Loughborough University

This item was submitted to Loughborough's Research Repository by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

The initial value problem for colliding plane waves: the linear case

PLEASE CITE THE PUBLISHED VERSION

PUBLISHER

© Miguel Santano-Roco

PUBLISHER STATEMENT

This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

Santano-Roco, Miguel. 2019. “The Initial Value Problem for Colliding Plane Waves: The Linear Case”. figshare. https://hdl.handle.net/2134/34753.

Loughborough University

University Library

Author/Filing Title ..... SANTANO-ROCA .....

.....

Class Mark ..... T .....

Please note that fines are charged on ALL overdue items.

FOR REFERENCE ONLY

0402861663

The Initial Value Problem for Colliding Plane Waves: The Linear Case

by

Miguel Santano-Roco

A Master's Thesis
Submitted in partial fulfilment of the requirements
for the award of

Master of Philosophy of Loughborough University

26 February 2003

© by Miguel Santano-Roco, 2003

Loughborough University
Library
Date June 04
Class
Acc No. 04028616 6

Contents

Abstract i

Keywords ii

Acknowledgements iii

Originality Statement iv

1 Introduction 1

1.1 Introductory material 1

1.2 Notation 4

1.3 Plane wave space-times 7

2 Colliding plane wave problem 11

2.1 The space-time structure 12

2.2 Field equations 14

2.3 Junction conditions and the initial value problem 16

2.4 The linear case 19

2.5 Riemann's method 20

3 Abel transform 22

4 Confirmations of the Abel transform method **27**

4.1 The Khan-Penrose solution 27

4.2 The Szekeres solution 28

4.3 Legendre functions 29

5 Double impulse and sandwich waves **36**

6 Initial value problem for other types of colliding waves **46**

6.1 Step wave 46

6.2 Sandwich Szekeres-like incident waves 47

7 The solution as a series expansion of hypergeometric functions **49**

8 Conclusion **55**

A Functional relations **57**

B Legendre functions **58**

C $\Pi(x|m)$ in terms of hypergeometric functions **60**

Bibliography **62**

Abstract

Einstein's equations are nonlinear, therefore, when gravitational waves meet, they must interact. This interaction process has been studied in detail for some cases, particularly those involving plane waves. To understand the structure of the space-time resulting from collisions of this type, many solutions have been generated. However, these have been obtained by first taking a "candidate" resulting space-time, and then extending it back to give rise to the originating waves. While these techniques are not too complex, it is not an easy task to obtain physically acceptable initial waves, and this is the greatest disadvantage of this indirect method.

The main aim of this thesis is to consider a direct approach, to find a method that can overcome the difficulties indicated above, giving rise to solutions from arbitrary colliding plane waves. A well posed initial value problem is formulated for the collinear case. This is achieved by making use of generalised Abel transforms. This method is successfully tested for some particularly well known cases. However, when it is applied to more general cases, a number of problems arise. Along the direct and inverse transformation process, there are several successive integrations involved, and it is in these integrations that the main difficulties appear, as the integrands themselves contain elliptic integrals.

Nevertheless, a final way is found to obtain a final solution, which gives the solution as a series expansion involving hypergeometric functions. Consequently, assuming we can obtain the spectral functions generated by the Abel transforms, the problem would be theoretically solved, although the calculations tend to become extremely complicated when arbitrary colliding waves are taken.

Keywords

Abel transform,
characteristic,
collision,
gravitation,
hypergeometric function,
initial value problem,
plane wave.

Acknowledgements

First of all, I would like to thank my supervisor, Professor Jerry Griffiths, for his kindness, continuous help and great patience. Without them, this work would simply not exist.

Gracias to my parents, Silvestre and Maruja, for their generosity and for being always there for me, supporting and encouraging me.

During my stay in Loughborough, I had the enormous luck to meet Mari Paz, who became not only my girlfriend, but my soulmate. Thanks for being so partial about me, and for believing in me even when I did not.

I am also thankful to Dr Carlos Sanz-Navarro for his help in numerous occasions, particularly in my first months in Loughborough, when I was completely lost. Thanks as well to Dr Rod Halburd, for his many help offers.

My gratitude also goes to those friends who made me feel I could count on them whenever I might need them. In particular, it goes to Oscar and to Ahping.

I must thank everybody at the Department of Mathematical Sciences for creating such a friendly environment, and for being always willing to give a hand.

Finally, I thankfully acknowledge the award of a University Research Studentship.

Originality Statement

This thesis is mostly an organised review of literature regarding the collision of plane gravitational waves in the linear case.

Although the author contributed with some original work –like that appearing in Section 4.3 and in Appendix C, or calculating terms here and there–, this thesis has been written making extensive use of notes provided by Professor Griffiths. He carried out the hard work of finding the principal results and helping me understand what he had done as well as what I was doing.

Chapter 1

Introduction

1.1 Introductory material

According to Einstein's general theory of relativity, the gravitational field equations, as well as the electromagnetic ones, posses wave-like solutions. However, the nonlinearity of the first ones imply that when two –or more– gravitational waves meet, they must interact. Since the gravitational waves do not just propagate in the space-time, but are disturbances of the space-time itself, the interaction of gravitational waves also produces a variation in the properties of the future region of the space-time after the interaction.

Apart from being an interesting mathematical feature of the field equations, gravitational waves have not always been believed to exist *physically*. However, since the controversial times when Weber began to claim he had detected gravitational radiation [32], until now, this scepticism has decreased significantly in the scientific community. Although no undoubted direct detection of gravitational radiation has yet been obtained , there has been found indirect evidence from the observations carried out by Taylor [30] on a binary pulsar. In that system, the energy loss indicated by the increase of angular frequency balances perfectly with the energy predicted to be radiated in form of gravitational waves. Nowadays, different projects are being carried out to detect gravitational radiation directly (an outline of these can be found in [31]).

It is expected that the gravitational waves that are likely to be detected will have originated in very distant parts of the galaxy, which implies that they will

only be observed after a great number of interactions. A much simpler approach to the study of gravitational waves interaction is given by the linearized theory, where the variations in spacetime are considered perturbations (thus neglecting higher order interactions). However, the most interesting effects, like the focussing of colliding waves (giving rise to singularities) are nonlinear, and can consequently be studied only in the framework given by the theory of exact collisions. It is this *exact* theory that will be used to obtain the results appearing in this thesis. Therefore, it may be seen how important it is to gain a deep understanding of the phenomenon of gravitational waves interaction.

An important property of gravitational waves is that, as they travel, their interaction with matter is much subtler than that of electromagnetic waves. Moreover, electromagnetic radiation has a predominantly dipole nature, whereas the gravitational radiation process is dominated by quadrupole moment variations. In other words, gravitational waves are consequence of coherent motions of great mass distributions, whereas electromagnetic waves typically originate in disordered collisions of charged particles, like gas concentrations. Therefore, not only could they reveal complementary information about some situation, but in most cases, the information provided by gravitational waves will be unattainable by other means.

Gaining understanding about the collision of gravitational waves and the underlying mathematical methods, can also shed more light in other areas, like in string theory. For instance, as pointed out by Feinstein [11], there is a very close relation between the solvability of the colliding wave problem and some problems in superstring theory (the equations involved take the same form).

In order to study the collision between gravitational waves, simple mathematical models have been constructed. The first ones were published by Szekeres [28, 29], and Khan and Penrose [21]. In those papers, exact solutions were found describing the situation where two plane gravitational waves with constant, aligned polarization axes propagating in a Minkowski background suffer a frontal collision[1]. A very important conclusion was found: the two waves mutually focus each other, creating a singularity in the future region.

Since the time when these first exact solutions where published, a great num-

[1]Note that we can always choose a reference frame for which the two waves can be observed to collide frontally.

ber of exact solutions describing the interaction of gravitational waves has been obtained (for a detailed review, see [14]). Nevertheless, most of these solutions have been achieved by finding firstly a family of solutions of the field equations in the interaction region. Then, the problem posed was to find, among the arbitrary parameters characterising a family, the ones satisfying the junction conditions corresponding to colliding plane waves. Finally, these functions were extended back in order to obtain the particular waves whose collision would give rise to the particular solution taken initially. However, most of these "initial" waves have physically unacceptable properties (infinite energy, etc.).

Thanks to the number of solutions generated by this "inverse" approach, the general structure of these solutions has now been generally understood. However, this does not seem to be the most appropriate way to approach the problem, as it requires a prior knowledge of the solution to find the waves which previously had to interact. A more satisfactory approach would be that of obtaining the solution from initially known waves. This direct approach (in opposition to the inverse one) has the advantage of enabling a free choice of initial waves, which means that one may restrict this choice to consider physically acceptable cases.

The study of the collision of plane gravitational waves is particularly important, as it can describe the situation involving realistic waves in an asymptotic limit. But even for this simple assumption of colliding plane waves, there is still much to be done, as no direct method was found to be practically applicable.

The direct approach to the study of collisions of plane gravitational waves is not new. A general method was developed by Szekeres [29] in one of the earliest publications dealing with the collision and further interaction of plane gravitational waves with constant, aligned polarizations in vacuum. This method, which makes use of Green's functions, works in theory, but is of little use when the approaching waves are taken arbitrarily, as the calculations involved happen to be too complicated. More recently, an alternative theoretical method to solve this initial value problem was given by Hauser and Ernst [17], which was developed by applying Abel transforms to the data on the junctions (in [18, 19, 20], they also dealt with the non-aligned case). These give rise to two spectral functions, which can be used to obtain the general solution.

In this thesis, the characteristic initial value problem for colliding plane gravitational waves with constant and aligned polarizations propagating in a flat

background is formulated, and the practical applicability of the Abel transform method is investigated in detail.

The Abel transform method is then implemented for some well known cases (Khan-Penrose solution, Szekeres solution, series of Legendre functions). However, when it is applied to more complex cases (double impulse and sandwich waves), difficulties are found when trying to obtain the exact solutions explicitly in terms of elementary functions. These difficulties arise in the final step of the process, when the final solution is being evaluated from the spectral functions produced by the Abel transform. Arguments are given which indicate that such solutions cannot be expressed in terms of elementary functions.

Finally, the general family of exact solutions in the interaction region is constructed for arbitrary initial data in terms of an infinite series of self-similar solutions of the main field equation. The solution has the form of an infinite series involving hypergeometric functions, whose coefficients are uniquely determined by a power expansion of each of the two spectral functions, thus avoiding the difficulties mentioned above.

The reader will notice that there are plenty of integrals which are expressed in the form of either logarithms or inverse trigonometric and hyperbolic functions. When these integrals were evaluated, a table of functional relations, like that appearing in Appendix A, was found useful, particularly to contrast results. Hence the inclusion of such a table.

1.2 Notation

Throughout this work, Greek indices will range from 0 to 3, whereas Latin ones, will take values between 1 and 3 or, when indicated, between 2 and 3.

We will consider a space-time described by a metric $g_{\mu\nu}$ with signature $(+,-,-,-)$, and a metric connection $\Gamma^{\lambda}_{\kappa\nu} = \frac{1}{2}g^{\lambda\alpha}(g_{\kappa\alpha,\nu} + g_{\alpha\nu,\kappa} - g_{\kappa\nu,\alpha})$. The Riemann curvature tensor is

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\lambda}_{\kappa\nu,\mu} - \Gamma^{\lambda}_{\kappa\mu,\nu} + \Gamma^{\beta}_{\kappa\nu}\Gamma^{\lambda}_{\beta\mu} - \Gamma^{\beta}_{\kappa\mu}\Gamma^{\lambda}_{\beta\nu}.$$

The symmetries of this tensor as can be found in many texts (for example, in

[23]), are given by

$$R_{\kappa\lambda(\mu\nu)} = 0, \qquad R_{(\kappa\lambda)\mu\nu} = 0, \qquad R_{\kappa[\lambda\mu\nu]} = 0,$$

where round brackets $(...)$ denote symmetrisation, and square brackets $[...]$, antisymmetrisation. These relations reduce the number of independent components from $4^4 = 256$ to 20. In most applications, the metric tensor is assumed to be C^1 and piecewise at least C^2, in order to make the curvature tensor be at least C^0 ("Lichnerowicz conditions", [22]).

The following decomposition can be made:

$$R_{\kappa\lambda\mu\nu} = C_{\kappa\lambda\mu\nu} + \frac{1}{2}(S_{\mu\nu}g_{\nu\kappa} - S_{\lambda\nu}g_{\mu\kappa} - S_{\kappa\mu}g_{\nu\lambda} + S_{\kappa\nu}g_{\mu\lambda}) + \frac{1}{12}R(g_{\lambda\mu}g_{\nu\kappa} - g_{\lambda\nu}g_{\mu\kappa}),$$

where

$R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$ is the Ricci tensor,

$R = R^{\alpha}{}_{\alpha} = R^{\alpha\beta}{}_{\alpha\beta}$ is the Ricci scalar,

$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ is the trace-free part of the Ricci tensor, and

$C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} - \frac{1}{2}(R_{\lambda\mu}g_{\nu\kappa} - R_{\lambda\nu}g_{\mu\kappa} - R_{\kappa\mu}g_{\nu\lambda} + R_{\kappa\nu}g_{\mu\lambda}) + \frac{1}{6}R(g_{\lambda\mu}g_{\kappa\nu} - g_{\lambda\nu}g_{\mu\kappa})$

is the Weyl tensor – the trace-free part of the Riemann tensor. This also possesses the symmetries of $R_{\kappa\lambda\mu\nu}$ and, consequently, is antisymmetric in any pair of indices. Note that the Weyl tensor has 10 independent components, $S_{\mu\nu}$ has another 9, and the remaining component is given by R.

According to Einstein's equations, the Ricci tensor is completely determined by the local distribution of mass-energy present,

$$R_{\mu\nu} = -8\pi\left(T_{\mu\nu} - \frac{1}{2}T^{\alpha}{}_{\alpha}\,g_{\mu\nu}\right),$$

whereas the Weyl tensor is the component of the curvature that is not determined by local sources. Therefore the Weyl tensor represents a free gravitational field, and includes the independent components of the curvature related to pure gravitational waves. We have chosen units such that $G = 1$, $c = 1$. It is also important to indicate that throughout the rest of this work, the cosmological constant is asumed to be $\Lambda = 0$ (for waves in spacetimes with non-zero Λ, one can see, for example, [26]).

Considering the timelike unit vector t_μ, and the spacelike unit vectors a_μ, b_μ, c_μ, all mutually orthogonal, there may be defined a null tetrad, specified by two

real and one complex null vectors,

$$l_\mu = \tfrac{1}{\sqrt{2}}\,(t_\mu - c_\mu) \qquad n_\mu = \tfrac{1}{\sqrt{2}}\,(t_\mu + c_\mu) \qquad m_\mu = \tfrac{1}{\sqrt{2}}\,(a_\mu - ib_\mu)\,.$$

These can be used to define the bivectors

$$U_{\mu\nu} = 2\bar{m}_{[\mu}n_{\nu]}; \qquad V_{\mu\nu} = 2l_{[\mu}m_{\nu]}; \qquad M_{\mu\nu} = l_{[\mu}n_{\nu]} - m_{[\mu}\bar{m}_{\nu]},$$

in terms of which, any antisymmetric tensor can be expressed. In particular, following the notation established by Newman and Penrose [24], the Weyl tensor can be expanded as

$$\begin{aligned} C_{\kappa\lambda\mu\nu} = \; & -\Psi_0 U_{\kappa\lambda}U_{\mu\nu} + 2\Psi_1(U_{\kappa\lambda}M_{\mu\nu} + M_{\kappa\lambda}U_{\mu\nu}) - \Psi_2(U_{\kappa\lambda}V_{\mu\nu} + 4M_{\kappa\lambda}M_{\mu\nu} + V_{\kappa\lambda}U_{\mu\nu}) \\ & +2\Psi_3(V_{\kappa\lambda}M_{\mu\nu} + M_{\kappa\lambda}V_{\mu\nu}) - \Psi_4 V_{\kappa\lambda}V_{\mu\nu} + \text{complex conjugate.} \end{aligned} \tag{1.1}$$

Its individual components can be defined by

$$\begin{aligned} \Psi_0 &= -C_{\kappa\lambda\mu\nu}l^\kappa m^\lambda l^\mu m^\nu\,, \\ \Psi_1 &= -C_{\kappa\lambda\mu\nu}l^\kappa n^\lambda l^\mu m^\nu\,, \\ \Psi_2 &= -\tfrac{1}{2}\,C_{\kappa\lambda\mu\nu}l^\kappa n^\lambda(l^\mu n^\nu - m^\mu\bar{m}^\nu)\,, \\ \Psi_3 &= -C_{\kappa\lambda\mu\nu}n^\kappa l^\lambda n^\mu\bar{m}^\nu\,, \\ \Psi_4 &= -C_{\kappa\lambda\mu\nu}n^\kappa\bar{m}^\lambda n^\mu\bar{m}^\nu\,, \end{aligned}$$

and they are going to play a significant role, as they can be interpreted [27] as follows:

- Ψ_0 denotes a transverse wave component in the n_μ direction
- Ψ_1 denotes a longitudinal wave component in the n_μ direction
- Ψ_2 denotes a Coulomb component
- Ψ_3 denotes a longitudinal wave component in the l_μ direction
- Ψ_4 denotes a transverse wave component in the l_μ direction.

In particular, we are going to be considering collisions involving transverse waves. Regarding these, a well known theorem states that no solution of Einstein's source-free field equations exist for which $\Psi_0 \neq 0$, $\Psi_4 \neq 0$ with $\Psi_1 = \Psi_2 = \Psi_3 = 0$. Since Ψ_0 and Ψ_4 represent gravitational wave components along n_μ and l_μ respectively, what this theorem implies is that two transverse gravitational waves cannot be superposed when they collide, as there must be a component of the Weyl tensor other than Ψ_0 and Ψ_4 different from zero[2].

[2]A similar theorem has been obtained stating that two longitudinal gravitational waves cannot be superposed [12].

1.3 Plane wave space-times

First of all, let us introduce the concept of *pp*-waves: these are plane-fronted gravitational waves with parallel rays, which are defined by the property that they admit a covariantly constant null tangent vector field (the *parallel rays*). Identifying l_μ with this field, one finds that $l_{\mu;\nu} = 0$ implies that the wavefronts are plane (as the congruence is expansion-free), and the rays orthogonal to the wavefronts are parallel.

A null coordinate u can be chosen such that $l_\mu = u_{,\mu}$, and the metric can then be written in the Kerr-Schild form

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r + H(u,X,Y)\,\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2\,, \tag{1.2}$$

where r is an affine parameter along the rays, and the coordinates X and Y span the wave surfaces. For space-times with the line element of the form of (1.2), the vacuum field equations reduce to

$$H_{XX} + H_{YY} = 0\,. \tag{1.3}$$

Using a natural tetrad, the only non-zero component of the Weyl tensor is

$$\Psi_4 = \frac{1}{4}(H_{XX} - H_{YY} + 2\,i\,H_{XY})\,. \tag{1.4}$$

Thus the gravitational wave propagating along l_μ can be specified by the single component $\Psi_4 = A\,e^{i\alpha}$, where A is regarded as its amplitude, and α as its polarization. The differential equation (1.3) is linear, consequently solutions with different expressions for H can be superposed, i.e. independent gravitational *pp*-waves travelling in the same direction do not interact [9, 2].

This class of *pp*-waves includes a possible family of sandwich waves, in which the Weyl tensor vanishes everywhere except for a finite range of u [33].

We are going to be working with *plane waves*, which are part of the class of *pp*-waves: a plane wave is a *pp*-wave in which the field components are the same at every point of the wavefront [8]. Therefore Ψ_4 must be independent of X and Y. Consequently, as may be seen from (1.4), $H(u,X,Y)$ must be quadratic in X and Y. This makes it possible to write

$$H(u,X,Y) = h_{11}\,X^2 + 2\,h_{12}\,XY + h_{22}\,Y^2\,.$$

The metric of a space-time containing a plane wave can be expressed in the Brinkmann form

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r + (h_{11}\,X^2 + 2\,h_{12}\,XY + h_{22}\,Y^2)\,\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2\,, \qquad (1.5)$$

where h_{ij} depends only on u.

Now consider the coordinate transformation

$$\begin{aligned} X &= Px + Ay \\ Y &= Bx + Qy \qquad (1.6) \\ r &= v + \frac{1}{2}(PP' + BB')\,x^2 + \frac{1}{2}(AP' + PA' + BQ' + QB')\,xy + \frac{1}{2}(AA' + QQ')\,y^2\,, \end{aligned}$$

(the null coordinate u remains unchanged) where P, A, Q, B depend only on u and are subject to

$$\begin{aligned} P'' + h_{11}P + h_{12}B &= 0 \\ A'' + h_{11}A + h_{12}Q &= 0 \\ Q'' + h_{12}A + h_{22}Q &= 0 \qquad (1.7) \\ B'' + h_{12}P + h_{22}B &= 0 \\ AP' - PA' - BQ' + QB' &= 0\ . \end{aligned}$$

This transforms (1.5) into the Rosen form of the line element,

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v - (P^2 + B^2)\,\mathrm{d}x^2 - 2\,(PA + QB)\,\mathrm{d}x\,\mathrm{d}y - (A^2 + Q^2)\,\mathrm{d}y^2\,. \qquad (1.8)$$

The curvature tensor component is, in terms of this metric,

$$\Psi_4 = -\frac{(QP'' - PQ'' - BA'' + AB'') - i\,(AP'' - PA'' + BQ'' - QB'')}{2\,(PQ - AB)}\,.$$

It is convenient to express (1.8) in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\ \mathrm{d}v - e^{-U}\left[\chi\,\mathrm{d}y^2 + \chi^{-1}(\mathrm{d}x - \omega\,\mathrm{d}y)^2\right]\,, \qquad (1.9)$$

where U, χ, ω are functions of u only. This is the form of the metric we are going to be using from next chapter onwards, and is achieved by writing

$$\begin{aligned} P^2 &= e^{-U}\,\sqrt{\omega^2 + \chi^2}\sqrt{\frac{\sqrt{\chi^2 + \omega^2}}{2\chi} + \frac{1}{2}}, \quad B^2 = -e^{-U}\,\sqrt{\omega^2 + \chi^2}\sqrt{\frac{\sqrt{\chi^2 + \omega^2}}{2\chi} - \frac{1}{2}}, \\ Q^2 &= e^{-U}\,\frac{1}{\sqrt{\omega^2 + \chi^2}}\sqrt{\frac{\sqrt{\chi^2 + \omega^2}}{2\chi} + \frac{1}{2}}, \quad A^2 = -e^{-U}\,\frac{1}{\sqrt{\omega^2 + \chi^2}}\sqrt{\frac{\sqrt{\chi^2 + \omega^2}}{2\chi} - \frac{1}{2}}, \end{aligned} \qquad (1.10)$$

where P, Q, A and B are expressed in terms of only three functions (U, χ and ω) because there is a redundant function in (1.8). It can be seen that the transformation inverse to (1.10) reads

$$\omega = -\frac{PA+QB}{P^2+B^2}\,, \qquad \chi = \frac{PQ-AB}{P^2+B^2}\,, \qquad e^{-U} = PQ - AB\,.$$

The symmetries of this metric, as shown in [8], include (but are not limited to) the Killing vectors

$$\xi_1 = \partial_x\,, \quad \xi_2 = \partial_y\,. \tag{1.11}$$

For a vacuum plane wave, H can be expressed by

$$H = h_{11}(X^2 - Y^2) + 2h_{12}XY\,,$$

and the line element can be written in the Brinkmann form

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r + \left(h_{11}\,(X^2 - Y^2) + 2\,h_{12}\,XY\right)\,\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2\,, \tag{1.12}$$

where h_{11} and h_{12} depend only on u. The only non-zero component of the Riemann tensor is

$$\Psi_4 = h_{11} + i\,h_{12} = h(u)\,e^{i\,\alpha(u)}\,. \tag{1.13}$$

Now consider the linear[3] case: the gravitational wave has a constant linear polarization if h_{12} is proportional to h_{11}. This can be expressed in the form

$$h_{11} = h(u)\cos\alpha\,, \qquad h_{12} = h(u)\sin\alpha\,,$$

where α is now a constant. In this case,

$$H = h(u)\left[(X^2 - Y^2)\cos\alpha + 2XY\,\sin\alpha\right]\,,$$

where $h(u)$ is arbitrary, and the constant α is the polarization. Thus

$$\Psi_4 = h(u)\,e^{i\,\alpha}\,.$$

If the gravitational wave possesses linear polarization, the X and Y axes can be rotated to make $\alpha = 0$ everywhere. This results in the line element (1.12)

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r + h(u)\,(X^2 - Y^2)\,\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2\,, \tag{1.14}$$

[3]Here, "linear" is used in analogy with some electromagnetic radiation terminology, to indicate that the wave is linearly polarized, i.e. with fixed polarization axes.

and $\Psi_4 = h(u)$, making this form particularly convenient to obtain the profile of the wave, that happens to be just $h(u)$, but only in the constant linear polarization case.

When we take a line element of the form (1.8), the constant linear polarization makes it possible that A and B can be taken to be zero. Consequently, the line element is written in a much simpler form,

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v - P^2\,\mathrm{d}x^2 - Q^2\,\mathrm{d}y^2\,, \tag{1.15}$$

Therefore, equation (1.6) is simplified to the transformation

$$\begin{aligned} X &= x\,P \\ Y &= y\,Q \\ r &= v + \frac{1}{2}x^2PP' + \frac{1}{2}y^2QQ'\,, \end{aligned}$$

where P and Q are again functions of u only, satisfying now

$$P'' + h(u)P = 0 \quad \text{and} \quad Q'' - h(u)Q = 0. \tag{1.16}$$

and the constraint equation

$$\frac{P''}{P} + \frac{Q''}{Q} = 0\,. \tag{1.17}$$

In this case,

$$\Psi_4 = \frac{1}{2}\left(\frac{Q''}{Q} - \frac{P''}{P}\right) = h(u)\,. \tag{1.18}$$

As an example of its convenience, let us consider an impulsive wave [21]. It is unfortunate that, since $h(u) \longrightarrow \delta(u)$, a distributional term appears in the metric,

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}r + \delta(u)(X^2 - Y^2)\,\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2\,.$$

However, one can see that this metric, when written in the Rosen form (1.8), is C^0, even when the function $h(u)$ is the Dirac delta:

$$\mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v - [1 - a\,u\,\Theta(u)]^2\,\mathrm{d}x^2 - [1 + a\,u\,\Theta(u)]^2\,\mathrm{d}y^2\,.$$

Chapter 2

Colliding plane wave problem

The colliding wave problem is to obtain the metric characterising the interaction region resulting from the collision of two given distinct plane shock waves.

The gravitational field equations are nonlinear, unlike Maxwell's equations, whose solutions can be superposed. This implies that, when the two gravitational waves meet, they must interact. (Notice that these colliding waves may initially be sandwich waves, as pointed out in the previous chapter).

We are going to study the problem corresponding to two colliding plane (gravitational) waves. Although this is an idealised model, the nonlinearity is considered completely, i.e. solutions, although not realistic, are *exact*. Therefore, some of the properties characterising this model should appear in a real collision.

It is assumed that the background region, into which the waves initially propagate, is part of a flat Minkowski spacetime.

Initially, the problem will be formulated for the general (nonlinear) case. Then, it will be restricted to the case in which the two waves have constant and aligned polarization states: we will start with the given profile function $h(u)$, which will make it possible to find the functions $P(u)$ and $Q(u)$ when the differential equations appearing in (1.16) are solved. Then, the metric functions $U(u)$ and $\chi(u)$ can be determined from their definitions.

2.1 The space-time structure

We will consider two plane shock waves in a head-on collision. The null coordinates u and v label each wavefront when $u, v = 0$, respectively. This situation clearly divides the space-time into four regions, as illustrated in Fig. 2.1 :

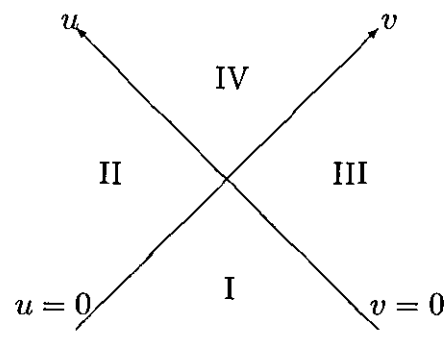


Figure 2.1: The structure of colliding plane wave space-times

These regions are:

- the initial space-time, where $u < 0$, $v < 0$, which is flat;
- the two regions containing the approaching plane shock waves are those where either $u \geq 0$, $v < 0$ or $v \geq 0$, $u < 0$;
- the interaction region, specified by $u, v > 0$.

Let us consider region II, where there is only a u-wave. It is convenient to align the direction of propagation of the colliding wave with the null vector $l_\mu = A^{-1} u_{,\mu}$. This vector is perpendicular to the plane spanned by ∂_x and ∂_y. Consequently, A is independent of x and y. Indeed, this function $A(u)$ arises whenever a transformation of the null coordinate u is made, i.e. when the null hypersurface is relabelled. When we apply the change $u \to \tilde{u} = \tilde{u}(u)$, the line element is also modified, as $\mathrm{d}u\,\mathrm{d}v \to \tilde{u}'(u)\,\mathrm{d}u\,\mathrm{d}v$, where $\tilde{u}'(u) = \mathrm{d}\tilde{u}/\mathrm{d}u = A^{-1}(u)$. If is found to be convenient to use the new metric function $M(u)$, given by

$$M(u) = -\log(\tilde{u}') = \log(A) \ .$$

Similarly, in region III, we can align the direction of propagation of the other wave with the null vector $n_\mu = B^{-1} v_{,\mu}$, which is also perpendicular to the plane

spanned by ∂_x and ∂_y. This would ensure that B is independent of x and y. We then could define a function $M(v)$ as

$$M(v) = -\log(\tilde{v}') = \log(B) \ .$$

It is important to point out that the functions $\tilde{u}$ and $\tilde{v}$ above are C^2 functions that, to ensure that the continuity is preserved and no coordinate singularity is introduced, must verify $\mathrm{d}\tilde{u}/\mathrm{d}u = 1$ on $u = \tilde{u} = 0$ and, respectively, $\mathrm{d}\tilde{v}/\mathrm{d}v = 1$ on $v = \tilde{v} = 0$.

In the indirect approach, it is almost always necessary to use this gauge freedom in order to construct explicit solutions. However, in the characteristic initial value problem we do not have to care about it, because this freedom is restricted from the beginning as the problem is formulated.

In region I ($u < 0,\ v < 0$) the line element takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 \ . \tag{2.1}$$

In region II ($u \geq 0,\ v < 0$), that contains one of the approaching waves, the metric can be expressed using the line element

$$\mathrm{d}s^2 = 2\,e^{-M_+(u)}\,\mathrm{d}u\,\mathrm{d}v - e^{-U_+}\left[\chi_+\mathrm{d}y^2 + {\chi_+}^{-1}(\mathrm{d}x - \omega_+\mathrm{d}y)^2\right] \ , \tag{2.2}$$

where the $_+$ subscript has been introduced, and will be used in the rest of the thesis, to indicate that a function, coefficient, etc, corresponds to region II, and consequently, depends only on u.

Region III will be described by a metric similar to (2.2), but with all the metric functions having a dependence on v only. In this case, a $_-$ subscript will denote dependence upon v only.

If there appears no subscript, then the function will be referred to region IV or the reference will be generic.

It has been indicated that regions II and III have the Killing vectors ∂_x and ∂_y in common. Consequently, the simplest assumption is to consider that region IV also has these symmetries; these are the kind of solutions we can search for as a first attempt. If no solution with this feature is found, one would have to try and look for solutions with less symmetries. Nevertheless, solutions of this type

are found to exist [29], at least for the vacuum and electro-vacuum cases, which justifies this prior assumption. Then, the metric may be expressed in the form

$$\mathrm{d}s^2 = 2\,e^{-M}\,\mathrm{d}u\,\mathrm{d}v - e^{-U}\left[\chi\,\mathrm{d}y^2 + \chi^{-1}(\mathrm{d}x - \omega\,\mathrm{d}y)^2\right]\,, \tag{2.3}$$

where now M, U, χ and ω depend on u and v.

Notice that the continuity of the metric requires that, at the junctions between regions I-II and I-III,

$$\begin{aligned} M(u=0, v\leq 0) &= M(u\leq 0, v=0) = 0\,,\\ \chi(u=0, v\leq 0) &= \chi(u\leq 0, v=0) = 1\,,\\ U(u=0, v\leq 0) &= U(u\leq 0, v=0) = 0\,,\\ \omega(u=0, v\leq 0) &= \omega(u\leq 0, v=0) = 0\,. \end{aligned}$$

2.2 Field equations

For the metric (2.3), the vacuum field equations are (see [14])

$$U_{uv} = \quad U_u U_v \tag{2.4}$$

$$2U_{vv} = U_v^2 + \frac{1}{\chi^2}(\chi_v^2 + \omega_v^2) - 2U_v M_v \tag{2.5}$$

$$2U_{uu} = U_u^2 + \frac{1}{\chi^2}(\chi_u^2 + \omega_u^2) - 2U_u M_u \tag{2.6}$$

$$\chi_{uv} = \frac{1}{\chi}(\chi_u\chi_v - \omega_u\omega_v) + \frac{1}{2}(U_u\chi_v + U_v\chi_u) \tag{2.7}$$

$$\omega_{uv} = \frac{1}{\chi}(\chi_u\omega_v + \chi_v\omega_u) + \frac{1}{2}(U_u\omega_v + U_v\omega_u) \tag{2.8}$$

$$2M_{uv} = -U_u U_v + \frac{(\omega_u\chi - \omega\chi_u)(\omega_v\chi - \omega\chi_v)}{\chi^2(\omega^2+\chi^2)} - \frac{(\omega_u + \chi_u)(\omega_v + \chi_v)}{\chi^2} \tag{2.9}$$

It follows immediately from (2.4) that

$$e^{-U} = f(u) + g(v) \tag{2.10}$$

for any arbitrary functions f and g satisfying $f + g > 0$.

It may be seen that (2.7) and (2.8) are the integrability conditions for (2.5), (2.6) and (2.9), i.e. if we find U, χ and ω satisfying (2.7) and (2.8), then there exists a function M that satisfies (2.5) and (2.6). The function M can be obtained,

once U, χ and ω are known, by integrating equations (2.5) and (2.6). Thus our aim is to solve the main field equations (2.5) and (2.6).

Now if we introduce the complex function

$$Z = \chi + i\omega\,, \tag{2.11}$$

the main equations (2.7) and (2.8) can be expressed by the single complex equation

$$(Z + \bar{Z})(2\,Z_{uv} - U_u Z_v - U_v Z_u) = 4\,Z_u Z_v\,, \tag{2.12}$$

which is just Ernst's equation, usually written in the coordinate-invariant form

$$(Z + \bar{Z})\nabla^2 Z = 2\,(\nabla Z)^2\,. \tag{2.13}$$

Thus, what will have to be done is to solve Ernst's equation subject to the appropriate initial conditions.

According to the definition given in [29], the scale-invariant components of the Weyl tensor, denoted by a superscript $^\circ$ below, are

$$\begin{aligned}
\Psi_0^\circ &= \Psi_0\, B^{-2}\,,\\
\Psi_1^\circ &= \Psi_1\, A^{-\frac{1}{2}}\, B^{-\frac{3}{2}}\,,\\
\Psi_2^\circ &= \Psi_2\, (A\,B)^{-1}\,,\\
\Psi_3^\circ &= \Psi_3\, B^{-\frac{1}{2}}\, A^{-\frac{3}{2}}\,,\\
\Psi_4^\circ &= \Psi_4\, A^{-2}\,,
\end{aligned}$$

where A and B are now functions of u and v. These scale-invariant components can also be explicitly evaluated in terms of the metric functions as

$$\begin{aligned}
\Psi_0^\circ &= \frac{\chi - i\omega}{\chi\sqrt{\chi^2+\omega^2}}(\chi_{vv} - U_v\chi_v + M_v\chi_v - 2\chi_v^2 + 2\omega_v^2)\\
&\qquad + \frac{\omega - i\chi}{\chi\sqrt{\chi^2+\omega^2}}(\omega_{vv} + U_v\omega_v - M_v\omega_v + 2\chi_v\omega_v)\\
\Psi_1^\circ &= 0\\
\Psi_2^\circ &= \frac{1}{2}M_{uv} + \frac{1}{4}\frac{(\chi+\omega)(\omega_v\chi_u - \omega_u\chi_v)}{\chi^2(\omega^2+\chi^2)}\\
\Psi_3^\circ &= 0\\
\Psi_4^\circ &= \frac{\chi - i\omega}{\chi\sqrt{\chi^2+\omega^2}}(\chi_{uu} - U_u\chi_u + M_u\chi_u - 2\chi_u^2 + 2\omega_u^2)\\
&\qquad + \frac{\omega - i\chi}{\chi\sqrt{\chi^2+\omega^2}}(\omega_{uu} + U_u\omega_u - M_u\omega_u + 2\chi_u\omega_u)\,.
\end{aligned} \tag{2.14}$$

From now on, we will be referring to these components and, for simplicity, the superscript $^\circ$ will be dropped.

2.3 Junction conditions and the initial value problem

For the the field equations to describe suitably the space-time regions I, II, III and IV as a whole, the metric functions U, χ, ω and M must satisfy some boundary conditions across the null hypersurfaces $u = 0$ and $v = 0$. These are usually taken to be Lichnerowicz conditions, which seems reasonable, because the curvature tensor comprises second derivatives of the metric tensor. However, these conditions exclude impulsive gravitational waves, which are shown to give rise to a geometrically acceptable space-time. O'Brien and Synge [25] proposed a less stringent set of conditions to be satisfied across null boundaries. Take $x^0 = u$, $x^1 = v$: the null hypersurfaces are denoted by $x^\flat$ =constant, where $\flat = 0, 1$, with $g_{\flat\flat} = 0$. Their conditions are that $g_{\mu\nu}$, $g^{\alpha\beta} g_{\alpha\beta,\flat}$ and $g^{\alpha\flat} g_{\alpha\beta,\flat}$, where $\mu, \nu, \alpha, \beta = 0, 1, 2, 3$ and $\alpha, \beta \neq \flat$, be continuous across $x^\flat$ =constant. It can be seen that the line element (1.8) corresponding to the impulsive wave satisfies the O'Brien-Synge conditions (see next section).

The same form of the line element (2.3) is taken for all four regions. However, the metric functions U, χ, ω and M must have different forms in each region. The O'Brien-Synge conditions infer that χ, ω and M must be continuous, and that U must be also C^1 across the null boundaries. Hence it is necessary that $f(u)$ and $g(v)$ be C^1 –at least. We want to describe a collision of two plane waves, therefore the metric functions U, χ, ω and M must be functions of u and v in region IV, of v only in region III, of u only in region II, and constant in region I.

Now, from the flat metric (2.1) in region I, we see that $f + g = 1$. The values

$$f = \tfrac{1}{2} \quad \text{for } u \leq 0\,, \qquad g = \tfrac{1}{2} \quad \text{for } v \leq 0 \tag{2.15}$$

are particularly suitable in order to keep a symmetric formulation. Moreover, it can be observed that, for continuity reasons, at the junctions with the flat region

we have

$$\begin{aligned}&\text{junction I-II}: \quad U_+(0)=U_{+u}(0)=0\,, \quad \chi_+(0)=1, \quad \omega_+(0)=0, \quad M_+(0)=0\\ &\text{junction I-III}: \quad U_-(0)=U_{-v}(0)=0\,, \quad \chi_-(0)=1, \quad \omega_-(0)=0, \quad M_-(0)=0.\end{aligned} \tag{2.16}$$

Consider in particular the junction between I and II. Notice that it is always possible in region II to make $M_+ = 0$ and, consequently, $M_{+u} = 0$. Then, equation (2.6) reads:

$$2\,U_{uu} = U_u^2 + \frac{1}{\chi^2}\,(\chi_u^2 + \omega_u^2)\,,$$

which clearly shows that U_{+u} and U_+ are monotonically increasing functions of u. Therefore, e^{-U_+} must be a monotonically decreasing function of u, which near the wavefront $u = 0$ can be expanded as

$$e^{-U_+} = 1 - (c_+\,u)^{n_+} + o(u^{n_+})\,, \tag{2.17}$$

where $c_+ > 0$ and $n_+ \geq 2$ are constants. Consequently, $f(u) = e^{-U_+} - \frac{1}{2}$ is a monotonically decreasing function of u at least in region II. According to (2.17),

$$f(u) = \frac{1}{2} - (c_+\,u)^{n_+} + o(u^{n_+})$$

near $u = 0$.

By means of the transformation $u \longrightarrow \tilde{u} = \tilde{u}(u)$, it is possible to put

$$f(u) = \frac{1}{2} - (c_+ u)^{n_+}\,\Theta(u)\,. \tag{2.18}$$

This relabelling of the null hypersurface is sometimes convenient to find solutions using the indirect approach. However, it is not appropriate for the initial value problem, as this function $f(u)$ is explicitly determined by the initial data, since it is given by $f(u) = P_+\,Q_+ - \frac{1}{2}$. This argumentation also applies to $g(v)$ in region III when (2.6) is replaced by (2.5). In order to avoid impulsive components in the Ricci tensor, $n_+, n_- \geq 2$. Here c_+ (or c_-, but not both at the same time) can be equated to 1 by means of further rescaling transformations, but n_+ and n_- cannot be changed, as this would modify the continuity of the metric.

If we take into account that the scale transformations which give rise to M cannot reverse the directions of u and v, we find that $f(u)$ and $g(v)$ must be monotonically decreasing, even in region IV. Moreover, since for any $u, v > 0$ there is a bijective relation with f, g respectively, these functions can be taken

as coordinates instead of u and v. Notice that, because f and g are decreasing functions, a singularity will occur when $f+g=0$. This may be either a curvature singularity or an unstable non-scalar curvature singularity, as analysed elsewhere (for a review, see [12]).

The characterization of M_+, written in terms of f, is given by

$$M_{+f} = -\frac{f''}{f'^2} + \frac{1}{2(f+\frac{1}{2})} - \frac{f+\frac{1}{2}}{2\chi_+^2}\left(\chi_{+f}^2 + \omega_{+f}^2\right), \tag{2.19}$$

which comes from (2.6). It is important to notice that this equation contains a singular term on the wavefront $u=0$, on which $f'=0$. Thus, to preserve the continuity of M_+ across $u=0$, it is necessary that, near the wavefront $u=0$, χ_+ and ω_+ verify the so-called *shock wave condition*

$$\lim_{u\to o}\left(\frac{\chi_{+u}^2+\omega_{+u}^2}{\chi_+^2}\right) = 2n_+(n_+-1)\,c_+^{n_+}\,u^{n_+-2}. \tag{2.20}$$

In terms of f, the shock wave condition reads

$$\lim_{f\to\frac{1}{2}}\left[\left(\tfrac{1}{2}-f\right)\frac{\chi_{+f}^2+\omega_{+f}^2}{\chi_+^2}\right] = 2k_+, \tag{2.21}$$

where $k_+ = 1-\frac{1}{n_+}$, i.e. $\frac{1}{2}\le k_+ < 1$. An equivalent equation can be found for g,

$$\lim_{g\to\frac{1}{2}}\left[\left(\tfrac{1}{2}-g\right)\frac{\chi_{-g}^2+\omega_{-g}^2}{\chi_-^2}\right] = 2k_-, \tag{2.22}$$

where $k_- = 1-\frac{1}{n_-}$, i.e. $\frac{1}{2}\le k_- < 1$. For a well posed initial value problem, these conditions are automatically satisfied.

The Ernst equation (2.12) can be expressed in f,g coordinates as

$$(Z+\bar{Z})[2(f+g)Z_{fg} - Z_f - Z_g] = 4(f+g)Z_fZ_g,$$

The initial value problem then consists of solving Ernst's equation for $Z(f,g)$ subject to

$$Z(f,\tfrac{1}{2}) = Z_+(f) = \chi_+(f) + i\omega_+(f), \quad Z(\tfrac{1}{2},g) = Z_-(g) = \chi_-(g) + i\omega_-(g),$$

and with $Z(\frac{1}{2},\frac{1}{2}) = 1$. Now the shock wave condition (2.21) reads

$$\lim_{f\to\frac{1}{2}}\left[\left(\frac{1}{2}-f\right)Z_{+f}\bar{Z}_{+f}\right] = 2k_+. \tag{2.23}$$

Therefore, the initial value problem will be solved when, from a given form of Ψ_0 and Ψ_4 defined on the junctions, we are able to obtain the metric functions corresponding to region IV. In the general case of vacuum plane waves, these initial data would be $\Psi_4 = h_+(u)\,e^{i\alpha_+(u)}$ and $\Psi_0 = h_-(v)\,e^{i\alpha_-(v)}$.

2.4 The linear case

In the colinear case (when the polarizations of both waves are constant and aligned), the x and y coordinates can be matched for each wave, so that in both cases their corresponding axes can be rotated to put $\omega = 0$ everywhere. Now equations (2.4)–(2.9) take the simpler form

$$\begin{aligned} U_{uv} &= U_u U_v \\ 2U_{vv} &= U_v^2 + \frac{\chi_v^2}{\chi^2} - 2\,U_v M_v \\ 2U_{uu} &= U_u^2 + \frac{\chi_u^2}{\chi^2} - 2\,U_u M_u \\ \chi_{uv} &= \frac{\chi_u \chi_v}{\chi} + \tfrac{1}{2}\,(U_u \chi_v + U_v \chi_u) \\ 2M_{uv} &= -U_u U_v + -\frac{\chi_u \chi_v}{\chi^2}\,, \end{aligned} \tag{2.24}$$

where we must not forget that the first equation was already solved in (2.10), finding $U = -\log(f(u) + g(v))$ for the arbitrary functions $f(u)$ and $g(v)$.

The Ernst equation, corresponding to the fourth equation above (for χ_{uv}), is now real. If we put $\chi = e^{-V}$ and rewrite this in terms of the coordinates f and g, it is expressed as the *linear* equation

$$(f+g)V_{fg} + \tfrac{1}{2}\,V_f + \tfrac{1}{2}\,V_g = 0\,, \tag{2.25}$$

which is an Euler-Poisson-Darboux equation with non-integer coefficients. The fact that this equation happens to be linear is very important, since it will allow us in the next chapter to build up a general solution by superposition. In this case, the initial data are given by two real functions $V_+(f)$ and $V_-(g)$, and the solution $V(f,g)$ in region IV must satisfy $V(f,\frac{1}{2}) = V_+(f)$ and $V(\frac{1}{2},g) = V_-(g)$, with $V(\frac{1}{2},\frac{1}{2}) = 0$. Now the shock wave condition is given by

$$\lim_{f,g\to\frac{1}{2}} \left[\left(\tfrac{1}{2} - f\right) V_f^2\right] = 2k_+\,, \qquad \text{and} \qquad \lim_{f,g\to\frac{1}{2}} \left[\left(\tfrac{1}{2} - g\right) V_g^2\right] = 2k_-\,. \tag{2.26}$$

It is convenient to notice that most of the colinear solutions now known have been achieved by means of an indirect approach, based on finding a family of solutions of the field equations in the interaction region, restricting its arbitrary parameters then to make it satisfy the boundary conditions (2.16) and, finally, extending them back to obtain the initial form of the colliding waves. However, it is not a straightforward task to find physically acceptable "initial" waves by these means.

2.5 Riemann's method

Shortly after the first solutions were published [28, 21], a general method concerning a direct approach, i.e. the derivation of a solution in the interaction region when the two initial waves are initially known, was given by Szekeres [29]. This produces an explicit integral expression for V by using Riemann's method. Let us see this a little more in detail: first notice that equation (2.25) may be equivalently written as

$$L[V] = V_{fg} + \frac{V_f}{2(f+g)} + \frac{V_g}{2(f+g)} = 0\,, \tag{2.27}$$

where the initial data are determined by the approaching waves:
$V(f, \frac{1}{2}) = V_+(f)$, $V(\frac{1}{2}, g) = V_-(g)$, and $V(\frac{1}{2}, \frac{1}{2}) = 0$. Because (2.27) is a linear hyperbolic equation, Riemann's method can be used to solve it explicitly for V. In order to do this, one must consider the equation

$$\tilde{L}[R] = R_{fg} - \left(\frac{R}{2(f+g)}\right)_{,f} - \left(\frac{R}{2(f+g)}\right)_{,g} = 0\,, \tag{2.28}$$

where R is any Riemann-Green function satisfying the conditions

$$\begin{aligned} R_f - \frac{R}{2(f+g)} &= 0 \quad \text{at } g = g_0\,, \\ R_g - \frac{R}{2(f+g)} &= 0 \quad \text{at } f = f_0\,, \\ R(f_0, g_0) &= 1\,. \end{aligned} \tag{2.29}$$

The solution of (2.27) is obtained by integrating $RL[V] - V\tilde{L}[R]$ over the rectangle $PNML$ of Fig. 2.2 which, by Green's theorem, can be evaluated as a line integral around the boundaries of the rectangle, resulting that [10]

$$V(f_0, g_0) = \int_M^L R\left[V_f + \frac{V}{2(f+1/2)}\right] \mathrm{d}f + \int_M^N R\left[V_g + \frac{V}{2(1/2+g)}\right] \mathrm{d}g\,. \tag{2.30}$$

A specific Riemann function satisfying (2.28) and (2.29) is

$$R(f, g; f_0, g_0) = \sqrt{\frac{f+g}{f_0+g_0}}\, P_{-\frac{1}{2}}\left[1 + \frac{2(f-f_0)(g-g_0)}{(f+g)(g_0+g_0)}\right]\,, \tag{2.31}$$

where $P_{-\frac{1}{2}}[\ldots]$ is the Legendre function of degree $-\frac{1}{2}$.

Therefore this method gives an integral expression for $V(f, g)$. However, the integrals that appear are in most cases simply too difficult to be evaluated explicitly in terms of elementary functions, which makes it rather impractical if we intend to consider arbitrary approaching waves.

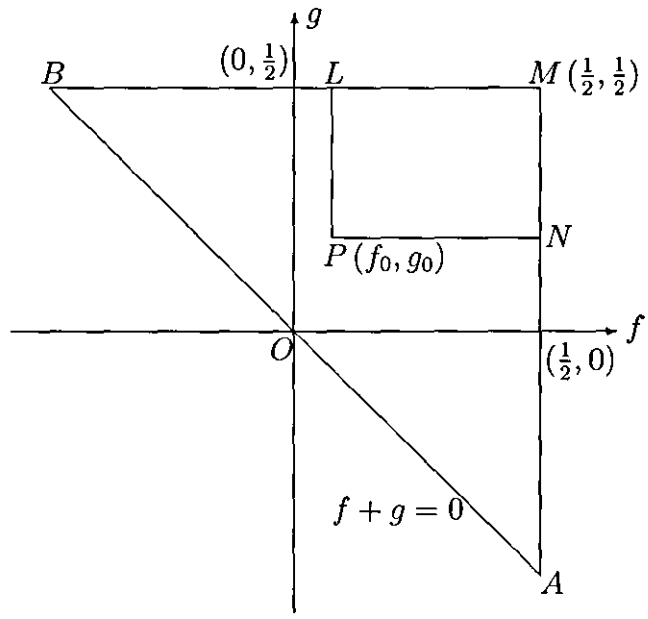


Figure 2.2: Region IV is represented in f, g coordinates by the region enclosed by the triangle. The side $\overline{AB}$ is the focussing singularity, and the sides $\overline{MA}$ and $\overline{MB}$ are the II-IV and III-IV boundaries, respectively. Given $V(f, \frac{1}{2}) = V_+(f)$ and $V(\frac{1}{2}, g) = V_-(g)$, $V(f, g)$ can be obtained explicitly at any point P by integrating around the rectangle $PNML$.

Chapter 3

Abel transform

This method of solving the E-P-D equation (2.25) was initially proposed by Hauser and Ernst [17]. It makes use of a generalised version of the Abel transform. First notice that (2.25) can be solved by considering a separable test solution in the form $V(f,g) = F(f)\,G(g)$, which results in

$$(f+g)\,F'(f)\,G'(g) + \frac{1}{2}F'(f)\,G(g) + \frac{1}{2}F(f)\,G'(g) = 0\,.$$

If we re-arrange terms, the above equation reads

$$2f + \frac{F(f)}{F'(f)} = -2g - \frac{G(g)}{G'(g)}\,,$$

i.e. there are two independent ODE's

$$2f + \frac{F(f)}{F'(f)} = k \quad \text{and} \quad -2g - \frac{G(g)}{G'(g)} = k\,,$$

where the parameter k depends neither on f nor on g. Consequently,

$$F(f) = \frac{c_1}{\sqrt{k-2f}} \quad \text{and} \quad G(g) = \frac{c_2}{\sqrt{k+2g}}\,.$$

Now, if we name $A = \dfrac{c_1\,c_2}{2}$ and $\sigma = \dfrac{k}{2}$, the separable solution is given by

$$V(f,g) = F(f)\,G(g) = \frac{A}{\sqrt{\sigma-f}\,\sqrt{\sigma+g}}\,.$$

Since equation (2.25) is linear, a superposition of such solutions will also be a solution. Therefore, replacing A by $A(\sigma)\sqrt{\sigma+\frac{1}{2}}$, and adding a similar expression

in which f and g are interchanged (the mathematical details are given in [17]), a general solution can be obtained in the form

$$V(f,g) = \int_f^{\frac{1}{2}} \frac{A(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma + \int_g^{\frac{1}{2}} \frac{B(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma+f}\sqrt{\sigma-g}}\,\mathrm{d}\sigma\,. \tag{3.1}$$

For this solution to satisfy the initial data, the following relations must be satisfied:

$$\lim_{g\to\frac{1}{2}} V(f,g) = V_+(f) = \int_f^{\frac{1}{2}} \frac{A(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma\,, \tag{3.2}$$

$$\lim_{f\to\frac{1}{2}} V(f,g) = V_-(g) = \int_g^{\frac{1}{2}} \frac{B(\sigma)}{\sqrt{\sigma-g}}\,\mathrm{d}\sigma\,. \tag{3.3}$$

Since the initial value problem is well posed, the shock wave conditions (2.21), written now in the form

$$\lim_{f\to\frac{1}{2}} \left[(\tfrac{1}{2}-f)V_+'^{\,2}\right] = 2k_+\,, \qquad \lim_{g\to\frac{1}{2}} \left[(\tfrac{1}{2}-g)V_-'^{\,2}\right] = 2k_-\,, \tag{3.4}$$

must be automatically satisfied.

Let us find $V_+'(f)$ near the wavefront $f=\frac{1}{2}$ to check that (3.2) satisfies the junction conditions:

$$\begin{aligned}
\frac{\mathrm{d}V_+}{\mathrm{d}f} &= \lim_{\varepsilon\to 0}\left[-\frac{A(f)}{\sqrt{f+\varepsilon-f}} + \int_{f+\varepsilon}^{\frac{1}{2}} A(\sigma)\,\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{1}{\sqrt{\sigma-f}}\right)\mathrm{d}\sigma\right] \\
&= \lim_{\varepsilon\to 0}\left[-\frac{A(f)}{\sqrt{\varepsilon}} - \int_{f+\varepsilon}^{\frac{1}{2}} A(\sigma)\,\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\sqrt{\sigma-f}}\right)\mathrm{d}\sigma\right] \\
&= \lim_{\varepsilon\to 0}\left[-\frac{A(f)}{\sqrt{\varepsilon}} - \left[\frac{A(\sigma)}{\sqrt{\sigma-f}}\right]_{f+\varepsilon}^{\frac{1}{2}} + \int_{f+\varepsilon}^{\frac{1}{2}} \frac{A'(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma\right] \\
&= -\frac{A(\frac{1}{2})}{\sqrt{\frac{1}{2}-f}} + \int_f^{\frac{1}{2}} \frac{A'(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma\,.
\end{aligned}$$

Equivalently, near the junction $g=\frac{1}{2}$,

$$\frac{\mathrm{d}V_-}{\mathrm{d}g} = -\frac{B(\frac{1}{2})}{\sqrt{\frac{1}{2}-g}} + \int_g^{\frac{1}{2}} \frac{B'(\sigma)}{\sqrt{\sigma-g}}\,\mathrm{d}\sigma\,.$$

Thus, if we substitute the expression above for $V_+'(f)$, and the corresponding expression for $V_-'(g)$, in (3.4), the junction conditions are preserved if

$$A(1/2) = \pm\sqrt{2k_+} \text{ and } A'(1/2) < \infty\,,$$

$$B(1/2) = \pm\sqrt{2k_-} \text{ and } B'(1/2) < \infty .$$

Now, for the method to be of use when it comes to solve the initial value problem, the functions $A(\sigma)$ and $B(\sigma)$ should be obtained explicitly from the initial data $V_+(f)$ and $V_-(g)$. In other words, we have to invert equations (3.2) and (3.3). To do so, consider the identity

$$\frac{1}{\pi}\int_\sigma^{\sigma'} \frac{\mathrm{d}f}{\sqrt{(\sigma'-f)(f-\sigma)}} = 1 , \tag{3.5}$$

in which $\sigma < f < \sigma' < \frac{1}{2}$. To derive this identity, notice that

$$(\sigma'-f)(f-\sigma) = \tfrac{1}{4}(\sigma'-\sigma)^2 - \left[f - \tfrac{1}{2}(\sigma'+\sigma)\right]^2 .$$

Thus, making the substitution $f - \frac{1}{2}(\sigma'+\sigma) = \alpha$ in the above integral, we have

$$\begin{aligned}
\int_{\alpha(\sigma)}^{\alpha(\sigma')} \frac{\mathrm{d}\alpha}{\sqrt{\frac{1}{4}(\sigma'-\sigma)^2 - \alpha^2}} &= \left[\sin^{-1}\left(\frac{2\alpha}{\sigma'-\sigma}\right)\right]_{\alpha(\sigma)}^{\alpha(\sigma')} \\
&= \left[\sin^{-1}\left(\frac{2f-(\sigma'+\sigma)}{\sigma'-\sigma}\right)\right]_{\sigma}^{\sigma'} \\
&= \sin^{-1}\left(\frac{\sigma'-\sigma}{\sigma'-\sigma}\right) - \sin^{-1}\left(\frac{\sigma-\sigma'}{\sigma'-\sigma}\right) \\
&= \pi .
\end{aligned} \tag{3.6}$$

This proves the identity (3.5). Now, multiplying both sides of (3.5) by an arbitrary function $A(\sigma')$ yields

$$\frac{1}{\pi}\int_\sigma^{\sigma'} \frac{A(\sigma')\,\mathrm{d}f}{\sqrt{(\sigma'-f)(f-\sigma)}} = A(\sigma') ,$$

and integrating this with respect to σ', changing the order of integration, gives

$$\frac{1}{\pi}\int_\sigma^{\frac{1}{2}} \mathrm{d}f \int_\sigma^{\sigma'} \frac{A(\sigma')\,\mathrm{d}\sigma'}{\sqrt{(\sigma'-f)(f-\sigma)}} = \int_\sigma^{\frac{1}{2}} A(\sigma')\,\mathrm{d}\sigma' .$$

Thus,

$$\begin{aligned}
\int_\sigma^{\frac{1}{2}} A(\sigma')\,\mathrm{d}\sigma' &= \frac{1}{\pi}\int_\sigma^{\frac{1}{2}} \frac{\mathrm{d}f}{\sqrt{f-\sigma}} \int_f^{\frac{1}{2}} \frac{A(\sigma')}{\sqrt{\sigma'-f}}\,d\sigma' \\
&= \frac{1}{\pi}\int_\sigma^{\frac{1}{2}} \frac{V_+(f)}{\sqrt{f-\sigma}}\mathrm{d}f ,
\end{aligned}$$

by virtue of (3.2).

If the expression above is differentiated with respect to σ, then

$$\begin{aligned} A(\sigma) &= -\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_{\sigma}^{\frac{1}{2}} A(\sigma')\,\mathrm{d}\sigma' \\ &= \frac{1}{\pi}\lim_{\alpha\to\sigma}\left[\frac{V_+(\sigma)}{\sqrt{\alpha-\sigma}} - \int_{\alpha}^{\frac{1}{2}} V_+(f)\,\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\sqrt{f-\sigma}}\right)\mathrm{d}f\right] \\ &= \frac{1}{\pi}\lim_{\alpha\to\sigma}\left(\frac{V_+(\sigma)}{\sqrt{\alpha-\sigma}} + \frac{V_+(f)}{\sqrt{f-\sigma}}\bigg|_{f=\frac{1}{2}} - \frac{V_+(\sigma)}{\sqrt{\alpha-\sigma}} - \int_{\alpha}^{\frac{1}{2}} \frac{V_+'(f)}{\sqrt{f-\sigma}}\,\mathrm{d}f\right) \\ &= -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{V_+'(f)}{\sqrt{f-\sigma}}\,\mathrm{d}f\,, \end{aligned} \tag{3.7}$$

since $V_+(\frac{1}{2}) = 0$. Consequently, if $V_+(f)$ is known, this integral determines the spectral function $A(\sigma)$. The same procedure can be followed to derive $B(\sigma)$ from $V_-'(g)$. Thus we have

$$\begin{aligned} V_+(f) &= \int_{f}^{\frac{1}{2}} \frac{A(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma\,, \quad A(\sigma) = -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{V_+'(f)}{\sqrt{f-\sigma}}\,\mathrm{d}f\,, \\ V_-(g) &= \int_{g}^{\frac{1}{2}} \frac{B(\sigma)}{\sqrt{\sigma-g}}\,\mathrm{d}\sigma\,, \quad B(\sigma) = -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{V_-'(g)}{\sqrt{g-\sigma}}\,\mathrm{d}f\,. \end{aligned} \tag{3.8}$$

These expressions are what we know as the *inverse* (on the left) and *direct* (on the right) *Abel transforms*. It is normally assumed that the functions in the integral are all regular. However, in this case V_+' and V_-' are unbounded on the wavefronts although, as outlined in the previous page, the relations still hold. Hauser and Ernst [17] have given a detailed mathematical proof of this fact. Because of this particularity of V', we should be referring to these equations as the *'generalised' Abel transforms* (although they take the exact same form as those 'non generalised').

Notice that it is because of the linearity of the Euler-Poisson-Darboux equation that we can use this method, based on the superposition of terms.

From what we have seen, if the incident waves are specified, i.e. if $f, V_+(f)$ are known in region II, and $g, V_-(g)$ are known in region III, this method allows us to obtain the solution in region IV by first determining the explicit form of $A(\sigma)$ and $B(\sigma)$, that we call the *spectral functions*, and substituting these into equation (3.1). However, this is a theoretical result. The applicability of the method will

be ultimately conditioned to the solvability of the emerging integrals, and it is this that will be considered next.

Chapter 4

Confirmations of the Abel transform method

A method has been derived for solving the initial value problem for colliding plane waves, at least in the linear case. Now it is time to test this method. For this, we are going to consider a couple of well known cases, the Khan-Penrose solution (single-impulse colliding waves) [21] and the Szekeres' family [29] of solutions, and also an extension to these, the combination of Legendre functions.

4.1 The Khan-Penrose solution

Consider first the idealized case of two colliding impulsive plane waves (presented in [21]). The function h to be taken is

$$h_+(u) = a\,\delta(u), \tag{4.1}$$

which yields, from (1.16),

$$P = 1 - a\,u\,\Theta(u) \quad \text{and} \quad Q = 1 + a\,u\,\Theta(u).$$

Therefore, in region II,

$$e^{-U_+} = 1 - a^2u^2, \quad \text{and} \quad e^{V_+} = \frac{1 - a\,u}{1 + a\,u}, \quad \text{i.e.} \quad V_+ = \log(1-au) - \log(1+au).$$

Now, since $e^{-U_+} = \frac{1}{2} + f$, one obtains $f = \frac{1}{2} - au^2$. Thus, in terms of f,

$$V_+(f) = \log\left(1 - \sqrt{\frac{1}{2} - f}\right) - \log\left(1 + \sqrt{\frac{1}{2} - f}\right).$$

Consequently,

$$\begin{aligned} A(\sigma) &= -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{V_+'(f)}{\sqrt{f-\sigma}}\,\mathrm{d}f = -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{\mathrm{d}f}{\sqrt{f-\sigma}\sqrt{\frac{1}{2}-f}(\frac{1}{2}+f)} \\ &= -\frac{2}{\pi\sqrt{\frac{1}{2}+\sigma}}\tan^{-1}\left[\sqrt{\frac{1}{2}+\sigma}\,\frac{\sqrt{\frac{1}{2}-f}}{\sqrt{f-\sigma}}\right]_{\sigma}^{\frac{1}{2}} = -\frac{2}{\pi\sqrt{\frac{1}{2}+\sigma}}\left[0-\frac{\pi}{2}\right] \\ &= -\frac{1}{\sqrt{\frac{1}{2}+\sigma}}, \end{aligned} \tag{4.2}$$

where the integral that appears has been calculated by means of two successive changes of variable: first $f-\frac{1}{2}(\frac{1}{2}+\sigma)=\frac{1}{2}(\frac{1}{2}-\sigma)\cos\phi$, and then, $t=\tan\left(\frac{\phi}{2}\right)$. Operating in an analogous way in region III, it is achieved

$$B(\sigma) = \frac{-1}{\sqrt{\frac{1}{2}+\sigma}}. \tag{4.3}$$

Therefore, according to the method described in the previous chapter, the solution in region IV (the interaction region) is given by the equation (3.1), which now becomes

$$\begin{aligned} V(f,g) &= \int_f^{\frac{1}{2}} \frac{A(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma + \int_g^{\frac{1}{2}} \frac{B(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma+f}\sqrt{\sigma-g}}\,\mathrm{d}\sigma \\ &= -\int_f^{\frac{1}{2}} \frac{\mathrm{d}\sigma}{\sqrt{\sigma-f}\sqrt{\sigma+g}} - \int_g^{\frac{1}{2}} \frac{\mathrm{d}\sigma}{\sqrt{\sigma+f}\sqrt{\sigma-g}} \\ &= -\left[2\log\left(\sqrt{\sigma-f}+\sqrt{\sigma+g}\right)\right]_f^{1/2} - \left[2\log\left(\sqrt{\sigma+f}+\sqrt{\sigma-g}\right)\right]_g^{1/2} \\ &= \log\left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right) + \log\left(\frac{\sqrt{\frac{1}{2}+f}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}+\sqrt{\frac{1}{2}-g}}\right), \end{aligned} \tag{4.4}$$

where the integrals have been evaluated changing variables to $a=\sqrt{\sigma-f}+\sqrt{\sigma+g}$ (for the first integral) and $b=\sqrt{\sigma+f}+\sqrt{\sigma-g}$ (for the second). This, indeed, is the Khan-Penrose solution.

4.2 The Szekeres solution

The original solution obtained by Szekeres [28] involved a profile function with a step $h_+(u)=\Theta(u)\,\sqrt{6}(1-u^4)^{-\frac{5}{2}}$. He later generalized this [29], obtaining incident

waves which can be characterized by [14]:

$$h_+(u) = c_+^2\, n_+^2\, \frac{k_+}{4}\left(1-\frac{2}{n_+}\right)\frac{(c_+u)^{n_+/2-2}}{[1-(c_+u)^{n_+}]^{3-2/n_+}}\,\Theta(u) \quad \text{for} \quad n_+ > 2, \tag{4.5}$$

in which $k_+ = \sqrt{2\left(1-\frac{1}{n_+}\right)}$.

This, in fact, is a mathematical generalisation of the Khan-Penrose solution, but with different profiles for the approaching waves. For simplicity, we will assume $c_+ = 1$, as Szekeres did. Operating as in the single impulse case above,

$$P = \begin{cases} 1 & \text{for } u \le 0 \\ \dfrac{\left(1-u^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1+u^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} & \text{for } 0 \le u \end{cases} \quad \text{and} \quad Q = \begin{cases} 1 & \text{for } u \le 0 \\ \dfrac{\left(1+u^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1-u^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} & \text{for } 0 \le u. \end{cases}$$

This can be greatly simplified if we write $f = \frac{1}{2} - u^{n_+}$, resulting in

$$V_+(f) = k_+ \log\left(1-\sqrt{\frac{1}{2}-f}\right) - k_+\log\left(1+\sqrt{\frac{1}{2}-f}\right).$$

Therefore, as we are integrating the same expression as in the Khan-Penrose solution, but multiplied by a factor k_+, we have

$$A(\sigma) = -\frac{k_+}{\pi}\int_\sigma^{\frac{1}{2}} \frac{\mathrm{d}f}{\sqrt{f-\sigma}\sqrt{\frac{1}{2}-f}(\frac{1}{2}+f)} = -\frac{k_+}{\sqrt{\frac{1}{2}+\sigma}} \tag{4.6}$$

and, analogously,

$$B(\sigma) = -\frac{k_-}{\sqrt{\frac{1}{2}+\sigma}} \tag{4.7}$$

Together, these give rise to

$$V(f,g) = k_+\log\left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right) + k_-\log\left(\frac{\sqrt{\frac{1}{2}+f}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}+\sqrt{\frac{1}{2}-g}}\right), \tag{4.8}$$

as required for the Szekeres family of solutions.

4.3 Legendre functions

Now introduce new variables t and z, given by

$$t = \sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g} + \sqrt{\tfrac{1}{2}-g}\sqrt{\tfrac{1}{2}+f}$$

$$z = \sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g} - \sqrt{\tfrac{1}{2}-g}\sqrt{\tfrac{1}{2}+f}\,.$$

The main equation

$$2(f+g)V_{fg} + V_f + V_g = 0$$

can be written as [14]

$$\Big((1-t^2)V_t\Big)_{,t} - \Big((1-z^2)V_z\Big)_{,z} = 0\,.$$

If we consider separable solutions of the form

$$V = T(t)\,Z(z)\,,$$

the prior equation reduces to the pair of Legendre equations

$$(1-t^2)T_{tt} - 2tT_t + n(n+1)T = 0$$
$$(1-z^2)Z_{zz} - 2zZ_z + n(n+1)Z = 0\,.$$

This gives rise to a class of solutions

$$V = \sum_n \Big(a_n P_n(t)P_n(z) + q_n Q_n(t)P_n(z) + p_n P_n(t)Q_n(z) + b_n Q_n(t)Q_n(z)\Big) \quad (4.9)$$

where $P_i(x)$, $Q_i(x)$ are the *i-th order* Legendre functions of first and second kind respectively.

In fact, the above examples are included in this family of solutions:

Khan-Penrose: $V = -2\,Q_0(t)\,P_0(z)\,,$

Szekeres: $V = -2\,(k_1+k_2)\,Q_0(t)\,P_0(z) - 2\,(k_1-k_2)\,P_0(t)\,Q_0(z)\,,$

which include only zero order terms. Another known solution [13] is obtained from the first order term

$$V = a\,Q_1(t)\,P_1(z) = az\left[\frac{t}{2}\log\left(\frac{1+t}{1-t}\right) - 1\right], \quad (4.10)$$

This solution is worth being considered because we already know that the Abel transform method can be easily applied to the zero order solutions above, and since there exist recurrence relations for the Legendre functions, if the method can also be applied for this first order solution, it will imply that it can be applied to generate any solution of the form $V = P_n(z)\,Q_n(t)$.

In region II, $z = t = \sqrt{\frac{1}{2} - f}$, and thus (4.10) is given here by

$$V_+(f) = \frac{a}{2}\left(\frac{1}{2} - f\right)\log\left(\frac{1+\sqrt{\frac{1}{2}-f}}{1-\sqrt{\frac{1}{2}-f}}\right) - a\sqrt{\frac{1}{2}-f}, \quad (4.11)$$

therefore

$$V'_+(f) = -\frac{a}{2}\log\left(\frac{1+\sqrt{\frac{1}{2}-f}}{1-\sqrt{\frac{1}{2}-f}}\right) + a\,\frac{f}{\left(\frac{1}{2}+f\right)\sqrt{\frac{1}{2}-f}}\,, \tag{4.12}$$

where the two terms are identified separately. Hence

$$A(\sigma) = \frac{a}{2\pi}\int_\sigma^{1/2}\log\left(\frac{1+\sqrt{\frac{1}{2}-f}}{1-\sqrt{\frac{1}{2}-f}}\right)\frac{\mathrm{d}f}{\sqrt{f-\sigma}} - \frac{a}{\pi}\int_\sigma^{1/2}\frac{f\,\mathrm{d}f}{\left(\frac{1}{2}+f\right)\sqrt{\frac{1}{2}-f}\sqrt{f-\sigma}} \tag{4.13}$$

$$= \frac{a}{2\pi}I_1 - \frac{a}{\pi}I_2\,.$$

Let us take firstly I_2:

$$I_2 = \int_\sigma^{1/2}\frac{f\,\mathrm{d}f}{\left(\frac{1}{2}+f\right)\sqrt{\frac{1}{2}-f}\sqrt{f-\sigma}} = \int_\sigma^{1/2}\frac{f\,\mathrm{d}f}{\left(\frac{1}{2}+f\right)\sqrt{\left[\frac{1}{2}\left(\frac{1}{2}-\sigma\right)\right]^2 - \left[f-\frac{1}{2}\left(\frac{1}{2}+\sigma\right)\right]^2}}\,.$$

Introducing the change of variable given by

$$f - \tfrac{1}{2}\left(\tfrac{1}{2}+\sigma\right) = \tfrac{1}{2}\left(\tfrac{1}{2}-\sigma\right)\sin\theta \quad \Rightarrow \mathrm{d}f = \tfrac{1}{2}\left(\tfrac{1}{2}-\sigma\right)\sin\theta$$

and changing the integration limits according to

$$\theta(f) = \sin^{-1}\left(\frac{2f-\left(\frac{1}{2}+\sigma\right)}{\frac{1}{2}-\sigma}\right)\,,$$

we get

$$I_2 = \int_{\theta(\sigma)}^{\theta(1/2)}\frac{\left(\frac{1}{2}+\sigma\right)+\left(\frac{1}{2}-\sigma\right)\cos\theta}{\left(\frac{3}{2}+\sigma\right)+\left(\frac{1}{2}-\sigma\right)\cos\theta}\,\mathrm{d}\theta = \int_{\theta(\sigma)}^{\theta(1/2)}\mathrm{d}\theta - \int_{\theta(\sigma)}^{\theta(1/2)}\frac{\mathrm{d}\theta}{\left(\frac{3}{2}+\sigma\right)+\left(\frac{1}{2}-\sigma\right)\cos\theta}\,.$$

The second integral above is solved by using a further change, namely $t = \tan(\theta/2)$, which finally results in

$$I_2 = \left[\sin^{-1}\left(\frac{2f-\left(\frac{1}{2}+\sigma\right)}{\frac{1}{2}-\sigma}\right)\right]_\sigma^{1/2}$$

$$-\frac{1}{\sqrt{\frac{1}{2}+\sigma}}\left[\tan^{-1}\left\{\frac{\left(\frac{3}{2}+\sigma\right)\tan\left(\frac{1}{2}\sin^{-1}\left[\frac{2f-\frac{1}{2}-\sigma}{\frac{1}{2}-\sigma}\right]\right)+\frac{1}{2}-\sigma}{2\sqrt{\frac{1}{2}+\sigma}}\right\}\right]_\sigma^{1/2}$$

$$= \pi\left(1-\frac{\frac{1}{2}}{\sqrt{\frac{1}{2}+\sigma}}\right)\,. \tag{4.14}$$

Now, to solve I_1, let us integrate by parts, taking

$$\mathrm{x} = \log\left(\frac{1+\sqrt{\frac{1}{2}-f}}{1-\sqrt{\frac{1}{2}-f}}\right) \quad \Rightarrow \quad \mathrm{dx} = -\frac{\mathrm{d}f}{\left(\frac{1}{2}+f\right)\sqrt{\frac{1}{2}-f}}$$

and

$$\mathrm{dy} = \frac{\mathrm{d}f}{\sqrt{f-\sigma}} \quad \Rightarrow \quad \mathrm{y} = 2\sqrt{f-\sigma}\,.$$

Consequently, I_1 is now given in a form that can be decomposed into two integrals we already know how to solve:

$$\begin{aligned} I_1 &= 2\int_{\sigma}^{1/2} \frac{\sqrt{f-\sigma}}{\sqrt{\frac{1}{2}-f}}\,\frac{\mathrm{d}f}{\frac{1}{2}+f} \\ &= 2\left[\int_{\sigma}^{1/2} \frac{\mathrm{d}f}{\sqrt{f-\sigma}\sqrt{\frac{1}{2}-f}} - \left(\frac{1}{2}+\sigma\right)\int_{\sigma}^{1/2} \frac{\mathrm{d}f}{\left(\frac{1}{2}+f\right)\sqrt{\frac{1}{2}-f}\sqrt{f-\sigma}}\right] \\ &= 2\pi\left(1-\frac{\frac{1}{2}+\sigma}{\sqrt{\frac{1}{2}+\sigma}}\right). \end{aligned} \tag{4.15}$$

Therefore

$$A(\sigma) = \frac{a}{2\pi}\,I_1 - \frac{a}{\pi}\,I_2 = -a\,\frac{\sigma}{\sqrt{\frac{1}{2}+\sigma}} \tag{4.16}$$

Before proceeding any further, it is convenient to check that no mistake has been made and everything works by applying the inverse Abel transform.

$$\int_f^{1/2} \frac{A(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma = -a\int_f^{1/2} \frac{\sigma}{\sqrt{\sigma-f}\sqrt{\frac{1}{2}+\sigma}}\,\mathrm{d}\sigma\,. \tag{4.17}$$

To evaluate this, let us put

$$\sigma + \frac{1}{2}\left(\frac{1}{2}-f\right) = \frac{1}{2}\left(\frac{1}{2}+f\right)\cosh\theta \quad \Rightarrow \quad \theta(\sigma) = \cosh^{-1}\left(\frac{2\sigma+\frac{1}{2}-f}{\frac{1}{2}+f}\right).$$

This makes

$$\begin{aligned} \int_f^{1/2} \frac{A(\sigma)}{\sqrt{\sigma-f}}\mathrm{d}\sigma &= -\frac{a}{2}\int_{\theta(f)}^{\theta(1/2)} \left[-\left(\tfrac{1}{2}-f\right)+\left(\tfrac{1}{2}+f\right)\cosh\theta\right]\mathrm{d}\theta \\ &= \frac{a}{2}\left(\tfrac{1}{2}-f\right)\log\left(\frac{1+\sqrt{\frac{1}{2}-f}}{1-\sqrt{\frac{1}{2}-f}}\right) - a\sqrt{\tfrac{1}{2}-f} \equiv V_+(f) \end{aligned}$$

Hence the expression $A(\sigma) = -a\,\dfrac{\sigma}{\sqrt{\frac{1}{2}+\sigma}}$ is correct.

Consider now region III: $f = \frac{1}{2} \quad \Rightarrow \quad t = -z = \sqrt{\frac{1}{2} - g}$. This particularizes (4.10) into

$$V_{-}(g) = -\frac{a}{2}\left(\tfrac{1}{2} - g\right) \log\left(\frac{1+\sqrt{\frac{1}{2}-g}}{1-\sqrt{\frac{1}{2}-g}}\right) + a\sqrt{\tfrac{1}{2}-g}\,, \tag{4.18}$$

which has the same form as that of $V_{+}(f)$ except for showing opposite signs and the variable g appearing instead of f. Therefore

$$B(\sigma) = -A(\sigma) = a\,\frac{\sigma}{\sqrt{\frac{1}{2}+\sigma}}\,. \tag{4.19}$$

Now the Abel transform method can be used to generate the first order solution (4.10) we are considering, which using the variables f and g instead of z and t takes the form

$$\begin{aligned} V(f,g) \;=\; & \frac{a}{2}(g-f)\log\left(\frac{1+\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}-g}}{1-\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}-g}}\right) \\ & -a\left(\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g}-\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}-g}\right). \end{aligned} \tag{4.20}$$

The Abel transform method states that

$$\begin{aligned} V(f,g) \;&=\; \int_f^{1/2} \frac{A(\sigma)\sqrt{\frac{1}{2}+\sigma}}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma + \int_g^{1/2} \frac{B(\sigma)\sqrt{\frac{1}{2}+\sigma}}{\sqrt{\sigma+f}\sqrt{\sigma-g}}\,\mathrm{d}\sigma \\ &=\; -a\int_f^{1/2} \frac{\sigma}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma + a\int_g^{1/2} \frac{\sigma}{\sqrt{\sigma+f}\sqrt{\sigma-g}}\,\mathrm{d}\sigma \\ &=\; X+Y\,. \end{aligned} \tag{4.21}$$

Applying the change of variable given by

$$\sigma + \frac{1}{2}(g-f) = \frac{1}{2}(g+f)\cosh\theta$$

to X, it gives rise to

$$\begin{aligned} X \;&=\; -\frac{a}{2}\int_{\theta(f)}^{\theta(1/2)} \left[(f-g)+(f+g)\cosh\theta\right]\mathrm{d}\theta \\ &=\; \frac{a}{2}(g-f)\log\left(\frac{1-f+g+2\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}+g}}{f+g}\right) - a\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}-g}\,. \end{aligned} \tag{4.22}$$

Analogously,

$$Y = \frac{a}{2}(f-g)\log\left(\frac{1+f-g+2\sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}-g}}{f+g}\right) + a\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}-g}\,. \tag{4.23}$$

Now

$$X+Y=\frac{a}{2}(g-f)\log\left[\frac{\left(1-f+g+2\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g}\right)\left(1-f+g+2\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g}\right)}{(f+g)^2}\right]$$
$$-a\left(\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g}-\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}-g}\right), \tag{4.24}$$

which is what we needed in order to prove (4.21): to see that (4.20) is exactly the same as (4.24), we must only multiply and divide what appears inside the logarithm in (4.20) by $\left(1-\sqrt{\tfrac{1}{2}-f}\sqrt{\tfrac{1}{2}+g}+\sqrt{\tfrac{1}{2}+f}\sqrt{\tfrac{1}{2}-g}\right)$.

Analogously, if we denote $A_n(\sigma)$ as the spectral function corresponding to $V_n = Q_n(t)\,P_n(z)$ (in region II), and $B_n(\sigma)$ as the spectral function corresponding to V_n in region III, it can easily be found that

$$A_0(\sigma)=\frac{1}{2\sqrt{\frac{1}{2}+\sigma}}, \qquad B_0(\sigma)=\frac{1}{2\sqrt{\frac{1}{2}+\sigma}},$$
$$A_1(\sigma)=-\frac{\sigma}{\sqrt{\frac{1}{2}+\sigma}}, \qquad B_1(\sigma)=\frac{\sigma}{\sqrt{\frac{1}{2}+\sigma}},$$
$$A_2(\sigma)=-\frac{12\sigma^2-1}{4\sqrt{\frac{1}{2}+\sigma}}, \qquad B_2(\sigma)=-\frac{12\sigma^2-1}{4\sqrt{\frac{1}{2}+\sigma}},$$
$$A_3(\sigma)=\frac{20\sigma^3-3\sigma}{2\sqrt{\frac{1}{2}+\sigma}}, \qquad B_3(\sigma)=-\frac{20\sigma^3-3\sigma}{2\sqrt{\frac{1}{2}+\sigma}},$$

and so on for arbitrary n (a list of the first few Legendre functions is provided in Appendix B). These can be applied to obtain the corresponding solution in region IV.

It would have been a desirable feature if the recurrence relations which characterise the Legendre functions could be used to generate the spectral functions. Unfortunately, no analytical expression for a general term involving only a few order terms could be found. Nevertheless, the existence of such recurrence relations for the functions V_{n+} and V_{n-} ensures the applicability of the Abel transform method in order to generate the spectral functions corresponding to the initial data. Moreover, it turns out to be very easy to generate spectral functions and their corresponding final solutions by computing algebra.

It is important to notice that to satisfy the plane wave condition (2.20) –and the equivalent for region III–, which requires the derivative of V with respect to

f and g to be unbounded on the wavefront, at least one Legendre function of the second kind must be included.

In general, solutions in the interaction region can be constructed as a linear combination of Legendre functions, and the corresponding spectral functions can be found as above. However, in practise this is only useful when the initial data V_+ and V_- consist only of a small number of this kind of term, as all the higher order terms should be taken into account, because they would all influence the continuity across the junctions.

Chapter 5

Double impulse and sandwich waves

As pointed out by Szekeres [29], it is important to consider the collision of two pairs of step functions of opposite amplitude, because in this situation, the energy content of each sandwich wave is finite, making this a physically acceptable limiting case. Here we are going to consider the more general case in which the amplitudes of each pair of waves do not have to be of equal sign or magnitude. Thus the function $h_+(u)$ must be of the form

$$h_+(u) = a\delta(u) + b\delta(u - u_1) \tag{5.1}$$

for some $u_1 \in (0, \frac{1}{|a|})$. The interval u_1 between the two impulses is assumed to be small. Now substituting this into the differential equations (1.16) results in

$$P = \begin{cases} 1 & \text{if } u \leq 0 \\ 1 - au & \text{if } 0 \leq u \leq u_1 \\ 1 - au - b(1 - au_1)(u - u_1) & \text{if } u_1 \leq u \end{cases}$$

and

$$Q = \begin{cases} 1 & \text{if } u \leq 0 \\ 1 + au & \text{if } 0 \leq u \leq u_1 \\ 1 + au + b(1 + au_1)(u - u_1) & \text{if } u_1 \leq u. \end{cases}$$

Thus

$$e^{-U_+} = \begin{cases} 1 & \text{if } u \leq 0 \\ 1 - a^2u^2 & \text{if } 0 \leq u \leq u_1 \\ 1 - a^2u^2 - (2ab + b^2 - b^2a^2u_1^2)(u - u_1)^2 & \text{if } u_1 \leq u. \end{cases},$$

or, alternatively,

$$e^{-U_+} = 1 - a^2u^2\,\Theta(u) - (2ab + b^2 - b^2a^2u_1^2)(u - u_1)^2\,\Theta(u - u_1)\,.$$

Hence, defining $f(u)$ by $e^{-U_+} = \frac{1}{2} + f(u)$, one obtains that

$$f(u) = \frac{1}{2} - a^2u^2\Theta(u) - (2ab + b^2 - b^2a^2u_1^2)\,(u - u_1)^2\Theta(u - u_1).$$

Notice that e^{-U_+} must be non-negative, and that a singularity occurs when $e^{-U_+} = 0$. This singularity, as indicated before, is due to the focussing effect of the wave. The initial restriction that $0 < u_1 < \frac{1}{|a|}$ is required to avoid the second wave appearing after the singularity. Also, since $e^{-U_+} = PQ$, and P and Q are piecewise linear functions, the singularity will then occur either when $P = 0$ or when $Q = 0$. For $u > u_1$,

$$\begin{aligned} P(u) &= 1 + bu_1 - abu_1^2 - (a + b - abu_1)\,u\,, \\ Q(u) &= 1 - bu_1 - abu_1^2 + (a + b + abu_1)\,u\,. \end{aligned}$$

Now consider the case when $a, b > 0$. The singularity occurs when $P = 0$ (it cannot come from $Q = 0$, as Q is now increasing), i.e. when

$$u = \frac{1 + bu_1 - abu_1^2}{a + b - abu_1}\,, \tag{5.2}$$

This value of u is less than $1/a$, as may be seen just by confirming that

$$a + bau_1(1 - au_1) < a + b(1 - au_1)\,,$$

which is true because, as indicated above, $au_1 < 1$. Thus, the distance at which the singularity is produced happens to be reduced when there are two waves whose amplitudes have the same sign, instead of just one, i.e. in this case, *the second wave enhances the focussing effect of the first wave.*

Now consider the case when the impulses have opposite sign. Take first $a > 0$, $b < 0$: when $\frac{-a}{1-au_1} < b < 0$, $P(u)$ is decreasing. Thus, for $\frac{-a}{1-a^2u_1^2} < b < 0$, equation (5.2) gives the placement of the focussing singularity[1]. Notice that now the focussing distance has increased compared to that of the first impulse alone.

[1]The range of b is reduced from $\frac{-a}{1-au_1} < b < 0$ to $\frac{-a}{1-a^2u_1^2} < b < 0$ because, as we will see soon, for $\frac{-a}{1-au_1} < b < \frac{-a}{1+au_1}$ both P and Q are decreasing functions of u, but P decreases faster than Q in this common range only when $b > \frac{-a}{1-a^2u_1^2}$.

However, when $b < \frac{-a}{1+a\,u_1}$, it is $Q(u)$ that decreases. Thus, for[2] $\frac{-a}{1-au_1} < b < \frac{-a}{1-a^2u_1^2}$, the focussing singularity occurs when

$$u = \frac{1 - bu_1 - abu_1^2}{a + b + abu_1}\,, \tag{5.3}$$

which is also greater than $1/a$.

Observe that, for $\frac{-a}{1-au_1} < b < \frac{-a}{1+au_1}$, both P and Q are decreasing. Hence the singularity will be reached either when $P = 0$ or when $Q = 0$, whichever comes first. To see what happens first, let us consider the derivatives of P and Q:

$$\frac{\mathrm{d}P}{\mathrm{d}u} = abu_1 - (a+b)\,,$$

$$\frac{\mathrm{d}Q}{\mathrm{d}u} = abu_1 + (a+b)\,.$$

Therefore

$$\frac{\mathrm{d}P}{\mathrm{d}u} > \frac{\mathrm{d}Q}{\mathrm{d}u} \iff \frac{\mathrm{d}P}{\mathrm{d}u} - \frac{\mathrm{d}Q}{\mathrm{d}u} = -2(a+b) > 0 \iff b < -a\,, \tag{5.4}$$

and, since for this range of b, both $\frac{\mathrm{d}P}{\mathrm{d}u}$ and $\frac{\mathrm{d}Q}{\mathrm{d}u}$ are negative, when $\frac{-a}{1-au_1} < b < -a$, $\left|\frac{\mathrm{d}Q}{\mathrm{d}u}\right| > \left|\frac{\mathrm{d}P}{\mathrm{d}u}\right|$, and vice-versa: when $-a < b < \frac{-a}{1+au_1}$, we have that $\left|\frac{\mathrm{d}P}{\mathrm{d}u}\right| > \left|\frac{\mathrm{d}Q}{\mathrm{d}u}\right|$. Now the intersection point of P and Q with the hypersurface $u = u_1$ are $P(u_1) = 1 - au_1$, and $Q(u_1) = 1 + au_1$, i.e. $Q(u_1) > P(u_1)$. Thus the equation $P = Q = 0 \Leftrightarrow P(u_i) = 0$, where $u_i = \frac{bu_1}{a+b}$ is the value of u such that $P(u_i) = Q(u_i)$, will give us the value of b above which P becomes zero before Q and below which Q becomes zero before P. This value of b happens to be $b = \frac{-a}{1-a^2u_1^2}$.

However, the important feature is that *when the sign of the second impulse differs from that of the first, the second wave reduces the focussing effect of the first one.*

It can be seen that P and Q cannot be both increasing, and consequently, although the focussing effect of the first impulse is reduced by a second one with opposite polarization, that effect does not vanish.

[2]The range of b is reduced for the same reason as argued in the footnote above.

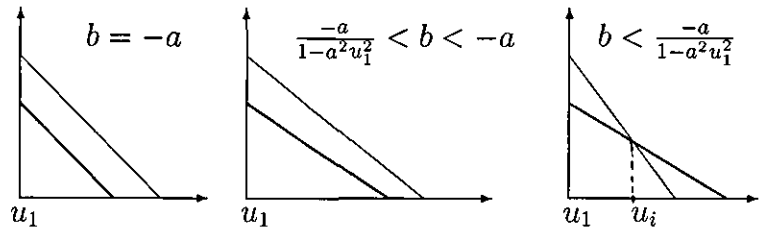


Figure 5.1: Representation of how P and Q vary according to the value of b: here, Q is represented by the thinner line. The vertical axis represents both P and Q, and the horizontal one, u. On the u-axis, u_i is the point of intersection between P and Q.

From the expressions of P and Q above, it is also obtained

$$e^{V_+} = \begin{cases} 1 & \text{if } u \leq 0 \\ \dfrac{1-au}{1+au} & \text{if } 0 \leq u \leq u_1 \\ \dfrac{1-au-b(1-au_1)(u-u_1)}{1+au+b(1+au_1)(u-u_1)} & \text{if } u_1 \leq u. \end{cases}$$

thus

$$V_+ = \log\left(\frac{1-au}{1+au}\right)\,\Theta(u) + \log\left[\frac{1-b\left(\frac{1-au_1}{1-au}\right)(u-u_1)}{1+b\left(\frac{1+au_1}{1+au}\right)(u-u_1)}\right]\,\Theta(u-u_1).$$

At this point, it is important to notice that it is not possible in general to use a transformation covering the whole of region II such that $e^{-U+} = \frac{1}{2} + f(u)$ with $f(u) = \frac{1}{2} - (c_+u)^{n+}$. This can always be done when there is *one* null hypersurface as a boundary, since it is achieved by relabelling it, but when the incident wave is limited by *two* null hypersurfaces, an appropriate transformation for one of them would not give the satisfactory result when applied to the other. Therefore, in these cases, it is going to be convenient to leave f in the form

$$f(u) = P(u)\,Q(u) - \tfrac{1}{2}\,,$$

with P and Q as resulting from solving (1.16). Thus, since $V = \log P - \log Q$, equation (3.7) reads now

$$A(\sigma) = \frac{1}{\pi}\int_0^{u_0} \frac{QP' - PQ'}{PQ\sqrt{PQ - \frac{1}{2} - \sigma}}\,\mathrm{d}u\,, \tag{5.5}$$

where u_0 is the positive root of $P(u)\,Q(u) - \frac{1}{2} - \sigma = 0$.

Notice also that in the intervals where $h_+(u) = 0$, P and Q are linear functions of u:

$$P = c_1 - c_2 u\,, \quad Q = c_3 + c_4 u$$

for some constants $c_i \quad (i = 1, 2, 3, 4)$. To solve the integral in (5.5) it is helpful to define the following constants:

$$k_1 = \tfrac{1}{2}\left(\tfrac{c_1}{c_2} - \tfrac{c_3}{c_4}\right)\,, \quad k_2 = \tfrac{1}{2}\left(\tfrac{c_1}{c_2} + \tfrac{c_3}{c_4}\right)\,, \quad k_3^2 = \tfrac{1}{c_2 c_4}\left(\tfrac{1}{2} + \sigma\right)\,.$$

This results in

$$\begin{aligned}
\int \frac{QP' - PQ'}{PQ\sqrt{PQ - c_2 c_4 k_3^2}}\,\mathrm{d}u &= -\frac{2k_2}{\sqrt{c_2 c_4}} \int \frac{\mathrm{d}u}{[k_2^2 - (u-k_1)^2\sqrt{k_2^2 - k_3^2 - (u-k_1)^2}]} \\
&= \frac{2k_2}{\sqrt{c_2 c_4}} \int \frac{\mathrm{d}\phi}{k_2^2 \sin^2\phi + k_3^2 \cos^2\phi} \\
&= \frac{2}{k_2\sqrt{c_2 c_4}} \int \frac{\sec^2\phi\,\mathrm{d}\phi}{(k_3^2/k_2^2) + \tan^2\phi} \qquad (5.6) \\
&= \frac{2}{k_3\sqrt{c_2 c_4}}\, \tan^{-1}\left(\frac{k_2}{k_3}\tan\phi\right)\,,
\end{aligned}$$

where the variable ϕ given by $u = k_1 + \sqrt{k_2^2 - k_3^2}\cos\phi$ has been introduced.

Therefore, the application of the method would require the inclusion of the term

$$\begin{aligned}
V_{+u} &= \frac{-2a}{1 - a^2 u^2}\,\Theta(u)\,\Theta(u_1 - u) \\
&\quad - \left(\frac{a + b\,(1 - a u_1)}{1 - au - b(1 - au_1)(u - u_1)} + \frac{a + b(1 + au_1)}{1 + au + b(1 + au_1)(u - u_1)}\right)\,\Theta(u - u_1).
\end{aligned}$$

into the integral used to obtain $A(\sigma)$, which results in the integral being split in two parts, one from 0 to u_1, and the other from u_1 to u_0. However these integrals have already been solved formally in (5.6), so it is only necessary to obtain the constants k_i (where $i = 1, 2, 3$) for each integral.

$$A(\sigma) = \frac{1}{\pi}\int_0^{u_1} \frac{Q_1 P_1' - P_1 Q_1'}{P_1 Q_1\sqrt{P_1 Q_1 - \frac{1}{2} - \sigma}}\,\mathrm{d}u + \frac{1}{\pi}\int_{u_1}^{u_0} \frac{Q_2 P_2' - P_2 Q_2'}{P_2 Q_2\sqrt{P_2 Q_2 - \frac{1}{2} - \sigma}}\,\mathrm{d}u\,, \quad (5.7)$$

where

$$\begin{aligned}
P_1 &= 1 - au\,, \qquad P_2 = 1 - au - b(1 - au_1)(u - u_1) \\
Q_1 &= 1 + au\,, \qquad Q_2 = 1 + au + b(1 + au_1)(u - u_1)\,.
\end{aligned}$$

For the first integral, $c_1 = c_3 = 1$ and $c_2 = c_4 = a$. Therefore $k_1 = 0$, $k_2 = 1/a$ and $k_3 = \frac{1}{a}\sqrt{\frac{1}{2} + \sigma}$, and the limits of integration change as $u = 0 \rightarrow \phi = \frac{\pi}{2}$ and $u = u_1 \rightarrow \phi = \phi_1$, where ϕ_1 is determined by

$$\tan\phi_1 = \frac{1}{au_1}\sqrt{\frac{1}{2} - \sigma - a^2u_1^2}\,.$$

For the second integral, $c_1 = 1 + bu_1(1 - au_1)$, $c_2 = a + b(1 - au_1)$, $c_3 = 1 - bu_1(1 + au_1)$, and $c_4 = a + b(1 + au_1)$. Thus

$$k_1 = bu_1\,\frac{2a+b(1-a^2u_1^2)}{(a+b)^2-a^2b^2u_1^2}\,,\quad k_2 = \frac{a+b(1-a^2u_1^2)}{(a+b)^2-a^2b^2u_1^2}\,,\quad k_3^2 = \frac{\frac{1}{2}+\sigma}{(a+b)^2-a^2b^2u_1^2}\,,$$

which put into (5.6) and setting the limits seen above for each integral, gives

$$A(\sigma) = \frac{2}{\pi\sqrt{\frac{1}{2}+\sigma}}\left[-\frac{\pi}{2} + \tan^{-1}\left(\frac{\sqrt{\frac{1}{2}-\sigma-a^2u_1^2}}{au_1\sqrt{\frac{1}{2}+\sigma}}\right) - \tan^{-1}\left(\frac{a+b(1-a^2u_1^2)}{a^2u_1}\,\frac{\sqrt{\frac{1}{2}-\sigma-a^2u_1^2}}{\sqrt{\frac{1}{2}+\sigma}}\right)\right]. \tag{5.8}$$

This has a very complicated dependence on σ, which makes it very difficult to continue, so we are first going to attempt to reconstruct the initial data using the transform

$$V_+(f) = \int_f^{\frac{1}{2}} \frac{A(\sigma)}{\sqrt{\sigma - f}}\,\mathrm{d}\sigma\,. \tag{5.9}$$

To do this, it is convenient to decompose the space-time into the different regions, which arise from having two pairs of colliding impulses. This is indicated in Fig 5.2.

For region II_a, (5.9) is

$$\int_f^{\frac{1}{2}} \frac{A(\sigma)}{\sqrt{\sigma - f}}\,\mathrm{d}\sigma = -\int_f^{\frac{1}{2}} \frac{\mathrm{d}\sigma}{\sqrt{\sigma - f}\sqrt{\sigma + \frac{1}{2}}} = -\left[\log\left(\sqrt{\sigma + \frac{1}{2}} + \sqrt{\sigma - f}\right)\right]_f^{1/2},$$

in which the integral has been solved analogously as in (4.4). This yields

$$V_+(f) = \log\left(\frac{1 - au}{1 + au}\right),$$

i.e. the initial data for the Khan-Penrose solution, as expected. However, the reconstruction of V_+ in region II_b implies the integration of (5.9) with the form of $A(\sigma)$ as given in (5.8), and

$$f = \frac{1}{2} - a^2u^2 - (2ab + b^2 - b^2a^2u_1^2)\,,$$

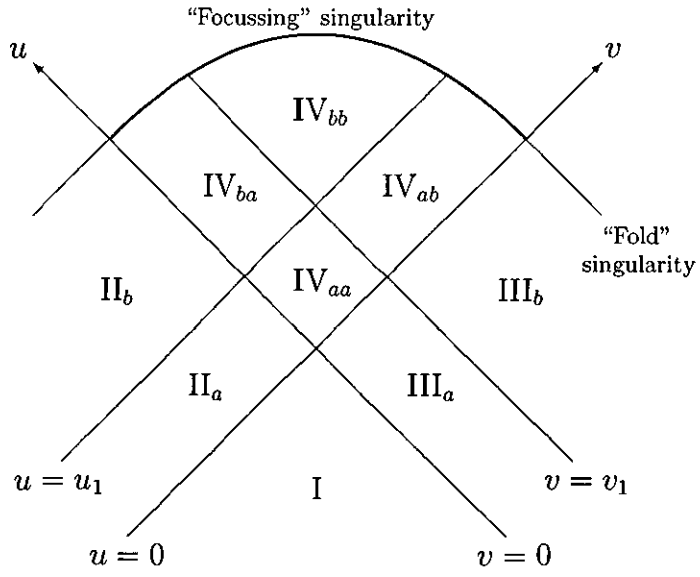


Figure 5.2: The structure of the space-time representing the collision of two pairs of impulsive gravitational waves located on the null hypersurfaces $u = 0$, $u = u_1$, $v = 0$ and $v = v_1$. Region I is part of Minkowski space. Regions II and III are also parts of Minkowski space between and behind the impulsive waves prior to their collision. The regions IV represent the interaction region. Region IV$_{aa}$ is given by the Khan-Penrose solution. The other parts of region IV need to be determined.

resulting in another complicated expression. As an alternative approach, the initial data can be given on the wavefront $f = f_1$, in order to avoid integrating over II$_a$. In this case, the resulting Abel transforms are given by

$$V_+(f) = V_+(f_1) + \int_f^{f_1} \frac{A(\sigma)}{\sqrt{\sigma - f}}\,\mathrm{d}\sigma\,, \qquad A(\sigma) = -\frac{1}{\pi}\int_\sigma^{f_1} \frac{V_+'(f)}{\sqrt{f - \sigma}}\,\mathrm{d}f\,. \tag{5.10}$$

For $0 \leq u \leq u_1$, the form of the functions are the same as those using in the first integral in equation (5.7), whereas for $u > u_1$, the whole expression (5.7) should be used. Thus we should obtain,

$$V_+(f) = \log\left(\frac{1 - au_1}{1 + au_1}\right) + \int_f^{f_1} \frac{A(\sigma)}{\sqrt{\sigma - f}}\,\mathrm{d}\sigma\,, \tag{5.11}$$

where

$$A(\sigma) = -\frac{2}{\pi\sqrt{\frac{1}{2} + \sigma}}\,\tan^{-1}\left(\frac{a + b(1 - a^2u_1^2)}{a^2u_1}\,\frac{\sqrt{\frac{1}{2} - \sigma - a^2u_1^2}}{\sqrt{\frac{1}{2} + \sigma}}\right)\,. \tag{5.12}$$

This can be simplified putting

$$c=\frac{a+b(1-a^2u_1^2)}{a^2u_1}\,,\quad f_1=\frac{1}{2}-a^2u_1^2\,,\quad x=\frac{\sqrt{f_1-\sigma}}{\sqrt{\frac{1}{2}+\sigma}}\,,\quad h=\frac{\sqrt{f_1-f}}{\sqrt{\frac{1}{2}+f}}\,,$$

yielding the following expression for the previous integral:

$$\int_f^{f_1}\frac{A(\sigma)}{\sqrt{\sigma-f}}\,\mathrm{d}\sigma=\frac{4}{\pi}\frac{\sqrt{\frac{1}{2}+f_1}}{\sqrt{\frac{1}{2}+f}}\int_f^{f_1}\frac{x\,\tan^{-1}(cx)}{\sqrt{h^2-x^2(1+x^2)}}\,\mathrm{d}x\,. \tag{5.13}$$

To evaluate this, we can make use of the identity

$$\tan^{-1}(cx)=\int_0^c\frac{x\,\mathrm{d}y}{1+x^2y^2}\,. \tag{5.14}$$

Therefore

$$\begin{aligned}\int_f^{f_1}\frac{x\,\tan^{-1}(cx)}{\sqrt{h^2-x^2(1+x^2)}}\,\mathrm{d}x&=\int_h^0\frac{x}{\sqrt{h^2-x^2(1+x^2)}}\left[\int_0^c\frac{x\,\mathrm{d}y}{1+x^2y^2}\right]\mathrm{d}x\\&=\int_0^c\left[\int_h^0\frac{x^2\,\mathrm{d}x}{\sqrt{h^2-x^2(1+x^2)(1+x^2y^2)}}\right]\mathrm{d}y\,,\end{aligned}$$

where the order of integration has been changed. The integral between brackets can be obtained by putting $x=h\,\sin\theta$:

$$\begin{aligned}\int_h^0\frac{x^2\,\mathrm{d}x}{\sqrt{h^2-x^2(1+x^2)(1+x^2y^2)}}&=\int_{\pi/2}^0\frac{h^2\sin^2\theta}{(1+h^2\sin^2\theta)(1+y^2h^2\sin^2\theta)}\,\mathrm{d}\theta\\&=-\frac{1}{1-y^2}\left[\int_{\pi/2}^0\frac{\mathrm{d}\theta}{1+h^2\sin^2\theta}-\int_{\pi/2}^0\frac{\mathrm{d}\theta}{1+y^2h^2\sin^2\theta}\right]\\&=\frac{\pi}{2(1-y^2)}\left[\frac{1}{\sqrt{1+h^2}}-\frac{1}{\sqrt{1+h^2y^2}}\right].\end{aligned}$$

Consequently,

$$\begin{aligned}\int_f^{f_1}\frac{x\,\tan^{-1}(cx)}{\sqrt{h^2-x^2(1+x^2)}}\,\mathrm{d}x&=\frac{\pi}{2\sqrt{1+h^2}}\int_0^c\frac{\mathrm{d}y}{1-y^2}-\frac{\pi}{2}\int_0^c\frac{\mathrm{d}y}{\sqrt{1+h^2y^2}(1-y^2)}\\&=\frac{\pi}{4\sqrt{1+h^2}}\left[\log\left(\frac{1+c}{1-c}\right)-\log\left(\frac{\sqrt{1+c^2h^2}+c\sqrt{1+h^2}}{\sqrt{1+c^2h^2}-c\sqrt{1+c^2h^2}}\right)\right],\end{aligned}$$

where the second integral has been evaluated introducing the variable ϕ, given by $hy=\sinh\phi$. With this, the initial data, $V_+(f)$, can be reconstructed:

$$\begin{aligned}V_+(f)&=\log\left(\frac{1-au_1}{1+au_1}\right)+\frac{4}{\pi}\frac{\sqrt{\frac{1}{2}+f_1}}{\sqrt{\frac{1}{2}+f}}\int_h^0\frac{x\,\tan^{-1}(cx)}{\sqrt{h^2-x^2(1+x^2)}}\,\mathrm{d}x\\&=\log\left[\left(\frac{1-au_1}{1+au_1}\right)\left(\frac{c+1}{c-1}\right)\left(\frac{c\sqrt{1+h^2}-\sqrt{1+c^2h^2}}{c\sqrt{1+h^2}+\sqrt{1+c^2h^2}}\right)\right].\end{aligned}$$

After obtaining $A(\sigma)$ corresponding to region IV_{ba}, in order to apply the method to find the solution in this region, it is also necessary to obtain the expression corresponding to $B(\sigma)$ here. Notice that, since we are considering this sub-region for which the initial value problem has been reformulated in terms of the data on $f = f_1$ and on $g = \frac{1}{2}$, $V_-(g)$ must be given by the solution obtained in region IV_{aa}, but fixing $f = f_1$. Therefore

$$V_-(g) = -\log\left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f_1}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f_1}}\right) - \log\left(\frac{\sqrt{\frac{1}{2}+f_1}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f_1}+\sqrt{\frac{1}{2}-g}}\right),$$

which gives rise to

$$V_-'(g) = -\frac{1}{f_1+g}\left(\frac{\sqrt{\frac{1}{2}-f_1}}{\sqrt{\frac{1}{2}+g}} + \frac{\sqrt{\frac{1}{2}+f_1}}{\sqrt{\frac{1}{2}-g}}\right).$$

Consequently,

$$B(\sigma) = \frac{\sqrt{\frac{1}{2}-f_1}}{\pi}\int_\sigma^{\frac{1}{2}} \frac{\mathrm{d}g}{(f_1+g)\sqrt{g-\sigma}\sqrt{\frac{1}{2}+g}} + \frac{\sqrt{\frac{1}{2}+f_1}}{\pi}\int_\sigma^{\frac{1}{2}} \frac{\mathrm{d}g}{(f_1+g)\sqrt{g-\sigma}\sqrt{\frac{1}{2}-g}}.$$

To evaluate this, it will be convenient use in the first integral the change of variable given by

$$g = \tfrac{1}{2}\left[(\tfrac{1}{2}+\sigma)\cosh\theta - (\tfrac{1}{2}-\sigma)\right],$$

and in the second integral,

$$g = \tfrac{1}{2}\left[(\tfrac{1}{2}+\sigma) - (\tfrac{1}{2}-\sigma)\cos\phi\right].$$

Applying these to the integral above, it now reads

$$\begin{aligned} B(\sigma) = \frac{au_1}{\pi}\int_0^{\cosh^{-1}\left(\frac{3-2\sigma}{1+2\sigma}\right)} & \frac{2\,\mathrm{d}\theta}{(\frac{1}{2}+\sigma)\cosh\theta + \frac{1}{2} + \sigma - 2a^2u_1^2} \\ & + \frac{\sqrt{1-a^2u_1^2}}{\pi}\int_0^{\pi} \frac{2\,\mathrm{d}\phi}{\frac{3}{2}+\sigma-2a^2u_1^2-(\frac{1}{2}-\sigma)\cos\phi} \\ = \frac{1}{\sqrt{f_1+\sigma}} & \left[\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{\frac{1}{2}-f_1}\sqrt{\frac{1}{2}-\sigma}}{\sqrt{f_1+\sigma}}\right)+1\right]. \end{aligned} \tag{5.15}$$

Now the solution in region IV_{ba} should be constructed as follows:

$$V(f,g) = V(f_1,\tfrac{1}{2}) + \int_f^{f_1} \frac{A(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma + \int_g^{\frac{1}{2}} \frac{B(\sigma)\sqrt{\sigma+f_1}}{\sqrt{\sigma-g}\sqrt{\sigma+f}}\,\mathrm{d}\sigma,$$

where

$$V(f_1, \tfrac{1}{2}) = \log\left(\frac{1-\sqrt{\frac{1}{2}-f_1}}{1+\sqrt{\frac{1}{2}-f_1}}\right),$$

$B(\sigma)$ is given by the expression above, and $A(\sigma)$ is the one given in (5.12), which may alternatively be written as

$$A(\sigma) = -\frac{2}{\pi\sqrt{\sigma+\frac{1}{2}}}\tan^{-1}\left(c\frac{\sqrt{f_1-\sigma}}{\sqrt{\sigma+\frac{1}{2}}}\right).$$

Using the same x, h, and c than those appearing in (5.13), and defining another term $\alpha = \frac{\sqrt{\frac{1}{2}-g}}{\sqrt{f_1+g}}$, we can write

$$\int_f^{f_1}\frac{A(\sigma)\sqrt{\sigma+\frac{1}{2}}}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma = \frac{4\left(\frac{1}{2}+f_1\right)}{\pi\sqrt{\frac{1}{2}+f}\sqrt{f_1+g}}\int_h^0\frac{x\tan^{-1}(cx)}{(1+x^2)\sqrt{h^2-x^2}\sqrt{1-\alpha^2x^2}}\,\mathrm{d}x,$$

in which the identity (5.14) can be inserted, resulting

$$\int_h^0\frac{x\tan^{-1}(cx)}{(1+x^2)\sqrt{h^2-x^2}\sqrt{1-\alpha^2x^2}}\,\mathrm{d}x$$

$$= \int_0^c\left[\int_h^0\frac{x^2\,\mathrm{d}x}{\sqrt{h^2-x^2}\sqrt{1-\alpha^2x^2}(1+x^2)(1+x^2y^2)}\right]\mathrm{d}y.$$

The integral between brackets is very similar to that appearing when $A(\sigma)$ was obtained, except for a term $\sqrt{1-\alpha^2x^2}$ in the denominator. Therefore the same decomposition used then is used now:

$$\frac{x^2}{(1+x^2)(1+x^2y^2)} = -\frac{1}{(1-y^2)(1+x^2)} + \frac{1}{(1-y^2)(1+x^2y^2)}.$$

Thus it can be seen that we will have to integrate expressions of the form

$$\int_h^0\frac{\mathrm{d}x}{\sqrt{h^2-x^2}(1+x^2y^2)\sqrt{1-\alpha^2x^2}} = \int_{\frac{\pi}{2}}^0\frac{\mathrm{d}\phi}{(1+h^2y^2\sin^2\phi)\sqrt{1-\alpha^2h^2\sin^2\phi}}$$

$$= -\Pi\left(-h^2y^2|\alpha^2h^2\right), \tag{5.16}$$

where $\Pi\left(-h^2y^2|\alpha^2h^2\right)$ is an elliptic integral of the third kind. Moreover, this still has to be integrated again with respect to y, which indicates it is extremely unlikely that the result could be expressed in terms of elementary functions.

Chapter 6

Initial value problem for other types of colliding waves

It is also appropriate to consider other types of colliding waves, to try to obtain physically well-behaved solutions. However, at the very best, the complications that emerge in doing so are of the same complexity as that described in the previous chapter. A couple of these unsuccessful cases are described below.

6.1 Step wave

The Abel transform method for the collision of single-impulse waves was satisfactorily tested. Therefore, a reasonably good idea could be to consider the collision involving exact step functions, as they are more realistic than the solutions of Szekeres class, which are unbounded.

Let us therefore consider the constant step profile function given by $h(u) = a^2\,\Theta(u)$. When introduced into the differential equations (1.16), this produces the results

$$P(u) = 1 + [\cos(au) - 1]\Theta(u)\,, \quad \text{and} \quad Q(u) = 1 + [\cosh(au) - 1]\Theta(u)\,.$$

Thus $e^{-U_+} = 1 + [\cos(au)\,\cosh(au) - 1]\,\Theta(u)$, and $e^{V_+} = 1 + \left[\dfrac{\cos(au)}{\cosh(au)} - 1\right]\Theta(u)$.

Therefore, for $u \geq 0$, $V_{+u} = -a[\tan(au) + \tanh(au)]$, and $f = \cos(au)\,\cosh(au) - \frac{1}{2}$.

Consequently, the spectral function is given by

$$A(\sigma) = -\frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}} \frac{V'_{+}(f)}{\sqrt{f-\sigma}}\mathrm{d}f = \frac{2a}{\pi}\int_{u_\sigma}^{u=f^{-1}(\frac{1}{2})=0} \frac{\tan(au)+\tanh(au)}{\sqrt{-\frac{1}{2}-\sigma+\cos(au)\cosh(au)}}\,\mathrm{d}u\,,$$

where $u_\sigma = f^{-1}(\sigma)$. We find two problems here: the lower integration limit complicates the expression, as it is not possible to invert $f = \cos(au)\,\cosh(au) - \frac{1}{2}$ in order to obtain $u = u(f)$ explicitly. Moreover, no way was found to evaluate the indefinite integral in terms of elementary functions, which makes it impossible to continue applying the method.

6.2 Sandwich Szekeres-like incident waves

One of the most interesting applications of a direct method is to find the solutions to more physically significant situations. A good starting point for this is to consider sandwich waves with relatively simple profiles. The collision of square waves presents the same difficulty as that appearing in the case above. Hence, it could be worth considering the sandwich waves composed of part of the Szekeres' family of solutions, because in this cause, the initial part of the interaction is known.

The Szekeres class of vacuum solutions [29] was characterized by the incident waves having their shape given by equation (4.5), consequently the function that has to be taken is

$$h(u) = c_+^2\, n_+^2\, \frac{k_+}{4}\left(1-\frac{2}{n_+}\right)\frac{(c_+u)^{n_+/2-2}}{[1-(c_+u)^{n_+}]^{3-2/n_+}}\,[\Theta(u)-\Theta(u-u_1)] \quad \text{for} \quad n_+ > 2.$$

This gives rise to

$$P = \begin{cases} 1 & \text{for } u \le 0 \\ \dfrac{\left(1-(c_+u)^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1+(c_+u)^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} & \text{for } 0 \le u \le u_1 \\ c_1 - c_2 u & \text{for } u_1 \le u \end{cases} ;\; Q = \begin{cases} 1 & \text{for } u \le 0 \\ \dfrac{\left(1+(c_+u)^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1-(c_+u)^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} & \text{for } 0 \le u \le u_1 \\ c_3 + c_4 u & \text{for } u_1 \le u, \end{cases}$$

where, to ensure continuity of P and Q across the junction, the constants c_{1-4} are given by

$$c_1 = \frac{\left(1-(c_+u_1)^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1+(c_+u_1)^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} + \frac{n_+}{2}u_1^{\frac{n_+}{2}+1}\left[\frac{k_++1}{2}\left(\frac{1-(c_+u_1)^{\frac{n_+}{2}}}{1+u_1^{\frac{n_+}{2}}}\right)^{\frac{k_+-1}{2}} + \frac{k_+-1}{2}\left(\frac{1-(c_+u_1)^{\frac{n_+}{2}}}{1+u_1^{\frac{n_+}{2}}}\right)^{\frac{k_++1}{2}}\right],$$

$$c_2 = \frac{n_+}{2}(c_+u_1)^{\frac{n_+}{2}}\left[\frac{k_++1}{2}\left(\frac{1-(c_+u_1)^{\frac{n_+}{2}}}{1+(c_+u)^{\frac{n_+}{2}}}\right)^{\frac{k_+-1}{2}} + \frac{k_+-1}{2}\left(\frac{1-(c_+u_1)^{\frac{n_+}{2}}}{1+(c_+u)^{\frac{n_+}{2}}}\right)^{\frac{k_++1}{2}}\right]$$

$$c_3 = \frac{\left(1+(c_+u_1)^{\frac{n_+}{2}}\right)^{\frac{k_++1}{2}}}{\left(1-(c_+u_1)^{\frac{n_+}{2}}\right)^{\frac{k_+-1}{2}}} + \frac{n_+}{2}u_1^{\frac{n_+}{2}+1}\left[\frac{k_++1}{2}\left(\frac{1+(c_+u_1)^{\frac{n_+}{2}}}{1-u_1^{\frac{n_+}{2}}}\right)^{\frac{k_+-1}{2}} + \frac{k_+-1}{2}\left(\frac{1+(c_+u_1)^{\frac{n_+}{2}}}{1-u_1^{\frac{n_+}{2}}}\right)^{\frac{k_++1}{2}}\right],$$

$$c_4 = \frac{n_+}{2}(c_+u_1)^{\frac{n_+}{2}}\left[\frac{k_++1}{2}\left(\frac{1+(c_+u_1)^{\frac{n_+}{2}}}{1-(c_+u)^{\frac{n_+}{2}}}\right)^{\frac{k_+-1}{2}} + \frac{k_+-1}{2}\left(\frac{1+(c_+u_1)^{\frac{n_+}{2}}}{1-(c_+u)^{\frac{n_+}{2}}}\right)^{\frac{k_++1}{2}}\right].$$

Therefore, for $u > u_1$ within region II, we have a flat region, and it is therefore possible to use (5.6) to evaluate a part of $A(\sigma)$ here. However, It may be observed that, for $u_1 \leq u$, P and Q have the same form as the ones appearing in the two-impulse waves collision, and similar complications arise now. Therefore, in order to proceed with the calculations, it is necessary to overcome the difficulties appearing in the two-impulse case.

Chapter 7

The solution as a series expansion of hypergeometric functions

Another way of approaching the problem is to consider the family of self-similar solutions of (2.25), described in [3, 4, 5]. This gives a general way of obtaining exact formal expressions for $V(f,g)$ corresponding to the interaction region. First of all, let us define new coordinates τ and ζ by

$$\tau = f+g\,, \qquad \zeta = \frac{1-f+g}{f+g}\,,$$

which are different from those originally considered in [3, 4, 5] in that f and g are replaced by $f-\frac{1}{2}$ and $g+\frac{1}{2}$ respectively, in order to adapt them to the problem considered here for the wave with wavefront $u=0$ corresponding to $f=\frac{1}{2}$. The hypersurface $f=\frac{1}{2}$ is expressed in these new coordinates as $\zeta=1$. Now consider the family of self-similar solutions

$$V(\tau,\zeta) = \tau^k\, H_k(\zeta)\,, \quad \text{i.e.} \quad V(f,g) = (f+g)^k\, H_k\left(\frac{1-f+g}{f+g}\right) \tag{7.1}$$

where k is any non-negative, real parameter. It can be seen that, in this case, the condition $V(f=\frac{1}{2})=0$ is equivalent to $H_k(1)=0$. The main field equation (2.25) can be written now as the O.D.E.

$$(1-\zeta^2)H_k'' + (2k-1)\zeta H_k' - k^2 H_k = 0\,. \tag{7.2}$$

This equation admits a class of solution which can be given in terms of standard Gauss hypergeometric functions by

$$H_k\left(\frac{1-f+g}{f+g}\right) = c_k\left(\frac{\frac{1}{2}-f}{f+g}\right)^{\frac{1}{2}+k} F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2}+k;\frac{f-\frac{1}{2}}{f+g}\right)\,, \tag{7.3}$$

The constant coefficients c_k may be chosen such that H_k satisfies the recursion relations

$$H_k(\zeta) = \int_1^\zeta H_{k-1}(\zeta')\,\mathrm{d}\zeta'\,,$$

i.e. making it generally possible to write

$$H_k'(\zeta) = H_{k-1}(\zeta)\,, \tag{7.4}$$

as stated in [4]. For integer values of k, explicit solutions can be generated using (7.4) from the initial solution $H_0(\zeta) = \cosh^{-1}\zeta$, which may be obtained by making $k = 0$ in (7.2) and imposing (7.1). In this case,

$$c_k = (-1)^k \frac{2^k\,\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})}\,,$$

although for an arbitrary k, the only requirement imposed by (7.4) is

$$c_k = -\frac{2}{k+\frac{1}{2}}\,c_{k-1}\,.$$

Since the main field equation is linear, an arbitrary linear combination of higher order terms will also be a solution.

Now, taking the case of a wave propagating in the opposite direction, i.e. the case corresponding to a wave with shock front on $g = \frac{1}{2}$, the expressions obtained are analogous to the ones considered for region II, and the corresponding solutions can also be combined in the same form, yielding a general solution in the form

$$V(f,g) = \sum_{k=0}^{\infty} a_k\,(f+g)^k\,H_k\!\left(\frac{1-f+g}{f+g}\right) + \sum_{k=0}^{\infty} b_k\,(f+g)^k\,H_k\!\left(\frac{1+f-g}{f+g}\right)\,, \tag{7.5}$$

where, in order to satisfy the shock wave conditions, it is necessary that $a_0 = -2\sqrt{2k_+}$ and $b_0 = -2\sqrt{2k_-}$ [1]. Notice that now, the shock wave conditions place constraints only on the coefficients with $k = 0$, unlike what happened in the expansion in terms of Legendre functions, in section 4.3, where a constraint appeared on all the coefficients. Particularly, taking only $k = 0$, if we make $a_0 = b_0 = -1$,

[1] The values of a_0 and b_0 have been corrected with respect to those appearing in [15].

$$
\begin{aligned}
V(f,g) &= -H_0\left(\frac{1-f+g}{f+g}\right) - H_0\left(\frac{1+f-g}{f+g}\right) \\
&= -\frac{\sqrt{\frac{1}{2}-f}}{\sqrt{f+g}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{f-\frac{1}{2}}{f+g}\right) - \frac{\sqrt{\frac{1}{2}-g}}{\sqrt{f+g}}\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{g-\frac{1}{2}}{f+g}\right) \\
&= 2\sinh^{-1}\left(-\frac{\sqrt{\frac{1}{2}-f}}{\sqrt{f+g}}\right) + 2\sinh^{-1}\left(-\frac{\sqrt{\frac{1}{2}-g}}{\sqrt{f+g}}\right) \\
&= \log\left(\frac{\sqrt{\frac{1}{2}+g}-\sqrt{\frac{1}{2}-f}}{\sqrt{\frac{1}{2}+g}+\sqrt{\frac{1}{2}-f}}\right) + \log\left(\frac{\sqrt{\frac{1}{2}+f}-\sqrt{\frac{1}{2}-g}}{\sqrt{\frac{1}{2}+f}+\sqrt{\frac{1}{2}-g}}\right)\,,
\end{aligned}
$$

which is exactly the Khan-Penrose Solution (note: on the second line, the constant c_0 has been replaced by its value, 1; also, to express the third line in terms of logarithms, it may be useful to use Appendix A).

The solution (7.5) above must coincide with the initial data specified on the hypersurfaces limiting region IV. Therefore

$$
V_+(f) = V\left(f,\tfrac{1}{2}\right) = \sum_{k=0}^{\infty} a_k \left(\tfrac{1}{2}+f\right)^k H_k\left(\frac{\frac{3}{2}-f}{\frac{1}{2}+f}\right)
$$

and, similarly,

$$
V_-(g) = V\left(\tfrac{1}{2},g\right) = \sum_{k=0}^{\infty} b_k \left(\tfrac{1}{2}+g\right)^k H_k\left(\frac{\frac{3}{2}-g}{\frac{1}{2}+g}\right)\,.
$$

Hence, if the initial data are re-written in terms of these a_k and b_k, the general solution can be easily obtained. The problem is how to find these sequences of constants a_k and b_k from $V_+(f)$ and $V_-(g)$.

Notice that to each V_{+k} corresponds a spectral function

$$
A_k(\sigma) = -\frac{1}{\pi}\int_\sigma^{1/2} \frac{V'_{+k}(f)}{\sqrt{f-\sigma}}\,\mathrm{d}f\,, \qquad \text{where} \qquad V_{+k}(f) = a_k\left(\tfrac{1}{2}+f\right)^k H_k\left(\frac{\frac{3}{2}-f}{\frac{1}{2}+f}\right)\,.
$$

The important point is that these particular solutions V_k can be obtained from the spectral functions in the form

$$
A_k(\sigma) = p_k \frac{(\frac{1}{2}-\sigma)^k}{\sqrt{\frac{1}{2}+\sigma}}\,, \qquad .
$$

where the constant p_k needs to be determined [5]. Thus, the part of $V_k(f,g)$ corresponding to $A_k(\sigma)$, that we may name $V_{k+}(f,g)$, is given by

$$\begin{aligned} V_{k+}(f,g) &= p_k \int_f^{1/2} \frac{(\frac{1}{2}-\sigma)^k}{\sqrt{\sigma-f}\sqrt{\sigma+g}}\,\mathrm{d}\sigma \\ &= p_k \frac{(\frac{1}{2}-f)^{\frac{1}{2}+k}}{\sqrt{f+g}} \int_0^1 t^{-\frac{1}{2}}(1-t)^k \left[1+\left(\frac{\frac{1}{2}-f}{f+g}\right)t\right]^{-\frac{1}{2}} \mathrm{d}t \\ &= p_k \frac{\Gamma(\frac{1}{2})\Gamma(1+k)}{\Gamma(\frac{3}{2}+k)} \frac{(\frac{1}{2}-f)^{\frac{1}{2}+k}}{\sqrt{f+g}} F\Big(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2}+k;-\tfrac{\frac{1}{2}-f}{f+g}\Big)\,, \end{aligned}$$

where relations found in [1] have been applied. This is clearly proportional to the expression in (7.3), and, comparing both equations, one finds that

$$p_k = (-1)^k \frac{2^k\,\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(1+k)}\,a_k = (-1)^k \frac{2^{k-1}}{\Gamma(1+k)}\,a_k\,. \tag{7.6}$$

Therefore the way to obtain the expression corresponding to the general solution consists of, having specified the initial data $V_+(f)$ on the junction, to obtain firstly the spectral function $A(\sigma)$, and then, to express it in the form

$$A(\sigma) = \sum_{k=0}^{\infty} A_k(\sigma) = \sum_{k=0}^{\infty} p_k \frac{(\frac{1}{2}-\sigma)^k}{\sqrt{\frac{1}{2}+\sigma}}\,,$$

equating the coefficients corresponding to the different powers of $(1-\sigma)$. Notice that to preserve the distinct wave condition, if the initial data are well posed, it must be automatically satisfied that $p_0 = -\sqrt{2k_+}$ (and similarly, when operating with $B(\sigma)$, $q_0 = -\sqrt{2k_-}$). After that, (7.6) gives the relation between p_k and a_k, so this would produce the coefficients which, when substituted into the general solution, gives it explicitly. Thus, we would have

$$\begin{aligned} V(f,g) &= \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(1+k)}{2^{k-1}} p_k\,(f+g)^k\,H_k\Big(\frac{1-f+g}{f+g}\Big) \\ &\quad + \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(1+k)}{2^{k-1}} q_k\,(f+g)^k\,H_k\Big(\frac{1+f-g}{f+g}\Big)\,, \end{aligned}$$

or, explicitly,

$$\begin{aligned} V(f,g) = &\sum_{k=0}^{\infty}(-1)^k \frac{2^{k+1}}{(2k+1)!!} \frac{(\frac{1}{2}-f)^{\frac{1}{2}+k}}{\sqrt{f+g}} \frac{\mathrm{d}^k}{\mathrm{d}\sigma^k}\Big[A(\sigma)\sqrt{\tfrac{1}{2}+\sigma}\Big]_{\sigma=\frac{1}{2}} F\Big(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2}+k;\tfrac{f-\frac{1}{2}}{f+g}\Big) \\ &+ \sum_{k=0}^{\infty}(-1)^k \frac{2^{k+1}}{(2k+1)!!} \frac{(\frac{1}{2}-g)^{\frac{1}{2}+k}}{\sqrt{f+g}} \frac{\mathrm{d}^k}{\mathrm{d}\sigma^k}\Big[B(\sigma)\sqrt{\tfrac{1}{2}+\sigma}\Big]_{\sigma=\frac{1}{2}} F\Big(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2}+k;\tfrac{g-\frac{1}{2}}{f+g}\Big)\,. \end{aligned} \tag{7.7}$$

Notice that the higher the k, the higher the differentiability of the corresponding terms on the wavefront. This has the clear advantage that we could work with a small number of terms, whereas in other series representations, like that using Legendre functions (see Section 4.3), all terms should be taken into account, as all of them would affect the differentiability. Consequently, it could be possible to observe an approximate behaviour of V by taking the first few terms in the expansion.

It is important to indicate that what has been obtained in this chapter is valid for 'global' spectral functions. What this means is that, if we decompose the regions, as was done in the two-impulse case, then the coordinate transformation needed to use the self-similar solutions given in [3, 4, 5] would replace f by $f - f_h$, where f_h is the value of f at the hypersurface on which the new function V_+ is defined (in the case of region IV_{ba} in the double-impulse, $f_h = f_1$), leaving $g \to g + \frac{1}{2}$ as before. Analogously, when V_- is defined on the hypersurface $g = g_h$, we should make $g \to g - g_h$ and $f \to f + \frac{1}{2}$. When the initial data are taken on different hypersurfaces, the spectral functions also take a different form accordingly (the reasoning is the same as that of Chapter 5)

This affects the expression of the final solution in that portion of the interaction region in the following form:

$$\begin{aligned} V(f,g) = \sum_{k=0}^{\infty}(-1)^k \frac{2^{k+1}}{(2k+1)!!} \frac{(f_h - f)^{\frac{1}{2}+k}}{\sqrt{f+g+\frac{1}{2}-f_h}} \frac{\mathrm{d}^k}{\mathrm{d}\sigma^k}\left[A(\sigma)\sqrt{\tfrac{1}{2}+\sigma}\right]_{\sigma=f_h} \\ \times F\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}+k; \tfrac{f-f_h}{f+g+\frac{1}{2}-f_h}\right) \\ + \sum_{k=0}^{\infty}(-1)^k \frac{2^{k+1}}{(2k+1)!!} \frac{(g_h - g)^{\frac{1}{2}+k}}{\sqrt{f+g+\frac{1}{2}-g_h}} \frac{\mathrm{d}^k}{\mathrm{d}\sigma^k}\left[B(\sigma)\sqrt{\tfrac{1}{2}+\sigma}\right]_{\sigma=g_h} \\ \times F\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}+k; \tfrac{g-g_h}{f+g+\frac{1}{2}-g_h}\right). \end{aligned} \tag{7.8}$$

Finally, it could be mentioned that it is possible to have the elliptic integral which appears in (5.16) expressed as a series involving Gauss' hypergeometric functions (see Appendix C). However, if we try to use this approach instead of the one we have just seen, we will observe that, to begin with, we will have to handle products of hypergeometric functions. Therefore, and since the aim of this thesis was to give a method as practically applicable as possible, it does not make much sense to follow a more difficult way, whose results are going to show a form

at least as complicated as that described in this chapter. Anyway, an exercise could be proposed to find these results and show the equivalence between the solutions produced in both cases.

Chapter 8

Conclusion

This thesis started by giving some basic definitions and results necessary to proceed through the following chapters. As well as outlining some important properties of the spacetimes under consideration, a brief historic overview of the different methods used and achievements obtained to find metrics subsequent to the interaction of plane waves was given, for looking back in time is the best way to understand the present situation.

The most important part of this thesis concerns the formulation and solution of the initial value problem corresponding to the interaction of plane gravitational waves in the linear case. Given the initial profile functions $h_+(u)$, in region II, and $h_-(v)$, in region III, the metric functions characterising the colliding wave metrics are taken in a particularly suitable form. From these h_+ and h_-, we obtain the corresponding functions $P_\pm$ and $Q_\pm$, which determine $U_\pm$ and $V_\pm$. These metric functions, V_+ and V_- play a particularly important role, as they are used as initial data on the characteristics. Our main concern is then to obtain V in the interaction region. To do it, we implement a theoretical method initially proposed by Hauser and Ernst, based on the use of Abel transforms. This gives a direct approach to the colliding plane waves problem, which is exactly what we want. From V_+ and V_- given on the wavefronts, the auxiliary spectral functions $A(\sigma)$ and $B(\sigma)$ are evaluated by means of the (direct) Abel transform, and thanks to these spectral functions, it is possible to find the function V in the interaction region as long as the calculations involved can be done.

It is convenient to point out that, although finding the functions $U(u,v)$ and $V(u,v)$ would be enough to have the equations necessary to evaluate the func-

tion M, which arises in the interaction region even for zero M_+ and M_- as a consequence of the field equations, it might not be possible to reach an explicit expression for this function.

The problem is theoretically solved for the general case, assuming the spectral functions can be found (sometimes difficulties appear at this stage which seem to be unsolvable, like in Section 6.1). However, when dealing with arbitrary incident waves, the expressions become extremely complicated. The difficulties found to obtain exact solutions for arbitrary cases have also been indicated. Finally, a way to overcome these difficulties is given, expressing the solution in terms of series of hypergeometric functions.

When we tried to apply the method to the double-impulse case, the results initially obtained were so complicated that made us desist to continue (the series comprises terms of the form indicated in (7.8). Nor have we found any particular realistic case whose solution had a relatively simple form. Consequently, although a method has been formulated which makes it possible to solve the initial value problem in the collision of plane, aligned gravitational waves, it has only been found to be practically applicable for the simplest cases.

Nevertheless, the theoretical implementation of the Abel transform method can be a powerful tool in order to study the properties of space-times resulting from the collision of plane gravitational waves in the linear case. A further step in doing so could be the implementation of the method in computer simulations. This could also be applied to other situations where the structure of the equations is similar (for instance, in some cases of higher dimensional metrics of colliding waves [16]).

However, as indicated above, this is only valid in the linear case, and no simple extension seems to exist enclosing the nonlinear case. Dealing with this, work is still being carried out by Alekseev and Griffiths [6, 7], involving a different approach.

Appendix A

Functional relations

$$\sin^{-1}(z) = -i \log\left(iz + \sqrt{1-z^2}\right) = -i \sinh^{-1}(iz) \tag{A.1}$$

$$\cos^{-1}(z) = -i \log\left(z + \sqrt{z^2-1}\right) = -i \cosh^{-1}(z) \tag{A.2}$$

$$\tan^{-1}(z) = -\frac{i}{2} \log\left(\frac{1+iz}{1-iz}\right) = -i \tanh^{-1}(z) \tag{A.3}$$

$$\sinh^{-1}(z) = \log\left(z + \sqrt{z^2+1}\right) = -i \sin^{-1}(iz) \tag{A.4}$$

$$\cosh^{-1}(z) = \log\left(z + \sqrt{z^2-1}\right) = i \cos^{-1}(z) \tag{A.5}$$

$$\tanh^{-1}(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) = -i \tan^{-1}(iz) \tag{A.6}$$

Appendix B

Legendre functions

Here is a list of some low order Legendre functions, that can be used to obtain the spectral functions $A(\sigma)$ and $B(\sigma)$ for the class of solutions given in (4.9):

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= -\frac{1}{2} + \frac{3}{2}x^2 \\
P_3(x) &= -\frac{3x}{2} + \frac{5}{2}x^3 \\
P_4(x) &= \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4 \\
\ldots &\quad \ldots \\
Q_0(x) &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \\
Q_1(x) &= -1 + \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) \\
Q_2(x) &= \frac{-3}{2}x + \frac{1}{4}\left(-1+3x^2\right)\log\left(\frac{1+x}{1-x}\right) \\
Q_3(x) &= \frac{2}{3} - \frac{5}{2}x^2 - \frac{3}{4}x\left(1-\frac{5}{3}x^2\right)\log\left(\frac{1+x}{1-x}\right) \\
Q_4(x) &= \frac{55}{24}x - \frac{35}{8}x^3 + \frac{3}{16}\left(1-10x^2+\frac{35}{3}x^4\right)\log\left(\frac{1+x}{1-x}\right) \\
\ldots &\quad \ldots
\end{aligned}
$$

However, as can be seen in section 4.3, operating by hand is a arduous process, and it is not difficult to make some mistake. This task can be greatly simplified with the aid of computer algebra. For example, if we want to find the $A(\sigma)$ corresponding to $V = P_n(z)\,Q_n(t)$ for some n, using *Mathematica*, we can type

$$\mathtt{A[n_] := -\frac{1}{\pi}\int_{\sigma}^{1/2} \frac{\partial_f \left(\mathtt{LegendreP}\left[n, \sqrt{\frac{1}{2}-f}\right]\mathtt{LegendreQ}\left[n, \sqrt{\frac{1}{2}-f}\right]\right)}{\sqrt{f-\sigma}}\,\mathtt{df}}$$

which defines the function $A(\sigma)$ depending on n, the order of the Legendre functions involved. The expression corresponding to $B(\sigma)$ would only differ from the one above in the different sign appearing in the argument of P_n, as in region III, $t = -z = \sqrt{\frac{1}{2} - g}$.[1]

[1]To be precise, the variable f should be replaced by g, although it does not make any change in the final expression.

Appendix C

$\Pi(x|m)$ in terms of hypergeometric functions

The elliptic integral of the third kind is defined as

$$\Pi(x|m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1 - x\,\sin^2\theta)\sqrt{1 - m\,\sin^2\theta}}\,. \tag{C.1}$$

Now, if we name $(1 - x\,\sin^2\theta)^{-1} = f(x)$, we find that the n-th derivative of f with respect to x is given by

$$f^{n)}(x) = n!\,(1 - x\,\sin^2\theta)^{-(n+1)}\,\sin^{2n}\theta\,.$$

Consequently, the function $f(x)$ may be expanded about $x = 0$ to obtain

$$f(x) = \sum_{n=0}^{\infty} x^n\,\sin^{2n}\theta\,,$$

which can be introduced into (C.1), resulting in

$$\Pi(x|m) = \sum_{n=0}^{\infty} x^n \int_0^{\pi/2} \frac{\sin^{2n}\theta}{\sqrt{1 - m\,\sin^2\theta}}\,\mathrm{d}\theta\,. \tag{C.2}$$

Defining the new variable $t = \sin^2\theta$, the above integral reads now

$$\int_0^{\pi/2} \frac{\sin^{2n}\theta}{\sqrt{1 - m\,\sin^2\theta}}\,\mathrm{d}\theta = \frac{1}{2}\int_0^1 \frac{t^{n-\frac{1}{2}}}{\sqrt{1 - t}\sqrt{1 - mt}}\,\mathrm{d}t\,, \tag{C.3}$$

which is $\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+n)}{2\,\Gamma(1+n)}$ times the integral representation of $F\left(\frac{1}{2}, \frac{1}{2} + n; 1 + n; m\right)$, as found in eq. 15.3.1 of [1][1].

[1]Simply replace the constants by the following values: $a = \frac{1}{2}$, $b = \frac{1}{2} + n$, $c = 1 + n$, $z = m$.

Therefore we have

$$\Pi(x|m) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} x^n \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)} F\left(\tfrac{1}{2}, \tfrac{1}{2}+n; 1+n; m\right) , \tag{C.4}$$

and the expression corresponding to (5.16) is obtained by making $x = -h^2 y^2$ and $m = \alpha^2 h^2$, resulting in

$$\Pi(-y^2 h^2|\alpha^2 h^2) = \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} x^n (-1)^n (hy)^{2n} \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)} F\left(\tfrac{1}{2}, \tfrac{1}{2}+n; 1+n; \alpha^2 h^2\right) . \tag{C.5}$$

Bibliography

[1] Abramowitz, M. and Segun, I. A. (1964), *Handbook of Mathematical Functions*, Dover Publications.

[2] Aichelburg, P. C. (1971), *Acta Phys. Austriaca*, **34**, 279-284.

[3] Alekseev, G. A. and Griffiths, J. B. (1995), *Phys. Rev. D* **52**, 4497-4502.

[4] Alekseev, G. A. and Griffiths, J. B. (1996), *Class. Quantum Grav.*, **13**, 2191-2209.

[5] Alekseev, G. A. and Griffiths, J. B. (1997), *Class. Quantum Grav.*, **14**, 2869-2880.

[6] Alekseev, G. A. and Griffiths, J. B. (2001) *Phys. Rev. Lett.*, **87**, 221101.

[7] Alekseev, G. A. and Griffiths, J. B. (2002), in preparation.

[8] Bondi, H., Pirani, F. A. E. and Robinson, I. (1959), *Proc. Roy. Soc. A*, **251**, 519-533.

[9] Bonnor, W. B. (1969), *Comm. Math. Phys*, **13**, 163-174.

[10] Courant, R. and Hilbert, D. (1989) *Methods of Mathematical Phyhsics (Vol. II)*, Interscience, Wiley Classic Edition.

[11] Feinstein, A. (2002), *Class. Quantum Grav.*, **19**, 5353-5360.

[12] Griffiths, J. B. (1975), *Math. Proc. Camb. Phil. Soc.*, **77**, 559-565.

[13] Griffiths, J. B. (1987), *Class. Quantum Grav.*, **3**, 957-966.

[14] Griffiths, J. B. (1991), *Colliding Plane Waves in General Relativity*, Oxford University Press.

[15] Griffiths, J. B. and Santano-Roco, M. (2002), *Class. Quantum Grav.*, **19**, 4273-4286.

[16] Gürses, M., Kahya, E. O. and Karasu, A. (2002), preprint, *gr-cq*/0204041.

[17] Hauser, I. and Ernst, F. J. (1989), *J. Math. Phys.*, **30**, 872-887.

[18] Hauser, I. and Ernst, F. J. (1989), *J. Math. Phys.*, **30**, 2322-2336.

[19] Hauser, I. and Ernst, F. J. (1990), *J. Math. Phys.*, **31**, 871-881.

[20] Hauser, I. and Ernst, F. J. (1991), *J. Math. Phys.*, **32**, 198-209.

[21] Khan, K. A. and Penrose, R. (1971), *Nature*, **229**, 185-186.

[22] Lichnerowicz, A. (1955), *Théories relativistes de la gravitation et de l'électromagnétisme.* Masson, Paris.

[23] Misner, C. W., Thorne, K. S. and Wheeler, J. A. (1973), *Gravitation*, Freeman.

[24] Newman, E. and Penrose, P. (1962), *J. Math. Phys.*, **3**, 566-578.

[25] O'Brien, S. and Synge, J. L. (1952), *Commun. Dublin Inst. Adv. Stud. A*, 9.

[26] Ortaggio, M. and Podolský, J. (2002), preprint, *gr-cq*/0209068.

[27] Szekeres, P. (1975), *J. Math. Phys*, **6**, 1387-1391.

[28] Szekeres, P. (1970), *Nature*, **228**, 1184-1184.

[29] Szekeres, P. (1972), *J. Math. Phys.*, **13**, 286-294.

[30] Taylor, J. H. (1994) *Reviews of Modern Physics*, **66**, 711-719.

[31] Thorne, K. S. (1997), preprint, *gr-qc*/9704042.

[32] Weber, J. (1969), *Phys. Rev. Lett.* **22**, 1320

[33] Yurtsever, U. (1989), *Phys. Rev. D*, **40**, 360-371.