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BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS WITH POLYNOMIAL GROWTH COEFFICIENTS

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ABSTRACT. In this paper we study the solvability of backward doubly stochastic differential equations (BDSDEs for short) with polynomial growth coefficients and their connections with SPDEs. The corresponding SPDE is in a very general form, which may depend on the derivative of the solution. We use Wiener-Sobolev compactness arguments to derive a strongly convergent subsequence of approximating SPDEs. For this, we prove some new estimates to the solution and its Malliavin derivative of the corresponding approximating BDSDEs. These estimates lead to the verifications of the conditions in the Wiener-Sobolev compactness theorem and the solvability of the BDSDEs and the SPDEs with polynomial growth coefficients.

1. Introduction. In this paper, we use the Malliavin calculus to study the solvability of BDSDEs with polynomial growth coefficients valued in a weighted $L^2(dx)$ space

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \\ &\quad - \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) d^\dagger \hat{B}_r - \int_s^T Z_r^{t,x} dW_r, \quad 0 \leq t \leq s \leq T. \end{aligned} \tag{1}$$

Here the Brownian motion \hat{B} could be a Q -Wiener process with values in a separable Hilbert space U and the stochastic integral with respect to \hat{B} is a backward Itô's integral. But for simplicity, we only consider the finite dimensional Brownian motion valued in \mathbb{R}^l . The other Brownian motion W is independent of \hat{B} and takes values in \mathbb{R}^d . The coefficients are given functions $h : \mathbb{R}^d \rightarrow \mathbb{R}^1$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ and $g : [0, T] \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1$. By the polynomially growing coefficient we mean that f in (1) is of a polynomial growth with power p , $p \geq 2$, with respect to

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the solution Y . Specific assumptions on f and g are given in the next section. And X is the solution of the SDE:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r, & s \geq t, \\ X_s^{t,x} = x, & 0 \leq s < t, \end{cases} \quad (2)$$

with $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^d$. Actually, BDSDE (1) and SDE (2) constitute a forward-backward stochastic differential system, and its connection with the classical solution of parabolic semilinear SPDE was first indicated in Pardoux and Peng [6]. In this paper, we consider the connection between them in the sense of weak solution of the following SPDE:

$$\begin{aligned} u(t, x) = & h(x) + \int_t^T [\mathcal{L}u(s, x) + f(s, x, u(s, x), (\sigma^* \nabla u)(s, x))] ds \\ & - \int_t^T g(s, x, u(s, x), (\sigma^* \nabla u)(s, x)) d^\dagger \hat{B}_s, \quad 0 \leq t \leq T. \end{aligned} \quad (3)$$

Here the second order differential operator \mathcal{L} is given by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.$$

This work is a further study of [11]. Different from the previous work, the corresponding SPDE we consider here involves the first order derivatives of the solution in both the drift and the diffusion terms. This causes difficulties in applying the Malliavin calculus method. Even in the case that g in (3) does not depend on ∇u , the estimates in [11] are not enough. We need some new estimates on the integrability and the continuity of Z and the Malliavin derivative of Z in the ρ -weighted space $L_\rho^m(dx, dP)$, $m > 2$. This kind of estimates was not given in the estimates derived from BDSDE (1), where only $L_\rho^2(dx, dP)$ can be given. The idea here is to associate Z with ∇Y , which can actually be the solution of another BDSDE if we differentiate BDSDE (1). We construct a sequence of approximating SPDEs with linear growth drift and deduce the desired estimates of their corresponding sequence of BDSDEs, then the new estimates can be transferred from the BDSDEs to the SPDEs. With these estimates, we verify that the approximating SPDEs satisfy the Wiener-Sobolev compactness theorem. As a consequence, we are able to get a strongly convergent subsequence of the solutions of approximating SPDEs and the approximating BDSDEs.

As we have shown in our previous works [8, 9, 11], if f and g in (3) are independent of the time variable, we can apply the time reverse transformation to (3) to obtain a SPDE with an initial value

$$\begin{cases} dv(t, x) = [\mathcal{L}v(t, x) + f(x, v(t, x), (\sigma^* \nabla v)(s, x))] dt \\ \quad + g(x, v(t, x), (\sigma^* \nabla v)(s, x)) dB_t, \\ v(0, x) = h(x), \end{cases} \quad (4)$$

where the Brownian motion B is the time reverse of \hat{B} . Then we can construct the stationary solution of SPDE (4) after extending the solvability from the finite time horizon to the infinite time horizon. But we don't intend to include this result here and only show the main differences when the finite time horizon BDSDE (1) and SPDE (3).

The rest of this paper is organized as follows. In Sections 2, we introduce some useful definitions and estimates. In section 3, some new estimates to the solutions of approximating BDSDEs are proved. A strongly convergent subsequence of the solutions of corresponding approximating SPDEs as well as approximating BDSDEs is derived. In Section 4, we finally prove the existence and uniqueness of the solution to BDSDE with the polynomial growth coefficient in our setting by weak convergence and strong convergence arguments, and demonstrate the correspondence between the BDSDEs and the SPDEs.

2. Preliminaries. As we know, when we consider such kind of BDSDEs and SPDEs with a general form, we usually construct a sequence of approximating BDSDEs and SPDEs. To get the strongly convergent subsequence is a key step. The Wiener-Sobolev compactness theorem, which is a natural but not trivial extension of Rellich-Kondrachov compactness theorem to stochastic case, is the method we use to get the strongly convergent subsequence. Here the Malliavin derivative plays a key role in the assumptions. We begin our preliminaries with Malliavin derivatives. For a smooth random variable F such that $F = f(W(h_1), \dots, W(h_n))$ with $n \in \mathbb{N}$, $h_1, \dots, h_n \in L^2([0, T])$ and $f \in C_p^\infty(\mathbb{R}^n)$, where $C_p^\infty(\mathbb{R}^n)$ is the set of infinitely differentiable functions whose differentials of any order all grow in a polynomial way. Let \mathbb{K} be the class of smooth random variables F . The derivative operator of F denoted by the stochastic process $\{D_t F, t \in [0, T]\}$ is then defined by (cf. [4])

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(W(h_1), \dots, W(h_n)) h_i(t).$$

We denote the domain of the operator D in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$ with the norm below

$$\|F\|_{1,2}^2 = E[|F|^2] + E[\|D_t F\|_{L^2([0,T])}^2].$$

We first recall a version of a Wiener-Sobolev compactness theorem in the space $L^2(\Omega \times [0, T] \times \mathcal{O}; \mathbb{R}^1)$ used in this paper. The theorem was proved by Bally and Saussereau [1]. See Da Prato, Malliavin and Nualart [2] and Peszat [7] for an earlier version of time and space independent case and Feng and Zhao [3] for a relative compactness result in the space $C([0, T], L^2(\Omega \times \mathcal{O}; \mathbb{R}^1))$.

Theorem 2.1. ([1]) *Let \mathcal{O} be a bounded domain in \mathbb{R}^d . Denote $C_c^k(\mathcal{O})$ the class of k -times differentiable functions with a compact support inside \mathcal{O} . For $\varphi \in C_c^k(\mathcal{O})$, we define $v^\varphi(s, \omega) = \int_{\mathcal{O}} v(s, x, \omega) \varphi(x) dx$. For a sequence $(u_n)_{n \in \mathbb{N}}$ in $L^2([0, T] \times \Omega; H^1(\mathcal{O}))$, assume that*

$$(1) \sup_n E[\int_0^T \|u_n(s, \cdot)\|_{H^1(\mathcal{O})}^2 ds] < \infty.$$

$$(2) \text{For all } \varphi \in C_c^k(\mathcal{O}) \text{ and } t \in [0, T], u_n^\varphi(s) \in \mathbb{D}^{1,2} \text{ and } \sup_n \int_0^T \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 ds < \infty.$$

$$(3) \text{For all } \varphi \in C_c^k(\mathcal{O}), \text{ the sequence } (E[u_n^\varphi])_{n \in \mathbb{N}} \text{ of } L^2([0, T]) \text{ satisfies}$$

$$(3i) \text{For any } \varepsilon > 0, \text{ there exists } 0 < \alpha < \beta < T \text{ s.t.}$$

$$\sup_n \int_{[0,T] \setminus (\alpha, \beta)} |E[u_n^\varphi(s)]|^2 ds < \varepsilon.$$

$$(3ii) \text{For any } 0 < \alpha < \beta < T \text{ and } h \in \mathbb{R}^1 \text{ s.t. } |h| < \min(\alpha, T - \beta), \text{ it holds}$$

$$\sup_n \int_\alpha^\beta |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds < C_p |h|.$$

(4) For all $\varphi \in C_c^k(\mathcal{O})$, the following conditions are satisfied:

(4i) For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ and $0 < \alpha' < \beta' < T$ s.t.

$$\sup_n E \left[\int_{[0,T]^2 \setminus (\alpha, \beta) \times (\alpha', \beta')} |D_\theta u_n^\varphi(s)|^2 d\theta ds \right] < \varepsilon.$$

(4ii) For any $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ and $h, h' \in \mathbb{R}^1$ s.t. $\max(|h|, |h'|) < \min(\alpha, \alpha', T - \beta, T - \beta')$, it holds that

$$\sup_n E \left[\int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^\varphi(s+h') - D_\theta u_n^\varphi(s)|^2 d\theta ds \right] < C_p(|h| + |h'|).$$

Then $(u_n)_{n \in \mathbb{N}}$ is relatively compact in $L^2(\Omega \times [0, T] \times \mathcal{O}; \mathbb{R}^1)$.

The conditions of Theorem 2.1 are not easy to verify, without the exception to our case when we apply the theorem to the approximating SPDEs of (3). We will utilize the correspondence between BDSDE and SPDE for the corresponding approximating BDSDEs to verify the conditions of Theorem 2.1.

Since we will consider the solution of BDSDE (1) in a weighted $L^2(dx)$ space which connects the weak solution of corresponding SPDE (3), a necessary equivalence between the norms of the BDSDE and SPDE solution spaces is needed. For this, we utilize the property of stochastic flow and always assume that

(A1): the diffusion coefficients $b \in C_{l,b}^2(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$;

(A2): there exists a constant $\varepsilon > 0$ s.t. $\sigma \sigma^*(x) \geq \varepsilon I_d$.

Here $C_{l,b}^k$, $k \geq 0$ denotes the set of C^k -functions for which the partial derivatives from the order 1 to k are bounded, but the functions themselves may not be bounded, and C_b^k denotes the set of C^k -functions for which the partial derivatives from the order 0 to k are bounded.

Lemma 2.2. (Generalized equivalence of norm principle [8]) Let X be the diffusion process defined in (2). If $s \in [t, T]$, $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ is independent of the σ -field $\mathcal{F}_{t,s}^W$ and $\varphi \rho^{-1} \in L^1(\Omega \times \mathbb{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ s.t.

$$cE \left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx \right] \leq E \left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho^{-1}(x) dx \right] \leq CE \left[\int_{\mathbb{R}^d} |\varphi(x)| \rho^{-1}(x) dx \right].$$

Moreover if $\Psi : \Omega \times [t, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^1$, $\Psi(s, \cdot)$ is independent of $\mathcal{F}_{t,s}^W$ and $\Psi \rho^{-1} \in L^1(\Omega \times [t, T] \times \mathbb{R}^d)$, then

$$\begin{aligned} cE \left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho^{-1}(x) dx ds \right] &\leq E \left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, X_s^{t,x})| \rho^{-1}(x) dx ds \right] \\ &\leq CE \left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho^{-1}(x) dx ds \right]. \end{aligned} \tag{5}$$

The inequality (5) implies the equivalence of norm between BDSDEs and SPDEs in their respective solution spaces when Ψ is regarded as the weak solution of SPDE. This equivalence of norm principle will be more clear after we clarify the definitions for the solution spaces of BDSDEs and SPDEs in the following. Both the solutions of BDSDEs and SPDEs are valued in a ρ^{-1} -weighted L^q , $q \geq 2$, space, denoted by L_ρ^q , where the weight function $\rho(x) = (1 + |x|)^q$, $q > d + 32p$.

BDSDE (1) takes values in $L_\rho^p(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$. For $t \leq s \leq T$ and $q \geq 2$, we denote by

- $M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ the space of all $\mathcal{F}_{s,T}^{\hat{B}} \vee \mathcal{F}_{t,s}^W$ measurable processes $f : \Omega \times [t, T] \rightarrow L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$ satisfying

$$\|f\|_{M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))} := \sqrt{E \int_t^T \|f(s)\|_{L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)}^2 ds} < \infty;$$

- $S^q([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))$ the space of all $\mathcal{F}_{s,T}^{\hat{B}} \vee \mathcal{F}_{t,s}^W$ measurable processes $f : \Omega \times [t, T] \rightarrow L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$ satisfying $s \mapsto f(s)$ is a.s. continuous and

$$\|f\|_{S^q([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1))} := \left(E \sup_{t \leq s \leq T} \|f(s)\|_{L_\rho^q(\mathbb{R}^d; \mathbb{R}^1)}^q \right)^{\frac{1}{q}} < \infty.$$

Definition 2.3. A pair of processes $(Y_s^{t,x}, Z_s^{t,x})$ is called a solution of BDSDE (1) if $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^p([t, T]; L_\rho^p(\mathbb{R}^d; \mathbb{R}^1)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (1) for a.e. $x \in \mathbb{R}^d$ a.s.

Correspondingly, we give the definition for weak solution of SPDE (3) in the sense of C_c^∞ test function.

Definition 2.4. A function u is called a weak solution of SPDE (3) if $(u, \sigma^* \nabla u) \in L^p([0, T]; L_\rho^p(\mathbb{R}^d; \mathbb{R}^1)) \times L^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ and for an arbitrary $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^1)$,

$$\begin{aligned} & \int_{\mathbb{R}^d} u(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} h(x) \varphi(x) dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} (\sigma^* \nabla u)(s, x) (\sigma^* \nabla \varphi)(x) dx ds \\ & - \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div}((b - \tilde{A})\varphi)(x) dx ds \\ = & \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), (\sigma^* \nabla u)(s, x)) \varphi(x) dx ds \\ & - \int_t^T \int_{\mathbb{R}^d} g(s, x, u(s, x), (\sigma^* \nabla u)(s, x)) \varphi(x) dx d^\dagger \hat{B}_s, \quad t \in [0, T]. \end{aligned}$$

Here $\tilde{A}_j := \frac{1}{2} \sum_{i=1}^d \frac{\partial(\sigma \sigma^*)_{ij}(x)}{\partial x_i}$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

We then assume the conditions to BDSDE (1). Given constants $\tilde{L} \geq 0$ and $0 \leq \alpha < \frac{\sqrt{2}}{2}$, for any $s, s_1, s_2 \in [0, T]$, $x, x_1, x_2, z, z_1, z_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^1$, we assume

(H1): $\partial_x h$ exists and $|\partial_x h| \leq \tilde{L}$;

(H2): there exists a function $f_0 : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^1$ with

$$\int_0^T \int_{\mathbb{R}^d} |f_0(s, x)|^{32p} \rho^{-1}(x) dx ds < \infty$$

s.t.

$$|f(s, x, y, z)| \leq \tilde{L}(|f_0(s, x)| + |y|^p + |z|),$$

f is locally Lipschitz on x and globally Lipschitz on z as follows:

$$\begin{aligned} |f(s, x_1, y, z) - f(s, x_2, y, z)| & \leq \tilde{L}(1 + |y|^p + |z|)|x_1 - x_2|, \\ |f(s, x, y, z_1) - f(s, x, y, z_2)| & \leq \tilde{L}|z_1 - z_2|, \end{aligned}$$

and there exists a constant $\mu \in \mathbb{R}^1$ s.t.

$$(y_1 - y_2)(f(s, x, y_1, z) - f(s, x, y_2, z)) \leq \mu|y_1 - y_2|^2;$$

(H3): the derivatives $\partial_x f, \partial_y f, \partial_z f$ exist and satisfy

$$\begin{aligned} |\partial_y f(s, x, y, z)| &\leq \tilde{L}(1 + |y|^{p-1}), \\ |\partial_x f(s, x_1, y_1, z_1) - \partial_x f(s, x_2, y_2, z_2)| \\ &\leq \tilde{L}(1 + |y|^p + |z|)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \\ |\partial_y f(s, x_1, y, z) - \partial_y f(s, x_2, y, z)| &\leq \tilde{L}(1 + |y|^{p-1})|x_1 - x_2|, \\ |\partial_y f(s, x, y_1, z) - \partial_y f(s, x, y_2, z)| &\leq \tilde{L}(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|, \\ |\partial_y f(s, x, y, z_1) - \partial_y f(s, x, y, z_2)| &\leq \tilde{L}|z_1 - z_2|, \\ |\partial_z f(s, x_1, y_1, z_1) - \partial_z f(s, x_2, y_2, z_2)| &\leq \tilde{L}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|); \end{aligned}$$

(H4): g is globally Lipschitz as follows:

$$|g(s_1, x_1, y_1, z_1) - g(s_2, x_2, y_2, z_2)| \leq \tilde{L}(|s_1 - s_2| + |x_1 - x_2| + |y_1 - y_2|) + \alpha|z_1 - z_2|,$$

and the derivatives $\partial_y g, \partial_z g$ exist and satisfy

$$\begin{aligned} |\partial_x g(s, x_1, y_1, z_1) - \partial_x g(s, x_2, y_2, z_2)| &\leq \tilde{L}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \\ |\partial_y g(s, x_1, y_1, z_1) - \partial_y g(s, x_2, y_2, z_2)| &\leq \tilde{L}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \\ |\partial_z g(s, x_1, y_1, z_1) - \partial_z g(s, x_2, y_2, z_2)| &\leq \tilde{L}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|). \end{aligned}$$

Remark 1. As indicated in the literatures (e.g. [5, 11]), the monotonic constant μ in (H2) can be assumed, without losing any generality, to be 0. To simplify the calculation, we always take $\mu = 0$ in the rest of the paper.

Then we construct a sequence $\{f_n\}_{n \in N}$ which converges to f a.s. For this, first set $f_n(s, x, y, z) = f(s, x, y, z)$ when $|y| \leq n$ and $f_n(s, x, y, z) = f(s, x, \frac{n+1}{|y|}y, z) + \partial_y f(s, x, \frac{n+1}{|y|}y, z)(y - \frac{n+1}{|y|}y)$ when $|y| \geq n + 1$. Then we use the standard partition and unity method to define f_n such that f_n has smooth enough connections and is monotone on the interval $[n, n + 1]$ and $[-n - 1, -n]$. For any $s \in [0, T]$, $x, x_1, x_2, z, z_1, z_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^1$, f_n satisfies the following conditions with the constant L depending on \tilde{L} :

(H2)': for the same f_0 in (H2),

$$|f_n(s, x, y, z)| \leq L(|f_0(s, x)| + (1 + (n + 1)^{p-1} \wedge |y|^{p-1})|y| + |z|),$$

f_n is locally Lipschitz on x and globally Lipschitz on z as follows:

$$\begin{aligned} |f_n(s, x_1, y, z) - f_n(s, x_2, y, z)| &\leq L(1 + |y|^p + |z|)|x_1 - x_2|, \\ |f_n(s, x, y, z_1) - f_n(s, x, y, z_2)| &\leq L|z_1 - z_2|, \end{aligned}$$

and there exists a constant $\mu \in \mathbb{R}^1$ s.t.

$$(y_1 - y_2)(f_n(s, x, y_1, z) - f_n(s, x, y_2, z)) \leq 0;$$

(H3)': the derivatives $\partial_x f_n, \partial_y f_n, \partial_z f_n$ exist and satisfy

$$\begin{aligned} |\partial_y f_n(s, x, y, z)| &\leq L(1 + |y|^{p-1}), \\ |\partial_x f_n(s, x_1, y_1, z_1) - \partial_x f_n(s, x_2, y_2, z_2)| \\ &\leq L(1 + |y|^p + |z|)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \\ |\partial_y f_n(s, x_1, y, z) - \partial_y f_n(s, x_2, y, z)| &\leq L(1 + |y|^{p-1})|x_1 - x_2|, \\ |\partial_y f_n(s, x, y_1, z) - \partial_y f_n(s, x, y_2, z)| &\leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|, \\ |\partial_y f_n(s, x, y, z_1) - \partial_y f_n(s, x, y, z_2)| &\leq L|z_1 - z_2|, \\ |\partial_z f_n(s, x_1, y_1, z_1) - \partial_z f_n(s, x_2, y_2, z_2)| &\leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|). \end{aligned}$$

Note that f_n is of linear growth in y for each $n \in \mathbb{N}$. We consider a sequence of BDSDEs of the linearly growing drift f_n :

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr \\ &\quad - \int_s^T \langle g(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}), d\hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \end{aligned} \quad (6)$$

For BDSDEs with linear growth coefficients, we have some estimates for which the reader can refer to [9, 11] for detailed proofs.

Proposition 1. *Let assumptions (A1), (A2), (H1), (H2)', (H3)', and (H4) be satisfied. Then BDSDE (6) has a unique solution $(Y^{t,\cdot,n}, Z^{t,\cdot,n}) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \times M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Moreover, for any $2 \leq m \leq 32p$,*

$$\begin{aligned} &\sup_n E \left[\sup_{s \in [t, T]} \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\ &\quad + \sup_n E \left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^{m-2} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right] \\ &\quad + \sup_n E \left[\left(\int_t^T \int_{\mathbb{R}^d} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right)^{\frac{m}{2}} \right] \\ &\leq C_p \left(\int_{\mathbb{R}^d} |h(x)|^m \rho^{-1}(x) dx + \int_t^T \int_{\mathbb{R}^d} |f_0(s, x)|^m \rho^{-1}(x) dx ds \right. \\ &\quad \left. + \int_t^T \int_{\mathbb{R}^d} |g(s, x, 0, 0)|^m \rho^{-1}(x) dx ds \right). \end{aligned}$$

Here and in the rest of this paper C_p is a generic constant depending only on given parameters.

Proposition 2. *Let assumptions (A1), (A2), (H1), (H2)', (H3)', and (H4) be satisfied. Then $u_n(t, x) := Y_t^{t,x,n}$ is the unique weak solution of the following SPDE*

$$\begin{aligned} u_n(t, x) &= h(x) + \int_t^T \{ \mathcal{L}u_n(s, x) + f_n(s, x, u_n(s, x), (\sigma^* \nabla u_n)(s, x)) \} ds \\ &\quad - \int_t^T \langle g(s, x, u_n(s, x), (\sigma^* \nabla u_n)(s, x)), d^\dagger \hat{B}_s \rangle, \quad 0 \leq t \leq T. \end{aligned} \quad (7)$$

Moreover,

$$u_n(s, X_s^{t,x}) = Y_s^{t,x,n}, \quad (\sigma^* \nabla u_n)(s, X_s^{t,x}) = Z_s^{t,x,n} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

3. The compactness of solutions to approximating equations. The main task in this section is to prove the relative compactness of the sequence of solutions to SPDEs (7) and BDSDEs (6). We need some preparations. First, as we indicate before, the Malliavin derivatives are used to obtain the strongly convergent subsequence of solutions of approximating equations. By (H3)', (H4) and the results of [1] or [6], the Malliavin derivative of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ in BDSDE (6) with respect to

Brownian motion \hat{B} exists and satisfies

$$\left\{ \begin{array}{l} D_\theta Y_s^{t,x,n} = g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, Z_\theta^{t,x,n}) \\ \quad + \int_s^\theta (\partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\ \quad \quad + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Z_r^{t,x,n}) dr \\ \quad - \int_s^\theta (\partial_y g(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\ \quad \quad + \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Z_r^{t,x,n}) d^\dagger \hat{B}_r \\ \quad - \int_s^\theta D_\theta Z_r^{t,x,n} dW_r, \quad 0 \leq \theta \leq T, \\ D_\theta Y_s^{t,x,n} = 0, \quad t \leq \theta < s. \end{array} \right. \quad (8)$$

Furthermore, we need the following estimates for the Malliavin derivatives.

Proposition 3. *Let assumptions (A1), (A2), and (H1)–(H4) be satisfied. For any $2 \leq m \leq 16$,*

$$\begin{aligned} & \sup_n \sup_{\theta \in [t,T]} E \left[\sup_{s \in [t,T]} \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\ & + \sup_n \sup_{\theta \in [t,T]} E \left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx ds \right] \\ & + \sup_n \sup_{\theta \in [t,T]} E \left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^{m-2} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right] \\ & + \sup_n \sup_{\theta \in [t,T]} E \left[\left(\int_t^T \int_{\mathbb{R}^d} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right)^{\frac{m}{2}} \right] \\ & \leq C_p \left(1 + \sup_{\theta \in [t,T]} \int_{\mathbb{R}^d} |g(\theta, x, 0, 0)|^m \rho^{-1}(x) dx \right. \\ & \quad \left. + \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Y_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] \right. \\ & \quad \left. + \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Z_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] \right). \end{aligned}$$

However, different from the case that f and g are independent of z in [11], the estimate for $\sup_n \sup_{s \in [t,T]} E \left[\int_{\mathbb{R}^d} |Z_s^{t,x,n}|^m \rho^{-1}(x) dx \right]$ is needed in Proposition 3. Since by the assumption on σ ,

$$|Z_s^{t,x,n}| = |\sigma^*(X_s^{t,x}) \nabla Y_s^{t,x,n}| \leq L |\nabla Y_s^{t,x,n}|.$$

So we only need to estimate $\nabla Y_s^{t,x,n}$ instead.

Noticing the smooth conditions (H1), (H3)', (H4) and the form of SPDE (7), we have

$$\begin{aligned}
& \nabla u^n(t, x) \\
= & \partial_x h(x) + \int_t^T \left\{ \mathcal{L} \nabla u^n(s, x) + \partial_x f_n(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \right. \\
& \quad + \partial_y f_n(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \nabla u^n(s, x) \\
& \quad + \partial_z f_n(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) ((\sigma^* \nabla^2 u^n)(s, x)) \\
& \quad \left. + (\nabla \sigma^* \nabla u^n)(s, x) + (\nabla \sigma \sigma^* \nabla^2 u^n)(s, x) + (\nabla u^n \nabla b)(s, x) \right\} ds \\
& - \int_t^T \left\{ \partial_x g(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \right. \\
& \quad + \partial_y g(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) \nabla u^n(s, x) \\
& \quad \left. + \partial_z g(s, x, u^n(s, x), (\sigma^* \nabla u^n)(s, x)) ((\sigma^* \nabla^2 u^n)(s, x) + (\nabla \sigma^* \nabla u^n)(s, x)) \right\} d^\dagger \hat{B}_s.
\end{aligned}$$

Set

$$\tilde{Y}_t^{t,x,n} = \nabla u^n(t, x), \quad \tilde{Z}_t^{t,x,n} = (\sigma^* \nabla^2 u^n)(t, x).$$

By the standard correspondence of BDSDE and SPDE (see e.g. [1] for details), we know that $(\tilde{Y}_r^{t,x,n}, \tilde{Z}_r^{t,x,n})_{t \leq r \leq T}$ is the solution of the following BDSDE:

$$\begin{aligned}
& \tilde{Y}_s^{t,x,n} \\
= & \partial_x h(X_T^{t,x}) + \int_s^T \left\{ \partial_x f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \right. \\
& \quad + \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} \\
& \quad + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) (\nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} + \tilde{Z}_r^{t,x,n}) \\
& \quad \left. + \nabla \sigma(X_r^{t,x}) \tilde{Z}_r^{t,x,n} + \nabla b(X_r^{t,x}) \tilde{Y}_r^{t,x,n} \right\} dr \\
& - \int_s^T \left\{ \partial_x g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \right. \\
& \quad + \partial_y g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} \\
& \quad \left. + \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) (\nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} + \tilde{Z}_r^{t,x,n}) \right\} d^\dagger \hat{B}_r \\
& - \int_s^T \tilde{Z}_r^{t,x,n} dW_r. \tag{9}
\end{aligned}$$

Proposition 4. *Let assumptions (A1), (A2), and (H1)–(H4) be satisfied. For any $2 \leq m \leq 16$,*

$$\begin{aligned}
& \sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx \right] + \sup_n E \left[\left(\int_t^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right)^{\frac{m}{2}} \right] \\
\leq & C_p \left(1 + \sqrt{\sup_n E \left[\left(\int_t^T \int_{\mathbb{R}^d} |Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right)^m \right]} \right).
\end{aligned}$$

Proof. We rewrite BDSDE (9) as

$$\begin{aligned}\tilde{Y}_s^{t,x,n} &= \partial_x h(X_T^{t,x}) + \int_s^T \left\{ a^t(r, \omega) + b^t(r, \omega) \tilde{Y}_r^{t,x,n} + c^t(r, \omega) \tilde{Z}_r^{t,x,n} \right\} dr \\ &\quad - \int_s^T \left\{ d^t(r, \omega) + e^t(r, \omega) \tilde{Y}_r^{t,x,n} + j^t(r, \omega) \tilde{Z}_r^{t,x,n} \right\} d^\dagger \hat{B}_r - \int_s^T \tilde{Z}_r^{t,x,n} dW_r,\end{aligned}\quad (10)$$

where

$$\begin{aligned}a^t(r, \omega) &= \partial_x f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}), \\ b^t(r, \omega) &= \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \\ &\quad + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) + \nabla b(X_r^{t,x}), \\ c^t(r, \omega) &= \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) + \nabla \sigma(X_r^{t,x}), \\ d^t(r, \omega) &= \partial_x g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}), \\ e^t(r, \omega) &= \partial_y g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \\ &\quad + \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}), \\ j^t(r, \omega) &= \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}).\end{aligned}$$

Taking integrations of (9) over \mathbb{R}^d and carrying out similar calculations as (A.6) in [1], we have

$$\begin{aligned}&E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx\right] \\ &\leq C_p E\left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t,x})|^m \rho^{-1}(x) dx\right] \\ &\quad + C_p E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|a^t(r, \omega)| + |b^t(r, \omega)| |\tilde{Y}_r^{t,x,n}| + |c^t(r, \omega)| |\tilde{Z}_r^{t,x,n}|) \rho^{-1}(x) dx dr\right)^m\right] \\ &\quad + (1 + \varepsilon) E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|d^t(r, \omega)| + |e^t(r, \omega)| |\tilde{Y}_r^{t,x,n}| \right.\right. \\ &\quad \left.\left. + |j^t(r, \omega)| |\tilde{Z}_r^{t,x,n}|)^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\ &\leq C_p E\left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t,x})|^m \rho^{-1}(x) dx\right] + C_p E\left[\left(\int_s^T \int_{\mathbb{R}^d} |a^t(r, \omega)| \rho^{-1}(x) dx dr\right)^m\right] \\ &\quad + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |b^t(r, \omega)|^2 \rho^{-1}(x) dx dr\right)^m\right]} \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} \\ &\quad + C_p (T-s)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] + C_p \\ &\quad + C_p E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\ &\quad + (1 + \varepsilon)^{\frac{m+2}{2}} \alpha^m E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right],\end{aligned}$$

where ε can be taken sufficiently small.

Note that

$$\begin{aligned}
& E\left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t,x})|^m \rho^{-1}(x) dx\right] \leq \int_{\mathbb{R}^d} |\partial_x h(x)|^m \rho^{-1}(x) dx < \infty \\
& E\left[\left(\int_s^T \int_{\mathbb{R}^d} |a^t(r, \omega)| \rho^{-1}(x) dx dr\right)^m\right] \\
\leq & C_p E\left[\int_s^T \int_{\mathbb{R}^d} (1 + |Y_r^{t,x,n}|^{pm} + |\tilde{Y}_r^{t,x,n}|^m) \rho^{-1}(x) dx dr\right] < \infty, \\
& E\left[\left(\int_s^T \int_{\mathbb{R}^d} |b^t(r, \omega)|^2 \rho^{-1}(x) dx dr\right)^m\right] \\
\leq & C_p E\left[\int_s^T \int_{\mathbb{R}^d} (1 + |Y_r^{t,x,n}|^{2(p-1)m}) \rho^{-1}(x) dx dr\right] < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx\right] \\
\leq & C_p + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} \\
& + C_p (T-s)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& + (1+\varepsilon)^{\frac{m+2}{2}} \alpha^m E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right]. \quad (11)
\end{aligned}$$

On the other hand, by B-D-G inequality and (10),

$$\begin{aligned}
& E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
\leq & C_p E\left[\left|\int_s^T \int_{\mathbb{R}^d} \tilde{Z}_r^{t,x,n} \rho^{-1}(x) dx dW_r\right|^m\right] \\
= & C_p E\left[\left|\int_{\mathbb{R}^d} \partial_x h(X_T^{t,x}) \rho^{-1}(x) dx - \int_{\mathbb{R}^d} \tilde{Y}_s^{t,x,n} \rho^{-1}(x) dx\right.\right. \\
& \quad \left.\left. + \int_s^T \int_{\mathbb{R}^d} \left\{a^t(r, \omega) + b^t(r, \omega)\tilde{Y}_r^{t,x,n} + c^t(r, \omega)\tilde{Z}_r^{t,x,n}\right\} \rho^{-1}(x) dx dr\right.\right. \\
& \quad \left.\left. - \int_s^T \int_{\mathbb{R}^d} \left\{d^t(r, \omega) + e^t(s, \omega)\tilde{Y}_r^{t,x,n} + j^t(r, \omega)\tilde{Z}_r^{t,x,n}\right\} \rho^{-1}(x) dx d\hat{B}_r\right|^m\right] \\
\leq & C_p E\left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t,x})|^m \rho^{-1}(x) dx\right] + 2^{\frac{m}{2}} (1+\varepsilon)^{\frac{q+2}{2}} E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx\right] \\
& + C_p E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|a^t(r, \omega)| + |b^t(r, \omega)| |\tilde{Y}_r^{t,x,n}| + |c^t(r, \omega)| |\tilde{Z}_r^{t,x,n}|) \rho^{-1}(x) dx dr\right)^m\right] \\
& + 2^{\frac{m}{2}} (1+\varepsilon)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|d^t(r, \omega)| + |e^t(r, \omega)| |\tilde{Y}_r^{t,x,n}| \right.\right. \\
& \quad \left.\left. + |j^t(r, \omega)| |\tilde{Z}_r^{t,x,n}|)^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right].
\end{aligned}$$

Noticing (11), we further have

$$\begin{aligned}
& E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \leq C_p \int_{\mathbb{R}^d} |\partial_x h(x)|^m \rho^{-1}(x) dx + C_p + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} \\
& \quad + C_p (T-s)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \quad + 2^{\frac{m}{2}} (1+\varepsilon)^{\frac{2m+2}{2}} \alpha^m E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] + C_p \\
& \quad + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} \\
& \quad + C_p (T-s)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \quad + C_p + C_p E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right] \\
& \quad + 2^{\frac{m}{2}} (1+\varepsilon)^{\frac{2m+2}{2}} \alpha^m E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \quad + C_p + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} \\
& \quad + C_p (T-s)^{\frac{m}{2}} E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \quad + 2^{\frac{m}{2}} (1+\varepsilon)^{\frac{2m+2}{2}} \alpha^m E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right]. \tag{12}
\end{aligned}$$

Since $\sigma(X_s^{t,x}) Z_s^{t,x,n} = (\sigma\sigma^*)(X_s^{t,x}) \tilde{Y}_s^{t,x,n} \geq \varepsilon \tilde{Y}_s^{t,x,n}$, by Proposition 1,

$$E\left[\left(\int_t^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right] \leq C_p E\left[\left(\int_t^T \int_{\mathbb{R}^d} |Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right] < \infty.$$

Thus by (11) and (12), we get for a sufficiently small $\delta > 0$,

$$\begin{aligned}
& \sup_n \sup_{s \in [T-\delta, T]} E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx\right] \\
& \quad + \sup_n E\left[\left(\int_{T-\delta}^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^{\frac{m}{2}}\right] \\
& \leq C_p + C_p \sqrt{\sup_n E\left[\left(\int_{T-\delta}^T \int_{\mathbb{R}^d} |Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^m\right]} < \infty.
\end{aligned}$$

Noticing that δ only depends on the given parameters, we can extend the above estimate to the interval $[t, T]$. Thus Proposition 4 follows. \diamond

From Proposition 4, it follows immediately that for $2 \leq m \leq 16$,

$$\begin{aligned} & \sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |Z_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\ & \leq C_p \sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx \right] < \infty. \end{aligned} \quad (13)$$

With (13), we can give the proof of Proposition 3.

Proof of Proposition 3. For $2 \leq m \leq 16$, applying Itô's formula to $e^{Kr} |D_\theta Y_r^{t,x,n}|^m$, we have

$$\begin{aligned} & e^{Ks} |D_\theta Y_s^{t,x,n}|^m + K \int_s^\theta e^{Kr} |D_\theta Y_r^{t,x,n}|^m dr \\ & + \frac{m(m-1)}{2} \int_s^\theta e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} |D_\theta Z_r^{t,x,n}|^2 dr \\ = & e^{K\theta} |g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, Z_\theta^{t,x,n})|^m \\ & + m \int_s^\theta e^{Kr} \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) |D_\theta Y_r^{t,x,n}|^m dr \\ & + m \int_s^\theta e^{Kr} \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} D_\theta Z_r^{t,x,n} |D_\theta Y_r^{t,x,n}|^{m-2} dr \\ & + \frac{m(m-1)}{2} \int_s^\theta e^{Kr} |(\partial_y g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\ & \quad + \partial_z g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Z_r^{t,x,n})|^2 |D_\theta Y_r^{t,x,n}|^{m-2} dr \\ & - m \int_s^\theta e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} \langle D_\theta Y_r^{t,x,n}, (\partial_y g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\ & \quad + \partial_z g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Z_r^{t,x,n}) d^\dagger \hat{B}_r \rangle \\ & - m \int_s^\theta e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} \langle D_\theta Y_r^{t,x,n}, D_\theta Z_r^{t,x,n} dW_r \rangle. \end{aligned} \quad (14)$$

From Conditions (H2)' and (H4), we know that for any $s \in [0, T]$, $y \in \mathbb{R}^1$, $x, z \in \mathbb{R}^d$,

$$\partial_y f_n(s, x, y, z) = \lim_{\varepsilon \rightarrow 0} \frac{f_n(s, x, y + \varepsilon, z) - f_n(s, x, y, z)}{\varepsilon} \leq 0,$$

and $|\partial_y g(s, x, y, z)| \leq L$. Therefore, we get from (14) that

$$\begin{aligned} & \int_{\mathbb{R}^d} e^{Ks} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx \\ & + (K - \frac{m(m+1)}{2} L^2 - \frac{m(m-1)}{2\varepsilon}) \int_s^\theta \int_{\mathbb{R}^d} e^{Kr} |D_\theta Y_r^{t,x,n}|^m \rho^{-1}(x) dx dr \\ & + \frac{m}{4} (2m-3 - (2m-2)\alpha^2 - (2m-2)\alpha^2\varepsilon) \\ & \times \int_s^\theta \int_{\mathbb{R}^d} e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} |D_\theta Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^d} e^{K\theta} |g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, Z_\theta^{t,x,n})|^m \rho^{-1}(x) dx \\
&\quad - m \int_s^\theta \int_{\mathbb{R}^d} e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} \langle D_\theta Y_r^{t,x,n}, (\partial_y g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\
&\quad \quad + \partial_z g_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) D_\theta Z_r^{t,x,n}) \rho^{-1}(x) dx d\hat{B}_r \rangle \\
&\quad - m \int_s^\theta \int_{\mathbb{R}^d} e^{Kr} |D_\theta Y_r^{t,x,n}|^{m-2} \langle D_\theta Y_r^{t,x,n}, D_\theta Z_r^{t,x,n} \rho^{-1}(x) dx dW_r \rangle. \tag{15}
\end{aligned}$$

Then, it follows from (H4), Proposition 1 and (13) that

$$\begin{aligned}
&\sup_n \sup_{\theta \in [t,T]} \sup_{s \in [t,T]} E \left[\int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&\quad + \sup_n \sup_{\theta \in [t,T]} E \left[\int_s^\theta \int_{\mathbb{R}^d} |D_\theta Y_r^{t,x,n}|^m \rho^{-1}(x) dx dr \right] \\
&\quad + \sup_n \sup_{\theta \in [t,T]} E \left[\int_s^\theta \int_{\mathbb{R}^d} |D_\theta Y_r^{t,x,n}|^{m-2} |D_\theta Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
&\leq C_p \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, Z_\theta^{t,x,n})|^m \rho^{-1}(x) dx \right] \\
&\leq C_p \sup_{\theta \in [t,T]} \int_{\mathbb{R}^d} |g(\theta, x, 0, 0)|^m \rho^{-1}(x) dx + C_p \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Y_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&\quad + C_p \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Z_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] < \infty.
\end{aligned}$$

Using the B-D-G inequality in (15), by the above formula we can further prove

$$\begin{aligned}
&\sup_n \sup_{\theta \in [t,T]} E \left[\sup_{s \in [t,T]} \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&\quad + \sup_n \sup_{\theta \in [t,T]} E \left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^m \rho^{-1}(x) dx ds \right] \\
&\quad + \sup_n \sup_{\theta \in [t,T]} E \left[\int_t^T \int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^{m-2} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right] \\
&\quad + \sup_n \sup_{\theta \in [t,T]} E \left[\left(\int_t^T \int_{\mathbb{R}^d} |D_\theta Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right)^{\frac{m}{2}} \right] \\
&\leq C_p + C_p \sup_{\theta \in [t,T]} \int_{\mathbb{R}^d} |g(\theta, x, 0, 0)|^m \rho^{-1}(x) dx \\
&\quad + C_p \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Y_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&\quad + C_p \sup_n \sup_{\theta \in [t,T]} E \left[\int_{\mathbb{R}^d} |Z_\theta^{t,x,n}|^m \rho^{-1}(x) dx \right] < \infty.
\end{aligned}$$

The proof is completed. \diamond

Lemma 3.1. *Let assumptions (A1), (A2), and (H1)–(H4) be satisfied. We have the following continuity dependence estimate:*

$$E\left[\int_{\mathbb{R}^d} |Z_s^{t+h,x,n} - Z_s^{t,x,n}|^4 \rho^{-1}(x) dx\right] \leq Ch^2,$$

for a constant $C > 0$.

Proof. First note that by a standard estimate, for $2 \leq m \leq 16p$ we have

$$\begin{aligned} & E\left[\int_{\mathbb{R}^d} |X_r^{s+h',x} - X_r^{s,x}|^m \rho^{-1}(x) dx\right] + E\left[\int_{\mathbb{R}^d} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^m \rho^{-1}(x) dx\right] \\ & + E\left[\left(\int_s^T \int_{\mathbb{R}^d} |Z_r^{s+h',x,n} - Z_r^{s,x,n}|^2 \rho^{-1}(x) dx\right)^{\frac{m}{2}}\right] \leq C_p h'^{\frac{m}{2}}. \end{aligned} \quad (16)$$

Since

$$\begin{aligned} |Z_s^{t+h,x,n} - Z_s^{t,x,n}| &= |\sigma^*(X_s^{t+h,x}) \tilde{Y}_s^{t+h,x,n} - \sigma^*(X_s^{t,x}) \tilde{Y}_s^{t,x,n}| \\ &\leq L|\tilde{Y}_s^{t+h,x,n} - \tilde{Y}_s^{t,x,n}| + L|\tilde{Y}_s^{t,x,n}| |X_s^{t+h,x} - X_s^{t,x}|, \end{aligned}$$

it turns out that

$$\begin{aligned} & E\left[\int_{\mathbb{R}^d} |Z_s^{t+h,x,n} - Z_s^{t,x,n}|^4 \rho^{-1}(x) dx\right] \\ & \leq C_p E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t+h,x,n} - \tilde{Y}_s^{t,x,n}|^4 \rho^{-1}(x) dx\right] \\ & \quad + C_p \sqrt{E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^8 \rho^{-1}(x) dx\right]} \sqrt{E\left[\int_{\mathbb{R}^d} |X_s^{t+h,x} - X_s^{t,x}|^8 \rho^{-1}(x) dx\right]}. \end{aligned} \quad (17)$$

Define

$$\begin{aligned} \overline{\tilde{Y}_s} &= \tilde{Y}_s^{t+h,x,n} - \tilde{Y}_s^{t,x,n}, \quad \overline{\tilde{Z}_s} = \tilde{Z}_s^{t+h,x,n} - \tilde{Z}_s^{t,x,n}, \\ \overline{X_s} &= X_s^{t+h,x,n} - X_s^{t,x,n}, \quad \overline{Y_s} = Y_s^{t+h,x,n} - Y_s^{t,x,n}, \quad \overline{Z_s} = Z_s^{t+h,x,n} - Z_s^{t,x,n}. \end{aligned}$$

Obviously, $(\overline{\tilde{Y}}, \overline{\tilde{Z}})$ satisfies the following equation:

$$\begin{aligned} \overline{\tilde{Y}_s} &= \partial_x h(X_T^{t+h,x}) - \partial_x h(X_T^{t,x}) \\ &+ \int_s^T \left\{ (a^{t+h}(r, x) - a^t(r, x)) + \tilde{Y}_r^{t,x,n} (b^{t+h}(r, x) - b^t(r, x)) \right. \\ &\quad \left. + \tilde{Z}_r^{t,x,n} (c^{t+h}(r, x) - c^t(r, x)) + b^{t+h}(r, x) \overline{\tilde{Y}_r} + c^{t+h}(r, x) \overline{\tilde{Z}_r} \right\} dr \\ &- \int_s^T \left\{ (d^{t+h}(r, x) - d^t(r, x)) + \tilde{Y}_r^{t,x,n} (e^{t+h}(r, x) - e^t(r, x)) \right. \\ &\quad \left. + \tilde{Z}_r^{t,x,n} (j^{t+h}(r, x) - j^t(r, x)) + e^{t+h}(r, x) \overline{\tilde{Y}_r} + j^{t+h}(r, x) \overline{\tilde{Z}_r} \right\} d^\dagger \hat{B}_r \\ &- \int_s^T \overline{\tilde{Z}_r} dW_r. \end{aligned}$$

By a similar computation as in the proof of Proposition 4, we have

$$\begin{aligned}
& \sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |\tilde{Y}_s|^4 \rho^{-1}(x) dx \right] + \sup_n E \left[\left(\int_t^T \int_{\mathbb{R}^d} |\tilde{Z}_r|^2 \rho^{-1}(x) dx dr \right)^2 \right] \\
\leq & C_p E \left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t+h,x}) - \partial_x h(X_T^{t,x})|^4 \rho^{-1}(x) dx \right] \\
& + C_p E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(a^{t+h}(r, x) - a^t(r, x)) + \tilde{Y}_r^{t,x,n}(b^{t+h}(r, x) - b^t(r, x)) \right. \right. \\
& \quad \left. \left. + \tilde{Z}_r^{t,x,n}(c^{t+h}(r, x) - c^t(r, x))| \rho^{-1}(x) dx dr \right)^4 \right] \\
& + C_p \sqrt{E \left[\left(\int_s^T \int_{\mathbb{R}^d} |b^{t+h}(r, x)|^2 \rho^{-1}(x) dx dr \right)^4 \right]} \sqrt{E \left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r|^2 \rho^{-1}(x) dx dr \right)^4 \right]} \\
& + C_p E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(d^{t+h}(r, x) - d^t(r, x)) + \tilde{Y}_r^{t,x,n}(e^{t+h}(r, x) - e^t(r, x)) \right. \right. \\
& \quad \left. \left. + \tilde{Z}_r^{t,x,n}(j^{t+h}(r, x) - j^t(r, x))|^2 \rho^{-1}(x) dx dr \right)^2 \right]. \tag{18}
\end{aligned}$$

Before we estimate each term on the right hand side of above inequality, we need do some calculations. Firstly,

$$\begin{aligned}
& E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(a^{t+h}(r, x) - a^t(r, x))| \rho^{-1}(x) dx dr \right)^4 \right] \\
\leq & C_p E \left[\int_s^T \int_{\mathbb{R}^d} (|\overline{X}_r|^4 + |\overline{Y}_r|^4) \rho^{-1}(x) dx dr \right] \\
& + C_p \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} |\overline{X}_r|^8 \rho^{-1}(x) dx dr \right]} \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^8 \rho^{-1}(x) dx dr \right]} \leq C_p h^2.
\end{aligned}$$

Next for the continuity dependence on b ,

$$\begin{aligned}
& E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(b^{t+h}(r, x) - b^t(r, x))|^2 \rho^{-1}(x) dx dr \right)^4 \right] \\
\leq & C_p E \left[\left(\int_s^T \int_{\mathbb{R}^d} |\partial_y f_n(r, X_r^{t+h,x}, Y_r^{t+h,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n}) \right. \right. \\
& \quad \left. \left. - \partial_y f_n(r, X_r^{t,x}, Y_r^{t+h,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n})|^2 \right. \right. \\
& \quad \left. \left. + |\partial_y f_n(r, X_r^{t,x}, Y_r^{t+h,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n}) \right. \right. \\
& \quad \left. \left. - \partial_y f_n(r, X_r^{t+h,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n})|^2 \right. \right. \\
& \quad \left. \left. + |\partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n}) \right. \right. \\
& \quad \left. \left. - \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n})|^2 \right. \right. \\
& \quad \left. \left. + |\nabla \sigma^*(X_r^{t,x})(\partial_z f_n(r, X_r^{t+h,x}, Y_r^{t+h,x,n}, \sigma^*(X_r^{t+h,x}) \tilde{Y}_r^{t+h,x,n}))|^2 \right. \right. \\
& \quad \left. \left. - \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n}))|^2 \right. \right. \\
& \quad \left. \left. + |\partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n})(\nabla \sigma^*(X_r^{t+h,x}) - \nabla \sigma^*(X_r^{t,x}))|^2 \right. \right. \\
& \quad \left. \left. + |\nabla b(X_r^{t+h,x}) - \nabla b(X_r^{t,x})|^2 \rho^{-1}(x) dx dr \right)^4 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq C_p E \left[\left(\int_s^T \int_{\mathbb{R}^d} (1 + |Y_r^{t+h,x,n}|^{p-1} + |Z_r^{t+h,x,n}|)^2 |\bar{X}_r|^2 \right. \right. \\
&\quad \left. \left. + (1 + |Y_r^{t+h,x,n}|^{p-1} + |Y_r^{t,x,n}|^{p-1})^2 |\bar{Y}_r|^2 \right. \right. \\
&\quad \left. \left. + |\bar{Z}_r|^2 + |\bar{X}_r|^2 + |\bar{Y}_r|^2 + |\bar{Z}_r|^2 + L^2 |\bar{X}_r|^2 + |\bar{X}_r|^2 \rho^{-1}(x) dx dr \right)^4 \right] \\
&\leq C_p \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} (1 + |Y_r^{t+h,x,n}|^{16p-16} + |Z_r^{t+h,x,n}|^{16}) \rho^{-1}(x) dx dr \right]} \\
&\quad \times \sqrt{E \left[\int_s^T |\bar{X}_r|^{16} \rho^{-1}(x) dx dr \right]} \\
&\quad + \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} (1 + |Y_r^{t+h,x,n}|^{16p-16} + |Y_r^{t,x,n}|^{16p-16}) \rho^{-1}(x) dx dr \right]} \\
&\quad \times \sqrt{E \left[\int_s^T |\bar{Y}_r|^{16} \rho^{-1}(x) dx dr \right]} \\
&\quad + E \left[\left(\int_s^T \int_{\mathbb{R}^d} |\bar{Z}_r|^2 \rho^{-1}(x) dx dr \right)^4 \right] \leq C_p h^4. \tag{19}
\end{aligned}$$

The continuity dependence on c is shown below:

$$\begin{aligned}
&E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(c^{t+h}(r, x) - c^t(r, x))|^2 \rho^{-1}(x) dx dr \right)^4 \right] \\
&\leq C_p E \left[\left(\int_s^T \int_{\mathbb{R}^d} |\bar{X}_r|^2 + |\bar{Y}_r|^2 + |\sigma^*(X_r^{t+h,x}) (\tilde{Y}_r^{t+h,x,n} - \tilde{Y}_r^{t,x,n})|^2 \right. \right. \\
&\quad \left. \left. + |\tilde{Y}_r^{t,x,n} (\sigma^*(X_r^{t+h,x}) - \sigma^*(X_r^{t,x}))|^2 \rho^{-1}(x) dx dr \right)^4 \right] \\
&\leq C_p \left(E \left[\int_s^T \int_{\mathbb{R}^d} (|\bar{X}_r|^8 + |\bar{Y}_r|^8) \rho^{-1}(x) dx dr \right] \right. \\
&\quad \left. + \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} |\bar{X}_r|^{16} \rho^{-1}(x) dx dr \right]} \sqrt{E \left[\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^{16} \rho^{-1}(x) dx dr \right]} \right) \\
&\leq C_p h^4. \tag{20}
\end{aligned}$$

We are ready to estimate each term of (18) with the help of (19)–(20). For the first term,

$$\begin{aligned}
&E \left[\int_{\mathbb{R}^d} |\partial_x h(X_T^{t+h,x}) - \partial_x h(X_T^{t,x})|^4 \rho^{-1}(x) dx \right] \\
&\leq C_p E \left[\int_{\mathbb{R}^d} |X_T^{t+h,x} - X_T^{t,x}|^4 \rho^{-1}(x) dx \right] \leq h^2.
\end{aligned}$$

For the second term, using the estimates in Proposition 4 we have

$$\begin{aligned}
&E \left[\left(\int_s^T \int_{\mathbb{R}^d} |(a^{t+h}(r, x) - a^t(r, x)) + \tilde{Y}_r^{t,x,n} (b^{t+h}(r, x) - b^t(r, x)) \right. \right. \\
&\quad \left. \left. + \tilde{Z}_r^{t,x,n} (c^{t+h}(r, x) - c^t(r, x))|^4 \rho^{-1}(x) dx dr \right) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq E\left[\left(\int_s^T \int_{\mathbb{R}^d} |(a^{t+h}(r, x) - a^t(r, x))| \rho^{-1}(x) dx dr\right)^4\right] \\
&\quad + E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}(b^{t+h}(r, x) - b^t(r, x))| \rho^{-1}(x) dx dr\right)^4\right] \\
&\quad + E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}(c^{t+h}(r, x) - c^t(r, x))| \rho^{-1}(x) dx dr\right)^4\right] \\
&\leq C_p h^2 + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\quad \times \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |b^{t+h}(r, x) - b^t(r, x)|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\quad + C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\quad \times \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |c^{t+h}(r, x) - c^t(r, x)|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\leq C_p h^2.
\end{aligned}$$

For the third term, $\sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |b^{t+h}(r, x)|^2 \rho^{-1}(x) dx dr\right)^4\right]} < \infty$ can be deduced as (19). So

$$\begin{aligned}
&\sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |b^{t+h}(r, x)|^2 \rho^{-1}(x) dx dr\right)^4\right]} \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\leq C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\leq C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} |(\sigma\sigma^*)(X_r^{t+h,x})\tilde{Y}_r|^2 \rho^{-1}(x) dx dr\right)^4\right]} \\
&\leq C_p \left(E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|\tilde{Y}_r^{t,x,n}|^2 |(\sigma\sigma^*)(X_r^{t+h,x}) - (\sigma\sigma^*)(X_r^{t,x})|^2 \right. \right. \right. \\
&\quad \left. \left. \left. + |\sigma(X_r^{t+h,x})Z_r^{t+h,x,n} - \sigma(X_r^{t,x})Z_r^{t,x,n}|^2) \rho^{-1}(x) dx dr\right)^4\right] \right)^{\frac{1}{2}} \\
&\leq C_p \sqrt{E\left[\left(\int_s^T \int_{\mathbb{R}^d} (|\tilde{Y}_r^{t,x,n}|^2 |\overline{X}_r|^2 + |\overline{Z}_r|^2 + |Z_r^{t,x,n}|^2 |\overline{X}_r|^2) \rho^{-1}(x) dx dr\right)^4\right]} \\
&\leq C_p \left(\sqrt{E\left[\int_s^T \int_{\mathbb{R}^d} |\overline{X}_r|^{16} \rho^{-1}(x) dx dr\right]} \sqrt{E\left[\int_s^T \int_{\mathbb{R}^d} |\tilde{Y}_r^{t,x,n}|^{16} \rho^{-1}(x) dx dr\right]} \right. \\
&\quad \left. + E\left[\left(\int_s^T \int_{\mathbb{R}^d} |\overline{Z}_r|^2 \rho^{-1}(x) dx dr\right)^4\right] \right. \\
&\quad \left. + \sqrt{E\left[\int_s^T \int_{\mathbb{R}^d} |\overline{X}_r|^{16} \rho^{-1}(x) dx dr\right]} \sqrt{E\left[\int_s^T \int_{\mathbb{R}^d} |Z_r^{t,x,n}|^{16} \rho^{-1}(x) dx dr\right]}\right)^{\frac{1}{2}} \\
&\leq C_p h^2.
\end{aligned}$$

The estimate for the forth term is similar to the second one, and by the Lipschitz conditions on $\partial_x g, \partial_y g, \partial_z g$ we have

$$\begin{aligned} & E\left[\left(\int_s^T \int_{\mathbb{R}^d} |(d^{t+h}(r, x) - d^t(r, x)) + \tilde{Y}_r^{t,x,n}(e^{t+h}(r, x) - e^t(r, x))\right.\right. \\ & \quad \left.\left. + \tilde{Z}_r^{t,x,n}(j^{t+h}(r, x) - j^t(r, x))|^2 \rho^{-1}(x) dx dr\right)^2\right] \\ & \leq C_p h^2. \end{aligned}$$

Eventually, we have

$$\begin{aligned} & \sup_n \sup_{s \in [t, T]} E\left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t+h,x,n} - \tilde{Y}_s^{t,x,n}|^4 \rho^{-1}(x) dx\right] \\ & + \sup_n E\left[\left(\int_t^T \int_{\mathbb{R}^d} |\tilde{Z}_s^{t+h,x,n} - \tilde{Z}_s^{t,x,n}|^2 \rho^{-1}(x) dx dr\right)^2\right] \leq C_p h^2. \end{aligned} \quad (21)$$

Then Lemma 3.1 follows immediately from (17) and (21). \diamond

Denote the Malliavin derivative of $(\tilde{Y}^{t,x,n}, \tilde{Z}^{t,x,n})$ by $(D\tilde{Y}^{t,x,n}, D\tilde{Z}^{t,x,n})$. Then $(D\tilde{Y}^{t,x,n}, D\tilde{Z}^{t,x,n})$ is the solution of the following BDSDE:

$$\begin{aligned} D_\theta \tilde{Y}_s^{t,x,n} = & \partial_x g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_\theta^{t,x,n}) \\ & + \partial_y g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_\theta^{t,x,n}) \tilde{Y}_\theta^{t,x,n} \\ & + \partial_z g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_\theta^{t,x,n}) (\nabla \sigma^*(X_\theta^{t,x}) \tilde{Y}_\theta^{t,x,n} + \tilde{Z}_\theta^{t,x,n}) \\ & + \int_s^T (\tilde{a}(r, \omega) + \tilde{b}(r, \omega) D_\theta \tilde{Y}_r^{t,x,n} + \tilde{c}(r, \omega) D_\theta \tilde{Z}_r^{t,x,n}) dr \\ & - \int_s^T (\tilde{d}(r, \omega) + \tilde{e}(r, \omega) D_\theta \tilde{Y}_r^{t,x,n} + \tilde{j}(r, \omega) D_\theta \tilde{Z}_r^{t,x,n}) d^\dagger \hat{B}_r \\ & - \int_s^T D_\theta \tilde{Z}_r^{t,x,n} dW_r, \end{aligned}$$

where

$$\begin{aligned} \tilde{a}(r, \omega) = & \partial_{xy} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\ & + \partial_{yy} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} D_\theta Y_r^{t,x,n} \\ & + \partial_{yz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} D_\theta Y_r^{t,x,n} \\ & + \partial_{yz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Z}_r^{t,x,n} D_\theta Y_r^{t,x,n}, \\ \tilde{b}(r, \omega) = & \partial_{xz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \sigma^*(X_\theta^{t,x}) + \nabla b(X_r^{t,x}) \\ & + \partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \\ & + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \\ & + \partial_{yz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} \sigma^*(X_\theta^{t,x}) \\ & + \partial_{zz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} \sigma^*(X_\theta^{t,x}) \\ & + \partial_{zz} f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Z}_r^{t,x,n} \sigma^*(X_\theta^{t,x}), \end{aligned}$$

$$\begin{aligned}
\tilde{c}(r, \omega) &= \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) + \nabla \sigma(X_r^{t,x}), \\
\tilde{d}(r, \omega) &= \partial_{xy} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) D_\theta Y_r^{t,x,n} \\
&\quad + \partial_{yy} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} D_\theta Y_r^{t,x,n} \\
&\quad + \partial_{yz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} D_\theta Y_r^{t,x,n} \\
&\quad + \partial_{yz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Z}_r^{t,x,n} D_\theta Y_r^{t,x,n}, \\
\tilde{e}(r, \omega) &= \partial_{xz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \sigma^*(X_\theta^{t,x}) \\
&\quad + \partial_y g_n(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \\
&\quad + \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \\
&\quad + \partial_{yz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Y}_r^{t,x,n} \sigma^*(X_\theta^{t,x}) \\
&\quad + \partial_{zz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \nabla \sigma^*(X_r^{t,x}) \tilde{Y}_r^{t,x,n} \sigma^*(X_\theta^{t,x}) \\
&\quad + \partial_{zz} g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}) \tilde{Z}_r^{t,x,n} \sigma^*(X_\theta^{t,x}), \\
\tilde{j}(r, \omega) &= \partial_z g(r, X_r^{t,x}, Y_r^{t,x,n}, \sigma^*(X_\theta^{t,x}) \tilde{Y}_r^{t,x,n}).
\end{aligned}$$

We further assume

(H5): $g(t, x, y, z)$ does not depend on z and the derivatives $\partial_{xy} f$, $\partial_{xz} f$, $\partial_{yy} f$, $\partial_{yz} f$, $\partial_{zz} f$, $\partial_{xy} g$, $\partial_{yy} g$ exist.

Obviously, f_n satisfies (H5).

Similar to the calculations of the estimate (10) in Proposition 4, we have

Proposition 5. *Let assumptions (A1), (A2), and (H1)–(H5) be satisfied. For $2 \leq m \leq 8$, the following estimate holds:*

$$\begin{aligned}
&\sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |D_\theta \tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&\quad + \sup_n \sup_{\theta \in [t, T]} E \left[\left(\int_t^T \int_{\mathbb{R}^d} |D_\theta \tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right)^{\frac{m}{2}} \right] \\
&\leq C_p \left(1 + \sqrt{\sup_n \sup_{\theta \in [t, T]} E \left[\left(\int_t^T \int_{\mathbb{R}^d} |D_\theta Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right)^m \right]} \right),
\end{aligned}$$

where C_p depends on $\sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |Y_s^{t,x,n}|^{4pm-8m} \rho^{-1}(x) dx \right]$,

$\sup_n \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |\tilde{Y}_s^{t,x,n}|^{2m} \rho^{-1}(x) dx \right]$, $\sup_n E \left[\left(\int_t^T \int_{\mathbb{R}^d} |\tilde{Z}_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right)^m \right]$ and

$\sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |D_\theta Y_s^{t,x,n}|^{4m} \rho^{-1}(x) dx \right]$.

From Proposition 5, we know that for $2 \leq m \leq 8$

$$\begin{aligned}
&\sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |D_\theta Z_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \\
&= \sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |\sigma^*(X_s^{t,x}) D_\theta \tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx \right] \quad (22) \\
&\leq C_p \sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E \left[\int_{\mathbb{R}^d} |D_\theta \tilde{Y}_s^{t,x,n}|^m \rho^{-1}(x) dx \right] < \infty.
\end{aligned}$$

Then we prove that a subsequence of $u_n(s, x)$ in SPDE (7) is relatively compact by Theorem 2.1.

Theorem 3.2. *Let assumptions (A1), (A2), and (H1)–(H5) be satisfied, $(Y^{t,\cdot,n}, Z^{t,\cdot,n})$ be the solution of BDSDE (6) and \mathcal{O} be a bounded domain in \mathbb{R}^d , then the sequence $u_n(s, x) := Y_s^{s,x,n}$ is relatively compact in $L^2(\Omega \times [0, T] \times \mathcal{O}; \mathbb{R}^1)$.*

Proof. We verify that u_n satisfies Conditions (1)–(4) in Theorem 2.1.

Step 1. It is not difficult to see that Condition (1) is satisfied. Actually, by Lemma 2.2, Propositions 1 and 2, it yields that

$$\begin{aligned} & \sup_n E \left[\int_0^T \|u_n(s, \cdot)\|_{H^1(\mathcal{O})}^2 ds \right] \\ & \leq C_p \sup_n E \left[\int_0^T \int_{\mathcal{O}} (|u_n(s, x)|^2 + |\nabla u_n(s, x)|^2) \rho^{-1}(x) dx ds \right] \\ & \leq C_p \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} (|Y_s^{0,x,n}|^2 + |Z_s^{0,x,n}|^2) \rho^{-1}(x) dx ds \right] < \infty. \end{aligned}$$

Step 2. We now check Condition (2). Note that $D_\theta u_n^\varphi(s) = \int_{\mathcal{O}} D_\theta u_n(s, x) \varphi(x) dx$. By Proposition 3, $D_\theta u_n(s, x) = D_\theta Y_s^{s,x,n}$ exists. We further prove $u_n^\varphi(s) \in \mathbb{D}^{1,2}$. Computing as Propositions 1 and 4, we have

$$\begin{aligned} & \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 \\ & \leq C_p E \left[\int_{\mathbb{R}^d} |u_n(s, x)|^2 \rho^{-1}(x) dx \right] + C_p E \left[\int_s^T \int_{\mathbb{R}^d} |D_\theta u_n(s, x)|^2 \rho^{-1}(x) dx d\theta \right] \\ & \leq C_p E \left[\int_{\mathbb{R}^d} |Y_s^{0,x,n}|^2 \rho^{-1}(x) dx \right] + C_p E \left[\int_s^T \int_{\mathbb{R}^d} |D_\theta Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta \right] \\ & \leq C_p \sup_{s \in [0, T]} E \left[\int_{\mathbb{R}^d} |Y_s^{0,x,n}|^2 \rho^{-1}(x) dx \right] + C_p \sup_{s \in [0, T]} \int_{\mathbb{R}^d} |g(s, x, 0)|^2 \rho^{-1}(x) dx \\ & \quad + C_p E \left[\int_s^T \int_{\mathbb{R}^d} |Y_\theta^{s,x,n}|^2 \rho^{-1}(x) dx d\theta \right] \\ & \leq C_p \sup_n \int_{\mathbb{R}^d} |h(x)|^2 \rho^{-1}(x) dx + C_p \int_0^T \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr \\ & \quad + C_p \sup_{s \in [0, T]} \int_{\mathbb{R}^d} |g(s, x, 0)|^2 \rho^{-1}(x) dx. \end{aligned} \tag{23}$$

The right hand side of the above inequality is bounded and independent of s and n , so

$$\sup_n \int_0^T \|u_n^\varphi(s)\|_{\mathbb{D}^{1,2}}^2 ds < \infty.$$

Step 3. Let us verify Condition (3). First (3i) follows immediately from (23). To see (3ii), assume $h > 0$ without losing any generality. From (6) and Cauchy-Schwarz

inequality, we have

$$\begin{aligned}
& \sup_n \int_{\alpha}^{\beta} |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds \\
& \leq C_p \sup_n \int_{\alpha}^{\beta} E \left[\int_{\mathbb{R}^d} |u_n(s+h, x) - u_n(s, x)|^2 \rho^{-1}(x) dx \right] ds \\
& \leq C_p \sup_n \int_{\alpha}^{\beta} E \left[\int_{\mathbb{R}^d} |Y_{s+h}^{0,x,n} - Y_s^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds \\
& \leq C_p \sup_n \int_{\alpha}^{\beta} E \left[\int_{\mathbb{R}^d} \int_s^{s+h} |f_n(r, X_r^{0,x}, Y_r^{0,x,n}, Z_r^{0,x,n})|^2 dr \rho^{-1}(x) dx \right] ds \\
& \quad + C_p \sup_n \int_{\alpha}^{\beta} E \left[\int_{\mathbb{R}^d} \int_s^{s+h} |g(r, X_r^{0,x}, Y_r^{0,x,n})|^2 dr \rho^{-1}(x) dx \right] ds \\
& \quad + C_p \sup_n \int_{\alpha}^{\beta} E \left[\int_{\mathbb{R}^d} \int_s^{s+h} |Z_r^{0,x,n}|^2 dr \rho^{-1}(x) dx \right] ds \\
& \leq C_p \int_{\alpha}^{\beta} \int_s^{s+h} \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr ds \\
& \quad + C_p \sup_n \int_{\alpha}^{\beta} \int_s^{s+h} \sup_{r \in [0, T]} (1 + E[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{2p} \rho^{-1}(x) dx]) dr ds \\
& \quad + C_p \sup_n \int_{\alpha}^{\beta} \int_s^{s+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr ds \\
& \quad + C_p \int_{\alpha}^{\beta} \int_s^{s+h} \sup_{r \in [0, T]} \int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx dr ds. \tag{24}
\end{aligned}$$

Note that by changing integration order,

$$\begin{aligned}
& \sup_n \int_{\alpha}^{\beta} \int_s^{s+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr ds \\
& = \sup_n \left(\int_{\alpha}^{\alpha+h} \int_{\alpha}^r E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr \right. \\
& \quad \left. + \int_{\alpha+h}^{\beta} \int_{r-h}^r E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr \right. \\
& \quad \left. + \int_{\beta}^{\beta+h} \int_{r-h}^{\beta} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] ds dr \right) \\
& = \sup_n \left((r - \alpha) \int_{\alpha}^{\alpha+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \right. \\
& \quad \left. + h \int_{\alpha+h}^{\beta} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \right. \\
& \quad \left. + (\beta + h - r) \int_{\beta}^{\beta+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \right) \\
& = C_p h \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right].
\end{aligned}$$

A similar calculation can be carried out to

$$\int_{\alpha}^{\beta} \int_s^{s+h} \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr ds.$$

Then it follows from (24) that

$$\begin{aligned} & \sup_n \int_{\alpha}^{\beta} |E[u_n^{\varphi}(s+h)] - E[u_n^{\varphi}(s)]|^2 ds \\ & \leq C_p h \int_0^T \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr \\ & \quad + C_p h \sup_n \sup_{r \in [0, T]} (1 + E[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{2p} \rho^{-1}(x) dx]) \\ & \quad + C_p h \sup_n E[\int_0^T \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr] + C_p h \sup_{r \in [0, T]} \int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx. \end{aligned}$$

Also noticing Condition (H2)' and Proposition 1, we conclude that (3ii) holds.

Step 4. We finally check Condition (4). For (4i), since by the equivalence of norm principle and (16) it turns out that

$$\begin{aligned} & \sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E[|D_{\theta} u_n^{\varphi}(s)|^2] \\ & \leq C_p \sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E[\int_{\mathbb{R}^d} |D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx] \\ & \leq C_p \sup_n \sup_{\theta \in [t, T]} \sup_{s \in [t, T]} E[\int_{\mathbb{R}^d} |D_{\theta} Y_s^{0,x,n}|^2 \rho^{-1}(x) dx] < \infty. \end{aligned}$$

So (4i) follows. To see (4ii), assume without losing any generality that $h, h' > 0$, then

$$\begin{aligned} & \sup_n E[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^{\varphi}(s+h') - D_{\theta} u_n^{\varphi}(s)|^2 d\theta ds] \\ & \leq C_p \sup_n E[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta+h} u_n(s, x) - D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds] \\ & \quad + C_p \sup_n E[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} u_n(s+h', x) - D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds]. \end{aligned} \tag{25}$$

For the first term on the right hand side of (25), by the equivalence of norm principle,

$$\begin{aligned} & \sup_n E[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta+h} u_n(s, x) - D_{\theta} u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds] \\ & = \sup_n E[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} Y_s^{s,x,n} - D_{\theta} Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds] \\ & \leq C_p \sup_n E[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} Y_s^{0,x,n} - D_{\theta} Y_s^{0,x,n}|^2 \rho^{-1}(x) dx d\theta ds]. \end{aligned} \tag{26}$$

By BDSDE (8) we know that

$$\begin{aligned}
& (D_{\theta+h} - D_\theta)Y_s^{0,x,n} \\
= & H(\theta, \theta + h) + \int_s^\theta (\partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})(D_{\theta+h} - D_\theta)Y_r^{t,x,n} \\
& + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})(D_{\theta+h} - D_\theta)Z_r^{t,x,n})dr \\
& - \int_s^\theta \partial_y g(r, X_r^{t,x}, Y_r^{t,x,n})(D_{\theta+h} - D_\theta)Y_r^{t,x,n} d^\dagger \hat{B}_r - \int_s^\theta (D_{\theta+h} - D_\theta)Z_r^{t,x,n} dW_r,
\end{aligned}$$

where

$$\begin{aligned}
& H(\theta, \theta + h) \\
= & g(\theta + h, X_{\theta+h}^{t,x}, Y_{\theta+h}^{t,x,n}) - g(\theta, X_\theta^{t,x}, Y_\theta^{t,x,n}) \\
& + \int_\theta^{\theta+h} (\partial_y f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})D_{\theta+h}Y_r^{t,x,n} \\
& + \partial_z f_n(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})D_{\theta+h}Z_r^{t,x,n})dr \\
& - \int_\theta^{\theta+h} \partial_y g(r, X_r^{t,x}, Y_r^{t,x,n})D_{\theta+h}Y_r^{t,x,n} d^\dagger \hat{B}_r - \int_\theta^{\theta+h} D_{\theta+h}Z_r^{t,x,n} dW_r.
\end{aligned}$$

Applying Itô's formula to $e^{Kr}|(D_{\theta+h} - D_\theta)Y_s^{0,x,n}|^2$ similarly as in (14)–(16), we have

$$\begin{aligned}
& E\left[\int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta)Y_s^{0,x,n}|^2 \rho^{-1}(x)dx\right] \\
& + E\left[\int_s^\theta \int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta)Y_r^{0,x,n}|^2 \rho^{-1}(x)dxdr\right] \\
& + E\left[\int_s^\theta \int_{\mathbb{R}^d} |(D_{\theta+h} - D_\theta)Z_r^{0,x,n}|^2 \rho^{-1}(x)dxdr\right] \\
\leq & C_p E\left[\int_{\mathbb{R}^d} |H(\theta, \theta + h)|^2 \rho^{-1}(x)dx\right]. \tag{27}
\end{aligned}$$

Next we prove that

$$\sup_n E\left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |H(\theta, \theta + h)|^2 \rho^{-1}(x)dx d\theta ds\right] \leq C_p h. \tag{28}$$

First note that

$$\begin{aligned}
& E\left[\int_{\mathbb{R}^d} |H(\theta, \theta + h)|^2 \rho^{-1}(x)dx\right] \\
\leq & C_p h^2 + C_p E\left[\int_{\mathbb{R}^d} |X_{\theta+h}^{0,x} - X_\theta^{0,x}|^2 \rho^{-1}(x)dx\right] \\
& + C_p E\left[\int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_\theta^{0,x,n}|^2 \rho^{-1}(x)dx\right] \\
& + C_p E\left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |\partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n}, Z_r^{0,x,n})|^2 |D_{\theta+h}Y_r^{0,x,n}|^2 \rho^{-1}(x)dxdr\right] \\
& + C_p \int_\theta^{\theta+h} \sup_{s \in [0, T]} \sup_{r \in [0, T]} E\left[\int_{\mathbb{R}^d} |D_s Y_r^{0,x,n}|^2 \rho^{-1}(x)dx\right] dr \\
& + C_p E\left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x)dxdr\right]. \tag{29}
\end{aligned}$$

We need to estimate each term in the above formula. From (2), we have

$$\begin{aligned}
& E \left[\int_{\mathbb{R}^d} |X_{\theta+h}^{0,x} - X_\theta^{0,x}|^2 \rho^{-1}(x) dx \right] \\
\leq & C_p E \left[\int_{\mathbb{R}^d} \left(\int_\theta^{\theta+h} |b(X_u^{0,x})| du \right)^2 \rho^{-1}(x) dx \right] \\
& + C_p \int_{\mathbb{R}^d} E \left[\int_\theta^{\theta+h} |\sigma(X_u^{0,x})|^2 du \right] \rho^{-1}(x) dx \\
\leq & C_p E \left[\int_{\mathbb{R}^d} h \int_\theta^{\theta+h} (1 + |X_u^{0,x}|)^2 du \rho^{-1}(x) dx \right] + C_p E \left[\int_{\mathbb{R}^d} \int_\theta^{\theta+h} L^2 du \rho^{-1}(x) dx \right] \\
\leq & C_p h \int_\theta^{\theta+h} E \left[\int_{\mathbb{R}^d} (1 + |X_u^{0,x}|)^2 \rho^{-1}(x) dx \right] du + C_p h E \left[\int_{\mathbb{R}^d} L^2 \rho^{-1}(x) dx \right] \leq C_p h.
\end{aligned}$$

By (H3)', Proposition 1 and Proposition 3, we have

$$\begin{aligned}
& E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |\partial_y f_n(r, X_r^{0,x}, Y_r^{0,x,n}, Z_r^{0,x,n})|^2 |D_{\theta+h} Y_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
\leq & C_p \int_\theta^{\theta+h} \sup_n \sup_{\theta \in [0, T]} \sup_{r \in [0, T]} \left(\sqrt{E \left[\int_{\mathbb{R}^d} (1 + |Y_r^{0,x,n}|^{4p-4}) \rho^{-1}(x) dx \right]} \right. \\
& \quad \times \sqrt{E \left[\int_{\mathbb{R}^d} |D_\theta Y_r^{0,x,n}|^4 \rho^{-1}(x) dx \right]} dr \\
\leq & C_p h.
\end{aligned}$$

By Proposition 3 again, we also have that

$$\int_\theta^{\theta+h} \sup_n \sup_{s \in [0, T]} \sup_{r \in [0, T]} E \left[\int_{\mathbb{R}^d} |D_s Y_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \leq C_p h.$$

Hence, from (29), to prove (28) is reduced to prove

$$\begin{aligned}
& \sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_\theta^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \\
& + \sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h. \quad (30)
\end{aligned}$$

From (6), we have

$$\begin{aligned}
& E \left[\int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_\theta^{0,x,n}|^2 \rho^{-1}(x) dx \right] \\
& \leq C_p E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |f_n(r, X_r^{0,x}, Y_r^{0,x,n}, Z_r^{0,x,n})|^2 \rho^{-1}(x) dx dr \right] \\
& \quad + C_p E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |g(r, X_r^{0,x}, Y_r^{0,x,n})|^2 \rho^{-1}(x) dx dr \right] \\
& \quad + C_p E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] \\
& \leq C_p \int_\theta^{\theta+h} \int_{\mathbb{R}^d} |f_0(r, x)|^2 \rho^{-1}(x) dx dr \\
& \quad + C_p \int_\theta^{\theta+h} \sup_n \sup_{r \in [0, T]} E \left[\int_{\mathbb{R}^d} (1 + |Y_r^{0,x,n}|^{2p}) \rho^{-1}(x) dx \right] dr \\
& \quad + C_p \int_\theta^{\theta+h} E \left[\int_{\mathbb{R}^d} |Z_r^{0,x,n}|^2 \rho^{-1}(x) dx \right] dr \\
& \quad + C_p \int_\theta^{\theta+h} \sup_{r \in [0, T]} E \left[\int_{\mathbb{R}^d} |g(r, x, 0)|^2 \rho^{-1}(x) dx \right] dr. \tag{31}
\end{aligned}$$

A similar calculation of changing the integrations order leads to

$$\begin{aligned}
& \sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_\theta^{\theta+h} E \left[\int_{\mathbb{R}^d} (|f_0(r, x)|^2 + |Z_r^{0,x,n}|^2) \rho^{-1}(x) dx \right] dr d\theta ds \\
& \leq C_p h \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} (|f_0(r, x)|^2 + |Z_r^{0,x,n}|^2) \rho^{-1}(x) dx dr \right].
\end{aligned}$$

Moreover, by Condition (H2)' and Proposition 1 we conclude from (31) that

$$\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |Y_{\theta+h}^{0,x,n} - Y_\theta^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h.$$

Furthermore, by changing the integrations order again and Proposition 3, we have

$$\sup_n \int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} E \left[\int_\theta^{\theta+h} \int_{\mathbb{R}^d} |D_s Z_r^{0,x,n}|^2 \rho^{-1}(x) dx dr \right] d\theta ds \leq C_p h.$$

Hence (30) follows. So (28) holds. By (26) and (27) we can deduce that

$$\sup_n E \left[\int_{\alpha+h'}^{\beta+h'} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta+h} u_n(s, x) - D_\theta u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds \right] \leq C_p h. \tag{32}$$

Next we deal with the second term on the right hand side of (25). Notice

$$\begin{aligned}
& \sup_n E \left[\int_\alpha^\beta \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_\theta u_n(s+h', x) - D_\theta u_n(s, x)|^2 \rho^{-1}(x) dx d\theta ds \right] \\
& \leq \sup_n 2E \left[\int_\alpha^\beta \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_\theta Y_{s+h'}^{s,x,n} - D_\theta Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right] \\
& \quad + \sup_n 2E \left[\int_\alpha^\beta \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_\theta Y_{s+h'}^{s+h',x,n} - D_\theta Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right]. \tag{33}
\end{aligned}$$

For the first term on the right hand side of (33), by (8), Lemma 2.2 and change of the integrations order, it is easy to see that

$$\begin{aligned}
& \sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathbb{R}^d} |D_{\theta} Y_{s+h'}^{s,x,n} - D_{\theta} Y_s^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right] \\
& \leq C_p \sup_{s \in [0, T-h']} \int_s^{s+h'} (1 + \sup_n E \left[\int_{\mathbb{R}^d} |Y_r^{0,x,n}|^{4p-4} \rho^{-1}(x) dx \right]) dr \\
& \quad + C_p \sup_{s \in [0, T-h']} \int_s^{s+h'} (1 + \sup_n \sup_{\theta \in [0, T]} E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{0,x,n}|^4 \rho^{-1}(x) dx \right]) dr \\
& \quad + C_p h' \sup_n \sup_{\theta \in [0, T]} E \left[\int_0^T \int_{\mathbb{R}^d} |D_{\theta} Z_r^{t,x,n}|^2 \rho^{-1}(x) dx dr \right] \leq C_p h'. \tag{34}
\end{aligned}$$

For the second term on the right hand side of (33), firstly from BDSDE (8) we know that

$$\begin{aligned}
& D_{\theta} (Y_{s+h'}^{s+h',x,n} - Y_{s+h'}^{s,x,n}) \\
& = J(s, s+h') + \int_{s+h'}^{\theta} \partial_y f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n}) D_{\theta} (Y_r^{s+h',x,n} - Y_r^{s,x,n}) dr \\
& \quad + \int_{s+h'}^{\theta} \partial_z f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n}) D_{\theta} (Z_r^{s+h',x,n} - Z_r^{s,x,n}) dr \\
& \quad - \int_{s+h'}^{\theta} \partial_y g(r, X_r^{s,x}, Y_r^{s,x,n}) D_{\theta} (Y_r^{s+h',x,n} - Y_r^{s,x,n}) d^{\dagger} \hat{B}_r \\
& \quad - \int_{s+h'}^{\theta} D_{\theta} (Z_r^{s+h',x,n} - Z_r^{s,x,n}) dW_r,
\end{aligned}$$

where

$$\begin{aligned}
& J(s, s+h') \\
& = g(\theta, X_{\theta}^{s+h',x}, Y_{\theta}^{s+h',x,n}) - g(\theta, X_{\theta}^{s,x}, Y_{\theta}^{s,x,n}) \\
& \quad + \int_{s+h'}^{\theta} (\partial_y f_n(r, X_r^{s+h',x}, Y_r^{s+h',x,n}, Z_r^{s+h',x,n}) \\
& \quad \quad - \partial_y f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n})) D_{\theta} Y_r^{s+h',x,n} dr \\
& \quad + \int_{s+h'}^{\theta} (\partial_z f_n(r, X_r^{s+h',x}, Y_r^{s+h',x,n}, Z_r^{s+h',x,n}) \\
& \quad \quad - \partial_z f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n})) D_{\theta} Z_r^{s+h',x,n} dr \\
& \quad - \int_{s+h'}^{\theta} (\partial_y g(r, X_r^{s+h',x}, Y_r^{s+h',x,n}) - \partial_y g(r, X_r^{s,x}, Y_r^{s,x,n})) D_{\theta} Y_r^{s+h',x,n} d^{\dagger} \hat{B}_r.
\end{aligned}$$

Applying Itô's formula to $e^{K\tau}|D_\theta(Y_r^{s+h',x,n} - Y_r^{s,x,n})|^2$ similarly as in (14)–(16), we have

$$\begin{aligned} & \sup_n E \left[\int_{\mathcal{O}} |D_\theta(Y_{s+h'}^{s+h',x,n} - Y_{s+h'}^{s,x,n})|^2 \rho^{-1}(x) dx \right] \\ & + \sup_n E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |D_\theta(Y_r^{s+h',x,n} - Y_r^{s,x,n})|^2 \rho^{-1}(x) dx dr \right] \\ & + \sup_n E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |D_\theta(Z_r^{s+h',x,n} - Z_r^{s,x,n})|^2 \rho^{-1}(x) dx dr \right] \\ & \leq C_p \sup_n E \left[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx \right]. \end{aligned} \quad (35)$$

So we only need to estimate $E[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx]$. Note that by Condition (H3)', (H4)-(H5),

$$\begin{aligned} & E \left[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx \right] \\ & \leq C_p E \left[\int_{\mathcal{O}} |g(\theta, X_\theta^{s+h',x}, Y_\theta^{s+h',x,n}) - g(\theta, X_\theta^{s,x}, Y_\theta^{s,x,n})|^2 \rho^{-1}(x) dx \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |\partial_y f_n(r, X_r^{s+h',x}, Y_r^{s+h',x,n}, Z_r^{s+h',x,n}) \right. \\ & \quad \left. - \partial_y f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n})|^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |\partial_z f_n(r, X_r^{s+h',x}, Y_r^{s+h',x,n}, Z_r^{s+h',x,n}) \right. \\ & \quad \left. - \partial_z f_n(r, X_r^{s,x}, Y_r^{s,x,n}, Z_r^{s,x,n})|^2 |D_\theta Z_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |\partial_y g(r, X_r^{s+h',x}, Y_r^{s+h',x,n}) \right. \\ & \quad \left. - \partial_y g(r, X_r^{s,x}, Y_r^{s,x,n})|^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & \leq C_p E \left[\int_{\mathcal{O}} |X_\theta^{s+h',x} - X_\theta^{s,x}|^2 \rho^{-1}(x) dx \right] + C_p E \left[\int_{\mathcal{O}} |Y_\theta^{s+h',x,n} - Y_\theta^{s,x,n}|^2 \rho^{-1}(x) dx \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^2 (1 + |Y_r^{s+h',x,n}|^{p-1})^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |Z_r^{s+h',x} - Z_r^{s,x}|^2 |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^2 (1 + |Y_r^{s+h',x,n}|^{p-2} + |Y_r^{s,x,n}|^{p-2})^2 \right. \\ & \quad \left. \times |D_\theta Y_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right] \\ & + C_p E \left[\int_{s+h'}^{\theta} \int_{\mathcal{O}} (|X_r^{s+h',x} - X_r^{s,x}|^2 + |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^2 + |Z_r^{s+h',x,n} - Z_r^{s,x,n}|^2) \right. \\ & \quad \left. \times |D_\theta Z_r^{s+h',x,n}|^2 \rho^{-1}(x) dx dr \right]. \end{aligned}$$

Noticing Lemma 3.1 and (22), we further have

$$\begin{aligned}
& E \left[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx \right] \\
\leq & C_p E \left[\int_{\mathcal{O}} |X_{\theta}^{s+h',x} - X_{\theta}^{s,x}|^2 \rho^{-1}(x) dx \right] \\
& + C_p E \left[\int_{\mathcal{O}} |Y_{\theta}^{s+h',x,n} - Y_{\theta}^{s,x,n}|^2 \rho^{-1}(x) dx \right] \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times (E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{s+h',x,n}|^8 \rho^{-1}(x) dx \right])^{\frac{1}{4}} \\
& \quad \times (E \left[\int_{\mathbb{R}^d} (1 + |Y_r^{s+h',x,n}|^{8p-16}) \rho^{-1}(x) dx \right])^{\frac{1}{4}} dr \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times (E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{s+h',x,n}|^8 \rho^{-1}(x) dx \right])^{\frac{1}{4}} \\
& \quad \times (E \left[\int_{\mathbb{R}^d} (1 + |Y_r^{s+h',x,n}|^{8p-8} + |Y_r^{s,x,n}|^{8p-16}) \rho^{-1}(x) dx \right])^{\frac{1}{4}} dr \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |Z_r^{s+h',x,n} - Z_r^{s,x,n}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times \sqrt{E \left[\int_{\mathbb{R}^d} |D_{\theta} Y_r^{s+h',x,n}|^4 \rho^{-1}(x) dx \right]} dr \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |X_r^{s+h',x} - X_r^{s,x}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times \sqrt{E \left[\int_{\mathbb{R}^d} |D_{\theta} Z_r^{s+h',x,n}|^4 \rho^{-1}(x) dx \right]} dr \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |Y_r^{s+h',x,n} - Y_r^{s,x,n}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times \sqrt{E \left[\int_{\mathbb{R}^d} |D_{\theta} Z_r^{s+h',x,n}|^4 \rho^{-1}(x) dx \right]} dr \\
& + C_p \int_{s+h'}^{\theta} \sqrt{E \left[\int_{\mathcal{O}} |Z_r^{s+h',x,n} - Z_r^{s,x,n}|^4 \rho^{-1}(x) dx \right]} \\
& \quad \times \sqrt{E \left[\int_{\mathbb{R}^d} |D_{\theta} Z_r^{s+h',x,n}|^4 \rho^{-1}(x) dx \right]} dr \leq C_p h'. \tag{36}
\end{aligned}$$

By (16), Lemmas 1 and 3, we know from (36) that

$$\sup_n E \left[\int_{\mathcal{O}} |J(s, s+h')|^2 \rho^{-1}(x) dx \right] \leq C_p h'. \tag{37}$$

Therefore, by (35) and (37) we have

$$\begin{aligned} & \sup_n E \left[\int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \int_{\mathcal{O}} |D_{\theta} Y_{s+h'}^{s+h',x,n} - D_{\theta} Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx d\theta ds \right] \\ & \leq C_p \int_{\alpha}^{\beta} \int_{\alpha'}^{\beta'} \sup_n E \left[\int_{\mathcal{O}} |D_{\theta} Y_{s+h'}^{s+h',x,n} - D_{\theta} Y_{s+h'}^{s,x,n}|^2 \rho^{-1}(x) dx \right] d\theta ds \leq C_p h'. \end{aligned} \quad (38)$$

Finally, by (25), (32)–(34) and (38), (4ii) holds. Theorem 3.2 is proved. \diamond

The above relative compactness of u_n holds in $L^2(\Omega \times [0, T]; L^2_{\rho}(\mathcal{O}; \mathbb{R}^1))$ for a bounded domain \mathcal{O} in \mathbb{R}^d rather than what we need in the space $L^2(\Omega \times [0, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$. But we can use the diagonal method (see [11]) and the estimate

$$\lim_{N \rightarrow \infty} \sup_n E \left[\int_0^T \int_{\mathbb{R}^d} |u_n(s, x)| I_{U_N^c}(x) \rho^{-1}(x) dx ds \right] = 0,$$

where $U_N = \{x \in \mathbb{R}^d : |x| \leq N\}$, to prove that a subsequence of u_n strongly converges in $L^2(\Omega \times [0, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$. Then we have

Theorem 3.3. *Let assumptions (A1), (A2), and (H1)–(H5) be satisfied, and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solution of BDSDEs (6), then there is a subsequence of $Y_s^{t,x,n}$, still denoted by $Y_s^{t,x,n}$, converging strongly to a limit $Y_s^{t,x}$ in $L^2(\Omega \times [t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$.*

Based on the form of BDSDE (6), we can further deduce a few consequences which are concluded in the following corollary. For the detailed proof procedure, one can refer to [10].

Corollary 1. *Let $(Y^{t,\cdot,n}, Z^{t,\cdot,n})$ be the solution of BDSDE (6), of which $Y^{t,\cdot,n}$ converges strongly to $Y^{t,\cdot}$ in $M^2([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$. Then $(Y^{t,\cdot,n}, Z^{t,\cdot,n})$ converges strongly to $(Y^{t,\cdot}, Z^{t,\cdot})$ in $S^2([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \times M^2([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$ and $Y^{t,\cdot} \in S^p([t, T]; L^p_{\rho}(\mathbb{R}^d; \mathbb{R}^1))$. Moreover, the claim that $Y_s^{t,x} = Y_s^{s,X_s^{t,x}}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s. holds.*

4. The solvability theorem. Now we are well prepared to use both weak convergence and strong convergence arguments to prove the main solvability theorem.

Theorem 4.1. *Let assumptions (A1), (A2), and (H1)–(H5) be satisfied. Then BDSDE (1) has a unique solution $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^p([t, T]; L^p_{\rho}(\mathbb{R}^d; \mathbb{R}^1)) \times M^2([t, T]; L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d))$.*

Proof. We start the proof from the weak convergence of BDSDEs (6). Taking $m = 2$ in Proposition 1, we know

$$\sup_n E \left[\int_t^T \int_{\mathbb{R}^d} |Y_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right] + \sup_n E \left[\int_t^T \int_{\mathbb{R}^d} |Z_s^{t,x,n}|^2 \rho^{-1}(x) dx ds \right] < \infty.$$

Define $U_s^{t,x,n} = f_n(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n})$ and $V_s^{t,x,n} = g(s, X_s^{t,x}, Y_s^{t,x,n})$, $s \geq t$. Using Lemma 1 again, we also have

$$\begin{aligned} & \sup_n E \left[\int_t^T \int_{\mathbb{R}^d} (|Y_s^{t,x,n}|^2 + |Z_s^{t,x,n}|^2 + |U_s^{t,x,n}|^2 + |V_s^{t,x,n}|^2) \rho^{-1}(x) dx ds \right] \\ & \leq \sup_n C_p E \left[\int_t^T \int_{\mathbb{R}^d} (1 + |f_0(s, X_s^{t,x})|^2 + |g(s, X_s^{t,x}, 0)|^2 \right. \\ & \quad \left. + |Y_s^{t,x,n}|^{2p} + |Z_s^{t,x,n}|^2) \rho^{-1}(x) dx ds \right] \\ & < \infty. \end{aligned}$$

Then, according to the Alaoglu lemma, we know that there exists a subsequence, still denoted by $(Y_s^{t,\cdot,n}, Z_s^{t,\cdot,n}, U_s^{t,\cdot,n}, V_s^{t,\cdot,n})$, converging weakly to a limit $(Y_s^{t,\cdot}, Z_s^{t,\cdot}, U_s^{t,\cdot}, V_s^{t,\cdot})$ in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1 \times \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^l)$, or equivalently $L^2(\Omega \times [t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^1) \times L_\rho^2(\mathbb{R}^d; \mathbb{R}^l))$. Now we take the weak limit in $L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$ to BDSDEs (6), we can verify that $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, V_s^{t,x})$ satisfies the following BDSDE:

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T U_r^{t,x} dr - \int_s^T \langle V_r^{t,x}, d\hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle.$$

For this, we need to check the weak convergence term by term. We only demonstrate the weak convergence of $\int_s^T \langle V_r^{t,x,n}, d\hat{B}_r \rangle$, and for the weak convergence of other terms one can refer to [11].

Take arbitrary $\mathcal{F}_{s,T}^{\hat{B}} \vee \mathcal{F}_{t,s}^W$ measurable $\eta \in L_\rho^2(\Omega \times [t, T] \times \mathbb{R}^d; \mathbb{R}^1)$. First note that η is $\mathcal{F}_{0,T-s}^B \vee \mathcal{F}_{t,s}^W$ measurable by setting $B_s = \hat{B}_{T-s} - \hat{B}_T$. For fixed $s \in [t, T]$ and $x \in \mathbb{R}^d$, $\eta(s, x) \in L^2(\Omega; \mathbb{R}^1)$ is $\mathcal{F}_{0,T-s}^B \vee \mathcal{F}_{t,s}^W$ measurable, hence there exist $\mathcal{F}_{0,T-s}^B$ measurable $\psi \in L^2(\Omega \times [t, T] \times \mathbb{R}^d \times [0, T-s]; \mathbb{R}^l)$ and $\mathcal{F}_{t,s}^W$ measurable $\phi \in L^2(\Omega \times [t, T] \times \mathbb{R}^d \times [t, T]; \mathbb{R}^d)$ s.t.

$$\eta(s, x) = E[\eta(s, x)] - \int_0^{T-t} \langle \psi(s, x, r), dB_r \rangle + \int_t^T \langle \phi(s, x, r), dW_r \rangle.$$

By the relationship between forward and backward Itô's integrals, we have

$$\int_t^T \langle \tilde{\psi}(s, x, r), d^\dagger \hat{B}_s \rangle = - \int_0^{T-t} \langle \psi(s, x, r), dB_r \rangle \quad \text{a.s.},$$

where $\tilde{\psi}(s, x, r) := \psi(s, x, T-r)$ is $\mathcal{F}_{s,T}^{\hat{B}}$ measurable. Therefore, $\tilde{\psi}$ and ϕ satisfy

$$\eta(s, x) = E[\eta(s, x)] + \int_t^T \langle \tilde{\psi}(s, x, r), d\hat{B}_r \rangle + \int_t^T \langle \phi(s, x, r), dW_r \rangle.$$

Noticing that for a.e. $s \in [t, T]$, $\tilde{\psi}(s, \cdot, \cdot) \in L^2(\Omega \times \mathbb{R}^d \times [t, T]; \mathbb{R}^l)$, $\phi(s, \cdot, \cdot) \in L^2(\Omega \times \mathbb{R}^d \times [t, T]; \mathbb{R}^d)$ and $\int_t^T \sup_n E[\int_s^T \int_{\mathbb{R}^d} |V_r^{t,x,n}|^2 \rho^{-1}(x) dx dr] ds < \infty$, by Lebesgue's dominated convergence theorem again, we obtain

$$\begin{aligned} & |E[\int_t^T \int_{\mathbb{R}^d} \int_s^T \langle V_r^{t,x,n} - V_r^{t,x}, d\hat{B}_r \rangle \eta(s, x) \rho^{-1}(x) dx ds]| \\ & = |\int_t^T \int_{\mathbb{R}^d} E[\int_s^T \langle V_r^{t,x,n} - V_r^{t,x}, \tilde{\psi}(s, x, r) \rangle dr] \rho^{-1}(x) dx ds| \\ & \leq \int_t^T |E[\int_s^T \int_{\mathbb{R}^d} \langle V_r^{t,x,n} - V_r^{t,x}, \tilde{\psi}(s, x, r) \rangle \rho^{-1}(x) dx dr]| ds \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

However, the weak convergence is not enough to prove the identification

$$U_s^{t,x} = f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}), \quad V_s^{t,x} = g(s, X_s^{t,x}, Y_s^{t,x}) \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

With the strongly convergent subsequence Y^n and Z^n obtained from Theorem 3.3 and Corollary 1, the identification is true. We do not involve the proofs here and one can refer to a similar proof in [11]. The identification also gives an end to the proof for the existence of solution of BDSDE (1). As for the uniqueness, it can be proved by a standard argument using Itô's formula. \diamond

For this kind of solvability problems, to obtain a strongly convergent subsequence is the key point. Once this is derived the solvability and the correspondence for BDSDE and SPDE can be obtained following a standard procedure. So the following corresponding result for SPDE is not a surprise and we omit its proof.

Theorem 4.2. *Define $u(t, x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of BDSDE (1) with the assumptions (A1), (A2), and (H1)–(H5), then $u(t, x)$ is the unique weak solution of SPDE (3). Moreover, let u be a representative in the equivalence class of the solution of the SPDE (3) in $S^p([t, T]; L_p^p(\mathbb{R}^d; \mathbb{R}^1))$ with $\sigma^* \nabla u \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$, then $u(t, x) = Y_t^{t,x}$ for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$ a.s. and*

$$u(s, X_s^{t,x}) = Y_s^{t,x}, \quad (\sigma^* \nabla u)(s, X_s^{t,x}) = Z_s^{t,x} \text{ for a.e. } s \in [t, T], \quad x \in \mathbb{R}^d \text{ a.s.}$$

REFERENCES

- [1] V. Bally and B. Saussereau, A relative compactness criterion in Wiener-Sobolev spaces and application to semi-linear stochastic PDEs, *J. Funct. Anal.*, **210** (2004), 465–515.
- [2] G. Da Prato, P. Malliavin and D. Nualart, Compact families of Wiener functionals, *C. R. Acad. Sci. Paris Ser. I Math.*, **315** (1992), 1287–1291.
- [3] C.R. Feng and H.Z. Zhao, Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding, *J. Funct. Anal.*, **262** (2012), 4377–4422.
- [4] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 1996.
- [5] E. Pardoux, BSDE's weak convergence and homogenization of semilinear PDE's, in *Nonlinear Analysis, Differential Equations and Control* (eds. F. Clarke and R. Stern), Kluwer Acad. Publ., Dordrecht (1999), 503–549.
- [6] E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, *Probab. Theory Rel.*, **98** (1994), 209–227.
- [7] S. Peszat, On a Sobolev space of functions of infinite number of variables, *Bull. Polish Acad. Sci. Math.*, **41** (1993), 55–60.
- [8] Q. Zhang and H.Z. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs, *J. Funct. Anal.*, **252** (2007), 171–219.
- [9] Q. Zhang and H.Z. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs with non-Lipschitz coefficients, *J. Differ. Equations*, **248** (2010), 953–991.
- [10] Q. Zhang and H.Z. Zhao, Probabilistic representation of weak solutions of partial differential equations with polynomial growth coefficients, *J. Theor. Probab.*, **25** (2012), 396–423.
- [11] Q. Zhang and H.Z. Zhao, SPDEs with polynomial growth coefficients and Malliavin calculus method, *Stoch. Proc. Appl.*, **123** (2013), 2228–2271.

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