A RIGOROUS GEOMETRIC DERIVATION OF THE CHIRAL ANOMALY IN CURVED BACKGROUNDS

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Abstract. We discuss the chiral anomaly for a Weyl field in a curved background and show that a novel index theorem for the Lorentzian Dirac operator can be applied to describe the gravitational chiral anomaly. A formula for the total charge generated by the gravitational and gauge field background is derived in a mathematically rigorous manner. It contains a term identical to the integrand in the Atiyah-Singer index theorem and another term involving the η-invariant of the Cauchy hypersurfaces.

1. Introduction

Anomalies in quantum field theory appear as a violation of conservation laws of currents, in other words currents that are classically preserved but whose quantum counterparts are not. These anomalies are of direct physical significance. A prominent example is the so-called chiral anomaly (also ABJ-anomaly or axial anomaly). It explains the rate of decay of the neutral pion into two photons ([1, 8]), $\pi_0 \to \gamma\gamma$. We would like to refer to the monograph [10] for further details and applications in quantum field theory. Apart from high energy physics it has also been proposed that this anomaly can be observed more directly in crystals (see [26], and also [31] for recent experimental evidence).

Whereas anomalies were first discovered in perturbative computations in quantum field theory their appearance is related to the Atiyah-Singer index theorem. Indeed the perturbative computation yields a term that looks precisely like the local Chern character form of the connection induced by an external electromagnetic field, and perturbative computations on curved space-times yield a term that is identical to the Atiyah-Singer integrand. A formal manipulation using the Euclidean formulation of quantum field theory and path integrals, the so-called Fujakawa method ([19]), provides a direct explanation of that. In curved backgrounds this method has also been applied ([20], see also [2]) but should be seen only as a formal manipulation as the process of Wick rotation, passing to Riemannian signature, can not be made sense of.

Another, more mathematically rigorous, way to understand anomalies is via a careful analysis of the second quantization procedure in external fields ([11]). This was done by Klaus and Scharf for fermions in an external field ([22, 23]). The one-particle time-evolution in an external field can mix the negative and positive energy solutions of the Dirac equation. This mixing can occur in such a way that the vacuum is mapped to a charged sector under the second quantized time-evolution ([22, 23, 25, 13]). The charge generated by this process relates to the Fredholm index of a certain operator constructed out of the positive energy projection

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and the time-evolution operator. This is the point of view we will take in this paper. We analyze the right-handed part of the second quantized massless Dirac field (the Weyl field) coupled to an external gravitational field and gauge field. For simplicity we will assume that the space-time is spatially compact. Our analysis shows that the total charge generated by the interaction with the external field can be expressed as the index of the Lorentzian Dirac operator which in turn is given as a sum of the Atiyah-Singer integrand and a correction term (similar to the one in the Atiyah-Patodi-Singer theorem for Riemannian manifolds with boundary) involving the η-invariant. This provides a direct and mathematically rigorous explanation of the terms appearing in the computations of the gravitational and mixed chiral anomalies.

We would like to remark that there is also another point of view on anomalies, where a Hadamard regularization procedure is applied to the singular currents. This procedure results in currents that are not preserved and whose divergence can in many cases shown to be the local Atiyah-Singer integrand. This is for example the point of view taken in [32] for the mixed and pure gravitational anomalies, and in [15] for the trace anomaly that also can be described by an index theorem ([14]). Whereas this divergence does capture aspects of renormalization in curved space-times the actual generation of charge in external fields is described by our formula, which differs then from the one above by the η-correction term.

We start by explaining in detail how the Weyl field is quantized in a globally hyperbolic space-time with spin structure. As is usual in quantum field theory in curved space-times we split this procedure into two parts (see for example [30] for an introduction). The field algebra can be constructed in a functorial (covariant) manner from the field equation. This was first done by Dimock ([17]) for the Dirac field.

In the second step we look at physical representations of the field algebra induced by states. The Hadamard condition singles out a class of states whose behaviour resembles that of finite energy states in Minkowski space-time. Following the seminal work of Duistermaat and Hörmander ([18]), Radzikowski ([28]) was able to show that the Hadamard condition is a microlocal one. Originally these theorems were established for the Klein-Gordon field, but they also hold for more general wave equations and higher spin fields such as the Dirac field ([29],[24]).

The use of microlocal analysis turned out to be very fruitful and lead for example to the possibility of the perturbative construction of interacting quantum field theories in curved space-times (see for example [12]). In our context the important fact is that Hadamard states differ only by smooth integral kernels. Quantities that are a priory singular and need regularization, such as the expectation value of the energy momentum tensor or the electric current, make sense as relative quantities between two Hadamard states. It is therefore possible to define the relative energy-momentum tensor and the relative current between two Hadamard states.

A particular example of a Hadamard state can be defined if the space-time is ultrastatic in a neighborhood of a smooth spacelike Cauchy hypersurface. Near this Cauchy hypersurface a one parameter group of time translations can be defined and it makes sense to define the vacuum state with respect to this time evolution by employing the usual frequency splitting procedure. We will think of this state as the vacuum state seen by an observer moving along the distinguished time-like Killing vector field near the Cauchy hypersurface. If there are two such Cauchy hypersurfaces we can compare the two vacua. Due to the interaction with the gravitational background and the external gauge field between the two Cauchy hypersurfaces it can happen that the difference of the expectation values of the electric current between the two states is non-zero. The difference of the expectation values of the total charge operator is given essentially by the index of the Lorentzian Dirac operator with Atiyah-Patodi-Singer boundary conditions. We then apply the index theorem proved by the authors in [7] to obtain an explicit formula for the charge generated by the background fields.

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2. The setup

We start by describing the classical setup which gives rise to the one-particle evolution. We will consider massless fermions with internal symmetries in an external field on a curved space-time which satisfy a Dirac equation. Let (X, g) be an even-dimensional Lorentzian manifold where g has signature $(+, -, \dots, -)$. We assume that (X, g) is globally hyperbolic meaning that there exist Cauchy hypersurfaces so that there will be a well-posed initial value problem for the Dirac equation. Then (X, g) is time-orientable and we fix one time-orientation. Moreover, we assume that (X, g) is spatially compact, i.e. that the Cauchy hypersurfaces are compact. We assume a spin structure on X is given so that the complex spinor bundle $SX \to X$ is defined.

Finally, to model the internal symmetries a Hermitian vector bundle $E \to X$ is fixed and, to model an external gauge field, a compactible connection ∇^E on E is given. Compatibility means that the Leibniz rule holds for ∇^E and the scalar product on E. For example, E could be a Hermitian line bundle; then ∇^E is a U(1)-connection and can describe the electromagnetic potential.

2.1. The Dirac operator.

Spinors are sections of the bundle $SX \otimes E$. Let ∇ be the connection on $SX \otimes E$ induced by the Levi-Civita connection ∇^S on SX and the connection ∇^E on E, i.e. $\nabla_X(\sigma \otimes e) = (\nabla^S_X \sigma) \otimes e + \sigma \otimes \nabla^E_X e$. The Dirac operator acts on spinors and is locally given by

$$i\nabla\!\!\!\!/ = i g^{\alpha\beta} \gamma_\alpha \nabla_\beta$$

where we used Einstein's summation convention and the coefficient matrices satisfy $\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}$. Here and henceforth $\{\cdot, \cdot\}$ denotes the anticommutator.

Since the dimension of X is even the spinor bundle splits into the subbundles of left-handed and right-handed spinors, $SX = S_L X \oplus S_R X$. The Dirac operator interchanges chirality, i.e. with respect to the splitting $SX \otimes E = S_L X \otimes E \oplus S_R X \otimes E$ we have

$$\nabla\!\!\!\!/ = \begin{pmatrix} 0 & \nabla\!\!\!\!/_R \\ \nabla\!\!\!\!/_L & 0 \end{pmatrix}.$$

Note that there is no mass term on the diagonal.

The bundle SX comes equipped with a natural nondegenerate but indefinite inner product $\langle \cdot, \cdot \rangle$. We use the convention that $\langle \cdot, \cdot \rangle$ is antilinear in the first argument and linear in the second. The subbundles $S_{L/R}X$ are isotropic with respect to this inner product. The same is of course true for the induced inner products on $SX \otimes E$ and $S_{L/R}X \otimes E$. Recall that the inner product on E is positive definite.

The Dirac operator is formally selfadjoint with respect to $\langle\cdot,\cdot\rangle$, i.e.

$$\int_X \langle i\not{\nabla} u, v\rangle \,\mathrm{dV} = \int_X \langle u, i\not{\nabla} v\rangle \,\mathrm{dV} \tag{1}$$

where $u, v \in C_0^\infty(X; SX \otimes E)$ and dV is the volume element on X induced by the Lorentzian metric.

The formally dual operator of $\not{\nabla}_R : C^\infty(X; S_RX \otimes E) \to C^\infty(X; S_LX \otimes E)$ is the operator $\not{\nabla}_R^* : C^\infty(X; S_L^*X \otimes E^*) \to C^\infty(X; S_R^*X \otimes E^*)$ on the dual bundle characterized by

$$\int_X (\not{\nabla}_R^* v)(u) \mathrm{dV} = \int_X v(\not{\nabla}_R u) \mathrm{dV} \tag{2}$$

where $u \in C_0^\infty(X; S_RX \otimes E)$ and $v \in C_0^\infty(X; S_L^*X \otimes E^*)$.

Example 2.1. If E is a Hermitian line bundle, then the curvature of the dual connection on E^* is the negative of that of E. Up to a factor i, the curvature is a real 2-form which describes the electro-magnetic field. Locally, if we write $\nabla_\alpha^E = \partial_\alpha + iA_\alpha$ then $\nabla_\alpha^{E^*} = \partial_\alpha - iA_\alpha$. For the Dirac operators this means $i\not{\nabla} = i\gamma^\alpha(\partial_\alpha + iA_\alpha) = i\gamma^\alpha\partial_\alpha - A_\alpha\gamma^\alpha$ while $i\not{\nabla}^* = i\gamma^\alpha\partial_\alpha + A_\alpha\gamma^\alpha$. Thus $i\not{\nabla}^*$ should be thought of as the operator with charge opposite to that of $i\not{\nabla}$.

2.2. **Dirac conjugation.** For $u \in S_RX_x \otimes E_x$ define $\overline{u} \in S_L^*X_x \otimes E_x^*$ by $\overline{u}(v) = \langle u, v\rangle$. The map $u \mapsto \overline{u}$, $S_RX \otimes E \to S_L^*X \otimes E^*$ is antilinear. Similarly we get an antilinear map $\bar{\cdot} : S_LX \otimes E \to S_R^*X \otimes E^*$ and the inverse maps $S_{R/L}^*X \otimes E^* \to S_{L/R}X \otimes E$ are also denoted by $\bar{\cdot}$. Equations (1) and (2) imply

$$i\not{\nabla}^*\overline{u} = \overline{(i\not{\nabla})u}. \tag{3}$$

Dirac conjugation is defined as the map

$$\Gamma : S_L^*X \otimes E^* \oplus S_RX \otimes E \to S_L^*X \otimes E^* \oplus S_RX \otimes E, \quad u \oplus v \mapsto \overline{v} \oplus \overline{u}. \tag{4}$$

Clearly Γ is antilinear and an involution, $\Gamma^2 = \mathbb{1}$.

2.3. **The Cauchy problem.** Denote the space of smooth solutions of $\not{\nabla}_R u = 0$ by $\mathrm{Sol}(\not{\nabla}_R) := \{u \in C^\infty(X; S_RX \otimes E) \mid \not{\nabla}_R u = 0\}$ and similarly for $\not{\nabla}_L$, $\not{\nabla}_R^*$, and $\not{\nabla}_L^*$. The Cauchy problem for the Dirac equation on globally hyperbolic manifolds is well posed (see e.g. [17, Thm. 2.3]). This means that for any smooth spacelike Cauchy hypersurface $\Sigma \subset X$ the restriction map

$$\rho_\Sigma : \mathrm{Sol}(\not{\nabla}_R) \to C^\infty(\Sigma; S_RX \otimes E), \quad u \mapsto u|_\Sigma,$$

is an isomorphism of topological vector spaces. If Σ and Σ' are two smooth spacelike Cauchy hypersurfaces of X, then we put

$$U_{\Sigma',\Sigma} := \rho_{\Sigma'} \circ (\rho_\Sigma)^{-1} : C^\infty(\Sigma; S_RX \otimes E) \to C^\infty(\Sigma'; S_RX \otimes E).$$

This evolution map $U_{\Sigma',\Sigma}$ extends to a unitary isomorphism

$$U_{\Sigma',\Sigma} : L^2(\Sigma; S_RX \otimes E) \to L^2(\Sigma'; S_RX \otimes E)\,. \tag{5}$$

Hence there is a unique Hilbert space completion $\mathbf{Sol}(\not{\nabla}_R)$ of $\mathrm{Sol}(\not{\nabla}_R)$ such that for each smooth spacelike Cauchy hypersurface $\Sigma \subset X$ the restriction map extends to a Hilbert space isometry

$$\rho_\Sigma : \mathbf{Sol}(\not{\nabla}_R) \to L^2(\Sigma; S_RX \otimes E),$$

see [6, Lemma 3.17]. We denote the scalar product on $\mathbf{Sol}(\not{\nabla}_R)$ by $(\cdot,\cdot)$. Similarly, we obtain the Hilbert space completions $\mathbf{Sol}(\not{\nabla}_L)$, $\mathbf{Sol}(\not{\nabla}_R^*)$, and $\mathbf{Sol}(\not{\nabla}_L^*)$ of $\mathrm{Sol}(\not{\nabla}_L)$, $\mathrm{Sol}(\not{\nabla}_R^*)$, and $\mathrm{Sol}(\not{\nabla}_L^*)$, respectively.

2.4. **The fermionic propagator.** Let $G_R : C_0^\infty(X; S_LX \otimes E) \to C^\infty(X; S_RX \otimes E)$ be the difference between retarded and advanced fundamental solutions of $i\not\nabla_R$. Then G_R maps onto the space of solutions $\mathrm{Sol}(\not\nabla_R)$ of the Dirac equation $\not\nabla_R u = 0$. This operator is sometimes called the fermionic propagator of $i\not\nabla_R$. Similarly, we obtain linear maps

$$G_L : C_0^\infty(X; S_RX \otimes E) \twoheadrightarrow \mathrm{Sol}(\not\nabla_L) \subset C^\infty(X; S_LX \otimes E),$$
$$G_{R,*} : C_0^\infty(X; S_R^*X \otimes E^*) \twoheadrightarrow \mathrm{Sol}(\not\nabla_R^*) \subset C^\infty(X; S_L^*X \otimes E^*),$$
$$G_{L,*} : C_0^\infty(X; S_L^*X \otimes E^*) \twoheadrightarrow \mathrm{Sol}(\not\nabla_L^*) \subset C^\infty(X; S_R^*X \otimes E^*).$$

Equation (3) implies

$$G_{R,*}\overline{u} = \overline{G_R u} \quad \text{and} \quad G_{L,*}\overline{v} = \overline{G_L v} \tag{6}$$

for all $u \in C_0^\infty(X; S_LX \otimes E)$ and $v \in C_0^\infty(X; S_RX \otimes E)$. Using an integration by parts, we can express the scalar product on the Hilbert space $\mathbf{Sol}(\not\nabla_R)$ as follows (compare [17, Prop. 2.2])

$$(G_R v, f) = -\int_X \langle v, f\rangle \,\mathrm{dV} \tag{7}$$

where $v \in C_0^\infty(X; S_LX \otimes E)$ and $f \in \mathbf{Sol}(\not\nabla_R)$. Analogous formulas hold for the scalar products on $\mathbf{Sol}(\not\nabla_L)$, $\mathbf{Sol}(\not\nabla_R^*)$, and $\mathbf{Sol}(\not\nabla_L^*)$, respectively.

By [5, Thm. 4.3] the fermionic propagators extend to operators on distributional sections, e.g.

$$G_R : C_0^{-\infty}(X; S_LX \otimes E) \to C^{-\infty}(X; S_RX \otimes E).$$

The image is precisely the space of distributional solutions of the Dirac equation.

2.5. **Fermionic propagator and Cauchy problem.** Let $\Sigma \subset X$ be a spacelike smooth Cauchy hypersurface with future-directed unit normal field n_Σ. Any $u \in C^{-\infty}(\Sigma; S_LX \otimes E)$ gives rise to a distribution $u\delta_\Sigma$ on X via $(u\delta_\Sigma)(v) = \int_\Sigma (\rho_\Sigma v)(u)\,\mathrm{dA}$ for all $v \in C^\infty(X; S_L^*X \otimes E^*)$. Since the support of $u\delta_\Sigma$ is contained in Σ and hence compact, we can apply G_R to $u\delta_\Sigma$. An argument using integration by parts shows that $f = G_R(u\delta_\Sigma) \in C^{-\infty}(X; S_RX \otimes E)$ is the solution of the Cauchy problem

$$i\not\nabla_R f = 0 \quad \text{and} \quad \rho_\Sigma f = \not n_\Sigma u. \tag{8}$$

Here $\not n_\Sigma = n_\Sigma^\alpha \gamma_\alpha$ denotes Clifford multiplication by n_Σ in accordance with Feynman's slash convention. The restriction $G_R : L^2(\Sigma; S_LX \otimes E) \subset C_0^{-\infty}(X; S_LX \otimes E) \to \mathbf{Sol}(\not\nabla_R)$ is therefore the inverse of the Hilbert space isometry $\not n_\Sigma \rho_\Sigma : \mathbf{Sol}(\not\nabla_R) \to L^2(\Sigma; S_LX \otimes E)$ and hence is itself a Hilbert space isometry.

3. States and the chiral anomaly

In order to describe the quantized Weyl field we start with the construction of the field algebra. It is essentially the CAR algebra associated with the space of solutions of the Dirac equation. More precisely, let $\mathcal{K}$ be the Hilbert space sum $\mathbf{Sol}(\not\nabla_R^*) \oplus \mathbf{Sol}(\not\nabla_R)$. We denote the scalar product by $(\cdot,\cdot)_\mathcal{K}$. This Hilbert space comes with the Dirac conjugation $\Gamma : \mathcal{K} \to \mathcal{K}$ as defined in (4). Then the selfdual CAR algebra associated with the pair $(\mathcal{K}, \Gamma)$ is the unital $*$-algebra generated by symbols $B(f)$ where $f \in \mathcal{K}$ and relations

$$f \mapsto B(f) \text{ is complex linear,} \tag{9}$$
$$\{B(f), B(g)\} = (\Gamma f, g)_\mathcal{K}, \tag{10}$$
$$B(f)^* = B(\Gamma f). \tag{11}$$

The CAR algebra admits a unique C^*-norm and we will denote its C^*-completion by $\mathcal{A}$. This is our field algebra.

3.1. **The field operators.** We define the field operators by

$$\Psi : C_0^\infty(X; S_R^*X \otimes E^*) \to \mathcal{A}, \quad \Psi(u) = B(G_{R,*}u \oplus 0),$$
$$\overline{\Psi} : C_0^\infty(X; S_LX \otimes E) \to \mathcal{A}, \quad \overline{\Psi}(v) = B(0 \oplus G_R v).$$

Relations (9)–(11) together with (6) and (7) imply that

$$u \mapsto \Psi(u) \text{ is complex linear,}$$
$$v \mapsto \overline{\Psi}(v) \text{ is complex linear,}$$
$$\{\Psi(u_1), \Psi(u_2)\} = 0, \quad \{\overline{\Psi}(v_1), \overline{\Psi}(v_2)\} = 0,$$
$$\{\overline{\Psi}(v), \Psi(u)\} = -i \int_X \langle \overline{v}, G_{R,*}u\rangle \,\mathrm{dV},$$
$$\Psi(u)^* = \overline{\Psi}(\overline{u}),$$
$$\not{\nabla}_R \Psi = 0, \quad \not{\nabla}_R^* \overline{\Psi} = 0. \tag{12}$$

Here $\not{\nabla}_R\Psi = 0$ is to be understood in the distributional sense, i.e. $\Psi(\not{\nabla}_R^* u) = 0$ for all $u \in C_0^\infty(X; S_L^*X \otimes E^*)$ and similarly $\overline{\Psi}(\not{\nabla}_R v) = 0$ for all $v \in C_0^\infty(X; S_RX \otimes E)$.

3.2. **The n-point functions.** A state ω on this algebra is of course uniquely determined by its n-point functions

$$\omega_n(f_1, \dots, f_n) := \omega(B(f_1) \cdots B(f_n)).$$

This also defines a distributional section $\tilde{\omega}_n \in C^{-\infty}(X^n; \boxtimes^n(S_RX \otimes E \oplus S_L^*X \otimes E^*))$ on the n-fold Cartesian product of X by

$$\tilde{\omega}_n(f_1 \otimes \cdots \otimes f_n) := \omega_n(\mathbf{G}f_1, \dots, \mathbf{G}f_n)$$

where we have used the notation $\mathbf{G} = G_{R,*} \oplus G_R$. We will also refer to this distribution as the n-point function.

A state is called quasi-free if for all $n = 1, 2, 3, \dots$ and $f_i \in \mathcal{K}$

$$\omega_{2n-1}(f_1, \dots, f_{2n-1}) = 0,$$
$$\omega_{2n}(f_1, \dots, f_{2n}) = (-1)^{\frac{n(n-1)}{2}} \sum \mathrm{sgn}(s) \sum_{j=1}^n \omega_2(f_{s(j)}, f_{s(j+n)})$$

where the sum is taken over all permutations s of $\{1, \dots, 2n\}$ such that

$$s(1) < s(2) < \cdots < s(n), \quad s(j) < s(j+n).$$

This means that the state is completely determined by its two-point function.

3.3. **Hadamard forms and relative currents.** A two-point function is said to be of Hadamard form if its wavefront set $\mathrm{WF}(\tilde{\omega}_2)$ satisfies

$$\mathrm{WF}(\tilde{\omega}_2) \subset \{(x_1, \xi_1, x_2, \xi_2) \in T^*(X \times X) \mid (x_1, \xi_1) \sim (x_2, -\xi_2),\ \xi_2 \text{ is future directed}\},$$

where $(x_1, \xi_1) \sim (x_2, \xi_2)$ means that these vectors are in the same orbit of the geodesic flow. It is known (see e.g. [29, 21] in the case of the Dirac field) that Hadamard forms are unique up to smooth kernels, i.e. if ω_1 and ω_2 are states with two-point functions of Hadamard form then

$$\tilde{\omega}_{1,2} := \tilde{\omega}_1 - \tilde{\omega}_2 \in C^\infty(X \times X; \boxtimes^2(S_R X \otimes E \oplus S_L^* X \otimes E^*)).$$

Note that $\tilde{\omega}_{1,2}$ is a smooth bi-solution of the Dirac equation. We can then define the *relative current* $J^{\omega_1,\omega_2} \in \Omega^1(X)$ as follows. Given a state ω we define $\hat{S}_\omega : C_0^\infty(X; S_R^* X \otimes E^*) \to C^{-\infty}(X; S_L^* X \otimes E^*)$ as the unique operator such that

$$\tilde{\omega}(\overline{\Psi}(v)\Psi(u)) = (\hat{S}_\omega u)(v)$$

for all $v \in C_0^\infty(X; S_L X \otimes E)$. Note that if ω has Hadamard form then we actually have $\hat{S}_\omega : C_0^\infty(X; S_R^* X \otimes E^*) \to C^\infty(X; S_L^* X \otimes E^*)$. We put $\hat{S} := \hat{S}_{\omega_1} - \hat{S}_{\omega_2}$. If both ω_1 and ω_2 have two-point functions of Hadamard form then the operator $\hat{S}$ has smooth integral kernel. We denote the integral kernel of $\hat{S}$ evaluated at $(x,y) \in M \times M$ by $\hat{s}(x,y) \in \mathrm{Hom}(S_R^* X_y \otimes E_y^*, S_L^* X_x \otimes E_x^*)$. Moreover, by (12), we obtain

$$\nabla\!\!\!/_R^* \circ \hat{S} = \hat{S} \circ \nabla\!\!\!/_R^* = 0.$$

For $\xi \in TX_x$ denote Clifford multiplication with ξ by $\xi\!\!\!/ \in \mathrm{Hom}(S_R X_x \otimes E_x, S_L X_x \otimes E_x)$ and its dual by $\xi\!\!\!/^* \in \mathrm{Hom}(S_L^* X_x \otimes E_x^*, S_R^* X_x \otimes E_x^*)$. Now $\xi\!\!\!/^* \circ \hat{s}(x,x) \in \mathrm{End}(S_R^* X_x \otimes E_x^*)$ and we can set

$$J^{\omega_1,\omega_2}(\xi) := \mathrm{tr}(\xi\!\!\!/^* \circ \hat{s}(x,x)).$$

Since $\hat{s}$ is smooth on $M \times M$, this defines a smooth one-form $J^{\omega_1,\omega_2} \in \Omega^1(X)$. In physics terminology, one could write this definition as

$$J_\mu^{\omega_1,\omega_2}(x) = \lim_{y \to x} \left(\omega_1(\overline{\Psi}^{\dot{A}}(x)(\gamma_\mu)_{\dot{A}}^B \Psi_B(y)) - \omega_2(\overline{\Psi}^{\dot{A}}(x)(\gamma_\mu)_{\dot{A}}^B \Psi_B(y)) \right),$$

where the Einstein summation convention was used on the dotted spinor index $\dot{A}$ and spinor index B. The relative current can be thought of as the expectation value $\omega_2(:\!J_\mu(x)\!:)$ of the normally ordered current operator $:\!J_\mu(x)\!:$, where the normal ordering has been done with respect to state ω_1. It follows directly from the definitions that

$$[\nabla\!\!\!/, f] = \nabla\!\!\!/ f \quad \text{and} \quad [f, \nabla\!\!\!/^*] = \nabla\!\!\!/ f^*$$

where $f \in C^\infty(X)$ is a function and ∇f its gradient vector field. For any smooth compactly supported function $f \in C_0^\infty(X)$ we get

$$\int_X f \delta J^{\omega_1,\omega_2} \,\mathrm{dV} = \int_X J^{\omega_1,\omega_2}(\nabla f)\,\mathrm{dV} = \mathrm{Tr}(\nabla\!\!\!/ f^* \hat{S}) = \mathrm{Tr}(f \nabla\!\!\!/_R^* \hat{S} - \nabla\!\!\!/_R^* f \hat{S}) = -\mathrm{Tr}(f \hat{S} \nabla\!\!\!/_R^*) = 0.$$

Here we have used the canonical trace Tr on the algebra of integral operators whose integral kernel have compact support in the first variable. Its properties are described in Appendix A.

This shows $\delta J^{\omega_1,\omega_2} = 0$. If Σ is a spacelike smooth Cauchy hypersurface and n_Σ the future-directed unit normal vector field along Σ, then we can integrate the relative current $J^{\omega_1,\omega_2}(n_\Sigma)$ along the Cauchy hypersurface. Since the current is co-closed the integral does not depend on the choice of Cauchy hypersurface and defines the *relative right-handed charge* $Q_R^{\omega_1,\omega_2}$ between two states

$$Q_R^{\omega_1,\omega_2} := \int_\Sigma J^{\omega_1,\omega_2}(n_\Sigma(x))\mathrm{dA}(x). \tag{13}$$

3.4. **Fock representations.** If $\mathcal{H}$ is a Hilbert space the fermionic Fock space $\mathcal{F}(\mathcal{H})$ is defined to be the Hilbert space direct sum

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \Lambda^k \mathcal{H}.$$

As usual the vector $1 \in \mathbb{C} = \Lambda^0\mathcal{H} \subset \mathcal{F}(\mathcal{H})$ is denoted by Ω and is called the *vacuum vector*. On this space we have the usual creation and annihilation operators

$$a^\dagger(v)h = v \wedge h,$$
$$a(v)h = \iota_v h.$$

Note that $a^\dagger(v)$ is linear in v, whereas $a(v)$ is antilinear in v.

Let $P : \mathcal{K} \to \mathcal{K}$ be an orthogonal projection such that

$$P + \Gamma P \Gamma = \mathbb{1}.$$

This induces an orthogonal splitting $\mathcal{K} = \mathcal{H} \oplus \Gamma\mathcal{H}$ where $\mathcal{H} = P\mathcal{K}$. We obtain a faithful $*$-representation π of $\mathcal{A}$ on the Fock space $\mathcal{F}(\mathcal{H})$ by setting

$$\pi(B(f)) := a^\dagger(Pf) + a(P\Gamma f).$$

This representation is called the Fock representation. One checks that for such a representation the state

$$\omega(\cdot) = \langle \Omega, \cdot\, \Omega\rangle$$

is a quasifree state and the Fock representation is canonically isomorphic to the GNS representation of ω. See [3] or [27] for a detailed discussion of CAR algebras and Fock representations.

3.5. **Fock representations constructed near $\boldsymbol{\Sigma}$.** If Σ is a spacelike smooth Cauchy hypersurface then the restriction map

$$\rho_\Sigma : \mathcal{K} = \mathbf{Sol}(\nabla\!\!\!\!/_R^*) \oplus \mathbf{Sol}(\nabla\!\!\!\!/_R) \to \mathcal{K}_\Sigma := L^2(\Sigma; S_L^*X \otimes E^*) \oplus L^2(\Sigma; S_R X \otimes E)$$

is an isomorphism of Hilbert spaces. We can therefore identify the Hilbert space $\mathcal{K}$ with conjugation Γ with $\mathcal{K}_\Sigma$ and conjugation $\Gamma|_\Sigma$. Moreover, we can identify $S_R X|_\Sigma$ with $S\Sigma$, the spinor bundle of Σ. Using the isomorphism $\not{n}_\Sigma : S_L X|_\Sigma \to S_R X|_\Sigma$ be also get an identification of $S_L X|_\Sigma$ with $S\Sigma$. The Dirac operator $\nabla\!\!\!\!/_\Sigma$ on Σ anticommutes with $\not{n}_\Sigma$ and is selfadjoint with respect to the positive definite scalar product $\langle \not{n}_\Sigma \cdot, \cdot\rangle$ on $S\Sigma$. This implies

$$\nabla\!\!\!\!/_\Sigma \overline{v} = -\overline{\nabla\!\!\!\!/_\Sigma^* v}. \tag{14}$$

In particular, conjugation $\bar{\cdot}$ maps $\nabla\!\!\!\!/_\Sigma$-eigenspaces for positive eigenvalues to $\nabla\!\!\!\!/_\Sigma^*$-eigenspaces for negative eigenvalues and vice versa. We consider the spectral projectors

$$p_\geq(\nabla\!\!\!\!/_\Sigma) := \chi_{[0,\infty)}(\nabla\!\!\!\!/_\Sigma) \quad \text{and} \quad p_>(\nabla\!\!\!\!/_\Sigma) := \chi_{(0,\infty)}(\nabla\!\!\!\!/_\Sigma)\,.$$

Similarly, we define $p_\leq(\nabla\!\!\!\!/_\Sigma)$, $p_<(\nabla\!\!\!\!/_\Sigma)$, and the corresponding projectors for $\nabla\!\!\!\!/_\Sigma^*$. Now

$$P := p_\geq(\nabla\!\!\!\!/_\Sigma^*) \oplus p_>(\nabla\!\!\!\!/_\Sigma)$$

is an orthogonal projection on $\mathcal{K}_\Sigma$ and (14) implies $P + \Gamma P\Gamma = \mathbb{1}$. Its Fock representation will then be modeled on the one-particle Hilbert space

$$\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^+ \oplus \mathcal{H}_\Sigma^-$$

where

$$\mathcal{H}^+_\Sigma = p_\geq(\not{\nabla}^*_\Sigma)L^2(\Sigma; S^*\Sigma \otimes E^*),$$
$$\mathcal{H}^-_\Sigma = p_>(\not{\nabla}_\Sigma)L^2(\Sigma; S\Sigma \otimes E) = \overline{p_<(\not{\nabla}^*_\Sigma)L^2(\Sigma; S^*\Sigma \otimes E^*)}\,.$$

The space $\mathcal{H}^+_\Sigma$ is the one-particle space for particles and $\mathcal{H}^-_\Sigma$ is the one-particle space for antiparticles. The Fock space then splits into a tensor product

$$\mathcal{F}(\mathcal{H}_\Sigma) = \mathcal{F}(\mathcal{H}^+_\Sigma)\hat{\otimes}\mathcal{F}(\mathcal{H}^-_\Sigma).$$

As usual we denote by c and $c^\dagger$ the creation and annihilation operators on $\mathcal{F}(\mathcal{H}^+_\Sigma)$ and by d and $d^\dagger$ the creation and annihilation operators on $\mathcal{F}(\mathcal{H}^-_\Sigma)$. Each of them acts on the entire Fock space by extending it by the identity map on the other tensor factor. The representation of $\mathcal{A}$ in this Fock representation is explicitly given by

$$\pi_\Sigma(\Psi(u)) = c^\dagger\big(p_\geq(\not{\nabla}^*_\Sigma)\rho_\Sigma G_{R,*}u\big) + d\big(p_>(\not{\nabla}_\Sigma)\rho_\Sigma G_R\overline{u}\big),$$
$$\pi_\Sigma(\overline{\Psi}(v)) = c\big(p_\geq(\not{\nabla}^*_\Sigma)\rho_\Sigma G_{R,*}\overline{v}\big) + d^\dagger\big(p_<(\not{\nabla}_\Sigma)\rho_\Sigma G_R v\big).$$

The notation

$$\pi_\Sigma(B(f)) = a^\dagger(P\rho_\Sigma f) + a(P\Gamma\rho_\Sigma f)$$

with the substitution $f = \mathbf{G}(u \oplus v)$ is however more compact.

It is well known (see for example [16, Sec. 4]) that the vacuum expectation value ω_Σ with respect to this representation is a Hadamard state, i.e. its two-point function is of Hadamard form, if the metric on X and ∇^E are of product type near Σ. This means that a neighborhood of Σ can be identified with $(-\varepsilon, \varepsilon) \times \Sigma$ in such a way that Σ corresponds to $\{0\} \times \Sigma$, that the metric takes the form $dt^2 - g_\Sigma$ where g_Σ is independent of t and that ∇^E is the pull-back of its restriction to Σ under the projection $(-\varepsilon, \varepsilon) \times \Sigma \to \Sigma$.

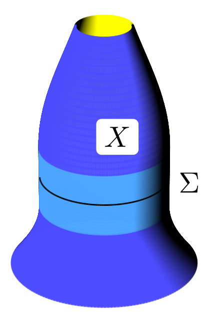

Fig. 1. *Manifold X with product structure near Σ*

In this case this representation is thought of as the preferred vacuum representation of an observer on Σ.

3.6. **Integral kernels of spectral projectors.** As before let $\hat{S}_\Sigma$ be the unique operator $C^\infty_0(X; S^*_R X \otimes E^*) \to C^\infty(X; S^*_L X \otimes E^*)$ such that

$$\omega_\Sigma(\overline{\Psi}(v)\Psi(u)) = \int_X \langle \overline{v}, \hat{S}_\Sigma u\rangle \,\mathrm{dV}$$

and $\hat{s}_\Sigma$ its integral kernel. Now let $u \in L^2(\Sigma; S^*_R X \otimes E^*)$. Then $G_{R,*}(u\delta_\Sigma) \in \mathbf{Sol}(\not{\nabla}^*_R)$ so that $\Psi(u) = B(G_{R,*}(u\delta_\Sigma) \oplus 0)$ is defined. Similarly, $\overline{\Psi}(v)$ is defined for $v \in L^2(\Sigma; S_L X \otimes E)$. We compute

$$
\begin{aligned}
\omega_\Sigma(\overline{\Psi}(v)\Psi(u)) &= \omega_\Sigma(B(0 \oplus G_R(v\delta_\Sigma))B(G_{R,*}(u\delta_\Sigma) \oplus 0)) \\
&= (\Omega, \pi_\Sigma(B(0 \oplus G_R(v\delta_\Sigma))B(G_{R,*}(u\delta_\Sigma) \oplus 0))\Omega)_{\mathcal{F}(\mathcal{H}_\Sigma)} \\
&= \Big(\pi_\Sigma(B(\overline{G_R(v\delta_\Sigma)} \oplus 0))\Omega, \pi_\Sigma(B(G_{R,*}(u\delta_\Sigma) \oplus 0))\Omega\Big)_{\mathcal{F}(\mathcal{H}_\Sigma)} \\
&= \Big(P\rho_\Sigma(\overline{G_R(v\delta_\Sigma)} \oplus 0) \wedge \Omega, P\rho_\Sigma(G_{R,*}(u\delta_\Sigma) \oplus 0) \wedge \Omega\Big)_{\mathcal{F}(\mathcal{H}_\Sigma)} \\
&\overset{(8)}{=} \big(p_\geq(\not{\nabla}^*_\Sigma)(\overline{\not{n}_\Sigma v}), p_\geq(\not{\nabla}^*_\Sigma)(\not{n}^*_\Sigma u)\big)_{\mathcal{H}^+_\Sigma} \\
&= \big(\not{n}^*_\Sigma \overline{v}, p_\geq(\not{\nabla}^*_\Sigma)(\not{n}^*_\Sigma u)\big)_{L^2(\Sigma)} \\
&= \int_\Sigma \langle \overline{v}, p_\geq(\not{\nabla}^*_\Sigma)(\not{n}^*_\Sigma u)\rangle \,\mathrm{dA}\,.
\end{aligned}
$$

In other words, the integral kernel of the projector $p_\geq(\not{\nabla}^*_\Sigma)$ coincides with $\hat{s}_\Sigma(y,x)\not{n}^*_\Sigma(x)$, restricted to $(y,x) \in \Sigma \times \Sigma$.

More generally, let Σ_1 and Σ_2 be two spacelike smooth Cauchy hypersurfaces and let U_{Σ_1,Σ_2} the unitary evolution operator defined in (5). For $u \in L^2(\Sigma_1; S^*_R X \otimes E^*)$ we have by (8) that $G_{R,*}((\not{n}^*_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} u)\delta_{\Sigma_2}) = G_{R,*}(\not{n}^*_{\Sigma_1} u\delta_{\Sigma_1})$ and hence $\Psi(\not{n}^*_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u) = \Psi(u)$ and similarly for $\overline{\Psi}$. If $v \in L^2(\Sigma_1; S_L X \otimes E)$ we have on the one hand

$$
\omega_{\Sigma_2}(\overline{\Psi}(v)\Psi(u)) = \int_{\Sigma_1 \times \Sigma_1} \langle \overline{v}(y), \hat{s}_{\Sigma_2}(y,x)u(x)\rangle \mathrm{dA}(x)\mathrm{dA}(y)
$$

and on the other hand

$$
\begin{aligned}
\omega_{\Sigma_2}(\overline{\Psi}(v)\Psi(u)) &= \omega_{\Sigma_2}(\overline{\Psi}(\not{n}_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} \not{n}_{\Sigma_1} v)\Psi(\not{n}^*_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u)) \\
&= \Big(\not{n}^*_{\Sigma_2} \overline{\not{n}_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} \not{n}_{\Sigma_1} v}, p_\geq(\not{\nabla}^*_{\Sigma_2})(\not{n}^*_{\Sigma_2} \not{n}^*_{\Sigma_2} U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u)\Big)_{L^2(\Sigma_2)} \\
&= \big(U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} \overline{v}, p_\geq(\not{\nabla}^*_{\Sigma_2})(U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u)\big)_{L^2(\Sigma_2)} \\
&= \big(\not{n}^*_{\Sigma_1} \overline{v}, U_{\Sigma_1,\Sigma_2} p_\geq(\not{\nabla}^*_{\Sigma_2})(U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u)\big)_{L^2(\Sigma_1)} \\
&= \int_{\Sigma_1} \big\langle \overline{v}, U_{\Sigma_1,\Sigma_2} p_\geq(\not{\nabla}^*_{\Sigma_2}) U^{-1}_{\Sigma_1,\Sigma_2} \not{n}^*_{\Sigma_1} u\big\rangle \,\mathrm{dA}\,.
\end{aligned}
$$

Hence the integral kernel of $U_{\Sigma_1,\Sigma_2} \circ p_\geq(\not{\nabla}^*_{\Sigma_2}) \circ U^{-1}_{\Sigma_1,\Sigma_2}$ coincides with $\hat{s}_{\Sigma_2}(y,x)\not{n}^*_{\Sigma_1}(x)$, restricted to $(y,x) \in \Sigma_1 \times \Sigma_1$.

3.7. **Relative charge between Fock states associated to Cauchy hypersurfaces.** Now we consider the following situation: let Σ_1 and Σ_2 be two spacelike smooth Cauchy hypersurfaces with Σ_1 lying in the past of Σ_2. We assume that the metric of X and the connection ∇^E of E have product structure near both hypersurfaces. We compute the relative charge for the Fock states ω_{Σ_1} and ω_{Σ_2}:

$$
Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = \int_{\Sigma_1} J^{\omega_{\Sigma_2},\omega_{\Sigma_1}}(n_{\Sigma_1})\,\mathrm{dA}
$$

$$
\begin{aligned}
&= \int_{\Sigma_1} \operatorname{tr}\left(\nu\!\!\!/^*_{\Sigma_1}(\hat{s}_{\Sigma_1}(x,x) - \hat{s}_{\Sigma_2}(x,x))\right) \mathrm{dA}(x) \\
&= \int_{\Sigma_1} \operatorname{tr}\left((\hat{s}_{\Sigma_1}(x,x) - \hat{s}_{\Sigma_2}(x,x))\nu\!\!\!/^*_{\Sigma_1}\right) \mathrm{dA}(x) \\
&= \operatorname{Tr}\big(p_{\geq}(\nabla\!\!\!\!/^*_{\Sigma_1}) - U_{\Sigma_1,\Sigma_2} p_{\geq}(\nabla\!\!\!\!/^*_{\Sigma_2}) U^{-1}_{\Sigma_1,\Sigma_2}\big)\,.
\end{aligned}
$$

It was shown in [7, Thm. 6.5] that the operator $p_{\geq}(\nabla\!\!\!\!/^*_{\Sigma_1}) - U_{\Sigma_1,\Sigma_2} p_{\geq}(\nabla\!\!\!\!/^*_{\Sigma_2}) U^{-1}_{\Sigma_1,\Sigma_2}$ has a smooth integral kernel. In particular, the operator is of trace class. It now follows from [4, Thm. 4.1] that the trace is an integer and equals an index, namely putting $U = U_{\Sigma_1,\Sigma_2}$ and $p_j = p_{\geq}(\nabla\!\!\!\!/^*_{\Sigma_j})$,

$$
\begin{aligned}
Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} &= \operatorname{ind}\big[U p_2 U^{-1} p_1 : p_1 L^2(\Sigma_1) \to U p_2 U^{-1} L^2(\Sigma_1)\big] \\
&= \operatorname{ind}\big[p_2 U^{-1} p_1 : p_1 L^2(\Sigma_1) \to p_2 U^{-1} L^2(\Sigma_1)\big]\,.
\end{aligned}
$$

By Theorem 4.1 and the concluding remark in [7] this index is given by

$$
Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = -\int_M \widehat{\mathrm{A}} \wedge \operatorname{ch}(\nabla^E) + \frac{h(\nabla\!\!\!\!/_{\Sigma_1}) + h(\nabla\!\!\!\!/_{\Sigma_2}) + \eta(\nabla\!\!\!\!/_{\Sigma_1}) - \eta(\nabla\!\!\!\!/_{\Sigma_2})}{2} - h(\nabla\!\!\!\!/_{\Sigma_2}) \tag{15}
$$

$$
= -\int_M \widehat{\mathrm{A}} \wedge \operatorname{ch}(\nabla^E) + \frac{h(\nabla\!\!\!\!/_{\Sigma_1}) - h(\nabla\!\!\!\!/_{\Sigma_2}) + \eta(\nabla\!\!\!\!/_{\Sigma_1}) - \eta(\nabla\!\!\!\!/_{\Sigma_2})}{2}\,. \tag{16}
$$

Here M is the region between Σ_1 and Σ_2, i.e. $M = J^+(\Sigma_1) \cap J^-(\Sigma_2)$ where J^+ and J^- denote the causal future and past, respectively. By $\widehat{\mathrm{A}}$ we denote the $\widehat{\mathrm{A}}$-form computed from the curvature of X and $\operatorname{ch}(\nabla^E)$ is the Chern-character form for the curvature of ∇^E, see [9, Sec. 4.1]. Hence $\widehat{\mathrm{A}}$ contains the contribution of gravitation to the relative charge and $\operatorname{ch}(\nabla^E)$ that of the external field.

Moreover, $\eta(\nabla\!\!\!\!/_{\Sigma})$ denotes the η-invariant of the Dirac operator on the Cauchy hypersurface Σ and $h(\nabla\!\!\!\!/_{\Sigma})$ the dimension of its kernel. The additional term $-h(\nabla\!\!\!\!/_{\Sigma_2})$ in (15) is caused by a different convention for the spectral projectors in [7] concerning the eigenvalue 0. There is no trangression boundary term because of the product-structure assumption near Σ_1 and Σ_2.

3.8. **Summary and example.** We summarize the results we have obtained.

Theorem 3.1. *Let X be an even-dimensional globally hyperbolic Lorentzian spin manifold, let $\Sigma_1, \Sigma_2 \subset X$ be two spacelike smooth Cauchy hypersurfaces with Σ_1 lying in the past of Σ_2. We put $M := J^+(\Sigma_1) \cap J^-(\Sigma_2)$.*

Let $E \to X$ be a Hermitian vector bundle with compatible connection ∇^E. We assume that the metric of X and ∇^E have product structure near Σ_1 and Σ_2.

Then the relative right-handed charge $Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}}$ as defined in (13) for the Fock states ω_{Σ_j} and the Dirac operator twisted with E is given by

$$
Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = -\int_M \widehat{\mathrm{A}} \wedge \operatorname{ch}(\nabla^E) + \frac{h(\nabla\!\!\!\!/_{\Sigma_1}) - h(\nabla\!\!\!\!/_{\Sigma_2}) + \eta(\nabla\!\!\!\!/_{\Sigma_1}) - \eta(\nabla\!\!\!\!/_{\Sigma_2})}{2}\,. \tag{17}
$$

Interchanging left-handed and right-handed spinors in the whole discussion we can also define the *relative left-handed charge* $Q_L^{\omega_{\Sigma_1},\omega_{\Sigma_2}}$. The projector defining the Fock representation is now given by $P = p_{>}(\nabla\!\!\!\!/^*_{\Sigma}) \oplus p_{\geq}(\nabla\!\!\!\!/_{\Sigma})$. Then a discussion analogous to the above yields

$$
Q_L^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = \int_M \widehat{\mathrm{A}} \wedge \operatorname{ch}(\nabla^E) + \frac{-h(\nabla\!\!\!\!/_{\Sigma_1}) + h(\nabla\!\!\!\!/_{\Sigma_2}) - \eta(\nabla\!\!\!\!/_{\Sigma_1}) + \eta(\nabla\!\!\!\!/_{\Sigma_2})}{2}\,. \tag{18}
$$

The exchange of chirality is equivalent to reversing the orientation of X. This explains the opposite sign for the contribution given by the integral. The induced orientations on Σ_1 and Σ_2 will also be reversed which results in a replacement of $\not{\nabla}_{\Sigma_j}$ by $-\not{\nabla}_{\Sigma_j}$. Hence the η-invariants get the opposite sign. The different convention concerning the eigenvalue 0 in the definition of the projection P is responsible for the opposite signs in the h-terms.

If we call $Q^{\omega_{\Sigma_1},\omega_{\Sigma_2}} := Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} + Q_L^{\omega_{\Sigma_1},\omega_{\Sigma_2}}$ the *relative total charge* and $Q_{\mathrm{chir}}^{\omega_{\Sigma_1},\omega_{\Sigma_2}} := Q_R^{\omega_{\Sigma_1},\omega_{\Sigma_2}} - Q_L^{\omega_{\Sigma_1},\omega_{\Sigma_2}}$ the *relative chiral charge* then (17) and (18) imply

$$Q^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = 0\,,$$

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = -2\int_M \widehat{\mathrm{A}} \wedge \mathrm{ch}(\nabla^E) + h(\not{\nabla}_{\Sigma_1}) - h(\not{\nabla}_{\Sigma_2}) + \eta(\not{\nabla}_{\Sigma_1}) - \eta(\not{\nabla}_{\Sigma_2})\,.$$

That is to say that the total charge is preserved in quantum field theory while the chiral charge is not. The chiral anomaly depends on the space-time curvature via $\widehat{\mathrm{A}}$, on the external field via $\mathrm{ch}(\nabla^E)$, and on spectral properties of the spatial Dirac operators via the h- and η-terms.

Example 3.2. The following example is well known but shows that the contributions of the η-invariant are essential to give a correct integer valued total charge. Let $X = \mathbb{R} \times S^1$ equipped with the metric $dt^2 - d\theta^2$ where θ denotes the standard coordinate on the circle $S^1 = \mathbb{R}/L\mathbb{Z}$ of length L. We twist with the topologically trivial complex line bundle E (indeed all complex line bundles on X are topologically trivial) and the connection $\nabla^E = \partial + iA$ where the electro-magnetic potential A is of the form $A = A_1(t)d\theta$. The (real) curvature form of ∇^E is given by $F = dA = \dot{A}_1 dt \wedge d\theta$. This describes an electric field with no magnetic component.

The surface X with the given metric has vanishing curvature. But even if it had nontrivial curvature it would not enter the formula for the relative charge because $\widehat{\mathrm{A}}$ has nonzero contributions only in dimensions divisible by 4. Thus $\widehat{\mathrm{A}} = 1$. In two dimensions the Chern character form is simply given by $\mathrm{ch}(\nabla^E) = \frac{1}{2\pi}F = \frac{\dot{A}_1}{2\pi} dt \wedge d\theta$.

For $t_1 < t_2$ and $\Sigma_j = \{t_j\} \times S^1$ we have $M = [t_1, t_2] \times S^1$ and the integral is given by

$$\int_M \widehat{\mathrm{A}} \wedge \mathrm{ch}(\nabla^E) = \int_{t_1}^{t_2} \int_{S^1} \frac{\dot{A}_1}{2\pi}\, d\theta\, dt = \frac{L}{2\pi}(A_1(t_2) - A_1(t_1))\,.$$

Now S^1 (and hence X) has two inequivalent spin structures, the *trivial* and the *nontrivial spin structure*. Spinors with respect to the trivial spin structure correspond to complex-valued functions. Spinors on S^1 with respect to the nontrivial spin structure correspond to functions u on $\mathbb{R}$ which are antiperiodic with period L, i.e. $u(\theta + L) = -u(\theta)$. The twisted Dirac operator on $\Sigma = \{t\} \times S^1$ takes the form $\not{\nabla}_\Sigma = i\partial_\theta - A_1(t)$. Its eigenvalues have multiplicity 1 and are given by

$$\frac{2\pi}{L}k - A_1(t)$$

for the trivial spin structure and

$$\frac{2\pi}{L}\left(k + \frac{1}{2}\right) - A_1(t)$$

for the nontrivial spin structure where $k \in \mathbb{Z}$. A little computation using Hurwitz ζ-functions shows for the trivial spin structure

$$2 \cdot \frac{L}{2\pi}A_1(t) + h(\not{\nabla}_\Sigma) + \eta(\not{\nabla}_\Sigma) = 2\left\lfloor \frac{L}{2\pi}A_1(t) \right\rfloor + 1$$

and hence

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = 2\left\lfloor \frac{L}{2\pi} A_1(t_1) \right\rfloor - 2\left\lfloor \frac{L}{2\pi} A_1(t_2) \right\rfloor .$$

For the nontrivial spin structure we obtain

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = 2\left\lfloor \frac{L}{2\pi} A_1(t_1) - \frac{1}{2} \right\rfloor - 2\left\lfloor \frac{L}{2\pi} A_1(t_2) - \frac{1}{2} \right\rfloor .$$

Example 3.3. Let $X = \mathbb{R} \times S^3$ with a metric of the form $dt^2 - g_t$ where g_t is a one-parameter family of Riemannian metrics on S^3. The manifold X has a unique spin structure. This time we let E be the trivial line bundle with trivial connection ∇^E so that there is no external field. It is shown [7, Sec. 5] that the family g_t and the Cauchy hypersurfaces $\Sigma_j = \{t_j\} \times S^3$ can be chosen in such a way that $h(\not{\nabla}_{\Sigma_1}) = h(\not{\nabla}_{\Sigma_2}) = 0$ and $\int_M \widehat{\mathrm{A}} + \frac{\eta(\not{\nabla}_{\Sigma_2}) - \eta(\not{\nabla}_{\Sigma_1})}{2} = 2$. Hence

$$Q_{\mathrm{chir}}^{\omega_{\Sigma_1},\omega_{\Sigma_2}} = -4 .$$

In this case the space-time metric, i.e. gravity, is causing the nontriviality of the chiral anomaly.

4. Outlook

Of course, the assumption on the space-time to be spatially compact is something that one would like to get rid of in order to better understand the local properties of anomalies and in order to be able to treat more models from general relativity. Technically, this assumption ensures that the relative charge is finite and can be expressed as an index. However, the relative current is locally defined and does make sense also on more general space-times. A local index theorem should be able to compute the local current generated by the external field. In case of noncompact Cauchy hypersurfaces which have compact quotients there is also the possibility to employ the concepts of L^2-indices, L^2-traces, and L^2-dimensions as introduced by von Neumann.

Appendix A. The canonical trace on the algebra of smoothing operators with compact support in one variable

Let X be a manifold with a positive volume density dV. Suppose that E is a complex vector bundle over X. Then any smooth section a of the bundle $E \boxtimes E^*$ will determine an operator $A : C_0^\infty(X; E) \to C^\infty(X; E)$ via

$$(Au)(x) = \int_M a(x,y) u(y) \,\mathrm{dV}(y).$$

If the integral kernel a has support contained in a set of the form $K \times X$, where $K \subset X$ is compact we say that a is compactly supported in the first variable. The set of operators with integral kernel that is compactly supported in the first variable shall be denoted by $\Psi_0^{-\infty}$. Note that this set is clearly an algebra and these operators map $C_0^\infty(X; E)$ to $C_0^\infty(X; E)$. A canonical trace can be defined on this algebra by

$$\mathrm{Tr}(A) := \int_X \mathrm{tr}\, a(x,x) \,\mathrm{dV}(x).$$

This trace cannot be directly interpreted as an L^2-trace because an operator in $\Psi_0^{-\infty}$ need not necessarily be L^2-bounded. However, given a compact exhaustion K_n of X and a sequence of compactly supported smooth functions χ_n with $\chi_n(x) = 1$ for all $x \in K_n$, we have

$$\mathrm{Tr}(A) = \lim_{n\to\infty} \mathrm{Tr}_{L^2}(A\chi_n).$$

The operator $A\chi_n$ has smooth integral kernel $a(x,y)\chi_n(y)$ which is compactly supported in both variables. Hence it is trace class by Mercer's theorem. For $A, B \in \Psi_0^{-\infty}$ the trace property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ follows immediately from Fubini's theorem.

Now let P be a differential operator acting on sections of E. If $A \in \Psi_0^{-\infty}$ then $PA, AP \in \Psi_0^{-\infty}$ because they have integral kernels $(P \otimes \mathbb{1}_{E^*})a$ and $(\mathbb{1}_E \otimes P^*)a$, respectively. The trace property $\mathrm{Tr}(PA) = \mathrm{Tr}(AP)$ also holds even though $P \notin \Psi_0^{-\infty}$. This can easily be checked using integration by parts and the fact that in local coordinates

$$\frac{\partial}{\partial x^i} a(x,x) = \frac{\partial}{\partial x^i} a(x,y)|_{y=x} + \frac{\partial}{\partial y^i} a(x,y)|_{y=x}.$$

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