SIGMA FORM OF THE SECOND PAINLEVÉ HIERARCHY

by

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CERTIFICATE OF ORIGINALITY

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgments or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

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Abstract

The main object of study of the thesis is the second Painlevé hierarchy, which appears as a reduction of the modified Korteweg- de Vries hierarchy.
In the first part we derive some explicit formulas for the corresponding Hamiltonians building on an earlier work by Mazzocco and Mo.
In the second part we derive the sigma form of the second Painlevé hierarchy, which is the main result of the thesis.
These results are applied to the study of the Bäcklund transformations and special solutions of this hierarchy.

Keywords: Sigma form, Painlevé equations, differential equations

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Contents

1 Introduction

The history of the Painlevé equations goes back to the late nineteenth century when Emile Picard [20], proposed the classification problem of finding all second order differential equation of the type

$$\frac{d^2w}{dz^2} = R\left(z, w, \frac{dw}{dz}\right),$$

where the function R is rational in $\frac{dw}{dz}$ and meromorphic in z and w, such that the general solution $w(z, c_1, c_2)$ satisfies the following two properties ([22]):

Painlevé-Kowalevski property: The solution $w(z, c_1, c_2)$ has no *movable critical points,* i.e. singular points other than poles which depend on the initial conditions.

Irreducibility: For generic values of the of the integration constants c_1, c_2 and of the parameters, the solution cannot be expressed in terms of elementary functions or classical transcendental functions.

All second order differential equations of this type (up to Möbius transformations of z and w) that satisfy the Painlevé-Kowalevski property were classified by Painevé and Gambier (see [22] and [5]). Only six of these equations, which are given in the so-called Painlevé-Gambier list[1], satisfy the irreducibility condition:

$$P_I : \ w'' = 6w^2 + z; \tag{1.1}$$

$$P_{II} : \ w'' = 2w^3 + zw + \alpha; \tag{1.2}$$

$$P_{III} : \ w'' = \frac{1}{zw}\left[z(w')^2 - ww' + \delta z + \beta w + \alpha w^3 + \gamma z w^4.\right] \tag{1.3}$$

[1]The last example is actually due to Richard Fuchs (son of the famous mathematician Lazarus Fuchs) who discovered what is nowadays called *sixth Painlevé equation* [4].

$$P_{IV}: \; w'' = \frac{1}{w}\left[\frac{1}{2}(w')^2 + \beta + 2(z^2-\alpha)w^2 + 4zw^3 + \frac{3}{2}w^4\right] \tag{1.4}$$

$$P_V: \; w'' = \left(\frac{1}{zw} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{z} + \delta\frac{w(w+1)}{w-1} \tag{1.5}$$

$$P_{VI}: \; w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \tag{1.6}$$
$$+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right).$$

In the above α, β, γ and δ are parameters taking values in the complex domain. Despite its beauty, this area of mathematics was then forgotten until 1970s when solutions to these equations appeared in several applications: non linear optics, P_{II}, [23]; Hele-Shaw geometry, P_{II}, [25]; random matrix theory, $P_I, \ldots, P_{VI}$, [7]; solutions of Einstein vacuum fields, P_{II}, P_V, [2]; $SU(2)$-Toda field equations, P_{III}, [26]; Einstein-Maxwell fields, P_V, [13]; twistor theory, P_{VI}, [16].

Owing to the proliferation of applications of the Painlevé equations, nowadays their solutions, which are called *Painlevé transcendents,* play the role of non linear special functions. They are so famous that a chapter has been dedicated to them in the Digital Library Project [3, 19] (replacement edition of the famous handbook of special functions by Abramowitz and Stegun).

It is interesting to observe that in most applications to date, the Painlevé equations do not appear in the original forms (1.1) to (1.6) but in the so called sigma forms here below, (1.14) to (1.19). The sigma forms where introduced in the eighties in the seminal papers by Jimbo, Miwa and Ueno ([11]) as the equations satisfied by the Hamiltonians of the Painlevé equations evaluated on the orbits. After that Okamoto systematically studied the Hamiltonian structure of the Painlevé equations and proved that all the Painlevé equations are related to the famous Toda equation through the sigma form ([18], [17], [8], [9]).

This is the reason why it is the sigma forms that appear in the physics literature. Indeed the main object of interest there are the correlation functions of certain system which usually consist of an algebra of operators acting on a function space. In order to be able to find the correlation functions one needs to equip the algebra of operators with some symmetries called Virasoro constraints. Loosely speaking, when there are combinations of the Virasoro constraints which annihilate the Toda equation, the problem of finding the correlation functions can be simplified. In these cases, often the problem simplifies to finding special solutions of the Toda equation, which then reduces to the sigma form of one of the Painlevé equations.

In order to explain the sigma forms of the Painlevé equations, we recall here the Hamiltonian formulation for all the Painlevé equations: using Hamiltonian notation, let us denote the solution to each Painlevé equation by $q(z)$, i.e. $w(z) = q(z)$. The Hamiltonian equations have the following form

$$\delta_k q = \frac{\partial H_k}{\partial p(z)}, \quad \delta_k p = -\frac{\partial H_k}{\partial q(z)}, \qquad k = I, II, III, IV, V, VI$$

where δ_k has different meaning according to which Painlevé equation we consider:

$$\delta_k = \begin{cases} \frac{\partial}{\partial z} & \text{for } k = I, II, IV, \\ z\frac{\partial}{\partial z} & \text{for } k = III, V, \\ z(z-1)\frac{\partial}{\partial z} & \text{for } k = VI, \end{cases}$$

and

$$H_I : \frac{1}{2}p^2 - 2q^2 - qz \tag{1.7}$$

$$H_{II} : \frac{1}{2}p^2 - (q^2 - z/2)p - (\alpha - 1/2)q \tag{1.8}$$

$$H_{III}: \frac{1}{2}[p^2q^2 - (2\eta_0 zq^2 + (2\theta_0+1)q - 2\eta_0 z)p + 2\eta_\infty(\theta_\infty - \theta_0)zq] \tag{1.9}$$

$$H_{IV}: 2qp^2 - [q^2 + 2zq + 2\theta_0]p + \theta_\infty q \tag{1.10}$$

$$H_V: \frac{1}{2}[q(q-1)^2p^2 - [\theta_0(q-1)^2 + \theta_1]q(q-1) - \eta_1 zq]p \tag{1.11}$$

$$+\frac{1}{4}[(\theta_0+\theta_1)^2 - \theta_\infty^2](q-1)$$

$$H_{VI}: \frac{1}{z(z-1)}[q(q-1)(q-z)p^2 - (\theta_0(q-1)(q-z) + \theta_1 q(q-z) + (\theta_1-1)q(q-1))p$$

$$+\frac{1}{4}((\theta_0+\theta_1+\theta_t-1)^2 - \theta_\infty^2)(q-t)]. \tag{1.12}$$

Here the new parameters $\theta_0, \theta_1, \theta_t, \theta_\infty$ are defined in terms of the one appearing in the Painlevé equations as follows: the parameters present and those in the original equation:

$$P_{III}: \ \alpha = 4\theta_0, \ \beta = 4(1-\theta_\infty) \ \gamma = 4, \ \delta = -4$$

$$P_{IV}: \ \alpha = 2\theta_\infty - 1, \ \beta = -8\theta_0^2,$$

$$P_V: \ \alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \ \beta = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2, \ \gamma = 1 - \theta_0 - \theta_1, \ \delta = 1/2$$

$$P_{VI}: \ \alpha = \frac{1}{2}(\theta_\infty - 1)^2, \ \beta = \frac{-1}{2}\theta_0^2, \ \gamma = \frac{1}{2}\theta_1^2, \ \delta = \frac{1}{2}(1-\theta_t^2)$$

We are now ready to explain the sigma form. For each Painlevé equation, define the following function:

$$\sigma_k(z) := H_k(p(z), q(z), z), \qquad k = I, II, III, IV, V, VI,$$

then $\sigma_k(z)$ satisfies the following second order ODEs (we drop the index k for ease of reading):

$$P_I^\sigma: (\sigma'')^2 + 4(\sigma')^3 + 2z\sigma' - 2\sigma = 0 \tag{1.13}$$

$$P_{II}^{\sigma} : (\sigma'')^2 + 4(\sigma')^3 + 2z\sigma'^2 - 2\sigma\sigma' - \frac{(\alpha - 1/2)^2}{4} = 0 \tag{1.14}$$

$$P_{III}^{\sigma} : (z\sigma'' - \sigma')^2 = 4(2\sigma - z\sigma')((\sigma')^2 - 4z^2) - 16\theta_0\theta_\infty 2\sigma' \tag{1.15}$$

$$+2(\theta_0^2 + \theta_\infty^2)((\sigma')^2 + 4z^2)$$

$$P_{IV}^{\sigma} : (\sigma'')^2 = 4(z\sigma' - \sigma)^2 - 4(\sigma' + \nu_0)(\sigma' + \nu_1)(\sigma' + \nu_2) \tag{1.16}$$

$$P_{V}^{\sigma} : (z\sigma'')^2 = (\sigma - z\sigma' + z(\sigma')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma')^2 \tag{1.17}$$

$$-4(\nu_0 + \sigma')(\nu_1 + \sigma')(\nu_2 + \sigma')(\nu_3 + \sigma')$$

$$P_{VI}^{\sigma} : \sigma'(z(z-1)\sigma'')^2 + [z\sigma'(z\sigma' - \sigma) - (\sigma')^2 - \nu_1\nu_2\nu_3\nu_4]^2 \tag{1.18}$$

$$= (\sigma' + \nu_1^2)(\sigma' + \nu_2^2)(\sigma' + \nu_3^2)(\sigma' + \nu_4^2),$$

where the new parameters $\nu_1, \nu_2, \nu_3, \nu_4$ appearing in these equations are related to $\theta_0, \theta_1, \theta_t, \theta_\infty$ as follows:

$$P_{IV} : \ \nu_1 = 4\theta_0, \ \nu_2 = 2(\theta_\infty + \theta_0)$$

$$P_V : \ \nu_1 = \frac{-\theta_0 - \theta_1 + \theta_\infty}{2}, \ \nu_2 = -\theta_0, \ \nu_3 = \frac{-\theta_0 + \theta_1 + \theta_\infty}{2}$$

$$P_{VI} : \ \nu_1 = \frac{\theta_1 + \theta_\infty}{2}, \ \nu_2 = \frac{\theta_t - \theta_\infty}{2}, \ \nu_3 = -\frac{\theta_0 + \theta_1}{2}, \ \nu_4 = \frac{\theta_1 - \theta_0}{2}.$$

In recent years many higher order analogues for the Painlevé equations have been discovered. In this dissertation we study the second Painlevé hierarchy [6, 1], an infinite sequence of nonlinear ordinary differential equations containing

$$P_{II}^{(1)} : \ w_z z - 2w^3 = zw + \alpha$$

as its simplest equation. The n-th element of the hierarchy is of order $2n$, depending on n parameters denoted by $t_1, \ldots, t_{n-1}$ and α_n:

$$P_{II}^{(n)} : (\frac{d}{dz} + 2w)L_n[w_z - w^2] + \sum_{k=1}^{n-1} t_{n-k}(\frac{d}{dz} + 2w)L_{n-k}[w_z - w^2] = zw + \alpha_n, \tag{1.19}$$

where L_k is the differential polynomial defined by the Lenard recursion

$$\frac{d}{dx}L_{k+1}[f] = \left(\frac{d^3}{dx^3} + 4f\frac{d}{dx} + 2f'\right) L_k[f], \quad k > 0, \tag{1.20}$$

commencing with $L_0[f] = \frac{1}{2}$.

Our main result is the explicit computation of the sigma form for this hierarchy:

Theorem 1.1. *For positive integer n, the sigma form is given by*

$$-2^{2n} \int p_n \left(\frac{d}{dz}\left[L_{n+1} + \sum_{k=1}^{n} t_{n-k}L_{n-k+1} - z\sigma'(z) - \sigma(z)\right]\right) dz + \alpha - \alpha^2 + \theta = 0 \tag{1.21}$$

with L_k denoting the k-th Lenard polynomial of argument $\sigma'(z) - \frac{t_{n-1}}{2}$ and p_n is a differential polynomial of $\sigma(z)$ given as

$$p_n = \frac{1}{2^{2n-1}}\left[L_n + \sum_{k=1}^{n-1} t_{n-k}L_{n-k} - \frac{z}{2}\right].$$

One considers $t_0 = -z$ and the constant θ has a value $1/4$ in the case $n = 1$ and zero otherwise.

Observe that while the appearance of an integral in the above expression may lead one to conclude that the higher sigma forms are integral, as opposed to differential, equations we shall show that the integrand is in fact an exact differential.

For example in the case n=1 one has the following

$$\int (\sigma'')[\sigma^{(3)} + 3(\sigma')^2 + 2z\sigma' - \sigma]dz - (\alpha - 1/2)^2 = 0.$$

This can be integrated exactly to give

$$\frac{1}{2}(\sigma'')^2 + (\sigma')^3 + z(\sigma')^2 - \sigma\sigma' + (\alpha - 1/2)^2 = 0.$$

This differs from the Okamoto form of $P_{II}^{(\sigma)}$ only by a choice of normalisation. Moreover in the case $n = 2$ integration by part on the above formula leads to one considering

$$\int g(z)dz = \int \frac{d}{dz}p_n \left(L_{n+1} + \sum_{k=1}^{n-1} t_{n-k}L_{n-k+1} - z\sigma'(z) - \sigma(z) + t_{n-1}z \right) dz.$$

Expanding the terms one then finds

$$\begin{gathered} 128 \int g(z)dz = -4z^2t_1 + 4zt_1^3 - 368t_1(\sigma')^4 + 192(\sigma')^5 + 16t_1\sigma'' + 8z(\sigma'')^2 \\ +28t_1^2(\sigma'')^2 + 4\sigma(2z - t_1^2 + 8t_1\sigma' - 12(\sigma')^2 - 4\sigma^{(3)}) + 16zt_1\sigma^{(3)} \\ -8t_1^3\sigma^{(3)} - 16(\sigma'')^2\sigma^{(3)} - 16t_1(\sigma^{(3)})^2 + (\sigma')^3(272t_1^2 - 32z + 160\sigma^{(3)}) \\ +16(\sigma')^2(15(\sigma'')^2 - 2t_1(3t_1^2 - 2z + 6\sigma^{(3)})) - 8\sigma^{(4)} - 32t_1\sigma''\sigma^{(4)} + 8(\sigma^{(4)})^2 \\ -8\sigma'(20t_1(\sigma'')^2 + (2z - t_1^2 - 4\sigma^{(3)})(2t_1^2 + \sigma^{(3)}) + \sigma''(6 - 12\sigma^{(4)})). \end{gathered}$$

In particular one can clearly see in the final term that the resulting differential equation is of the forth order (as to be expected). The two above integrals gave concrete examples that the integral sign in theorem (1.1) is an artefact of its construction; the integrand is an exact differential for all n.

In order to find this sigma form we first prove a result about the Hamiltonians of the second Painlevé hierarchy. The canonical coordinates and Hamiltonians for the second Painlevé hierarchy were found by Mazzocco and Mo [15]. As we shall see in Section 3, their formula for the Hamiltonians in terms of the canonical coordinates was rather complicated and involves generating functions. In this dissertation we provide a simpler explicit formula which does not involve generating functions nor recursion:

Theorem 1.2. *The Hamiltonian for $P_{II}^{(n)}$ is given in terms of the Mazzocco-Mo canonical coordinates by the following formula:*

$$\begin{aligned}
H^{(n)} &= 2^{2n} \sum_{\sum k.m_k = n+1} (-1)^{\sum m_k} \binom{\sum m_k}{m_1, \ldots, m_n} \prod_{k=1}^{n} p_k^{m_k} + 2p_1 z \\
&- \sum_{k=1}^{n} p_k \sum_{r+s=n+k,\ r,s\neq n} \frac{q_r q_s}{2^{2n}} - \sum_{r+s=n} \frac{q_r q_s}{2^{2n}} - \frac{p_n q_n^2}{2^{2n}} + \frac{(1-2\alpha)q_n}{2^{2n}} \\
&+ \sum_{j=1}^{n-1} t_j \sum_{\sum r.m_r = j+1} 2^{2j+1} (-1)^{\sum m_r} \binom{\sum m_r}{m_1, \ldots, m_n} \prod_{r=1}^{n} p_r^{m_r} \\
&+ \sum_{j=[n/2+1]}^{n-1} t_j^2 \sum_{\sum r.m_r = 2j-n+1} 2^{2(2j-n)} (-1)^{\sum m_r} \binom{\sum m_r}{m_1, \ldots, m_n} \prod_{r=1}^{n} p_r^{m_r} \\
&+ \sum_{s=1}^{n-1} t_a \sum_{b=n-a, a\neq b}^{n-1} t_b \sum_{\sum r.m_r = a+b-n+1} 2^{(a+b-n)+1} (-1)^{\sum m_r} \binom{\sum m_r}{m_1, \ldots, m_n} \prod_{r=1}^{n} p_r^{m_r},
\end{aligned}$$

where the summation over $\sum k.m_k = n+1$, indicates that one sums over all choices of integers $m_1, m_2, \ldots, m_n$ such that the sum of the products $k.m_k$ is equal to $n+1$.

2 Background

In this section we review the derivation of the $P_{II}^{(n)}$ hierarchy and its Hamiltonian structure. Of particular importance is the choice of canonical coordinates made by the authors of [15]. The formulas, given in terms of the solution of $P_{II}^{(n)}$, are critical for the derivation of the sigma form. We shall end with a synopsis of the Bäcklund transformations for this hierarchy. We will need there to derive some simple applications of our new sigma form. All material in this section is taken from [15] and [21].

2.1 Derivation of the P_{II} hierarchy

The second Painlevé hierarchy is obtained from the modified Korteweg-de Vries hierarchy (mKdV) by a self similarity reduction ([14], [21]):

$$\frac{\partial}{\partial T_{l+1}} v + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right) L_l[v_x - v^2] = 0 \tag{2.1}$$

for positive integer, n, and L_k is the kth Lenard polynomial defined by equation (1.20). The integrability of this recursion, beyond the first two iterations is non obvious and is due to Lax ([12]).

Example 2.1. *Here we show the first three Lenard differential polynomials:*

$$L_1[f] = f,$$

$$L_2[f] = f'' + 3f^2,$$

$$L_3[f] = f^{(4)} + 10ff'' + 5(f')^2 + 10f^3.$$

The n-th member of the second Painlevé hierarchy is obtained as the equation satisfied for the solutions of the n-th element of the mKdV hierarchy which are stationary under the Virasoro symmetry generator

$$\sum_{l=0}^{n}(2l+1)T_{l+1}\frac{\partial}{\partial T_{l+1}},$$

namely solutions of the equation

$$\sum_{l=0}^{n}(2l+1)T_{l+1}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+2v\right)L_l[v_x-v^2]=0.$$

One can clearly take a first integral of such a solution to obtain

$$\sum_{l=0}^{n}(2l+1)T_{l+1}\left(\frac{\partial}{\partial x}+2v\right)L_l[v_x-v^2]=\alpha. \tag{2.2}$$

Observe that the case $n=1$ in (2.1) allows one to deduce that $T_1=-x$. Thus (2.2) is an ordinary differential equation in x with $n+1$ parameters $\alpha, T_2, \ldots, T_{n+1}$. Then if one makes the change of variables

$$v(x,T_{n+1})=\frac{w(z)}{((2n+1)T_{n+1})^{(2n+1)^{-1}}},\ z=\frac{x}{((2n+1)T_{n+1})^{(2n+1)^{-1}}},$$

$$L_k[v_x-v^2]=\frac{L_k[w_z-w^2]}{((2n+1)T_{n+1})^{2k(2n+1)^{-1}}},$$

$$t_0=-z,\ t_k=\frac{(2k+1)T_{k+1}}{((2n+1)T_{n+1})^{(2k+1)(2n+1)^{-1}}},$$

brings (2.2) to (1.19), which of course is $P_{II}^{(n)}$.

2.2 Hamiltonian of $P_{II}^{(n)}$

The Hamiltonian for $P_{II}^{(n)}$ is constructed by first considering the isomonodromic deformation given in [15], [14] and [10]. This is given by the compatibility of the following system :

$$\frac{\partial \Phi}{\partial z} = \mathbf{B}\Phi = \begin{pmatrix} -\lambda & w \\ w & \lambda \end{pmatrix} \Phi$$

$$\frac{\partial \Phi}{\partial \lambda} = \mathbf{A}^{(n)}\Phi = \frac{1}{\lambda}\left[\begin{pmatrix} -\lambda z & -\alpha \\ -\alpha & \lambda z \end{pmatrix} + \mathbf{M}^{(n)} + \sum_{l=1}^{n-1} t_{n-l}\mathbf{M}^{(n-l)} \right] \Phi$$

with

$$\mathbf{M}^{(n)} = \begin{pmatrix} \sum_{j=1}^{2l+1} A_j^{(l)}\lambda^j & \sum_{j=1}^{2l} B_j^{(l)}\lambda^j \\ \sum_{j=1}^{2l} C_j^{(l)} & -\sum_{j=1}^{(n)} A_j^{(l)}\lambda^j \end{pmatrix}$$

and the entries are given by

$$A_{2l+1}^{(l)} = 4^l, \; A_{2k} = 0, \; \forall k = 0, \ldots, l$$

$$A_{2k+1}^{(l)} = 2^{2k+1}\left[L_{l-k}[w_z - w^2] - \frac{d}{dz}\left(\frac{d}{dz} + 2w\right) L_{l-k-1}[w_z - w^2] \right], \; k = 0, \ldots, l-1$$

$$B_{2k+1}^{(l)} = 2^{2k+1}\frac{d}{dz}\left(\frac{d}{dz} + 2w\right) L_{l-k-1}[w_z - w^2], \; k = 0, \ldots, l-1$$

$$B_{2k}^{(l)} = -4^k \left(\frac{d}{dz} + 2w\right) L_{l-k}[w_z - w^2], \; k = 1, \ldots, l$$

$$C_k^{(l)} = (-1)^k B_k^{(l)}.$$

The compatibility equation is

$$\frac{\partial \mathbf{A}^{(n)}}{\partial z} - \frac{\partial \mathbf{B}}{\partial \lambda} = [\mathbf{B}, \mathbf{A}^{(n)}], \tag{2.3}$$

which is satisfied if and only if $w(z)$ solves $P_{II}^{(n)}$ (see [14]). It is convenient to rewrite the matrix $\mathbf{A}^{(n)}$ in the following way:

$$\mathbf{A}^{(n)} = \begin{pmatrix} \sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k} & \sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1} + \sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k} \\ \sum_{k=0}^{n} b_{2k}^{(n)}\lambda^{2k-1} - \sum_{k=0}^{n-1} b_{2k+1}^{(n)}\lambda^{2k} & -\sum_{k=0}^{n} a_{2k+1}^{(n)}\lambda^{2k} \end{pmatrix}$$

with its entries given by

$$a_{2k+1}^{(n)} = \sum_{l=1}^{n} t_l A_{2k+1}^{(l)}, \ k = 1, \ldots, n, \ a_1^{(n)} = \sum_{l=1}^{(n)} t_l A_1^{(l)} - z \tag{2.4}$$

$$b_{2k+1}^{(n)} = \sum_{l=1}^{n} t_l B_{2k+1}^{(l)}, \ k = 0, \ldots, n-1 \tag{2.5}$$

$$b_{2k}^{(n)} = \sum_{l=1}^{n} t_l B_{2k}^{(l)}, k = 1, \ldots, n, \ b_0^{(n)} = -\alpha \tag{2.6}$$

where we take $t_n = 1$.

To compute the Hamiltonian, one uses the Kostant-Kirillov Poisson structure on the coadjoint orbit of this twisted loop algebra. This is possible as every element in a Lie algebra (which our loop algebra is) defines a linear function on its dual. Indeed this allows one to than identify the dual of the dual of this Lie algebra with the Lie algebra itself. To use this to compute the Poisson bracket of two functions, say F and H on this dual space recall that the Poisson bracket of two functions is itself a function. This is required to be able to specify how this acts on elements of the dual, say x. The differential of these functions, dF and dH, exist in the dual to the dual, which we have already identified with the initial Lie algebra. In this we have the Lie bracket as a natural operation. Thus one can then write

$$\{F, H\}(x) = \langle x, [dF, dH] \rangle,$$

with $\langle x, dF \rangle$ being the action of linear functions on the Lie algebra on the elements of the dual.

Mazzocco and Mo then showed that the vector field defined by (2.3), is in fact Hamiltonian with Hamiltonian function given by

$$H^{(n)} := \frac{1}{2^{2n+1}} Tr[Res_\lambda(\lambda^{1-2n}(\mathbf{A}^{(n)})^2)].$$

By residue we mean the coefficient of $1/\lambda$. Writing this in terms of the coefficients of $\mathbf{A}^{(n)}$, and making a slight modification to account for the explicit dependance of p_n upon z, one arrives at

$$H^{(n)} = \frac{-1}{4^n}\left[\sum_{l=0}^{n-1} a_{2l+1}^{(n)} a_{2(n-l)-1}^{(n)} - \sum_{l=0}^{n-1} b_{2l+1}^{(n)} b_{2(n-l)-1}^{(n)} + \sum_{l=0}^{n} b_{2l}^{(n)} b_{2(n-l)}^{(n)}\right] + \frac{q_n}{4^n}. \tag{2.7}$$

The authors in [15] gave an expression for the canonical coordinates in terms of the entries of the matrix $\mathbf{A}^{(n)}$.

2.3 Coadjoint orbit interpretation of the isomonodromic problem for $P_{II}^{(n)}$

Now we shall present a detailed exposition of work undertaken by Mazzocco and Mo in [15]. This pertains to the use of the isomonodromic problem for the second Painlevé hierarchy to derive the Hamiltonians for this equation, and the choice of canonical coordinates for these systems.

2.3.1 Isomonodromic compatibility and coadjoint orbits

Following [15], we need to introduce some notation. Let $f = f[w]$ be a function of $w(z)$ and its derivatives, z and λ. Then

$$\partial_z f = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial w} w_z + \frac{\partial f}{\partial w_z} w_{zz} + \dots.$$

Now we introduce the derivative of f with respect to z, while keeping w fixed i.e.

$$\partial_z^w f := \frac{\partial f}{\partial z}.$$

So defined, we now turn to the isomonodromy problem defined initially in section (2). With the matrices $\mathbf{A}^{(n)}$ and $\mathbf{B}$ as defined one has the equality

$$\partial_z^w \mathbf{A}^{(n)} = \frac{\partial \mathbf{B}}{\partial \lambda}.$$

Consequently, the compatibility equation (2.3) can be written as

$$(\partial_z - \partial_z^w)\mathbf{A}^{(n)} = [\mathbf{B}, \mathbf{A}^{(n)}]. \tag{2.8}$$

The evolution along $(\partial_z - \partial_z^{(w)})$ is what we shall interpret as a vector field. Of course this requires an appropriate algebra.

Let LG be the group of smooth maps f from S^1 to $SL(2,\mathbb{C})$ such that

$$f(\lambda)\sigma_1(f(-\lambda))^{-1} = I,\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the presence of σ above is used to preserve standard notation for Pauli matrices and should not be confused with the sigma form. Now consider $L_{2n+2}G$, the subgroup comprised of maps of the form $f = I + \lambda^{-2n-2} f_\infty$, were f_∞ is holomorphic outside S^1. The Lie algebra of this subgroup can be represented as follows:

$$\mathfrak{g}_{2n+2} = \left\{ X(\lambda) = \sum_{-\infty}^{-2n-2} X_i \lambda^i | X_i \in \mathfrak{sl}(2,\mathbb{C}),\ X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\}.$$

One can construct the quotient of these groups $LG/L_{2n+2}G$ and its Lie algebra is given by

$$\mathfrak{g} = \left\{ X(\lambda) = \sum_{-\infty}^{\infty} X_i \lambda^i | X_i \in \mathfrak{sl}(2,\mathbb{C}),\ X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\} / \mathfrak{g}_{2n+2}$$

The Lie bracket is defined as

$$[X(\lambda), \tilde{X}(\lambda)] = \sum_{-2n-1}^{\infty} \left(\sum_{k=-2n-1}^{i+2n+1} [X_k, X_{i-k}] \right) \lambda^i \; mod \; g_{2n+2}.$$

This commutator clearly lies in $\mathfrak{g}$, and it satisfies the Jacobi identitiy. We also require a dual space. Consider the following:

$$\mathfrak{g}^* = \left\{ \Xi(\lambda) = \sum_{-\infty}^{\infty} \Xi_i \lambda^i | \Xi_i \in \mathfrak{sl}(2, \mathbb{C}),\; \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right\} / \mathfrak{g}^*_{2n+2},$$

$$\mathfrak{g}^*_{2n+2} = \left\{ X(\lambda) = \sum_{2n+1}^{\infty} X_i \lambda^i | X_i \in \mathfrak{sl}(2, \mathbb{C}),\; X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\}.$$

By using the pairing

$$\langle X(\lambda), \Xi(\lambda) \rangle := Tr(Res(X(\lambda)\Xi(\lambda)))$$

we make these into the dual spaces of the above algebra. Note by *Res* we mean to take the residue i.e. the coefficient of λ^{-1}.
From this we now consider a subalgebra

$$\mathfrak{g}_- = \left\{ X(\lambda) = \sum_{-\infty}^{-1} X_i \lambda^i | X_i \in \mathfrak{sl}(2, \mathbb{C}),\; X(\lambda)\sigma_1 = \sigma_1 X(-\lambda) \right\} / \mathfrak{g}_{2n+2}$$

and its dual

$$\mathfrak{g}^*_- = \left\{ \Xi(\lambda) = \sum_{0}^{\infty} \Xi_i \lambda^i | \Xi_i \in \mathfrak{sl}(2, \mathbb{C}),\; \Xi(\lambda)\sigma_1 = -\sigma_1 \Xi(-\lambda) \right\} / \mathfrak{g}^*_{2n+2}.$$

We have of course the coadjoint action; an element $X \in \mathfrak{g}$ acts on an element $\Xi \in \mathfrak{g}^*$ by

$$\langle ad^*_X \Xi, Y \rangle = -\langle \Xi, [X, Y] \rangle = \langle [X, \Xi], Y \rangle$$

for all $Y \in \mathfrak{g}$. Restricting this action to the subalgebra $\mathfrak{g}_-$, and its dual we have

$$[X_-, \Xi]_+ = ad_{X_-}\Xi, \quad \Xi \in \mathfrak{g}_-^*, \; X_- \in \mathfrak{g}_-$$

with $()_+$ denoting the projection to $\mathfrak{g}_-^*$ and $()_-$ denoting the projection onto $\mathfrak{g}_-$. This preamble allows one to deduce the following result:

Lemma 2.2. *With* $\mathbf{A}^{(n)}$ *and* $\mathbf{B}$ *as given in equation (2.3), one can re-write (2.3) as*

$$(\partial_z - \partial_z^{(n)})\mathbf{A}^{(n)} = ad_B^* A$$

were $B = \left(\frac{\mathbf{A}^{(n)}\lambda^{-2n+1}}{4^n}\right)_- \in \mathfrak{g}_-$ *and* $A = \left(\mathbf{A}^{(n)}\right)_+ \in \mathfrak{g}_-^*$.

Thus our evolution equation, (2.8), can now naturally be viewed as occuring on a coadjoint orbit.

What is important now is the Poisson structure, which we give an overview of in section two, and the canonical coordinates on this structure.

2.3.2 The Painlevé property and canonical coordinates

From the standard Poisson structure as sketched above, there is a standard process one can use to build a set of coordinates. One does this by considering the spectral curve

$$\Gamma(\mu, \lambda) = \{det(\mu - \mathbf{A}^{(n)}) = 0\} = \{\mu^2 = -det(\mathbf{A}^{(n)})\}.$$

Upon some further consideration (see [15]), one can show that the coordinates q_k can be given by taking the roots of the equation

$$\sum_{k=0}^{n-1}(b_{2k+1}^{(n)} + a_{2k+1}^{(n)})\lambda^{2k} + a_{2n+1}^{(n)}\lambda^{2n+1} = 0 \tag{2.9}$$

and the p_k are given by

$$p_j = \sum_{k=0}^{n} b_{2k}^{(n)} q_j^{2k+1}. \tag{2.10}$$

The issue is that the q_k are obtained by taking roots of (2.9). As such it is quite possible that these coordinates do not posses the Painlevé property, as the process of taking such a root may introduce new essential singularities. The advantage these coordinates have is that, as defined, the are guaranteed to be canonical i.e.

$$\{p_j, p_k\} = 0, \ \{q_j, q_k\} = 0, \ \{p_j, q_k\} = \delta_{ij}.$$

We therefore seek coordinates that both have this feature, and the Painlevé property.

In [15], Mazzocco and Mo found the following answer:

$$p_k = \frac{a^{(n)}_{2(n-k)+1} + b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}}, \tag{2.11}$$

$$q_k = \sum_{j=1}^{n} -b^{(n)}_{2j} \left[\left(\sum_{i=0}^{n} \left[\frac{a^{(n)}_{2(n-k)+1} + b^{(n)}_{2(n-k)+1}}{a^{(n)}_{2n+1}} \right] \lambda^{2i} \right)^{-1} \right]_{2j-2k}. \tag{2.12}$$

Their presentation in terms of the entries of $\mathbf{A}^{(n)}$ means that one can easily check the canonicity of these new coordinates. From the canonical Poisson structure one can easily compute the differentials of the series $a^{(n)}_{2k+1}$, $b^{(n)}_{2k+1}$ and $b^{(n)}_{2k}$. Indeed one has

$$da^{(n)}_{2k+1} = \frac{1}{2}(E_{11} - E_{12})\lambda^{-(2k+1)}, \ 0 \le k \le n$$

$$da^{(n)}_{2k+1} = \frac{1}{2}(-E_{12} + E_{21})\lambda^{-(2k+1)}, \ 0 \le k \le n-1$$

$$db^{(n)}_{2k} = \frac{1}{2}(E_{12} + E_{21})\lambda^{-2k}, \ 0 \le k \le n$$

with E_{ij} being the the 2×2 matrix whose entries are zero except that in the position ij, which is one. With this one can easily compute the Poisson bracket between these series:

$$\{a^{(n)}_{2k+1}, b^{(n)}_{2l+1}\} = -b^{(n)}_{2(k+l+1)} \tag{2.13}$$

$$\{a^{(n)}_{2k+1}, b^{(n)}_{2l}\} = -b^{(n)}_{2(k+l)+1} \tag{2.14}$$

$$\{b^{(n)}_{2k}, b^{(n)}_{2l+1}\} = a^{(n)}_{2(k+l)+1} \tag{2.15}$$

with all other brackets vanishing. The above differentials reduce the computation of these quantities to the computation of commutators.
One can now easily check the canonicity of these new coordinates.

2.3.3 The canonicity of the new coordinates

Theorem 2.3. *The coordinates, given by the formulas (2.11) and (2.12), are canonical with respect to the standard Poisson structure on the coadjoint orbit.*

Proof. We begin with the bracket exclusively in the p_k. From (2.13) this is given by

$$\{p_k, p_l\} = \left(\frac{1}{a^{(n)}_{2n+1}}\right)^2 (\{b^{(n)}_{2(n-l)+1}, a^{(n)}_{2(n-k)+1}\} - \{b^{(n)}_{2(n-k)+1}, a^{(n)}_{2(n-l)+1}\}) = 0 \tag{2.16}$$

The final equality is obvious from (2.15).
Let us now consider the mixed bracket. First notice that the coordinate q_k can in fact be written as

$$q_k = \sum_{j=1}^{n} -b^{(n)}_{2j} \left[\left(\sum_{i=0}^{n} [p_k]\, \lambda^{2i}\right)^{-1}\right]_{2j-2k}.$$

For the sake of brevity we shall refer to $\left[(\sum_{i=0}^{n} [p_k]\, \lambda^{2i})^{-1}\right]_{2j-2k}$ as $P(p_k)$. Indeed, it should be clear that this expression is a polynomial in p_k. Consequently one can write the mixed bracket as

$$\{p_k, q_l\} = -\sum_{j=1}^{n} \{p_k, b_{2j}^{(n)}\} P(p_l) - \sum_{j=1}^{n} b_{2j}^{(n)} \{p_k, P(p_l)\}.$$

Clearly (3.9) implies that the second term on the right hand side of the above equation vanishes. Thus we need only consider

$$-\sum_{j=1}^{n} \{p_k, b_{2j}^{(n)}\} P(p_l).$$

The Poisson bracket in the above becomes $-p_{k-j}$. Thus we consider

$$\sum_{j=l}^{n} p_{k-j} \left[\left(\sum_{i=0}^{n} p_l \lambda^{2i} \right)^{-1} \right]_{2j-2l}.$$

Observe that we can sum over the reduced range $l, \ldots, n$ as there are no negative powers in the inverse series. This series is in fact given by the coefficient of the expression λ^{2k-2l} in

$$\left(\sum_{i=0}^{n} p_k \lambda^{2i} \right) \left(\sum_{j=0}^{n} p_l \lambda^{2j} \right)^{-1} = 1.$$

Thus one sees that

$$\{p_k, q_l\} = \delta_{kl}. \tag{2.17}$$

Finally we need to compute the bracket in terms of the q_k exclusively. This is of course given by

$$\{q_k, q_l\} = \sum_{i,j=1}^{n} \{b_{2j}^{(n)} P(p_k), b_{2i}^{(n)} P(p_l)\}.$$

Though expanding the bracket would usually lead to numerous terms, many of these will vanish owing to our earlier assertion regarding the bracket exclusively in the p_k and the fact that $\{b_{2k}^{(n)}, b_{2l}^{(n)}\} = 0$. Therefore we can write

$$\{q_k, q_l\} = \sum_{i,j=1}^{n} b_{2j}^{(n)}\{P(p_k), b_{2i}^{(n)}\}P(p_l) - b_{2i}^{(n)}\{P(p_l), b_{2j}^{(n)}\}P(p_k)$$

$$= \sum_{i,j=1}^{n} b_{2j}^{(n)}\{P(p_k), b_{2i}^{(n)}\}P(p_l) - b_{2j}^{(n)}\{P(p_l), b_{2i}^{(n)}\}P(p_k).$$

This last equality is arrived at by noticing that the formula is symmetric in i and j. This in effect reduces our consideration to

$$\sum_{i=1}^{n}\{P(p_k), b_{2i}^{(n)}\}P(p_l) - \{P(p_l), b_{2i}^{(n)}\}P(p_k). \tag{2.18}$$

One can compute the Poisson brackets as follows:

$$\{P(p_k), b_{2i}^{(n)}\} = \left[\left(\sum_{i=0}^{n} p_k\lambda^{2j}\right)^{-2} \sum_{s=0}^{n}\{p_s, b_{2i}^{(n)}\}\lambda^{2s}\right]_{2j-2k}$$

$$= \left[\left(\sum_{i=0}^{n} p_k\lambda^{2j}\right)^{-2} \sum_{s=0}^{n} p_s\lambda^{2s}\right]_{2j-2k-2i}.$$

One obtains this equality by using earlier results, particularly the bracket between the p_k and the $b_{2i}^{(n)}$ and by manipulating indices. From this one can see that

$$\sum_{i=1}^{n}\{P(p_k), b_{2i}^{(n)}\}P(p_l) = \left[\left(\sum_{i=0}^{n} p_k\lambda^{2j}\right)^{-3} \sum_{s=0}^{n} p_s\lambda^{2s}\right]_{2j-2k-2l}.$$

By implementing the same procedure *verbatim* on the second term in (2.18) one concludes that

$$\sum_{i=1}^{n}\{P(p_l), b_{2i}^{(n)}\}P(p_k) = \left[\left(\sum_{i=0}^{n} p_k\lambda^{2j}\right)^{-3}\sum_{s=0}^{n} p_s\lambda^{2s}\right]_{2j-2k-2l}.$$

Hence one can conclude that

$$\{q_k, q_l\} = 0.$$

□

Thus we have demonstrated that the new coordinates are still canonical with respect to the natural Poisson structure. Moreover as they are polynomial in the series $a_{2k+1}^{(n)}$ etc. which are themselves differential polynomials in the solution of $P_{II}^{(n)}$ ($a_{2n+1}^{(n)}$ is a constant), they clearly possess the Painlevé property.

2.4 The canonical coordinates in terms of solutions of $P_{II}^{(n)}$

The coordinates can be written in terms of solutions to $P_{II}^{(n)}$:

$$p_k = \frac{1}{2^{2k-1}}\left(L_k[w_z - w^2] + \sum_{j=1}^{k} t_{n-j}L_{k-j}[w_z - w^2]\right), \tag{2.19}$$

$$q_k = \sum_{j=1}^{n} -b_{2j}^{(n)}\left[\left(\sum_{i=0}^{n}\left[\frac{a_{2(n-k)+1}^{(n)} + b_{2(n-k)+1}^{(n)}}{a_{2n+1}^{(n)}}\right]\lambda^{2i}\right)\right]_{2j-2k}. \tag{2.20}$$

In the above presentation of q_k one can clearly use the equations (2.4), (2.5) and (2.6) to write the $a_{2k+1}^{(n)}$, $b_{2k+1}^{(n)}$ and the $b_{2k}^{(n)}$ in terms of $w(z)$. The notation $[\lambda]_{m-n}$ indicates one takes the coefficient of λ^{m-n} in the formal series $k(\lambda)$.

These formulas shall be fundamental in writing the canonical coordinates in terms of sigma and its derivatives.
The last question is to be able to write the entries of $\mathbf{A}^{(n)}$ in terms of these coordinates. In this regard we have the following result.

Theorem 2.4. *[15] Consider the polynomials in λ*

$$A = \sum_{k=0}^{n} a_{2k+1}^{(n)} \lambda^{2k+1},\ B_o = \sum_{k=0}^{n} b_{2k+1}^{(n)} \lambda^{2k+1},\ B_e = \sum_{k=1}^{n} b_{2k}^{(n)} \lambda^{2k}$$

$$Q = \sum_{j=1}^{n} q_j \lambda^{2j},\ P = \sum_{j=1}^{n} p_j \lambda^{-2j},\ T = \frac{-z}{(2\lambda)^{2n}} + \sum_{i=1}^{n-1} t_i (2\lambda)^{2i-2n}.$$

Then the following relations between them hold

$$A = \left[\frac{1}{4}(2\lambda)^{2n+1} \left(1 + P + \frac{(1+T)^2}{1+P} \right) - (2\lambda)^{-2n-1} Q^2 (1+P) \right]_+ \tag{2.21}$$

$$B_o = \left[\frac{1}{4}(2\lambda)^{2n+1} \left(1 + P - \frac{(1+T)^2}{1+P} \right) + (2\lambda)^{-2n-1} Q^2 (1+P) \right]_+ \tag{2.22}$$

$$B_e = -\lambda^2 \left[\lambda^{-2} Q (1+P) \right]_+ . \tag{2.23}$$

The notation $[f]_+$ indicates to take only the polynomial part of the series f.

With this one can compute the Hamiltonian explicitly for any given n. However this form is not appropriate to give a formula in terms of p_k and q_k for general n, which is necessary for the derivation of sigma forms.

2.5 Sigma form of P_{II}

The sigma form of the second Painlevé equation is a known quantity derived by Okamoto in [18]. The procedure he used is a template for the one we shall develop for the entire hierarchy and so we shall recall it.
Starting form H_{II} given in equation (1.8), one computes the first and second

derivative of this Hamiltonian. One does this while considering that we make the identification $H_{II} = \sigma(z)$. Thus one deduces

$$\sigma'(z) = \frac{1}{2}p$$

$$\sigma''(z) = 2qp + \left(\alpha - \frac{1}{2}\right).$$

One can solve the above system in the sense that one can write expressions for the canonical coordinates in terms $\sigma(z)$ and its derivatives. One can then substitute these expressions into $H_{II} = \sigma(z)$, and one easily arrives at (1.14). To find the sigma form of the hierarchy we shall implement a similar procedure. The main difference with the higher order members is the need to solve a system of polynomial equations for the $2n$ canonical coordinates. The feasibility of this is not apparent. As we shall show, the choice of coordinates made by the authors of [15] allows for the development of systematic approach to produce the sigma form for the whole hierarchy.

2.6 Special solutions and Bäcklund transformations of the second Painlevé hierarchy

We shall now examine the results of Clarkson, Joshi and Pickering presented in [21]. In this they obtain not only Bäcklund transformations for the second Painlevé hierarchy, but also first integrals corresponding to certain values of its parameter.

Note that the results in [21] are obtained under the hypothesis that the additional parameters, $t_1, \ldots, t_{n-1}$ vanish in (1.19). Therefore all details that follow shall also be subject to this consideration.

2.6.1 The Bäcklund transformation

The method to be used to obtain these transformation is known as a truncated Painlevé expansion. To begin, one takes equation (1.19) ($P_{II}^{(n)}$) and makes the following change of variables

$$w(z) = \frac{1}{2}\frac{\psi''}{\psi'}.$$

Upon performing this change (1.19) becomes

$$\left(\frac{d}{dx} + \frac{\psi''}{\psi'}\right) L_n[\frac{1}{2}S(\psi)] - \frac{1}{2}\frac{\psi''}{\psi'}x - \alpha = 0.$$

As before $L_n[f]$ refers to the n-th Lenard polynomial and $S(\psi)$ is the Schwartzian derivative:

$$S(\psi) = \frac{d}{dx}\left(\frac{\psi''}{\psi'}\right) - \frac{1}{2}\left(\frac{\psi''}{\psi'}\right)^2.$$

The appearance of the Schwartzian derivative is fortuitous; it is invariant under Möbius transformations thus one can apply them freely. Indeed setting $\psi = -1/\phi$, one has

$$w(z) = \frac{-\phi'}{\phi} + \frac{1}{2}\frac{\phi''}{\phi'}.$$

This new function satisfies the equation

$$\left(\frac{d}{dx} + \frac{\phi''}{\phi'} - 2\frac{\phi'}{\phi}\right) L_n[\frac{1}{2}S(\phi)] - \frac{1}{2}\left(\frac{\phi''}{\phi'} - 2\frac{\phi'}{\phi}\right) x - \alpha = 0. \tag{2.24}$$

If one now sets

$$\tilde{w}(z) = \frac{1}{2}\frac{\phi''}{\phi'} \tag{2.25}$$

such that

$$w(z) = -\frac{\phi'}{\phi} + \tilde{w}(z), \tag{2.26}$$

the $\tilde{w}(z)$ satisfies

$$\left(\frac{d}{dx}+\tilde{w}-2\frac{\phi'}{\phi}\right)L_n[\tilde{w}'-\tilde{w}^2]-\frac{1}{2}\left(\tilde{w}-2\frac{\phi'}{\phi}\right)x-\alpha=0. \tag{2.27}$$

In equation (2.27) there are two possibilities. Either the new function $\tilde{w}(z)$ satisfies the equation $P_{II}^{(n)}$ with the same parameter α as that satisfied by $w(z)$ or it does not. The former situation in fact leads one to the special solutions; we shall suspend considering it for now.

Given that we now accept the latter situation let us suppose that $\tilde{w}$ satisfies (1.19) with the parameter $\tilde{\alpha}$. We shall use the the notation $P_{II}^{(n)}[\tilde{w},\tilde{\alpha}]$ to denote this equation. The consequence of this is that equation (2.27) can now be written as

$$(2L_n[\tilde{w}'-\tilde{w}^2]-x)\frac{\phi'}{\phi}+\alpha-\tilde{\alpha}=0. \tag{2.28}$$

The key question is what is the relationship between α and $\tilde{\alpha}$. To establish this one needs to differentiate (2.28). Upon doing so one has

$$\left(2\frac{d}{dx}L_n[\tilde{w}'-\tilde{w}^2]-1\right)\frac{\phi'}{\phi}+(2L_n[\tilde{w}'-\tilde{w}^2]-x)\left(\frac{\phi''}{\phi}-\left(\frac{\phi'}{\phi}\right)^2\right)=0.$$

Observing

$$\frac{\phi''}{\phi}=\frac{\phi''\phi'}{\phi\phi'}$$

one can use equations (2.25) and (2.28) to write an equation exclusively in terms of $\tilde{w}$. The result is the following:

$$\left(\frac{d}{dx}+2\tilde{w}\right)L_n[\tilde{w}'-\tilde{w}^2]+\tilde{w}x-\frac{1}{2}(\alpha-\tilde{\alpha}-1)=0. \tag{2.29}$$

Recall that, by hypothesis, $\tilde{w}(z)$ satisfies $P_{II}^{(n)}[\tilde{w}, \tilde{\alpha}]$. It must hold that (2.29) is compatible with this. For this to be so, one must have the relation

$$\alpha = 1 - \tilde{\alpha}. \tag{2.30}$$

Having now obtained a relation between the transformed and untransformed parameter we can now derive the transformation itself.

Returning to equation (2.28), one can rearrange to solve for the rational expression in ϕ. This is of course given by

$$\frac{\phi'}{\phi} = \frac{\tilde{\alpha} - \alpha}{2L_n[\tilde{w}' - \tilde{w}^2] - x}.$$

Using (2.30) and (2.26) one can then write this as

$$w = \tilde{w} + \frac{2\tilde{\alpha} - 1}{x - 2L_n[\tilde{w}' - \tilde{w}^2]}. \tag{2.31}$$

This is the Bäcklund transformation. This in effect maps the pair $(w(z), \alpha)$ to the pair $(\tilde{w}(z), \tilde{\alpha})$.
One can generalise the transformation. Indeed consider equation (1.19) i.e. $P_{II}^{(n)}[w, \alpha]$. It is obvious that the function $-w(z)$ satisfies the equation $P_{II}^{(n)}[-w(z), -\alpha]$. Thus, applying this symmetry to the transformation one arrives at

$$w = \tilde{w} + \frac{2\tilde{\alpha} \mp 1}{x - 2L_n[\pm\tilde{w}' - \tilde{w}^2]}. \tag{2.32}$$

Later in this thesis, we shall rediscover this transformation. Indeed we shall show that it relates to certain canonical transformations of Hamiltonian systems.
We now wish to move on to special solutions of equation (1.19).

2.6.2 First integrals of $P_{II}^{(n)}$ for half integer values of the parameter

Returning to equation (2.27) we now assume that the transformed solution, $\tilde{w}(z)$, satisfies the equation $P_{II}^{(n)}[w(z), \alpha]$. As such now splitting (2.27) in terms of powers of ϕ one arrives at

$$2L_n[\tilde{w}' - \tilde{w}^2] - x = 0. \tag{2.33}$$

The above is a differential equation of order one less than $P_{II}^{(n)}$. Moreover, it is only compatible with it when $\alpha = 1/2$. In that sense we can consider (2.33) as a special solution to $P_{II}^{(n)}$ for this given value of the parameter. Note that in the case $n = 1$, $L_1[\tilde{w}' - \tilde{w}^2] = \tilde{w}' - \tilde{w}^2$ and thus (2.33) represents a Riccati equation that can be linearised and can be solved by standard methods.
The key observation os that one can combine this solution for $\alpha = 1/2$ with our Bäcklund transformation to generate solutions for all half integer values of the parameter. For the sake of simplicity, the best way to proceed is to choose the positive parity in equation (2.32). Then one can use the discrete symmetry $(w, \alpha) \to (-w, -\alpha)$ to write the transform as

$$w = -\tilde{w} - \frac{2\tilde{\alpha} - 1}{x - 2L_n[\tilde{w}' - \tilde{w}^2]}, \quad \alpha = \tilde{\alpha} - 1$$

The above transform clearly takes solutions with parameter α and maps them to those with parameter $\alpha + 1$. From here one can use the discrete, parity transform to generate those solutions that correspond to the negative half integers. To provide illustration, we shall produce some of these special solutions.

2.6.3 Examples

Example 2.5 ($n = 1$)**.**

From equation (2.33), it is obvious that in the case of $n = 1$ the base special solution is given by

$$I^{(1)}_{\frac{1}{2}} = w' - w^2 - \frac{1}{2}x = 0.$$

The solution in the case of $\alpha = -1/2$ is clearly given by

$$I^{(1)}_{-\frac{1}{2}} = -w' - w^2 - \frac{1}{2}x = 0.$$

Now using the transformation one can easily produce the solutions in the case of $\alpha = \pm 3/2$:

$$I^{(1)}_{\pm\frac{3}{2}} = (w')^3 \mp \left(w^2 + \frac{1}{2}x\right)(w')^2 - \left(w^4 + xw^2 \pm 4w + \frac{1}{2}x^2\right)$$

$$\pm w^6 \pm \frac{3}{2}xw^4 + 4w^3 \pm \frac{3}{4}x^2w^2 + 2xw \pm \frac{1}{8}x^3 \pm 2 = 0$$

Note the above is obtained by substituting our formula for the transformation into $I^{(1)}_{\frac{1}{2}}$ and multiplying out rational parts. There is no inconsistency here; (2.33) only gives a solution when $\alpha = 1/2$. We are therefore not multiplying by zero (as it might appear). As the value of α ascends in absolute value the number of terms in the solutions increases dramatically. Indeed for $\alpha = \pm 5/2$ the expression has 38 terms. Thus implementation in computer algebra is recommended to produce more integrals.

Example 2.6 ($n = 2$)**.**

For the second member of the hierarchy, the special solution for $\alpha = \pm 1/2$ is given by

$$I^{(2)}_{\pm\frac{1}{2}} = w''' \pm (w')^2 \pm 2ww' - 6w^2w' \pm w^4 \mp \frac{1}{2}x = 0.$$

From here the number of terms in the integrals grows rapidly. For $\alpha = \pm 3/2$, these have 63 terms. From there the next integrals have 281 and 318 terms.

Having now reviewed the pertinent results that we shall build on, we move to original results in the next section. Specifically we shall discuss Hamiltonians of the second Painlevé hierarchy, and derive new closed form formulas for these.

3 Closed forms of the Hamiltonians of $P_{II}^{(n)}$

The main result of the paper of Mazzocco and Mo ([15]) was a formula for the Hamiltonian of $P_{II}^{(n)}$, given by equation (2.7). As shown in equations (2.11) and (2.12), the $a_{2k+1}^{(n)}$, $b_{2k+1}^{(n)}$ and $b_{2k}^{(n)}$ are all polynomials of the canonical coordinates. These are defined by the non trivial power series (2.21), (2.22) and (2.23).

3.1 Closed form Hamiltonians

Our aim is to prove the following theorems:

Theorem 3.1. *The Hamiltonian for $P_{II}^{(n)}$, with all $t_k = 0$, is given in terms of the canonical coordinates by the following formula:*

$$\begin{aligned} H_{t=0}^{(n)} &= 2^{2n} \sum_{\sum k.m_k = n+1} (-1)^{\sum m_k} \binom{\sum m_k}{m_1, \ldots, m_n} \prod_{k=1}^{n} p_k^{m_k} + 2p_1 z \\ &- \sum_{k=1}^{n} p_k \sum_{r+s=n+k,\ r,s\neq n} \frac{q_r q_s}{2^{2n}} - \sum_{r+s=n} \frac{q_r q_s}{2^{2n}} - \frac{p_n q_n^2}{2^{2n}} + \frac{(1-2\alpha) q_n}{2^{2n}}. \end{aligned}$$

Here

$$\binom{\sum m_k}{m_1, \ldots, m_n} = \frac{(\sum m_k)!}{(m_1)! \ldots (m_n)!}$$

is the multinomial coefficient , and the first summation here means that one sums over all n-tuples m_k such that the sum of the products $k.m_k$ is equal to $n + 1$.

Theorem 3.2. *For $n \in \mathbb{N}$, the Hamiltonian of $P_{II}^{(n)}$ is given by*

$$H^{(n)} = H^{(n)}|_{t=0} + \sum_{j=1}^{n-1} t_j \sum_{\sum r.m_r = j+1} 2^{2j+1} (-1)^{\sum m_r} \binom{\sum m_r}{m_1, \ldots, m_n} \prod_{r=1}^{n} p_r^{m_r}$$

$$+\sum_{j=[n/2+1]}^{n-1} t_j^2 \sum_{\sum r.m_r=2j-n+1} 2^{2(2j-n)}(-1)^{\sum m_r}\binom{\sum m_r}{m_1,\ldots,m_n}\prod_{r=1}^{n} p_r^{m_r}$$

$$+\sum_{a=1}^{n-1} t_a \sum_{b=n-a,a\neq b}^{n-1} t_b \sum_{\sum r.m_r=a+b-n+1} 2^{(a+b-n)+1}(-1)^{\sum m_r}\binom{\sum m_r}{m_1,\ldots,m_n}\prod_{r=1}^{n} p_r^{m_r}$$

where $H^{(n)}|_{t=0}$ denotes the part of the Hamiltonian independent of the parameters t_k given in theorem (3.1) and $[r]$ denotes the integer part of r.

In the car $n = 1$ the above formulas produce the following

$$H = 4p_1^2 + \frac{1}{4}q_1 + \frac{1}{4}p_1 q_1^2 + 2p_1 z - \frac{1}{2}q_1\alpha$$

. One can the use Hamilton's equations to eliminate p_1 and then setting $w = \frac{1}{4}q_1$, $w(z)$ will satisfy P_{II}. Note this additional constant factor in the relation $w = \frac{1}{4}q_1$ is the reason for the above differing form the classical form of the P_{II} Hamiltonian. This reflects a choice of normalisation made by Mazzocco and Mo ([15]).

The proof of these shall be given in section (3.4). The former gives an expression for the Hamiltonian under the hypothesis that the parameters t_k vanish. This has typically been the situation considered when the hierarchy has been discussed previously (as in [21]). The latter details the dependence of the Hamiltonian upon these parameters.

One observation that this representation allows one to make is that, in full generality, $H^{(n)}$ contains no products of the coordinates q_j and the parameters t_k. It would be difficult to make this same observation from the original formula for $H^{(n)}$ (2.7) for a specific n. From this one can deduce that the Hamilton's equations for the p_k are independent of the t_k parameters.

This begin with the following lemmas:

Lemma 3.3. *Assume that all $t_k = 0$, then for positive integer n and for $1 \leq j \leq n-1$ we have*

$$\frac{\partial H^{(n)}}{\partial q_j} = \frac{1}{2^{2j-1}}[b^{(n)}_{2(n-j)} + p_j q_n] = \frac{1}{2^{2j-1}}\left[-q_{n-j} - \sum_{k=1}^{j-1} p_k q_{n-j+k}\right].$$

For $j = n$ one has

$$\frac{\partial H^{(n)}}{\partial q_n} = \frac{-1}{2^{2n-1}}[(\alpha - \frac{1}{2}) - p_n q_n].$$

Moreover, for p_j with $j \in \{2, \ldots, n\}$ we have

$$\begin{aligned} \frac{\partial H^{(n)}}{\partial p_j} &= \sum_{\sum k.mk=n-j+1} 2^{2n}(-1)^{\sum m_k}\binom{\sum m_k + 1}{m_1, \ldots, m_n}\prod_{k=1}^{n} p_k^{m_k} \\ &- \sum_{r+s=n+j} \frac{q_r q_s}{2^{2n}} + \frac{q_j q_n}{2^{2n-1}}. \end{aligned}$$

For $j = 1$ one has

$$\begin{aligned} \frac{\partial H^{(n)}}{\partial p_1} &= 2z + \sum_{\sum k.mk=n} 2^{2n}(-1)^{\sum m_k}\binom{\sum m_k + 1}{m_1, \ldots, m_n}\prod_{k=1}^{n} p_k^{m_k} \\ &- \sum_{r+s=n+1} \frac{q_r q_s}{2^{2n}} + \frac{q_1 q_n}{2^{2n-1}}. \end{aligned}$$

Lemma 3.4. *For $k \in \{1, \ldots, n-1\}$ and for general t_k, one has*

$$\frac{\partial H^{(n)}}{\partial t_k} = 2^{2k+1}\sum_{\sum r.m_r=k+1}(-1)^{\sum m_r}\binom{\sum m_r}{m_1, \ldots, m_n}\prod_{r=1}^{n} p_r^{m_r}$$

$$+ \sum_{j=n-k}^{n-1} 2^{2(j+k-n)+1} t_j \sum_{\sum r.m_r=j+k-n+1}(-1)^{\sum m_r}\binom{\sum m_r}{m_1, \ldots, m_n}\prod_{r=1}^{n} p_r^{m_r}.$$

For the sake readability we shall spilt the proof of the above formulas in to two natural cases; the case were the parameters t_k vanish and the case of general parameters. We commence with the former.

3.2 Derivatives of the $a_{2k+1}^{(n)}$ and the $b_{2k}^{(n)}$

For clarity, in this section we assume that all the t_k identically vanish. With this in mind we shall proceed to derive formulas for the $b_{2k}^{(n)}$, and the derivatives of the functions $a_{2k+1}^{(n)}$. Note that we have excluded the $b_{2k+1}^{(n)}$. The reason for this is the formula relating the $b_{2k_1}^{(n)}$ to the $a_{2k+1}^{(n)}$

$$a_{2k+1}^{(n)} + b_{2k+1}^{(n)} = 2^{2n} p_{n-k}$$

given in [15].

With this one can now write the Hamiltonian as

$$H^{(n)} = \frac{-1}{4^n}\left[\sum_{l=0}^{n-1} 2^{2n+1}(a_{2l+1}^{(n)} p_{l+1}) - 4^{2n} p_{n-l} p_{l+1} + \sum_{l=0}^{n} b_{2l}^{(n)} b_{2(n-l)}^{(n)}\right] + \frac{q_n}{4^n}. \quad (3.1)$$

We have now eliminated the dependence of $H^{(n)}$ upon the $b_{2k+1}^{(n)}$. Moreover the resulting formula also contains no quadratic terms in the $a_{2k+1}^{(n)}$, which shall become important as we move forward.

To find $b_{2k}^{(n)}$, we can use formula (2.23). Direct computations bring us to the following lemma.

Lemma 3.5. *The formula (2.23) implies that the $b_{2k}^{(n)}$ can be given explicitly as*

$$b_{2i}^{(n)} = -q_i - \sum_{k=1}^{n-i} p_k q_{k+i}. \quad (3.2)$$

One just uses the definitions in theorem (2.4) to expand (2.23). From this it is quite simple of isolate the coefficient of λ^{2k}, which of course brings one to the claimed expression.

For the $a_{2k+1}^{(n)}$, we cannot easily derive a similar result. However, as we are interested in initially constructing the gradient we only require the derivatives.

Lemma 3.6. *With the* $a_{2k+1}^{(n)}$ *given by the coefficient of* λ^{2k+1} *in (2.21), we have*

$$\frac{\partial a_{2i+1}^{(n)}}{\partial q_j} = \frac{-1}{4^n}\left[q_{n+1-j+i} + \sum_{l=1}^{j-i-1} p_l q_{n+1-j+i+l}\right] \tag{3.3}$$

for $j \leq i$ *and* 0 *otherwise. Under the same hypothesis one has*

$$\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} = \sum_{\sum k.m_k = n-(i+j)} \prod_{k=1}^{n} p_k^{m_k} C_{m_1,\ldots,m_k} - \sum_{r+s=n+i+j+1} \frac{q_r q_s}{2^{2n+1}} \tag{3.4}$$

for $i+j<n$*, and again* 0 *otherwise.The coefficient* $C_{m_1,\ldots,m_n}$ *is given by*

$$C_{m_1,\ldots,m_n} = (m_j+1)2^{2n-1}(-1)^{\sum m_k+1}\binom{\sum m_k+1}{m_1,\ldots,m_j+1,\ldots,m_n}.$$

In all these the indices take the range $1,\ldots,n$*.*

Proof. The justification of these formulas are all very similar. As such we shall give the details for the case of derivatives with respect to p_k only.

The $a_{2k+1}^{(n)}$ are defined by the coefficient of λ^{2k+1} in the power series in formula (2.21). Observe that in order to expand this formula we need to consider the rational term. We use the formalism $(1+P)^{-1} = \sum_{i=0}^{\infty}(-P)^i$. While this is formally an infinite sum, one can easily show that the terms of P^k are of minimum degree $-2k$ in λ. As we only require the polynomial part, and as all terms of $(1+T)^2$ are of maximal degree 0 in λ, we may replace the infinite sum with $\sum_{i=0}^{n}(-P)^i$.

Moreover, from theorem (2.3) is is clear that the terms of $(1+T)^2$, with the exception of the constant term, are of minimum degree $-2n$ in λ. In combination with the fact that the non-constant terms of $\sum_{i=0}^{n}(-P)^i$ are of maximal degree -2 in λ, we can eliminate the terms $\frac{2T}{1+P}$ and $\frac{T^2}{1+P}$. Thus we are in fact required to only find the coefficient of λ^{2i+1} in

$$\left[\frac{1}{4}(2\lambda)^{2n+1}(2+\sum_{i=2}^{n}(-P)^i)-(2\lambda)^{-2n-1}Q^2(1+P)\right]_+ . \tag{3.5}$$

Fixing now an i and j and expanding one has, using the definition of P and Q,

$$2^{2n-1}\sum_{k=2}^{n^2}\lambda^{2n+1-2k}\sum_{\sum r.m_r=k}\prod_{r=1}^{n}p_r^{m_r}(-1)^{\sum m_r}\binom{\sum m_r}{m_1,\ldots,m_n} \tag{3.6}$$

$$-(2)^{-2n-1}\lambda^{-2n-2j-1}\sum_{r,s=1}^{n}2q_rq_sp_j\lambda^{2(r+s)}.$$

The above is arrived at using the multinomial formula, which yields

$$P^k=\sum_{m_1+\ldots m_k=k}\prod_{r=1}^{n}p_r^{m_r}\lambda^{-2(\sum r.m_r)}\binom{k}{m_1,\ldots,m_n}.$$

Taking the sum over k and parsing in powers of λ we have

$$\sum_{i=2}^{n}(-P)^k=\sum_{k=2}^{n^2}\lambda^{-2k}\sum_{\sum r.m_r=k}\prod_{r=1}^{n}p_r^{m_r}(-1)^{\sum m_r}\binom{\sum m_r}{m_1,\ldots,m_n}.$$

Clearly only terms that are non-zero upon differentiating with respect to p_j are those that explicitly depend on it. These are of course those summands for which $m_j\geq 1$. We can shift the indexing variables to represent only these terms, which allow one to rewrite (3.6) as

$$2^{2n-1}\sum_{k=2}^{n^2}\lambda^{2n+1-2k}\sum_{\sum r.m_r=k-j}\prod_{r\neq j}^{n}p_r^{m_r}p_j^{m_j+1}(-1)^{\sum m_r+1}\binom{\sum m_r+1}{m_1,\ldots,m_j+1,\ldots,m_n} \tag{3.7}$$
$$-(2)^{-2n-1}\lambda^{-2n-2j-1}\sum_{r,s=1}^{n}2q_rq_sp_j\lambda^{2(r+s)}.$$

The coefficient of λ^{2i+1} is given when

$$r+s=n+i+j+1,$$

$$k=n-i.$$

Thus upon differentiation by p_j we arrive at

$$\frac{\partial a_{2i+1}^{(n)}}{\partial p_j}=\sum_{\sum k.m_k=n-(i+j)}\prod_{k=1}^{n}p_k^{m_k}C_{m_1,\ldots,m_k}-\sum_{r,s,\ r+s=n+i+j+1}\frac{q_rq_s}{2^{2n+1}}$$

which is of course precisely what was claimed.
Finally, consider the case when $i+j\geq n$. This, in conjunction with the equation $r+s=n+i+j+1$, implies that $r+s\geq 2n+1$ which is a contradiction as $r,s\in\{1,\ldots,n\}$. Hence there are no contributions dependent upon the q variables when $i+j\geq n$. Also, under the same hypothesis, all solutions of $\sum r.m_r=n-i-j$ are either non existent or trivial. Thus both statements of the lemma are proved. $\square$

We omit formal justification of the other derivative of $a_{2k+1}^{(n)}$ as the proof is very similar to the above.

3.3 Proof of lemma (3.3)

Our aim is compute the derivatives of $H^{(n)}$ with respect to the p_j and q_j. We shall give the details for the derivative with respect to the p_j. The others are too similar to warrant detailed discussion.

Proof. One has that $\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} = 0$ if $i + j \leq n$ and $\frac{\partial b_{2i}^{(n)}}{\partial p_j} = 0$ if $i + j > n$. Thus now taking the derivative of the formula (3.1) one has

$$\frac{\partial H^{(n)}}{\partial p_j} = -\left[\sum_{i=0}^{n-j-1} 2\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} p_{i+1} - 2^{2n+1} p_{n-j+1} + 2a_{2j-1} + \frac{1}{2^{2n-1}} \sum_{i=0}^{n-j} \frac{\partial b_{2i}^{(n)}}{\partial p_j} b_{2(n-j)}^{(n)}\right]. \tag{3.8}$$

To rationalise the size of expressions we shall expand the terms of (3.8) individually and then sum them. Therefore, using lemma (3.6) one has

$$\sum_{i=0}^{n-j-1} 2\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} p_{i+1} =$$

$$\sum_{i=0}^{n-j-1} p_{i+1} \left(\sum_{\sum k.m_k = n-(i+j)} 2C_{m_1,\ldots,m_n} \prod_{k=1}^{n} p_k^{m_k} - \sum_{r+s=n+i+j+1} \frac{q_r q_s}{2^{2n}} \right).$$

We can in fact absorb the p_{i+1} factor into the secondary summations by adjusting certain indices.

$$\sum_{i=0}^{n-j-1} 2\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} p_{i+1} =$$

$$\sum_{i=0}^{n-j-1} \sum_{\sum k.m_k = n-j+1} 2^{2n} (-1)^{\sum m_k} \binom{\sum m_k}{m_1, \ldots, m_i - 1, \ldots, m_n} \prod_{k=1}^{n} p_k^{m_k}$$

$$- \sum_{i=0}^{n-j-1} \sum_{r+s=n+i+j+1} \frac{p_{i+1} q_r q_s}{2^{2n}}.$$

We can in fact go further and absorb the outer sum into the bracket. To illustrate we shall provide an example. Consider the case $n = 7$ and $j = 2$. One of the solutions to $\sum k.m_k = n - j + 1$ is given by $m_1 = 1$, $m_2 = 1$ and $m_3 = 1$. One can write the term $p_1p_2p_3$ in three ways. Those are $p_1.(p_2p_3)$, $p_2.(p_1p_3)$ and $p_3.(p_1p_2)$. These are contributed by the summands corresponding to when $i = 0$, $i = 1$ and $i = 2$.
In general the number of appearances is equal to the number of non-zero m_k. Thus to evaluate the coefficient of the summation we need to compute

$$\sum_{m_k>0} \binom{\sum m_k}{m_1, \ldots, m_i - 1, \ldots, m_n}.$$

Using the definition, and putting over a common denominator one has

$$\sum_{m_k>0} \binom{\sum m_k}{m_1, \ldots, m_i - 1, \ldots, m_n} = (\sum m_k)! \left[\sum_{m_k>0} \frac{m_k}{m_1! \ldots m_n!}\right].$$

Note that the sum over all nonzero m_k of the m_k themselves is equal to just the sum of the m_k for all k in the appropriate range. Thus we have

$$\sum_{m_k>0} \binom{\sum m_k}{m_1, \ldots, m_i - 1, \ldots, m_n} = \sum m_k (\binom{\sum m_k}{m_1, \ldots, m_n})$$

after inclusion of the $-2^{2n+1}p_{n-j+1}$ term,

$$\sum_{i=0}^{n-j-1} 2\frac{\partial a_{2i+1}^{(n)}}{\partial p_j} p_{i+1} - 2^{2n+1} p_{n-j+1} = \tag{3.9}$$

$$\sum_{\sum k.m_k = n-j+1} 2^{2n}(-1)^{\sum m_k} \sum m_k \binom{\sum m_k}{m_1, \ldots, m_n} \prod_{k=1}^{n} p_k - \sum_{i=0}^{n-j-1} \sum_{r+s=n+i+j+1} \frac{p_{i+1}q_r q_s}{2^{2n}}.$$

For $2a_{2j-1}^{(n)}$ one can expand it using the procedure given in lemma (3.6).

Therefore it can be written as

$$2a_{2j-1}^{(n)} = \tag{3.10}$$

$$\sum_{\sum k.m_k=n-j+1} (-1)^{\sum m_k} 2^{2n} \binom{\sum m_k}{m_1,\dots,m_n} \prod_{k=1}^{n} p_k^{m_k} - \sum_{s+r-t=n+j} \frac{p_t q_t q_r}{2^{2n}} - \sum_{r+s=n+j} \frac{q_r q_s}{2^{2n}}.$$

Finally we have the term in the $b_{2i}^{(n)}$ and its derivative. Using lemma (3.5) we can expand giving

$$\frac{1}{2^{2n-1}} \left(\sum_{i=0}^{n-j} \frac{\partial b_{2i}^{(n)}}{\partial p_j} b_{2(n-j)}^{(n)} \right) = \frac{1}{2^{2n-1}} \left(\sum_{i=1}^{n-j} q_{i+j} \sum_{k=1}^{i} p_k q_{n-i+k} + \sum_{i=1}^{n-j} q_{i+j} q_{n-i} \right). \tag{3.11}$$

To arrive at the formula in the statement one needs to sum (3.9), (3.10) and (3.11). Upon doing so certian cancelations shall occur that are not immediately obvious. Consider

$$\frac{1}{2^{2n-1}} \sum_{i=1}^{n-j} q_{i+j} \sum_{k=1}^{i} p_k q_{n-i+k} - \frac{1}{2^{2n}} \sum_{s+r-t=n+j} p_t q_r q_s - \sum_{i=0}^{n-j-1} p_{i+1} \sum_{r+s=n+i+j+1} \frac{q_r q_s}{2^{2n}}.$$

This in fact vanishes. One can check this by observing that each of the sums has the same number of terms, $\frac{(n-j)(n-j-1)}{2}$, and the sum of the q indices minus the p index is identical in each expression. The differing sign and numerical coefficient then clearly lead to the cancelation. Next consider

$$\frac{1}{2^{2n-1}} \sum_{i=1}^{n-j} q_{i+j} q_{n-j} - \sum_{r+s=n+j} \frac{q_r q_s}{2^{2n}}.$$

Notice that the term $\frac{q_j q_n}{2^{2n}}$ only occurs once in the sum of the $q_{i+j} q_{n-i}$ but twice in the other term. One can therefore re-write this combination as

$$\frac{1}{2_{2n}} \left[\sum_{i=1}^{n-j} q_{i+j} q_{n-i} - q_j q_n \right].$$

One can now simplify the process of summing (3.9), (3.10) and (3.11). This leads one to

$$\begin{aligned}\frac{\partial H^{(n)}}{\partial p_j} &= \sum_{\sum k.mk=n-j+1} 2^{2n}(-1)^{\sum m_k}\binom{\sum m_k+1}{m_1,\ldots,m_n}\prod_{k=1}^{n} p_k^{m_k}\\ &- \sum_{r+s=n+j}\frac{q_r q_s}{2^{2n}}+\frac{q_j q_n}{2^{2n-1}}.\end{aligned}$$

For the case $j=1$, one can repeat the argument used above to obtain all terms in the above formula, with the exception of $2z$. To show why such a term appears in this derivative one must examine the formula for the series A given in equation (2.21).
In the expression given in that theorem one has a expression (in λ) in

$$2^{2n-1}\lambda^{2n+1}(1+T)^2\sum_{i=0}^{n}(-P)^i.$$

One has that $T=\frac{z}{(2\lambda)^{2n}}$. Hence when one expands this, we find the only polynomial term that is explicitly dependent upon z is $z\lambda^0$. Thus, this appears in the formula for $a_1^{(n)}$. Examining now the formula for the linearised Hamiltonian given in equation (3.1), one has the product $2a_1^{(n)}p_1$. Hence, when one expands this one will clearly have the the product $2zp_1$, and thus when one computes $\frac{\partial H^{(n)}}{\partial p_1}$, one will have the term $2z$ as part of it. $\square$

3.4 Proof of theorem 3.1 and 3.2

Proof. With both lemma (3.3) and (3.4) now proved, one has all the information to construct the gradient of $H^{(n)}$.The proof is simple. We construct the gradient of $H^{(n)}$. For this we treat the p_k, q_k and t_k as coordinates. Once these have been computed one can easily establish the compatibility conditions;

$$\frac{\partial^2 H^{(n)}}{\partial r\partial s}=\frac{\partial^2 H^{(n)}}{\partial s\partial r}$$

with r and s being any one of the p_k, q_k or t_k.
From this one can then solve all the differential equations of the for $\frac{\partial H^{(n)}}{\partial s}$ with s being one of p_s q_s and t_s. These then can be solved leading to $3n-1$ arbitrary functions that need to be determined. One does this by substituting each solution into the other equations and deducing the remaining terms. The solution obtained by this process is not necessarily unique; one can add an arbitrary constant term to the Hamiltonian and still satisfy the compatibility conditions .
To rule out the existence of such a constant term let us examine the formula (2.7). One notices immediately that if their is a constant term, then t is contained within the series $a^{(n)}_{2k+1}$, $b^{(n)}_{2k+1}$ or $b^{(n)}_{2k}$. If one look at the defining formulas for these, the equations (2.21), (2.22) and (2.23), it becomes clear that $b^{(n)}_{2k}$ has no constant term.
From this observation, one should then examine a slightly different formula for the Hamiltonian, given as formula (3.1). This contains only products of the $a^{(n)}_{2k}$ with the coordinates p_k. As such this will clearly yield no constant term. □

From this we now wish to demonstrate how one can obtain elements of the Painlevé hierarchy form these Hamiltonians.

3.5 Example: $P^{(2)}_{II}$

We shall now demonstrate the reverse computation of the equations of the Painlevé two hierarchy from our Hamiltonians. The process is identical for each member. Though for higher members the number of calculations is increased. Thus to best illustrate we shall give details for the case of $P^{(2)}_{II}$.
For the subsequent calculations we shall assume that $t_1 = 0$. Starting from the formula in theorem (3.1) one can use this to compute Hamilton's equations for p_1, p_2, q_1 and q_2. $P^{(2)}_{II}$ is a fourth order differential equation. Thus

taking the equation for q_2 and differentiating 3 additional times one has

$$q_2^{(4)} = 16\alpha + (z + 40p_1^2)q_2 + \frac{5}{2^5}q_1q_2^2 + \frac{5}{2^4}p_1q_1^3 + \frac{3}{2^{13}}q_2^5. \tag{3.12}$$

We aim to express the above entirely in terms of q_2. From Hamilton's equations we can write

$$p_1 = \frac{1}{2^5}\left(\frac{-q_2^2}{2^4} + q_2'\right), \; q_1 = 8p_1'$$

$$p_2 = \frac{1}{2^4}\left(\frac{q_1'}{2} + 24p_1^2 - z\right).$$

These are all easily expressible in terms of q_2. After doing so and substituting into (3.12) one gets

$$q_2^{(4)} = 16\alpha - \frac{3}{2^{15}}q_2^5 + q_2\left(z + \frac{5}{2^7}(q_2')^2\right) + \frac{5}{2^7}q_2^2q_2''.$$

From this, if one makes the simple change of variables $q_2 = 4^2w(z)$ one obtains

$$w^{(4)} = \alpha + wz - 6w^5 + 10(w')^2w + 10w^2w''$$

which is exactly $P_{II}^{(2)}$.

It should be clear from the calculations above how this process can be adapted to higher order members of the hierarchy. We now shall shift focus. Specifically we shall now examine the problem of deriving the so called sigma forms for the Painlevé two hierarchy.

4 The sigma form of $P_{II}^{(n)}$

With the information obtained above we are now in a position to derive the sigma form of $P_{II}^{(n)}$. Consider $H^{(n)}$. It is a function of the coordinates p_k, q_k, the parameters t_k and z. We now shift perspective. We consider $\sigma(z)$ a function of z and the t_k alone. The connection between these objects is that they coincide on the trajectories of $H^{(n)}$.
Our ultimate aim is to obtain the differential equation satisfied by $\sigma(z)$. Indeed it takes the particular form:

Theorem 4.1. *The sigma form of $P_{II}^{(n)}$, for all positive integer n, is a non-linear differential equation given by*

$$-2^{2n}\int p_n\left(\frac{d}{dz}\left[L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'-\sigma(z)\right]+t_{n-1}\right)dz+\alpha-\alpha^2+\theta=0$$

with $L_k = L_k\left[\sigma'(z)-\frac{t_{n-1}}{2}\right]$ defined by equation (1.20), and
$p_n = \frac{1}{2^{2n-1}}L_n\left[\sigma'(z)-\frac{t_{n-1}}{2}\right]+\sum_{k=1}^{n-1}t_{n-k}L_{n-k}\left[\sigma'(z)-\frac{t_{n-1}}{2}\right]-\frac{z}{2}$. One has $t_0=-z$ and the constant θ has the value $1/4$ for $n=1$ and zero otherwise. The parameter α is that of $P_{II}^{(n)}$.

One way to proceed is to adapt the Okamoto method, used in the calculation of the sigma form of P_{II}, to our case. Recall this involves calculating the canonical coordinates as differential polynomials of σ and then substitution into $H^{(n)}=\sigma(z)$.
The issue one has with attempting apply the same process in the more general case is that one will have to solve a system of $2n$ polynomial equations. Not only may this not be possible, it may not be feasible for large n without computer algebra.
Our alternative is to use Hamilton's equations to find a transformation between $w(z)$ and $\sigma(z)$. This will allow us to write the canonical coordinates as

differential polynomials of $\sigma(z)$. We shall then use this information to derive a differential equation in terms of $\sigma(z)$ alone.

4.1 The canonical coordinates as differential polynomials of $\sigma(z)$

As an initial step we consider $\frac{dH^{(n)}}{dz}$, which we now consider in terms of sigma. Using the results of the previous section one has

$$\frac{d\sigma(z)}{dz} = \{H^{(n)}, H^{(n)}\} + \frac{\partial H^{(n)}}{\partial z} = 2p_1$$

with the above equality holding for arbitrary n. This result shall prove crucial moving forward.

Recall now the Mazzocco and Mo formalism. It allows one to write the canonical coordinates in terms of $w(z)$, the solution of $P_{II}^{(n)}$, as

$$p_k = \frac{1}{2^{2k-1}}\left(L_k[w_z - w^2] + \sum_{j=1}^{k} t_{n-j} L_{k-j}[w_z - w^2]\right). \tag{4.1}$$

In particular, examining the formula for p_1 one can deduce that

$$p_1 = \frac{1}{2}\left(w_z - w^2 + \frac{t_{n-1}}{2}\right) = \frac{1}{2}\sigma'(z).$$

Thus we have a transformation between $\sigma'(z)$ and $w(z)$. Moreover $w_z - w^2$ forms the argument of the Lenard polynomials that define the p_k.

Lemma 4.2. *The canonical coordinates, p_k for $2 \leq k \leq n$, of $H^{(n)}$ can be written as*

$$p_k = \frac{1}{2^{2k-1}}\left(L_k\left[\sigma'(z) - \frac{t_{n-1}}{2}\right] + \sum_{j=1}^{k} t_{n-j} L_{k-j}\left[\sigma'(z) - \frac{t_{n-1}}{2}\right]\right)$$

with $t_0 = -z$, $\sigma(z) = H^{(n)}$, and $L_k[f]$ being the kth Lenard differential polynomial. The q_k are given as

$$q_{n-j} = 2^{2n-1}\frac{dp_j}{dz} - \sum_{k=1}^{j-1} p_k q_{n-j+k}, \quad j \in \{1, \ldots, n-1\} \tag{4.2}$$

with $q_{n-1} = 4^{n-1}\sigma''(z)$. Moreover one has

$$q_n = \frac{\frac{d}{dz}\left[L_n\left[\sigma'(z) - \frac{t_{n-1}}{2}\right] + \sum_{k=1}^{n-1} t_{n-k}L_{n-k}\left[\sigma'(z) - \frac{t_{n-1}}{2}\right]\right] - \alpha}{-p_n}.$$

Proof. The formula for p_k follows from above, the formula for q_k is recursive. To find its starting point we take the first derivative of $H^{(n)}$, given by the case of $k = 1$ in equation (4.1), and differentiate again. One then finds

$$\frac{d}{dz}2p_1 = \frac{1}{2^{2n-2}}q_{n-1}.$$

This can be verified by using our extensive formulas covering the Hamiltonian derivatives. Combine this with Hamilton's equations i.e.

$$\frac{dp_j}{dz} = -\frac{\partial H^{(n)}}{\partial q_j}$$

and one will obtain the given recursion. The final claim relates to the specific form of q_n. One computes the derivative of p_n with respect to z resulting in

$$\frac{dp_n}{dz} = \frac{1}{2^{2n-1}}\left(\frac{d}{dz}\left[L_n\left[\sigma'(z) - \frac{t_{n-1}}{2}\right] + \sum_{k=1}^{n-1} t_{n-k}L_{n-k}\left[\sigma'(z) - \frac{t_{n-1}}{2}\right]\right] - \frac{1}{2}\right).$$

Then using the formula for q_n given in lemma (3.3) one can solve for the claimed form. $\square$

We can now construct the sigma form.

4.2 Proof of theorem 4.1

Proof. This above lemma gives a recursive formula that can be implemented to compute the coordinates q_k explicitly in terms of $\sigma(z)$ and its derivatives. It should be noted that this recursion can be solved, should direct computation of these quantities be required. We shall commence with the proof of the general formula, and shall at the end observe the alterations that occur in the case of $n = 1$.

The way the canonical coordinates were defined by Mazzocco and Mo allows one to observe that

$$q_n = 4^n w(z).$$

If one now uses the second claim in lemma (4.2) this clearly means that we can write the solution if $P_{II}^{(n)}$ as a rational function of $\sigma(z)$ and its derivatives:

$$w(z) = \frac{\frac{d}{dz}\left[L_n\left[\sigma'(z) - \frac{t_{n-1}}{2}\right] + \sum_{k=1}^{n-1} t_{n-k} L_{n-k}\left[\sigma'(z) - \frac{t_{n-1}}{2}\right]\right] - \alpha}{-2^{2n} p_n}. \quad (4.3)$$

Note that in the case of $n = 1$ one has that $t_0 = -z$. Consequently one will have the term $-(\alpha - 1/2)$ in the numerator of (4.3). Observe that substitution of this expression into $P_{II}^{(n)}$ shall solve it automatically. This formula is the key piece that will enable the final derivation of the sigma form.

Commencing with the case of $k = 1$ in formula (4.1), in conjunction with our assertion that $\sigma'(z) = 2p_1$, we arrive at

$$w_z - w^2 - \sigma'(z) + \frac{t_{n-1}}{2} = 0. \quad (4.4)$$

Using the abbreviation $f(z) = L_n\left[\sigma'(z) - \frac{t_{n-1}}{2}\right] + \sum_{k=1}^{n-1} t_{n-k} L_{n-k}\left[\sigma'(z) - \frac{t_{n-1}}{2}\right]$ (4.4) can be written as

$$\frac{-2(f(z) - z/2)f''(z) + (f'(z))^2 - f'(z) + \alpha - \alpha^2}{4(f(z) - z/2)^2} = \sigma'(z) - \frac{t_{n-1}}{2}$$

for $n \geq 2$, and

$$\frac{-2f(z)f''(z) - (f'z)^2 + \alpha - \alpha^2 - \theta}{4(f(z))^2} = \sigma'(z) + \frac{z}{2}$$

for $n = 1$.

The appearance of θ in the numerator for the $n = 1$ case is a consequence of the $-(\alpha - 1/2)$ in the numerator of (4.3). We can clear the rational part of this equation if we assume that $f(z) - \frac{z}{2} \neq 0$. We shall in fact show in the next section that if this is not true, then this corresponds to a special solution of $P_{II}^{(n)}$, which fixes the value of α. Thus we lose little generality by adopting this assumption. This means that, conceptually, one can write (4.4) as

$$g(\sigma, \sigma', \dots) = \alpha^2 - \alpha + \theta$$

We wish to rewrite the function g. To do this we will differentiate the above, and then divide by a (non-zero) factor i.e.

$$\frac{d}{dz} g(\sigma, \sigma', \dots) = \frac{d}{dz}[h(\sigma, \dots)m(\sigma, \dots)] = \frac{d}{dz}(\alpha^2 - \alpha + \theta)$$

In the process one in principle looses information regarding the constant term $\alpha^2 - \alpha$. However, having already determined the constant, this will be recovered upon integration.

We proceed with the case for $n \geq 2$. The case for $n = 1$ contains very minor differences that do not warrant detailed discussion. Indeed the following procedure is valid only with limited arithmetical adjustments.

Observe that, using lemma (4.2) one can write $p_n = 2^{-(2n-1)}(f(z) - \frac{z}{2})$. Then upon taking the derivative with respect to z and dividing by p_n (4.4) becomes

$$2f'''(z) + 4\sigma''(z)f(z) - 2z\sigma''(z) + 8\sigma'(z)f'(z) \tag{4.5}$$

$$-4\sigma'(z) - 4t_{n-1}(f'(z) - \frac{1}{2}) = 0.$$

Using the definition of $f(z)$, (4.5) can be written as

$$\left(\frac{d^3}{dz^3}+4(\sigma'(z)-\frac{t_{n-1}}{2})\frac{d}{dz}+2\sigma''(z)\right)\left[L_n+\sum_{k=1}^{n-1}t_{n-k}L_{n-k}\right].$$

$$-\frac{d}{dz}(z\sigma'(z)+\sigma(z))+t_{n-1}=0$$

Using the equation (4.1), in conjunction with the fact that the parameters t_k are casimirs, one can conclude that

$$\left(\frac{d^3}{dz^3}+4(\sigma'(z)-\frac{t_{n-1}}{2})\frac{d}{dz}+2\sigma''(z)\right)$$

is in fact the Lenard recursion operator. This acts on the polynomials L_k in the manner given in equation (1.20), thus (4.5) can be written as

$$\left(\frac{d}{dz}\left[L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'(z)-\sigma(z)\right]+t_{n-1}\right)=0. \tag{4.6}$$

It may occur that one should integrate the result to obtain

$$L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'(z)-\sigma(z)+t_{n-1}z=C$$

with C being a constant. The issue is that we have no way of determining information about the constant. Note that (4.6) (aside from dividing through by p_n) is mathematical equivalent to (4.4). Between the two equations we have only applied arithmetic and properties of the Lenard operators to re-write the equation in a more truncated form.

To recover an equation equivalent to (4.4) form (4.6) we must reverse the operations that we performed in obtaining this new representation. This was a division by the function $-2p_n$ (which we can freely multiply (4.6) by as we have already assumed $p_n \neq 0$) and a differentiation. This of course gives all non-constant terms. From (4.4), we know the exact form of the constant term, $\alpha - \alpha^2 - \theta$. Thus reversing these brings (4.6) to an equation that is equivalent to (4.4), but is not in the claimed form, given in the statement of theorem (4.1). $\square$

While the equation in theorem (4.1) is given in the form of an integral, the integrand is in fact an exact differential. In that sense it is a nonlinear differential equation in terms of $\sigma(z)$. This is yet another remarkable occurrence of the integrability of the Lenard functions. To demonstrate our equation we shall give the explicit example of the cases $n = 1, 2$.

Example 4.3 (n=1)**.**

Starting form theorem (4.1) we wish to demonstrate that our formula, up to normalisation recovers the Okamoto sigma form (1.14). Thus taking the equation form (4.1) and integrating by parts we consider

$$\int \frac{d}{dz}(\sigma')[L_2[\sigma' + z/2] - zL_1[\sigma' + z/2] - \sigma]dz - (\alpha - 1/2)^2 = 0. \qquad (4.7)$$

One can then write (4.7) as

$$\int (\sigma'')[\sigma^{(3)} + 3(\sigma')^2 + 2z\sigma' - \sigma]dz - (\alpha - 1/2)^2 = 0.$$

This can be integrated exactly to give

$$\frac{1}{2}(\sigma'')^2 + (\sigma')^3 + z(\sigma')^2 - \sigma\sigma' + (\alpha - 1/2)^2 = 0.$$

This is exactly the Okamoto equation ([18]) with a different choice of normalisation owing to the choice made by Mazzocco and Mo when deriving the hierarchy Hamiltonians.

Example 4.4 (n=2)**.**

From the formula in theorem (4.1) one can clearly integrate by parts to obtain the amended equation

$$-2p_n(L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'(z)-\sigma(z)+t_{n-1}z) \tag{4.8}$$

$$+2\int\frac{d}{dz}p_n\left(L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'(z)-\sigma(z)+t_{n-1}z\right)dz+\alpha-\alpha^2=0$$

with p_n given by lemma (4.2) and L_k being the kth Lenard polynomial of base $\sigma'(z)-\frac{t_{n-1}}{2}$. Clearly we have a differential equation for $\sigma(z)$ if the integrand

$$g(z)=\frac{d}{dz}p_n\left(L_{n+1}+\sum_{k=1}^{n-1}t_{n-k}L_{n-k+1}-z\sigma'(z)-\sigma(z)+t_{n-1}z\right) \tag{4.9}$$

is an exact differential. This is what we shall see explicitly for $n=2$.
We need to compute the relevant quantities i.e. L_3, L_2 and p_2. We therefore obtain

$$p_2=\frac{1}{8}\left(\sigma^{(3)}(z)+3(\sigma')^2-2t_1\sigma'(z)-\frac{t_1^2}{2}-\frac{z}{2}\right),$$

$$L_2=\sigma^{(3)}(z)+3(\sigma'(z))^2-3t_1\sigma'(z)+\frac{3t_1^2}{4},$$

$$L_3=\sigma^{(5)}(z)+10\sigma^{(3)}(z)\sigma'(z)+5(\sigma''(z))^2+10(\sigma'(z))^3,$$

$$-5t_1\sigma^{(3)}(z)-15t_1(\sigma'(z))^2+\frac{15}{2}t_1^2\sigma'(z)-\frac{5}{4}t_1^3.$$

From these we can then form the following quantities:

$$\frac{d}{dz}p_2=\frac{1}{8}\left(\sigma^{(4)}(z)+6\sigma'(z)\sigma''(z)-2t_1\sigma''(z)-\frac{1}{2}\right)$$

$$L_3+t_1L_2-z\sigma'(z)-\sigma(z)+t_1z=$$

$$\sigma^{(5)}(z)+10\sigma^{(3)}(z)\sigma'(z)+5(\sigma''(z))^2+10(\sigma'(z))^3$$

$$-4t_1(\sigma^{(3)}(z)+3(\sigma'(z))^2)+\frac{9}{2}t_1^2\sigma'(z)-\frac{t_1^3}{2}-z\sigma'(z)-\sigma(z)+t_1z.$$

One can now compute the quantity $g(z)$, (4.9), for the case $n=2$. One can now show that it is indeed an exact differential. To abbreviate we drop the argument from σ and its derivatives, and thus we can write

$$128\int g(z)dz = -4z^2t_1+4zt_1^3-368t_1(\sigma')^4+192(\sigma')^5+16t_1\sigma''+8z(\sigma'')^2$$

$$+28t_1^2(\sigma'')^2+4\sigma(2z-t_1^2+8t_1\sigma'-12(\sigma')^2-4\sigma^{(3)})+16zt_1\sigma^{(3)}$$

$$-8t_1^3\sigma^{(3)}-16(\sigma'')^2\sigma^{(3)}-16t_1(\sigma^{(3)})^2+(\sigma')^3(272t_1^2-32z+160\sigma^{(3)})$$

$$+16(\sigma')^2(15(\sigma'')^2-2t_1(3t_1^2-2z+6\sigma^{(3)}))-8\sigma^{(4)}-32t_1\sigma''\sigma^{(4)}+8(\sigma^{(4)})^2$$

$$-8\sigma'(20t_1(\sigma'')^2+(2z-t_1^2-4\sigma^{(3)})(2t_1^2+\sigma^{(3)})+\sigma''(6-12\sigma^{(4)})).$$

As one can see, even the simplest case of our new formula leads to expressions of substantial size. Thus implementation of our formula is only practical in computer algebra.

From this we move on to some natural applications of both the sigma form and our explicit Hamiltonian expression.

5 Special solutions and Bäcklund transformations

The first application is perhaps the most natural; canonical transformations of $H^{(n)}$. Our starting point for these are two transformations of the Painlevé hierarchy detailed in [21]. We shall briefly introduce and review these transformation and their action upon $P_{II}^{(n)}$. We shall show that they give rise to canonical transformation of the Hamiltonian system defined by $H^{(n)}$ as given in theorem (3.1).

Our second application, is an examination of the case when the coordinate p_n vanishes. Recall that this not vanishing was a key assumption in our derivation of the sigma form. We shall discover that this leads very naturally to a special solution of $P_{II}^{(n)}$. Moreover one can, using the work of Shimomura ([24]), show that this solution is equivalent to a solution of a member a Painlevé one hierarchy.

5.1 Bäcklund Transformations

For this section we shall impose the condition that all the additional parameter t_k vanish. We do so in order to draw relevant comparisons to and deductions from the work in [21], in which the same condition is also applied. Consider the parameter space of the second Painlevé hierarchy. Given our above hypothesis related to the additional parameters this is completely determined by, to use the language of [27], the essential parameter α.
On this space acts an affine Weyl group, in this case $A_{(1)}$. They key question is to determine how this action manifests upon the equation itself. In a series of papers, [17], Okamoto showed that this is in fact given by Bäcklund transformations.

In [21] Clarkson, Pickering and Joshi studied Bäcklund transformations of the P_{II} hierarchy. They begin with the observation that for P_{II} itself, with solution $w(z)$ and parameter α, has a transform given by

$$w(z) = \tilde{w}(z) + \frac{2\tilde{\alpha} \mp 1}{z \mp 2w_z + 2w^2}, \quad \alpha = \pm 1 - \tilde{\alpha} \tag{5.1}$$

with $\tilde{w}(z)$ solving P_{II} with parameter $\tilde{\alpha}$. From this, they proceed to generalise this transformation to all members of the P_{II} hierarchy. These are given, using our notation, by

$$w(z) = \tilde{w}(z) + \frac{2\tilde{\alpha} \mp 1}{z - L_n[\pm\tilde{w}_z - \tilde{w}^2]}, \tag{5.2}$$

with $\alpha = \pm 1 - \tilde{\alpha}$. Recall that the function, p_n (the co-ordinate of the Hamiltonian system of $P_{II}^{(n)}$) is given by equation (4.1). Thus apply the transformation one can replace every instance of $w(z)$ by $\pm\tilde{w}(z)$ in the formula. As before $\tilde{w}(z)$ solves $P_{II}^{(n)}$ with parameter $\tilde{\alpha}$. Observe that the indeterminate ± 1 factor present in the above formulas is in fact a manifestation of the trivial transformation $(w(z), \alpha) \to (-w(z), -\alpha)$, possessed by all members of the hierarchy. This can be seen in the fact that the terms of the functions

$$\left(\frac{d}{dz} + 2w\right) L_k[w_z - w^2]$$

considered as polynomials in $w(z)$ and its derivatives, are of odd degree. Observe that some combination of the above transformations bring the Painlevé two equation with parameter α to that with parameter $\alpha \pm k$ with k being an integer. This fact is exploited in [21] to produce special solutions for the P_{II} hierarchy for half integer values of the parameter.

Our result, is that for the trivial transformation $(w(z), \alpha) \to (-w(z), -\alpha)$, which we call parity, and for that given by equation (5.1), which we call affine, we can construct canonical transformations for the system $H^{(n)}$. We provide explicit formulas for how the coordinates transform, and write the new Hamiltonians.

5.1.1 The affine transformation

As explained above this transformation is given by (5.1). Writing this in terms of $w(z)$ explicitly one has the formula (5.2).
This transformation has the property that it preserves the form $w_z - w^2$ i.e.

$$w_z - w^2 = \tilde{w}_z - \tilde{w}^2.$$

Conveniently, the canonical coordinates as defined by equations (4.1) and (4.2) are given as Lenard polynomials of this form. We can therefore conclude immediately that

$$\tilde{p}_k = p_k, \ k = 1, \ldots, n \tag{5.3}$$

$$\tilde{q}_k = q_k, \ k = 1, \ldots, n-1. \tag{5.4}$$

The omission of q_n from the above statement reflects that q_n is explicitly dependent upon α. Thus the only work one must do to establish the details of the canonical transformation is to deduce how this transforms.
Recall that q_n is linearly proportional to $w(z)$. Thus dividing (5.2) through by 4^n, and choosing a parity one can write

$$\tilde{q}_n = q_n - \frac{2\tilde{\alpha} - 1}{-p_n}. \tag{5.5}$$

It is easy to show that these new coordinates, as given by equations (5.3), (5.4) and (5.5) are canonical.
What remains is to detail how $H^{(n)}$ itself transforms. For the majority of terms in formula of theorem (3.1) this is trivial. Indeed we need only restrict our attention to those terms dependent upon q_n. These are given by

$$\frac{1}{2^{2n}}(p_n q_n^2 + (1 - 2\alpha)q_n).$$

Substituting our above assertions into this and calculating this can be written in the new variables as

$$\frac{1}{2^{2n}}(\tilde{p}_n\tilde{q}_n^2 + (1-2\tilde{\alpha})\tilde{q}_n).$$

These terms are also those explicitly dependent upon the parameter α. Thus we can conclude that

$$\tilde{H}^{(n)}(\tilde{p},\tilde{q},z,\tilde{\alpha}) = H^{(n)}(\tilde{p},\tilde{q},z,\tilde{\alpha}) \tag{5.6}$$

i.e. the new Hamiltonian is the old Hamiltonian with the old coordinates exchanged one for one with the new, but with the new parameter $\tilde{\alpha}$.

While computing this canonical transformation is of some intrinsic worth, our ultimate aim is to demonstrate what is the corresponding change on the sigma form. In the case of the affine transformation we have the relation

$$w_z - w^2 = \tilde{w}_z - \tilde{w}^2$$

between the transformed and untransformed solution to $P_{II}^{(n)}$. If one then uses the procedure of section 4 on this transformed system this relation implies that

$$\tilde{\sigma}'(z) = \sigma'(z).$$

Thus in this case our sigma form is changed by at most a constant. Moreover if we consider the differential equation satisfied by $\tilde{\sigma}(z)$, then owing to equation (5.6), it will be identical to that satisfied by $\sigma(z)$ with α replaced by $\tilde{\alpha}$. In that sense one can describe this as a sigma invariant transformation.

5.1.2 The parity transformation

This transformation is given by, at the level of $P_{II}^{(n)}$, the change $\tilde{w}(z) = -w(z)$ in the solution and the corresponding change $\tilde{\alpha} = -\alpha$ in the parameter.

The coordinate q_n is linearly dependent upon $w(z)$ (given explicitly as $q_n = 4^n w(z)$). Form this we immediately deduce that the canonical transformation is, in part, given by

$$\tilde{q}_n = -q_n.$$

From equations (4.1) and (4.2) one can see that, in terms of the solution to $P_{II}^{(n)}$, the p_k and q_k are given in terms of the differential polynomial $w_z - w^2$. While applying the transform $\tilde{w} = -w$ is a trivial affair, we want to be able to compute the new coordinates in terms of the old.

Theorem 5.1. *The Bäcklund transformation $\tilde{w} = -w$ is given in the canonical coordinates by*

$$\tilde{p}_k = \frac{1}{2^{2k-1}} \left[L_k \left[-2p_1 - \frac{q_n^2}{4^{n-1}} \right] + \delta_{kn} \frac{z}{2} \right], \ k = 1, \ldots, n \tag{5.7}$$

$$\tilde{q}_{n-j} = 2^{2n-1} \frac{dp_j}{dz} - \sum_{k=1}^{j-1} \tilde{p}_k \tilde{q}_{n-j+k}, \ j = 1, \ldots, n-2. \tag{5.8}$$

Proof. Starting for the transformation for q_n, which is given, and differentiating this formula using the formula in theorem (3.1) one obtains

$$\frac{d\tilde{q}_n}{dz} = -2^{2n+1} p_1 - \frac{q_n^2}{2^{2n}}.$$

As this is a canonical transformation we require Hamilton's equations for the new coordinates to be given by those for the old with the new substituted. In the case of the above this implies that

$$\frac{d\tilde{q}_n}{dz} = 2^{2n+1} \tilde{p}_1 + \frac{\tilde{q}_n^2}{2^{2n}}.$$

Thus from these one can deduce that

$$\tilde{p}_1 = -p_1 - \frac{q_n^2}{2^{4n}}. \tag{5.9}$$

Using the equation (4.1) we can write the above in terms of $w(z)$ (note this is the solution before transformation) as $\tilde{p}_1 = \frac{1}{2}(-w_z - w^2)$. This observation in combination with the formulas (4.1) and (4.2) yields the formulas in the statement. $\square$

For the purposes of computing these formulas (say, in computer algebra), the fact that the Lenard polynomials $L_k[f]$ are obtained by a process of both differentiation and integration means that direct computation of (5.7) is not recommended. Instead one should produce a generic formula for $L_k[f]$, which are differential polynomials. Then one can substitute $f = -p_1 - \frac{q_n^2}{2^{4n}}$ and use the results of theorem (3.1) to evaluate the derivatives properly.

While the above gives a way to compute the transformed coordinates in terms of the old, it does not guarantee the canonicity of these coordinates. We demonstrate now the canonicity of the transformation in the case $n = 2$ explicitly.

Example 5.2 $(n = 2)$**.**

Our theorem above and theorem (3.1) for $n = 2$ leads to the following transformation of coordinates:

$$\tilde{p}_1 = -p_1 - \frac{q_2^2}{2^8};$$

$$\tilde{p}_2 = -\frac{z}{8} + p_1^2 - p_2 - \frac{q_1 q_2}{128} - \frac{p_1 q_2^2}{256};$$

$$\tilde{q}_1 = -q_1 - 2p_1 q_2 - \frac{q_2^3}{256};$$

$$\tilde{q}_2 = -q_2.$$

From these expressions one can directly compute the Poisson brackets:

$$\{\tilde{p}_1, \tilde{q}_1\} = 1, \quad \{\tilde{p}_2, \tilde{q}_2\} = 1;$$

$$\{\tilde{p}_1, \tilde{p}_2\} = 0, \quad \{\tilde{p}_1, \tilde{q}_2\} = 0;$$

$$\{\tilde{q}_1, \tilde{p}_2\} = 0, \quad \{\tilde{q}_1, \tilde{q}_2\} = 0.$$

The veracity of these formulas is predicated on the original coordinates being canonical. This of course was a key result of the work of Mazzocco and Mo ([15]).

For the case of general n, we can also use the work in [15]. Recall that Mazzocco and Mo constructed the canonical coordinates, given by equations (4.1) and (4.2), related to $P_{II}^{(n)}(w(z), \alpha)$. Now it is evident, in particular from the work in [21], that if $w(z)$ satisfies this equation, then $-w(z)$ satisfies the $P_{II}^{(n)}$ with parameter $-\alpha$. Thus one can repeat their procedure and arrive at equations (5.7) and (5.8). Moreover, this ensures that they are canonical.

The final consideration is to determine how the Hamiltonian transforms. By our supposition, the Hamilton's equations for the new coordinates are the same as the old, just with the new coordinates substituted. All that remains is to calculate how α changes.

From (5.7), we have that

$$\tilde{p}_n = \frac{1}{2^{2n-1}} \left[L_n[-w_z - w^2] - \frac{z}{2} \right].$$

We shall, in order to rationalise the size of subsequent expressions invoke the above Lenard polynomial as L_n^-. Taking the derivative of the above with respect to z, and using $P_{II}^{(n)}$ one can obtain

$$\frac{d}{dz}\tilde{p}_n = \frac{1}{2^{2n-1}}[2w(z)L_n^- - zw(z) - \alpha - \frac{1}{2}]$$

$$= \frac{1}{2^{2n-1}}[\tilde{q}_n\tilde{p}_n - (\frac{1}{2} + \alpha)].$$

This is precisely what one would expect, apart form the substitution of $-\alpha$ for α in the original formula (though, realistically, this would also be expected). Consequently we can conclude that the Hamiltonian transforms as follows:

$$\tilde{H}^{(n)}(\tilde{p}, \tilde{q}, z, \alpha) = H^{(n)}(\tilde{p}, \tilde{q}, z, -\alpha). \tag{5.10}$$

That is the new Hamiltonian is identical to the old (when written in the new coordinates), with only a change in α given by a change in parity.

As for the affine case we want to establish the corresponding change in the sigma form. For the differential equation, one can apply the procedure of section 4 starting with equation (5.10). The consequence is an equation, for $\tilde{\sigma}(z)$, identical to that found in theorem (4.1), with $-\alpha$ substituted for α. Thus our transformation leaves that equation invariant to a large extent. Unlike in the affine case the relationship between $\tilde{\sigma}(z)$ and $\sigma(z)$ is not trivial. From the details of the canonical transformation, specifically equation (5.9) one can deduce

$$\tilde{\sigma}'(z) = -\sigma'(z) - \frac{\partial_z L_n[\sigma'(z)] - \alpha}{z - 2L_n[\sigma'(z)]}.$$

This is a non trivial differential equation. In effect it would be of equivalent complexity to solve the equations for σ and $\tilde{\sigma}$ given by theorem (4.1) and compare them directly, as to solving the above. We include it here to complete our treatment of this case, as the same for the affine case.

5.2 A special solution of $P_{II}^{(n)}$

The issue at hand is that of the assumption made in the proof of the new sigma form. That is, in the general situation, one may assume that the coordinate p_n does not vanish identically.

The first test that one can apply is to examine the Hamilton's equation for p_n. This is of course directly related to the derivative $\frac{\partial H^{(n)}}{\partial q_n}$. Therefore consider the Hamilton equation for p_n. This is given by

$$\frac{dp_n}{dz} = \frac{1}{2^{2n-1}} \left(\alpha - \frac{1}{2} - p_n q_n \right).$$

From this it is obvious that a necessary condition for the vanishing of p_n is that the parameter α must take the value $\frac{1}{2}$. As such we can conclude that for generic α our assumption was indeed valid.
Having settled this, it is natural to consider this special case. If p_n does vanish then this implies that $w(z)$ satisfies

$$L_n \left[w_z - w^2 \right] + \sum_{k=1}^{n-1} t_{n-k} L_{n-k} \left[w_z - w^2 \right] = \frac{z}{2}. \tag{5.11}$$

Importantly any function that satisfies the above equation will solve $P_{II}^{(n)}$, (1.19), upon direct substitution, for $\alpha = 1/2$. We have consequently obtained a solution of $P_{II}^{(n)}$ determined exclusively by (5.1). This is itself a differential equation for $w(z)$ of order one less than $P_{II}^{(n)}$ i.e. a special solution. Note that these special solutions are not in themselves new. In [21] Clarkson et al derived a method to compute special solutions for the hierarchy for all half integer values of the parameter α. Our construction of these solutions is new however, as it depends on the explicit formulas for the Hamiltonians computed in section three. We shall now add an additional observation about these solutions.

5.2.1 Connection to P_I hierarchy

Consider our special solution in the case $n = 2$. Explicitly, this is given as

$$L_2[w_z - w^2] = \frac{1}{2}.$$

Now if one performs the transformation $-y = w_z - w^2$ one obtains

$$-y'' + 3y^2 = \frac{1}{2}z$$

which after some simple rescaling is the P_I equation.
In [24], Shimomura defined a Painlevé one hierarchy in the following manner

$$Dd_{n+1}[y] = (D^3 - 8yD - 4y)d_n[y]$$

$$P_I^{(n)} : d_{n+1}[y] + 4z = 0. \tag{5.12}$$

It is clear that in a similar manner, our special solutions can be mapped to this hierarchy. We have therefore established the following result:

Proposition 5.3. *For any solution, $y(z)$ to the n member of the Painlevé one hierarchy (5.12), The solution $w(z)$ defined by $-y = w_z - w^2$ corresponds to a special solution of the $n+1$th member of the Painlevé two hierarchy.*

In particular the change of variables between these hierarchies is given by $-y = w_z - w^2$. This is a Ricatti equation and therefore can in principle always be solved.

6 Conclusion

In this thesis we have presented 2 main results:

- Closed form formulas for the Hamiltonian of the second Painlevé hierarchy, theorems (3.1) and (3.2).
- A sigma form of this hierarchy, theorem (4.1).

We have also discussed the canonicity of the Bäcklund transformations and some special solutions of the second Painlevé hierarchy.

The new formulas for the Hamiltonian was an extension of the work done by Mazzocco and Mo, [15], who had obtained expression for these quantities in terms of certain non-trivial power series. By recognising that these series simplified upon differentiation, we constructed the gradient of these Hamiltonians. From this, deduction of the full formula was relatively straight forward.

The purpose of finding these new expression was to allow the computation of the sigma forms of the higher members of the Painlevé two hierarchy. Okamoto ([18]) had obtained this for the first member by a procedure that required intimate knowledge of this Hamiltonian. Having obtained equally detailed expressions for the higher members, we were able to derive sigma forms in these cases.

We then wanted to apply our new results. The first was to study the Bäcklund transformation of $P_{II}^{(n)}$ and show that these very naturally lead to canonical transformation of the Hamiltonian systems. In doing so we were able to detail the transformation explicitly. Importantly, we observed the effect that these transformations had on the sigma form. Indeed, this was also computed explicitly.

A more interesting application relates to a special solution to the Painlevé hierarchy. This solution, while already known ([21]), was particularly simple to derive given our new expressions of the Hamiltonians. We then combined this with the work of Shimomura ([24]), and concluded that these correspond to solutions to members of a Painlevé one hierarchy. This raises questions about the connection between these hierarchies and could be a future line of inquiry.

Given the appearance of the sigma forms for the classical Painlevé equations in physical contexts, it is natural to ask if these new higher forms also have such utility. Another natural question is to find the sigma form of other Painlevé hierarchies. Given the similarity between the second and fourth equation, one could reason that a similar process to the one described above could be used to obtain the sigma forms for that hierarchy as well.

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