Position-dependent random maps in one and higher dimensions

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POSITION DEPENDENT RANDOM MAPS IN ONE AND HIGHER DIMENSIONS

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Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. In this paper, we study random maps with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$. Sufficient conditions for the existence of an absolutely continuous invariant measure for random map with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$ are the main results of this note.

1. Introduction

Let $\tau_1, \tau_2, ..., \tau_K$ be a collection of transformations from X to X. Usually, the random map T is defined by choosing τ_k with constant probability p_k, $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$. The ergodic theory of such dynamical systems was studied in [9] and in [8] (See also [7]).

There is a rich literature on random maps with position dependent probabilities with $\tau_1, \tau_2, ..., \tau_K$ being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations $\tau_1, \tau_2, ..., \tau_K$ and position dependent probabilities $p_k(x)$, $k = 1, ..., K$, $p_k(x) > 0$, $\sum_{k=1}^{K} p_k(x) = 1$, i.e., the p_k's are functions of position. We point out that studying such dynamical systems was first introduced in [4] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [4] are applicable only when $\tau_1, \tau_2, ..., \tau_K$ are C^2 expanding transformations (see [4] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map T on $[a, b]$ under milder conditions (see section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map T on S, where S is a bounded domain of $\mathbb{R}^n$ (see section 6, Condition (C)) .

The paper is organized in the following way: In section 2, following the ideas of [4], we formulate the definition of a random map T with position dependent probabilities and introduce its Perron-Frobenius operator. In section 3, we prove properties of the Perron-Frobenius operator of T. In section 4, we prove the existence of an absolutely continuous invariant measure for T on $[a, b]$. In section 5, we give an example of a random map T which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A)

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and (B). In section 6, we prove the existence of an absolutely continuous invariant measure for T on a bounded domain of $\mathbb{R}^n$. In section 7, we give an example of a random map in $\mathbb{R}^n$ that preserves an absolutely continuous invariant measure.

2. PRELIMINARIES

Let $(X, \mathfrak{B}, \lambda)$ be a measure space, where λ is an underlying measure. Let $\tau_k : X \to X$, $k = 1, ..., K$ be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of X : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map $T = \{\tau_1, ...\tau_K; p_1(x), ...p_K(x)\}$ as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x)\chi_A(\tau_k(x)), \tag{2.1}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^{K}$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and χ_A denotes the characteristic function of the set A. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(X, \mathfrak{B})$ defined by

$$\begin{aligned} \mathbb{P}_*\mu(A) &= \int \mathbb{P}(x, A) d\mu(x) = \sum_{k=1}^{K} \int p_k(x)\chi_A(\tau_k(x)) d\mu(x) \\ &= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) d\mu(x) \end{aligned} \tag{2.2}$$

We say that measure μ is T-invariant iff $\mathbb{P}_*\mu = \mu$, i.e.,

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x), \qquad , A \in \mathfrak{B}. \tag{2.3}$$

If μ has density f with respect to λ, the $\mathbb{P}_*\mu$ has also a density which we denote by $P_T f$. By change of variables, we obtain

$$\begin{aligned} \int_A P_T f(x) d\lambda(x) &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) f(x) d\lambda(x) \\ &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_A p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda(x) \end{aligned} \tag{2.4}$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ. Since this holds for any measurable set A we obtain an a.e. equality:

$$(P_T f)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} \chi_{\tau_k(I_i)}(x) \tag{2.5}$$

or

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k}(p_k f)(x) \tag{2.6}$$

where P_{τ_k} is the Perron-Frobenius operator corresponding to the transformation τ_k (see [1] for details). We call P_T the Perron-Frobenius of the random map T. The main tool in this paper is the Perron-Frobenius of T which has very useful properties.

3. Properties of the Perron-Frobenius operator of T

The properties of P_T resemble the properties of the classical Perron-Frobenius operator of a single transformation.

Lemma 3.1. *P_T satisfies the following properties:*
(i) P_T is linear;
(ii) P_T is non-negative; i.e., $f \geq 0 \implies P_T f \geq 0$;
(iii) $P_T f = f \Leftrightarrow mu = f \cdot \lambda$ is T-invariant;
(iv) $\|P_T f\|_1 \leq \|f\|_1$, where $\|.\|_1$ denotes the L^1 norm;
(v) $P_{T\circ R} = P_T \circ P_R$. In particular, $P_T^N f = P_{T^N} f$.

Proof. The proofs of (i)-(iv) are analogous to that for single transformation. For the proof of (v), let T and R be two random maps corresponding to $\{\tau_1, \tau_2, ..., \tau_K; p_1, p_2, ..., p_K\}$ and $\{\zeta_1, \zeta_2, ..., \zeta_L; r_1, r_2, ..., r_L\}$ respectively. We define $\{\tau_k\}_{k=1}^K$ and $\{\zeta_l\}_{l=1}^L$ on a common partition $\mathcal{P}$. We have

$$
\begin{aligned}
P_R(P_T f) &= P_R\left(\sum_{k=1}^K P_{\tau_k}(p_k f)\right) = \sum_{l=1}^L \sum_{k=1}^K P_{\zeta_l}\left(r_l P_{\tau_k}(p_k f)\right) \\
&= \sum_{l=1}^L \sum_{k=1}^K \sum_{i=1}^q r_l(\zeta_{l,i}^{-1})[P_{\tau_k}(p_k f)](\zeta_{l,i}^{-1}) \frac{1}{J_{\zeta,l,i}(\zeta_{l,i}^{-1})} \chi_{\zeta_{l,i}(I_i)} \\
&= \sum_{k=1}^K \sum_{l=1}^L \sum_{j=1}^q \sum_{i=1}^q r_l(\zeta_{l,i}^{-1}) p_k(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1}) f(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1}) \\
&\qquad \times \frac{1}{J_{\tau,k,j}(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1})} \frac{1}{J_{\zeta,l,i}(\zeta_{l,i}^{-1})} \chi_{\tau_k(I_j)}(\zeta_{l,i}^{-1}) \chi_{\zeta_{l,i}(I_i)} \\
&= \sum_{k=1}^K \sum_{l=1}^L P_{\tau_k \circ \zeta_l}\left(p_k(\zeta_l) r_l f\right) = P_{T\circ R} f.
\end{aligned} \tag{3.1}
$$

□

4. The existence of absolutely continuous invariant measure on $[a, b]$

Let $(I, \mathfrak{B}, \lambda)$ be a measure space, where λ is normalized Lebesgue measure on $I = [a, b]$. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one and differentiable, non-singular transformations on a partition $\mathcal{P}$ of I : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Denote by $V(\cdot)$ the standard one dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on I equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_1$.

Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}$, $k = 1, \ldots, K$. We assume that
Condition (A): $\sum_{k=1}^K g_k(x) < \alpha < 1$, $x \in I$, and
Condition (B): $g_k \in BV(I)$, $k = 1, \ldots, K$.

Under the above conditions our goal is to prove:

$$V_I P_T^n f \leq A V_I f + B\|f\|_1 \tag{4.1}$$

for some $n \geq 1$, where $0 < A < 1$ and $B > 0$. The inequality (4.1) guarantees the existence of a T-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of operator P_T with all the consequences of this fact, see [1]. We will need a number of lemmas:

Lemma 4.1. *Let $f \in BV(I)$. Suppose $\tau : I \to J$ is differentiable and $\tau'(x) \neq 0$, $x \in I$. Set $\phi = \tau^{-1}$ and let $g(x) = \frac{p(x)}{|\tau'(x)|} \in BV(I)$. Then*

$$V_J(f(\phi)g(\phi)) \leq (V_I f + \sup_I f)(V_I g + \sup_I g).$$

Proof. First, note that we have dropped all the k, i indices to simplify the notation. Then, the proof follows in the same way as in Lemma 3 of [9]. □

Lemma 4.2. *Let T satisfy conditions (A) and (B). Then for any $f \in BV(I)$,*

$$V_I P_T f \leq A V_I f + B\|f\|_1, \tag{4.2}$$

where

$$A = 3\alpha + \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{I_i} g_k;$$

and

$$B = 2\beta\alpha + \beta \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{I_i} g_k,$$

where $\beta = \max_{1\leq i\leq q}(\lambda(I_i))^{-1}$.

Proof. First, we will refine partition $\mathcal{P}$ to satisfy additional condition. Let $\eta > 0$ be such that $\sum_{k=1}^{K}(g_k(x) + \varepsilon_k) < \alpha$ whenever $|\varepsilon_k| < \eta$, $k = 1, \ldots, K$. Since g_k, $k = 1, \ldots, K$ are of bounded variation we can find a finite partition $\mathcal{K}$ such that for any $k = 1, \ldots, K$

$$|g_k(x) - g_k(y)| < \eta,$$

for x, y in the same element of $\mathcal{K}$. Instead of the partition $\mathcal{P}$ we consider a join $\mathcal{P} \vee \mathcal{K}$. Without restricting generality of our considerations, we can assume that this is our original partition $\mathcal{P}$. Then, we have

$$\max_{1\leq i\leq q} \sum_{k=1}^{K} \sup_{x\in I_i} g_k(x) < \alpha. \tag{4.3}$$

We have $V_I(P_T f) = V_I(\sum_{k=1}^{K} P_{\tau_k}(p_k f))$. We will estimate this variation. Let $\phi_{k,i} = \tau_{k,i}^{-1}$, $k = 1, \ldots, K$, $i = 1, \ldots, q$. We have

$$\begin{aligned} V_I\left(\sum_{k=1}^{K} P_{\tau_k}(p_k f)\right) &= V_I\left(\sum_{k=1}^{K}\sum_{i=1}^{q} f(\phi_{k,i}) g_k(\phi_{k,i}) \chi_{\tau_k(I_i)}\right) \\ &\leq \sum_{k=1}^{K}\sum_{i=1}^{q} [|f(a_{i-1})||g_k(a_{i-1})| + |f(a_i)||g_k(a_i)|] \qquad (4.4) \\ &\quad + \sum_{k=1}^{K}\sum_{i=1}^{q} V_{\tau_k(I_i)}[f(\phi_{k,i}) g_k(\phi_{k,i})]. \end{aligned}$$

First, we estimate the first sum on the right hand side of (4.4):

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{i=1}^{q}[|f(a_{i-1})||g_k(a_{i-1})| + |f(a_i)||g_k(a_i)|] \\
&= \sum_{i=1}^{q}\left[|f(a_{i-1})|\left(\sum_{k=1}^{K}|g_k(a_{i-1})|\right) + |f(a_i)|\left(\sum_{k=1}^{K}|g_k(a_i)|\right)\right] \\
&\le \alpha\left(\sum_{i=1}^{q}(|f(a_{i-1})| + |f(a_i)|)\right) \\
&\le \alpha\left(\sum_{i=1}^{q}\left(V_{I_i}f + (\lambda(I_i))^{-1}\int_{I_i} f d\lambda\right)\right) = \alpha\left(V_I f + \beta\|f\|_1\right).
\end{aligned} \tag{4.5}
$$

We now estimate the second sum on the right hand side of (4.4). Using Lemma 4.1 we obtain:

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{i=1}^{q}V_{\tau_k(I_i)}[f(\phi_{k,i})g_k(\phi_{k,i})] \le \sum_{k=1}^{K}\sum_{i=1}^{q}\left(V_{I_i}f + \sup_{I_i} f\right)\left(V_{I_i}g_k + \sup_{I_i} g_k\right) \\
&\le \sum_{i=1}^{q}\left(2V_{I_i}f + \beta\int_{I_i} f d\lambda\right)\left(\max_{1\le i\le q}\sum_{k=1}^{K}\left(V_{I_i}g_k + \sup_{I_i} g_k\right)\right) \\
&\le (2V_I f + \beta\|f\|_1)\left(\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k + \alpha\right).
\end{aligned} \tag{4.6}
$$

Thus, using (4.5) and (4.6), we obtain

$$
V_I P_T f \le \left(3\alpha + \max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k\right)V_I f + \left(2\beta\alpha + \beta\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k\right)\|f\|_1. \tag{4.7}
$$

□

In the following two lemmas we show that constants α and $\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k$ decrease when we consider higher iterations T^n instead of T. The constant β obviously increases, but this is not important.

Lemma 4.3. *Let T be a random map which satisfies condition (A). Then, for $x \in I$,*

$$
\sum_{w\in\{1,2,\dots,K\}^N}\frac{p_w(x)}{|T_w'(x)|} < \alpha^N, \tag{4.8}
$$

where $T_w(x) = \tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)$ and $p_w(x) = p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x)$, define random map T^N.

Proof. We have

$$
T^N(x) = \tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)
$$

with probability

$$
p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x).
$$

The maps defining T^N may be indexed by $w \in \{1,2,\dots,K\}^N$. Set

$$
T_w(x) = \tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)
$$

where $w=(k_1,...,k_N)$, and

$$p_w(x)=p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x).$$

Then,

$$T'_w(x)=\tau'_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\tau'_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots\tau'_{k_1}(x).$$

Suppose that T satisfies condition (A). We will prove (4.8) using induction on N. For $N=1$, we have

$$\sum_{w\in\{1,2,...,K\}}\frac{p_w(x)}{|T'_w(x)|}<\alpha \tag{4.9}$$

by condition (A). Assume (4.8) is true for $N-1$. Then,

(4.10)

$$\begin{aligned}\sum_{w\in\{1,2,...,K\}^N}\frac{p_w(x)}{|T'_w(x)|}&=\sum_{\overline{w}\in\{1,2,...,K\}^{N-1}}\sum_{k=1}^{K}\frac{p_k(x)p_{\overline{w}}(\tau_k(x))}{|\tau'_k(x)||T'_{\overline{w}}(\tau_k(x))|}\\&\le\left(\sum_{k=1}^{K}\frac{p_k(x)}{|\tau'_k(x)|}\right)\left(\sum_{\overline{w}\in\{1,2,...,K\}^{N-1}}\frac{p_{\overline{w}}(\tau_k(x))}{|T'_{\overline{w}}(\tau_k(x))|}\right)<\alpha\cdot\alpha^{N-1}=\alpha^N.\end{aligned}$$

□

Lemma 4.4. *Let $g_w=\frac{p_w}{|T'_w|}$, where T_w and p_w are defined in Lemma 4.3, $w\in\{1,...,K\}^n$. Define*

$$W_1\equiv\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k,$$

and

$$W_n\equiv\max_{J\in\mathcal{P}^{(n)}}\sum_{w\in\{1,...,K\}^n}V_Jg_w,$$

where $\mathcal{P}^{(n)}$ is the common monotonicity partition for all T_w. Then, for all $n\ge1$

$$W_n\le n\alpha^{n-1}W_1, \tag{4.11}$$

where α is defined in condition (A).

Proof. We prove the lemma by induction on n. For $n=1$ the lemma is true by definition of W_n. Assume that the lemma is true for n, i.e.,

$$W_n\le n\alpha^{n-1}W_1. \tag{4.12}$$

Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < ... < x_l$ be a sequence of points in J. Then

(4.13)

$$\begin{aligned}
\sum_w \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| &= \sum_{j=0}^{l-1} \sum_{w \in \{1,\dots,K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)| \\
&\le \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)| \\
&\le \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j)| \\
&\quad + \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)| \\
&\le \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \sum_{\overline{w} \in \{1,\dots,K\}^n} g_{\overline{w}}(\tau_k(x_{j+1})) \\
&\quad + \sum_{j=0}^{l-1} \sum_{k=1}^{K} g_k(x_j) \sum_{\overline{w} \in \{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))| \\
&\le \alpha^n \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \\
&\quad + \alpha \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))| \\
&\le \alpha^n W_1 + \alpha W_n \le \alpha^n W_1 + n\alpha^n W_1 = (n+1)\alpha^n W_1.
\end{aligned}$$

We used condition (A) and lemma 4.3. □

Theorem 4.5. *Let T be a random map which satisfies conditions (A) and (B). Then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(I)$, see* [1].

Proof. Let N be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

$$\sum_{w \in \{1,\dots,K\}^N} g_w(x) < \alpha^N, \qquad x \in I.$$

We refine the partition $\mathcal{P}^{(N)}$ like in the proof of Lemma 4.2, to have

$$\max_{J \in \mathcal{P}^N} \sum_{w \in \{1,\dots,K\}^N} \sup_J g_w < \alpha^N.$$

Then, by lemma 4.2, we get

$$\|P_T^N f\|_{BV} \le A_N \|f\|_{BV} + B_N \|f\|_1, \tag{4.14}$$

where $B_N = \beta_N(2\alpha^N + W_N)$, $\beta_N = \max_{J \in \mathcal{P}^N} (\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]). □

Remark 4.6. It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \ge 1$.

Remark 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [6]. The proof of [6], which uses graph theoretic methods, goes through analogously in our case; i.e., when T is a random map with position dependent probabilities.

5. Example

We present an example of a random map T which does not satisfy the conditions of [4]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B).

Example 5.1. Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where

$$\tau_1(x) = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2} \\ x & \text{for } \frac{1}{2} < x \le 1 \end{cases}, \tag{5.1}$$

$$\tau_2(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} < x \le 1 \end{cases}; \tag{5.2}$$

and

$$p_1(x) = \begin{cases} \frac{2}{3} & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{1}{3} & \text{for } \frac{1}{2} < x \le 1 \end{cases}, \tag{5.3}$$

$$p_2(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{2}{3} & \text{for } \frac{1}{2} < x \le 1 \end{cases}. \tag{5.4}$$

Then, $\sum_{k=1}^{2} g_k(x) = \frac{2}{3} < 1$. Therefore, T satisfies conditions (A) and (B). Consequently, by theorem 4.5, T preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that τ_1, τ_2 are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} : \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. For such maps the Perron-Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

$$P_{\tau_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \qquad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{5.5}$$

Their invariant densities are $f_{\tau_1} = [0, 2]$ and $f_{\tau_2} = [2, 0]$. The Perron-Frobenius operator of the random map T is given by:

$$P_T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{5.6}$$

If the invariant density of T is $f = [f_1, f_2]$, normalized by $f_1 + f_2 = 2$ and satisfying equation $fP_T = f$, then $f_1 = \frac{2}{3}$ and $f_2 = \frac{4}{3}$.

6. The existence of absolutely continuous invariant measure in $\mathbb{R}^n$

Let S be a bounded region in $\mathbb{R}^n$ and λ_n be Lebesgue measure on S. Let $\tau_k : S \to S$, $k = 1, ..., K$ be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of S : $\mathcal{P} = \{S_1, ..., S_q\}$ and $\tau_{k,i} = \tau_k |_{S_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Let each S_i be a bounded closed domain having a piecewise C^2 boundary of finite $(n-1)$-dimensional measure. We assume that the faces of ∂S_i meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$

are piecewise C^1 functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at x. We assume:

Condition (C):

$$\max_{1\leq i\leq q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$

Let $\sup_{x\in\tau_{k,i}(S_i)}\|D\tau_{k,i}^{-1}(x)\| := \sigma_{k,i}$ and $\sup_{x\in S_i} p_k(x) := \pi_{k,i}$. Using smoothness of $D\tau_{k,i}^{-1}$'s and p_k's we can refine partition $\mathcal{P}$ to satisfy

Condition (C'):

$$\sum_{k=1}^{K}\max_{1\leq i\leq q}\sigma_{k,i}\pi_{k,i} < \sigma < 1$$

.

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T=\{\tau_1,...,\tau_K;p_1,...,p_K\}$. The main tool of this section is the multi-dimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f)=\int_{\mathbb{R}^n}\|Df\| = \sup\{\int_{\mathbb{R}^n} f\operatorname{div}(g)d\lambda_n : g=(g_1,...,g_n)\in C_0^1(\mathbb{R}^n,\mathbb{R}^n)\},$$

where $f\in L_1(\mathbb{R}^n)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^n,\mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3], Remark 2.14: If $f=0$ outside a closed domain A whose boundary is Lipschitz continuous, $f_{|A}$ is continuous, $f_{|\mathrm{int}(A)}$ is C^1, then

$$V(f)=\int_{\mathrm{int}(A)}\|Df\|d\lambda_n+\int_{\partial A}|f|d\lambda_{n-1},$$

where λ_{n-1} is the $n-1$-dimensional measure on the boundary of A. In this section we shall consider the Banach space (see [3], Remark 1.12),

$$BV(S)=\{f\in L_1(S): V(f)<+\infty\},$$

with the norm $\|f\|_{BV}=V(f)+\|f\|_1$. We adapt the following two lemmas from [5]. The proofs of Lemma 6.1 and Lemma 6.2 are exactly the same as in [5].

Lemma 6.1. *Consider $S_i\in\mathcal{P}$. Let x be a point in ∂S_i and $y=\tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let $J_{k,i}$ be the Jacobian of $\tau_{k|S_i}$ at x and $J_{k,i}^0$ be the Jacobian of $\tau_{k|\partial S_i}$ at x. Then*

$$\frac{J_{k,i}^0}{J_{k,i}}\leq\sigma_{k,i}.$$

□

Let us fix $1\leq i\leq q$. Let Z denote the set of singular points of ∂S_i. Let us construct at any $x\in Z$ the largest cone having a vertex at x and which lies completely in S_i. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S_i)=\min_{x\in Z}\theta(x).$$

Since the faces of ∂S_i meet at angles bounded away from 0, $\beta(S_i)>0$. Let $\alpha(S_i)=\pi/2+\beta(S_i)$ and

$$a(S_i)=|\cos(\alpha(S_i))|.$$

Now we will construct a C^1 field of segments L_y, $y \in \partial S_i$, every L_y being a central ray of a regular cone contained in S_i, with angle subtended at the vertex y greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$, where the minimal angle $\beta(S_i)$ is attained, defining L_y to be central rays of the largest regular cones contained in S_i. Then we extend this field of segments to C^1 field we want, making L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of L_y, $y \in \partial S_i$. By the compactness of ∂S_i we have

$$\delta(S_i) = \inf_{y\in\partial S_i} \delta(y) > 0.$$

Now, we shorten L_y of our field, making them all of the length $\delta(S_i)$.

Lemma 6.2. *For any S_i, $i = 1, \dots, q$, if f is a C^1 function on S_i, then*

$$\int_{\partial S_i} f(y) d\lambda_{n-1}(y) \le \frac{1}{a(S_i)} \left(\frac{1}{\delta(S_i)} \int_{S_i} f d\lambda_n + V_{\mathrm{Int(S_i)}}(f) \right).$$

□

Our main technical result is the following :

Theorem 6.3. *If T is a random map which satisfies Condition (C), then*

$$V(P_T f) \le \sigma(1 + 1/a)V(f) + (M + \frac{\sigma}{a\delta})\|f\|_1,$$

where $a = \min\{a(S_i) : i = 1, \dots, q\} > 0$, $\delta = \min\{\delta S_i, : i = 1, \dots, q\} > 0$, $M_{k,i} = \sup_{x\in S_i}(Dp_k(x) - \dfrac{DJ_{k,i}}{J_{k,i}} p_k(x))$ *and* $M = \sum_{k=1}^{K} \max_{1\le i\le q} M_{k,i}$.

Proof. We have $V(P_T f) \le \sum_{k=1}^{K} V(P_{\tau_k}(p_k f))$. We first estimate $V(P_{\tau_k}(p_k f))$. Let $F_{k,i} = \frac{f(\tau_{k,i}^{-1})p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}$, and $R_{k,i} = \tau_{k,i}(S_i)$, $i = 1, \dots, q$, $k = 1, \dots, K$. Then,

(6.1)
$$\begin{aligned}\int_{\mathbb{R}^n} \|DP_{\tau_k}(p_k f)\| d\lambda_n &\le \sum_{i=1}^{q} \int_{\mathbb{R}^n} \|D(F_{k,i}\chi_{R_i})\| d\lambda_n \\ &\le \sum_{i=1}^{q} \left(\int_{\mathbb{R}^n} \|D(F_{k,i})\chi_{R_i}\| d\lambda_n + \int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\| d\lambda_n \right).\end{aligned}$$

Now, for the first integral we have,

(6.2)
$$\begin{aligned}&\int_{\mathbb{R}^n} \|D(F_{k,i})\chi_{R_i}\| d\lambda_n = \int_{R_i} \|D(F_{k,i}p_k)\| d\lambda_n \\ &\le \int_{R_i} \|D(f(\tau_{k,i}^{-1}))\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\| d\lambda_n + \int_{R_i} \|f(\tau_{k,i}^{-1}) D\left(\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\right)\| d\lambda_n \\ &\le \int_{R_i} \|Df(\tau_{k,i}^{-1})\|\|D\tau_{k,i}^{-1}\| \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_n + \int_{R_i} \|f(\tau_{k,i}^{-1})\| \frac{M_k}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_n \\ &\le \sigma_{k,i}\pi_{k,i} \int_{S_i} \|Df\| d\lambda_n + M_k \int_{S_i} \|f\| d\lambda_n.\end{aligned}$$

For the second integral we have,

(6.3)
$$\int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\| d\lambda_n = \int_{\partial R_i} |f(\tau_{k,i}^{-1}))| \frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda_{n-1} = \int_{\partial S_i} |f| p_k \frac{J_{k,i}^0}{J_{k,i}} d\lambda_{n-1}.$$

By Lemma 4.3, $\frac{J_{k,i}^0}{J_{k,i}} \leq \sigma_{k,i}$. Using Lemma 4.2, we get:

(6.4)
$$\begin{aligned}\int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\| d\lambda_n &\leq \sigma_{k,i}\pi_{k,i} \int_{\partial S_i} |f| d\lambda_{n-1} \\ &\leq \frac{\sigma_{k,i}\pi_{k,i}}{a} V_{S_i}(f) + \frac{\sigma_{k,i}\pi_{k,i}}{a\delta} \int_{S_i} |f| d\lambda n.\end{aligned}$$

Using Condition (C'), summing first over i, we obtain

$$V(P_{\tau_k}(p_k f)) \leq (\max_{1\leq i\leq q} \sigma_{k,i}\pi_{k,i})(1+1/a)V(f) + (\max_{1\leq i\leq q} M_{k,i} + \frac{\max_{1\leq i\leq q} \sigma_{k,i}\pi_{k,i}}{a\delta})\|f\|_1,$$

and then, summing over k we obtain

$$V(P_T f) \leq \sigma(1+1/a)V(f) + (M + \frac{\sigma}{a\delta})\|f\|_1.$$

□

Theorem 6.4. *Let T be a random map which satisfies condition (C). If $\sigma(1 + 1/a) < 1$, then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(S)$, see* [1].

Proof. The proof of the theorem follows by the standard technique (see [1]). □

7. Example in $\mathbb{R}^2$

In this section, We present an example of a random map which satisfies condition (C) of theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

Example 7.1. Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where $\tau_1, \tau_2 : I^2 \to I^2$ defined by:

(7.1)
$$\tau_1(x_1, x_2) = \begin{cases} (3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{0 \leq x_1, x_2 \leq \frac{1}{3}\} \\ (3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{\frac{1}{3} < x_1 \leq \frac{2}{3}; 0 \leq x_2 \leq \frac{1}{3}\} \\ (3x_1 - 2, 2x_2) & \text{for } (x_1, x_2) \in S_3 = \{\frac{2}{3} < x_1 \leq 1; 0 \leq x_2 \leq \frac{1}{3}\} \\ (3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{0 < x_1 \leq \frac{1}{3}; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\ (3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{\frac{1}{3} < x_1, x_2 \leq \frac{2}{3}\} \\ (3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_6 = \{\frac{2}{3} < x_1 \leq 1; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\ (3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{0 \leq x_1 \leq \frac{1}{3}; \frac{2}{3} < x_2 \leq 1\} \\ (3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{\frac{1}{3} < x_1 \leq \frac{2}{3}; \frac{2}{3} < x_2 \leq 1\} \\ (3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_9 = \{\frac{2}{3} < x_1 \leq 1; \frac{2}{3} < x_2 \leq 1\} \end{cases},$$

$$\tau_2(x_1,x_2)=\begin{cases}(3x_1,3x_2) & \text{for } (x_1,x_2)\in S_1\\ (2-3x_1,3x_2) & \text{for } (x_1,x_2)\in S_2\\ (3x_1-2,3x_2) & \text{for } (x_1,x_2)\in S_3\\ (3x_1,3x_2-1) & \text{for } (x_1,x_2)\in S_4\\ (2-3x_1,3x_2-1) & \text{for } (x_1,x_2)\in S_5\\ (3x_1-2,3x_2-1) & \text{for } (x_1,x_2)\in S_6\\ (3x_1,3x_2-2) & \text{for } (x_1,x_2)\in S_7\\ (2-3x_1,3x_2-2) & \text{for } (x_1,x_2)\in S_8\\ (3x_1-2,3x_2-2) & \text{for } (x_1,x_2)\in S_9\end{cases}, \tag{7.2}$$

and

$$p_1(x)=\begin{cases}0.215 & \text{for } (x_1,x_2)\in S_1\\ 0.216 & \text{for } (x_1,x_2)\in S_2\\ 0.216 & \text{for } (x_1,x_2)\in S_3\\ 0.216 & \text{for } (x_1,x_2)\in S_4\\ 0.215 & \text{for } (x_1,x_2)\in S_5\\ 0.216 & \text{for } (x_1,x_2)\in S_6\\ 0.216 & \text{for } (x_1,x_2)\in S_7\\ 0.216 & \text{for } (x_1,x_2)\in S_8\\ 0.215 & \text{for } (x_1,x_2)\in S_9\end{cases},\quad p_2(x)=\begin{cases}0.785 & \text{for } (x_1,x_2)\in S_1\\ 0.784 & \text{for } (x_1,x_2)\in S_2\\ 0.784 & \text{for } (x_1,x_2)\in S_3\\ 0.784 & \text{for } (x_1,x_2)\in S_4\\ 0.785 & \text{for } (x_1,x_2)\in S_5\\ 0.784 & \text{for } (x_1,x_2)\in S_6\\ 0.784 & \text{for } (x_1,x_2)\in S_7\\ 0.784 & \text{for } (x_1,x_2)\in S_8\\ 0.785 & \text{for } (x_1,x_2)\in S_9\end{cases} \tag{7.3}$$

The derivative matrix of $(\tau_{1,i})^{-1}$, is

$$\begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{2}\end{pmatrix}, \tag{7.4}$$

and the derivative matrix of $(\tau_{2,i})^{-1}$, is

$$\begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix}-\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix}. \tag{7.5}$$

Therefore, the Euclidean matrix norm, $\|D(\tau_{1,i})^{-1}\|$ is $\frac{\sqrt{2}}{3}$, or $\frac{\sqrt{13}}{6}$ and the Euclidean matrix norm, $\|D(\tau_{2,i})^{-1}\|$ is $\frac{\sqrt{2}}{3}$. Then

$$\max_{1\le i\le q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| \le 0.216\frac{\sqrt{13}}{6}+0.785\frac{\sqrt{2}}{3}.$$

For this partition $\mathcal{P}$, we have $a=1$, which implies

$$\sigma(1+1/a)=2(0.216\frac{\sqrt{13}}{6}+0.785\frac{\sqrt{2}}{3})\approx 0.9998<1.$$

Therefore, by theorem 6.4, the random map T admits an absolutely continuous invariant measure. Notice that τ_1,τ_2 are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P}=\{S_1,S_2,\dots,S_9\}$. For such maps the Perron-Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron-Frobenius operator of the random map T is represented by the following matrix

$$M=\Pi_1 M_1+\Pi_2 M_2, \tag{7.6}$$

where M_1, M_2 are the matrices of P_{τ_1} and P_{τ_2} respectively, and Π_1, Π_2 are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then, M is given by

(7.7)

$$M = p_1\mathbf{Id}_9 \times \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

$$+ p_2\mathbf{Id}_9 \times \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix} = \begin{pmatrix} a & a & a & a & a & a & b & b & b \\ c & c & c & c & c & c & d & d & d \\ c & c & c & c & c & c & d & d & d \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \end{pmatrix},$$

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.215, 0.216, 0.216, 0.216, 0.215)$, $p_2 = (0.785, 0.784, 0.784, 0.784, 0.785, 0.784, 0.784, 0.784, 0.785)$, $\mathbf{Id}_9$ is 9×9 identity matrix and

$$\begin{aligned} a &= 0.12306 \\ b &= 0.087222 \\ c &= 0.12311 \\ d &= 0.087111 \\ e &= 0.11111. \end{aligned}$$

The invariant density of T is

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_{|S_i}, \quad i = 1, 2, \ldots, 9, \tag{7.8}$$

normalized by

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9, \tag{7.9}$$

and satisfying equation $fM = f$. Then, $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{6.29739}$ and $f_7 = f_8 = f_9 = \frac{0.29739}{3} f_1$.

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