Embedded Eigenvalues and Neumann-Wigner Potentials for Relativistic Schrödinger Operators

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Abstract

We construct Neumann-Wigner type potentials for the massive relativistic Schrödinger operator in one and three dimensions for which an eigenvalue embedded in the absolutely continuous spectrum exists. First we consider the relativistic variants of the original example by von Neumann and Wigner, and as a second example we discuss the potential due to Moses and Tuan. We show that in the non-relativistic limit these potentials converge to the classical Neumann-Wigner potentials. For the massless operator in one dimension we construct two families of potentials, different by the parities of the (generalized) eigenfunctions, for which an eigenvalue equal to zero or a 0-resonance exists, dependent on the rate of decay of the corresponding eigenfunctions.

Key-words: relativistic Schrödinger operator, non-local operator, Neumann-Wigner potentials, embedded eigenvalues, resonances

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1 Introduction

In the theory of classical Schrödinger operators $H = -\frac{1}{2}\Delta + V$, featuring the Laplacian Δ and a multiplication operator V called potential, a remarkable result says that eigenvalues embedded in the absolutely continuous spectrum may occur for carefully chosen potentials [18, 6]. A first example has been proposed by von Neumann and Wigner in the early days of quantum mechanics [17], constructing an oscillating potential for which the reflected wave and the transmitted wave combine through tunneling to a finite wave-function at eigenvalue equal to 1 in appropriately chosen units. This is a rotationally symmetric potential on $\mathbb{R}^3$ given by

$$V_{\mathrm{NW}}(x) = -32\frac{\sin|x|\left(g(|x|)^3\cos|x| - 3g(|x|)^2\sin^3|x| + g(|x|)\cos|x| + \sin^3|x|\right)}{(1+g(|x|)^2)^2}, \qquad (1.1)$$

where $g(|x|) = 2|x| - \sin 2|x|$, and the corresponding eigenfunction is

$$u_{\mathrm{NW}}(x) = \frac{\sin|x|}{|x|(1+g(|x|)^2)}. \tag{1.2}$$

In physics, generalized eigenfunctions for such positive eigenvalues are known as scattering states, and they correspond to quantum states with finite life-time; they have been realized even experimentally, see [2]. Since this example many generalizations of Neumann-Wigner type potentials have been obtained. In particular, the set of embedded eigenvalues is not necessarily a small set, Simon has shown that examples can be constructed for which there is a dense set of positive eigenvalues [21], see also [16, 19]. In spite of this, the possibility of existence of embedded eigenvalues is a delicate problem. Indeed, a fundamental result by Kato shows that if $V(x) = o(1/|x|)$, then no embedded eigenvalues exist [14]. Since

$$V_{\mathrm{NW}}(x) \simeq -\frac{8\sin 2|x|}{|x|} + O(|x|^{-2}), \quad \text{as } |x| \to \infty,$$

clearly there is only a narrow margin separating potentials for which embedded eigenvalues exist from potentials for which they can be ruled out.

In this paper we consider the relativistic Schrödinger operator

$$H = (-\Delta + m^2)^{1/2} - m + V \tag{1.3}$$

on $L^2(\mathbb{R}^d)$, with rest mass $m \geq 0$. The spectral properties of this operator and its variants have been much studied, see e.g. [23, 8, 3, 9, 10]. In contrast with the classical case, for relativistic Schrödinger operators no similar result to von Neumann and Wigner seems to have been established before. In the classical cases the main idea underlying the construction is to rewrite the eigenvalue equation and seek a suitable potential $V = \lambda + \frac{1}{2}\frac{\Delta u}{u}$, where λ is an eigenvalue and u is a corresponding eigenfunction. In order the potential V to be non-singular, the zeroes of u need to be matched with the zeroes of Δu. When, however, the operator $(-\Delta + m^2)^{1/2}$ is used instead of the Laplacian, one has to cope with the difficulty of controlling the zeroes of functions transformed under a non-local operator. In the present paper we develop a new technique dealing with this issue and obtain explicit formulae.

Our results are as follows. Assuming $m > 0$, in Section 2.1 we construct a Neumann-Wigner type potential in one and three dimensions for which the relativistic Schrödinger operator (1.3) has a positive eigenvalue equal to $\sqrt{1+m^2} - m$. These potentials are smooth and decay at infinity like $1/|x|$. This can be compared with a result in [13] where we show that when the potential is $o(1/|x|)$ and some further conditions hold, no embedded eigenvalues exist. In Section 2.2 we show that in the non-relativistic limit the potentials obtained in Section 2.1 converge uniformly to the classical Neumann-Wigner potentials in the C^2-norm, thus our result can be seen as a relativistic counterpart of the original Hamiltonian. In Section 2.3 we construct a second example of a potential for which a positive eigenvalue exists, and whose classical variant is originally due to Moses and Tuan [15]. This potential has a less regular behaviour than the original Neumann-Wigner potential and needs a more delicate technique.

Next we consider the massless case $m = 0$ of the operator (1.3). In this case it is known that for $d = 3$ and $m = 0$ the conditions that $|V|$, $|x \cdot \nabla V|$ and $|x \cdot \nabla(x \cdot \nabla V)|$ are bounded by $C(1+x^2)^{-1/2}$, with a small $C > 0$, jointly imply that H has no non-negative eigenvalue

[20]. In Section 2.4 we construct a family of potentials V_ν in one dimension, for which the massless relativistic Schrödinger operator has an eigenvalue equal to zero corresponding to an even eigenfunction, and a family $\tilde{V}_\nu$ for which there is a zero eigenvalue corresponding to an odd eigenfunction. In either case the eigenvalues become resonances if ν is small enough. These results are of interest also since by the results obtained in [11, 12], it would follow for the massless operator with a decaying potential having *negative* eigenvalues that the corresponding eigenfunctions decay at a rate $1/|x|^2$, while in our cases they decay like $1/|x|$ or slower. Since the fall-off of eigenfunctions for decaying potentials depends on the distance of the eigenvalue from the edge of the continuous spectrum, it is interesting to see that when this distance drops to zero, the decay of eigenfunctions goes through a regime change slowing them down, and we are able to say precisely to which rate. Our examples for the massive and massless operators also complement the explicit formulae recently obtained in developing a calculus for the fractional Laplace operator [5].

2 Existence of positive eigenvalues

2.1 Neumann-Wigner type potential for relativistic Schrödinger operators

We consider the relativistic Schrödinger operator on $L^2(\mathbb{R})$ as given by (1.3), and assume $m > 0$. Denote $p = -i\frac{d}{dx}$ and define the following functions:

$$g(x) := 2x - \sin(2x), \qquad h(x) := \frac{1}{1 + g(x)^2}$$
$$f(x) := \left(\sqrt{(p+1)^2 + m^2} + \sqrt{(p-1)^2 + m^2}\right) h(x)$$

and

$$u(x) := f(x) \sin x \tag{2.1}$$
$$V(x) := \lambda - \frac{1}{u(x)}\left(\sqrt{p^2 + m^2} - m\right) u(x), \tag{2.2}$$

where $\lambda := \sqrt{1 + m^2} - m > 0$.

Theorem 2.1. *Let H be given by (1.3) and V by (2.2). If $m \geq 146$, then V is a real-valued smooth potential with the property that $V(x) = O(1/|x|)$, and λ and u satisfy the eigenvalue equation*

$$Hu = \lambda u, \qquad u \in D(H). \tag{2.3}$$

Remark 2.2.

(1) The condition $m \geq 146$ is inessential, and the restriction can be removed by scaling so that the result applies for all $m > 0$. For $a > 0$ let $(U_a g)(x) = a^{1/2} g(ax)$. Then H is unitary equivalent to

$$U_a H U_a^{-1} = \frac{1}{a}\left(\sqrt{p^2 + (am)^2} - am + aV(ax)\right). \tag{2.4}$$

By using Theorem 2.1 we can construct a smooth decaying potential V such that (2.4) has a positive eigenvalue for any a with $am > 146$.

(2) Formally it is clear that u and V satisfy the eigenvalue equation (2.3). A main difficulty is that since in (2.2) the denominator u has zeroes in $x = n\pi$, $n \in \mathbb{N}$, the numerator should vanish at the same points in order V to be continuous. However, in the numerator we have u under the non-local operator $(p^2+m^2)^{1/2}$ and in general there is no straightforward way to control the zeroes of such functions. This problem is solved by Theorem 2.1 in the present setting.

We can use this basic result to derive a result in three dimensions.

Corollary 2.3. *Let $m \geq 146$, write $W(x) = V(|x|)$, $x \in \mathbb{R}^3$, and define*

$$H_{\mathrm{r}} = \sqrt{-\Delta + m^2} - m + W(x), \tag{2.5}$$

acting on $L^2(\mathbb{R}^3)$. Then

$$v(x) = \frac{u(|x|)}{\sqrt{4\pi}|x|}$$

is in $D(H_{\mathrm{r}})$ and satisfies the eigenvalue equation $H_{\mathrm{r}} v = \lambda v$ with the same eigenvalue $\lambda = \sqrt{1+m^2} - m$.

2.2 Non-relativistic limit

Next we show that in the non-relativistic limit the potentials, eigenvalues and eigenfunctions constructed in the previous section converge to the expressions obtained by von Neumann and Wigner. To show this, we restore the speed of light $c > 0$ as a parameter in the operator. Define

$$f_c(x) := \frac{1}{2mc}\left(\sqrt{(p+1)^2 + m^2c^2} + \sqrt{(p-1)^2 + m^2c^2}\right) h(x), \tag{2.6}$$

$$u_c(x) := f_c(x) \sin x, \tag{2.7}$$

$$\lambda_c := c\left(\sqrt{1+m^2c^2} - mc\right), \tag{2.8}$$

$$V_c(x) := \lambda_c - c\frac{\left(\sqrt{p^2+m^2c^2} - mc\right) u_c(x)}{u_c(x)}. \tag{2.9}$$

Then we define the relativistic Hamiltonian with c by

$$H_c := \sqrt{c^2p^2 + m^2c^4} - mc^2 + V_c(x). \tag{2.10}$$

Note that we keep using a system of units in which Planck's constant is $\hbar = 1$. Then by Theorem 2.1 we see that the eigenvalue equation

$$H_c u_c = \lambda_c u_c \tag{2.11}$$

holds for all $c > 146/m$.

Theorem 2.4. *For every fixed $m > 0$ we have the following non-relativistic limit:*

$$\lim_{c\to\infty} u_c(x) = \sin(x)h(x) =: u_\infty(x), \qquad \textit{uniformly in } C^2(\mathbb{R}), \tag{2.12}$$

$$\lim_{c\to\infty} \lambda_c = \frac{1}{2m}, \tag{2.13}$$

$$\lim_{c\to\infty} W_c(x) = \frac{1}{2m}\Big(1 - \frac{p^2 u_\infty(x)}{u_\infty(x)}\Big), \qquad x \in \mathbb{R}\setminus\pi\mathbb{N}. \tag{2.14}$$

In the three-dimensional case we retrieve the expressions (1.1)-(1.2). With a similar notation as in (2.6)-(2.10) we obtain

Corollary 2.5. *For every fixed $m > 0$ we have $\lim_{c\to\infty}\lambda_c = \frac{1}{2m}$, $\lim_{c\to\infty} v_c(x) = u_{\mathrm{NW}}(x)$, uniformly in $C^2(\mathbb{R}^3)$, and $\lim_{c\to\infty} W_c(x) = V_{\mathrm{NW}}(x)$, for all $x \in \mathbb{R}^3$.*

2.3 Moses-Tuan type potential

In [15], Moses and Tuan presented another example of a potential and eigenfunction for which an eigenvalue equal to 1 occurs. Their observation is the following. Write

$$u_{\mathrm{MT}}(x) = \frac{\sin|x|}{|x|(1+g(|x|))}, \qquad x \in \mathbb{R}^3$$

$$V_{\mathrm{MT}}(x) = \frac{-32\sin|x|((|x|+1/2)\cos|x| - \sin|x|)}{(1+g(|x|))^2}.$$

Then $(-\Delta + V_{\mathrm{MT}}(x))u_{\mathrm{MT}}(x) = u_{\mathrm{MT}}(x)$ holds, for all $x \in \mathbb{R}^3$. In this section we construct the relativistic variant of this example.

Let

$$\tilde{h}(x) = \frac{1}{1+g(|x|)}, \quad x \in \mathbb{R}, \tag{2.15}$$

and write $p = -id/dx$ as before. Define

$$\tilde{f}(x) = (\sqrt{(p+1)^2+m^2} + \sqrt{(p-1)^2+m^2})\tilde{h}(x), \tag{2.16}$$

$$\tilde{u}(x) = \tilde{f}(x)\sin x,$$

$$\tilde{V}(x) = \lambda - \frac{1}{\tilde{u}(x)}(\sqrt{p^2+m^2} - m)\tilde{f}(x), \tag{2.17}$$

$$\tilde{H} = \sqrt{p^2+m^2} - m + \tilde{V}(x)$$

where $\lambda = \sqrt{1+m^2}$. Since $\tilde{h} \in D(p^3) \subset L^2(\mathbb{R})$, $\tilde{f}$ in (2.16) is defined as a function in $L^2(\mathbb{R})$.

Theorem 2.6. *If $m > 34$, then $\tilde{V}(x)$ is a continuous function with the property that $\tilde{V}(x) = O(1/|x|)$, and λ and $\tilde{u}$ satisfy*

$$\tilde{H}\tilde{u} = \lambda\tilde{u}, \qquad \tilde{u} \in D(\tilde{H}). \tag{2.18}$$

Theorem 2.6 can be extended to the three dimensional case.

Corollary 2.7. *If $m > 34$, write $\tilde{W}(x) = \tilde{V}(|x|)$, $x \in \mathbb{R}^3$, and define*

$$\tilde{H}_{\mathrm{r}} := \sqrt{-\Delta + m^2} - m + \tilde{W}(x), \tag{2.19}$$

acting on $L^2(\mathbb{R}^3)$. Then

$$\tilde{v}(x) = \frac{\tilde{u}(|x|)}{\sqrt{4\pi}|x|} \tag{2.20}$$

is in $D(\tilde{H}_{\mathrm{r}})$ and satisfies the eigenvalue equation $\tilde{H}_{\mathrm{r}}\tilde{v} = \lambda\tilde{v}$ with $\lambda = \sqrt{1+m^2} - m$.

The non-relativistic limit yielding u_{MT} and V_{MT} can be derived in a similar way as in Theorem 2.4; the details are left to the reader.

2.4 Massless case and zero eigenvalue

In this section we consider the massless relativistic Schrödinger operator

$$H^0(V) := \sqrt{-d^2/dx^2} + V(x),$$

on $L^2(\mathbb{R})$. One might expect that there is a similar way to construct a potential having a positive eigenvalue for H^0, however, our method used to establishing Theorems 2.1 and 2.4 does not work for the massless case. The reason for this can be appreciated more directly by using a Feynman-Kac-type description and recalling that the processes generated by the massive and massless operators differ essentially, see [12]. Instead, we obtain two families of potentials for which $H^0(V)$ has an eigenvalue equal to zero or a 0-resonance. Recall the hypergeometric function ${}_2F_1$, see e.g. [1].

Theorem 2.8. *Let $\nu > 0$ and define*

$$\begin{aligned} V_\nu(x) &:= -\frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(\nu)}(1+x^2)^{-1/2}{}_2F_1(1, \tfrac{1}{2}+\nu, \tfrac{1}{2}, -x^2) \\ u_\nu(x) &:= \frac{1}{(1+x^2)^\nu}. \end{aligned}$$

(1) If $0 < \nu < 1/2$, then $V_\nu(x) = O(1/|x|)$ and u_ν satisfies

$$\sqrt{-\frac{d^2}{dx^2}}u_\nu + V_\nu u_\nu = 0, \tag{2.21}$$

in distributional sense.

(2) If $\nu = \frac{1}{2}$, the same eigenvalue equation holds with

$$V_{1/2}(x) := -\frac{1}{\pi}\left(\frac{1}{\sqrt{1+x^2}} - \frac{2|x|\operatorname{arcsinh}|x|}{1+x^2}\right) \quad \text{and} \quad u_{1/2}(x) := \frac{1}{\sqrt{1+x^2}},$$

and we have $V_{1/2}(x) = O(\log|x|/|x|)$.

(3) If $1/2 < \nu < 1$, then $V_\nu(x) = O(1/|x|^{2-2\nu})$, and the eigenvalue equation (2.21) holds.

Remark 2.9. Note that $V_\nu(x) \in C^\infty(\mathbb{R})$. Since $u_\nu \in L^2(\mathbb{R})$ only for $\nu > \frac{1}{4}$, Theorem 2.8 implies that $H^0(V_\nu)$ has an eigenvalue equal to zero if $\frac{1}{4} < \nu \leq 1$, and a 0-resonance if $0 < \nu \leq \frac{1}{4}$.

Since u_ν is an even function, Theorem 2.8 can not be extended to three dimensions. The following result gives odd zero-energy eigenfunctions for another family of potentials.

Theorem 2.10. *Define*

$$\widetilde{V}_\nu(x) := \begin{cases} -\frac{2(1-2\nu)\Gamma\left(\nu-\frac{1}{2}\right)}{(1-\nu)\sqrt{\pi}\Gamma(\nu-1)}(1+x^2)^\nu \; {}_2F_1\left(2, \frac{1}{2}+\nu; \frac{3}{2}; -x^2\right), & \text{if } \nu \neq 1 \\ -\frac{2}{1+x^2}, & \text{if } \nu = 1, \end{cases}$$

$$v_\nu(x) := \frac{x}{(1+x^2)^\nu}.$$

Then $\sqrt{-\frac{d^2}{dx^2}}v_\nu + \widetilde{V}_\nu v_\nu = 0$ *holds in distributional sense, and*

$$\widetilde{V}_\nu(x) = \begin{cases} O(1/|x|), & \text{if } \frac{1}{2} < \nu < \frac{3}{2}, \nu \neq 1 \\ O(1/|x|^2), & \text{if } \nu = 1 \\ O(\log|x|/|x|), & \text{if } \nu = \frac{3}{2} \\ O(1/|x|^{4-2\nu}), & \text{if } \frac{3}{2} < \nu < 2. \end{cases} \tag{2.22}$$

Remark 2.11.

(1) Since $v_\nu(x) = O(1/|x|^{2\nu-1})$, we have $v_\nu \in L^2(\mathbb{R})$ only for $\nu > \frac{3}{4}$. Thus Theorem 2.10 implies that $H^0(\widetilde{V}_\nu)$ has a zero-energy eigenvalue if $\nu > \frac{3}{4}$, and it has a zero-energy resonance whenever $\frac{1}{2} < \nu \leq \frac{3}{4}$.

(2) A special situation occurs for $\nu = 1$. In this case $H^0(\widetilde{V}_1)$ has a zero-energy eigenvalue, and $\widetilde{V}_1(x) = -\frac{2}{1+x^2}$ is a smooth, short-range and strictly negative potential. Note that this is the only case when $\widetilde{V}_\nu$ is short-range.

(3) Since v_ν is an odd smooth function, by taking its radial part as in Corollary 2.3, the conclusion of Theorem 2.10 can be extended to three dimensions.

(4) We note that both V_ν and $\widetilde{V}_\nu$ have a finite number of zeroes. For $0 < \nu < 1/2$, we have that $|x|V_\nu(x)$ tends to a positive number given below by (3.66) as $|x| \to \infty$, i.e., $V_\nu(x)$ has no zeroes beyond large enough $|x|$, and since ${}_2F_1$ is an analytic function, there is no accumulation point of the zeroes of V_ν. A similar argument applies for the other ranges of ν and for $\widetilde{V}_\nu$. In fact, we conjecture that each of these functions has at most one zero.

3 Proofs

3.1 Proof of Theorem 2.1

We start by showing some properties of h.

Lemma 3.1. *We have that $h \in C^\infty(\mathbb{R})$ and the estimates*

$$\frac{1}{6}\frac{1}{1+x^2} < h(x) < \frac{1}{x^2+2/3} \tag{3.1}$$
$$|h'(x)| \le 8h(x)^{3/2} \tag{3.2}$$
$$|h''(x)| \le 120h(x)^{3/2}, \tag{3.3}$$

hold for all $x \in \mathbb{R}$.

Proof. The fact $h \in C^\infty(\mathbb{R})$ and estimate (3.1) are elementary. Note that $g'(x) = 4\sin^2 x$, thus $|g'(x)| \le 4$ and $|g''(x)| \le 4$. Since $h' = -2g'gh^2$, we have

$$|h'(x)| \le 8h(x)\frac{|g(x)|}{1+g(x)^2} = 8h(x)\frac{|g(x)|}{\sqrt{1+g(x)^2}}\frac{1}{\sqrt{1+g(x)^2}} \le 8h(x)^{3/2}. \tag{3.4}$$

Also, we have $h'' = -8(g')^2h^3 + 6(g')^2h^2 - 2gg''h^2$, and thus

$$\begin{aligned}
|h''(x)| &\le 2|(4h(x)-3)(g'(x))^2h(x)^2| + 2|g''(x)g(x)h(x)^2| \\
&\le 2|(3+h(x))(g'(x))^2h(x)^2| + 2|g''(x)g(x)h(x)^2| \\
&\le 8h(x)(4(3+h(x))h(x) + |g(x)|h(x)) \\
&\le 8h(x)(4(3+1)h(x)^{1/2} + h(x)^{1/2}) \\
&\le 120h(x)^{3/2}.
\end{aligned} \tag{3.5}$$

□

We write for simplicity $p := -i\frac{d}{dx}$ and $\langle x\rangle := (1+x^2)^{1/2}$. The nth derivative of h will be denoted by $h^{(n)}$.

Lemma 3.2. *For every $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that*

$$|h^{(n)}(x)| \le C_n \langle x\rangle^{-3}, \qquad x \in \mathbb{R}. \tag{3.6}$$

In particular, we have that $h, f, u \in \cap_{n=1}^\infty D(p^n)$.

Proof. Note that $|g(x)| \le 2\langle x\rangle$. Since $g'(x) = 4\sin^2 x$, it is clear that $g^{(n)}(x)$ is bounded for all $n \ge 1$. We show (3.6) by induction on n. For $n = 1$ we have $h'(x) = -2h(x)^2g(x)g'(x)$, which is bounded by $36\langle x\rangle^{-3}$ and (3.6) holds. Suppose that the claim holds for $k \le n$. We estimate $(h^2)^{(k)}$ and $(gg')^{(k)}$. By the assumption, we have for $k \in \mathbb{N}$ that

$$\begin{aligned}
|(h^2)^{(k)}(x)| &= \left|2h(x)h^{(k)}(x) + \sum_{j=1}^{k-1}\binom{k}{j}h^{(j)}(x)h^{(k-j)}(x)\right| \\
&\le 3\langle x\rangle^{-2}C_k\langle x\rangle^{-3} + \left|\sum_{j=1}^{k-1}\binom{k}{j}C_jC_{k-j}\langle x\rangle^{-3}\langle x\rangle^{-3}\right| \\
&\le C_{1,k}\langle x\rangle^{-5},
\end{aligned} \tag{3.7}$$

where $C_{1,k}$ is a constant. Since all derivatives of g are bounded, the estimate $|(gg')^{(k)}(x)| \leq C_{2,k} \langle x \rangle$, $k = 0, 1, 2, \ldots$, holds with a suitable constant $C_{2,k}$. Thus we have

$$\begin{aligned} |h^{(n+1)}(x)| &= |(h')^{(n)}(x)| = 2|(h^2 gg')^{(n)}(x)| \\ &\leq 2h^2(x)(gg')^{(n)}(x) + 2\sum_{k=0}^{n-1} \binom{n}{k} \left|(h^2)^{(n-k)}(x)\,(gg')^{(k)}(x)\right|. \end{aligned} \tag{3.8}$$

The first term in (3.8) is of order $\langle x \rangle^{-3}$, and the second of order $\langle x \rangle^{-4}$. Thus $|h^{(n+1)}(x)|$ is bounded by $C_{n+1} \langle x \rangle^{-3}$ with a constant C_{n+1}, which completes the induction step. The bound (3.6) implies that $h \in D(p^n)$ for all $n \in \mathbb{N}$. Hence we obtain by functional calculus that $f, u \in D(p^n)$ for all n. □

Note that since h is real and even, the eigenfunction u is real and odd, and the potential V is real and even. We write

$$\omega(p) := \sqrt{p^2 + m^2} - m \quad \text{and} \quad \omega_0(p) := \sqrt{p^2 + m^2}.$$

Lemma 3.3. *We have that*

$$\omega(p)u(x) = \lambda u(x) + \sin(x)(\omega(p+1) - \lambda)f(x) - 2e^{-ix}h'(x) \tag{3.9}$$

$$V(x) = -\frac{(\omega(p+1) - \lambda)f(x)}{f(x)} + \frac{2h'(x)e^{-ix}}{f(x)\sin x}. \tag{3.10}$$

Proof. By functional calculus it is readily seen that the equality $e^{-ix}pe^{ix} = p + 1$ gives $e^{-ix}\omega(p)e^{ix} = \omega(p+1)$. Using this, we obtain

$$\begin{aligned} \omega(p)u = \omega(p)\frac{1}{2i}(e^{ix} - e^{-ix})f &= \frac{1}{2i}(e^{ix}\omega(p+1) - e^{-ix}\omega(p-1))f \\ &= \sin x\omega(p+1)f + \frac{1}{2i}e^{-ix}(\omega(p+1) - \omega(p-1))f \\ &= \lambda u(x) + \sin x(\omega(p+1) - \lambda)f + \frac{e^{-ix}}{2i}(\omega(p+1) - \omega(p-1))f \\ &= \lambda u(x) + \sin x(\omega(p+1) - \lambda)f + \frac{e^{-ix}}{2i}(\omega_0(p+1) - \omega_0(p-1))f. \end{aligned} \tag{3.11}$$

By the definition of f, we furthermore get that

$$\begin{aligned} (\omega_0(p+1) - \omega_0(p-1))f(x) &= (\omega_0(p+1) - \omega_0(p-1))(\omega_0(p+1) + \omega_0(p-1))h(x) \\ &= (\omega_0(p+1)^2 - \omega_0(p-1)^2)h(x) \\ &= 4(-i)h'(x). \end{aligned} \tag{3.12}$$

□

Since $h' = -2g'gh = -8\sin^2 xg(x)h(x)^2$, the potential can be written as

$$V(x) = -\frac{1}{f(x)}\big(\omega(p+1) - \lambda\big)f(x) - \frac{16e^{-ix}}{f(x)}g(x)h(x)^2 \sin x. \tag{3.13}$$

The following lemma makes the crucial steps for proving the main statement. In Propositions 3.10 and 3.12 below we will prove that conditions (P.1)-(P.2) in the lemma hold.

Lemma 3.4. *If*

(P.1) there exists a constant $C > 0$ *such that* $f(x) \geq C(1+x^2)^{-1}$ *for all* $x \in \mathbb{R}$,

(P.2) $(\omega(p+1) - \lambda)f(x) = f(x)O(|x|^{-1})$ *as* $x \to \infty$,

then Theorem 2.1 follows.

Proof. Note that the eigenvalue equation (2.3) is equivalent to (2.2) and (3.13). Since by Lemma 3.2 we have $f \in \cap_{n=1}^{\infty} D(p^n)$, it follows that $\omega(p+1)f \in \cap_{n=1}^{\infty} D(p^n)$, in particular, $\omega(p+1)f \in C^{\infty}(\mathbb{R})$. By (P.1) the denominator $f(x)$ in (3.13) has no zeroes, and thus V has no singularity. Since $f(x)$, $\omega(p+1)f(x)$ and $g(x)h(x)e^{-ix}\sin x$ are smooth functions, V is also smooth. By (P.2) the first term of (3.13) is of order $\langle x\rangle^{-1}$, and by (P.1) and Lemma 3.1 the second term of (3.13) is also of order $\langle x\rangle^{-1}$. Hence $V(x)$ is of order $\langle x\rangle^{-1}$, which goes to zero as $|x| \to \infty$. □

Let $K : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Formally, the operator $K(p)$ can be defined by the convolution

$$K(p)g(x) = \frac{1}{\sqrt{2\pi}}(\widehat{K} * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{K}(x-y)g(y)dy, \tag{3.14}$$

where $\widehat{K}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} K(k)e^{-ikx}dk$ denotes Fourier transform.

Lemma 3.5. *The following bounds hold:*

$$\sqrt{\frac{2}{\pi}}(1-e^{-1})\frac{e^{-m|x|}}{\sqrt{1+2m|x|}} \leq \widehat{\omega_0^{-1}}(x) \leq \frac{e^{-m|x|}}{\sqrt{m|x|}}, \quad x \in \mathbb{R}. \tag{3.15}$$

Proof. The integral formula

$$\widehat{\omega_0^{-1}}(x) = \sqrt{\frac{2}{\pi}} K_0(m|x|) \tag{3.16}$$

is well-known, where K_0 denotes the modified Bessel function of the second kind [22, p183]. For $z > 0$ by a change of variable we obtain

$$K_0(z) = \int_0^{\infty} \exp(-z\cosh t)dt = e^{-z}\int_0^{\infty} \frac{e^{-s}}{\sqrt{s(s+2z)}}ds, \tag{3.17}$$

giving

$$\begin{aligned} K_0(z) \geq e^{-z}\int_0^1 \frac{e^{-s}}{\sqrt{s(s+2z)}}ds &\geq e^{-z}\int_0^1 \frac{e^{-s}}{\sqrt{1+2z}}ds \\ &\geq (1-e^{-1})e^{-z}\sqrt{1+2z}. \end{aligned} \tag{3.18}$$

Hence the lower bound in (3.15) follows. To get the upper bound, we estimate

$$K_0(z) \leq e^{-z}\int_0^{\infty} \frac{e^{-s}}{\sqrt{2sz}}ds = \frac{e^{-z}}{\sqrt{2z}}\int_0^{\infty} \frac{e^{-s}}{\sqrt{s}}ds = \frac{e^{-z}}{\sqrt{2z}}\sqrt{\pi}. \tag{3.19}$$

□

Lemma 3.6. *If $m > 18$, then for all $x \in \mathbb{R}$,*

$$\omega_0(p)^{-1}h(x) \geq \frac{1}{25m}\langle x\rangle^{-2}. \tag{3.20}$$

Proof. By Lemmas 3.1 and 3.5, we have

$$\begin{aligned}
\omega_0(p)^{-1}h(x) &\geq \frac{1-e^{-1}}{\pi}\int_{\mathbb{R}}\frac{e^{-m|x-y|}}{\sqrt{1+2m|x-y|}}\frac{1}{6}\frac{1}{1+y^2}dy \\
&= \frac{1-e^{-1}}{6\pi}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{1+2m|y|}}\frac{1}{1+(x+y)^2}dy \\
&= \frac{1-e^{-1}}{6\pi}\frac{1}{1+x^2}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{1+2m|y|}}\frac{1+x^2}{1+(x+y)^2}dy \\
&\geq \frac{1-e^{-1}}{6\pi}\frac{1}{1+x^2}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{1+2m|y|}}\left(\inf_{x\in\mathbb{R}}\frac{1+x^2}{1+(x+y)^2}\right)dy. \qquad (3.21)
\end{aligned}$$

We estimate the integral by using that

$$\inf_{x\in\mathbb{R}}\frac{1+x^2}{1+(x+y)^2} = \frac{2}{2+y^2+\sqrt{4y^2+y^4}} \geq \frac{1}{(1+|y|)^2}, \tag{3.22}$$

and

$$\begin{aligned}
\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{1+2m|y|}}\frac{1}{(1+|y|)^2}dy &= \frac{2}{m}\int_0^\infty\frac{e^{-s}}{\sqrt{1+2s}}\frac{1}{(1+m^{-1}s)^2}ds \\
&\geq \frac{2}{m}\int_0^\infty\frac{e^{-s}}{\sqrt{1+2s}(1+(s/18))^2}ds \geq \frac{6}{5m}. \qquad (3.23)
\end{aligned}$$

Thus finally we have

$$\omega_0(p)^{-1}h(x) \geq \frac{1-e^{-1}}{6\pi}\frac{6}{5m}\frac{1}{1+x^2}, \tag{3.24}$$

which proves the claim. □

Lemma 3.7. *If $m > 0$, then for all $x \in \mathbb{R}$,*

$$\omega_0(p)^{-1}h(x) \leq \frac{3+4m^2}{\sqrt{2}m^3}\langle x\rangle^{-2}. \tag{3.25}$$

Proof. Lemmas 3.1 and 3.5 imply

$$\begin{aligned}
\omega_0(p)^{-1}h(x) &\leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-m|x-y|}}{\sqrt{m|x-y|}}\frac{1}{y^2+2/3}dy \\
&\leq \frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{m|y|}}\left(\sup_{x\in\mathbb{R}}\frac{1+x^2}{(x+y)^2+2/3}\right)dy, \qquad (3.26)
\end{aligned}$$

where it is elementary that

$$\sup_{x\in\mathbb{R}} \frac{1+x^2}{(x+y)^2+\frac{2}{3}} = \frac{5+3y^2+\sqrt{1+30y^2+9y^4}}{4} \leq 2y^2+2. \tag{3.27}$$

Hence we have

$$\omega_0(p)^{-1}h(x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\int_{\mathbb{R}} \frac{e^{-m|y|}}{\sqrt{m|y|}}(2y^2+2)dy = \frac{3+4m^2}{\sqrt{2}m^3}\frac{1}{1+x^2}. \tag{3.28}$$

□

Lemma 3.8. *If $m > 10$, we have for all $x \in \mathbb{R}$ that*

$$\left|\omega_0(p)^{-1}p^2h(x)\right| \leq \frac{700}{m}\langle x\rangle^{-3}. \tag{3.29}$$

Proof. By Lemma 3.1 and 3.5, it follows that

$$\begin{aligned}
\left|\omega_0(p)^{-1}p^2h(x)\right| &= \left|\omega_0(p)^{-1}h''(x)\right| \\
&\leq \frac{120}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{-m|x-y|}}{\sqrt{m|x-y|}}h(x)^{3/2}dy \\
&\leq \frac{120}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{-m|y|}}{\sqrt{m|y|}}\frac{1}{((x+y)^2+\frac{2}{3})^{3/2}}dy \\
&\leq \frac{120}{\sqrt{2\pi}}\frac{1}{(1+x^2)^{3/2}}\int_{\mathbb{R}} \frac{e^{-m|y|}}{\sqrt{m|y|}}\sup_{x\in\mathbb{R}}\left(\frac{1+x^2}{(x+y)^2+\frac{2}{3}}\right)^{3/2}dy \\
&\leq \frac{120}{\sqrt{2\pi}}\frac{1}{(1+x^2)^{3/2}}\int_{\mathbb{R}} \frac{e^{-m|y|}}{\sqrt{m|y|}}(2y^2+2)^{3/2}dy \\
&= \frac{240}{\sqrt{2\pi}}\frac{1}{m}\frac{1}{(1+x^2)^{3/2}}\int_0^{\infty} \frac{e^{-s}}{\sqrt{s}}(2(s/m)^2+2)^{3/2}ds \\
&\leq \frac{240}{\sqrt{2\pi}}\frac{1}{m}\frac{1}{(1+x^2)^{3/2}}\int_0^{\infty} \frac{e^{-s}}{\sqrt{s}}(2(s/10)^2+2)^{2}ds,
\end{aligned} \tag{3.30}$$

where in the last inequality we used the assumption that $m > 10$. By computing the integral, the claim follows. □

Lemma 3.9. *If $m \geq 146$, we have for all $x \in \mathbb{R}$,*

$$\omega_0(p)h(x) \geq \langle x\rangle^{-2}, \tag{3.31}$$

Proof. We split up the expression like

$$\omega_0(p)h(x) = \frac{p^2+m^2}{\sqrt{p^2+m^2}}h(x) = \frac{m^2}{\sqrt{p^2+m^2}}h(x) + \frac{p^2}{\sqrt{p^2+m^2}}h(x). \tag{3.32}$$

By Lemmas 3.6 and 3.8, for $m \geq 146$ we have

$$\begin{aligned}\sqrt{p^2+m^2}h(x) &\geq \frac{1}{25m}(1+x^2)^{-1} - \frac{700}{m}(1+x^2)^{-3/2} \\ &\geq \left(\frac{1}{25m} - \frac{700}{m}\right)(1+x^2)^{-1} \geq \frac{1}{1+x^2}.\end{aligned} \tag{3.33}$$

□

Now we turn to proving conditions (P.1) and (P.2) in Lemma 3.4.

Proposition 3.10. *If $m \geq 146$, then for all $x \in \mathbb{R}$ we have*

$$f(x) \geq 2\langle x\rangle^{-2}. \tag{3.34}$$

Proof. Define

$$T(k) := \sqrt{(k+1)^2+m^2} + \sqrt{(k-1)^2+m^2} - 2\sqrt{k^2+m^2},$$

and note that $T(k) \to 0$ as $k \to \infty$. The Fourier transform of $T(k)$ is

$$\widehat{T}(x) = 2\widehat{\omega_0}(x)(\cos x - 1).$$

From (3.17), by noting that $\omega_0(k) = (k^2+m^2)\omega_0^{-1}(k)$, we have

$$\begin{aligned}\widehat{\omega_0}(x) &= \left(-\frac{d^2}{dx^2} + m^2\right)\widehat{\omega_0^{-1}}(x) \\ &= -\sqrt{\frac{2}{\pi}}m^2 \int_0^\infty \exp(-m|x|\cosh t)\sinh^2 t dt \leq 0.\end{aligned} \tag{3.35}$$

Hence $\widehat{T}(x)$ is non-negative and $T(p)$ is positivity preserving. Therefore, by the definition of f and Lemma 3.9 we have

$$\begin{aligned}f(x) &= T(p)h(x) + 2\sqrt{p^2+m^2}h(x) \\ &= (\widehat{T} * h)(x) + 2\sqrt{p^2+m^2}h(x) \geq 2\sqrt{p^2+m^2}h(x) \geq \frac{2}{1+x^2}.\end{aligned} \tag{3.36}$$

□

We use the extra shorthands $\omega_\pm(p) := \sqrt{(p\pm 1)^2+m^2}$ and $\lambda_0 := \sqrt{1+m^2}$.

Lemma 3.11. *For every $w \in D(p^2)$ and almost every $x \in \mathbb{R}$ we have*

$$|(\omega_+(p) - \lambda_0)w(x)| \leq \omega_0^{-1}|(p^2+2p)w(x)|, \tag{3.37}$$

$$|(\omega_-(p) - \lambda_0)w(x)| \leq \omega_0^{-1}|(p^2-2p)w(x)|. \tag{3.38}$$

Proof. Let $w \in D(p^2)$. Then

$$\begin{aligned} |(\omega_+(p) - \lambda_0)w(x)| &= |(\omega_+(p) + \lambda_0)^{-1}(\omega_+(p)^2 - \lambda_0^2)w(x)| \\ &= |(\omega_+(p) + \lambda_0)^{-1}(p^2 + 2p)w(x)| \\ &= |e^{-ix}(\omega_0 + \lambda_0)^{-1}e^{ix}(p^2 + 2p)w(x)| \\ &\leq (\omega_0 + \lambda_0)^{-1}|(p^2 + 2p)w(x)|, \end{aligned} \tag{3.39}$$

where in the last step we used that $(\omega_0 + \lambda_0)^{-1}$ is positivity preserving. Moreover, since also ω_0^{-1} is positivity preserving, for any non-negative function $s(x)$ we have

$$0 \leq \frac{1}{\omega_0 + \lambda_0} s(x) = \omega_0^{-1}s(x) - \frac{\lambda_0}{\omega_0 + \lambda_0}\omega_0^{-1}s(x) \leq \omega_0^{-1}s(x). \tag{3.40}$$

From (3.39)-(3.40) we obtain (3.37). The estimate (3.38) can be shown similarly. ◻

Proposition 3.12. *If $m \geq 146$, then for $|x| \to \infty$ we have that*

$$\frac{1}{f(x)} \left| \left(\sqrt{(p+1)^2 + m^2} - \sqrt{1 + m^2} \right) f(x) \right| = O(|x|^{-1}).$$

Proof. We apply Lemma 3.11 to $w(x) = f(x) = (\omega_+(p) + \omega_-(p))h(x)$. Thus

$$\begin{aligned} &|(\omega_+(p) - \lambda_0)f(x)| \\ &\quad \leq \omega_0^{-1}|(p^2 + 2p)f(x)| \\ &\quad \leq \omega_0^{-1}|(\omega_+ + \omega_-)(p^2 + 2p)h(x)| \\ &\quad \leq \omega_0^{-1}\Big(|(\omega_+ - \lambda_0)(p^2 + 2p)h(x)| + |(\omega_- - \lambda_0)(p^2 + 2p)h(x)| \\ &\qquad + \lambda_0|(p^2 + 2p)h(x)| \Big) \\ &\quad \leq \omega_0^{-1}\Big(\omega_0^{-1}|(p^2 + 2p)(p^2 + 2p)h(x)| + \omega_0^{-1}|(p^2 - 2p)(p^2 + 2p)h(x)| \\ &\qquad + \lambda_0|(p^2 + 2p)h(x)| \Big) \\ &\quad = \omega_0^{-2}\Big(|(p^2 + 2p)^2 h(x)| + |(p^4 - 4p^2)h(x)| \Big) + \lambda_0\omega_0^{-1}|(p^2 + 2p)h(x)|, \end{aligned} \tag{3.41}$$

using (3.37)-(3.38) in the last inequality. From Lemma 3.1 we know that

$$|h^{(3)}(x)| \leq C \langle x \rangle^{-3} \quad \text{and} \quad |h^{(4)}(x)| \leq C \langle x \rangle^{-3},$$

where $C = \max\{C_1, C_2\}$. Thus

$$\text{r.h.s.}\,(3.41) \leq C'\omega_0^{-2} \langle x \rangle^{-3} + \lambda_0 C'\omega_0^{-1} \langle x \rangle^{-3}, \tag{3.42}$$

with some $C' > 0$. As shown in the proof of Lemma 3.8, there exists a constant $D_1 > 0$ such that $\omega_0^{-1} \langle x \rangle^{-3} < D_1 \langle x \rangle^{-3}$. This implies that there exists $D_2 > 0$ such that

$$\text{r.h.s.}\,(3.41) \leq D_2 \langle x \rangle^{-3}. \tag{3.43}$$

Hence, by Proposition 3.10 we have

$$\frac{|(\omega_+(p) - \lambda_0)f(x)|}{f(x)} \leq \frac{D_2}{2} \langle x \rangle^{-1}$$

i.e., of the order $O(|x|^{-1})$. ◻

3.2 Proof of Corollary 2.3

Let
$$L^2_{\mathrm{r}}(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3) : f(x) = f(|x|),\, x \in \mathbb{R}^3\} \subset L^2(\mathbb{R}^3)$$
be the closed subspace of rotationally invariant square integrable functions on $\mathbb{R}^3$. Consider the unitary transform
$$U : f(|x|) \mapsto \sqrt{4\pi} r f(r), \quad r \in \mathbb{R}^+, \tag{3.44}$$
from $L^2_{\mathrm{r}}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^+)$. Using that $-\Delta$ on $L^2_{\mathrm{r}}(\mathbb{R}^3)$ has the form $D_{\mathrm{r}} := -r^{-2}(d/dr)r^2(d/dr)$, we have
$$U D_{\mathrm{r}} U^* = -\frac{d^2}{dr^2}, \tag{3.45}$$
i.e., the Laplacian in one dimension on $[0, \infty)$, with Dirichlet boundary condition at 0. Note that $(Uv)(r) = u(r)$, $r \in \mathbb{R}^+$. Since $u(r)$ is smooth and odd (Theorem 2.1) in $r \in \mathbb{R}$, we have $u \in D(-d^2/dr^2)$. Moreover, since V is bounded, it follows that $D(H_{\mathrm{r}}) = D((-\Delta + m^2)^{1/2}) \supset D(-\Delta)$. Thus, $v \in D(H_{\mathrm{r}})$ and the equality
$$\begin{aligned}(\sqrt{-\Delta + m^2}\, v)(x) &= (U^* U \sqrt{-\Delta + m^2}\, U^* U v)(x) \\ &= (U^* \sqrt{-d^2/dr^2 + m^2} u)(x)\end{aligned} \tag{3.46}$$
holds. By Theorem 2.1, the right hand side of (3.46) equals
$$U^*(\lambda - V(r))u)(x) = (\lambda - W(x))v(x). \tag{3.47}$$
Hence v satisfies the eigenvalue equation $H_{\mathrm{r}} v = \lambda v$.

3.3 Proof of Theorem 2.6

From the definition of $\tilde{h}$, we can show the following result directly.

Lemma 3.13. *$\tilde{h} \in D(p^3)$ and the estimates*
$$c_1 \langle x \rangle^{-1} \le \tilde{h}(x) \le c_2 \langle x \rangle^{-1}, \qquad x \in \mathbb{R} \tag{3.48}$$
$$|\tilde{h}^{(j)}(x)| \le c_3 \langle x \rangle^{-2}, \qquad j = 1, 2, 3. \tag{3.49}$$
hold with $c_1 = 0.26$, $c_2 = 1.02$, $c_3 = 2.2$.

Lemma 3.14. *If*

(Q.1) there exists a constant $C > 0$ such that $\tilde{f}(x) \ge C \langle x \rangle^{-1}$ for all $x \in \mathbb{R}$,

(Q.2) $(\omega(p+1) - \lambda)\tilde{f}(x) = O(\langle x \rangle^{-2})$,

then Theorem 2.6 follows.

Proof. By Lemma 3.13, $\tilde{f}$ and $(\omega(p+1)-\lambda)\tilde{f}$ are continuous. By a similar argument as in the proof of Lemma 3.3, we have

$$\tilde{V}(x) = -\frac{(\omega(p+1)-\lambda)\tilde{f}(x)}{\tilde{f}(x)} + \frac{2\tilde{h}'(x)}{\sin x}\frac{e^{-ix}}{\tilde{f}(x)}. \tag{3.50}$$

Notice that (2.18) is equivalent to (3.50). By a direct computation we get that $\tilde{h}'(x) = -4\mathrm{sgn}(x)\sin^2 x\,\tilde{h}(x)^2$. Hence we have

$$\tilde{V}(x) = -\frac{(\omega(p+1)-\lambda)\tilde{f}(x)}{\tilde{f}(x)} - 8\tilde{h}(x)^2\frac{e^{-ix}}{\tilde{f}(x)}\sin|x|. \tag{3.51}$$

Assumption (Q.1) implies that the denominator of (3.51) is strictly positive, and hence $\tilde{V}(x)$ is continuous. Moreover, Lemma 3.13 and assumption (Q.2) yield that $\tilde{V} = O(\langle x\rangle^{-1})$. $\square$

To prove (Q.1) in the previous lemma, we need some further preparation.

Lemma 3.15. *If $m > 20$, then for all $x \in \mathbb{R}$ we have*

$$\omega_0(p)^{-1}\tilde{h}(x) \geq \frac{c_1}{10m}\langle x\rangle^{-1}. \tag{3.52}$$

Proof. The proof can be obtained by the argument used in the proof of Lemma 3.6. By Lemma 3.13 we have

$$\begin{aligned}
\omega_0(p)^{-1}\tilde{h}(x) &\geq \frac{1-e^{-1}}{\pi}c_1\langle x\rangle \int_{\mathbb{R}} \frac{e^{-m|y|}}{\sqrt{1+2m|y|}}\left(\inf_{x\in\mathbb{R}}\frac{1+x^2}{1+(x+y)^2}\right)^{1/2} dy \\
&\geq 2\frac{1-e^{-1}}{\pi}c_1\langle x\rangle \int_0^\infty \frac{e^{-s}}{\sqrt{1+2s}}\frac{1}{1+s/20}\frac{ds}{m} \\
&\geq \frac{c_1}{10m}\langle x\rangle^{-1}.
\end{aligned}$$

$\square$

Lemma 3.16. *For all $m > 0$ and $x \in \mathbb{R}$,*

$$\omega_0(p)^{-1}\tilde{h}(x) \leq c_2\left(\frac{2}{m}+\frac{1}{m^2}\right)\langle x\rangle^{-1}.$$

Proof. Similarly as above, by Lemma 3.13 we have

$$\begin{aligned}
\omega_0(p)^{-1}\tilde{h}(x) &\leq \frac{c_2}{\sqrt{2\pi}}\langle x\rangle^{-1}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{m|y|}}\sqrt{2y^2+2}dy = \frac{2c_2}{\sqrt{\pi}}\langle x\rangle^{-1}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}\sqrt{s^2/m^2+1}\frac{ds}{m} \\
&\leq \frac{2c_2}{\sqrt{\pi}m}\langle x\rangle^{-1}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}(s/m+1)ds = c_2\left(\frac{2}{m}+\frac{1}{m^2}\right)\langle x\rangle^{-1}.
\end{aligned}$$

$\square$

Lemma 3.17. *For $m > 0$, the estimate*

$$|\omega_0(p)^{-1}p^2\tilde{h}(x)| \le \frac{c_3\sqrt{2}}{m}\left(2+\frac{3}{4m^2}\right)\langle x\rangle^{-2}$$

holds.

Proof. As in the proof of Lemma 3.8, we have

$$\begin{aligned}|\omega_0(p)^{-1}p^2\tilde{h}(x)| &\le c_3\omega_0(p)^{-1}\langle x\rangle^{-2}\\ &\le c_3\frac{1}{\sqrt{2\pi}}\langle x\rangle^{-2}\int_{\mathbb{R}}\frac{e^{-m|y|}}{\sqrt{m|y|}}\left(\sup_{x\in\mathbb{R}}\frac{1+x^2}{1+(x+y)^2}\right)\\ &= \frac{c_3\sqrt{2}}{m}\left(2+\frac{3}{4m^2}\right)\langle x\rangle^{-2}.\end{aligned}$$

□

Lemma 3.18. *If $m > 34$, then*

$$\omega_0(p)\tilde{h}(x) \ge \langle x\rangle^{-1}. \tag{3.53}$$

Proof. By Lemmas 3.15 and 3.17 we have

$$\begin{aligned}\omega_0\tilde{h}(x) &= m^2\omega_0(p)^{-1}\tilde{h}(x) + \omega_0(p)^{-1}p^2\tilde{h}(x)\\ &\ge m^2\omega_0(p)^{-1}\tilde{h}(x) - \omega_0(p)^{-1}|p^2\tilde{h}(x)|\\ &\ge \left(\frac{1}{10}mc_1 - \frac{\sqrt{2}}{m}c_3\left(2+\frac{3}{4m^2}\right)\right)\langle x\rangle^{-1}.\end{aligned}$$

Since $m > 34$, we have that $\frac{1}{10}mc_1 - \frac{\sqrt{2}}{m}c_3\left(2+\frac{3}{4m^2}\right) > 1$, which completes the proof. □

Hence follows (Q.1) in Lemma (3.14).

Proposition 3.19. *If $m > 34$, for all $x \in \mathbb{R}$, we have*

$$\tilde{f}(x) \ge 2\langle x\rangle^{-1}.$$

Proof. Similar to the proof of Proposition 3.10. □

In order to prove (Q.2) we will use a limiting argument. Let $j \in C_0^\infty(\mathbb{R})$ be a function such that $j(x) \ge 0$ and $\int_{\mathbb{R}} j(x)dx = 1$, and set $j_n(x) = nj(nx)$. Write

$$\begin{aligned}\tilde{h}_n(x) &:= (j_n * \tilde{h})(x),\\ \tilde{f}_n(x) &:= (\omega_+(p)+\omega_-(p))\tilde{h}_n(x)\end{aligned}$$

Since $\tilde{h}$ and $\tilde{f}$ are continuous and decaying, the sequences $\tilde{h}_n(x)$ and $\tilde{f}_n(x)$ converge uniformly to $\tilde{h}(x)$ and $\tilde{f}$, respectively, as $n \to \infty$. For the same reason, moreover we have

$$\lim_{n\to\infty} D\tilde{f}_n(x) = D\tilde{f}(x),$$

for $D = \omega_0, \omega_+(p), \omega_-(p)$. Now we can show (Q.2).

Proposition 3.20. *For $m > 0$,*

$$(\omega(p+1)-\lambda)\tilde{f}(x) = O(\langle x\rangle^{-2}), \qquad x \in \mathbb{R} \tag{3.54}$$

holds.

Proof. By the above observations on convergence, we have

$$\left|(\omega_+(p)-\lambda_0)\tilde{f}(x)\right| = \lim_{n\to\infty}\left|(\omega_+(p)-\lambda_0)\tilde{f}_n(x)\right|,$$

Note that $\tilde{f}_n \in D(p^n)$ for every $n \in \mathbb{N}$. By using Lemma 3.11, we have

$$\begin{aligned}\left|(\omega_+(p)-\lambda_0)\tilde{f}_n(x)\right| &\le \omega_0^{-1}|(p^2+2p)\tilde{f}_n(x)| \\ &= \omega_0^{-1}|(\omega_+ + \omega_-)(p^2+2p)\tilde{h}_n(x)| \\ &\le \omega_0^{-2}(|(p^2+2p)^2\tilde{h}_n(x)| + |(p^2-2p)(p^2+2p)\tilde{h}_n(x)|) \\ &\le \omega_0^{-2}(2|p^4\tilde{h}_n(x)| + 4|p^3\tilde{h}_n(x)| + 8|p^2\tilde{h}_n(x)|).\end{aligned}$$

From the fact that $\tilde{h} \in D(p^3)$, we have that $|p^3\tilde{h}_n|$ converges to $|p^3\tilde{h}|$ in L^2 norm. Hence $\omega_0^{-2}|p^3\tilde{h}_n|$ goes to $\omega_0^{-2}|p^3\tilde{h}|$ in L^2 sense. By taking a subsequence n_j,

$$\omega_0^{-2}|p^3\tilde{h}_{n_j}|(x) \to \omega_0^{-2}|p^3\tilde{h}|(x) \quad \text{for a.e. } x \in \mathbb{R},$$

as $j \to \infty$. Similarly, $\omega_0^{-2}|p^2\tilde{h}_{n_j}|(x)$ goes to $\omega_0^{-2}|p^2\tilde{h}|(x)$ for a.e. $x \in \mathbb{R}$. Next we consider the term

$$p^4\tilde{h}_n = (j_n * \tilde{h}^{(3)})'.$$

In the remainder of the proof we denote $g(|x|)$ by $\widetilde{g}(x)$. Writing

$$\tilde{h}^{(3)} = -\widetilde{g}^{(3)}\tilde{h}^2 + J_1,$$

we obtain $J_1 = -4\widetilde{g}''\tilde{h}^2\tilde{h}'' - 2\widetilde{g}'((\tilde{h}')^2 + \tilde{h}\tilde{h}'') \in D(p)$. Thus we have

$$\omega_0^{-2}|j_{n_j} * J_1'|(x) \to \omega_0^{-2}|J_1'|(x), \qquad \text{a.e. } x \in \mathbb{R}$$

as $j \to \infty$. Hence

$$\left|(\omega_+(p)-\lambda_0)\tilde{f}(x)\right| \le 4\omega_0^{-2}|p^3\tilde{h}(x)| + 8\omega_0^{-2}|p^2\tilde{h}(x)| + 2\omega_0^{-2}|J_1'(x)| \tag{3.55}$$
$$+ 2\limsup_{j\to\infty}\omega_0^{-2}|(j_{n_j} * (\widetilde{g}^{(3)}\tilde{h}^2))'|(x), \tag{3.56}$$

for a.e. x. By differentiation in distributional sense, we get $\widetilde{g}^{(3)}(x) = 8\cos(2x)\mathrm{sgn}(x)$. Write $J_2 := 8\cos(2x)\tilde{h}^2$. Then we have

$$(j_n * (\widetilde{g}^{(3)}\tilde{h}^2))' = (j_n * \mathrm{sgn}(x)J_2)' = 2J_2(0)j_n(x) + j_n * (\mathrm{sgn}(x)J_2').$$

Again, by noting that $J_2' \in L^2(\mathbb{R})$, we get that $|j_{n_j} * (\mathrm{sgn}(x)J_2')(x)| \to |J_2'(x)|$, for a.e. x. Thus

$$(3.56) \le 4|J_2(0)| \limsup_{j\to\infty} \omega_0^{-2} j_{n_j}(x) + 2\omega_0^{-2}|J_2'(x)|.$$

Since $\omega_0^{-2} = (p^2+m^2)^{-1}$, it follows that

$$\lim_{n\to\infty} \omega_0^{-2} j_n(x) = \frac{e^{-m|x|}}{2m}.$$

Thus

$$\text{l.h.s. } (3.55) \le \omega_0^{-2} J_3(x) + \frac{e^{-m|x|}}{2m},$$

where

$$J_3 := 4|p^3\tilde{h}(x)| + 8|p^2\tilde{h}(x)| + 2|J_1'(x)| + 2|J_2'(x)|.$$

By the definition of J_1, J_2 and $\tilde{h}$, we have

$$J_3(x) \le \frac{C}{1+x^2} \tag{3.57}$$

for some $C > 0$. Hence, by the same argument as in the proof of Lemma 3.16, we obtain that

$$\omega_0^{-2} J_3(x) = O(\langle x\rangle^{-2}).$$

Clearly, $e^{-m|x|}/2m = O(\langle x\rangle^{-2})$, and therefore we conclude that $|(\omega_+ - \lambda)\tilde{f}(x)| = O(\langle x\rangle^{-2})$. $\square$

3.4 Proof of Theorem 2.4

The limit (2.13) is elementary. We show that $f_c(x)$ converges to $h(x)$ in the uniform norm of $C^2(\mathbb{R})$. Lemma 3.2 implies that $|k|^n\widehat{h}(k) \in L^2(\mathbb{R})$ for all $n = 0, 1, \ldots$ By this fact and the Hölder inequality we have

$$\begin{aligned}\int_{\mathbb{R}} |k|^n|\widehat{h}(k)|dk &= \int_{-1}^{1} |k|^n|\widehat{h}(k)|dk + \int_{|k|\ge 1} |k|^{-1}|k|^{n+1}|\widehat{h}(k)|dk \\ &\le \Big(\int_{-1}^{1} |k|^{2n}dk\Big)^{1/2} \|\widehat{h}\|_{L^2} + \Big(\int_{|k|>1} |k|^{-2}dk\Big)^{1/2} \|p^{n+1}\widehat{h}\|_{L^2} < \infty.\end{aligned}$$

Hence $k^n\widehat{h} \in L^1(\mathbb{R})$, for all $n \ge 0$. By using Fourier transforms, we have

$$\sup_{x\in\mathbb{R}} |f_c(x) - h(x)| \le \|\widehat{f_c} - \widehat{h}\|_{L^1}, \tag{3.58}$$

and by the definition of f_c we obtain the bound

$$\begin{aligned}|\widehat{f_c}(k) - \widehat{h}(k)| &= \frac{1}{2m}\left|\left(\sqrt{\frac{1}{c^2}(k+1)^2 + m^2} + \sqrt{\frac{1}{c^2}(k-1)^2 + m^2} - 2m\right)\widehat{h}(k)\right| \\ &\le \frac{1}{2m}\frac{2k^2+2}{mc^2}|\widehat{h}(k)|. \end{aligned} \tag{3.59}$$

This estimate and $(k^2+1)\widehat{h} \in L^1$ imply that the right hand side of (3.58) goes to zero as $c \to \infty$. Thus

$$\sup_{x\in\mathbb{R}} |u_c(x) - h(x)\sin x| \le \|\widehat{f}_c - \widehat{h}\|_{L^1} \to 0 \quad \text{as } c \to \infty. \tag{3.60}$$

Similarly, we can show that f_c' and f_c'' are uniformly convergent to h' and h'', respectively. Hence $u_c(x)$ converges to $h(x)\sin x$ in the uniform norm of $C^2(\mathbb{R})$.

Next we show that

$$\left(\sqrt{-c^2\tfrac{d^2}{dx^2} + m^2c^4} - mc^2\right) u_c(x) \to -\frac{1}{2m}\frac{d^2}{dx^2}u_\infty(x), \tag{3.61}$$

uniformly as $c \to \infty$. Using Fourier transforms, we get

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|\left(\sqrt{-c^2\tfrac{d^2}{dx^2} + m^2c^4} - mc^2\right) u_c(x) + \frac{1}{2m}\frac{d^2}{dx^2}u_\infty(x)\right| \\
&\quad\le \left\|(\sqrt{c^2k^2+m^2c^4} - mc^2)\widehat{u}_c - \frac{k^2}{2m}\widehat{u}_\infty\right\|_{L^1} \\
&\quad\le \left\|\left(\sqrt{c^2k^2+m^2c^4} - mc^2 - \frac{k^2}{2m}\right)\widehat{u}_c\right\|_{L^1} + \left\|\frac{k^2}{2m}(\widehat{u}_c - \widehat{u}_\infty)\right\|_{L^1}.
\end{aligned} \tag{3.62}$$

The estimate

$$\left|\sqrt{c^2k^2+m^2c^4} - mc^2 - \frac{k^2}{2m}\right| = \left| - \frac{k^4}{2mc^2}\frac{1}{(m+\sqrt{c^{-2}k^2+m^2})^2}\right| \le \frac{k^4}{4m^2c^2}$$

gives

$$(3.62) \le \frac{1}{4m^2c^2}\|k^4\widehat{u}_c\|_{L^1} + \frac{1}{2m}\|k^2(\widehat{u}_c - \widehat{u}_\infty)\|_{L^1}.$$

Note that $\widehat{u}_c(k) = (\widehat{f}_c(k+1) - \widehat{f}_c(k-1))/2i$ holds identically. Thus

$$k^4|\widehat{u}_c(k)| \le k^4(|\widehat{f}_c(k+1)| + |\widehat{f}_c(k-1)|).$$

From (3.59) we have $|\widehat{f}_c(k)| \le (1 + (k^2+1)/m^2c^2)|\widehat{h}(k)|$. This and $k^6\widehat{h} \in L^1(\mathbb{R})$ imply that

$$\lim_{c\to\infty}\frac{1}{c^2}\|k^4\widehat{u}_c\|_{L^1} = 0.$$

Similarly, we can show that $\|k^2(\widehat{u}_c - \widehat{u}_\infty)\|_{L^1}$ goes to zero as $c \to \infty$. Hence (3.61) holds.

By Proposition 3.10, $f_c(x)$ is strictly positive and thus $u_c(x)$ has zeroes only at $\pi\mathbb{N}$. Therefore by (3.60)-(3.61) we conclude

$$\lim_{c\to\infty} V_c(x) = \frac{1}{2m} - \frac{1}{2m}\frac{-\frac{d^2}{dx^2}u_\infty(x)}{u_\infty(x)}, \tag{3.63}$$

for all $x \in \mathbb{R}\setminus\pi\mathbb{N}$. The proof of Corollary 2.5 can be done similarly.

3.5 Proof of Theorems 2.8 and 2.10

Proof of Theorem 2.8. It is known, see [7, p11, (7)], that

$$\widehat{u}_\nu(k) = \frac{2^{1-\nu}}{\Gamma(\nu)}|k|^{\nu-\frac{1}{2}}K_{\nu-\frac{1}{2}}(|k|). \tag{3.64}$$

Using the integral formula [7, p61, (9)], we have

$$\sqrt{-d^2/dx^2}\, u_\nu(x) = (\widehat{|k|\hat{u}_\nu})(x) = \frac{2\Gamma(\frac{1}{2}+\nu)}{\sqrt{\pi}\Gamma(\nu)}\, {}_2F_1\left(1, \tfrac{1}{2}+\nu; \tfrac{1}{2}; -x^2\right). \tag{3.65}$$

Thus (2.21) follows. Next we show that $V_\nu(x) = O(|x|^{-1})$ whenever $\nu < 1/2$. In this case, by Pfaff transformation [1, Th. 2.2.5] it follows that

$$ {}_2F_1\left(1, \tfrac{1}{2}+\nu; \tfrac{1}{2}; -x^2\right) = (1+x^2)^{-\nu-\frac{1}{2}}{}_2F_1(\tfrac{1}{2}+\nu, -\tfrac{1}{2}; \tfrac{1}{2}; \tfrac{x^2}{1+x^2}).$$

From the definition of V_ν we have

$$\sqrt{-d^2/dx^2}\, u_\nu(x) = -V_\nu(x)u_\nu(x).$$

With a constant $C > 0$ we obtain

$$\lim_{|x|\to\infty} |x|V_\nu(x) = C\lim_{z\uparrow 1} {}_2F_1(\tfrac{1}{2}+\nu, -\tfrac{1}{2}; \tfrac{1}{2}; z).$$

Using Gauss's formula [1, Th. 2.2.2], the limit at the right hand side can be computed to be

$$ {}_2F_1(\tfrac{1}{2}+\nu, -\tfrac{1}{2}; \tfrac{1}{2}; 1) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)}{\Gamma(-\nu)}, \tag{3.66}$$

where we used that $0 < \nu < 1/2$. Hence $V(x) = O(1/|x|)$. Similarly, if $1/2 < \nu < 1$, by Pfaff transformation we have

$$ {}_2F_1\left(1, \tfrac{1}{2}+\nu; \tfrac{1}{2}; -x^2\right) = (1+x^2)^{-1}{}_2F_1(1, -\nu; \tfrac{1}{2}; \tfrac{x^2}{1+x^2}).$$

Hence,

$$V_\nu(x) = -\frac{2\Gamma(\frac{1}{2}+\nu)}{\sqrt{\pi}\Gamma(\nu)}(1+x^2)^{-1+\nu}{}_2F_1(1, -\nu; \tfrac{1}{2}; \tfrac{x^2}{1+x^2}),$$

which is of order $O(1/|x|^{2-2\nu})$, and (3) follows.

For $\nu = 1/2$ we have $\widehat{u}_{1/2}(k) = (2/\pi)^{1/2}K_0(|k|)$. Hence,

$$\sqrt{-d^2/dx^2}\, u_{1/2}(x) = (|k|\widehat{\widehat{u}_{1/2}(k)})(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^\infty K_0(k)\sin kx dk. \tag{3.67}$$

This integral can be computed explicitly [7, p93, (51)], and we obtain

$$(3.67) = \frac{2}{\pi}\left(\frac{1}{x^2+1} - \frac{|x|\sinh^{-1}|x|}{(x^2+1)^{3/2}}\right). \tag{3.68}$$

It is straightforward to show that (3.67) is of order $O(\log|x|/x)$. ∎

Proof of Theorem 2.10. For $\nu = 1$ we derive the equation

$$\sqrt{-d^2/dx^2}\, v_1 = \frac{2}{(1+x^2)^2} = \widetilde{V}_1(x) v_1. \tag{3.69}$$

Thus the eigenvalue equation and part line 2 in (2.22) hold. Note that $v_\nu = (2-2\nu)^{-1}(d/dx)u_\nu$, whenever $\nu \neq 1$. By (3.65) we have

$$\begin{aligned}\sqrt{-d^2/dx^2}\, v_\nu &= \frac{\Gamma(\frac{1}{2}+\nu)}{(1-\nu)\sqrt{\pi}\Gamma(\nu)} \frac{d}{dx}\, {}_2F_1\left(1, \tfrac{1}{2}+\nu; \tfrac{1}{2}; -x^2\right) \\ &= \frac{2(1-2\nu)}{(1-\nu)\sqrt{\pi}} \frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu-1)} x\, {}_2F_1\left(2, \tfrac{1}{2}+\nu; \tfrac{3}{2}; -x^2\right) \\ &= \widetilde{V}_\nu(x) v_\nu. \end{aligned} \tag{3.70}$$

Thus the eigenvalue equation follows. Next we show lines 1, 3 and 4 in (2.22). By Pfaff transformation we obtain

$$\begin{aligned}{}_2F_1\left(2, \tfrac{1}{2}+\nu; \tfrac{3}{2}; -x^2\right) &= (1+x^2)^{-2}\, {}_2F_1\left(2, 1-\nu; \tfrac{3}{2}; \tfrac{x^2}{1+x^2}\right) \\ &= (1+x^2)^{-(\frac{1}{2}+\nu)}\, {}_2F_1\left(\tfrac{1}{2}+\nu, -\tfrac{1}{2}; \tfrac{3}{2}; \tfrac{x^2}{1+x^2}\right).\end{aligned}$$

By another use of the Gauss formula we see that the limits $\lim_{z\uparrow 1} {}_2F_1(2, 1-\nu; 3/2; z)$ and $\lim_{z\uparrow 1} {}_2F_1(1/2+\nu, -1/2; 3/2; z)$ are finite whenever $\nu > 3/2$ and $\nu < 3/2$, respectively. Hence the expressions in lines 1 and 4 hold. Consider now the case $\nu = 3/2$. Making use of (3.67), we have

$$\sqrt{-d^2/dx^2}\, u_{3/2}(x) = \frac{2}{\pi}\left(\frac{1}{x^2+1} - \frac{x \sinh^{-1} x}{(x^2+1)^{3/2}}\right),$$

and thus

$$\sqrt{-d^2/dx^2}\, v_{3/2} = -\frac{2}{\pi}\frac{d}{dx}\left(\frac{1}{x^2+1} - \frac{x \sinh^{-1} x}{(x^2+1)^{3/2}}\right).$$

Combining this and (3.70), we obtain

$$\begin{aligned}\widetilde{V}_{3/2}(x) &= \frac{1}{v_{3/2}(x)}\sqrt{-d^2/dx^2}\, v_{3/2}(x) \\ &= -\frac{2}{\pi}\frac{(1+x^2)^{3/2}}{x}\frac{d}{dx}\left(\frac{1}{x^2+1} - \frac{x \sinh^{-1} x}{(x^2+1)^{3/2}}\right).\end{aligned} \tag{3.71}$$

It is then direct to show that (3.71) is of order $O(\log|x|/x)$ as $|x| \to \infty$. □

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References

[1] G.E. Andrews, R. Askey, R. Roy: *Special Functions*, Cambridge University Press, 1999

[2] F. Capasso et al: Observation of an electronic bound state above a potential well, *Nature* **358**, 565-567, 1992

[3] R. Carmona, W.C. Masters, B. Simon: Relativistic Schrödinger operators: asymptotic behaviour of the eigenfunctions, *J. Funct. Anal.* **91**, 117-142, 1990

[4] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon: *Schrödinger Operators. With Applications to Quantum Mechanics and Global Geometry*, Springer, 1987

[5] B. Dyda, A. Kuznetsov, M. Kwaśnicki: Fractional Laplace operator and Meijer G-function, arXiv:1509.08529, 2015

[6] M.S.P. Eastham, H. Kalf: *Schrödinger-Type Operators with Continuous Spectra*, Pitman, 1982

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi: *Tables of Integral Transforms*, vol. I, McGraw-Hill, 1954

[8] I.W. Herbst: Spectral theory of the operator $(p^2+m^2)^{1/2}-Ze^2/r$, *Commun. Math. Phys.* **53**, 285-294, 1977

[9] F. Hiroshima, T. Ichinose, J. Lőrinczi: Probabilistic representation and fall-off of bound states of relativistic Schrödinger operators with spin 1/2, *Publ. RIMS* **49**, 189-214, 2013

[10] F. Hiroshima, T. Ichinose, J. Lőrinczi: Kato's inequality for magnetic relativistic Schrödinger operators, preprint, arXiv:1604.03933, 2015 (submitted for publication)

[11] K. Kaleta, J. Lőrinczi: Pointwise estimates of the eigenfunctions and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes, *Ann. Probab.* **43**, 1350-1398, 2015

[12] K. Kaleta, J. Lőrinczi: Fall-off of eigenfunctions of non-local Schrödinger operators with decaying potentials, arXiv:1503.03508, 2015 (submitted for publication)

[13] J. Lőrinczi, I. Sasaki: Non-existence of embedded eigenvalues for a class of non-local Schrödinger operators, preprint, 2016

[14] T. Kato: Growth properties of solutions of the reduced wave equation with a variable coefficient, *Commun. Pure Appl. Math.* **12**, 403-425, 1959

[15] H. E. Moses and S. E. Tuan: Potentials with zero scattering phase, *Nuovo Cimento* **13**, 197-206, 1959

[16] S.N. Naboko: Dense point spectra of Schrödinger and Dirac operators, *Teoret. Mat. Fiz.* **68**, 18-28, 1986

[17] J. von Neumann, E. Wigner: Über merkwürdige diskrete Eigenwerte, *Z. Physik* **30**, 465-467, 1929

[18] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, vol. 3: Scattering Theory*, Academic Press, 1979

[19] S. Richard, J. Uchiyama, T. Umeda: Schrödinger operators with n positive eigenvalues: an explicit construction involving complex-valued potentials, *Proc. Japan Acad. Ser. A Math. Sci.* **92**, 7-12, 2016

[20] S. Richard, T. Umeda: Low energy spectral and scattering theory for relativistic Schrödinger operators, to appear in *Hokkaido Math. J.*, available at arXiv:1208.2006

[21] B. Simon: Some Schrödinger operators with dense point spectrum, *Proc. AMS* **125**, 203-208, 1997

[22] G.N. Watson: *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1922

[23] R.A. Weder: Spectral properties of one-body relativistic spin-zero Hamiltonians, *Ann. IHP* **20**, 211-220, 1974