Modeling internal solitary waves on the Australian North West Shelf

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Abstract:

The transformation of nonlinear internal waves and the development of internal solitary waves on the Australian North West Shelf is studied numerically in the framework of the generalized Korteweg–de Vries equation. This model contains both nonlinearity (quadratic and cubic), the Coriolis effect, depth variation and horizontal variability of the density stratification. The computed results demonstrate that a wide variety of nonlinear wave shapes can be explained by the synergetic action of cubic nonlinearity and the variability of the hydrology along the wave path.

1. Introduction

Peter Holloway was a pioneer in the study of internal waves on the Australian North West Shelf (NWS). A wide variety of nonlinear waves of solitary and bore-like shapes have been observed during two decades of his productive work in this area [Holloway, 1983, 1984, 1985, 1987, 1988, 1992]. These studies have demonstrated the important role of nonlinearity in the formation of the baroclinic wave field. As follows from Holloway's numerous observations, a typical picture of the evolution of the quasi-sinusoidal semi-diurnal internal tide is similar to that of a simple (Riemann) wave, with incipient shock wave formation at the leading or trailing faces, and subsequent disintegration into packets of short period solitary waves. Some examples of such observations include table-like solitons, or groups of isolated short-scale solitons as demonstrated in Fig. 1. Nonlinear internal waves on the NWS have

been observed from satellites [Baines, 1981; GOA, 2004] and Fig. 2 shows isolated waves with quasi-plane fronts, and groups of short-scale waves with curvilinear fronts.



Fig. 1. Displacement of the 25°C isotherm observed on 2 April (left) and 25 March (right) 1992 (from [Holloway et al, 1999]).



Fig. 2. Satellite image of internal waves on the Australian North West Shelf [GOA, 2004].

Statistical analysis of large-amplitude internal waves has been carried out by Pelinovsky et al. (1995), where it was shown that such waves up to 25 m amplitudes can be described by Poisson statistics. Various numerical models have been applied to explain the internal tide in this area. Models such as the Princeton Ocean Model based on the nonlinear primitive equations shows a high degree of spatial variability in the amplitude and phase of internal wave currents and vertical displacements [Craig, 1988; Holloway, 1996; Holloway and Barnes, 1998]. Weakly nonlinear and dispersive models based on generalizations of the Korteweg–de Vries (KdV) equation demonstrate the appearance of intense short-scale waves of soliton-like type from the internal tide on the NWS [Smyth & Holloway, 1988; Holloway

et al, 1997, 1999, 2001]. The results of numerical simulations within the framework of these models demonstrate the important role of nonlinearity, dispersion, the Earth's rotation, bottom slope and horizontal variability of density and current stratification for the wave shape formation. These combined effects lead to existence of a wide variety of nonlinear waves; forward and backward shocks, solitary waves of positive and negative polarities, table-like (or thick) solitons are very often observed on the NWS.

This paper deals with the further study of soliton dynamics on the NWS. The main goal of this study is to determine the soliton life-time taking into account the spatial variability of hydrology and the depth variation in the coastal zone. The basic generalized KdV equation is briefly described in section 2. The coefficients of the model are calculated using the hydrological data on the NWS obtained by Peter Holloway; they are presented in section 3. Our results of numerical simulation of the solitary wave dynamics on the NWS are presented in section 4. It is demonstrated that the variability of the hydrology on the shelf leads to the strong variability of the nonlinear wave shapes. The results obtained are summarized in the conclusion.

2. Generalized Korteweg-de Vries equation

Initially the classical KdV model was a very popular model to describe the properties of the observed internal waves in 80th years, see for instance, the review papers by Grimshaw (1983) and Ostrovsky & Stepanyants (1989). Then, with increasing and more detailed internal wave observations, it was found that several further factors should be included in the model. First of all, variable depth effect leads to wave amplitude variation and the possible changing of the soliton polarity; this effect has been studied in the South China Sea [Liu et al, 1998; Cai et al, 2002; Zhao et al, 2003]. The second one is the horizontal variability of the density and shear flow stratification which is important, in particular, for the NSW [Holloway et al, 1997]. The third one is the cubic nonlinear effects which are comparable with quadratic nonlinearity in tropical conditions [Holloway et al, 1999]. And the last one is the Coriolis effect due to the Earth's rotation which influences the number and amplitudes of the solitary wave disturbances generated from the internal tide [Gerkema, 1996]. These factors are included in the theoretical and numerical model which can be titled as the generalized KdV equation [Holloway et al, 1999, 2001]. The basic equation of this model is

$$\frac{\partial\varsigma}{\partial x} + \left(\frac{\alpha Q}{c^2}\varsigma + \frac{\alpha_1 Q^2}{c^2}\varsigma^2\right)\frac{\partial\varsigma}{\partial s} + \frac{\beta}{c^4}\frac{\partial^3\varsigma}{\partial s^3} = \frac{f^2}{2c}\int_{-\infty}^s \varsigma ds, \qquad (1)$$

where $\xi(x, s)$ is the wave function determining the time evolution of the vertical displacement on the isopycnal surface $\xi(x, z, t)$

$$\xi(x,z,t) = \Phi(z,x)Q(x)\varsigma(x,t) + T(z,x)Q^{2}(x)\xi^{2}(x,t).$$
(2)

Here $\Phi(z, x)$ is the modal function of long internal waves in the linear approximation. It is found from the eigenvalue problem, given the buoyancy frequency N(z, x) and water depth H(x),

$$\frac{\partial^2 \Phi}{\partial z^2} + \frac{N^2(z,x)}{c^2(x)} \Phi = 0, \qquad \Phi(z=0) = 0, \qquad \Phi(z=-H(x)) = 0, \tag{3}$$

where z = 0 corresponds to the free surface and z = -H(x) to the seafloor. The modal function is normalized on its maximum, so that $\Phi(z_m) = 1$. The solution of the eigenvalue problem (3) determines also the speed of long linear internal waves, c(x).

The coefficients of Eq. (1) are expressed in terms of integrals containing the modal function:

$$\beta(x) = \frac{c}{2} \frac{\int_{-H}^{0} \Phi^2 dz}{\int_{-H}^{0} (\partial \Phi / \partial z)^2 dz}$$
 (4)

$$\alpha(x) = \frac{3c}{2} \frac{\int_{-H}^{0} (\partial \Phi / \partial z)^3 dz}{\int_{-H}^{0} (\partial \Phi / \partial z)^2 dz},$$
(5)

$$Q(x) = \sqrt{\frac{\left(Mc^3\right)}{Mc^3}}, \qquad M(x) = \int_{-H}^{0} \left(\partial \Phi / \partial z\right)^2 dz, \qquad (6)$$

The term with a subscript "0" is for a value at the point x = 0 corresponding to the incident wave. Finally in Eq. (1) f is the Coriolis parameter ($f = 2\Omega \sin\theta$ with Ω being the frequency of the Earth's rotation and θ being the local latitude [Grimshaw et al., 1998]), and s is the "running time" which is determined as

$$s = \int_{0}^{x} \frac{dx}{c} - t..$$
 (7)

The coefficient of the cubic nonlinear term is

$$\alpha_{1}(x) = \frac{\int_{-H}^{0} \left\{ 3c \left[3\left(\frac{\partial T}{\partial z}\right) - 2\left(\frac{\partial \Phi}{\partial z}\right)^{2} \right] \left(\frac{\partial \Phi}{\partial z}\right)^{2} - \frac{\alpha^{2}}{c} \left(\frac{\partial \Phi}{\partial z}\right)^{2} + \alpha \left[5\left(\frac{\partial \Phi}{\partial z}\right)^{2} - 4\frac{\partial T}{\partial z} \right] \left(\frac{\partial \Phi}{\partial z}\right)^{2} dz, \tag{8}$$

and the nonlinear correction to the mode can be found from

$$\frac{\partial^2 T}{\partial z^2} + \frac{N^2}{c^2} T = -\frac{\alpha}{c^2} \frac{\partial^2 \Phi}{\partial z^2} + \frac{3}{2} \frac{\partial}{\partial z} \left[\left(\frac{\partial \Phi}{\partial z} \right)^2 \right]$$
(9)

with the boundary conditions: T(-H) = T(0) = 0 and $T(z_m) = 0$. Note that we use so-called time-like Korteweg–de Vries Eq. (1) in the form were spatial and time variables are interchanged. This form is relevant for the observational data analysis of time series measured at fixed spatial places, while the KdV equation in its standard form is relevant for the initial value problem when the snapshot of wave profile is available.

Before proceeding further, let us note that strictly speaking, Eq. (1) should include a term due to frictional effects (see, e.g. [Holloway et al, 1997, 1999]). However, the dissipative decay length has been estimated as typically about 100 wavelengths [Craig, 1991; Holloway, 1997, 1999], or about 100 km for the NWS. Since here we are considering internal solitary wave evolution for distances which are essentially less than this, dissipative effects have been omitted, since we expect that the role of friction will then be less than that of the nonlinearity and the horizontal variability of the hydrology.

The "initial condition" at x = 0 for the generalized KdV Eq. (1) corresponds to the incident wave, $\xi(x = 0, t)$, which should be determined from observations of the vertical displacement, $\xi(x = 0, z, t)$ with the use of the truncated series (2). The boundary conditions (in time) for the generalized KdV equation are periodic, with the semidiurnal period of an internal wave, that is 12 hr. Equation (1) has two integrals used to control the numeric simulation

$$I_1 = \int_0^T \zeta(s, x) ds = \int_0^T \xi(t, x = 0) dt = \text{const}, \qquad I_2 = \int_0^T \zeta^2(s, x) ds = \int_0^T \xi^2(t, x = 0) dt = \text{const}.$$
(10)

Note that the first integral, which can be interpreted as related to mass conservation, is identically zero if the Coriolis parameter $f \neq 0$. The second integral is that for wave action flux conservation.

This model is valid for long internal waves of weak and moderate amplitudes, and one simple criterion for this is the weakness of the nonlinear correction to the linear speed; this criterion is checked in our computations. Recently, Small and Hornby (2005) made some comparisons of calculations obtained within the framework of the extended KdV equation (but without the Coriolis term) and a fully nonlinear model for moderate-amplitude internal waves, and showed that the extended KdV equation "works" well to describe the nonlinear wave evolution.

3. Hydrology of the Australian North West Shelf used for simulations

Most of Holloway's observations of internal waves were made in the area of NWS shown in Fig. 3. In the deep water (marked by the symbol C13 in the figure), the buoyancy frequency profile has one large maximum at depth 40 m, which corresponds to the existence of a sharp pycnocline (Fig. 4a). In the transition zone from deep to shallow water the pycnocline spreads and at depths less than 250 m, the pycnocline disappears which results in an almost uniform buoyancy frequency profile in average (Figs. 4b, c). In shallow water the buoyancy frequency profile contains two smooth pycnoclines with the biggest one near the seafloor (Fig. 4d). Such large variations of the density stratification lead to a variety of nonlinear wave shapes, as has been noted many times in Holloway's papers.



Fig. 3. Map of the region of observations from the Australian North West Shelf showing mooring locations Break and Shelf and temperature-salinity measurement locations C1 to C13 (from Holloway et al, 1999).

According to our theoretical model, such a variety of nonlinear wave shapes is related to the large variability of the coefficients of the generalized KdV equation through the horizontal variability of the buoyancy frequency profile. In particular, the nonlinear coefficients (α and α_1) vary significantly, even changing sign, and this induces the complicated dynamics of the nonlinear wave field. The computed coefficients of the generalized KdV equation are presented in Fig. 5 along the line of CTD stations shown in Fig. 3 (length 100 km, depth is varied from 1400 to 60 m). The linear speed of wave propagation monotonically decreases with depth from 1.4 m/s to 0.3 m/s. The coefficient Q in Eq. (6) describing the linear amplification effect due to the hydrology variability varies from 1 to 4, and therefore even within the linear theory of long waves the wave amplitude may grow up 4 times when the wave approaches the cost.



Fig. 4. Buoyancy profile at depths 1380 m (a), 240 m (b), 120 m (c) and 70 m (d)

But in reality, the nonlinear and dispersive effects have a more important effect on the wave amplitude and shape. The dispersion coefficient decreases significantly with depth (through several orders of magnitude), and the role of the nonlinear terms increase. The coefficient of quadratic nonlinearity, α , is negative for deep water, and positive for shallow water passing through a zero value twice. The coefficient of cubic nonlinearity, α_1 being small for deep water is positive for intermediate depth and negative for shallow water. Such a large variability of the coefficients leads to the strong variability of the wave shapes, as will be seen in the next section.



Fig. 5. Variation of the coefficients of the generalized KdV equation for conditions of the NWS.

4. Computed wave evolution on the Australian North West Shelf

Solitary waves (solitons) are very often observed on the NWS. As discussed above, the horizontal variability of the density stratification should result in a strong transformation of an internal solitary wave. This effect is investigated here. The numerical model is initialized with the "thick" soliton – the soliton solution of Eq. (1) without the Coriolis term (such equation is called the extended KdV equation, or Gardner equation),

$$\varsigma(s) = \frac{A}{1 + B\cosh[\gamma(s - s_0)]}, \qquad A = \frac{6\beta\gamma^2}{\alpha c^2}, \qquad \gamma = \sqrt{\frac{\alpha^2 c^2}{6\alpha_1 \beta}(B^2 - 1)}.$$
(11)

The soliton amplitude is (not to be confused with the parameter *A*)

$$a = \frac{A}{1+B} = \frac{\alpha}{\alpha_1} (B-1).$$
(12)

The whole soliton family can be characterized by two parameters: amplitude, *a*, and phase, s_0 . Equation (1) is then solved in the periodic domain in time (that is, in *s*) with a period of 12 h corresponding to the semidiurnal tidal cycle, and for convenience of graphic presentation, the initial phase is chosen as $s_0 = 6.2$ h. The initial wave amplitude is widely varied, but here we present results for the amplitude a = 10 m only. In the following figures the wave displacement is shown at the depth where the maximum of the linear mode is located. The value $\xi(t, z_m, x) = Q(x)\xi(t, x)$ is shown at different distances from the deepest point (point C13 in Fig. 3 at depth 1400 m) in the onshore direction. The full wave profile at any depth can be obtained from Eq. (2).

The first run is done within the framework of the KdV equation when the cubic nonlinear term and the Coriolis effect are both ignored. In this case the soliton (11) transforms to the KdV soliton

$$\zeta(s) = a \operatorname{sech}^{2} \left[\sqrt{\frac{\alpha a}{12\beta}} c(s - s_{0}) \right].$$
(13)

The wave evolution for this case is displayed in the left panel of Fig. 6. After 40 km of distance, when dispersion has reduced significantly, the initial depression solitary wave transforms to an oscillatory wave train (similar to an Airy-function). The wave amplitude has increased, but not four times as predicted by the linear long wave theory; this is a consequence of the influence of dispersion on the wave packet. Because the quadratic nonlinear term in shallow water is positive, an elevation solitary wave is formed at a distance of 70 km.

The second set of simulations for the NWS with the Coriolis effect now taken into account did not bring essential qualitative changes in the picture of wave profiles (see right panel in Fig. 6). Meanwhile, the leading wave height (trough-to-crest) becomes higher than 6 m. The reason for this is that in the absence of both dispersion and the Coriolis effect, the maximum amplitude (relative to Q) of a wave is conserved, but the presence of a non-zero Coriolis effect removes this constraint.

The third set of simulations takes account of the cubic nonlinear term. Cubic nonlinearity changes the wave shape radically. Compared with the KdV model, the number of waves is less and their amplification is not so high (see left panel in Fig. 7). Here also one can see the transformation of a depression wave to elevation waves due to change in sign of the coefficient of quadratic nonlinearity. Further, importantly, the wave amplitudes within the group are not ordered, as is the KdV case when the cubic nonlinearity was ignored. The Coriolis term influences the wave transformation very weakly (see right panel in Fig. 7).



Fig. 6. Wave evolution when cubic nonlinearity is ignored (left/right – without/with the Coriolis effect). Numbers indicate distance in km.

The fourth simulation was conducted for a quasi-cnoidal wave of height 10 m in the framework of the generalized KdV Eq. (1); the results are shown in Fig. 8. Here the initial perturbation was created by the linear superposition of nine solitons less the constant pedestal, so that the total mass of the perturbation was zero in accordance with Eq. (10). Again, one can clearly see the changing of the polarity of the internal waves with distance, due to the change of sign of the coefficient of quadratic nonlinearity. The wave height is increased in shallow water up to four times, and the resulting wave shapes show the complicated character of the wave field.



Fig. 7. Wave evolution taking account of cubic nonlinearity (left/right – without/with the Coriolis effect). Numbers indicate distance in km.



Fig. 8. Periodic wave evolution on the Australian North West Shelf. Numbers indicate distance in km.

There are groups of solitons of positive polarity and different heights, as well as shock-like disturbances. These profiles look very similar to those observed in the ocean (see, e.g. right panel in Fig. 1).

5. Conclusion

This paper is dedicated to the memory of Peter Holloway, who stimulated our study of internal waves on the NWS. The wide variety of observed internal solitary wave shapes can be explained theoretically by the synergetic action of cubic nonlinearity and the strong horizontal variability of the density stratification. Our numerical simulations presented here demonstrate such variability in the nonlinear wave profiles.

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