THE COLLISION OF PLANE WAVES IN THE GENERAL THEORY OF RELATIVITY

by

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ABSTRACT

The problem of colliding plane waves in General Relativity is discussed and all known exact solutions of the Einstein-Maxwell equations corresponding to various collisions are reviewed. These include collisions involving combinations of electromagnetic and gravitational waves with both collinear and non-collinear polarization.

It is pointed out how the collision problem may be simplified by a suitable choice of reference frame. In this way incoming waves approach from spatially opposite directions and the plane symmetry of the waves enable the spacetime to be considered to consist of four regions. One of these regions contains both waves as they interact subsequent to the collision. A solution of the collision problem may be uniquely determined by solving the field equations for this region subject to appropriate junction conditions at the regional boundaries.

To facilitate this review, the formalism of Newman and Penrose is utilized and using this it is shown how the field equations may be more appropriately formulated for the treatment of the collision problem. Furthermore, the formalism allows a ready interpretation of the geometry of the spacetime congruences. More precisely, the congruence geometry is described by certain scalar functions which arise in the formalism. The colliding fields may each be considered to define physically a congruence in spacetime and the focussing effect which each field induces on the congruences of the other may then be used to interpret the development of irregularities in the various solutions published.

Real curvature singularities develop in all the solutions discussed in this thesis except in the case of colliding electromagnetic waves, for which only a single highly specialized solution exists. Moreover, it is shown that for more realistic electromagnetic wave collisions, Weyl curvature necessarily develops in the interaction region. A theorem, due to Tipler, which is discussed in the context of the results given, requires that this curvature must become infinite and consequently realistic, planewave electromagnetic collisions cannot be singularity free.

CONTENTS

C O N T E N T S (continued)

1 Introduction

In the General Theory of Relativity much difficulty arises when attempts are made to draw physically important conclusions from the basic assumptions of the theory. This is largely due to the non-linearity of the field equations. One way in which insight into the implications of the equations may be obtained is to construct suitable exact solutions and study their properties.

The non-linear features of the theory can be especially highlighted and studied when solutions are found for waves which collide. This is because the collisions do not give rise to simple superpositions but instead the non-linearity of the theory causes the waves to interact. Unfortunately the complexity of the field equations generally requires the imposition of certain symmetries in order to find exact solutions at all. This simplifying procedure has made it possible for a number of exact solutions to be discovered corresponding to colliding plane waves. These solutions merit special consideration because they nearly all possess a characteristic future closing singularity. Moreover, the factors which lead to the development of these irregularities have been a matter of discussion for some time and a certain amount of controversy has naturally arisen over them. It is intended that this thesis should help in clarifying the theory at this stage.

We proceed initially by developing the idea of the spacetime congruence since using this a geometrically flavoured interpretation may be given to the solutions. More precisely, we give a quantitative description of the various combinations of collisions which can occur. In each case the collision and subsequent interaction is interpreted in terms of the focussing effect which each field induces on the congruences defined by the other. The exact solutions reviewed in this thesis confirm the fact that where singularities occur, they do so on the hypersurfaces onto which the null ray congruences are focussed.

Chapters have also been included in order to set up the collision problem and the more appropriate formulation of the field equations, initiated by Szekeres (1972) is given. A short résumé of gravitational wave polarization, both constant and variable is included since examples of both occur in the literature cited.

The exact solutions are then reviewed and expressed in a common notation. In most cases we have checked the solutions by direct substitution into the field equations and we have stated why this has not been practical in those few cases where such a verification is absent. In each case our review follows a fairly general format which includes expressions for the Weyl curvature components. We have calculated these explicitly for the numerous cases where they have been omitted in the original references or given in assymptotic form.

Additionally, where singularities arise in the solutions they have been classified as either real or co-ordinate dependent by utilizing the Polynomial Curvature Scalars and applying them appropriately. This has allowed us to clarify the status of the singularity structure behind the main singularity in a number of results. In the case of more specialized solutions, such as those with non-collinear polarization or with mixed fields, we have pointed out how these may reduce to other less specialized solutions in limiting cases. Relevant boundary conditions which are satisfied by the solutions are also referred to.

In some cases we have found it appropriate to restate results in the more concise form of theorems since they are sufficiently general as to profit from such a rationalization. Generally these theorems may then be used to derive new solutions using metric coefficients from others which are less specialized.

Following this review, the singularity problem is discussed in detail and particular attention is drawn to the case of colliding plane electromagnetic waves. The only relevant result available in the literature for this case is a highly idealized particular solution, included in our review and due to Bell and Szekeres (1974). Furthermore, it is the only singularity free planewave collision known. According to a theorem of Tipler (1980), however, all planewave collisions satisfying certain criteria must necessarily develop singularities. Actually the solution of Bell and Szekeres fails to meet these requirements because of the idealized wave profiles it possesses. We have shown, however, that it is a direct consequence of the field equations themselves, that Weyl curvature must generally develop within the interaction region of all colliding electro-magnetic waves with more realistic profiles. Thus such collisions are at least potentially singular as Tipler's theorem demands. Unfortunately, we have not yet been able to obtain a more general solution of the field equations which satisfy the appropriate boundary conditions for this case and consequently have been unable to verify Tipler's theorem with a concrete example. Nonetheless, we are reasonably convinced that when such solutions are obtained, they too will be singular.

It is interesting to note that Tipler's work implicates plane symmetry itself to be the factor responsible for the emergence of singularities in the solutions reviewed. It is therefore probably unlikely that such irregular behaviour would actually appear in the real world where such a high degree of symmetry is not expected to occur.

This thesis is concluded with a summary of the present knowledge and suggestions for further work in the field of wave collisions.

2 RELEVANT CONCEPTS AND METHODS

In order to facilitate the discussion of planewave collisions it will be convenient to make use of certain relevant topics which will be briefly introduced in the following sections.

§ 2.1 Notation and Conventions

We begin by introducing the notations and conventions which have been adopted within this thesis.

The form $A_{,\mu}$ or, when confusion with tensor indices is not possible, A_{μ} will be used to denote partial derivatives. Covariant derivatives will be distinguished by a semi-colon thus $A_{;\mu}$. It is assumed that spacetime is a pseudo-Riemannian Manifold having a symmetric affine connection $\Gamma^{\lambda}_{\mu\nu}$ and a metric $g_{\mu\nu}$ with signature (−−−+). The Riemann curvature tensor is given by

$$R_{\alpha\ \mu\nu}^{\ \lambda} = \Gamma^{\lambda}_{\alpha\mu,\nu} - \Gamma^{\lambda}_{\alpha\nu,\mu} + \Gamma^{\beta}_{\alpha\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu}\Gamma^{\lambda}_{\beta\nu}$$

here the greek letters, denoting tensor indices take values 1, 2, 3, 4.

The covariant symmetric Ricci tensor and the Ricci scalar are formed as follows

$$R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$$

$$R = R^{\alpha}_{\ \alpha}$$

The Weyl tensor is given by

$$C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - \frac{1}{2}\left(R_{\lambda\mu}g_{\nu\kappa} - R_{\lambda\nu}g_{\mu\kappa} - R_{\kappa\mu}g_{\nu\lambda} + R_{\kappa\nu}g_{\mu\lambda}\right) + \frac{R}{6}\left(g_{\lambda\mu}g_{\nu\kappa} - g_{\lambda\nu}g_{\mu\kappa}\right)$$

Einstein's gravitational field equations are given by

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -kT_{\mu\nu}$$

Where $T_{\mu\nu}$ is the Stress-Energy tenscr and k is defined by

$$k = \frac{8\pi G}{c^4}$$

Square and rcund brackets will be used tc denote symmetrization and antisymmetrization respectively, e.g.

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$$

$$T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right)$$

The symbols $\theta(x)$ and $\delta(x)$ will be used to denote the Heaviside unit step function and the Dirac distribution respectively.

§ 2.2 Congruences in Spacetime

Congruences in a regicn of spacetime are three parameter families of curves such that there exists a unique curve passing thrcugh each point of the region defined by

$$x^{\mu} = x^{\mu}(y^{a}, r), \quad (a = 1,2,3) \qquad \text{.... (2.1)}$$

where the y^a are the three parameters identifying particular curves of the congruence and r is a parameter along each. A tangent vector to each congruence is given by

$$k^{\mu} = \frac{\partial x^{\mu}}{\partial r} \qquad \text{---- (2.2)}$$

and the congruence is null iff

$$k^{\mu} k_{\mu} = 0$$

The congruences are gecdesic if their tangent vectors are parallelly propagated alcng them, i.e.

$$k^{\mu}{}_{;\nu} k^{\nu} = \lambda(r) k^{\mu} \qquad \text{---- (2.3)}$$

The parameter r is affine along the geodesic congruence iff $\lambda = 0$

§ 2.3 Tetrads

At any point in spacetime it is possible to define four linearly independent basis vectors

$$e_i^{\ \mu}$$

where i(=1, 2, 3, 4) labels each vector.

These may be chosen to satisfy the orthogonality relations

$$g_{\mu\nu} e_i^{\ \mu} e_j^{\ \nu} = \eta_{ij} \qquad \text{----} (2.4)$$

where η_{ij} is a constant matrix which can be interpreted in terms of the frame components of the metric tensor. Its inverse is denoted by η^{ij} and these constant matrices may be used to raise and lower tetrad indices.

Any vector may be represented in terms of its tetrad components thus

$$A^{\mu} = A^{i} e_i^{\ \mu} \qquad \text{----} (2.5)$$

and similarly for any tensor

$$T^{\kappa\lambda\ldots\ldots}{}_{\mu\nu\ldots\ldots} = T^{kl\ldots\ldots}{}_{mn\ldots\ldots}\, e_k^{\ \kappa} e_l^{\ \lambda} e^{m}{}_{\mu} e^{n}{}_{\nu}\ldots \qquad (2.6)$$

In this thesis the notation of Newman and Penrose (1962) is adopted. This has also been described by Pirani (1964) and Carmeli (1977). In this notation, tetrad vectors are labelled separately as

$$e_i^{\ \mu} = (l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}) \qquad (2.7)$$

where l^{μ} and n^{μ} are real, future pointing null vectors. The vector m^{μ} with its complex conjugate $\bar{m}^{\mu}$ are complex pseudo null vectors. They are also required to satisfy the relations

$$l^{\mu} n_{\mu} = 1, \quad m^{\mu} \bar{m}_{\mu} = -1, \quad l^{\mu} m_{\mu} = n^{\mu} m_{\mu} = 0 \qquad (2.8)$$

and thus

$$\eta_{ij} = e_i{}^{\mu} e_{j\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

From this it follows that the metric tensor can be written

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \quad (2.9)$$

It is often convenient, when using the Newman Penrose formalism, to adapt a null tetrad so as to align the tetrad vectors to given congruences at each event. A transformation which preserves the null l^μ direction (null rotation) is given by Pirani, (1964).

$$\begin{aligned} l^{\mu'} &= R\, l^\mu \\ m^{\mu'} &= (RT l^\mu + m^\mu)\, e^{iS} \qquad (2.10) \\ n^{\mu'} &= R^{-1} n^\mu + \bar{T} m^\mu + T \bar{m}^\mu + RT\bar{T} l^\mu \end{aligned}$$

Where R and S are real and T is complex. Using this it is possible to transform the n^μ vector into any other null vector satisfying the orthogonality relations (2.8)

§ 2.4 Spin Coefficients

The spin coefficients are the tetrad components of the covariant derivatives of the tetrad vectors. They can conveniently be defined in the form

$$
\begin{aligned}
l_{\mu;\nu} = {} & (\gamma+\bar{\gamma}) l_\mu l_\nu + (\epsilon+\bar{\epsilon}) l_\mu n_\nu - (\alpha+\bar{\beta}) l_\mu m_\nu - (\bar{\alpha}+\beta) l_\mu \bar{m}_\nu \\
& - \bar{\tau} m_\mu l_\nu - \bar{\kappa} m_\mu n_\nu + \bar{\sigma} m_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu \\
& - \tau \bar{m}_\mu l_\nu - \kappa \bar{m}_\mu n_\nu + \rho \bar{m}_\mu m_\nu + \sigma \bar{m}_\mu \bar{m}_\nu
\end{aligned}
\qquad (2.11)
$$

$$
\begin{aligned}
n_{\mu;\nu} = {} & -(\gamma+\bar{\gamma}) n_\mu l_\nu - (\epsilon+\bar{\epsilon}) n_\mu n_\nu + (\alpha+\bar{\beta}) m_\nu n_\mu + (\bar{\alpha}+\beta) n_\mu \bar{m}_\nu \\
& + \nu m_\mu l_\nu + \pi m_\mu n_\nu - \lambda m_\mu m_\nu - \mu m_\mu \bar{m}_\nu \\
& + \bar{\nu} \bar{m}_\mu l_\nu + \bar{\pi} \bar{m}_\mu n_\nu - \bar{\lambda} \bar{m}_\mu \bar{m}_\nu - \bar{\mu} \bar{m}_\mu m_\nu
\end{aligned}
\qquad (2.12)
$$

$$
\begin{aligned}
m_{\mu;\nu} = {} & \bar{\nu} l_\mu l_\nu + \bar{\pi} l_\mu n_\nu - \bar{\mu} l_\mu m_\nu - \bar{\lambda} l_\mu \bar{m}_\nu \\
& - \tau n_\mu l_\nu - \kappa n_\mu n_\nu + \rho n_\mu m_\nu + \sigma n_\mu \bar{m}_\nu \\
& + (\gamma-\bar{\gamma}) m_\mu l_\nu + (\epsilon-\bar{\epsilon}) m_\mu n_\nu - (\alpha-\bar{\beta}) m_\mu m_\nu - (\beta-\bar{\alpha}) m_\mu \bar{m}_\nu
\end{aligned}
\qquad (2.13)
$$

With this definition, some of the spin coeffieients have a simple geometic interpretation which will be discussed in section (2.5)

§ 2.5 The Optical Scalars

Equation (2.11) may be contracted to yield the instrinsic derivative

$$
l^{\mu}{}_{;\nu} l^{\nu} = (\epsilon+\bar{\epsilon}) l^{\mu} - \bar{\kappa} m^{\mu} - \kappa \bar{m}^{\mu} \qquad (2.14)
$$

Comparing this with the equation (2.3) implies that the null congruence tangent to l^μ is geodesic iff $\kappa = 0$ Consequently κ is referred to as the refraction since it measures the deviation of the congruence from

the geodesic. In addition if the parameter along the congruence is affine then
$$(\epsilon + \bar{\epsilon}) = 0$$

Further, the $Re\rho$, $Im\rho$ and $|\sigma|$ define the contraction twist and shear of the congruence for small change in affine apramete[r] respectively and $\frac{1}{2}$ arg σ defines the shear axis. These quantities are often referred to as the Optical Scalars (Ehlers & Kundt, 1962) and are given explicitly for $\kappa = (\epsilon + \bar{\epsilon}) = 0$ as follows

$$\text{expansion} = -Re\rho = \frac{1}{2} l^{\mu}{}_{;\mu}$$

$$\text{twist} = Im\rho = \left(\frac{1}{2} l_{[\mu;\nu]} l^{\mu;\nu}\right)^{1/2}$$

$$\text{shear} = |\sigma| = \left(\frac{1}{2} l_{(\mu;\nu)} l^{\mu;\nu} - \frac{1}{4}\{l^{\mu}{}_{;\mu}\}^{2}\right)^{1/2}$$

For the congruence tangent to n^{μ} the spin coefficients $-\mu, -\lambda, -\nu, -\gamma$ correspond to ρ, σ, κ, ϵ respectively.

§ 2.6 Components of the curvature and Ricci tensors

In the Newman Penrose formalism distinct labels are given to the tetrad components of various tensors.

Corresponding to the ten real independent components of the Weyl tensor, there are five independent complex tetrad components which are labelled as follows

$$\Psi_0 = -C_{\kappa\lambda\mu\nu} l^{\kappa} m^{\lambda} l^{\mu} m^{\nu}$$

$$\Psi_1 = -C_{\kappa\lambda\mu\nu} l^{\kappa} n^{\lambda} l^{\mu} m^{\nu}$$

$$\Psi_2 = -\frac{1}{2} C_{\kappa\lambda\mu\nu} l^{\kappa} n^{\lambda} (l^{\mu} n^{\nu} - m^{\mu} \bar{m}^{\nu}) \qquad (2.15)$$

$$\Psi_3 = -C_{\kappa\lambda\mu\nu} n^{\kappa} l^{\lambda} n^{\mu} \bar{m}^{\nu}$$

$$\Psi_4 = -C_{\kappa\lambda\mu\nu} n^{\kappa} \bar{m}^{\lambda} n^{\mu} \bar{m}^{\nu}$$

Szekeres (1965) has given a physical interpretation to these components as follows: Ψ_0 and Ψ_1 (or Ψ_4 and Ψ_3) describe transverse and longitudinal gravitational wave components in the n^μ (or l^μ) direction respectively. Ψ_2 denotes a coulomb component.

The Ricci tensor may also be written in terms of the Ricci scalar and nine components of an Hermitian three by three square matrix

$$\Phi_{AB} \qquad (A,B = 0,1,2.)$$

which are the tetrad components of the trace free part of the Ricci tensor as follows:

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2} R_{\mu\nu} l^\mu l^\nu \\
\Phi_{11} &= -\frac{1}{4} R_{\mu\nu} (l^\mu n^\nu + m^\mu \bar{m}^\nu) \\
\Phi_{22} &= -\frac{1}{2} R_{\mu\nu} n^\mu n^\nu \\
\Phi_{01} &= -\frac{1}{2} R_{\mu\nu} l^\mu m^\nu = \bar{\Phi}_{10} \\
\Phi_{02} &= -\frac{1}{2} R_{\mu\nu} m^\mu m^\nu = \bar{\Phi}_{20} \\
\Phi_{12} &= -\frac{1}{2} R_{\mu\nu} n^\mu m^\nu = \bar{\Phi}_{21} \\
\Lambda &= \frac{R}{24} .
\end{aligned} \qquad (2.16)$$

For electromagnetic fields it is possible to use the gravitational field equations to put

$$\Phi_{AB} = \frac{k}{4\pi} \Phi_A \bar{\Phi}_B \qquad (A,B = 0,1,2)$$

where the Φ_A are the tetrad components of the Electromagnetic Field tensor $F_{\mu\nu}$ defined by

$$\Phi_0 = F_{\mu\nu} l^\mu m^\nu$$

$$\Phi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu) \qquad (2.17)$$

$$\Phi_2 = F_{\mu\nu} \bar{m}^\mu n^\nu .$$

§ 2.7 The Newman Penrose Formalism

We now introduce the Intrinsic derivative operators of Newman and Penrose. These are the covariant derivatives taken in the direction of the tetrad vectors and are defined by

$$D(\ ) = (\ )_{;\mu} l^\mu$$

$$\Delta(\ ) = (\ )_{;\mu} n^\mu$$

$$\delta(\ ) = (\ )_{;\mu} m^\mu$$

$$\bar{\delta}(\ ) = (\ )_{;\mu} \bar{m}^\mu$$

When operating on scalar functions $\phi(x^\mu)$ these operators are generally non commutative. A complete set of commutator relations are defined in the Newman Penrose formalism and are given by

$$(\Delta D - D\Delta)\phi = (\gamma + \bar{\gamma}) D\phi + (\epsilon + \bar{\epsilon})\Delta\phi - (\bar{\tau} + \pi)\delta\phi - (\tau + \bar{\pi})\bar{\delta}\phi$$

$$(\delta D - D\delta)\phi = (\bar{\alpha} + \beta - \bar{\pi}) D\phi + \kappa\Delta\phi - \sigma\bar{\delta}\phi - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta\phi$$

$$(\delta\Delta - \Delta\delta)\phi = -\bar{\nu} D\phi + (\tau - \bar{\alpha} - \beta)\Delta\phi + \bar{\lambda}\bar{\delta}\phi + (\mu - \gamma + \bar{\gamma})\delta\phi$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\phi = (\bar{\mu} - \mu) D\phi + (\bar{\rho} - \rho)\Delta\phi + (\beta - \bar{\alpha})\bar{\delta}\phi + (\alpha - \bar{\beta})\delta\phi \qquad \ldots (2.18)$$

By substituting the tetrad vectors into the Ricci identity

$$A_{\lambda;\mu\nu} - A_{\lambda;\nu\mu} = A_\kappa R^\kappa{}_{\lambda\mu\nu}$$

it is possible to generate a series of complex scalar equations relating the intrinsic derivatives of the spin coefficients and the components of the Ricci and Weyl tensors. These are essentially the independent components of the Ricci identity and form the first set of the Newman Penrose identities. The second set consists of the Bianchi identities which are written in terms of the tetrad components of the Weyl and Ricci tensors, their intrinsic derivatives and the spin coefficients. The full set of identities may be obtained from the literature cited.

In the Newman Penrose formalism Maxwell's equations for source free (electrovac) fieldsmay be written in the form

$$\begin{aligned} D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - k\Phi_2 \\ D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \\ \delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\ \delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 \end{aligned} \qquad (2.19)$$

§ 2.8 The Polynomial Curvature Scalars

In this thesis we are concerned with the description of regions of spacetime where certain types of plane waves collide and interact. It is an interesting consequence of these collisions that in some cases singularities can develop in the region of interaction. Furthermore, in many cases the singular structure which develops is coordinate independent and accordingly is interpreted as a physically real consequence of the collision.

In order to distinguish between those singularities which are real and those which occur as a result of coordinate choice, use may be made of the fourteen Polynomial Curvature Scalars (Kramer et al 1980) symbolized by

$$I_A \, , \quad (A = 1, 2, \ldots \, 13, 14)$$

These fall into three groups: the Pure Weyl Scalars, the Pure Ricci scalars and the Mixed scalars. In constructing a computer algorithm for the calculation of the Weyl and Ricci tensors, relative to a complex null tetrad, Campbell and Wainwright (1977) have shown that the Newman Penrose formalism allows a simplified representation of the Polynomial Curvature Scalars. In this representation, the scalars are expressed in terms of the tetrad components of various tensors. The Weyl scalars are given by

$$I_1 - iI_2 = 16\left(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4\right)$$

$$I_3 - iI_4 = 96\left(\Psi_2\left[\Psi_2^2 - 2\Psi_1\Psi_3 - \Psi_0\Psi_4\right] + \Psi_0\Psi_3^2 + \Psi_1\Psi_4^2\right)$$

The Ricci and Mixed scalars may all be obtained from the references cited.

3 PLANE WAVES

Following Ehlers and Kundt (1962) and also Pirani (1964), algebraically special type N gravitational fields will be identified as plane fronted if they have vanishing twist, shear and expansion*.

§ 3.1 The Planewave Spacetime Metric

A suitable metric for the discussion of Planewaves, due originally to Brinkmann(1923) and described by Peres (1959) and Takeno (1961), is given by

$$ds^2 = 2\,dx^4 dx^1 + H(x^4, x^2, x^3)(dx^4)^2 - (dx^2)^2 - (dx^3)^2 \qquad (3.1)$$

Choosing a null tetrad in the form

$$\begin{aligned} l_\mu &= \delta^4_\mu , & l^\mu &= \delta^\mu_1 , \\ n_\mu &= \delta^1_\mu + \frac{H}{2}\delta^4_\mu , & n^\mu &= -\frac{H}{2}\delta^\mu_1 + \delta^\mu_4 , \\ m_\mu &= \frac{1}{\sqrt{2}}\left(\delta^2_\mu - i\delta^3_\mu\right), & m^\mu &= \frac{-1}{\sqrt{2}}\left(\delta^\mu_2 - i\delta^\mu_3\right). \end{aligned} \qquad (3.2)$$

then the only non zero spin coefficient is ν and thus the congruence tangent to n^μ is not geodesic. The congruence tangent to l^μ which is aligned with the field, however, has parallel rays etc. Consequently the field is identified as plane fronted with null coordinate x^1.
Choosing $(x^1, x^2, x^3, x^4) = (v, x, y, u)$
then the non zero components of the Ricci & Weyl tensors are given respectively by

$$\Phi_{22} = \frac{1}{4}\left(H_{xx} + H_{yy}\right) \qquad (3.3)$$

$$\Psi_4 = \frac{1}{4}\left(H_{xx} - H_{yy}\right) \qquad (3.4)$$

---

* cf Bondi, Pirani and Robinson (1959)

It is sometimes convenient to write H in the form:

$$H(u,x,y) = h(u)(x^2+y^2) + p(u)f(x,y) \qquad (3.5)$$

where $f(x,y)$ is an arbitrary solution of Laplace's equation
Comparison of (3.3),(3.4) and (3.5) implies that

$$\begin{aligned} h(u) &= \Phi_{22} \\ p(u) &= \frac{2}{f_{xx}}\Psi_4 \end{aligned} \qquad (3.6)$$

It is now clear that the presence of the functions $h(u)$ and $p(u)$ in the metric define the type of wave which is propagating. In particular if h(u) = 0, then the metric (3.1) describes a plane gravitational wave and equation (3.5) may now be re-written in the more general form

$$H(u,x,y) = g(u)(x^2-y^2) + 2b(u)xy + c(u)x + d(u)y + e(u) \qquad \text{----} (3.7)$$

A linear transformation

$$\begin{aligned} u &= U \\ x &= X + \alpha(u) \\ y &= Y + \beta(u) \\ v &= V + \alpha' X + \beta' Y + \gamma(u) \end{aligned} \qquad (3.8)$$

where primes denote the differential coefficient with respect to u, puts the metric (3.1) in the form

$$ds^2 = 2dUdV + dU^2[g(u)(X^2-Y^2) + 2b(u)XY] - dX^2 - dY^2 \qquad \text{----} (3.9)$$

provided α, β and γ satisfy

$$\alpha'' + g\alpha + b\beta + \frac{c}{2} = 0$$

$$\beta'' - g\beta + b\alpha + \frac{d}{2} = 0 \qquad (3.10)$$

$$2\gamma' - (\alpha')^2 - (\beta')^2 + g(\alpha^2 - \beta^2) + 2b\alpha\beta + c\alpha + d\beta + e = 0$$

Equation (3.9) is then the general metric defining a plane gravitational wave. For reasons which we shall shortly give, it is appropriate to regard the function $b(u)$ as a polarization function for the wave. It does not, however, appear possible to set $b = 0$ by a further transformation unless $\frac{b}{g}$ is constant.
When this condition is satisfied, however, the coordinate rotation

$$X = x\cos\theta - y\sin\theta$$

$$Y = x\sin\theta + y\cos\theta \qquad (3.11)$$

where $\theta = \frac{1}{2}\tan^{-1} b/2g$,
gives the metric

$$ds^2 = 2du\,dV + g(u)(x^2 - y^2)du^2 - dx^2 - dy^2 \qquad (3.12)$$

When transformations of the form (3.11) are possible (i.e. when $\frac{b}{g}$ is constant) the wave has a <u>constant polarization</u>. Otherwise it has a <u>variable polarization</u>.

The polarization of a gravitational wave may be interpreted in terms of the shearing of the congruence orthogonal to the direction of propagation of the wave. Essentially a ring of test particles in the x, y plane would begin, with time to form into an ellipse as the wave propagates perpendicularly through it. In the case of a wave with constant polarization, the x and y coordinates may be rotated into alignment with the major and minor axes by rotations of the type (3.11). Variably polarized waves will have shear axes which rotate with time.

In a similar way if $p(u)f(x,y)=0$ in equation (3.5), then it is always possible to write

$$H(u,x,y) = h(u)(x^2+y^2)$$

In this case the metric (3.1) describes a pure electromagnetic planewave.

§ 3.2 The Metric in Rosen Form

The metric, due to Rosen (1937)

$$ds^2 = \lambda du dv - g_{ij}\,dx^i dx^j \tag{3.13}$$

where the g_{ij} are functions of u or v only, is more convenient for the discussion of planewave collisions.

Equation (3.9) may be transformed into the metric (3.13) by the transformation (Kramer et al 1980)

$$\begin{aligned} u &= U \\ x &= \alpha X + \beta Y \\ y &= \delta X + \gamma Y \\ v &= V + \tfrac{1}{2}(\alpha\alpha' + \delta\delta')X^2 + \tfrac{1}{2}(\beta\beta' + \gamma\gamma')Y^2 + (\beta\alpha' + \gamma\delta')XY \end{aligned} \tag{3.14}$$

where α, β, γ and δ are functions of u chosen to satisfy

$$\begin{aligned} \alpha'' + g\alpha + b\delta &= 0 \\ \beta'' + g\beta + b\gamma &= 0 \\ \gamma'' - g\gamma + b\beta &= 0 \\ \delta'' - g\delta + b\alpha &= 0 \end{aligned} \tag{3.15}$$

and the condition

$$\beta\alpha' + \gamma\delta' = \alpha\beta' + \delta\gamma'.$$

The metric now takes the form

$$ds^2 = 2\,dU\,dV - (\alpha^2+\delta^2)dX^2 - (\beta^2+\gamma^2)dY^2 - 2(\alpha\beta+\gamma\delta)dX\,dY \qquad (3.16)$$

In the case of constant polarization (3.14) transforms the metric (3.12) into the simplified form of (3.16) with $\beta = \delta = b = 0$,

$$ds^2 = 2\,dU\,dV - \alpha^2 dX^2 - \gamma^2 dY^2 \qquad (3.17)$$

where now (3.14) corresponds to the transformation of Khan and Penrose (1971).

In the case of a pure electromagnetic wave where $\rho(u) f(x,y) = 0$ the metric

$$ds^2 = 2\,du\,dv + h(u)(x^2+y^2)du^2 - dx^2 - dy^2$$

can be transformed into the form

$$ds^2 = 2\,dU\,dV - \alpha^2(dX^2 + dY^2)$$

using the transformation

$$u = U$$

$$v = V + \tfrac{1}{2}(X^2+Y^2)\alpha\alpha'$$

$$x = \alpha X$$

$$y = \alpha Y$$

where $\alpha = \alpha(u),\ \alpha'' = -h\alpha$ and primes denote differentiation with respect to u.

§ 3.3 Plane Wave Profiles

Although the profile of a plane wave may be arbitrary, many of the solutions in the literature corresponding to plane wave collisions have generally been for reasons of relative simplicity, either impulsive or shock wave types. The imposition of these profiles is helpful in the process of obtaining exact solutions from the otherwise very complicated field equations. However, whether these solutions are physically realistic is a matter which turns out to have some relevance.

In equation (3.7), by setting

$$g(u) = \begin{cases} a\,\delta(u) & \text{for impulsive waves} \\ a\,\theta(u) & \text{for shock waves} \end{cases}$$

the Brinkmann metric given by (3.12), for gravitational waves, under transformations of the type (3.14) yield the respective Rosen metrics

$$ds^2 = 2\,du\,dv - (1+au)^2 dx^2 - (1-au)^2 dy^2 \tag{3.18}$$

$$ds^2 = 2\,du\,dv - \cosh^2 au\, dx^2 - \cos^2 au\, dy^2 \tag{3.19}$$

for impulsive and shock waves respectively. Electromagnetic shock waves are similarly described by setting $\rho = 0$ & $h = a\theta(u)$ in (3.5) Transforming to Rosen form gives the corresponding metric as

$$ds^2 = 2\,du\,dv - \cos^2 au\,(dx^2 + dy^2) \tag{3.20}$$

The metrics (3.18), (3.19) and (3.20) are singular on $u = 1/a$, $u = \pi/2a$ and $u = \pi/2a$ respectively. However, these are clearly coordinate singularities since they can be removed by transforming back into the Brinkmann form which is regular everywhere.

4 THE COLLISION PROBLEM

We now consider the spacetime containing two planewaves in collision. It is possible to make a Lorentz transformation to a frame of reference in which the two fields approach each other from exactly opposite spatial directions. A field of tetrads may now be introduced such that the l^{μ} and n^{μ} vectors are aligned with the propagation vectors of the two waves. Two null coordinates, u and v may also be chosen so that the wavefronts of the two waves prior to the collision are represented by the hypersurfaces $u = 0$, $v = 0$.

The spacetime may be represented as in Fig 4.1 and the collision problem is set up by specifying the metrics in regions I, II and III. Region I is usually taken to be flat ($H = 0$) and regions II and III describe incoming planewaves of various types. Possible metrics for these regions have been described in Chapter 3. In terms of the Petrov classification (Petrov 1954), the planewave metric (3.1) describes a type N gravitational field and consequently with appropriate choice of tetrad the gravitational waves in regions II and II are defined by the Weyl components Ψ_4 and Ψ_0 respectively. The Ricci components Φ_{22} and Φ_{00} respectively represent plane electromagnetic waves in these regions (see Fig. 4.1).

The problem now is to determine the field in region IV. This is to be done by solving the field equations in this region subject to the appropriate Junction Conditions for the II/IV and III/IV boundaries. Prior to collision, the spacetime is determined and thus a unique solution is expected in region IV. An appropriate uniqueness theorem has been obtained by Penrose (1980).

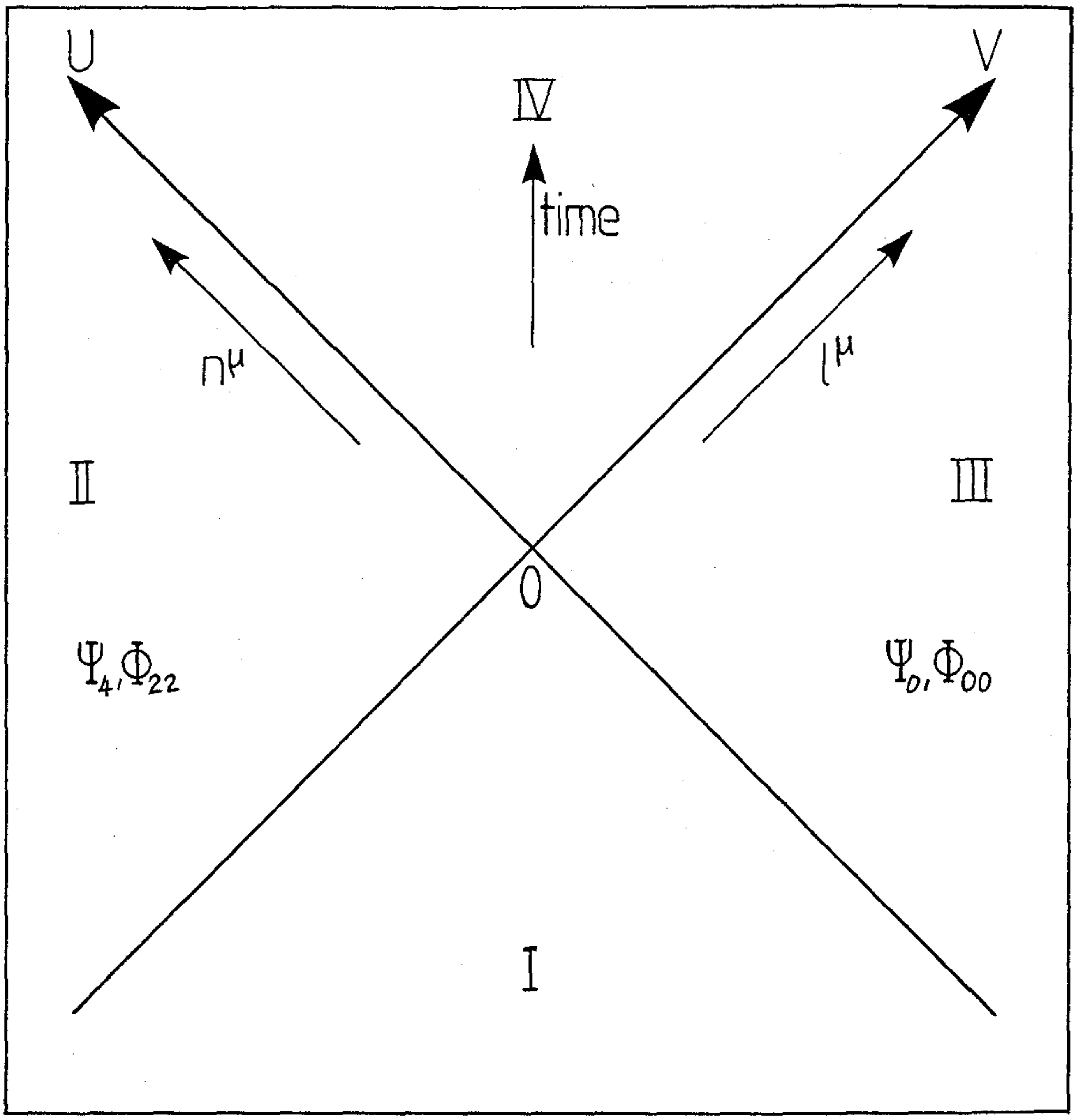

fig 4.1

It is appropriate to use the Rosen form for the metrics in regions II and III rather than the Brinkmann alternative. This is because it is convenient to continue to use the same two null coordinates in region IV also.

Suitable boundary conditions for joining solutions in the four regions are those of Lichnerowicz (1955), Darmois (1927) or those of O'Brien and Synge (1952). In fact the Lichnerowicz and Darmois conditions have been shown to be essentially equivalent (Bonner and Vickers 1981). Many solutions published satisfy all of these but the conditions of Lichnerowicz and Darmois do not permit impulsive gravitational waves. Appropriate conditions in this case are those of O'Brien and Synge (1952) (cf. Robson 1973). Bell and Szekeres (1974) have shown that, in the case of colliding electromagnetic waves the Lichnerowicz conditions cannot be satisfied and therefore in this case, these conditions must be relaxed to those of O'Brien and Synge. The O'Brien Synge conditions allow impulsive gravitational waves to be generated by the collision.

Since similar forms of the metric are considered in all three regions and the wavefronts of the two waves are given by u = constant, v = constant, the fact that the O'Brien Synge junction conditions are not covariant does not restrict their imposition in this case.

5 GEOMETRICAL CONSIDERATIONS

In this thesis we are concerned with solutions which describe the interaction region IV for various colliding waves. It is possible to establish a geometrical interpretation of these solutions via the spin coefficient formalism and the optical scalars.

§ 5.1 The Focussing of Congruences

Penrose (1966) has suggested that null geodesics suffer a focussing effect when traversing regions of spacetime possessing Weyl and/or Ricci curvature. Moreover, he has proposed that such focussing may be used as a measure of the total energy flux (matter plus - gravitational) across the ray. Such focussing of null geodesics may be seen to be of relevance to the collision problem in a manner which will now be described.

The tetrad vector l^{μ} is first aligned with a null geodesic congruence and an affine paramatrization is assumed ($\chi = \epsilon + \bar{\epsilon} = 0$). Furthermore, if the tetrad is chosen so that $\epsilon = 0$ then under these conditions the first two equations of the Newman-Penrose formalism are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \qquad (5.1)$$

$$D\sigma = (\rho + \bar{\rho})\sigma + \Psi_0 \qquad (5.2)$$

The above equations show that initially shear, twist and expansion free congruences (i.e. $\rho = \sigma = 0$), will remain so, provided they extend through regions of spacetime where $\Phi_{00} = \Psi_0 = 0$. However, if the congruence extends in to a region where $\Phi_{00} > 0$ (e.g. it meets an opposing electromagnetic wave), it will, from (5.1) start to contract. This is interpreted as a pure focussing of the congruence since it remains shear free. It may be noticed that since Φ_{00} is necessarily positive for an electromagnetic wave, geodesic congruences meeting it must contract rather than expand or at least reduce their rate of expansion.

In a similar way when the congruence enters a region where $\Psi_0 \neq 0$ (e.g. it meets an opposing gravitational wave) then in this region, from (5.2), it will start to shear. This in turn, however, from (5.1) will induce a corresponding contraction and effectively the congruence is astigmatically focussed. Again the term $\sigma\bar{\sigma}$ in (5.1) is necessarily positive, so opposing geodesic congruences must contract.

It turns out that where singularities occur in colliding planewave solutions, they do so on the hypersurfaces on which congruences are focussed. However, in the case of the single planewave metrics described by equations (3.18),(3.19) and (3.20), the actual congruences which are focussed by the fields they extend into, are not themselves defined by a physical field. The singularities in these metrics are therefore Coordinate Singularities. Furthermore, when calculated, the appropriate Polynomial Curvature Scalars are everywhere zero in the regions which these metrics describe.

§ 5.2 Types of Collisions

Since we are here concerned with plane wave collisions it is assumed that both incoming waves initially follow expansion free and shear free null geodesic congruences. This survey is also restricted to solutions describing collisions involving gravitational and electromagnetic waves, we have therefore four possibilities.

(i) Gravitational waves colliding with gravitational waves:

If the gravitational waves are considered to follow null geodesics after the collision, then both congruences are induced to shear and hence contract. They may therefore be considered to mutually focus one another astigmatically.

(ii) Electromagnetic waves colliding with electromagnetic waves:

In this case, after the collision both congruences are induced to contract and hence are considered to mutually focus each other. However, in the Bell-Szekeres solution, the step electromagnetic waves develop impulsive gravitational waves due to the collision which lie on the II/IV, III/IV boundaries. These impulsive waves introduce shear into the interaction region and astigmatic focussing results. This effect is not due to the nature of the initial conditions but turns out to be a necessary consequence of the field equations which prohibit pure focussing for a collision of this kind (cf. § 9.1)

(iii) Electromagnetic waves colliding with gravitational waves:

Here the congruences associated with both waves are induced to contract. Those which are defined by the gravitational wave remain shear free but those associated with the electromagnetic wave are induced to shear. (It is this shear which essentially develops the contraction along these congruences). However, according to a theorem of Mariot (1954) and Robinson (1961), null electromagnetic fields necessarily propagate along shear free null geodesic congruences. Thus, in the interaction region, the electromagnetic field is non null. The development of the shear induces a partial reflection or backscattering of the wave as it collides with the gravitational wave. This can be seen from Maxwell's equations (8.2) (c) and (d) directly; extra Maxwell components are necessarily induced following the development of shear terms (λ or σ respectively).

(iv) Mixed electromagnetic-gravitational wave collisions:

Opposing congruences traverse regions containing both Weyl and Ricci curvature in the spacetime and consequently contraction and shear develop along both sets of congruences.

These qualitative features have all been confirmed by the exact solutions described later and the focussing properties discussed are found to be associated with the development of singularities in the interaction region. In particular, Szekeres (1965) has given a theorem which requires that Vacuum solutions for the region IV must necessarily develop a Ψ_2 Weyl component. The components which define the waves also continue into the interaction region and collectively these components become unbounded on the spacelike hypersurface on which the two waves are mutually focussed. The appearance of the Ψ_2 term ensures that certain Polynomial curvature scalars also become infinite on these hypersurfaces and coordinate free singularities develop. The Rosen metrics in Regions II and III are also singular on the hypersurfaces on which the opposing congruences focus. However, only one of these opposing congruences for each region, is defined physically by a given field. Accordingly no interaction terms are present in these regions and the singularities are thus coordinate related.

6 THE FIELD EQUATIONS

The field equations for the interaction region are now derived and expressed in terms of the metric coefficients. The derivation follows that of Szekeres (1972) and the generalizations of Griffiths (1976) in order to account for solutions involving Einstein-Maxwell fields.

It is possible in Region IV to choose two null vectors which are aligned with the propogation vectors of the two waves. Provided that the fields continueto follow twist free null geodesics these can be given by:

$$A\ell_\mu = u_\mu\,, \qquad B n_\mu = v_\mu \tag{6.1}$$

where A and B are arbitrary functions of the coordinates and the wavefronts of the two waves are given by u = constant and v = constant. Thus u and v are two null coordinates associated with the two waves. Choosing

$$(x^1, x^2, x^3, x^4) = (u, x, y, v)$$

then

$$\ell_\mu = A^{-1}\delta^1_\mu\,, \quad n_\mu = B^{-1}\delta^4_\mu \tag{6.2}$$

The orthogonality relations (2.8) imply that general expressions for the tetrad vectors are

$$\begin{aligned} \ell^\mu &= (0, Y^2, Y^3, B) \\ n^\mu &= (A, X^2, X^3, 0) \\ m^\mu &= (0, \xi^2, \xi^3, 0) \end{aligned} \tag{6.3}$$

Where Y^i, X^i and ξ^i, (i = 2, 3) are functions of the coordinates to be determined. The ξ^i are complex.

We now follow Szekeres (1972) in the assumption that since the metric in regions I, II and III has no dependence on the x and y coordinates, no such dependence is expected in region IV. Accordingly, the assumption is made that the metric components and spin coefficients are functions of u and v only so that when applied to these quantities the intrinsic differential operators become:

$$
\begin{aligned}
D &\equiv \frac{\partial}{\partial x^\nu} l^\nu &= B\frac{\partial}{\partial v} \\
\Delta &\equiv \frac{\partial}{\partial x^\nu} n^\nu &= A\frac{\partial}{\partial u} \\
\delta &\equiv \frac{\partial}{\partial x^\nu} m^\nu &= 0
\end{aligned}
\tag{6.4}
$$

The assumption above is justified in that is successfully leads to exact solutions.

The coordinates may now be related to the tetrad and spin coefficients by substituting the coordinates into the commutation relations of Newman and Penrose to give the metric equations as follows

$$
\begin{aligned}
&DA = -(\epsilon+\bar{\epsilon})A \\
&\Delta B = (\gamma+\bar{\gamma})B \\
&\Delta Y^i - DX^i = (\gamma+\bar{\gamma})Y^i + (\epsilon+\bar{\epsilon})X^i - 4\alpha\xi^i - 4\bar{\alpha}\bar{\xi}^i \\
&D\xi^i = (\rho+\epsilon-\bar{\epsilon})\xi^i + \sigma\bar{\xi}^i \\
&\Delta\xi^i = -(\mu+\bar{\gamma}-\gamma)\xi^i - \bar{\lambda}\bar{\xi}^i \\
&i = 2,3 \quad \& \quad \kappa=\nu=0,\ \rho=\bar{\rho},\ \mu=\bar{\mu},\ \beta=\bar{\alpha},\ \bar{\tau}=\pi=2\alpha
\end{aligned}
\tag{6.5}
$$

Szekeres introduces the redefined quantities which are invariant under scale transformations of the form (2.10) for T = S = 0. These are

$$
\begin{aligned}
&\rho^\circ = \rho B^{-1},\quad \sigma^\circ = \sigma B^{-1},\quad \mu^\circ = \mu A^{-1},\quad \lambda^\circ = \lambda A^{-1}, \\
&E = i(\bar{\epsilon}-\epsilon)B^{-1},\ G^\circ = i(\bar{\gamma}-\gamma)A^{-1},\ \alpha^\circ = \alpha\,(AB)^{-1/2}, \\
&Y^{\circ i} = Y^i B^{-1},\ X^{\circ i} = X^i A^{-1},\quad \Psi_0^\circ = \Psi_0 B^{-2}, \\
&\Psi_1^\circ = \Psi_1 A^{-1/2} B^{-3/2},\ \Psi_2^\circ = \Psi_2 (AB)^{-1},\ \Psi_3^\circ = \Psi_3 B^{-1/2} A^{-3/2}, \\
&\Psi_4^\circ = \Psi_4 A^{-2},\ \Phi_{00}^\circ = \Phi_{00} B^{-2},\ \Phi_{01}^\circ = \Phi_{01} A^{-1/2} B^{-3/2}, \\
&\Phi_{02}^\circ = \Phi_{02} (AB)^{-1},\ \Phi_{11}^\circ = \Phi_{11} (AB)^{-1},\ \Phi_{12}^\circ = \Phi_{12} B^{-1/2} A^{-3/2}, \\
&\Phi_{22}^\circ = \Phi_{22} A^{-2}
\end{aligned}
\tag{6.6}
$$

The first two metric equations can be written

$$
\begin{aligned}
(\epsilon+\bar{\epsilon}) &= -B\,(\operatorname{Log} A)_{,v} \\
(\gamma+\bar{\gamma}) &= A\,(\log B)_{,u}
\end{aligned}
\tag{6.7}
$$

Only scale invariant quantities will be used in future and consequently it is now convenient to drop the invariance indicator $(^{\circ})$. Writing $e^{M}=AB$, the metric and field equations may now be written as follows

$$Y^{i}_{,u}-X^{i}_{,v}=-4e^{-\frac{M}{2}}(\alpha\xi^{i}+\bar{\alpha}\bar{\xi}^{i}) \qquad (6.8)\ (a)$$

$$\xi^{i}_{,v}=(\rho+iE)\xi^{i}+\sigma\bar{\xi}^{i} \qquad (b)$$

$$\xi^{i}_{,u}=-(\mu-iG)\xi^{i}-\bar{\lambda}\bar{\xi}^{i} \qquad (c)$$

$$\rho_{,v}=\rho^{2}+\sigma\bar{\sigma}-\rho M_{,v}+\Phi_{00} \qquad (d)$$

$$\rho_{,u}=-2\rho\mu-4\alpha\bar{\alpha}-\Phi_{11} \qquad (e)$$

$$\mu_{,v}=2\rho\mu+4\alpha\bar{\alpha}+\Phi_{11} \qquad (f)$$

$$\mu_{,u}=-\mu^{2}-\lambda\bar{\lambda}-\mu M_{,u}-\Phi_{22} \qquad (g)$$

$$\sigma_{,v}=\sigma(2\rho-M_{,v}+2iE)+\Psi_{0} \qquad (h)$$

$$\sigma_{,u}=-\sigma(\mu-2iG)-\rho\bar{\lambda}-4\bar{\alpha}^{2}-\Phi_{02} \qquad (i)$$

$$\lambda_{,v}=\lambda(\rho-2iE)+\mu\bar{\sigma}+4\alpha^{2}+\Phi_{20} \qquad (j)$$

$$\lambda_{,u}=-\lambda(2\mu+M_{,u}+2iG)-\Psi_{4} \qquad (k)$$

$$\alpha_{,v}=\alpha(3\rho-\tfrac{1}{2}M_{,v}-iE)+\bar{\alpha}\bar{\sigma}+\Phi_{10} \qquad (l)$$

$$\alpha_{,u}=-\alpha(3\mu+\tfrac{1}{2}M_{,u}+iG)-\bar{\alpha}\lambda-\Phi_{21} \qquad (m)$$

$$\frac{1}{2}M_{,uv}=-\frac{1}{2}i(G_{,v}-E_{,u})+12\alpha\bar{\alpha}+\rho\mu-\sigma\lambda+2\Phi_{11} \qquad (n)$$

$$\Psi_1 = 2\rho\bar{\alpha} - 2\sigma\alpha + \Phi_{01} \qquad (o)$$

$$\Phi_2 = \rho\mu - \sigma\lambda + \Phi_{11} \qquad (p)$$

$$\Psi_3 = 2\mu\alpha - 2\lambda\bar{\alpha} + \Phi_{21} \qquad (q)$$

Using the method of Szekeres (1972) it is possible to choose

$$\xi^2 = e^{(U-V)/2}\{1/2 \cosh W\}^{1/2} e^{i\theta} \qquad (6.9)\ (a)$$

$$\xi^3 = e^{(U+V)/2}\{1/2 \cosh W\}^{1/2} e^{i\phi} \qquad (b)$$

where $\tanh W = \cos(\theta - \phi)$ (c)

using the identity

$$\cosh^2 W/2 + \sinh^2 W/2 = \cosh W$$

Then (6.9) can be written in the form

$$\xi^2 = 2^{-1/2} e^{(U-V)/2}(\cosh W/2 + i \sinh W/2) \qquad (6.10)(a)$$

$$\xi^3 = 2^{-1/2} e^{(U+V)/2}(\sinh W/2 + i \cosh W/2) \qquad (b)$$

Equations (6.8) (b) and (c) now give

$$\rho = \frac{1}{2} U_v \quad , \quad \mu = \frac{1}{2} U_u , \qquad (6.11)\ (a)$$

$$\varepsilon = -\frac{1}{2} V_v \sinh W , \quad G = -\frac{1}{2} V_u \sinh W , \qquad (b)$$

$$\sigma = \frac{1}{2} i W_v - \frac{1}{2} V_v \cosh W , \quad \lambda = \frac{1}{2} i W_u + \frac{1}{2} V_u \cosh W \qquad (c)$$

The expressions (6.11) may now be substituted into equations (6.8) (d) - (g), (i), (j) and (n) to give

$$U_{uv} = U_u U_v - 8\alpha\bar{\alpha} - 2\,\Phi_{11} \qquad (6.12)\ (a)$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2 \cosh^2 W - 2U_v M_v + 4\Phi_{00} \qquad (b)$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 \cosh^2 W - 2U_u M_u + 4\Phi_{22} \qquad (c)$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W + 8i(\bar{\alpha}^2 - \alpha^2)$$
$$+ 2i(\Phi_{02} - \Phi_{20}) \qquad (d)$$

$$(2V_{uv} - U_u V_v - U_v V_u)\cosh W + 2(V_v W_u + V_u W_v)\sinh W$$
$$= 8(\bar{\alpha}^2 + \alpha^2) + 2(\Phi_{02} + \Phi_{20}) \qquad (e)$$

$$2M_{uv} = -U_u U_v + W_u W_v + V_u V_v \cosh^2 W + 48\alpha\bar{\alpha} + 8\Phi_{11} \quad (f)$$

It should be noted here that equations (6.12) (d) and (e) are integrability conditions for the remaining equations. This means that if suitable expressions are obtained for V and W, satisfying these integrability conditions, then a function M exists satisfying the equations which remain.

Equations (6.8) (k), (h), (o), (p) and (q) may be used to compute the curvature components as follows

$$\Psi_0 = -\frac{1}{2}\left\{V_{vv}\cosh W + 2V_v W_v \sinh W - V_v(U_v - M_v)\cosh W\right\}$$
$$+\frac{i}{2}\left\{W_{vv} - W_v(U_v - M_v) - V_v^2 \cosh W \sinh W\right\} \qquad (6.13)$$

$$\Psi_1 = U_v\bar{\alpha} - \alpha(iW_v - V_v\cosh W) + \Phi_{01} \qquad (6.14)$$

$$\Psi_2 = \frac{1}{2}M_{uv} - 12\alpha\bar{\alpha} - \Phi_{11} - \frac{i}{4}(V_u W_v - V_v W_u)\cosh W$$

$$----(6.15)$$

$$\Psi_3 = \Phi_{21} - U_u\alpha - \bar{\alpha}(iW_u + V_u\cosh W) \qquad (6.16)$$

$$\Psi_4 = -\frac{1}{2}\{V_{uu}\cosh W + 2V_uW_u\sinh W - V_u(U_u - M_u)\cosh W\}$$

$$-\frac{i}{2}\{W_{uu} - W_u(U_u - M_u) - V_u^2\cosh W\sinh W\} \qquad (6.17)$$

Szekeres' approach was to notice that since in region I all spin coefficients are zero, then equations (6.8) (l) and (m) with Φ_{01} and Φ_{21} both zero imply $\alpha = 0$ everywhere. Available transformations can then be used to put X^i and Y^i simultaneously zero. When this is the case the metric takes the Rosen form

$$ds^2 = 2e^{-M}du\,dv - e^{-U}(e^{V}\cosh W dx^2 + e^{-V}\cosh W dy^2 - 2\sinh W dx\,dy). \quad (6.18)$$

With the metric in this form, the conditions of Lichnerowicz and Darmois are satisfied if U, V, W and M are continuous and have continuous first partial derivatives across the regional junctions.

The O'Brien-Synge conditions differ, however, and when applied require continuity of U, V, W and M along with a continuous first partial derivative of U only, across regional junctions (Robson 1973).

7 EXACT VACUUM SOLUTIONS

In this thesis we are concerned with reviewing the known exact solutions rather than with the techniques by which they were derived. To this end, the known solutions will generally be considered in detail only with regard to their structure in the interaction region for which the Rosen metric coefficients will be given in each case.

When dealing with vacuum spacetimes we note the simplifying requirement

$$\Lambda = \Phi_{AB} = 0$$

§ 7.1 Colliding Gravitational Waves with Colinear Polarization

Comparing the metric (6.18) with (3.16) then

$$\sinh W = \frac{\alpha\beta + \gamma\delta}{\delta\beta - \gamma\alpha} \tag{7.1}$$

According to the discussion in Chapter 3, in the case of constant polarization, it is always possible to put (7.1) identically equal to zero $(i.e.\ \beta = \delta = 0)$ by the transformation (3.11).

Colliding waves with constant colinear polarization are therefore also characterized by the fact that a suitable coordinate rotation can always be made such that W = 0 in the wave regions II and III simultaneously. Accordingly, in this case, since W = 0 on the boundaries u = 0, V = 0 then W must be zero throughout the interaction region also.

The field equations then reduce to

$$U = -\log_e\left(f(u) + g(v)\right) \tag{7.2)(a}$$

$$2U_{uu} - U_u^2 + 2U_u M_u = V_u^2 \tag{b}$$

$$2U_{vv} - U_v^2 + 2U_v M_v = V_v^2 \tag{c}$$

$$2M_{uv} + U_{uv} - V_u V_v = 0 \tag{d}$$

$$2V_{uv} - U_u V_v - U_v V_u = 0 \tag{e}$$

where f(u) and g(v) are arbitrary functions and 7.2 (a) has been obtained by integrating equation (6.12) (a).

The scale invariant curvature components become

$$\Psi_0 = -\frac{1}{2}\left[V_{vv} - V_v(U_v - M_v)\right] \qquad (7.3)\ (a)$$

$$\Psi_4 = -\frac{1}{2}\left[V_{uu} - V_u(U_u - M_u)\right] \qquad (b)$$

$$\Psi_2 = \frac{1}{2}M_{uv}\,, \quad \Psi_1 = \Psi_3 = 0 \qquad (c),(d)$$

§ 7.2 The Solution of Khan and Penrose

The first solution of the vacuum equations (7.2) was given by Khan and Penrose (1971). It represents specifically the collision of two impulsive gravitational waves with profiles given by (cf. § 3.3)

$$g(u) = \delta(u)$$

in region II and

$$g(v) = \delta(v)$$

in region III

In Rosen form, the solution in the interaction region is given by

$$e^{-U} = 1 - u^2 - v^2 \qquad (7.4)\ (a)$$

$$e^{-V} = \frac{\{(1-v^2)^{1/2} - u\}\{(1-u^2)^{1/2} - v\}}{\{(1-v^2)^{1/2} + u\}\{(1-u^2)^{1/2} + v\}} \qquad (b)$$

$$e^{M} = \frac{(1-u^2)^{1/2}(1-v^2)^{1/2}\{uv + (1-u^2)^{1/2}(1-v^2)^{1/2}\}^2}{(1-u^2-v^2)^{3/2}} \qquad (c)$$

The scale invariant curvature components are given in the various regions as

Region I: $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$

Regions II and III: all components are zero except on the boundaries. These are described by

$$v \geq 0, \quad u < 0 \quad : \Psi_0 = \delta(v) \tag{7.5 (a)}$$

$$u \geq 0, \quad v < 0 \quad : \Psi_4 = \delta(u) \tag{b}$$

Region IV

$$\Psi_0 = \frac{\delta(v)}{(1-u^2)^{1/2}} + \frac{3u(1-u^2)\{uv+(1-v^2)^{1/2}(1-u^2)^{1/2}\}}{(1-u^2-v^2)^2(1-v^2)} \tag{7.6 (a)}$$

$$\Psi_4 = \frac{\delta(u)}{(1-v^2)^{1/2}} + \frac{3v(1-v^2)\{uv+(1-v^2)^{1/2}(1-u^2)^{1/2}\}}{(1-u^2-v^2)^2(1-u^2)} \tag{b}$$

$$\Psi_2 = \frac{\{uv+(1-u^2)^{1/2}(1-v^2)^{1/2}\}^2 - uv(1-u^2)^{1/2}(1-v^2)^{1/2}}{(1-u^2-v^2)^2(1-u^2)^{1/2}(1-v^2)^{1/2}} \tag{c}$$

It can be seen from the metric (3.18) describing the region II that a singularity is present on the null hypersurface u = 1 (a similar singularity occurs on v = 1 in region III). This is clearly a coordinate singularity since, except for the boundaries, the spacetime is flat. Evaluation of the Weyl polynomial curvature invariants (§2.8) indicates that in region IV, these singularities on the null hypersurfaces u = 1, v = 1 are essential. This is in contradiction to a statement of Griffiths (1976). A further singularity, which is again essential, occurs on the spacelike hypersurface (see Fig. 7.1)

$$f(u) + g(v) = 0$$

and given by

$$u^2 + v^2 = 1$$

This is the hypersurface on which the two waves mutually focus each other. It may be noticed finally that the boundary conditions employed are those of O'Brien and Synge (cf. Chapter 4).

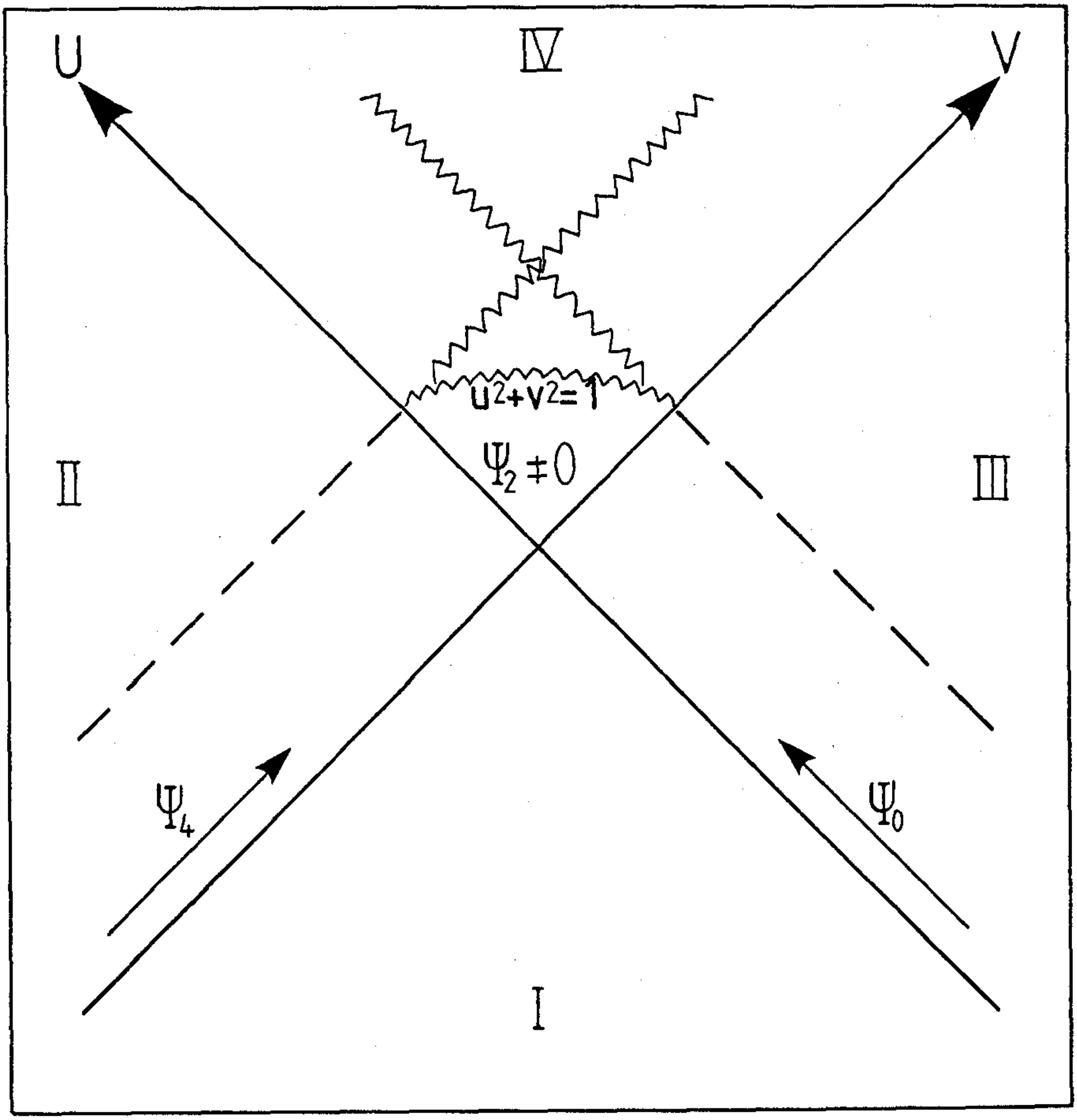

fig 7.1

§ 7.3 The Particular Solution of Szekeres

A solution describing the collision of Gravitational shock waves was later given by Szekeres (1970). Regions II and III are described by metrics of the form (3.19), with u replaced by v in region III. In the interaction region the solution takes the Rosen form

$$e^{-U} = 1 - u^4 - v^4 \qquad (7.7)\ (a)$$

$$V = \sqrt{6}\left\{\tanh^{-1}\left(\frac{1/2-f}{1/2+g}\right)^{1/2} + \tanh^{-1}\left(\frac{1/2-g}{1/2+f}\right)^{1/2}\right\} \qquad (b)$$

where

$$\left.\begin{aligned} f &= \tfrac{1}{2} - u^4 \\ g &= \tfrac{1}{2} - v^4 \end{aligned}\right\} \qquad (c)$$

$$M = \frac{3}{4}\log_e\left[(1-u^4)(1-v^4)\right] + U + 3\tanh^{-1}\left[\frac{u^2v^2}{(1-u^4)^{1/2}(1-v^4)^{1/2}}\right] \qquad \ldots.(d)$$

Substitution of the metric coefficients and appropriate derivatives into (7.3) gives the non vanishing curvature components in the regions as follows:

Region I : All components zero

Region II : $\Psi_4 = \sqrt{6}(1-u^4)^{-2}$ (7.8) (a)

Region III: $\Psi_0 = \sqrt{6}(1-v^4)^{-2}$ (b)

Region IV : $\Psi_4 = \sqrt{6}.R.N.(1-u^4-v^4)^{-2}(1-u^4)^{-3/2}$ (c)

$$\Psi_0 = \sqrt{6}.R.N(1-u^4-v^4)^{-2}(1-v)^{-3/2} \qquad (d)$$

$$\Psi_2 = \frac{2uv\left[3R^2 - 2u^2v^2(1-u^4)^{1/2}(1-v^4)^{1/2}\right]}{(1-u^4-v^4)^2(1-u^4)^{1/2}(1-v^4)^{1/2}} \qquad (e)$$

where

$$R = \left(u^2v^2 + [1-u^4]^{1/2}[1-v^4]^{1/2}\right)$$

$$N = \left(1 - [u^2-v^2]^2 + 8u^2v^2[1-u^4]^{1/2}[1-v^4]^{1/2}\right)$$

On the boundaries we have

for $u = 0,\ v < 0\ :\ \Psi_0 = \Psi_2 = 0\ ,\ \Psi_4 = \sqrt{6}$

for $v = 0,\ u < 0\ :\ \Psi_4 = \Psi_2 = 0\ ,\ \Psi_0 = \sqrt{6}$

for $u = 0,\ v > 0\ :\ \Psi_0 = \Psi_2 = 0\ ,\ \Psi_4 = \sqrt{6}(1-v^4)^{-1/2}$

for $v = 0,\ u > 0\ :\ \Psi_4 = \Psi_2 = 0\ ,\ \Psi_0 = \sqrt{6}(1-u^4)^{-1/2}$

The solution in the interaction region is qualitatively similar to that of Khan and Penrose in that essential singularities occur on the null hypersurfaces u = 1, v = 1, and on the spacelike hypersurface

$$u^4 + v^4 = 1$$

The polynomial curvature scalars remain finite in the wave regions II and III as discussed in section (5.2) and consequently, the singularities on u = 1, v = 1 in region IV, do not extend back into these regions.

The appearance of the coulomb component Ψ_2 in the interaction region for this solution and that of Khan and Penrose is not unexpected as indicated in section (5.2) also. Physically it is interpreted as a consequence of the scattering of the waves due to the collision. The various regions are represented by the diagram in Fig. 7.2.

In this case, the boundary conditions of Lichnerowicz are satisfied.

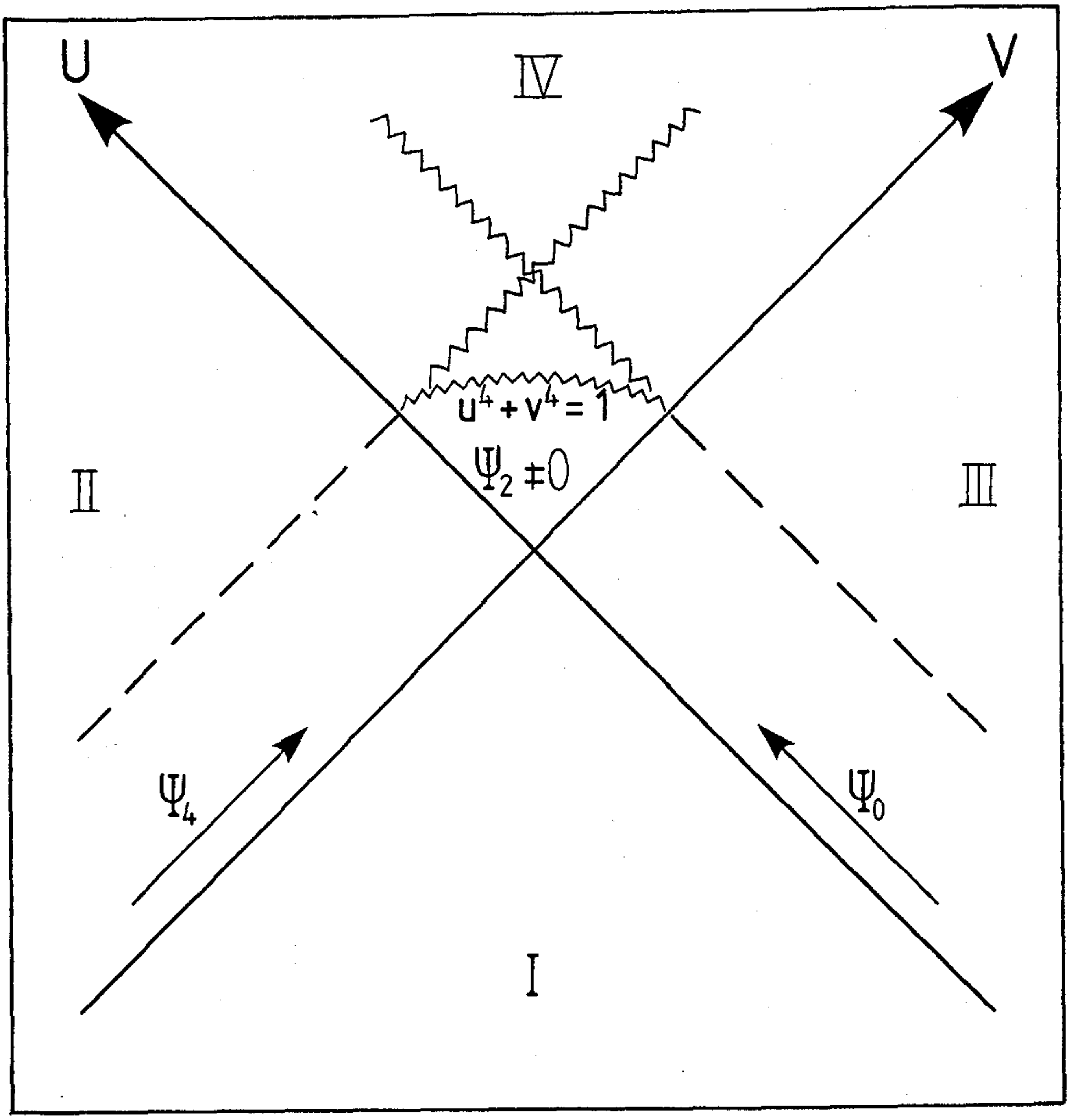

fig 7.2

§ 7.4 The Generalised Solution of Szekeres

Szekeres (1972) later generalised the initial solution given by (7.7) as follows:

With collinear polarization the field equations relevant are those of section (7.1) with

$$U = -\log_e\left(f(u) + g(v)\right) \qquad (7.9)\,(a)$$

The integrability equation (6.12) (e) can now be solved for some V, and to do this it is convenient to change the variables of V as follows treating $V = V(f, g)$

$$V_u = \frac{\partial V}{\partial f}\frac{df}{du}\ , \qquad V_v = \frac{\partial V}{\partial g}\frac{dg}{dv} \qquad (b)$$

$$V_{uv} = \frac{\partial^2 V}{\partial f \partial g}\frac{df}{du}\frac{dg}{dv} \qquad (c)$$

Equation (6.12)(e) becomes using (7.9) (a), (b), and (c)

$$2(f+g)V_{fg} + V_f + V_g = 0 \qquad (7.10)$$

The Euler-Darboux equation.

Although this is essentially the approach adopted by Szekeres to obtain the particular solution (7.7), in this later work he gives a general solution of (7.10) as

$$V(f,g) = -(f+g)^{-1/2}\left[\int_f^{1/2} P_{-1/2}\left(1 + \frac{2(\xi - f)(\frac{1}{2} - g)}{(\xi + \frac{1}{2})(f+g)}\right)\frac{d}{d\xi}\left(\sqrt{\tfrac{1}{2}+\xi}\, V_1(\xi)\right)d\xi\right.$$

$$\left. + \int_g^{1/2} P_{-1/2}\left(1 + \frac{2(\frac{1}{2} - f)(\eta - g)}{(\eta + 1/2)(f+g)}\right)\frac{d}{d\eta}\left(\sqrt{\tfrac{1}{2}+\eta}\, V_2(\eta)\right)d\eta\right] \qquad (7.11)$$

The starting functions $V_1(f)$ and $V_2(g)$ determine the waves up to the region of interaction.

Unfortunately, as Szekeres has pointed out, it is very difficult to perform the integrations in (7.11). Instead he gives a solution obtained by trial and error but with the appropriate behaviour at $f=\frac{1}{2}$ and $g=\frac{1}{2}$. This is given by:

$$V(f,g) = k_1 \tanh^{-1}\left\{\frac{\frac{1}{2}-f}{\frac{1}{2}+g}\right\}^{1/2} + k_2 \tanh^{-1}\left\{\frac{\frac{1}{2}-g}{\frac{1}{2}+f}\right\}^{1/2} \qquad (7.12)$$

where now $f = \frac{1}{2} - (au)^{n_1}\,\theta(u)$

$$g = \frac{1}{2} - (bv)^{n_2}\,\theta(v)$$

and $n_i \geq 1$

U is given directly from equation 6.12 (a), thereby allowing equations (6.12) (b) and (c) to be explicitly integrated giving*

$$M = \left[1 - k_1k_2 - \tfrac{1}{4}(k_1-k_2)^2\right]\log_e\left(1 - a^{n_1}u^{n_1} - b^{n_2}v^{n_2}\right)^{1/2}$$

$$+\frac{1}{4}k_1^2\log_e\left(1-b^{n_2}v^{n_2}\right)^{1/2} + \frac{1}{4}k_2^2\log_e\left(1-a^{n_1}u^{n_1}\right)^{1/2}$$

$$+\frac{1}{2}k_1k_2\log_e\left[\left(a^{n_1}b^{n_2}u^{n_1}v^{n_2}\right)^{1/2} + \left(1-a^{n_1}u^{n_1}\right)^{1/2}\left(1-b^{n_2}v^{n_2}\right)^{1/2}\right] \qquad \text{.....}\ (7.13)$$

The general solution for the interaction region, in Rosen form can be expressed by the metric coefficients:

$$e^{-U} = 1 - a^{n_1}u^{n_1} - b^{n_2}v^{n_2} \qquad (7.14)\ (a)$$

---

* Using the relationship

$$\tanh^{-1}x = \frac{1}{2}\log\frac{(1+x)}{(1-x)} \quad \text{for } x<1$$

$$e^{V} = \left[\frac{\{1-(bv)^{n_2}\}^{1/2} + (au)^{n_1/2}}{\{1-(bv)^{n_2}\}^{1/2} - (au)^{n_1/2}}\right]^{k_1/2} \cdot \left[\frac{\{1-(au)^{n_1}\}^{1/2} + (bv)^{n_2/2}}{\{1-(au)^{n_1}\}^{1/2} - (bv)^{n_2/2}}\right]^{k_2/2}$$

..... (b)

$$e^{M} = [1-(au)^{n_1} - (bv)^{n_2}]^{\alpha} . [(1-b^{n_2}v^{n_2})^{k_1^2} . (1-a^{n_1}u^{n_1})^{k_2^2}]^{1/8}$$

$$\times [(a^{n_1}u^{n_1}b^{n_2}v^{n_2})^{1/2} + (1-a^{n_1}u^{n_1})^{1/2} . (1-b^{n_2}v^{n_2})^{1/2}]^{\frac{k_1 k_2}{2}}$$

.... (c)

where $\alpha = \frac{1}{2}[1 - k_1 k_2 - \frac{1}{4}(k_1 - k_2)^2]$ .... (d)

By considering the metric in region II (or III) the expressions simplify to functions of u only (or v only). Equation (6.12) (b) may be used to show that

$$k_i^2 = 8(1 - 1/n_i) \qquad (7.15)$$

$$(i = 1,2)$$

It may be noticed also that for $k_1 = k_2 = -2$, $n_1 = n_2 = 2$ and with $a = b = 1$, the solution reduces to that of Khan and Penrose for impulsive gravitational waves given in section (7.2). With $k_1 = k_2 = -\sqrt{6}$ and $n_1 = n_2 = 4$, the solution reduces to the initial solution of Szekeres given in section (7.3) describing the collision of gravitational shockwaves.

Although (7.12) is an explicit solution of (7.10), it only describes a particular class of collinear collisions for incoming waves, defined in region II by

$$V(f) = k_1 \tanh^{-1}(\tfrac{1}{2} - f)^{1/2}$$

In the Brinkmann form (3.12), these have profiles[†]

$$g(u') = -\frac{\frac{k_1}{8}\left(\frac{k_1^2}{4}-1\right)}{(\tfrac{1}{2}+f)(\tfrac{1}{2}-f)^{3/2}}\left(\frac{\tfrac{1}{2}-f}{\tfrac{1}{2}+f}\right)^{k_1^2/4} \qquad (7.16)$$

and in order that the metric be appropriately behaved through regions I and II, it turns out that f is not arbitrary. It is therefore not possible to choose an appropriate profile $g(u')$ from which, in principle, f may be deduced*. Instead f is necessarily a continuously decreasing function from $f(0)=\frac{1}{2}$. In a similar way g also decreases and the essential singularity which occurs on $f+g=0$ is inevitable.

† Equation (7.16) does not reduce to the expression given by Halil (1979) which is incorrect.

* Thus only specific forms of $g(u')$ are included in this family of solutions. In particular the rectangular shock fronted profile described by the metric (3.19) is not included in (7.16).

§ 7.5 The Solution of Nutku and Halil

A solution, describing the collision of impulsive gravitational waves with non collinear, but constant polarization has been given by Nutku and Halil (1977). The technique used to generate this solution involves transforming the field equations into a form that is similar to those for axially symmetric gravitational fields. This similarity has also been pointed out by Fischer (1980). These equations are then solved using Ernst's (1968) technique before retransformation to the appropriate form for colliding plane waves.

The solution takes the following Rosen form in the interaction region:

$$e^{-U} = 1 - u^2 - v^2, \quad \sinh W = -\frac{i(\eta - \bar{\eta})}{1 - \eta\bar{\eta}},$$

$$e^{V} = \sqrt{\frac{(1-\eta)(1-\bar{\eta})}{(1+\eta)(1+\bar{\eta})}}, \tag{7.17}$$

$$e^{-M} = \frac{(1-\eta\bar{\eta})(1-u^2-v^2)^{-1/2}}{(1-u^2)^{1/2}(1-v^2)^{1/2}},$$

where $\eta = e^{i\alpha} u\sqrt{(1-v^2)} + e^{i\beta} v\sqrt{(1-u^2)}$

and $(\alpha-\beta)$ is a measure of the relative polarization. It may be noticed that from (7.17) with $\alpha=\beta=0$, ($\sinh W = 0$) the solution reduces to that of Khan and Penrose (1971) for a collinear collision.

Accordingly the solution is interpreted as the collision of two impulsive gravitational waves, initially with constant polarization but with polarization vectors inclined at an angle of $(\alpha-\beta)$. The polarization vectors may be thought of as unit vectors directed along the major axes of the ellipsoids formed by shearing congruence bundles in regions II and III. α and β are the angles which these vectors make with a coordinate system in the two space orthogonal to the direction of propogation of the waves.

This solution has been further generalised by Halil as discussed in

§ 7.6. The singularity structure can be deduced from the discussion given there.

§ 7.6 The Solution of Halil

Utilizing the same technique as in § 7.5 Halil has obtained solutions which generalize the results of Szekeres (1972) to include constant relative polarization. The solution is given by (Halil 1979)

$$e^{-U} = 1 - p^2 - q^2$$

$$e^{-M} = \frac{(1-|\eta|^2)^{K^2}(1-p^2-q^2)^{-1/2}}{(1-p^2)^{K^2/2}(1-q^2)^{K^2/2}}$$

$$\sinh W = \left[\frac{\eta^K + \bar{\eta}^K}{4|\eta|^K}\right]\cdot\left[\left(\frac{1+|\eta|}{1-|\eta|}\right)^K - \left(\frac{1-|\eta|}{1+|\eta|}\right)^K\right] \tag{7.18}$$

$$e^{2V} = \frac{N^+(2i|\eta|^K + \eta^K - \bar{\eta}^K) + N^-(2i|\eta|^K - \eta^K + \bar{\eta}^K)}{N^+(2i|\eta|^K - \eta^K + \bar{\eta}^K) + N^-(2i|\eta|^K + \eta^K - \bar{\eta}^K)}$$

where

$$N^{\pm} = (1 \pm |\eta|)^{2K}$$

$$p = u^n \theta(u)$$

$$q = v^n \theta(v)$$

and

$$\eta = e^{i\alpha} p(1-q^2)^{1/2} + e^{i\beta} q(1-p^2)^{1/2}$$

$$K^2 = 2 - 1/n$$

K relates the generalized solution above with the Szekeres class through (7.15) by $2K = k_i$ and $2n = n_i$.

The expressions generated when these metric coefficients are substituted into the field equations are exceedingly complicated and the author of this thesis makes no claim to have been successful in verifying the result in this way. However, the solution clearly reduces to the appropriate collinearly polarized results of Khan and Penrose (1971), (i.e. $\alpha=\beta=\pi/2$, $K=n=1$) and Szekeres (1972), (i.e. $\alpha=\beta=\pi/2K, n=1, K=\sqrt{\frac{3}{2}}$)

This being the case, and until it can be shown otherwise (using computer techniques for example) we shall assume, for the purpose of continuity, that the solution satisfies the field equations.

In the more general case for $\alpha=\beta=\pi/2K$, the solution can be written in the reduced form

$$
\begin{aligned}
e^{-U} &= 1-p^2-q^2 \\
e^{-M} &= \frac{(1-p^2-q^2)^{2K^2-1/2}\,(1-p^2)^{-K^2/2}\,(1-q^2)^{-K^2/2}}{\left(1-p^2-q^2+2p^2q^2+2pq\sqrt{1-p^2}\sqrt{1-q^2}\right)^{K^2}} \\
e^{V} &= \left(\frac{\sqrt{1-q^2}+p}{\sqrt{1-q^2}-p}\right)^K\left(\frac{\sqrt{1-p^2}+q}{\sqrt{1-p^2}-q}\right)^K \\
W &= 0
\end{aligned}
\tag{7.19}
$$

which is a Szekeres (1972) solution in which $n_1=n_2=2n$, $K_1=K_2=2K$ and $a=b=1$

Further generalizations, which include relative polarization and for which $K_1\neq K_2$, have not yet been found.

The curvature components Ψ_0 and Ψ_4 given by Halil, unfortunately do not reduce to the appropriate expressions given by Szekeres (1972) and are therefore incorrect. It is, however, understandable that such errors might arise in view of the complexity of the calculations involved.

Halil also claims that in addition to the singularities which arise in the Szekeres (1972) solutions, further irregularities appear on the hypersurfaces $|\eta|=0$ and $1-|\eta|^2=0$ which depend on the relative polarization angle thus:

$$\left.\begin{aligned} |\eta| = 0 \quad &\Rightarrow p^2+q^2-2p^2q^2 = -2pq\sqrt{1-p^2}\sqrt{1-q^2}\cos(\alpha-\beta) \\ 1-|\eta|^2 = 0 \quad &\Rightarrow 1-p^2-q^2+2p^2q^2 = \ 2pq\sqrt{1-p^2}\sqrt{1-q^2}\cos(\alpha-\beta) \end{aligned}\right\} \quad (7.20)$$

The minus sign on the right hand side of the first equation does not appear in the account given by Halil (1979) on P. 125.

§ 7.7 The Collision of Plane Gravitational Waves with Variable Polarization

Utilizing a method due to Geroch, Panov (1980), has been able to construct new exact solutions describing variably polarized, colliding gravitational waves. However, we have seen fit to express this result in the form of a theorem since it is sufficiently general as to profit from such a restatement.

Theorem

Given any vacuum solution describing an interaction of collinear, constantly polarized gravitational waves, with metric of the form (6.18) where $M = M(u,v)$, $V = V(u,v)$, $W = 0$ and $U = U(u,v)$, then a new solution with metric

$$ds^2 = 2e^{-M'}du\,dv - e^{-U'}\left(e^{V'}\cosh W' dx^2 + e^{-V'}\cosh W' dy^2\right) - 2\sinh W' dx\,dy \qquad (7.21)$$

can be constructed where the new functions

$$M'(u,v) \quad , \quad V'(u,v) \; , \quad U'(u,v) \;\; 4 \;\; W'(u,v)$$

are defined by

$$\begin{aligned} U' &= U \\ e^{-2V'} &= X^2 e^{-2V} + \alpha^2 \sin^2 2\theta \end{aligned} \qquad (7.22)$$

$$e^{-M'} = X e^{-M}$$

$$\sinh W' = e^{V} \alpha X^{-1} \sin 2\theta$$

$$X = \cos^2\theta + e^{2(V-U)} \sin^2\theta$$

where θ is a constant and

$$\alpha = \int_0^u e^{-U}(V_u - U_u)\,du - \int_0^v e^{-U}(V_v - U_v)\Big|_{u=0} dv$$

The non zero curvature components, which are not given by Panov (1980) are defined by:

$$\mathcal{R}e\,\Psi_0' = \frac{1}{\cosh W'}\Big[\Psi_0 + \frac{\cosh^2 W'}{2}\Big\{T_{vv} + T_v\big(Y_v - U_v + M_v - T_v\big)\Big\}$$

$$+ \frac{\sinh^2 W'}{2}\Big\{Y_{vv} - 2\big(V_v - T_v - Y_v \tanh^2 W'\big)Y_v$$

$$- \big(Y_v T_v + A_v[U_v - M_v]\big)\Big\} + \tanh^2 W' . Y_v^2\Big] \qquad (7.23)$$

$$\mathcal{I}m\,\Psi_0' = -\frac{1}{2}\tanh W'\Big[2\Psi_0 - Y_{vv} - Y_v^2 \operatorname{sech}^2 W'$$

$$+ A_v(U_v - M_v) - T_v Y_v$$

$$+ \cosh^2 W'\big(V_v - Y_v \tanh^2 W' - T_v\big)^2\Big] \qquad (7.24)$$

$$Re\,\Psi_4' = \frac{1}{\cosh W'}\left[\Psi_4 + \frac{\cosh^2 W'}{2}\left\{T_{uu} + T_u(V_u - U_u + M_u - T_u)\right\}\right.$$

$$+ \frac{\sinh^2 W'}{2}\left\{Y_{uu} - 2(V_u - T_u - Y_u \tanh^2 W')Y_u\right.$$

$$\left.\left. - (Y_u T_u + A_u[U_u - M_u])\right\} + Y_u^2 \tanh^2 W'\right] \qquad (7.25)$$

$$Im\,\Psi_4' = -\frac{1}{2}\tanh W'\left[2\Psi_4 - Y_{uu} - Y_v^2 \operatorname{sech}^2 W'\right.$$

$$\left. + A_u(U_u - M_u) - T_u Y_u + \cosh^2 W'(V_u - \tanh^2 W' Y_u - T_u)^2\right] \qquad (7.26)$$

$$Re\,\Psi_2' = \Psi_2 - \frac{1}{2} T_{uv} \qquad (7.27)$$

$$Im\,\Psi_2' = -\frac{\sinh W'}{4}\left[\xi_v(V_u - T_u) - \xi_u(V_v - T_v)\right] \qquad (7.28)$$

where

$$T = \log_e X, \quad A = \log_e \frac{\alpha}{X},$$

$$Y = A + V, \quad \xi = \log_e \alpha .$$

It may be noticed that when the polarization parameter θ is chosen to be zero then;

$$T_{vv} = T_{uu} = \sinh W' = \tanh W' = 0 \quad \& \quad \cosh W' = 1$$

and consequently the curvature components reduce to

$$Re\,\Psi_0' = \Psi_0 \quad , \quad Im\,\Psi_0' = 0$$

$$Re\,\Psi_4' = \Psi_4 \quad , \quad Im\,\Psi_4' = 0$$

$$Re\,\Psi_2' = \Psi_2 \quad , \quad Im\,\Psi_2' = 0$$

A coordinate transformation of the form (3.11), when applied to the metric (7.21) in region II, yields the new metric coefficient

$$2dXdY\left[\sin\phi\cos\phi\left(\frac{e^{V-U}}{X} - Xe^{-V-U} - \frac{e^{V-U}\alpha^2\sin^2 2\theta}{X}\right)\right.$$
$$\left. + (\cos^2\phi - \sin^2\phi)(X^{-1}.e^{V-U}.\alpha\sin 2\theta)\right]$$

It is not possible to choose ϕ such that this coefficient is zero for all u. Hence, the gravitational wave in region II must have variable polarization.

Clearly in this solution, singularities qualitatively similar to those found in the generalized Szekeres family of solutions appear. In particular, there is the real singularity occurring on the spacelike hypersurface

$$f+g=0$$

confirming the qualitative results of Sbytov (1976), which predict its presence in spite of arbitrary wave polarizations. Coordinate singularities are again identified in the wave regions II and III where $f=-\frac{1}{2}$, $g=-\frac{1}{2}$ respectively. (cf. § 7.2, § 7.3 and § 7.4).

No other particular or general solutions corresponding to variably polarized, colliding waves have yet been found.

§ 7.8 The Collision of Plane Gravitational Waves without Singularities: An Incorrect Solution

Stoyanov (1979) has asserted that singularity free planewave collisions are possible and in order to demonstrate this he gives the following singularity free "solution".

$$U = -\log_e(f+g)$$

$$V = aU \qquad (7.29)$$

$$M = (a^2-1).\frac{U}{2}\;, \quad W=0$$

where $f = \frac{1}{2} + u\theta(u)$, $g = \frac{1}{2} + v\theta(v)$

and a is a constant. However, it has been pointed out by Nutku (1981) that this metric, in the interaction region is in fact a Kasner solution (Kasner 1921) in a different coordinate system. He has also stated the required coordinate transformation demonstrating this equivalence.

More seriously Nutku (1981) has also pointed out that the matter tensor must be non zero on the boundaries of the interaction region so that Stoyanov's "solution" is not a solution of the vacuum field equations. This criticism has also been discussed by Tipler (1980). Stoyanov's "solution" satisfies neither the Lichnerowicz conditions nor the O'Brien-Synge conditions.

In fact it can be seen that

$$\begin{aligned} U_u &= -\theta(u)\left(1+u\theta(u)+v\theta(v)\right)^{-1} \\ U_v &= -\theta(v)\left(1+u\theta(u)+v\theta(v)\right)^{-1} \end{aligned} \qquad (7.30)$$

and

$$\left.\begin{aligned} 2\Phi_{22} &= -\delta(u) = -R_{11} \\ 2\Phi_{00} &= -\delta(v) = -R_{44} \end{aligned}\right\} \qquad (7.31)$$

(The negative signs in the last equations have been omitted by Nutku (1981)).

The Stoyanov "solution" thus requires null matter with negative energy density to be generated by the collision. In fact it is the presence of this negative energy matter which prevents the occurrence of a singularity, since it induces congruences crossing it to expand rather than contract (see Chapter 5). This "solution" must therefore be dismissed as unphysical.

§ 7.9 The Generation of New Exact Solutions: An Incorrect Theorem

Techniques for generating new exact solutions from those already existing have been given from time to time. Some of these, where relevant, are included in this work (Panov, 1979 and 1980, Nutku and Halil, 1977). However, Ray (1980) has given an incorrect theorem for the generation of new solutions describing the collision of gravitational waves with a relative polarization.

These are derived, according to Ray, by use of either a Szekeres (1972) or Stoyanov (1979) metric. The theorem may be stated thus:

To any colliding planewave metric defined by

$$ds^2 = 2e^{-M'}du\,dv - e^{-U'}(e^{V'}dx^2 + e^{-V'}dy^2) \tag{7.32}$$

there is associated a new solution with $W = W(V) \neq 0$:

$$ds^2 = 2e^{-M}du\,dv - e^{-U}(e^{V}\cosh W dx^2 + e^{-V}\cosh W dy^2)$$
$$- 2\sinh W dx\,dy \tag{7.33}$$

where

$$\left.\begin{aligned} \int \frac{dW}{\cosh W(A^2\cosh^2 W - 1)^{1/2}} &= \pm V \\ \int \frac{A\cosh W dW}{(A^2\cosh^2 W - 1)^{1/2}} &= \pm V' \end{aligned}\right\} \tag{7.34}$$

Thus given a solution of the type (7.32) then apparently with $M = M'$, $U = U'$

and using

$$\left.\begin{aligned} \tanh V &= \cos\alpha \tanh V' \\ \tanh W &= \tan\alpha \sinh V \end{aligned}\right\} \tag{7.35}$$

where $\cos\alpha = A^{-1}$, then a new solution with $W \neq 0$ is generated; (7.35) are obtained by integration of (7.34)

However, this is not the case and as pointed out by Halilsoy (1981), (7.35) are just the conditions required for a coordinate rotation of the form (3.11) with $\theta = \frac{\alpha}{2}$ to diagonalize the metric everywhere. The solution is thus not new but a constant collinear solution disguised by rotated space coordinates defined by the single parameter α (cf. § 7.1). In the case of non-collinear but constant polarization it is not possible to diagonalize the metric in all the regions simultaneously by rotations defined in this way. Instead, incoming waves are described by metrics which may be diagonalized separately by rotations defined by different parameters in regions II and III. It is this feature which distinguishes a non-trivial solution of the type (7.33)

8. EXACT EINSTEIN MAXWELL SOLUTIONS

We consider first the case of the collision of two null electromagnetic plane waves. In terms of fig. 4.1 a wave in region III with propogation vector l^μ is defined, in terms of the Newman Penrose formalism by the scale invariant Maxwell component Φ_0^o. A similar wave in region II, with propogation vector n^μ will be defined by Φ_2^o. Using the collective relations (6.6), the Maxwell scale invariant components may be defined as follows:

$$\begin{aligned}\Phi_0^o &= B^{-1}\Phi_0 \\ \Phi_1^o &= \tfrac{1}{2}(AB)^{-1}\Phi_1 \\ \Phi_2^o &= A^{-1}\Phi_2\end{aligned} \tag{8.1}$$

Since we will use scale invariant quantities only, we again drop the invariance indicator. Maxwell's equations (2.19) reduce to

$$\Phi_{1,v} = (2\rho - \tfrac{1}{2}M_{,v})\Phi_1 \tag{8.2)(a}$$

$$\Phi_{1,u} = -(2\mu + \tfrac{1}{2}M_{,u})\Phi_1 \tag{b}$$

$$\Phi_{2,v} = (\rho - iE)\Phi_2 + 4\alpha\Phi_1 - \lambda\Phi_0 \tag{c}$$

$$\Phi_{0,u} = -(\mu - iG)\Phi_0 - 4\bar{\alpha}\Phi_1 + \sigma\Phi_2 \tag{d}$$

Equations (8.2) (a) and (b) imply that Φ_1 is zero in region IV since it is zero in all the other regions. The Ricci components Φ_{10} and Φ_{21} are consequently zero everywhere and it follows from the field equations (6.8) (l) and (m) that again $\alpha = 0$ throughout the spacetime.

With the above conditions, the metric for the region IV will again be of the form (6.18) and the relevant field equations are those given by (6.12) (a) to (f) in conjunction with the Maxwell equations (8.2) (c) and (d) which become

$$\Phi_{2,v} = \frac{1}{2}(U_v + iV_v \sinh W)\Phi_2 - \frac{1}{2}(iW_u + V_u \cosh W)\Phi_0 \tag{8.3)(a}$$

$$\Phi_{0,u} = \frac{1}{2}(U_u - iV_u \sinh W)\Phi_0 + \frac{1}{2}(iW_v - V_v \cosh W)\Phi_2 \tag{b}$$

§ 8.1 The Solution of Bell and Szekeres

An exact solution, satisfying the field equations discussed above has been given by Bell and Szekeres (1974) and in the interaction region the metric takes the Rosen (6.18) form

$$\begin{aligned} e^{-U} &= \cos(au - bv).\cos(au + bv) \\ e^{V} &= \frac{\cos(au - bv)}{\cos(au + bv)} \\ M &= W = 0 \end{aligned} \tag{8.4}$$

where

$$\Phi_0 = \sqrt{\frac{4\pi}{k}}.b \quad , \quad \Phi_2 = \sqrt{\frac{4\pi}{k}}.a$$

This solution describes the collision of a pair of electromagnetic shock waves in regions II and III described by metrics of the form (3.20) (with suitable choice of coordinate for the argument of the cosine term).

The only non-zero components of the Weyl curvature tensor are

$$\left.\begin{aligned} \Psi_0 &= -\,\delta(v)\,\theta(u)\,b\tan au \\ \Psi_4 &= -\,\delta(u)\,\theta(v)\,a\tan bv \end{aligned}\right\} \tag{8.5}$$

The metric is thus conformally flat in the interiors of all regions but there are δ- function discontinuities in the curvature tensor on the II - IV and III - IV boundaries. These may be interpreted as impulsive gravitational waves generated by the collision. The discontinuities are permitted by use of the O'Brien and Synge junction conditions (cf. Chapter 4). Bell and Szekeres have shown that these discontinuities necessarily arise so that the Lichnerowicz conditions cannot be satisfied in this case.

In region IV both contraction and shear develop and it is reasonable to suppose that the shear is induced by the impulsive gravitational wave and the contraction by both the electromagnetic field and the shear. It might thus be argued that solutions describing colliding electromagnetic waves with more general profiles would, due to the absence of impulsive gravitational waves, remain shear free. However, this turnsout not to be the case as we have argued in Chapter 9 and it seems that pure focussing is prohibited in these collisions as a consequence of the field equations rather than wave profile. Curvature components thus necessarily develop at least on the boundary of the interaction region if not within it.

The solution (8.4) is singular on

$$\pm(au+bv)=\pi/2 \qquad \& \qquad \pm(au-bv)=\pi/2$$

However, except for the impulsive waves on $u=0$, $v=0$ the curvature components are zero and the Maxwell field components are everywhere finite on these hypersurfaces. These singularities are thus coordinate singularities and a coordinate transformation which removes them is given by Bell and Szekeres. (Cf. Fig. 8.1)

The solution in the interaction region IV is therefore a conformally flat, non null Einstein-Maxwell field. Thus according to a theorem of Tariq and Tupper (1974) it must be transformable to the solution of Bertotti and Robinson (Bertotti 1959, Robinson 1959).

A further solution may be obtained trivially from that of Bell and Szekeres by assuming $V=0$ and $W\neq 0$ region IV in which case equation (6.12) (d) is identical in form to that of (6.12) (e). Hence with Φ_0 real and Φ_2 imaginary the solution region IV is

$$e^{-U} = \cos(au-bv)\cdot\cos(au+bv)$$

$$e^{W} = \frac{\cos(au - bv)}{\cos(au + bv)}$$

$$M = V = 0$$

In this case, incoming waves are perpendicularly polarized and the polarization of the induced impulsive gravitational waves is also different. This again will be transformable to the Bertotti-Robinson solution. Indeed it seems likely that for the geometry of the interaction region to deviate from that of Bertotti and Robinson at all, then solutions describing colliding electromagnetic waves with more general profiles must be found. No such generalizations have yet appeared in the literature.

This lack of available solutions is unfortunate since the Bell-Szekeres result is the only example of a singularity-free planewave collision. (However, cf. Chapter 9).

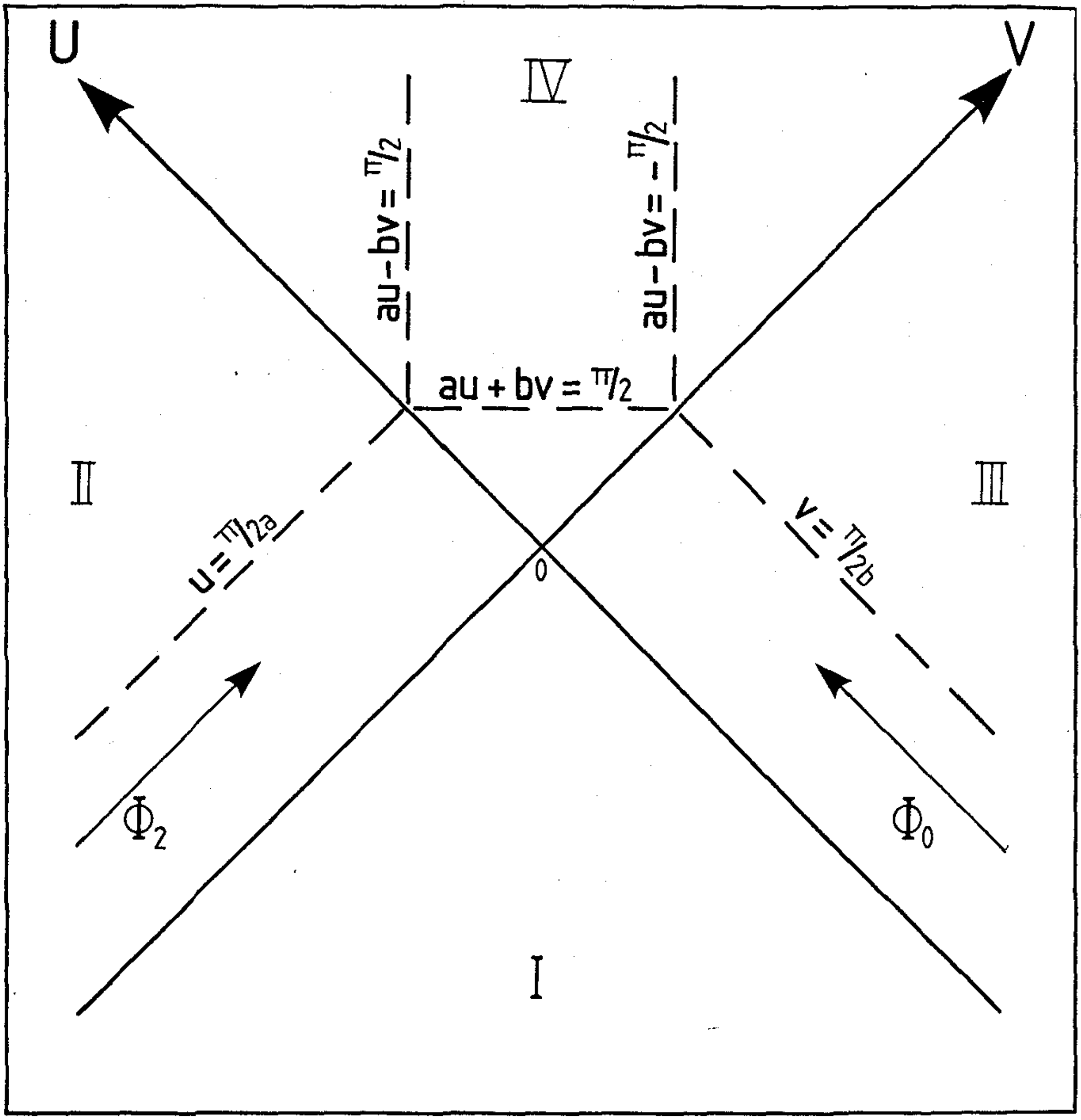

fig 8.1

§ 8.2 The Gürses-Halilsoy Extension

An extension of the Bell-Szekeres solution with some interesting features has been given by Gurses and Halilsoy (1982). Here a number of Bell-Szekeres type shock waves are superposed in a manner which allows the essentially conformal structure of the Bell-Szekeres solution to be retained. Impulsive gravitational waves are similarly generated by the collision(s) and these divide the interaction region into a number of subregions with conformally flat interiors. More precisely, if in region II there is an electromagnetic wave described by the metric (3.5) with $p = 0$ and

$$h(u) = \sum_{i=1}^{M} A_i\,\theta(u-u_i), \qquad (M = \text{number of superpositions})$$

then, after appropriate transformation to Rosen form, the metric may be written as in equation (3.17) with $\alpha = \gamma = \cos P$

where $$P = \sum_{i=1}^{M} a_i\,(u-u_i)\,\theta(u-u_i)$$

such a solution is possible iff

$$u_i > u_j$$

$$A_i = 2a_i \sum_{j=1}^{i} a_j - a_i^2 \qquad (8.6)$$

$$\sum_{i<j} a_i\,(u_j - u_i) = m_j \pi$$

A similar argument applies to region III where the quantities B_i , b_i , Q and N correspond to A_i , a_i , P and M of region II respectively. Q is given by

$$Q = \sum_{i=1}^{N} b_i (v-v_i)\,\theta(v-v_i)$$

These sets of constraints can then be used to show that in region IV the following solution satisfies the Einstein-Maxwell field equations:

$$e^{-U} = \cos(P+Q)\cos(P-Q)$$

$$e^{V} = \frac{\cos(P+Q)}{\cos(P-Q)}$$

$$M = W = 0 \qquad (8.7)$$

$$\Phi_{22} = \sum_{i=1}^{M} A_i\,\theta(u-u_i)$$

$$\Phi_{00} = \sum_{i=1}^{N} B_i\,\theta(v-v_i)$$

The non zero Weyl components are given by

$$\Psi_4 = -\sum_{i=1}^{M} a_i\,\delta(u-u_i)\tan Q$$
$$\Psi_0 = -\sum_{i=1}^{N} b_i\,\delta(v-v_i)\tan P \qquad (8.8)$$

The somewhat idealized solution of Bell and Szekeres has been discussed in the previous section and in connection with singular behaviour it is further discussed in Chapter 9. In addition to each subregion being isometric with the Bell-Szekeres solution, this extension suggests that curious impulsive gravitational wave behaviour can occur when sandwich wave collisions are considered. However, the discussion given in Chapter 9 would suggest that such behaviour is unlikely to occur in a collision involving quasi-rectangular waves with smooth leading and trailing edges. The extension is thus a novel theoretical development but yields no new insight into the electromagnetic collision problem.

§ 8.3 Mixed Field Collisions

Utilizing a method due to Enss (1967), which enables new Einstein-Maxwell solutions to be constructed from existing vacuum solutions, Panov (1979 a and b) has obtained new solutions for colliding waves. These are interpreted as collisions between fields with both gravitational and electromagnetic components. In his first paper (1979 a) he generalizes the Khan and Penrose solution and in the second (1979 b) he generalizes the Szekeres class described in § 7.4. However, in both of these papers the same general solution is used and in keeping with the comments made in § 7.7, which also apply here, it is more appropriate to restate the results of Panov in the theorem below.

Theorem:

Given any vacuum solution describing an interaction ofcollinear, constantly polarized gravitational waves with metric of the form (6.18) where

$$M = M(u,v)\ ,\quad V = V(u,v)\ ,\quad U = U(u,v)\ \ \&\ \ W = 0$$

then a new Einstein-Maxwell solution, given by

$$ds^2 = 2e^{-M'}du\,dv - e^{-U'}(e^{V'}dx^2 + e^{-V'}dy^2)$$

can be constructed in terms of the new functions

$M'(u,v)$, $V'(u,v)$, $U'(u,v)$ and defined by:

$$\begin{aligned} U' &= U \\ e^{-M'} &= e^{-M}X^2 \\ e^{V'} &= e^{V}X^2 \\ X &= \cos^2\theta + e^{-(U+V)}\sin^2\theta \end{aligned} \qquad (8.9)$$

with Maxwell components given by

$$\Phi_0 = i.\sqrt{\frac{4\pi}{k}} \cdot \cot\theta . e^{(U+V)/2} . \frac{X_v}{X}$$

$$\Phi_2 = i.\sqrt{\frac{4\pi}{k}} \cdot \cot\theta . e^{(U+V)/2} . \frac{X_u}{X}$$

The curvature components, not given explicitly by Panov are defined by

$$\Psi_0' = \Psi_0(1-2\xi) + \xi\left[U_{vv} + U_v(V_v+M_v) + (U_v+V_v)^2(3\xi-2)\right]$$

$$\Psi_4' = \Psi_4(1-2\xi) + \xi\left[U_{uu} + U_u(V_u+M_u) + (U_u+V_u)^2(3\xi-2)\right]$$

$$\Psi_2' = \Psi_2 + \xi\left[V_uV_v - V_{uv} + \xi(U_u+V_u)(U_v+V_v)\right]$$

$$\xi = (1 + e^{(U+V)}\cot^2\theta)^{-1}$$

$$\ldots\ldots\ldots \qquad (8.10)$$

This solution describes focussing electromagnetic-gravitational waves. The presence of the electromagnetic field does not destroy the singularity on the hypersurface f + g = 0 which is present in the vacuum solutions, but supplements it. It may be noted that in the limiting case of $\theta = 0$ the solution reduces to the appropriate vacuum metric. The curvature components we have given above also reduce accordingly. The solution does not reduce to that of Bell and Szekeres. Although there appears to be little to distinguish the two papers cited in terms of fundamental content there are differences of emphasis. In particular Panov (1979 a) obtains a generalized solution by appropriate substitution of the Khan and Penrose (1971) (unprimed) metric coefficients in the stated (primed) solution (8.9). He has interpreted this as the collision and interaction of a pair of plane waves consisting of gravitational-electromagnetic pulses on $u=0, v<0$ and $v=0, u<0$ along with plane gravitational radiation on $0<u<1, v<0$ and $0<v<1, u<0$. Furthermore, from this he evaluates asymptotic expressions for the curvature tensor in order to demonstrate the presence of the singularity mentioned above. We might point out that this may be inferred directly from the curvature components we have calculated since combined products of these will always contain unprimed products of components which are known to become infinite on the relevant hypersurfaces.

§ 8.4. Colliding Electromagnetic and Gravitational Waves

Exact solutions for the collision of a gravitational wave with an electromagnetic wave have been given by Griffiths. These have recently been encompassed in a generalization, also due to Griffiths, which is discussed in § 8.5.

In terms of Figure 4.1 and the Newman Penrose formalism, the colliding waves are defined by the scalar components Ψ_4 and Ψ_0 in regions II and II respectively. In particular the collision between an impulsive gravitational wave and a shock electromagnetic wave can be described by the Rosen metric (Griffiths 1975)

$$e^{-U} = \cos^2 bv - a^2u^2$$

$$e^{V} = \frac{1 + au}{1 - au}$$

$$e^{M} = \frac{\left(\cos^2 bv - a^2u^2\right)^{1/2}}{\cos bv\,\left(1 - a^2u^2\right)^{1/2}}$$

$$\Phi_0 = \sqrt{\frac{4\pi}{k}}\;.\; b\cos bv\left(\cos^2 bv - a^2u^2\right)^{-1/2} \tag{8.11}$$

$$\Phi_1 = \sqrt{\frac{4\pi}{k}}\;,\; \frac{-a\sin bv\left(\cos^2 bv - a^2u^2\right)^{-1/2}}{\left(1 - a^2u^2\right)}$$

The waves are described in regions II and III by metrics (3.18) and (3.20) respectively. The solution illustrates the geometrical properties remarked upon in Chapter 5.

After the collision the spin coefficients λ, μ and ρ become non zero and the null congruences associated with both waves experience contraction. In addition the electromagnetic wave shears ($\lambda \neq 0$). The field equations (8.2) (c) and (6.8) (p), with $\rho \neq 0$ and $\lambda \neq 0$ imply that Φ_2 and Ψ_2 are non zero in region IV. Furthermore, since $V = V(u)$ we have from equation (6.13) that Ψ_0 is zero everywhere.

The presence of the Φ_2 term in the interaction region indicates a partial reflection or scattering of the incoming electromagnetic wave. This feature can be predicted from the Mariot Robinson theorem as discussed in Chapter 5. (See also Penrose 1972).

The remaining curvature components are given by

$$\Psi_2 = \frac{-a^2 b u \theta(u)\theta(v) \sin(\theta(v)bv)\cos(\theta(v)bv)}{(\cos^2(\theta(v)bv) - a^2u^2\theta(u))^2} \tag{8.12}$$

$$\Psi_4 = \frac{-a\delta(u)}{(1-a^2u^2\theta(u))} + \frac{3a^3u\theta(u)\, \sin^2(bv\theta(v))}{(1-a^2u^2\theta(u))^2(\cos^2(vb\theta(v)) - a^2u^2\theta(u))}$$

The Maxwell components and the curvature components above, when substituted into the Polynomial Curvature Scalars, define real singularities in the interaction region on the null hypersurface $u = 1/a$ and the spacelike hypersurface $\cos^2 bv - a^2u^2 = 0$ (See Fig. 8.2). The boundary conditions satisfied are those of O'Brien and Synge.

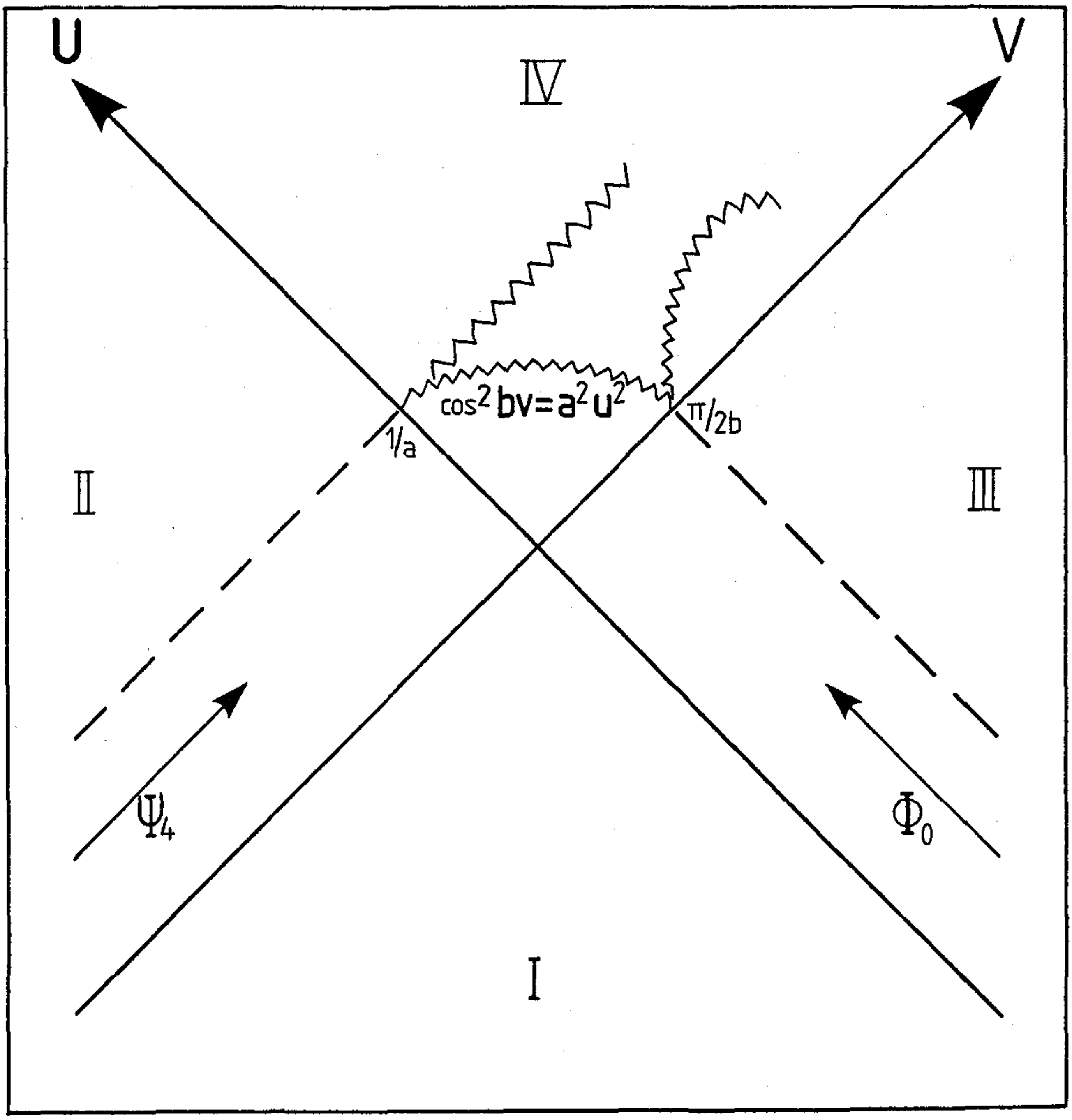

fig 8.2

A particular solution which may be verified by direct substitution into the field equations and describing the collision of electromagnetic and gravitational waves when both have shock profiles has the Rosen form (Griffiths 1976)

$$e^{-U} = \cos au . \cosh au - \sin^2 bv$$

$$e^{V} = \cosh au\,(\cos au)^{-1}$$

$$e^{-M} = \cos bv\,(\cos au \cosh au)^{1/2} e^{U/2} \qquad (8.13)$$

$$\Phi_0 = \sqrt{\frac{4\pi}{k}} . \, b\cos bv \, . \, e^{U/2}$$

$$\Phi_2 = -\frac{a}{2}\sqrt{\frac{4\pi}{k}} . \, \sin bv . (\tan au + \tanh au) . e^{U/2}$$

$$W = 0$$

The metric in regions II and III is defined by (3.19) and (3.20) respectively and these regions are joined to region IV via the Lichnerowicz conditions. The solution (8.13) is quantitively similar to the previous solution (8.11) in that real curvature singularities occur in region IV on $u = \pi/2a$ and on the spacelike hypersurface

$$\cos au \cosh au - \sin^2 bv = 0$$

The non zero curvature components (omitted by Griffiths (1976)) are given by

$$\Psi_2 = \frac{1}{2}\, ab\,\theta(u)\theta(v)\sin(bv\theta(v))\cos(bv\theta(v))\,\Omega$$

$$\Psi_4 = -a^2\theta(u)\left\{1 - \frac{3T^+}{4}\left(T^- - \Omega\right)\right\} \tag{8.14}$$

where

$$\Omega = \frac{\cos(au\theta(u))\sinh(au\theta(u)) - \cosh(au\theta(u))\sin(au\theta(u))}{\cos(au\theta(u))\cosh(au\theta(u)) - \sin^2(bv\theta(v))}$$

$$T^{\pm} = \tanh(au\theta(u)) \pm \tan(au\theta(u))$$

In conjunction with the Maxwell components given in the solution, the Polynomial Curvature Scalars indicate that both singularities in the interaction region are essential.

§ 8.5 A General Solution for the Collision of Electromagnetic and Gravitational Waves

A general solution (Griffiths 1983) for the collision between arbitrary gravitational and electromagnetic waves is now available. In order to comply with the previous particular solutions it is useful to assume that the gravitational wave lies in region II and the electromagnetic wave in the region III. In this way the fields are again described by those Newman Penrose scalar components discussed in § 8.4. It is assumed that, in region IV,

$$V = V(u) \quad \& \quad W = W(u) \tag{8.15}$$

Although this procedure leads to exact solutions, the assumption is not unqualified. The congruences to which the tetrad vector field n^{μ} is tangent will both contract and shear with increasing u-coordinate (cf. Fig. 4.1) i.e. $\mu(u) \neq 0$, $\lambda(u) \neq 0$ and hence in region II we have the functional forms for V and W indicated by the relation (8.15) above. In contrast, the congruence to which ℓ^{μ} is tangent, although developing contraction, remains shear free. The spacetime in region IV is uniquely determined and the assumption that V and W retain the same functional (v-independent) form in the interaction region is then reasonable.

The solution may be written,

$$e^{-U} = f(u) + g(v)$$

$$e^{-M} = \frac{g_v . \left(\frac{1}{2} + f\right)^{1/2}}{\left(\frac{1}{2} - g\right)^{1/2} . \left(f + g\right)^{1/2}} \tag{8.16}$$

$$\Phi_0 = \sqrt{\frac{4\pi}{k}} \cdot \frac{1}{2} . g_v . e^{i\theta} . (f+g)^{-1/2} . \left(\tfrac{1}{2} - g\right)^{-1/2}$$

$$\Phi_2 = \sqrt{\frac{4\pi}{k}} . \frac{1}{2} . \; e^{i\theta} . (V_u \cosh W + i W_u) . (f+g)^{-1/2} . \left(\tfrac{1}{2} - g\right)^{1/2}$$

V and W are chosen to satisfy the (field) equation

$$W_u^2 + V_u^2 \cosh^2 W + 2 f_{uu} \left(\tfrac{1}{2} - f\right)^{-1} - f_u^2 \left(\tfrac{1}{2} + f\right)^{-2} = 0 \tag{8.17}$$

and θ is a function of u only, that must satisfy the equation

$$\theta_u = -\frac{1}{2} V_u \sinh W$$

The electromagnetic field is then determined in terms of the known functions up to an arbitrary constant phase.

The non-vanishing scale invariant components of the Weyl tensor in the region IV are

$$\Psi_4 = -\frac{1}{2}\left(i\,W_{uu} + V_{uu}\cosh W\right) - V_u W_u \sinh W + \frac{1}{2}\,i\,V_u^2 \sinh W \cosh W - \frac{f_u}{4}\left\{\frac{3}{f+g} - \frac{1}{\frac{1}{2}+f}\right\}\left\{i\,W_u + V_u \cosh W\right\} \qquad (8.19)$$

$$\Psi_2 = -\frac{1}{4}\, f_u . g_v . \frac{1}{(f+g)^2} \qquad (8.20)$$

The solution may be interpreted as follows: prior to the collision the incoming gravitational wave is specified by the functions $f(u)$, $V(u)$ and $W(u)$ whilst in region III the electromagnetic wave is specified by the function $g(v)$. After the collision the electromagnetic wave, described by the Φ_0 component, continues through into region IV where the field is additionally described by a Φ_2 component. This develops as a result of the partial reflection of the electromagnetic wave off the gravitational wave. This effect was discussed in Chapter 5. The gravitational wave also continues into the interaction region where the spacetime is additionally described by the development of a Ψ_2 component. The Polynomial curvature scalars indicate that the singularity which develops on the hypersurface

$$f + g = 0$$

is an essential curvature singularity. As remarked upon earlier, the previous particular solutions for this class of collision are included in this generalization when appropriate forms for $f(u)$, $V(u)$, $W(u)$ and $g(v)$ are chosen.

9 DISCUSSION

Following the examination of relevant theoretical details and a discussion of the congruence geometry of null rays, a number of exact vacuum and electrovac solutions have been reviewed. These correspond to the collision and subsequent interaction of various plane waves on a flat background.

The interpretation of these solutions in terms of the focussing effect on the ray congruences in the interaction region and analysis of the Polynomial Curvature Scalars there, indicate that in all cases except that of Bell and Szekeres (cf. § 8.2) and the unphysical solution of Stoyanov, a real curvature singularity is induced by the collision. This occurs on the spacelike hypersurface on which the waves mutually focus each other. Colliding electromagnetic waves, as described by the solution of Bell and Szekeres thus provide the only available example of a singularity free plane wave collision in a non-expanding background.

A number of workers have attempted to isolate those factors which lead to irregularities in the other solutions. For example, Stoyanov (1979) has argued that incoming waves with non smooth wavefronts lead to the development of the characteristic singularity. However, since we have shown in § 7.7 that the solution given by him in support of this conjecture is incorrect, we reject this possibility. In any case, a theorem due to Tipler, which we will give in the next section points to the plane symmetry of the solutions as the factor responsible for the development of the singularities. Unfortunately, it is just this feature which simplifies the field equations such that solutions can be obtained at all. Furthermore, there is no proof, as yet, which suggests that non planar wave collisions will develop singularities and it has been suggested that they may not (Khan and Penrose 1971, Sbytov 1976).

§ 9.1 Singularities and Tipler's Theorem

For the purpose of this work it is necessary only to assume the minimum condition for a singularity free spacetime (Hawking, S. W and Ellis, G. F. R. 1973). In this way if a spacetime is timelike or null geodesically incomplete, it will be assumed to possess a singularity. A manifold is geodesically complete if all geodesics on the manifold are complete and a complete geodesic is one which has an affine parameter r such that r takes all values.

Theorem (Tipler 1980)

Let (M,g) be a spacetime with g at least C^2 and suppose (M,g) has two globally defined commuting spacelike killing vector fields S_1^α and S_2^α which together generate plane symmetry. If (1) the null convergence condition holds; (2) at least one of the six Newman-Penrose quantities Ψ_0, Ψ_4, Φ_{00}, Φ_{44}, σ, λ is non zero at some point p in (M,g); and (3) through the point p there is a spacelike partial Cauchy surface S, which is everywhere tangent to S_1^α and S_2^α and S is noncompact in the spacelike direction normal to S_1^α and S_2^α ; then (M,g) is null incomplete.

Accordingly, it should be expected that any colliding plane-fronted waves, satisfying the condition of the theorem, will necessarily develop singularities either in the past or the future of the collision. It is, however, interesting to note that singularities still occur in the solutions given when in some cases the metric coefficients violate the C^2 requirement. It may be that in some sense this requirement is too restrictive. However, the shock waves in the Bell-Szekeres type solution(s) are directly responsible for the fact that no curvature components develop within region IV and hence it remains singularity free (Note also that $\Psi_2 = \frac{-1}{4}\left(U_v U_u - V_u V_v\right) = 0$ everywhere).

Unfortunately no exact solutions corresponding to colliding electromagnetic waves, with smooth wavefronts is available at present (we have not yet been successful at finding one either) but it seems likely that they too will develop the irregularities predicted by Tipler's theorem. They will at least develop Weyl curvature which may be seen by first assuming that two such smooth fronted waves collide. Equations (6.8)(d), (g), (i), (j) (h), (k) and (p) become:

$$\rho_{,v} = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \qquad (9.1)\ (a)$$

$$\mu_{,u} = -\mu^2 - \lambda\bar{\lambda} - \Phi_{22} \qquad \ldots (b)$$

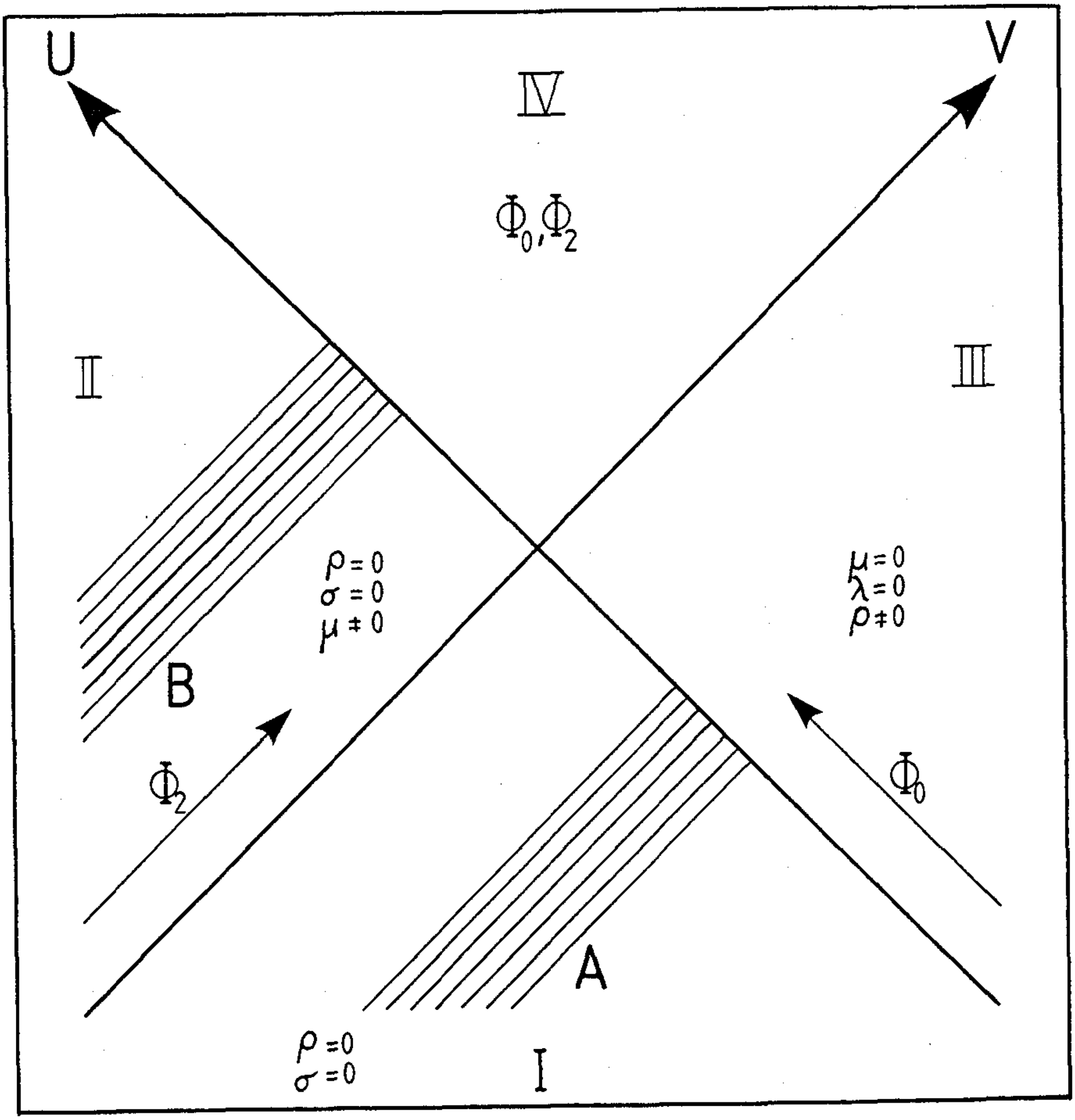
U
V
IV
Φ_0, Φ_2
II
III
$\rho = 0$
$\sigma = 0$
$\mu \neq 0$
$\mu = 0$
$\lambda = 0$
$\rho \neq 0$
B
Φ_2
Φ_0
A
$\rho = 0$
$\sigma = 0$
I

fig 9.1

$$\sigma_{,u} = -\sigma(\mu - 2iG) - \rho\tilde{\lambda} - \tilde{\Phi}_{02} \qquad (c)$$

$$\lambda_{,v} = \lambda(\rho - 2iE) + \mu\bar{\sigma} + \tilde{\Phi}_{20} \qquad (d)$$

$$\sigma_{,v} = 2\sigma(\rho + iE) + \Psi_0 \qquad (e)$$

$$\lambda_{,u} = -2\lambda(\mu + iG) - \Psi_4 \qquad (f)$$

$$\Psi_2 = \rho\mu - \sigma\lambda \qquad (g)$$

The collision may be described using Fig. 9.1. The congruence A is not defined physically by a field and consequently when it extends into region III, from equation (9.1)(a), it contracts but does not develop shear. Note that equation (9.1)(e) implies that $\sigma_{,v} = 2\sigma(\rho + iE)$ and hence by uniqueness $\sigma = 0$ in region III also. However, in region IV, $\tilde{\Phi}_0$ and $\tilde{\Phi}_2$ are non zero and thus from equation (9.1) (a), (d), (b) and (c) we have that ρ, λ, μ and σ become non zero. Therefore congruence B experiences both contraction and shear as it extends into region IV. This necessarily requires the emergence of Weyl curvature, Ψ_0 and Ψ_4, otherwise equation (9.1) (e) and (f) would again give $\sigma = \lambda = 0$ in region IV, contradicting their development described previously. Furthermore equation (9.1) (g) suggests that in general a coulomb component will also arise.

There is thus legitimate reason to suppose that smooth profile solutions will possess curvature which is potentially unbounded in region IV. In the absence of exact solutions we are therefore inclined to accept Tipler's theorem and the associated assertion that plane symmetry is the significant factor in the emergence of singularities.

The fact that the singularities encountered in this thesis occur only in the future is related to the background. (cf. § 9.2). Their time inverse, however, can be interpreted in terms of plane waves emerging from a past singularity.

§ 9.2 Other Related Solutions

A number of other planewave collisions have also been considered. For example Griffiths (1976) has obtained collineary polarized solutions for both colliding neutrino fields and colliding neutrino and electromagnetic waves. Exact solutions describing the collision of neutrino and gravitational waves have not yet been found. Scalar waves have been considered by Wu (1982).

The singularity behaviour and causal structure of expanding vacuum, plane symmetric backgrounds containing gravitational waves have been discussed by Centrella and Matzner (1982). Here oncoming waves propogate along null congruences which share in the expansion of the spacetime and the singularity required by Tipler's theorem occurs in the past of the collision. When the waves interact, the induced shear in turn reduces this expansion which, however, remains essentially positive. In this way focussing is avoided and the induced curvature in the interaction region dies off. This is in contrast to the initially flat space-time arenas of the collisions we have cited. These have null rays with initially vanishing expansion and shear. Any convergence following the development of shear, due to the interaction leads to singularities in the future (unless the interaction region is conformally flat in its interior).

§ 9.3 Summary of Present Knowledge and Further Work To Be Done

The diagram in Fig. 9.2 provides an overview of the current state of knowledge for exact solutions of colliding planewaves. Solutions which appear higher up in the diagram are more general. Arrows indicate where one solution reduces to another and broken arrows where a solution reduces to only the restricted class of another.

Clearly any new solutions corresponding to colliding electromagnetic waves, not transformable to the solution of Bell and Szekeres would be of great importance in clarifying the theory at this stage, especially with regard to Tipler's theorem. Unfortunately, to do this the field equations (6.12e), (8.3a) and(8.3b) must be solved simultaneously subject to the appropriate boundary conditions. We have not yet been successful in obtaining any solutions of these by trial and error methods and it is likely that other techniques from the theory of differential equations may be more appropriate here or, alternatively, a reformulation, perhaps on the lines developed by Fischer (1980).

Collisions of plane gravitational with plane electromagnetic waves are now completely generalized. Still required, however, are more solutions for colliding constant and variably polarized waves and in particular solutions with more realistic wavefronts. Ideally these solutions would reduce, in the appropriate limit to the more general solution of Szekeres (1972) (and include solutions for which $k_1 \neq k_2$).

More ambitious projects could be to include collisions in fluid filled, electromagnetic or expanding backgrounds. Such work has already been initiated (cf. § 9.2). Wainwright (1979) has constructed a gravitational wave pulse in a fluid filled, spatially homogeneous background but no solutions describing collisions of these pulses have been obtained.

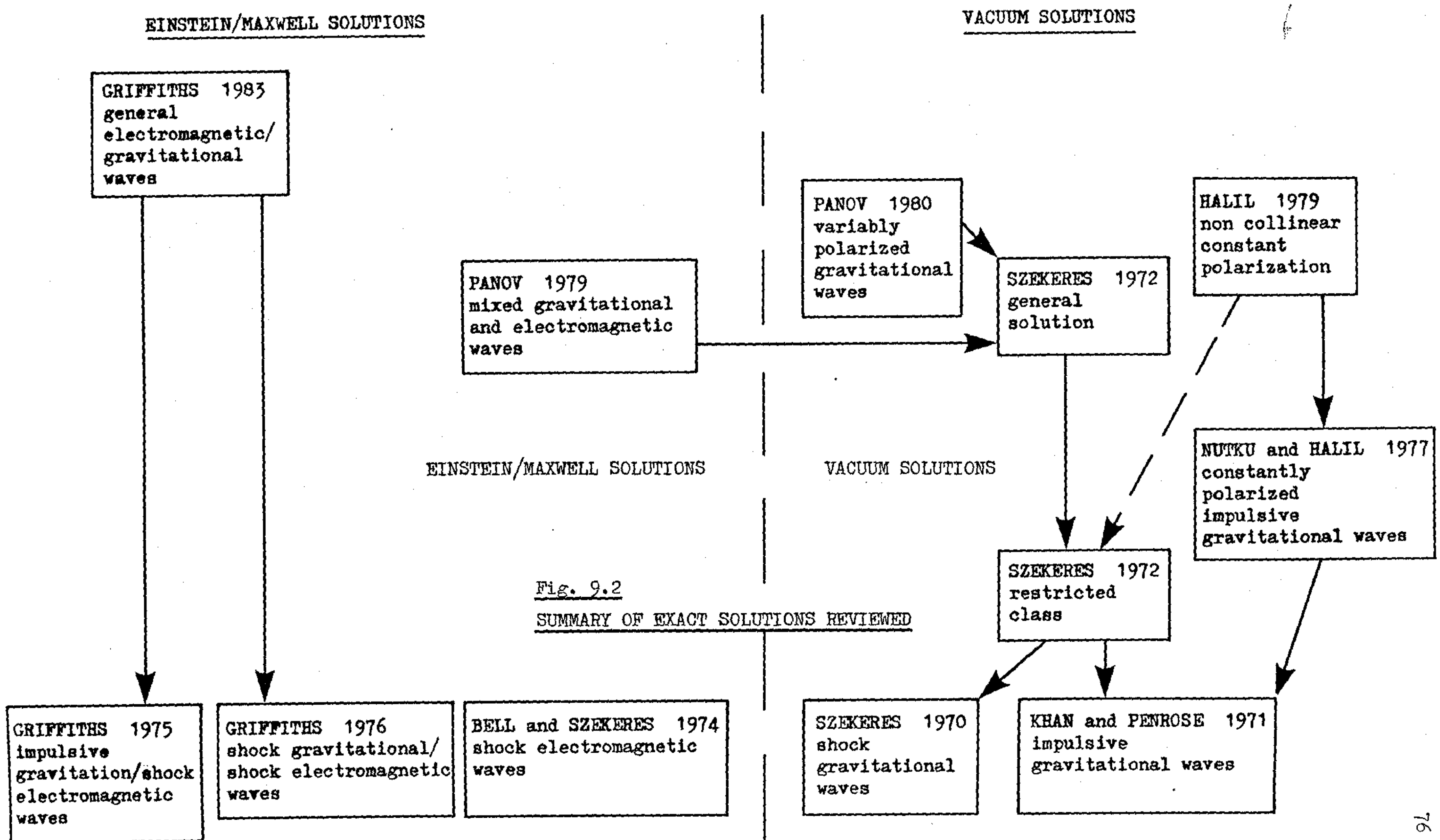

Fig. 9.2

SUMMARY OF EXACT SOLUTIONS REVIEWED

In classical electromagnetic field theory, planewaves illustrate all the properties of realistic wave collisions. However, in the General Theory of Relativity, plane symmetry can give rise to misleading ideas about the structure of the interaction zone following a collision of waves. In particular, the singular structure which develops is unlikely to occur in real collisions where this high degree of symmetry is absent. This inevitably must lead to a search for solutions in which the imposed plane symmetry is relaxed. However, this will require a reformulation of the field equations significantly different than that given in Chapter 6.

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