton conjugate wavenumber  $\tilde{k}^+$  are found as  $k^- = k(u^-)$ ,  $\tilde{k}^+ = \tilde{k}(u^+)$ . By definition, they must be real. The speeds of the dispersive shock edges are found from the expressions

$$s^- = \frac{\partial \omega_0}{\partial k}(u^-, k^-), \quad s^+ = \frac{\tilde{\omega}_s(u^+, \tilde{k}^+)}{\tilde{k}^+}, \quad (84)$$

where  $\tilde{\omega}_s(\bar{u}, \tilde{k}) = -i\omega_0(\bar{u}, i\tilde{k})$ . The second set of parameters  $\{\tilde{k}^-, s^-; k^+, s^+\}_2$  associated with the reversed, positive dispersion pattern, is obtained from the same system (82) – (84) in which one replaces “–” with “+” which is equivalent to a respective replacing of the parameter subscripts directly in the set  $\{\cdot\}_1$ . The valid set is selected by the reality condition for the wavenumber and by the “entropy” conditions,

$$s^- < V(u^-), \quad s^+ > V(u^+), \quad s^+ > s^-. \quad (85)$$

In principle, the solution can switch between the sets  $\{\cdot\}_1$  and  $\{\cdot\}_2$  depending on the initial conditions. “Switching” of the edge parameter sets implies the reversion of the spatial

structure of the dispersive shock. Another possibility is that for some domain of the initial data  $(u^+, u^-)$  both sets  $\{\cdot\}_1$  and  $\{\cdot\}_2$  fail to satisfy either reality or “entropy” condition. This would imply that a genuine global solution to the dispersionless limit equations is available for such initial data and no breaking occurs. This solution, of course, is a classical rarefaction wave.

We also briefly address the issue of the modulational stability. Our main mathematical assumption in this paper is that of the hyperbolicity of the Whitham system for the solutions under study. This ensures *global* modulational stability of the dispersive shock. For non-integrable dynamics, establishing the region of hyperbolicity for the modulation system is a separate, often technically involved, problem. However, in a more restricted context of the dispersive shock description, some effective necessary conditions of global modulational stability can be formulated in the following way.

We evaluate the the frequency and conjugate frequency at the respective boundaries of the dispersive shock transition using the solutions of the ordinary differential equations (82), (83). We denote these frequencies as  $\omega_0[u^-, u^+]$  and  $\tilde{\omega}_s[u^-, u^+]$ . Then, given the strict hyperbolicity of the dispersionless limit, the necessary criterion for global modulational stability of the dispersive shock is given by the conditions

$$\text{Im } \omega_0[u^-, u^+] = 0, \quad \text{Im } \tilde{\omega}_s[u^-, u^+] = 0, \quad (86)$$

which are equivalent to the natural requirement for the speeds  $s^-, s^+$  to be real. The conditions (86) define the domain  $D$  in the initial data plane  $(u^+, u^-)$  which corresponds to modulationally stable solutions. It is not clear if the formulated criterion is the sufficient condition so the obtained region  $D[u^+, u^-]$  could be corrected by some additional conditions.

We note that, for non-integrable equations some additional restrictions on admissible values of initial data can occur due to existence conditions for the single-phase travelling wave solutions, for instance, due to the presence of the upper bound for the soliton amplitude.

**Example:** *Decay of a step problem for the defocusing mKdV equation*

As the next simplest example of an effective construction of the dispersive shock transition in the simple-wave led dispersive equation we consider the step resolution problem for the defocusing mKdV (mKdV(d)) equation

$$u_t - u^2 u_x + u_{xxx} = 0, \quad (87)$$

$$u(x, 0) = u_- \text{ for } x < 0; \quad u_+ \text{ for } x > 0. \quad (88)$$

Equation (87) is an exactly integrable equation and can be treated by the IST method. It belongs to the defocusing NLS hierarchy (self-adjoint spectral operator) and its Whitham system is known to be hyperbolic [14]. Actually, the modulation system for mKdV(d) equation in Riemann invariants is identical to that for the KdV equation (this fact had been established in [41] by a direct calculation before the methods of finite-gap modulation theory became available). That, however, does not imply that the physical modulation solutions for KdV and mKdV(d) in terms of initial data  $u^+$  and  $u^-$  will be the same or even have same properties. We will demonstrate now how major qualitative and quantitative characteristics of the modulation solution can be derived from the dispersive shock conditions (82) – (84) without derivation of the Whitham system and analysis of its full solution.

We infer from (87) that  $V(\bar{u}) = -\bar{u}^2$ ,  $\omega_0(\bar{u}, k) = -k\bar{u}^2 - k^3$ . Then using (82) – (84) we readily obtain two possible sets for the edge parameters:

$$k_1^- = \tilde{k}_1^+ = \sqrt{\frac{2}{3}(u_+^2 - u_-^2)}, \quad s_1^- = u_-^2 - 2u_+^2, \quad s_1^+ = -\frac{1}{3}u_+^2 - \frac{2}{3}u_-^2, \quad (89)$$

$$k_2^+ = \tilde{k}_2^- = \sqrt{\frac{2}{3}(u_-^2 - u_+^2)}, \quad s_2^- = -\frac{1}{3}u_-^2 - \frac{2}{3}u_+^2, \quad s_2^+ = u_+^2 - 2u_-^2. \quad (90)$$

There are three cases to consider.

(i) Let  $u_+^2 - u_-^2 > 0$ . Then the second set (90) must be discarded by failure to satisfy the reality condition for the wavenumbers. The first set (89) satisfies both the reality condition and the “entropy” conditions. Thus, for  $u_+^2 - u_-^2 > 0$  one gets the dispersive shock with the negative dispersion wave pattern. The width of the dispersive shock is  $(s^+ - s^-)t = \frac{5}{3}(u_+^2 - u_-^2)t$ . The amplitude of the lead soliton is found from the relationship between the velocity and the amplitude for the mKdV(d) solitons moving on the background  $\bar{u}$ :  $c_s = -\bar{u}^2 + a_s/3$ . Since by the kinematic condition (62)  $c_s(u^+, a^+) = s^+$  one gets  $a^+ = 2(u_+^2 - u_-^2)$ . One should also note that mKdV(d) equation (87) can have soliton solutions of different polarities for  $u$ , depending on the initial conditions.

(ii)  $u_+^2 = u_-^2$ , which implies  $u_+ = u_-$  (trivial case) or  $u_+ = -u_-$ . The width of the dispersive shock becomes zero, which implies that symmetric initial discontinuity with  $u_+ = -u_-$  does not break and propagates as a whole with the velocity  $-u_+^2$ . This corresponds to the exact solution of the mKdV(d) equation in the form of a smooth kink (in the Whitham approximation its width is equal to zero, hence the propagating discontinuity).

(iii) If  $u_+^2 - u_-^2 < 0$ , the first set of parameters (89) fails to satisfy the reality condition for the wavenumbers while the second set (90) does not pass the “entropy” test. This implies that there exists a genuine global solution to the dispersionless limit which does not require a dispersive shock. This solution is a rarefaction wave  $x/t = -u^2$  confined to the interval  $-u_-^2 t \leq x \leq -u_+^2 t$ .

Of course, these results could have been obtained with the aid of Miura transform which maps solutions  $\{u\}_{KdV} \mapsto \{-u^2\}_{mKdV(d)}$ . For the modulation solution this implies in the current notations a simple change  $(u^+; u^-)_{KdV} \mapsto (-u_+^2; -u_-^2)_{mKdV(d)}$ . Our aim here, however, was to demonstrate effectiveness of the dispersive shock conditions rather than to study particular equation.

## 5 Bi-directional dispersive hydrodynamics

### 5.1 General setting and formulation of the problem

We now generalise the obtained results to a physically important case of  $2 \times 2$  strictly hyperbolic systems modified by weak dispersion. We represent such a system in a symbolic form

$$\phi_t = \mathbf{K}_4(\phi, \phi_x, \phi_{xx}, \phi_{xt} \dots). \quad (91)$$

where  $\phi$  is 2-vector,  $\mathbf{K}_4$  is vector differential operator of fourth order with respect to spatial/mixed derivatives so that the system (91) has real linear dispersion relation  $\omega = \omega_0(k)$



so that  $\omega_0 \rightarrow k$  as  $k \ll 1$  (weak dispersion). We assume that the system (91) has at least three independent conservation laws

$$\frac{\partial P_j}{\partial t} + \frac{\partial Q_j}{\partial x} = 0, \quad j = 1, 2, 3. \quad (92)$$

For convenience of explanation we associate two conserving densities  $P_{1,2}$  with the “gas” density  $\rho$  and the momentum  $\rho u$  and assume that the dispersionless limit of (91) has the form of the gas-dynamic Euler equations for the isentropic ideal gas

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \quad (93)$$

where  $p(\rho)$  is the pressure in the corresponding gas dynamics. It should be emphasized that generally speaking the assumption about the gas dynamic ‘core’ of the system (91) is just a convenient (and in many cases physically relevant) way to convey our ideas – actually there is no need to restrict oneself with this particular form of the dispersionless limit. The only property of the dispersionless limit system which will be used below is that it can be represented in the Riemann form (which is always the case for quasilinear  $2 \times 2$  systems). So actually one should put quotation marks for the “density”, “velocity” and “pressure” in this section.

As earlier, we assume that system (91) supports the single-phase periodic travelling wave solutions for  $\phi = (\rho, u)$ :

$$\phi(x, t) = \phi(x - ct), \quad \phi(\theta + 2\pi/k) = \phi(\theta), \quad (94)$$

which are parametrised by four constants, say  $\bar{\rho}, \bar{u}, k, a$ , where  $\bar{\rho}$  and  $\bar{u}$  are the mean density and mean velocity respectively, and  $k$  and  $a$  are, as usual, the wavenumber and the amplitude. We assume that the potential curve  $G(\lambda)$  in the travelling wave equation  $\lambda_\theta^2 = -G(\lambda)$  has generic properties outlined in Section 4 (see (78), (79)) such that the solution (94) admits the limiting transitions to a linear wave as  $a \rightarrow 0$  and to a solitary wave as  $k \rightarrow 0$ .

The described class of systems (91) is quite broad and includes some known integrable models such as defocusing nonlinear Schrödinger equation and Kaup-Boussinesq system [42], [43]. As physically important examples of bi-directional non-integrable systems that possess the above general properties (including the gas dynamics form of the dispersionless limit) one can indicate the Green-Naghdi system for fully nonlinear shallow water gravity waves [44] and its multi-layer generalisations [24], the systems for nonlinear ion-acoustic and magnetoacoustic waves in collisionless plasma [45], [16], and many others.

We consider initial data for the system (91) in the form of a step for the variables  $\rho$  and  $u$ :

$$t = 0 : \quad \rho = \rho^-, \quad u = u^- \text{ for } x < 0; \quad \rho = \rho^+, \quad u = u^+ \text{ for } x > 0, \quad (95)$$

where  $\rho^\pm$  and  $u^\pm$  are constants.

Analytical studies of the decay of an initial discontinuity problem in integrable dispersive wave equations (see for instance [46], [9], [43]) as well as direct numerical simulations for non-integrable systems ( see [26] [25], [13] and references therein) suggest that the asymptotic solution for the decay of an arbitrary initial discontinuity problem generally consists of three constant states separated by two expanding waves: centred rarefaction wave(s) and/or dispersive shock(s), which is quite natural taking into account the “two-wave” nature of the

system (91). The structure and the qualitative properties of the dispersive shock are the same as in the case of simple-wave lead dispersive equations (see Section 4) so we will model such a dispersive shock by the expansion fan solution of the corresponding Whitham system, which can be represented in the form

$$\frac{\partial}{\partial t} \bar{P}_j(\bar{\rho}, \bar{u}, k, a) + \frac{\partial}{\partial x} \bar{Q}_j(\bar{\rho}, \bar{u}, k, a) = 0, \quad j = 1, 2, 3, \quad (96)$$

$$\frac{\partial}{\partial t} k + \frac{\partial}{\partial x} \omega(\bar{\rho}, \bar{u}, k, a) = 0, \quad (97)$$

where  $\bar{\rho}$  and  $\bar{u}$  are the density and velocity averaged over the family (94),  $k$  is the wavenumber and  $a$  is the wave amplitude; also as follows from the formulation  $\bar{P}_1 = \bar{\rho}$ ,  $\bar{P}_2 = \bar{\rho}\bar{u}$ . Using the arguments presented in Section 3.1, we *postulate* the following fundamental property of the Whitham equations (96) for weakly dispersive nonlinear wave equations (91):

**The averaged “dispersive-hydrodynamic” conservation laws (96) admit exact reductions to the dispersionless limit system (93) for  $a = 0$  or  $k = 0$ :**

$$\bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \quad (\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + p(\bar{\rho}))_x = 0, \quad (98)$$

so that the “energy” equation for  $\bar{P}_3$  becomes a consequence of (98). Of course, in concrete instances this physically transparent property can be easily verified by direct calculation using the asymptotic behaviour of the potential curve  $G(\lambda)$  for the nearly linear and nearly soliton configurations (see (78), (79)).

The wave number conservation law (97) in the linear limit assumes the form (cf. (46)):

$$a = 0, \quad k_t + (\omega_0(\bar{\rho}, \bar{u}, k))_x = 0 \quad (99)$$

where  $\omega = \omega_0(\bar{\rho}, \bar{u}, k)$  is the linear dispersion relation for the system (91) obtained by considering an infinitesimally small perturbation of the ground state  $\rho = \bar{\rho}$ ,  $u = \bar{u}$ . Owing to the two-wave nature of the system (91) there are two branches of the linear dispersion relation. To avoid unnecessary complication, it will be assumed that, unless otherwise specified, we consider the branch corresponding to the right-propagating waves.

Now we can formulate the natural boundary conditions of the Gurevich-Pitaevskii type for a bi-directional Whitham system (96), (97). Again, for simplicity of presentation we assume the negative dispersion wave pattern and later will remove this restriction in the formulation of the transition conditions. In the decay of an initial discontinuity problem we are interested in the similarity solutions of the modulation equations so we present the natural matching problem for modulation variables at once in the form analogous to (36)

$$\begin{aligned} x = s^- t : \quad & a = 0, \quad \bar{\rho} = \rho^-, \quad \bar{u} = u^-, \\ x = s^+ t : \quad & k = 0, \quad \bar{\rho} = \rho^+, \quad \bar{u} = u^+, \end{aligned} \quad (100)$$

where the dependencies of the edge speeds  $s^\pm$  on the initial jump parameters  $\rho^+, \rho^-, u^+, u^-$  are to be found.

It is clear that the similarity solution of the fourth order Whitham system is parametrised by only three constants. Therefore, one must impose an additional restriction on the values of the parameters  $\rho^\pm, u^\pm$  to select the family of admissible jumps across the dispersive shock. Of course, the necessity of an additional jump condition for the dispersive shock generated

in the decay of an initial discontinuity is inherently implied by the two-wave nature of the system under consideration. Finding this restriction is equivalent to the extracting the family of the initial discontinuities resolving into a single dispersive shock propagating in a given direction (we associate the direction of the dispersive shock propagation with the direction of the corresponding linear characteristic). In dissipative gas dynamics, such admissible discontinuities are selected by the Rankine-Hugoniot conditions following from the balance of mass, momentum and total energy across the shock. So, before we proceed with the determination of the dispersive shock edges as we did in the simple-wave led equations, our task is to obtain a replacement for the classical shock curve in the bi-directional dispersive hydrodynamics which would have a form of a relationship

$$\Phi(\rho^+, \rho^-, u^+, u^-) = 0 \quad (101)$$

We note that in the conservative dispersive shock transition, the pressure  $p$  is the function of the density only (the thermodynamic entropy does not change in the course of the dispersive shock propagation) so, in contrast to the classical shock theory [48], the pressure should not appear in the transition relation (101).

## 5.2 Dispersive shock curve and “entropy” conditions

We proceed with the dispersive shock curve using a simple method recently proposed in [47]. The idea is, using the conservative nature of the dissipationless dispersive shocks to “reverse” the dispersive shock solution in time. It is then “natural” to expect that the dispersive shock would convert into a rarefaction wave of the dispersionless Euler system (93). One can then “extract” the required transition relation from simple analytic solution for this rarefaction wave which is easily constructed using Riemann invariants (see for instance [48], [34]). Of course, such an intuitive argument requires mathematical justification since it is not obvious why the “inverse” resolution should happen through a single rarefaction wave.

Our construction is based on the following geometric consideration in the characteristic  $(x, t)$ -plane. First we represent the Euler system (93) in the Riemann invariant form

$$r_t + V_r(r, l)r_x = 0, \quad l_t + V_l(r, l)l_x = 0, \quad (102)$$

where

$$r = u + \int_{\rho_0}^{\rho} \frac{\sigma(\rho')}{\rho'} d\rho', \quad l = u - \int_{\rho_0}^{\rho} \frac{\sigma(\rho')}{\rho'} d\rho'; \quad V_{r,l} = u \pm \sigma(\rho). \quad (103)$$

Here  $\rho_0 = \text{const}$  and  $\sigma(\rho) = (dp/d\rho)^{1/2}$  is the “sound speed”. The initial conditions (95) are rewritten in terms of the Riemann invariants as

$$r = r^-, \quad l = l^- \quad \text{for } x < 0; \quad r = r^+, \quad l = l^+ \quad \text{for } x > 0, \quad (104)$$

where  $r^\pm = r(\rho^\pm, u^\pm)$ ,  $l^\pm = l(\rho^\pm, u^\pm)$ . The corresponding four characteristic directions at  $t = 0$  are  $V_r^\pm \equiv V_r(r^\pm, l^\pm)$  and  $V_l^\pm \equiv V_l(r^\pm, l^\pm)$ .

We consider the “right-propagating” dispersive shock, which occurs when the characteristics of the  $r$ -family intersect i.e. when  $V_r^- > V_r^+$ . Then, assuming the modulation description of the dispersive shock with the aid of the expansion fan solution of the Whitham equations, one can observe that, due to (assumed) hyperbolicity the solution of the GP problem (96),

(97), (100) can be, in principle, constructed geometrically using characteristics. The continuity matching (100) for the mean values in such a construction is replaced with the equivalent continuity matching for the Whitham and Euler characteristics (see [33]). One can also observe that although such a construction can be very complicated for  $t > 0$ , it can be readily realised for negative  $t$ . Indeed, since the Whitham expansion fan has zero width at  $t = 0$  (hyperbolicity) the initial conditions (104) are specified entirely in the external, “dispersionless” domain of the space-time of the GP problem. Thus the global solution of the GP problem can be continued backwards along the characteristics of the  $2 \times 2$  dispersionless limit equations (102). By construction, this solution must (i) be three-parametric (see (101)) and (ii) satisfy the inequality  $V_r^- > V_r^+$  at  $t = 0$ .

There is a unique three-parametric solution to the hyperbolic system (102) in the lower  $(xt)$  half-plane satisfying the described restrictions. This solution is a centred expansion fan (in  $(-x, -t)$  - coordinates) given by the expressions

$$l = l_0 = \text{constant} \quad (105)$$

$$\begin{aligned} r &= r^-; & x &> a_2 t; \\ V_r(r, l_0) &= x/t; & a_1 t &\leq x \leq a_2 t; \\ r &= r^+; & x &< a_1 t. \end{aligned} \quad (106)$$

Here

$$a_1 = V_r^+, \quad a_2 = V_r^-, \quad a_2 > a_1. \quad (107)$$

The solution (105) – (107) is characterised by three parameters  $(r^+, r^-, l_0)$  and *it exists for all  $t < 0$* . Since this solution represents a continuation, along the characteristics of the *full* solution to the GP problem, one can regard this solution evaluated at a fixed moment  $t_0 < 0$  as new initial conditions for the same GP problem. Therefore, one can extract from it the global restriction (101) imposed on possible values of the hydrodynamic variables at the opposite sides of the dispersive shock. This restriction follows from (105) and has the form

$$l^- = l^+, \quad (108)$$

which in view of the relationships (103) can be represented in an explicit gas-dynamic form,

$$u^- - u^+ = \int_{\rho^+}^{\rho^-} \frac{\sigma(\rho)}{\rho} d\rho. \quad (109)$$

Given the state in front of the dispersive shock  $(\rho^+, u^+)$  this relation yields all admissible states  $(\rho^-, u^-)$  behind it i.e. it represents the equation of the  $\rho$ - $u$  diagram of the dispersive shock.

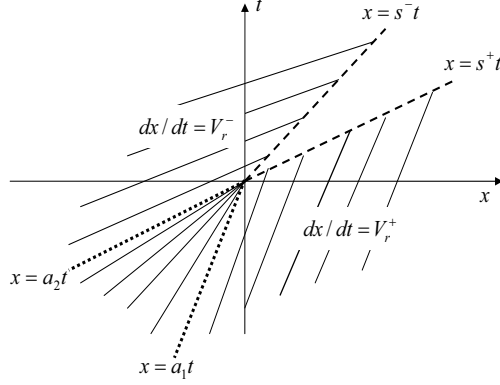
One can see that the whole above construction is subject to the additional inequalities

$$V_l^- < s^- < V_r^-, \quad V_r^+ < s^+, \quad s^+ > s^-. \quad (110)$$

These inequalities ensure fulfillment of our original requirement about the single-wave resolution of a step (i.e. three-parametric solution). Indeed, the number of parameters characterising the solution of the hyperbolic system in some domain is equal to the number of

families of characteristics transferring given initial or boundary data *into* this domain (see for instance [48]). The inequalities (110) require that only three of the four families of classical characteristics  $x/t = V_{l,r}^\pm$  (namely,  $x/t = V_r^-$ ,  $x/t = V_r^+$ , and  $x/t = V_l^+$ ) transfer initial data (104) from the  $x$ -axis into the dispersive shock domain (see Fig.4). One can see that the inequalities (110) represent an extension of the “entropy” conditions (73) to the bi-directional case.

a)



b)

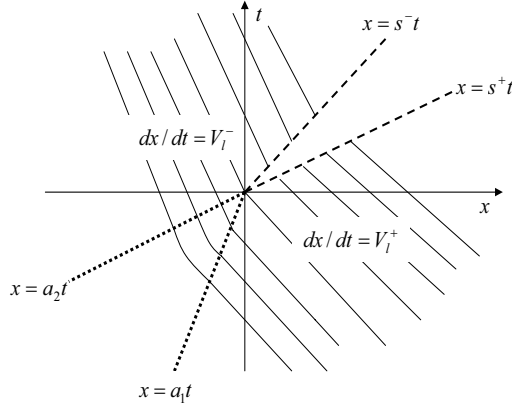


Figure 4: Qualitative behaviour of characteristics in the “simple-wave” decay in dispersive hydrodynamics. Broken lines: the dispersive shock boundaries. Dotted lines: the “mirror” expansion fan boundaries. (a) Families  $dx/dt = V_r^\pm$  transfers values of  $r$ , (b) Families  $dx/dt = V_l^\pm$  transfers values of  $l$ .

Now we discuss briefly the meaning of the obtained transition relation (109). One can see that it coincides with the relationship between any two pairs  $\rho$ ,  $u$  in the simple wave solution of the isentropic gas dynamics (see for instance [48]). A nontrivial fact is that the corresponding wave is the simple wave of *compression* which breaks after a certain time interval. Contrastingly, the similarity solution of the Whitham equations satisfying the relationship (105) and describing the expanding dispersive shock does not break. This solution essentially represents a *compression fan*, which does not exist in classical gas dynamics. It

is natural to call the dispersive shock satisfying the relationship (105) *a simple dispersive shock*. We emphasize that, according to the matching conditions (100) the relationship (109) is only valid *for the boundary values* of the modulation parameters  $\bar{\rho}$  and  $\bar{u}$  and, of course, does not hold within the dispersive shock region.

In the solutions of the GP problem for integrable systems, the simple dispersive shock condition is a mere consequence of the constancy of one of the Riemann invariants of the Whitham system (see [46], [43] for instance). In non-integrable case, when the Riemann invariants are not available for the Whitham system, the condition (109) is not obvious at all. Also, one should keep in mind that inequalities (110) represent a necessary part of the transition conditions.

We recall that the above consideration was concerned with the dispersive shocks propagating to the right. For the left-propagating dispersive shocks an analogous system of relations would include the zero jump condition for the classical Riemann invariant  $r$  instead of  $l$  in (105) and the “entropy” inequalities analogous to (110) would have the form

$$V_l^+ < s^+ < V_r^+, \quad s^- < V_l^-, \quad s^+ > s^-. \quad (111)$$

The simple dispersive shock curve in the form (109) has been proposed for the first time, using intuitive physical arguments, by Gurevich and Meshcherkin [26] in the context of collisionless shocks in plasma. Later, this condition has been interpreted in [33] in terms of the “local Riemann invariant” transport through the Whitham zone.

We note in conclusion that, in a more general case, when the dispersionless limit has the form other than isentropic gas dynamics, one should use the simple dispersive shock curve in the invariant form (108).

### 5.3 Speeds of the dispersive shock edges

We now proceed with the dispersive shock edges similarly to Sections 3.2, 3.3. We consider a characteristic equation for the Whitham system (96), (97)

$$b_1^{(i)} d_i \bar{\rho} + b_2^{(i)} d_i \bar{u} + b_3^{(i)} d_i k + b_4^{(i)} d_i a = 0, \quad (112)$$

where  $\mathbf{b}^{(i)}(\bar{\rho}, \bar{u}, k, a)$  is the left eigenvector of the coefficient matrix  $A(\bar{\rho}, \bar{u}, k, a)$  of the modulation system (96), (97) and  $i$  is a number of the characteristic. Let  $a = 0$  on this characteristic. Then for the remaining three variables we have the ordinary differential equation

$$b_1^{(i)}(\bar{\rho}, \bar{u}, k, 0) d_i \bar{\rho} + b_2^{(i)}(\bar{\rho}, \bar{u}, k, 0) d_i \bar{u} + b_3^{(i)}(\bar{\rho}, \bar{u}, k, 0) d_i k = 0 \quad (113)$$

which necessarily is a characteristic equation for the reduced as  $a = 0$  modulation system (96), (97):

$$a = 0, \quad \bar{\rho}_t + (\bar{\rho} \bar{u})_x = 0, \quad (\bar{\rho} \bar{u})_t + (\bar{\rho} \bar{u}^2 + p(\bar{\rho}))_x = 0, \quad k_t + (\omega_0(\bar{\rho}, \bar{u}, k))_x = 0. \quad (114)$$

Now we impose an additional constraint

$$F(\bar{\rho}, \bar{u}) = C_0, \quad (115)$$

where  $C_0$  is a constant. It will be shown later that this constraint leads to a self-consistent set of the transition conditions. One can see that Eq. (115) is consistent with the reduction

(114) and, therefore, is compatible with (113) iff the function  $F(\bar{\rho}, \bar{u})$  is one of the Riemann invariants  $l, r$  defined by (103). Setting Eq. (115) into Eq. (113) we obtain a 2-form which can be integrated to give the local relationship between admissible values of  $\bar{\rho}$  and  $k$  on the characteristic,

$$\Phi_i(\bar{u}, k; C_0) = C_1, \quad (116)$$

$C_1$  being a constant of integration.

Alternatively, the relationship (116) can be obtained by a direct substitution of Eq. (115) into the system (114) and then, by looking for the solution in the form  $k(\bar{u})$ . Substitution of Eq. (115) into (114) yields (cf. (46)):

$$a = 0, \quad \bar{u}_t + V(\bar{u})\bar{u}_x = 0, \quad k_t + (\omega_0(\bar{u}, k))_x = 0, \quad (117)$$

where (see Eq. (103))

$$V(\bar{u}) \equiv V_r(\rho(\bar{u}), \bar{u}), \quad \omega_0(\bar{u}, k) \equiv \omega_0(\bar{\rho}(\bar{u}), \bar{u}, k). \quad (118)$$

Thus, the problem essentially reduces to the simple-wave led case considered in Sections 3, 4 and, therefore, yields the ordinary differential equation (82) for  $k(\bar{u})$ . In our case, however, this equation is additionally parametrised by the unidentified (yet) function  $F(\bar{\rho}, \bar{u})$  and a constant  $C_0$  in (115) defining the dependence  $\bar{\rho}(\bar{u})$  in (118). These can be found by applying the matching conditions (100) considered in the four-dimensional space of the field variables  $\bar{\rho}, \bar{u}, k, a$ . Since the integral (115) does not contain  $a$  and  $k$  it must apply to both boundaries  $a = 0$  and  $k = 0$ , which requires  $F(\rho^-, u^-) = F(\rho^+, u^+)$ . Then, to be consistent with the dispersive shock curve (108) one must identify  $F(\bar{\rho}, \bar{u}) \equiv l(\bar{\rho}, \bar{u})$ , which immediately yields  $C_0 = l^- = l^+$ . Now, the trailing edge speed  $s^-$  is found with the aid of Eq. (84a).

The leading edge edge is handled in exactly the same way, by reducing the “hydrodynamic” part of the modulation system for  $k = 0$  to the simple wave equation with the aid of the constraint (115). Then one proceeds with the conjugate variables as in Section 3.3 to get a general simple-wave analog of the ordinary differential equation (59) i.e. Eq. (83) for  $\tilde{k}(\bar{u})$ . At last, using the dispersive shock curve (108) the function  $F(\bar{\rho}, \bar{u})$  in (118) is identified with the same classical Riemann invariant  $l$ , which justifies self-consistency of the whole construction. Then the leading edge speed is found via the conjugate wavenumber  $\tilde{k}$  by Eqs. (84b), (83) where all the necessary ingredients are given by Eq. (118).

The whole construction is subject to the “entropy conditions” (85) which should be complemented by an additional inequality  $s^- > V_l^-$  (see (110)) occurring due to the presence of the fourth characteristic family in the bi-directional problems (see [47] for an example where this inequality is violated).

It is clear that analogous formulas can be obtained for the left- propagating dispersive shock: one should just replace in Eq. (118)  $V_r(\bar{u})$  with  $V_l(\bar{u})$  and  $l(\bar{\rho}, \bar{u}) = C_0$  with  $r(\bar{\rho}, \bar{u}) = C_0^*$  where  $C_0^* = r^- = r^+$  according to the corresponding dispersive shock curve. Also, one should use the “left” branch of the linear dispersion relation and the “entropy” conditions (111).

Some additional details of the derivation of the trailing and the leading edge curves for bi-directional systems can be found in [47].

## 5.4 Example: fully nonlinear ion-acoustic collisionless shock in plasma

As an example of an effective construction of the dispersive shock conditions in a non-integrable system we consider classical system of equations describing finite-amplitude ion-acoustic waves in a two-temperature ( $T_e \gg T_i$ ) collisionless plasma (see for instance [45])

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ u_t + uu_x + \varphi_x &= 0, \\ \varphi_{xx} &= e^\varphi - \rho.\end{aligned}\tag{119}$$

Here  $\rho$  and  $u$  are the ion density and velocity and  $\varphi$  is the electric potential; all dependent variables are dimensionless. We note that the system does not contain the time derivative of  $\varphi$  so  $\varphi$  is not a “real” dependent variable as concerns the  $2 \times 2$  representation (91). A direct numerical simulation of the decay of an initial discontinuity in Eqs. (119) has been performed in [26].

The system (119) supports periodic travelling waves allowing for linear and solitary wave limits and also possesses (at least) four conservation laws [38]. Thus we can apply the methods developed in this paper for obtaining the dispersive shock conditions. We consider the right-propagating simple dispersive shock.

In the dispersionless limit  $\varphi = \ln \rho$ , which yields the Euler isentropic gas-dynamic equations (93) with the equation of state  $p(\rho) = \rho$ . The corresponding Riemann invariants and characteristic velocities are (see (102), (103)):

$$l = u - \ln \rho, \quad r = u + \ln \rho, \quad V_l = u - 1, \quad V_r = u + 1, \tag{120}$$

while the linear dispersion relation for the right-propagating modulated waves has the form

$$\omega_0(k, \bar{\rho}, \bar{u}) = k[\bar{u} + (1 + k^2/\bar{\rho})^{-1/2}]. \tag{121}$$

The dispersive shock transition curve (109) then assumes the form  $u^- - u^+ = \ln(\rho^-/\rho^+)$ . Without loss of generality we put  $u^+ = 0$ ,  $\rho^+ = 1$ . Then the relationship between  $\bar{\rho}$  and  $\bar{u}$  in Eq. (118) becomes  $\bar{u} = \ln \bar{\rho}$ . As a result, we get all the necessary ingredients for the basic ordinary differential equations (82), (83):

$$V(\bar{u}) = \bar{u} + 1, \quad \omega_0(\bar{u}, k) = k[\bar{u} + (1 + k^2/e^{\bar{u}})^{-1/2}]. \tag{122}$$

Then the equation (82) after elementary transformations, assumes the form with separated variables

$$\frac{d\alpha}{d\bar{u}} = -\frac{(1 + \alpha)^2 \alpha}{2(1 + \alpha + \alpha^2)}, \quad \alpha(1) = 1, \tag{123}$$

where  $\alpha = (1 + k^2/e^{\bar{u}})^{-1/2}$ . Integrating (123) we get

$$\bar{u} + 2 \ln \alpha + \frac{1 - \alpha}{1 + \alpha} = 0. \tag{124}$$

Now, using the formula (84a) we obtain a simple equation determining velocity of the trailing edge  $s^-$  in terms of the density ratio across the dispersive shock  $d = \rho^-/\rho^+ = \rho^-$ ,

$$\ln d + \frac{2}{3} \ln(s^- - \ln d) = \frac{(s^- - \ln d)^{1/3} - 1}{(s^- - \ln d)^{1/3} + 1}. \tag{125}$$



The leading edge is handled in a completely analogous way. The solitary wave dispersion relation is obtained from the linear dispersion relation (122) as  $\tilde{\omega}_s(\bar{u}, \tilde{k}) = -i\omega_0(\bar{u}, i\tilde{k})$  which is to say

$$\tilde{\omega}_s(\bar{u}, \tilde{k}) = \tilde{k}[\bar{u} + (1 - \tilde{k}^2/e^{\bar{u}})^{-1/2}]. \quad (126)$$

Then, integrating (83) we obtain  $\tilde{k}(\bar{u})$  (it is also convenient to introduce  $\tilde{\alpha} = (1 - \tilde{k}^2/e^{\bar{u}})^{-1/2}$  as an intermediate variable instead of  $\tilde{k}$ ). Setting it into (84b) we eventually get the equation for the leading edge

$$2 \ln s^+ - \frac{s^+ - 1}{s^+ + 1} = \ln d. \quad (127)$$

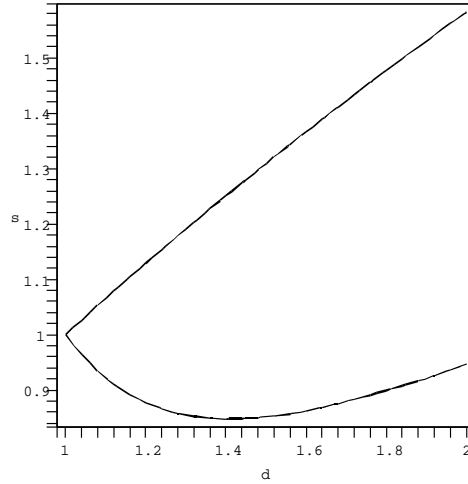


Figure 5: Speeds of the edges of the ion-acoustic dispersive shock transition versus density ratio  $d = \rho^-/\rho^+$  across the shock; upper graph:  $s^+$  – leading edge, lower graph:  $s^-$  – trailing edge

It can be easily verified that the “entropy” conditions (110) are satisfied for all values of the density ratio across the dispersive shock. The curves  $s^+(d)$  and  $s^-(d)$  are presented in Fig.5 and demonstrate very good agreement with the results of direct numerical simulation of the decay of an initial discontinuity for the system (119) presented in [26] (see Fig. 6 in this paper analogous to our Fig.5). From theoretical point of view such an agreement can be regarded as a strong indication of validity of the modulation theory in a certain class of non-integrable initial value problems where rigorous derivation of the Whitham asymptotic as a zero-dispersion limit is not available.

The weakly nonlinear asymptotic decompositions of (125) and (127) for small  $\delta = d - 1 \ll 1$ , have the form

$$s^- \approx 1 - \delta, \quad s^+ \approx 1 + \frac{2}{3}\delta, \quad (128)$$

an, as a matter of fact, agree with the Gurevich-Pitaevskii solution for the KdV solution (30) which is another confirmation of validity of the obtained modulation solution. Curiously, as is clearly seen from Fig. 5, the fully nonlinear dynamics of the leading (solitary wave) edge  $s^+$  is very well approximated by the weakly nonlinear asymptotic (128) in a broad

range of the density ratios  $d$  while the speed of the trailing (harmonic) edge  $s^-$  demonstrates significant qualitative and quantitative deviations from its weakly nonlinear counterpart even for quite moderate values of  $d$ .

A detailed comparison of the modulation transition conditions obtained here with results of direct numerical simulation for the (non-integrable) Green – Naghdi system describing fully nonlinear shallow-water waves will be published in [50].

## 6 Some restrictions

We first address the accuracy of the obtained results in the context of the original (dispersive) system and outline some restrictions of our analysis of the dispersive shocks. Our main assumption was about modelling the a dispersive shock with the aid of the expansion fan solution of the Whitham equations. The Whitham equations, being obtained by an asymptotic procedure, have inherent accuracy restrictions on their applicability to certain wave regimes. In our (GP) formulation, in contrast to the formal zero-dispersion limit approach, the small dispersion parameter determining the accuracy of the Whitham approximation is assumed to occur *in the solutions* and is defined by the ratio of the typical wavelength to the characteristic scale of the modulation variations (in our case, the width of the dispersive shock  $L \sim t$ ), i.e.

$$\epsilon \sim (kt)^{-1} \ll 1. \quad (129)$$

which implies that that modulation description of the dispersive shock is valid asymptotically as  $t \gg 1$ . The Whitham method of averaging conservation laws [16] also requires that the relative variations of modulation parameters over the characteristic wavelength scale  $\sim k^{-1}$  be  $\mathcal{O}(\epsilon)$  which yields the criterion

$$|k_x|k^{-1} \sim t^{-1}. \quad (130)$$

One can now observe that, in the modulation solution for the dispersive shock, for *any*  $t$ , however large, there is a certain vicinity of the of the leading edge where both criteria (129), (130) are violated due to small (and rapidly changing) values of  $k$ . This is basically a reflection of non-uniformity occurring as  $k \rightarrow 0$  in the formal perturbation decompositions (see e.g. [49]) equivalent to leading order to the Whitham method. So a natural question of applicability of the modulation description to the dynamics of the leading edge of the dispersive shock arises. While this question does not appear in the the rigorous zero-dispersion limit approach [11] in which the Whitham equations are *derived* as a certain asymptotics in the initial-value problem and  $\epsilon$  does not depend on  $t$ , it has to be addressed in our case when the Whitham description is *assumed* on heuristic foundations in the absence of rich integrable structure.

We first determine the behaviour of the function  $k(x)$  in the vicinity of the leading edge of the simple dispersive shock for fixed  $t$ . It is instructive to make first an estimate for the KdV case, which can be made by using the wave number conservation law in the self-similar form  $-sk' + \omega' = 0$ ,  $s = x/t$ . Now setting the asymptotic decompositions for  $k$  and  $\omega$  as  $m' \ll 1$ , which are readily obtained from (6) we get for the vicinity of the leading edge  $s^+ - s \ll 1$  the following asymptotic behaviour:  $k \sim k_0 / \ln \delta^{-1}$ ,  $\omega - ks^+ \sim k\delta$ ,  $\delta \sim (s^+ - s)(\ln 1/(s^+ - s))^{-1}$ . It is clear that this behaviour is automatically generalised to other nonlinear dispersive systems for which the potential curve  $G(u)$  in the travelling waves has the asymptotic behaviour (78), (79) in the nearly linear and nearly soliton configurations.

Now, using the obtained asymptotics for  $k$ , we estimate the relative widths  $\sigma = \Delta x/L$  of the vicinities of the leading edge  $x^+ = s^+t$  where the criteria (129), (130) are violated. For (129) we get  $\sigma_1 \sim \exp(-t)$  while for (130) we get  $\sigma_2 \sim t^{-1} \ln t$ , i.e.  $\sigma_2 \gg \sigma_1$ . Still,  $\sigma_2(t) \rightarrow 0$  as  $t \rightarrow \infty$  and, therefore, the Whitham description of the dispersive shock is asymptotically as  $t \gg 1$  valid for all  $s^-t < x < s^+t$ .

The modulation approach used in this paper requires the existence of the single-phase periodic solutions. For non-integrable systems such solutions often exist only within a certain domain of parameters. Typically the role of the “critical” parameter is played by the wave amplitude, so that for  $a > a_{cr}$  the periodic solution (or a solitary wave) ceases to exist because of occurrence of breaking or cusp-type singularities. This usually violates the “single-flow” type assumptions used in the derivation of the original system. For instance, for ion-acoustic system (119) the critical solitary wave amplitude for the potential  $\varphi$  is  $a_{cr} \approx 1.3$  [3]. For  $a_s > a_{cr}$  the “hydrodynamic” single-flow description of the two-temperature plasma becomes not applicable and a more general, kinetic plasma theory should be used. The critical value for the density ratio across the ion-acoustic dispersive shock  $\Delta_{cr}$  can be found by setting the critical value of the solitary wave speed  $c_s(a_{cr}) \approx 1.6$  [26] into Eq. (127):  $s^+(\Delta_{cr}) \approx 1.6$ , which yields  $\Delta_{cr} \approx 2.0$ . Another possible restriction on the application of the obtained transition conditions partially addressed in Section 4 is imposed by the requirement of the modulational stability of the dispersive shock solution. For some equations the type of the modulation system can change from hyperbolic to elliptic depending on the initial data. One of the examples is provided by the so-called Kaup-Boussinesq (integrable) system describing bi-directional shallow water waves where the linear dispersion relation yields complex frequencies if the wavenumber exceeds a certain value [43]. The corresponding restrictions on the jump values across the dispersive shock can be found from the conditions (86).

## 7 Conclusions

A new method has been proposed to analyse the dispersive shock transition for a broad class of weakly dispersive nonlinear wave equations. No assumptions of integrability have been made which, in particular, allows one to apply the developed method to problems of fully nonlinear dispersive wave dynamics. The transition conditions have been derived by assuming the description of the dispersive shock with the aid of the expansion fan solutions of the associated modulation (Whitham) equations. The Whitham system was assumed to be hyperbolic for the solutions of our interest, which seems a reasonable assumption for the outlined class dispersive systems (at least for a certain type of initial data). This assumption can also be viewed as an inference from direct numerical simulations.

The analysis has been performed using the generalisation of the so-called Gurevich-Pitaevskii problem formulated originally for the averaged KdV equation. It has been shown that the Gurevich-Pitaevskii type natural boundary conditions for the mean flow in the dispersive shock region can be translated into information about degeneration of the families of characteristics along the dispersive shock boundaries. The latter implies certain restrictions on admissible values of the hydrodynamic and wave parameters at these boundaries. It has been shown that, for the problem of an initial step resolution, these restrictions can be found without presence of the Riemann invariant structure for the Whitham system.

As a result, we have derived a complete set of transition conditions for dispersive shocks linking two different constant hydrodynamic flows. The analysis has been performed as for single-wave equations so for bi-directional systems. The transition conditions have been derived in a general form and include: i) a “dispersive” analog of the traditional jump conditions; ii) the equations for the boundary curves of the dispersive shock; iii) a set of inequalities similar to the entropy conditions of traditional gas dynamics.

Remarkably, the whole set of the transition conditions is constructed in terms of the linear dispersion relation and the dispersionless nonlinear characteristic velocities of the system under study. The obtained conditions allow one to “fit” a dispersive shock into the classical dispersionless solution without complicated analysis of the internal structure of the dispersive shock similarly to classical shock theory in traditional gas dynamics. The method can be useful not only for the analysis of non-integrable dispersive systems where full solution is not available but also for integrable systems when one is interested only in main physical parameters of the dispersive shock transition. As an additional bonus, the developed method allows for obtaining the lead solitary wave amplitude, the major parameter observed in experiments.

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