RIESZ MEANS OF THE COUNTING FUNCTION OF THE LAPLACE OPERATOR ON COMPACT MANIFOLDS OF NON-POSITIVE CURVATURE

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ABSTRACT. Let (M, g) be a compact, d-dimensional Riemannian manifold without boundary. Suppose further that (M, g) is either two dimensional and has no conjugate points or (M, g) has non-positive sectional curvature. The goal of this note is to show that the long time parametrix obtained for such manifolds by Bérard can be used to prove a logarithmic improvement for the remainder term of the Riesz means of the counting function of the Laplace operator.

Let (M, g) be a closed Riemannian manifold. Denote by Δ the (geometric) Laplace operator on functions, given in local coordinates by

$$\Delta = -\sum_{i,k=1}^{d} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ik} \frac{\partial}{\partial x_k} \right),$$

where $|g|$ is the determinant of the metric g, g^{ik} are entries of the inverse metric. As usual we define the space of square integrable functions, $L^2(M)$, as the completion of the space of smooth functions, $C^\infty(M)$, with respect to the norm induced by the inner product

$$\langle f, g \rangle = \int_M f(x)\, \overline{g(x)}\, \mathrm{d}\mu(x).$$

Here $\mathrm{d}\mu$ denotes the Riemannian volume element of M, which is given in local coordinates by $\sqrt{|g|}\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d$. The Laplace operator is self-adjoint, non-negative and has compact resolvent. Therefore there is an orthonormal basis $\{\varphi_i \in L^2(M) : i \in \mathbb{N}_0\}$ consisting of eigenfunctions

$$\Delta \varphi_i = \lambda_i^2 \varphi_i, \quad \varphi_i \in C^\infty(M)$$

which is ordered such that $0 = \lambda_0 \leq \lambda_1 \leq \ldots$. Let $e_\lambda(x, y)$ be defined as the finite sum $\sum_{\lambda_i < \lambda} \varphi_i(x)\overline{\varphi_i(y)}$. The restriction of e_λ to the diagonal is called the local counting function and we will denote it by $N_x(\lambda) = e_\lambda(x, x)$. Integration of $N_x(\lambda)$ over M gives the counting function of the Laplace operator

$$N(\lambda) = \#\{i \mid \lambda_i < \lambda\}.$$

The local Weyl law states that

$$N_x(\lambda) = \frac{\omega_d}{(2\pi)^d} \lambda^d + O(\lambda^{d-1}),$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. This result was proved by Levitan in the case of closed Riemannian manifolds. His work was based on the study of the cosine transform of the spectral function of the Laplace operator. In 1968 Hörmander [9] generalised it to the case of pseudo-differential operators of order m. The term $O(\lambda^{d-1})$ can not be improved in general as the example of S^d shows. Obtaining better estimates of this error term under additional geometric assumptions is still an active area of research. Various improvements are known in the case of compact manifolds with negative sectional curvature or under assumptions on the nature of the dynamics of the geodesic flow. For example, it was shown by Duistermaat and Guillemin [3] that the assumption of the set of periodic trajectories in the cosphere bundle having Liouville measure zero implies that the $O\left(\lambda^{d-1}\right)$ may be replaced by $o\left(\lambda^{d-1}\right)$. The most significant for the purposes of this paper is the result of Bérard [1], who in 1977 obtained a logarithmic improvement for manifolds with non-positive sectional curvature:

$$N_x(\lambda) = \frac{\mathrm{Vol}(B_x^*)}{(2\pi)^d}\lambda^d + O\left(\frac{\lambda^{d-1}}{\log\lambda}\right),$$

where B_x^* is the unit ball in T_x^*M, i.e. $\mathrm{Vol}(B_x^*) = \mathrm{Vol}(B^d)$.

It is well known that regularized versions of the counting function have better asymptotic expansions. An example of a regularized counting function is the k-th Riesz means

$$R_k N_x(\lambda) = k\lambda^{-1}\int_0^\lambda (1-\tau\lambda^{-1})^{k-1} N_x(\tau)\,\mathrm{d}\tau, \quad k = 1, 2, \ldots.$$

It shown by Hörmander in [8] and [9] that the k-th Riesz means admits an asymptotic expansion with an error term of order $O\left(\lambda^{d-k-1}\right)$. Safarov showed in [13] that the assumption that the set of periodic trajectories in the cosphere bundle under the geodesic flow has Liouville measure zero implies that this error term may be replaced by $o\left(\lambda^{d-k-1}\right)$.

Another well known asymptotic expansion is that of the mollified counting function (see e.g. Duistermaat and Guillemin [3]). Namely, let $\rho \in S(\mathbb{R})$, $\hat\rho \in C_0^\infty(\mathbb{R})$, $\hat\rho(\xi) = 1$ for all ξ in a neighbourhood of zero, then

$$\rho * N_x(\lambda) \sim \sum_{i=0}^{\infty} a_i(x)\lambda^{d-i}. \tag{1}$$

whenever the support of $\hat\rho$ is sufficiently small. Here the coefficients are local densities and are related directly to the local heat kernel coefficients of the Laplace operator. Weyl's asymptotic formula is equivalent to the fact that

$$a_0(x) = \frac{\mathrm{Vol}(B^d)}{(2\pi)^d}.$$

It is known (see for example [13]) that the existence of such a full asymptotic expansion of the mollified counting function (1), independent of the geometric context, is enough to conclude that for $k < d$ we have

$$R_k N_x(\lambda) = \sum_{i=0}^{k} \frac{k!(d-i)!}{(d-i+k)!} a_i(x)\lambda^{d-i} + O\left(\lambda^{d-k-1}\right), \quad \text{as } \lambda \to +\infty. \tag{2}$$

(see also [5] for the relation of these coefficients to the heat kernel coefficients). For surfaces ($d = 2$) of constant negative curvature the Selberg trace formula can be used to prove a logarithmic improvement of this formula (see [6]):

$$R_1 N(\lambda) = \frac{\mathrm{Vol}(M)}{12\pi}\lambda^2 + a_2 + O((\log\lambda)^{-2}). \tag{3}$$

A direct combination of Bérard's asymptotic formula, the expansion (3), and the method described in [13] yields in case $k < d$:

$$R_k N_x(\lambda) = \sum_{i=0}^{k+1} \frac{k!(d-i)!}{(d-i+k)!} a_i(x)\lambda^{d-i} + O\left(\frac{\lambda^{d-k-1}}{\log\lambda}\right), \quad \text{as } \lambda \to +\infty.$$

In the case $k = 1$ and $d = 2$ this does however not reduce to Hejhal's estimate (3) but is weaker by a factor of $\log\lambda$. The purpose of this paper is to improve the above error estimate to cover (3) and thus to generalize this estimate to the case of possibly non-constant curvature, higher dimension, and higher Riesz means. Our main result is the following.

Theorem 0.1. *Let (M, g) be a compact d-dimensional smooth Riemannian manifold. Assume that either $d = 2$ and M has no conjugate points or that M has non-positive sectional curvature. Then for $k \in \{0, 1, 2, \ldots, d-1\}$*

$$R_k N_x(\lambda) = \sum_{i=0}^{k+1} \frac{k!(d-i)!}{(d-i+k)!} a_i(x) \lambda^{d-i} + O\left(\frac{\lambda^{d-k-1}}{(\log \lambda)^{k+1}}\right), \quad \text{as } \lambda \to +\infty,$$

where the densities $a_i(x)$ are the same as in (1) and may be calculated explicitly. For $k \geq d$ one has

$$R_k N_x(\lambda) = \sum_{i=0}^{d} \frac{k!(d-i)!}{(d-i+k)!} a_i(x) \lambda^{d-i} + O\left(\lambda^{-1+\varepsilon}\right), \quad \text{as } \lambda \to +\infty,$$

for $\varepsilon > 0$.

Of course integration with respect to x over M yields corresponding estimates for the counting function.

Apart from the theoretical significance estimates of the form given in Theorem 0.1 have practical applications in numerical computations of eigenvalues. Some algorithms, such as the method of particular solutions ([4, 2], or [14] on manifolds) produce a list of eigenvalues of the Laplace operator and one would then like to have a method to check whether or not eigenvalues are missing in this list. Whereas the error estimate in Weyl's law is too large to detect a missing eigenvalue, the error estimates in some of the higher Riesz means will be sensitive to a change in the counting function by a positive integer. Averaged versions of the Weyl law are being used in numerical computations of large sets of eigenvalues, for example of Maass eigenvalues (see e.g. [11] where such Weyl laws are derived from the Selberg trace formula for certain congruence groups and subsequently being used in this context). This method is sometimes referred to as Turing's method.

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1. Proof and main estimates

Throughout the text $\rho \in \mathcal{S}(\mathbb{R})$ will be be a real valued Schwartz function such that

(i): $\text{supp}(\hat\rho) \subset [-1, 1]$,

(ii): $\hat\rho(\xi) = 1$ for all $|\xi| < 1/2$, and

(iii): ρ is even.

Here $\hat\rho$ denotes the Fourier transform of ρ defined by

$$\hat\rho(\xi) = \int_{\mathbb{R}} \rho(t) e^{-it\xi} \, \mathrm{d}t.$$

For $T > 0$ we denote by ρ_T the rescaled function $\rho_T(t) = T\rho(Tt)$, so that $\hat\rho_T(t) = \hat\rho(t/T)$. We therefore have $\text{supp}(\hat\rho_T) \in [-T, T]$.
Then

$$N_x(\lambda) = N_x * \rho_\epsilon(\lambda) + [N_x * (\rho_T - \rho_\epsilon)](\lambda) + [N_x * (\delta - \rho_T)](\lambda), \tag{4}$$

where $\epsilon > 0$ is a small parameter, which is smaller than the injectivity radius at x. Under the stated assumptions the first term has a full asymptotic expansion i.e

$$N_x * \rho_\epsilon(\mu) \sim \sum_{i=0}^{\infty} a_i(x) \mu^{d-i} \tag{5}$$

for $\mu \to \infty$ as it was proved in [3] and [10]. Because N_x is supported on the positive semi-axes and ρ_ϵ is the Schwartz function $N_x * \rho_\epsilon(\mu)$ is rapidly decreasing as $\mu \to -\infty$.

The k-th Riesz means of the local counting function is then given by

$$
\begin{aligned}
R_k N_x(\lambda) &= \int_{-\infty}^{\lambda} (1-\tau\lambda^{-1})^k \, \mathrm{d}N_x(\tau) = \lambda^{-k} \int_{-\infty}^{\lambda} (\lambda-\tau)^k \, \mathrm{d}N(\tau) \\
&= \lambda^{-k} k! \; \chi_+^k * N_x'(\lambda) = \lambda^{-k} k! \; \chi_+^{k-1} * N_x(\lambda).
\end{aligned} \tag{6}
$$

Here $\chi_+^\alpha(r)$ is the analytic continuation of $r_+^\alpha/\Gamma(\alpha+1)$ in the parameter α, as described for example in [7]. Let us apply the Riesz means operator to (4):

$$
\begin{aligned}
(R_k N_x)(\lambda) &= \lambda^{-k} k! \Big[\left(\chi_+^{k-1} * N_x * \rho_\epsilon \right)(\lambda) + \left(\chi_+^{k-1} * N_x * (\rho_T - \rho_\epsilon) \right)(\lambda) \\
&+ \left(\chi_+^{k-1} * N_x * (\delta - \rho_T) \right)(\lambda) \Big].
\end{aligned} \tag{7}
$$

Convolution with the distribution χ_+^k may be understood as a repeated integral from $-\infty$ to λ, therefore

$$
\left[\chi_+^{k-1} * (N_x * \rho_\epsilon) \right](\lambda) = \int_{-\infty}^{\lambda} \int_{-\infty}^{\lambda_1} \cdots \int_{-\infty}^{\lambda_{k-1}} (N_x * \rho_\epsilon)(\lambda_k) \, \mathrm{d}\lambda_k \ldots \mathrm{d}\lambda_2 \mathrm{d}\lambda_1, \quad \text{for } k \geq 1.
$$

Together with the full asymptotic expansion (5) we obtain

$$
\lambda^{-k} k! \left[\chi_+^{k-1} * (N_x * \rho_\epsilon) \right](\lambda) = \sum_{i=0}^{d} \frac{(d-i)! a_i(x)}{(d-i+k)!} \lambda^{d-i} + O(\lambda^{-1+\varepsilon}) \tag{8}
$$

for any $\varepsilon > 0$ as $\lambda \to \infty$.

Our main result is derived from the following estimates of the various terms appearing in (4) and (7). For convenience we use the following notation

$$
\langle \lambda \rangle = (1 + |\lambda|).
$$

Proposition 1.1. *For fixed $\epsilon > 0$ and $k \geq 0$ there exists a constant $c > 0$ such that for all $T \geq 1$ we have*

$$
|(N_x * \chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda)| \leq e^{cT} \langle \lambda \rangle^{\frac{d-1}{2}}.
$$

Proposition 1.2. *For fixed $\epsilon > 0$ there exists $c > 0$ such that for all $T \geq 1$ we have*

$$
|N_x * (\delta - \rho_T)(\lambda)| \leq \frac{c}{T} \langle \lambda \rangle^{d-1} + e^{cT} \langle \lambda \rangle^{\frac{d-1}{2}}.
$$

We postpone the proof of these two propositions for the moment and show how they imply the result. The splitting (4), the expansion (5), Proposition 1.1 in the case $k = 0$ together with Proposition 1.2 immediately imply that there exists a $c > 0$ such that for $T \geq 1$:

$$
|N_x(\lambda) - H(\lambda) \sum_{i=0}^{d-1} a_i(x) \lambda^{d-i}| \leq \frac{c}{T} \langle \lambda \rangle^{d-1} + e^{cT} \langle \lambda \rangle^{\frac{d-1}{2}}. \tag{9}
$$

Here $H(\lambda)$ is the Heaviside step function defined by $H(\lambda) = 1$ for $\lambda \geq 0$ and $H(\lambda) = 0$ for $\lambda < 0$. Now we use a proposition derived by Safarov in [13], which we slightly adapt to our situation.

Proposition 1.3. *Suppose that $\nu_1, \nu_2 \in \mathbb{R}$ and suppose $a_0, a_1, \ldots, a_d \in \mathbb{R}$. Then there exists a constant $C > 0$ depending only on ρ, ν_1, ν_2, and $(a_i)_{i=1,\ldots,d}$ such that the following statement holds. Suppose N is a function of locally bounded variation that is supported in $[0, \infty)$, and assume that:*

$$
|N(\lambda) - H(\lambda) \sum_{i=0}^{d} a_i \lambda^{d-i}| \leq C_1 \langle \lambda \rangle^{\nu_1} + C_2 \langle \lambda \rangle^{\nu_2}.
$$

Then, for all $T \geq 1$ and $\lambda \geq 0$:

$$\left| \int_{-\infty}^{\lambda} N(\mu) - N * \rho_T(\mu) \, \mathrm{d}\mu \right| \leq \frac{C}{T}(C_1 \langle\lambda\rangle^{\nu_1} + C_2 \langle\lambda\rangle^{\nu_2} + 1).$$

Proof. Our assumptions on ρ imply that $\int \rho(t)\mathrm{d}t = 1$ and $\int \rho(t)t^k\mathrm{d}t = 0$ for all $k \in \mathbb{N}$. We have,

$$\begin{aligned}
&\int_{-\infty}^{\lambda} (N(\mu) - N * \rho_T(\mu)) \, \mathrm{d}\mu \\
&= \int \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} N(\tau)\mathrm{d}\tau\mathrm{d}t = \int \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} (N(\tau) - \sum_{i=0}^{d} a_i \tau^{d-i})\mathrm{d}\tau\mathrm{d}t \\
&= \int \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} (N(\tau) - H(\tau)\sum_{i=0}^{d} a_i \tau^{d-i})\mathrm{d}\tau\mathrm{d}t \\
&\quad - \int \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} H(-\tau)(\sum_{i=0}^{d} a_i \tau^{d-i})\mathrm{d}\tau\mathrm{d}t.
\end{aligned}$$

Since ρ is rapidly decreasing the modulus of the last term is bounded by $\frac{C_3}{T}$ for some $C_3 > 0$ depending only on ρ and the a_i. Therefore,

$$\left| \int_{-\infty}^{\lambda} (N(\mu) - N * \rho_T(\mu)) \, \mathrm{d}\mu \right| \leq \int \left| \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} (C_1 \langle\tau\rangle^{\nu_1} + C_2 \langle\tau\rangle^{\nu_2}) \, \mathrm{d}\tau \right| \mathrm{d}t + \frac{C_3}{T}.$$

Using the triangle inequality and the fact that $\langle\tau + \lambda\rangle^{\nu} \leq \langle\tau\rangle^{|\nu|}\langle\lambda\rangle^{\nu}$ one obtains

$$\int \left| \rho(t) \int_{\lambda - T^{-1}t}^{\lambda} \langle\tau\rangle^{\nu}\mathrm{d}\tau \right| \mathrm{d}t \leq \int \left| \rho(t) \int_{0}^{T^{-1}t} \langle\tau - \lambda\rangle^{\nu}\mathrm{d}\tau \right| \mathrm{d}t \leq T^{-1} \left(\int |\rho(t)t| \, \langle t\rangle^{|\nu|}\mathrm{d}t \right) \langle\lambda\rangle^{\nu}.$$

This shows the proposition with for all $\lambda \geq 0$ with constant

$$C = \int |\rho(t)t| \, \langle t\rangle^{\max\{\nu_1,\nu_2\}}\mathrm{d}t + C_3.$$

□

Repeated application of Proposition 1.3 to the estimate (9), using Proposition 1.1 and (5), shows that for any integer $k \geq 0$ there exists a $c > 0$ such that for $T \geq 1$:

$$\left| \left[\chi_+^{k-1} * (N_x * (\delta - \rho_T)) \right] (\lambda) \right| \leq \frac{c}{T^{k+1}} \langle\lambda\rangle^{d-1} + e^{cT} \langle\lambda\rangle^{\frac{d-1}{2}}.$$

When we substitute this estimate into (7), use the expansion (8), and Proposition 1.1 we get that

$$\left| R_k N_x(\lambda) - \sum_{i=0}^{d-1} \frac{k!(d-i)!}{(d-i+k)!} a_i \lambda^{d-i} \right| \leq \frac{c}{T^{k+1}} \langle\lambda\rangle^{d-1-k} + e^{cT} \langle\lambda\rangle^{\frac{d-1}{2}}$$

This estimate is valid for $T \geq 1$. If we take $T = \alpha \log \lambda$ for some small $\alpha > 0$ of obtain Theorem 0.1 for large λ.

The Laplace operator is non-negative and thus the local counting function is supported on the positive half line $\lambda \geq 0$. Let us define

$$N_x^{odd}(\lambda) := N_x(\lambda) - N_x(-\lambda), \quad N_x^{neg}(\lambda) := N_x(-\lambda).$$

It is clear that the functions just defined sum up to N_x. The convolution, $[(N_x^{odd}) * (\rho_T - \rho_\epsilon)](\lambda)$, admits the same asymptotics as $[(N_x) * (\rho_T - \rho_\epsilon)](\lambda)$ as $\lambda \to \infty$. Moreover,

$[(N_x^{odd}) * (\rho_T - \rho_\epsilon)](\lambda)$ is the restriction to the diagonal of the integral kernel of the operator

$$(2\pi)^{-1} \int_{\mathbb{R}} (\hat{\rho}_T(t) - \hat{\rho}_\epsilon(t)) t^{-1} \sin(t\lambda) \cos(t\sqrt{\Delta})\, \mathrm{d}t. \tag{10}$$

We have

$$\chi_+^{k-1} * (N_x * (\rho_T - \rho_\epsilon))(\lambda) = \left(N_x * \left(\chi_+^{k-1} * (\rho_T - \rho_\epsilon)\right)\right)(\lambda). \tag{11}$$

Let as usual $p_{\alpha,\beta}$ be the Schwartz space semi-norms defined by $p_{\alpha,\beta}(f) = \sup_x |x^\alpha \partial_x^\beta f|$. Then it easy to check that for all $T > 1$ we have

$$p_{\alpha,\beta}(\hat{\rho}_T - \hat{\rho}_\epsilon) \le C_{\alpha,\beta} T^\alpha.$$

Thus, since N_x is supported in the half line, $N_x * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda) = O(e^{cT}\lambda^{-\infty})$ as $\lambda \to -\infty$ for any $c > 0$. Hence, $N_x * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda)$ and $N_x^{odd} * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda)$ differ by a function of order $O(e^{cT}\lambda^{-\infty})$ as $\lambda \to \infty$ for any $c > 0$. We use the identity

$$\widehat{\chi_+^{k-1}}(\xi) = \frac{(-i)^k}{\sqrt{2\pi}} (\xi - i0)^{-k},$$

to express the function $N_x^{odd} * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda)$ as the restriction to the diagonal of the integral kernel of

$$(2\pi)^{-1} \mathrm{Re} \int_{\mathbb{R}} \frac{\hat{\rho}_T(t) - \hat{\rho}_\epsilon(t)}{(it)^{k+1}} e^{it\lambda} \cos(t\sqrt{\Delta})\, \mathrm{d}t.$$

To estimate this integral we will as usual be exploiting the properties of a suitable parametrix for the operator $e^{-it\sqrt{\Delta}}$. In our case we will use the parametrix for $\cos(t\sqrt{\Delta})$ that was constructed by Bérard in [1] and which we describe in the following. Let $\pi : \tilde{M} \to M$ be the universal cover of M and let Γ be its group of automorphisms, so that $M \cong \tilde{M}/\Gamma$. Denote by $\tilde{C}(t, x, y)$ the integral kernel of $\cos(t\sqrt{\tilde{\Delta}})$, where $\tilde{\Delta}$ is the Laplace operator on the non-compact complete manifold $\tilde{M}$, then

$$\tilde{C}(t, x, y) = C_0 \sum_{l=0}^{N} (-1)^l 4^{-l} u_l(x, y) |t| \left. \frac{\left(d(x, y)^2 - t^2\right)_-^{l-\alpha}}{\Gamma(l + 1 - \alpha)} \right|_{\alpha = \frac{d+1}{2}} + \tilde{\epsilon}_N(t, x, y),$$

where $d(x, y)$ denotes the distance between x and y on $\tilde{M}$. The regularizations of the distributions $x^\alpha/\Gamma(\frac{\alpha+1}{2})$, $x^\alpha/\Gamma(\alpha)$, $x^{-\alpha}$ are described for example in [7]. Moreover the functions u_l and $\Delta_y^m u_l$ all have at most exponential growth as $d(x, y)$ tends to infinity, i.e. for all l, m exists a $c > 0$ such that

$$|\Delta_y^m u_l(x, y)| \le c\, e^{c\, d(x,y)}, \tag{12}$$

Moreover, for $N \ge \lfloor d/2 \rfloor + 3$, the error term is continuous and bounded uniformly in x and y as follows

$$|\tilde{\epsilon}_N(t, x, y)| \le c_N\, e^{c_N |t|}. \tag{13}$$

The distributional kernel of $\cos(t\sqrt{\Delta})$ on M is given by:

$$C(t, x, y) = \sum_{\gamma \in \Gamma} \tilde{C}(t, x, \gamma y). \tag{14}$$

The integral kernel of $\cos(t\sqrt{\tilde{\Delta}})$ has the finite propagation speed property, i.e. it is supported where $d(x, y) \le t$. This property implies that for fixed x and y the number of terms in (14) is finite for every t. In fact, the assumptions on the curvature for $d > 2$ and on the absence of conjugate points for $d = 2$ imply that the number of terms in (14) is $O(e^{c|t|})$ for some $c > 0$ which depends on the geometry of the manifold. The integral kernels of $\cos(t\sqrt{\Delta})$, $(\rho_T - \rho_\epsilon)(t)$ and $\mathrm{Re}((it)^{-k-1} e^{it\lambda})$ are even functions with respect to the independent variable t. Therefore we may restrict our integration to the positive semi-axes and in order to get a bound on (10) we need to estimate terms of the following form:

$$\operatorname{Re}\int_0^\infty \frac{\hat\rho_T(t)-\hat\rho_\epsilon(t)}{(it)^k}\left.\frac{\left(d(x,\gamma x)^2-t^2\right)_-^{l-\alpha}}{\Gamma(l+1-\alpha)}\right|_{\alpha=\frac{d+1}{2}} e^{it\lambda}\,\mathrm{d}t \tag{15}$$

Note that for $\gamma=\mathrm{id}$ and odd dimension $d=2m-1$ the distribution $t^{l+\alpha}/\Gamma(l+\alpha+1)\big|_{\alpha=-m}$ is supported at the origin for $l<m$, therefore in this case the pairing above is 0, since $\hat\rho_T-\hat\rho_\epsilon$ is supported for $|t|>\epsilon/2$ otherwise for $\gamma=\mathrm{id}$ we have the Fourier transform of a $C_0^\infty(-T,T)$ function. In the following we use the notation $O_T(g(x))$ for $O(g(x))$ in case the implied constant can be chosen independent of T, i.e. $f(x,T)=O_T(g(x))$ if $|f(x,T)|\le C|g(x)|$ with C not dependent on T. The estimates will be based on the following two Lemata. Assume that η is an even real valued Schwartz function such that $\hat\eta\in C_0^\infty([-1,1])$ is an even Schwartz function and let $\hat\eta_T$ be the rescaled function $\hat\eta_T(t)=\hat\eta(t/T)$.

Lemma 1.4. *Let $\epsilon>0$ and $k\ge 0$. If $k>0$ we suppose furthermore that $\hat\eta-1$ vanishes of order k at zero. Then,*

$$\int_0^\infty (\hat\eta_T(t)-\hat\eta_\epsilon(t))\,t^{-k}e^{it\lambda}\mathrm{d}t = O_T(\lambda^{-\infty})$$

as $\lambda\to\infty$ for $T\ge 1$.

Proof. One checks directly that for any integer $m>0$ the L^1-norm of $\frac{\mathrm{d}^m}{\mathrm{d}t^m}(\hat\eta_T(t)-\hat\eta_\epsilon(t))t^{-k})$ is bounded in T for $T\ge 1$. The Lemma then follows by integration by parts. ☐

Lemma 1.5. *For any fixed $\epsilon>0$, $k\ge 0$, and $m\in\mathbb{R}$ there exists an $L>0$ such that*

$$T^{-L}\int_0^\infty \hat\eta_T(t)\,t^{-k}\,\frac{(R^2-t^2)_-^m}{\Gamma(m+1)}\,e^{i\lambda t}\,dt = O_{T,R}\left(1+|\lambda|^{-m-1}\right)$$

as $|\lambda|\to\infty$ for all $T\ge 1$ and R with $T\ge R>\epsilon$.

Proof. For $m\ge 0$ the estimate follows for $L=2m+1-k$ immediately from the support properties of $\hat\eta_T$ and the fact that $\hat\eta_T$ is bounded. It remains to show the estimate in case $m<0$. Therefore, assume $m<0$. Let $\phi_T\in C_0^\infty(\mathbb{R}_+)$ such that $\phi_T=0$ in a neighborhood of zero, $\phi_T=1$ on $[R,T]$ and $\phi_T=0$ on $[2T,\infty)$, such that $\phi_T^{(\beta)}\le C_\beta$ uniformly in T and d for $T\ge 1$ and $T\ge R>\epsilon$. Then one shows easily that the Schwartz semi-norms of $\phi_T\hat\eta_T t^{-k}(t+R)^m$ satisfy

$$p_{\alpha,\beta}\left(\phi_T\hat\eta_T t^{-k}(t+R)^m\right)\le C_{\alpha,\beta}T^\alpha$$

for $T\ge 1$ and $T\ge R>\epsilon$, where $C_{\alpha,\beta}$ is independent of T and R. If ψ_T is the Fourier transform of the function $\phi_T\hat\eta_T t^{-k}(t+R)^m$ we therefore have

$$p_{\alpha,\beta}\left(\psi_T\right)\le \tilde C_{\alpha,\beta}T^{\beta+2}$$

On the other hand the Fourier transform of

$$\hat\eta_T t^{-k}\frac{(R^2-t^2)_-^m}{\Gamma(m+1)}$$

is the convolution of ψ_T with the Fourier transform of $\frac{(R-t)_-^m}{\Gamma(m+1)}$. The Fourier transform of the distribution $\frac{(R-t)_-^m}{\Gamma(m+1)}$ can be computed explicitly and is a locally integrable function of order $O_R(\lambda^{-m-1})$ as $\lambda\to\infty$ To estimate this convolution we use the well known inequality

$$(1+|\mu-\lambda|)^{-m-1}\le(1+|\mu|)^{|m+1|}(1+|\lambda|)^{-m-1}, \tag{16}$$

and the fact that

$$\int \psi_T(\mu)(1+|\mu|)^{|m+1|}\mathrm{d}\mu$$

can be bounded by a multiple of $\sup_\mu \langle\mu\rangle^q |\psi_T(\mu)|$ for all $q > (|m+1|+1)$. It follows that the convolution is of order $O_{T,R}(T^2\lambda^{-m-1})$ and we may therefore choose $L = 2$. $\square$

Proof of Proposition 1.1:
For each $T \geq 1$ the number of non-zero terms in the sum (14) in finite, the number of terms grows at most exponentially fast with T. Moreover, the estimate (12) implies that there is a constant c_l independent on x such that

$$|u_l(x,\gamma x)| \leq c_l \exp(c_l T) \tag{17}$$

on the support of $\widehat{\rho_T}$. The above Lemmata, applied with $\eta(\xi) = \hat{\rho}(\xi)$, together with these growth estimates show that for some $c > 0$

$$N_x^{odd} * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda) \leq e^{cT}\langle\lambda\rangle^{\frac{d-1}{2}}, \tag{18}$$

for $T \geq 1$, This implies the statement of the Proposition. $\square$

Proof of Proposition 1.2:
We would now like to estimate $N_x * (\delta - \rho_T)$. This can be done as follows using a Fourier Tauberian theorem. Let $\tilde{\rho} \in \mathcal{S}(\mathbb{R})$ be a non-negative even Schwartz function such that the Fourier transform $\hat{\tilde{\rho}}$ is supported in the interval $[-1/2, 1/2]$ and such that $\hat{\tilde{\rho}}(0) = 1$. Let $\tilde{\rho}_T$ be the rescaled function defined by $\tilde{\rho}(t) = T\tilde{\rho}(Tt)$. Following Safarov ([12]) we define $\tilde{\rho}_{1,0} \in \mathcal{S}(\mathbb{R})$ by

$$\tilde{\rho}_{1,0}(t) = \int_t^\infty \tau\tilde{\rho}(\tau)\mathrm{d}\tau,$$

so that

$$\hat{\tilde{\rho}}_{1,0}(\xi) = -\frac{1}{\xi}\frac{d}{d\xi}\hat{\tilde{\rho}}(\xi).$$

Define $\tilde{\rho}_{T,0}$ by $\tilde{\rho}_{T,0}(t) = T\tilde{\rho}_{1,0}(Tt)$. Lemma 1.4 and Lemma 1.5 applied with $\eta(\xi) = \hat{\tilde{\rho}}_{1,0}$ and $k = 0$, together with the above growth estimates then imply the bound

$$N'_x * \tilde{\rho}_{T,0}(\lambda) \leq e^{c_1 T}\langle\lambda\rangle^{\frac{d-1}{2}} + c_1\langle\lambda\rangle^{d-1} \tag{19}$$

for all $T \geq 1$. The term $c_1\langle\lambda\rangle^{d-1}$ appears here because of the contribution of the identity element in Γ.

Under these conditions the Fourier Tauberian Theorem 1.3 in [12] states that

$$|N_x * (\delta - \tilde{\rho}_T)(\lambda)| \leq \frac{C}{T}N'_x * \tilde{\rho}_{T,0}(\lambda),$$

for all $T \geq 1$ with a constant $C > 0$ that does not depend on T. Therefore, there exists $c > 0$ such that for all $T \geq 1$:

$$|N_x * (\delta - \tilde{\rho}_T)(\lambda)| \leq \frac{c}{T}\langle\lambda\rangle^{d-1} + e^{cT}\langle\lambda\rangle^{\frac{d-1}{2}}.$$

The support properties of the Fourier transforms of ρ_T and $\tilde{\rho}_T$ imply that $\rho_T * \tilde{\rho}_T = \tilde{\rho}_T$. Hence, we have

$$|N_x * (\delta - \rho_T)(\lambda)| \leq |N_x * (\delta - \tilde{\rho}_T)(\lambda)| + |(N_x * (\delta - \tilde{\rho}_T) * \rho_T)(\lambda)|.$$

Using $\langle\lambda - \mu\rangle^\alpha \leq \langle\lambda\rangle^{|\alpha|}\langle\mu\rangle^\alpha$ one derives the bound

$$\int |\rho_T(\mu)|\langle\lambda - \mu\rangle^\alpha d\mu = O_T(\langle\lambda\rangle^\alpha).$$

Therefore,

$$|N_x * (\delta - \tilde{\rho}_T) * \rho_T(\lambda)| \leq \frac{C}{T}\left(c_3\langle\lambda\rangle^{d-1} + c_2 e^{c_1 T}\langle\lambda\rangle^{\frac{d-1}{2}}\right),$$

Summarizing, there exists a $c > 0$ such that for all $T \geq 1$ we have

$$[N_x * (\delta - \rho_T)] \leq \frac{c}{T} \langle \lambda \rangle^{d-1} + e^{cT} \langle \lambda \rangle^{\frac{d-1}{2}}. \tag{20}$$

This implies the proposition. ☐

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