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Quasi-linear PDEs and Forward-Backward Stochastic Differential Equations: Weak Solutions

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Abstract

In this paper, we study the existence, uniqueness and the probabilistic representation of the weak solutions of quasi-linear parabolic and elliptic partial differential equations (PDEs) in the Sobolev space $H_\rho^1(\mathbb{R}^d)$. For this, we study first the solutions of forward-backward stochastic differential equations (FBSDEs) with smooth coefficients, regularity of solutions and their connection with classical solutions of quasi-linear parabolic PDEs. Then using the approximation procedure, we establish their convergence in the Sobolev space to the solutions of the FBSDEs in the space $L_\rho^2(\mathbb{R}^d; \mathbb{R}^d) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$. This gives a connection with the weak solutions of quasi-linear parabolic PDEs. Finally, we study the unique weak solutions of quasi-linear elliptic PDEs using the solutions of the FBSDEs on infinite horizon.

Keywords: forward-backward stochastic differential equations; weak solutions; quasi-linear partial differential equations; probabilistic representation; parabolic; elliptic; infinite horizon.

1 Introduction

In this paper, we study the existence, uniqueness and the probabilistic representation of solutions of systems of quasi-linear second order parabolic (or elliptic) partial differential equations (PDEs). Consider the following parabolic type PDEs:

$$\frac{\partial}{\partial t}u + \mathcal{L}u + f(t, x, u, \sigma^*(t, x, u)\nabla u) = 0, \quad u(T, x) = h(x), \quad (1.1)$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^k$ with $u(t)$ being in an appropriate Sobolev space, which will be made clear later, and \mathcal{L} is a second order differential operator defined by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(t, x, u(t, x)) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x, u(t, x)) \frac{\partial}{\partial x_i}. \quad (1.2)$$

Quasi-linear PDEs arise in many physical and engineering problems and have been subject to intensive studies ([4], [9], [10], [14]). Due to the complexity of the equations, there are many difficulties in both analytic and probabilistic approaches.

The aim of this work is to study the weak solutions of (1.1) through a probabilistic approach by studying the forward-backward stochastic differential equations (FBSDEs):

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \quad 0 \leq t \leq T. \end{cases} \quad (1.3)$$

It is well-known that a solution of a linear parabolic (or elliptic) PDEs can be formulated as the expectation of a functional of the solutions of some stochastic differential equations, known as the Feynman-Kac formula. By introducing a kind of backward stochastic differential equations (BSDEs), Pardoux-Peng [20, 21] and Peng [24] obtained a probabilistic interpretation for a semi-linear parabolic (or elliptic) PDEs. The probabilistic interpretation:

$$u(t, x) = Y_t^{t,x} \quad (1.4)$$

establishes the connection between the classical and viscosity solutions of PDEs and the solutions of BSDEs (or FBSDEs), and provides a new insight into studying non-linear PDEs. Probabilistic representation of weak solutions of semi-linear PDEs in a Sobolev space was studied by Barles-Lesigne [6], Bally-Matoussi [5] and Zhang-Zhao [31, 32, 33, 34], and for Hamilton-Jacobi-Bellman equations by Wei-Wu-Zhao [27]. For the quasi-linear case, there are a few results about the viscosity solutions (Pardoux-Tang [23], Wu-Yu [28]). As far as we know, this paper is the first result on FBSDEs in $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$ and their connection with quasi-linear PDE (1.3) in the Sobolev space $H^1_\rho(\mathbb{R}^d)$.

FBSDEs were first considered by Antonelli who obtained an existence and uniqueness result over a small time duration by using the *Contraction Mapping Method* ([1]). Ma-Protter-Yong introduced the *Four Step Scheme* and proved the existence and uniqueness under some regularity assumptions on the coefficients and non-degeneracy of the forward equation in [15]. Several other results on a more general form of FBSDEs (σ allowed to depend on z) are given by Hu-Peng [11], Peng-Wu [26], based on stochastic Hamiltonian systems, under certain monotone conditions. Yong [29] generalized these results by introducing a more flexible type of monotone condition. Using homotopic technique, Yong developed a *Continuation Method* in [30]. Recently, Ma-Wu-Zhang-Zhang [16] integrated all these existing methods, and provided a unified approach.

Comparing all these works on FBSDEs, the balance between the regularity of the coefficients and the time duration is still a challenging problem. In fact, under Lipschitz conditions, one can only get an existence and uniqueness result over a small time duration (local result) by using a *Contraction Mapping Method* (e.g. Delarue [8]). For an arbitrary time duration (global result), one should consider more complicated assumptions by the *Four Step Scheme* or the *Continuation Method*. In this work we use a purely probabilistic method to study the FBSDEs instead of applying a PDEs approach. The advantage is that we can push the probabilistic method to solve FBSDEs beyond what analytic methods can offer, e.g. the infinite horizon case (Section 5). Our approach does not depend on results of PDEs. In Section 2, by using a *Contraction Mapping Method*, we give a global result under either of two classes of monotone-Lipschitz conditions. Meanwhile, the *Continuation Method* (Hu-Peng [11], Peng-Wu [26], Yong [29, 30]) can deal with some general FBSDEs (σ can depend on z). However, one cannot obtain the regularity of $Y_s^{t,x}$ (Lemmas 3.1 and 3.3) when σ depends on z , which is a necessary step

to connect with PDEs. On the other hand, the conditions from our method are weaker than conditions offered by the *Continuation Method* when σ is independent of z . Moreover, our result on infinite horizon FBSDEs gives the solutions of quasi-linear PDEs.

The difficulty of solving FBSDE (1.3) lies in the coupling between the forward and backward equations. It is a circular dependence of solutions of both equations, which need to be solved simultaneously rather than one after another. Such a difficulty was also pointed out in Pardoux-Tang [23]. We will construct mappings based on a monotonicity assumption. In fact, such assumptions were used in many other works such as Hu-Peng [11], Pardoux-Tang [23], Peng-Shi [25], Peng-Wu [26]. In Delarue [8], he assumed globally Lipschitz conditions with σ being non-degenerate with the help of the PDE method. But our work mainly uses the probabilistic method to solve the FBSDEs and we do not need σ being non-degenerate. A monotonicity condition is given in a weak sense and weak solutions of PDEs are obtained by solving FBSDEs in a function space.

In this paper, we study the solutions of PDE (1.1) in both classical sense (Section 3) and in the sense of weak solutions in a Sobolev space (Section 4). The latter is the main purpose of this paper. To do this, we need to study the classical solutions and use an approximation procedure to obtain weak solutions. Moreover, the norm equivalence result (Lemma 4.5), which plays an important role in this analysis, is new for FBSDEs. Finally, let us consider the following FBSDEs on the infinite horizon when all the coefficients in (1.3) are independent of t ,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}, Y_r^{t,x}) dW_r, \\ e^{-Ks} Y_s^{t,x} = \int_s^\infty e^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r. \end{cases} \quad (1.5)$$

Backward doubly stochastic differential equations (BDSDEs) and BSDEs of infinite horizon were studied by [31, 32, 34] and the stationary solutions for semi-linear stochastic partial differential equations (SPDEs) and PDEs were obtained. It is easy to see that the stationary solutions of parabolic PDEs turn out to be the solutions of elliptic type PDEs. Having this in mind, we extend results in [31] to FBSDE (1.5) by the Picard iteration procedure and prove that $Y_t^{t,x}$ is independent of t and gives the weak solutions of the following quasi-linear elliptic PDEs:

$$\mathcal{L}u + f(x, u, \sigma^*(x, u) \nabla u) = 0. \quad (1.6)$$

2 FBSDEs on finite horizon

Let (Ω, \mathcal{F}, P) be a probability space, and $T > 0$ be fixed. Let $\{W_t, 0 \leq t \leq T\}$ be a d -dimensional standard Brownian motion on (Ω, \mathcal{F}, P) , and \mathcal{N} denote the P -null sets of \mathcal{F} . For $t \leq s \leq T$, we define $\mathcal{F}_{t,s} = \sigma\{W_r - W_t; t \leq r \leq s\} \vee \mathcal{N}$, $\mathcal{F}_s = \mathcal{F}_{0,s}$.

Definition 2.1. Let \mathbb{S} be a Banach space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ -field \mathcal{S} . For $K \in \mathbb{R}^+$, we denote by $M^{2,-K}([0, \infty); \mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\phi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

- (i) $\phi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$;

$$(ii) \|\phi(s)\|_{M^{2,-K}([0,\infty);\mathbb{S})}^2 := \mathbb{E}[\int_0^\infty e^{-Ks} \|\phi(s)\|_{\mathbb{S}}^2 ds] < \infty.$$

Also we denote by $S^{2,-K}([0,\infty);\mathbb{S})$ the set of $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}/\mathcal{S}$ -measurable random processes $\{\psi(s)\}_{s \geq 0}$ with values on \mathbb{S} satisfying

(i) $\psi(s) : \Omega \rightarrow \mathbb{S}$ is \mathcal{F}_s -measurable for $s \geq 0$ and $s \mapsto \psi(s, \omega)$ is continuous in \mathbb{S} , P -a.s.;

$$(ii) \|\psi(s)\|_{S^{2,-K}([0,\infty);\mathbb{S})}^2 := \mathbb{E}[\sup_{s \geq 0} e^{-Ks} \|\psi(s)\|_{\mathbb{S}}^2] < \infty.$$

Similarly, for $K \in \mathbb{R}^+$, we can also define spaces $M^{2,K}([0,\infty);\mathbb{S})$, $S^{2,K}([0,\infty);\mathbb{S})$. When $K = 0$, and finite horizon $[0, T]$, we simply denote them by $M^2([0, T]; \mathbb{S})$ and $S^2([0, T]; \mathbb{S})$.

Remark 2.2. In this paper, we always take the Banach space \mathbb{S} to be Hilbert space $L_\rho^2(\mathbb{R}^d, \mathbb{R}^k)$ space with the inner product $\langle u_1, u_2 \rangle_{L_\rho^2} = \int_{\mathbb{R}^d} u_1(x) \cdot u_2(x) \rho^{-1}(x) dx$, a ρ -weighted L^2 space (or weighted Sobolev space). Here $\rho(x) = (1 + |x|^2)^q$, $q \geq 2$, is a weight function and $u_1(x) \cdot u_2(x)$ is the inner product of the Euclidean space \mathbb{R}^k . It is easy to see that $\rho(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous positive function satisfying $\int_{\mathbb{R}^d} |x|^p \rho^{-1}(x) dx < \infty$ for any $p \in (2, 2q - 1)$.

Now we consider the FBSDEs with finite horizon $[t, T]$, for $0 \leq t \leq T$,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad t \leq s \leq T, \end{cases} \quad (2.1)$$

where the functions $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$, $h : \mathbb{R}^d \rightarrow \mathbb{R}^k$. We also assume that b , σ , f and h are Borel measurable functions, $X_t^{t,x} = x$ is the initial point in \mathbb{R}^d , and for $0 \leq s \leq t$, we regulate $X_s^{t,x} = x$.

Definition 2.3. The process $(X^{\cdot,\cdot}, Y^{\cdot,\cdot}, Z^{\cdot,\cdot}) \in S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ is called a solution of Eq. (2.1) if for any $\varphi \in C_c^0(\mathbb{R}^d; \mathbb{R}^d)$ and $\tilde{\varphi} \in C_c^0(\mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{cases} \int_{\mathbb{R}^d} X_s^{t,x} \cdot \varphi(x) dx = \int_{\mathbb{R}^d} x \cdot \varphi(x) dx + \int_t^s \int_{\mathbb{R}^d} b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \cdot \varphi(x) dx dr \\ \quad + \int_t^s \int_{\mathbb{R}^d} \sigma(r, X_r^{t,x}, Y_r^{t,x}) \cdot \varphi(x) dx dW_r, \\ \int_{\mathbb{R}^d} Y_s^{t,x} \cdot \tilde{\varphi}(x) dx = \int_{\mathbb{R}^d} h(X_T^{t,x}) \cdot \tilde{\varphi}(x) dx + \int_s^T \int_{\mathbb{R}^d} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \cdot \tilde{\varphi}(x) dx dr \\ \quad - \int_s^T \left\langle \int_{\mathbb{R}^d} Z_r^{t,x} \tilde{\varphi}(x) dx, dW_r \right\rangle \quad \mathbb{P} - a.s.. \end{cases} \quad (2.2)$$

In this section, we consider two classes of monotone-Lipschitz conditions to study FBSDE (2.1) over an arbitrary time duration. Denote $g = (b, \sigma, h)$. Assume

(A.1): There exist constants $L \geq 0$, $\mu > 0$ satisfying $2\mu - K - 2L^2 - 7L - 1 > 0$ where $K > 2L^3 + L^2 + 5L + 1$ such that for any $t \in [0, T]$, $X_1, X_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$, $Y_1, Y_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$, $Z_1, Z_2 \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$, the function g and f satisfy

$$\|g(t, X_1(x), Y_1(x), Z_1(x)) - g(t, X_2(x), Y_2(x), Z_2(x))\|_{L_\rho^2}^2$$

$$\begin{aligned}
&\leq L(\|X_1(x) - X_2(x)\|_{L_\rho^2}^2 + \|Y_1(x) - Y_2(x)\|_{L_\rho^2}^2 + \|Z_1(x) - Z_2(x)\|_{L_\rho^2}^2); \\
&\quad \|f(t, X_1(x), Y_1(x), Z_1(x)) - f(t, X_2(x), Y_1(x), Z_2(x))\|_{L_\rho^2}^2 \\
&\leq L(\|X_1(x) - X_2(x)\|_{L_\rho^2}^2 + \|Z_1(x) - Z_2(x)\|_{L_\rho^2}^2); \\
&\quad \|f(t, 0, Y_1(x), 0)\|_{L_\rho^2}^2 \leq L(1 + \|Y_1(x)\|_{L_\rho^2}^2); \\
&\langle Y_1(x) - Y_2(x), f(t, X_1(x), Y_1(x), Z_1(x)) - f(t, X_1(x), Y_2(x), Z_1(x)) \rangle_{L_\rho^2} \leq -\mu \|Y_1(x) - Y_2(x)\|_{L_\rho^2}^2.
\end{aligned}$$

And for any $t \in [0, T]$, $(x, z) \in \mathbb{R}^d \times \mathbb{R}^{k \times d}$, the function $v \mapsto f(t, x, v, z)$ is continuous.

(A.2): The function $g = (f, \sigma, h)$ satisfies the Lipschitz condition in the same sense as in **(A.1)**, $b(t, x, y, z)$ is uniformly Lipschitz continuous w.r.t. (y, z) in the same sense as in **(A.1)**. There exists a constant $\mu > 0$ with $2\mu - K - L - \max\{4L + 1, 2L^2\} > 0$, where $K > 2L^2 + 7L + 1$, such that

$$\begin{aligned}
\langle X_1(x) - X_2(x), b(t, X_1(x), Y_1(x), Z_1(x)) - b(t, X_2(x), Y_1(x), Z_1(x)) \rangle_{L_\rho^2} &\leq -\mu \|X_1(x) - X_2(x)\|_{L_\rho^2}^2, \\
\|b(t, X_1(x), 0, 0)\|_{L_\rho^2}^2 &\leq L(1 + \|X_1(x)\|_{L_\rho^2}^2).
\end{aligned}$$

And for any $t \in [0, T]$, $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, the function $x \mapsto b(t, x, y, z)$ is continuous;

(A.3): The following holds

$$\int_0^T (|b(s, 0, 0, 0)|^2 + \|\sigma(s, 0, 0)\|^2 + |f(s, 0, 0, 0)|^2) ds < \infty,$$

where the Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $|x|$, and the matrix norm of $z \in \mathbb{R}^{k \times d}$ by $\|z\| := \sqrt{\text{tr}(zz^*)}$.

Before we give the existence and uniqueness results of FBSDE (2.1), we prove the following lemma as a preparation.

Lemma 2.4. *Under conditions **(A.1)** (or **(A.2)**) and **(A.3)**, if there exists $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying (2.2) for $t \leq s \leq T$, then $(X(\cdot), Y(\cdot)) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$, and $(X_s(x), Y_s(x), Z_s(x))$ is a solution of (2.1).*

Proof. In the following, we only prove our result under the conditions **(A.1)** and **(A.3)**, the other one can be done similarly. The proof is similar to [31]. Let us first see $Y_s(\cdot)$ is continuous with respect to s in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$. Since $(X_s(x), Y_s(x), Z_s(x))$ satisfies (2.2) for $t \leq s \leq T$, therefore

$$\begin{aligned}
\int_{\mathbb{R}^d} |Y_{s+\Delta s}(x) - Y_s(x)|^2 \rho^{-1}(x) dx &\leq C_p \int_{\mathbb{R}^d} \int_s^{s+\Delta s} |f(r, X_r(x), Y_r(x), Z_r(x))|^2 dr \rho^{-1}(x) dx \\
&\quad + C_p \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} Z_r(x) dW_r \right|^2 \rho^{-1}(x) dx.
\end{aligned}$$

For the stochastic integral part, it is trivial to see that for $0 \leq \Delta s \leq T - s$,

$$\left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \leq \sup_{0 \leq \Delta s \leq T-s} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 a.s..$$

And we can deduce that $\int_{\mathbb{R}^d} \sup_{0 \leq \Delta s \leq T-s} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx < \infty$ a.s. by Burkholder-Davis-Gundy's inequality and $Z(\cdot) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. So by the dominated convergence theorem, $\lim_{\Delta s \rightarrow 0+} \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx = 0$ a.s.. Similarly we can prove $\lim_{\Delta s \rightarrow 0-} \int_{\mathbb{R}^d} \left| \int_s^{s+\Delta s} \langle Z_r(x), dW_r \rangle \right|^2 \rho^{-1}(x) dx = 0$ for $t < s \leq T$. So $Y_s(\cdot)$ is continuous w.r.t. s in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$. Similarly, we can check that $X_s(\cdot)$ is continuous w.r.t. s in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)$. From conditions **(A.1)**, **(A.3)** and $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, we have that for a.e. $x \in \mathbb{R}^d$, $b(r, X_r(x), Y_r(x), Z_r(x))$, $\sigma(r, X_r(x), Y_r(x))$ and $f(r, X_r(x), Y_r(x), Z_r(x))$ are mean square integrable. We use the generalized Itô's formula to $\psi_M(X_r(x))$ and $\psi_M(Y_r(x))$, $\psi_M(x) = x^2 I_{\{-M \leq x < M\}} + 2M(x-M) I_{\{x \geq M\}} - 2M(x+M) I_{\{x < -M\}}$, take the spatial integration $\rho^{-1}(x) dx$ on both sides and apply stochastic Fubini theorem. Then we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi_M(X_s(x)) \rho^{-1}(x) dx \\ \leq & \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + \int_t^s \int_{\mathbb{R}^d} \psi'_M(X_r(x)) b(r, 0, 0, 0) \rho^{-1}(x) dx dr \\ & + \int_t^s \int_{\mathbb{R}^d} \psi'_M(X_r(x)) (b(r, X_r(x), Y_r(x), Z_r(x)) - b(r, 0, 0, 0)) \rho^{-1}(x) dx dr \\ & + 2 \int_t^s \int_{\mathbb{R}^d} \|\sigma(r, X_r(x), Y_r(x)) - \sigma(r, 0, 0)\|^2 \rho^{-1}(x) dx dr + 2 \int_t^s \int_{\mathbb{R}^d} \|\sigma(r, 0, 0)\|^2 \rho^{-1}(x) dx dr \\ & + \int_t^s \left\langle \int_{\mathbb{R}^d} \psi'_M(X_r(x)) \sigma(r, X_r(x), Y_r(x)) \rho^{-1}(x) dx, dW_r \right\rangle \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d} \psi_M(Y_s(x)) \rho^{-1}(x) dx \\ \leq & \int_{\mathbb{R}^d} \psi_M(h(X_T(x))) \rho^{-1}(x) dx + \int_s^T \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) f(r, 0, Y_r(x), 0) \rho^{-1}(x) dx dr \\ & + \int_s^T \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) (f(r, X_r(x), Y_r(x), Z_r(x)) - f(r, 0, Y_r(x), 0)) \rho^{-1}(x) dx dr \\ & - \int_s^T \left\langle \int_{\mathbb{R}^d} \psi'_M(Y_r(x)) Z_r(x) \rho^{-1}(x) dx, dW_r \right\rangle. \end{aligned}$$

Noting that $|\psi'_M(X_r(x))|^2 \leq 4|X_r(x)|^2$ and using Burkholder-Davis-Gundy's inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \psi_M(X_s(x)) \rho^{-1}(x) dx \\ \leq & \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + C \mathbb{E} \int_t^T \int_{\mathbb{R}^d} |b(r, 0, 0, 0)|^2 \rho^{-1}(x) dx dr + C \mathbb{E} \int_t^T |X_r(x)|^2 dr \\ & + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\ & + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2) \rho^{-1}(x) dx dr + C \mathbb{E} \int_t^T \|\sigma(r, 0, 0)\|^2 dr \end{aligned}$$

$$\begin{aligned}
& + C\mathbb{E}\sqrt{\int_t^T \left(\int_{\mathbb{R}^d} |\psi'_M(X_s(x))|^2 \rho^{-1}(x) dx \right) \left(\int_{\mathbb{R}^d} \|\sigma(r, X_r(x), Y_r(x))\|^2 \rho^{-1}(x) dx \right) dr} \\
& \leq \int_{\mathbb{R}^d} x^2 \rho^{-1}(x) dx + C\mathbb{E} \int_t^T (|b(r, 0, 0, 0)|^2 + \|\sigma(r, 0, 0)\|^2) dr \\
& \quad + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{5} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\psi'_M(X_s(x))|^2 \rho^{-1}(x) dx.
\end{aligned}$$

Since $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, taking the limit as $M \rightarrow \infty$ and applying the monotone convergence theorem, we have $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |X_s(x)|^2 \rho^{-1}(x) dx < \infty$. Due to $\psi_M(h(X_T(x))) \leq |h(X_T(x))|^2$ and $|\psi'_M(Y_r(x))|^2 \leq 4|Y_r(x)|^2$, by the similar estimate we have

$$\begin{aligned}
\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \psi_M(Y_s(x)) \rho^{-1}(x) dx & \leq C_L \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |X_s(x)|^2 \rho^{-1}(x) dx + C|h(0)|^2 \\
& \quad + C_L \mathbb{E} \int_t^T \int_{\mathbb{R}^d} (|X_r(x)|^2 + |Y_r(x)|^2 + \|Z_r(x)\|^2) \rho^{-1}(x) dx dr \\
& \quad + \frac{1}{5} \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |\psi'_M(Y_s(x))|^2 \rho^{-1}(x) dx.
\end{aligned}$$

Similarly, taking $M \rightarrow \infty$, we can see that $\mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} |Y_r(x)|^2 \rho^{-1} dx < \infty$. So $(X(\cdot), Y(\cdot)) \in S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$ follows. That is to say $(X(\cdot), Y(\cdot), Z(\cdot))$ is a solution of (2.1). \square

Now we present the existence and uniqueness results.

Theorem 2.5. *Under conditions (A.1) and (A.3), (2.1) has a unique solution.*

Proof. The proof is based on a contraction mapping from $M^{2, -K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2, -K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2, -K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ to itself. From this, we obtain a unique $M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ -valued solution since the two norms $\|\cdot\|_{M^{2, -K}([t, T]; \mathbb{S})}$ and $\|\cdot\|_{M^2([t, T]; \mathbb{S})}$ are equivalent. By Lemma 2.4 we know that the solution is in $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as well.

Before we prove the theorem, let us introduce the method to construct the solution $(X^{t, \cdot}, Y^{t, \cdot}, Z^{t, \cdot}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Consider the BSDE

$$Y_s^{t, x} = h(X_T^{t, x}) + \int_s^T f(r, X_r^{t, x}, Y_r^{t, x}, Z_r^{t, x}) dr - \int_s^T Z_r^{t, x} dW_r, \quad (2.3)$$

where $X_s^{t, x}$ is a diffusion process given by the solution of the SDE

$$X_s^{t, x} = x + \int_t^s b(r, X_r^{t, x}) dr + \int_t^s \sigma(r, X_r^{t, x}) dW_r. \quad (2.4)$$

Observe that the functions b and σ are time-dependent, so the forward SDE (2.4) is different from those in [5], [22], [31]. However, there exists a unique solution for SDE (2.4) (see [18] or [12]). For BSDE (2.3), we can use a similar method as in the proof of Theorem 3.5 in [31] to prove that there exists a unique solution $(Y^{\cdot,t}, Z^{\cdot,t}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Step 1: Construct the following mapping

$$\begin{aligned} \Xi &: M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ &\rightarrow M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ &(X^{\cdot,t}, Y^{\cdot,t}, Z^{\cdot,t}) \mapsto (\bar{X}^{\cdot,t}, \bar{Y}^{\cdot,t}, \bar{Z}^{\cdot,t}), \end{aligned}$$

Given $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, $(\bar{X}_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})$ is defined as follows: for any $s \in [t, T]$,

$$\bar{X}_s^{t,x} = x + \int_t^s b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) dW_r, \quad (2.5)$$

and

$$\bar{Y}_s^{t,x} = h(\bar{X}_T^{t,x}) + \int_s^T f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) dr - \int_s^T \bar{Z}_r^{t,x} dW_r. \quad (2.6)$$

We will prove that the map Ξ is a contraction. To this end, consider $(X^{\cdot,t}, Y^{\cdot,t}, Z^{\cdot,t}), (U^{\cdot,t}, V^{\cdot,t}, \mathcal{W}^{\cdot,t}) \in M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Set

$$(\bar{X}^{\cdot,t}, \bar{Y}^{\cdot,t}, \bar{Z}^{\cdot,t}) = \Xi(X^{\cdot,t}, Y^{\cdot,t}, Z^{\cdot,t}), \quad (\bar{U}^{\cdot,t}, \bar{V}^{\cdot,t}, \bar{\mathcal{W}}^{\cdot,t}) = \Xi(U^{\cdot,t}, V^{\cdot,t}, \mathcal{W}^{\cdot,t}).$$

Applying Itô's formula to $e^{-Ks} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$, taking spatial integration $\rho^{-1}(x) dx$ on both sides, applying stochastic Fubini theorem and taking expectation we get

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx + K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{X}_r^{t,x} - \bar{U}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ &= \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) - \sigma(r, \bar{U}_r^{t,x}, V_r^{t,x})\|^2 \rho^{-1}(x) dx dr \\ &\quad + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{X}_r^{t,x} - \bar{U}_r^{t,x}, \\ &\quad \quad b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - b(r, \bar{U}_r^{t,x}, V_r^{t,x}, \mathcal{W}_r^{t,x}) \rangle \rho^{-1}(x) dx dr. \end{aligned} \quad (2.7)$$

The first term on the RHS of (2.7) can be estimated by the Lipschitz condition. The second one can be estimated by using the Cauchy-Schwarz inequality, the Lipschitz condition and Young's inequality. It turn out that

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx + K \|\bar{X}^{\cdot,t} - \bar{U}^{\cdot,t}\|_{M^{2,-K}([t,T]; L_\rho^2)}^2 \\ &\leq (5L + \frac{1}{4}) \|\bar{X}^{\cdot,t} - \bar{U}^{\cdot,t}\|_{M^{2,-K}([t,T]; L_\rho^2)}^2 + (L + \frac{1}{4}) \|Y^{\cdot,t} - V^{\cdot,t}\|_{M^{2,-K}([t,T]; L_\rho^2)}^2 \\ &\quad + \frac{1}{4} \|Z^{\cdot,t} - \mathcal{W}^{\cdot,t}\|_{M^{2,-K}([t,T]; L_\rho^2)}^2. \end{aligned} \quad (2.8)$$

For BSDE (2.6), applying Itô's formula to $e^{-Ks}|\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$, we have

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{-Kt} |\bar{Y}_t^{t,x} - \bar{V}_t^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \|\bar{Z}_r^{t,x} - \bar{\mathcal{W}}_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\
& - K \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |\bar{Y}_r^{t,x} - \bar{V}_r^{t,x}|^2 \rho^{-1}(x) dx dr \\
& = \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx + 2 \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} \langle \bar{Y}_r^{t,x} - \bar{V}_r^{t,x}, \\
& \quad f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) - f(r, \bar{U}_r^{t,x}, \bar{V}_r^{t,x}, \bar{\mathcal{W}}_r^{t,x}) \rangle \rho^{-1}(x) dx dr.
\end{aligned} \tag{2.9}$$

Note that, we can also have following estimate from (2.7)

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
& \leq (2L^2 + L + \frac{1}{2L}) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 + (L + \frac{1}{2L}) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 \\
& \quad + \frac{1}{2L} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2.
\end{aligned}$$

By the Lipschitz condition and the above result, the first term on the RHS of (2.9) can be estimated as

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\
& \leq L \mathbb{E} \int_{\mathbb{R}^d} e^{-KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx \\
& \leq (2L^3 + L^2 + \frac{1}{2}) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 + (L^2 + \frac{1}{2}) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 \\
& \quad + \frac{1}{2} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2.
\end{aligned}$$

And we can use the monotonicity condition and the Lipschitz condition of f , the Cauchy-Schwarz inequality and Young's inequality to estimate the second term. It turn out that (2.9) gives

$$\begin{aligned}
& \|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 - K \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 \\
& \leq (2L^3 + L^2 + \frac{1}{2}) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 + (L^2 + \frac{1}{2}) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 \\
& \quad + \frac{1}{2} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 + (-2\mu + 5L) \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 \\
& \quad + \frac{1}{5} \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2 + \frac{1}{5} \|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2.
\end{aligned} \tag{2.10}$$

Step 2: Now let us construct the contraction mapping. To simplify notation, we denote

$$\begin{aligned}
\bar{A} &= \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2, A = \|X^{t,\cdot} - U^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2, \\
\bar{B} &= \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2, B = \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2, \\
\bar{C} &= \|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2, C = \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,-K}([t,T];L_\rho^2)}^2.
\end{aligned} \tag{2.11}$$

Then (2.8) and (2.10) lead to

$$(K - 2L^3 - L^2 - 5L - \frac{19}{20})\bar{A} + (2\mu - K - 5L)\bar{B} + \frac{4}{5}\bar{C} \leq (\frac{3}{4} + L + L^2)B + \frac{3}{4}C.$$

It turns out that

$$\left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(\frac{2\mu - K - 5L}{\frac{4}{5}} \right) \bar{B} + \bar{C} \leq \frac{15}{16} \left\{ \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}.$$

We assume $1 + \frac{4}{3}L + \frac{4}{3}L^2 \leq \frac{2\mu - K - 5L}{\frac{4}{5}}$ and $K - 2L^3 - L^2 - 5L - \frac{19}{20} > 0$, then we have

$$\begin{aligned} & \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) \bar{A} + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) \bar{B} + \bar{C} \\ & \leq \frac{15}{16} \left\{ \left(\frac{K - 2L^3 - L^2 - 5L - \frac{19}{20}}{\frac{4}{5}} \right) A + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}. \end{aligned}$$

Thus the map Ξ is a contraction from $M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Note that the two norms $\|\cdot\|_{M^{2,-K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent. Consequently, Banach's fixed point theorem leads to that (2.1) has a unique solution $(X^{\cdot, \cdot}, Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. By Lemma 2.4, the solution $(X^{\cdot, \cdot}, Y^{\cdot, \cdot}, Z^{\cdot, \cdot})$ is in $S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as well.

Finally, for $s \in [0, t]$, we regulate $X_s^{t,x} = x$, and (2.1) is equivalent to the following,

$$X_s^{t,x} = x, \quad Y_s^x = Y_t^{t,x} + \int_s^t f(r, x, Y_r^x, 0) dr, \quad Z_s^x = 0. \quad (2.12)$$

Here $Y_t^{t,x}$ is an \mathcal{F}_t -measurable random vector, and therefore is deterministic. In this case, (2.12) is a simple BSDE. By a similar method, we can obtain process $(X_\cdot, Y_\cdot, Z_\cdot) \in S^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, is the unique solution of (2.12). To unify the notation, we define $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) = (X_s^x, Y_s^x, Z_s^x)$ when $s \in [0, t]$ and extend the solution to $[0, T]$. \square

Theorem 2.6. *Under conditions (A.2) and (A.3), (2.1) has a unique solution.*

Proof. It is natural to consider $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ norm to set up our contraction mapping since the norms $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent as well. In this case, after applying Itô's formula to the forward equation, the coefficient of \bar{A} is $-K - 5L - \frac{1}{5}$ which is definitely negative. So we should introduce the monotonicity condition in (A.2) to cover this negative part, which could be positive if μ is big enough. On the other hand, the way we treat $|h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x})|^2$ is also different from the one in the proof of Theorem 2.5. In fact, we require $2\mu - K - 2L^2 - L \geq 0$ to enable us to estimate $|h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x})|^2$, so that the desired contraction can be obtained. The proof is similar to that of Theorem 2.5, so we only give a sketch here.

Construct following mapping

$$\Xi : M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$$

$$\begin{aligned} &\rightarrow M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ &(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}), \end{aligned}$$

there the mapping is exactly the same as in the proof of Theorem 2.5.

For forward SDE (2.5), applying Itô's formula to $e^{Ks}|\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$, taking integration $\rho^{-1}(x)dx$, applying stochastic Fubini theorem and taking expectation, using the Cauchy-Schwarz inequality, monotone-Lipschitz condition **(A.2)** and Young's inequality, we have

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx + (2\mu - K - 5L) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 \\ &\leq (L + \frac{1}{4}) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{4} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2. \end{aligned} \quad (2.13)$$

For BSDE (2.6), we apply Itô's formula to $e^{Ks}|\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$. In order to estimate the term $\mathbb{E} \int_{\mathbb{R}^d} e^{KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx$, we need the following result that is different from (2.13)

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} e^{KT} |\bar{X}_T^{t,x} - \bar{U}_T^{t,x}|^2 \rho^{-1}(x) dx + (2\mu - K - 2L^2 - L) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 \\ &\leq (L + \frac{1}{2L}) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{2L} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2. \end{aligned}$$

As $2\mu - K - 2L^2 - L \geq 0$, so

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^d} e^{KT} \left| h(\bar{X}_T^{t,x}) - h(\bar{U}_T^{t,x}) \right|^2 \rho^{-1}(x) dx \\ &\leq (\frac{1}{2} + L^2) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{2} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2. \end{aligned}$$

Similarly, we can use the Cauchy-Schwarz inequality and Young's inequality to estimate the other terms. Finally, from BSDE (2.6), we have

$$\begin{aligned} &\|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + K \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 \\ &\leq (\frac{1}{2} + L^2) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{2} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 \\ &\quad + 5L \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{5} \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 \\ &\quad + \frac{1}{5} \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2 + \frac{1}{5} \|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,K}([t,T]; L_\rho^2)}^2. \end{aligned} \quad (2.14)$$

Now we construct the contraction mapping. We adopt the similar notation as in (2.11) with a replacement of the space $M^{2,-K}([t, T]; L_\rho^2)$ by $M^{2,K}([t, T]; L_\rho^2)$. Then (2.13) and (2.14) lead to

$$(2\mu - K - 5L - \frac{1}{5})\bar{A} + (K - 5L - \frac{1}{5})\bar{B} + \frac{4}{5}\bar{C} \leq (\frac{3}{4} + L + L^2)B + \frac{3}{4}C.$$

As $1 + \frac{4}{3}L + \frac{4}{3}L^2 \leq \frac{K-5L-\frac{1}{5}}{\frac{4}{5}}$ and $2\mu - K - 5L - \frac{1}{5} > 0$, so we have

$$\left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) \bar{A} + (1 + \frac{4}{3}L + \frac{4}{3}L^2) \bar{B} + \bar{C}$$

$$\leq \frac{15}{16} \left\{ \left(\frac{2\mu - K - 5L - \frac{1}{5}}{\frac{4}{5}} \right) A + \left(1 + \frac{4}{3}L + \frac{4}{3}L^2 \right) B + C \right\}.$$

Thus the map Ξ is a contraction from $M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,K}([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Note that the two norms $\|\cdot\|_{M^{2,K}([t, T]; \cdot)}$ and $\|\cdot\|_{M^2([t, T]; \cdot)}$ are equivalent. Consequently, Banach's fixed point theorem leads to that (2.1) has a unique solution $(X^{\cdot, \cdot}, Y^{\cdot, \cdot}, Z^{\cdot, \cdot}) \in M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([t, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Finally following a similar proof of Theorem 2.5 we can extend this result from $[t, T]$ to $[0, T]$. \square

3 Regularity of solutions

The purpose of this section is to find the unique classical solution of parabolic PDE (1.1) through the results of FBSDE (1.3). For this, we strengthen our conditions in L_ρ^2 sense to the pointwise sense and study the regularity of the solution of FBSDE (1.3), and show that $u(t, x)$ in (1.4) which is expressed in terms of the solution of FBSDE (1.3) solves quasi-linear parabolic PDE (1.1). Note when the function b depends on z , the regularity problem has not been solved.

Let us first repeat some notation. For $r \in \mathbb{N}$, $C^r(\mathbb{R}^m; \mathbb{R}^n)$, $C_{l,b}^r(\mathbb{R}^m; \mathbb{R}^n)$ denote respectively the set of functions of class C^r from \mathbb{R}^m into \mathbb{R}^n , the set of C^r -functions whose partial derivatives of order less than or equal to r are bounded (and hence the function itself grows at most linearly at infinity). And we set the following conditions:

(B.0): For any $s \in [0, T]$, $b(s, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $\sigma(s, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$; $f(s, \cdot, \cdot, \cdot) \in C_{l,b}^3(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $h \in C_{l,b}^3(\mathbb{R}^d; \mathbb{R}^k)$.

(B.1): Denote $g = (b, \sigma, h)$. Assume there exists a constant $L \geq 0$ such that for any $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k \times d}$,

$$\begin{aligned} |g(t, x_1, y_1) - g(t, x_2, y_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ |f(t, x_1, y_1, z_1) - f(t, x_2, y_1, z_2)|^2 &\leq L(|x_1 - x_2|^2 + \|z_1 - z_2\|^2), \\ |f(t, 0, y, 0)|^2 &\leq L(1 + |y|^2). \end{aligned}$$

There exists positive constants μ , C_L and C'_L , with C_L , C'_L only depending on L , such $2\mu > K + C_L$ and $K > C'_L$ such that

$$\langle y_1 - y_2, f(t, x, y_1, z) - f(t, x, y_2, z) \rangle \leq -\mu|y_1 - y_2|^2,$$

(B.2): Function $g = (f, \sigma, h)$ satisfies the Lipschitz condition in the sense of **(B.1)**, $b(t, x, y)$ is uniformly Lipschitz continuous w.r.t. y in the sense of **(A.1)**. Moreover, there exists positive constants μ , C_L and C'_L , where C_L , C'_L only depending on L , with $2\mu > K + C_L$ and $K > C'_L$ such that

$$\langle x_1 - x_2, b(t, x_1, y_1) - b(t, x_2, y_1) \rangle \leq -\mu|x_1 - x_2|^2, \quad |b(t, x_1, 0)|^2 \leq L(1 + |x_1|^2).$$

(B.3): For some constant $p > 2$,

$$\int_0^T (|b(s, 0, 0)|^p + \|\sigma(s, 0, 0)\|^p + |f(s, 0, 0, 0)|^p) ds < \infty.$$

Lemma 3.1. *Under conditions (B.1) (or (B.2)) and (B.3), FBSDE (1.3) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$. Moreover, there exists $C_{p,L,\mu,T} > 0$ depending on p, L, μ and T such that*

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p). \quad (3.1)$$

Proof. By using a similar method to that of the proof of Theorem 2.5 (or Theorem 2.6), it is easy to see that, for any $t \in [0, T]$, $x \in \mathbb{R}^d$, FBSDE (1.3) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \in S^2([0, T]; \mathbb{R}^d) \otimes S^2([0, T]; \mathbb{R}^k) \otimes M^2([0, T]; \mathbb{R}^{k \times d})$. In the following, we only consider conditions (B.1) and (B.3). The result still holds under conditions (B.2) and (B.3).

Step 1: First we apply Itô's formula to $(|X_r^{t,x}|^2)^{\frac{p}{2}}$ to yield that

$$\begin{aligned} |X_s^{t,x}|^p &= |x|^p + p \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle dr \\ &\quad + \frac{p}{2} \int_t^s |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\ &\quad + \frac{p}{2} (p-2) \int_t^s |X_r^{t,x}|^{p-4} \langle \sigma \sigma^*(r, X_r^{t,x}, Y_r^{t,x}) X_r^{t,x}, X_r^{t,x} \rangle dr \\ &\quad + p \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle. \end{aligned} \quad (3.2)$$

As the stochastic integral has zero expectation, using Hölder's inequality and Young's inequality, we can deduce that

$$\begin{aligned} \mathbb{E} |X_s^{t,x}|^p &\leq |x|^p + C_{p,L} \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr + C_p \mathbb{E} \int_t^T (|b(r, 0, 0)|^p + \|\sigma(r, 0, 0)\|^p) dr \\ &\leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \end{aligned}$$

Moreover, the last term in (3.2) can be estimated by the Burkholder-Davis-Gundy inequality as follows

$$\begin{aligned} &p \mathbb{E} \sup_{t \leq s \leq T} \left| \int_t^s |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \rangle \right| \\ &\leq C_p \mathbb{E} \sqrt{\int_t^T |X_r^{t,x}|^p |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr} \\ &\leq C_p \mathbb{E} \sqrt{\sup_{0 \leq r \leq T} |X_r^{t,x}|^p \int_t^T |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr} \\ &\leq \frac{C_p}{N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + C_{p,L,N} \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr + C_p \mathbb{E} \int_t^T \|\sigma(r, 0, 0)\|^p dr. \end{aligned}$$

Here we can choose N such that $\frac{C_p}{N} < \frac{1}{2}$. Therefore, from (3.2) we have

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (3.3)$$

Next, we apply Itô's formula to $(|Y_r^{t,x}|^2)^{\frac{p}{2}}$ to have

$$\begin{aligned} & |Y_s^{t,x}|^p + \frac{p}{2} \int_s^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr + \frac{p}{2}(p-2) \int_s^T |Y_r^{t,x}|^{p-4} \langle Z_r^{t,x} (Z_r^{t,x})^* Y_r^{t,x}, Y_r^{t,x} \rangle dr \\ = & |h(X_T^{t,x})|^p + p \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle dr \\ & - p \int_s^T |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, Z_r^{t,x} dW_r \rangle. \end{aligned} \quad (3.4)$$

By Hölder's inequality and Young's inequality, we have that

$$\mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) + \frac{p}{8} \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr.$$

Taking $s = t$ in (3.4), immediately we have

$$\begin{aligned} & \frac{p}{2} \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\ \leq & C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right) + \frac{p}{8} \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr. \end{aligned}$$

From above two inequalities we have that

$$\mathbb{E}|Y_s^{t,x}|^p + \frac{p}{4} \mathbb{E} \int_t^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right).$$

By the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (3.5)$$

From (3.3) and (3.5), we obtain

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \quad (3.6)$$

Step 2: To estimate $\mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr$, we apply Itô's formula to $e^{-pKr} |\bar{X}_r|^p$ and $e^{-pKr} |\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$. Note that the stochastic integral has zero expectation, so we have

$$\begin{aligned} & \mathbb{E} e^{-KT} |X_T^{t,x}|^p + K \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^p dr \\ = & e^{-Kt} |x|^p + p \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^{p-2} \langle X_r^{t,x}, b(r, X_r^{t,x}, Y_r^{t,x}) \rangle dr \\ & + \frac{1}{2} p(p-1) \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr, \end{aligned} \quad (3.7)$$

and

$$\mathbb{E} e^{-Kt} |Y_t^{t,x}|^p - K \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^p dr + \frac{1}{2} p(p-1) \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr$$

$$= \mathbb{E} e^{-KT} |h(X_T^{t,x})|^p + p \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^{p-2} \langle Y_r^{t,x}, f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \rangle dr. \quad (3.8)$$

Denote

$$\begin{aligned} \gamma : &= p\mu - K - 4pL - \frac{p-2}{16} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} - L(p-1)(1+\varepsilon) \\ &\quad - \left[\frac{1}{4L} + L(p-1)(1+\varepsilon) \right] (1+\varepsilon)L^{\frac{p}{2}}, \\ \beta : &= K - 4pL - \frac{p}{8} + \frac{1}{8} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8} - L(p-1)(1+\varepsilon) \\ &\quad - \left[2pL^2 + \frac{p}{4L} - \frac{1}{4L} + \varepsilon + L(p-1)^2(1+\varepsilon) \right] (1+\varepsilon)L^{\frac{p}{2}}. \end{aligned}$$

From (3.7) and (3.8), using a similar method as in the proof of Theorem 2.5, we have that

$$\begin{aligned} &\gamma \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^p dr + \beta \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^p dr \\ &+ \left(\frac{1}{2}p(p-1) - \frac{p}{16} \right) \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr + \mathbb{E} e^{-KT} |X_T^{t,x}|^p + \mathbb{E} e^{-Kt} |Y_t^{t,x}|^p \\ &\leq C_{p,L} e^{-Kt} |x|^p + C_{p,L} \int_t^T e^{-Kr} dr. \end{aligned} \quad (3.9)$$

Here note $\frac{1}{2}p(p-1) - \frac{p}{16} > 0$. In addition, if we assume that $2\mu > K + L^2 + 10L + 1$ and $K > 4L^3 + L^2 + 10L + 1$, then there exists a constant $p \in (2, \infty)$ such that $\gamma, \beta > 0$ and (3.9) immediate leads to

$$\mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x}|^p dr + \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x}|^p dr \leq C_{p,L,\mu} e^{-Kt} |x|^p + C_{p,L,\mu} \int_t^T e^{-Kr} dr.$$

Note that

$$e^{-KT} \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \leq \mathbb{E} \int_t^T e^{-Kr} (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr.$$

So we have

$$\begin{aligned} \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr &\leq C_{p,L,\mu} e^{-K(t-T)} |x|^p + C_{p,L,\mu} e^{KT} \int_t^T e^{-Kr} dr \\ &\leq C_{p,L,\mu,T} (1 + |x|^p). \end{aligned} \quad (3.10)$$

From (3.6) and (3.10) we have

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L,\mu,T} (1 + |x|^p).$$

Following a similar procedure as in *Step 2* of the proof of Theorem 2.5, we can extend our result from $s \in [t, T]$ to $s \in [0, T]$ so that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p \leq C_{p,L,\mu,T} (1 + |x|^p).$$

Finally, we consider

$$\begin{aligned} \int_0^T \|Z_r^{t,x}\|^2 dr &= |h(X_T^{t,x})|^2 - |Y_0^{t,x}|^2 \\ &\quad + 2 \int_0^T \langle f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), Y_r^{t,x} \rangle dr - 2 \int_0^T \langle Z_r^{t,x}, Y_r^{t,x} dW_r \rangle. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \\ &\leq C_p \mathbb{E} \left(|h(X_T^{t,x})|^p + |Y_0^{t,x}|^p + \left| \int_0^T \langle Z_r^{t,x}, Y_r^{t,x} dW_r \rangle \right|^{\frac{p}{2}} + \left| \int_0^T \langle f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), Y_r^{t,x} \rangle dr \right|^{\frac{p}{2}} \right) \\ &\leq C_{p,L,T} \mathbb{E} \sup_{0 \leq s \leq T} (|X_s^{t,x}|^p + |Y_s^{t,x}|^p) + \frac{1}{2} \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} + C_p \int_0^T |f(r, 0, 0, 0)|^p dr \\ &\leq C_{p,L,\mu,T} (1 + |x|^p) + \frac{1}{2} \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}}. \end{aligned}$$

So (3.1) follows. \square

Remark 3.2. In Step 2 of the proof of Lemma 3.1, alternatively, we can use the Gronwall inequality to obtain the same result (3.1). But the key estimate to make it work is (3.9). We can rewrite the forward SDE part in (1.3) as follows,

$$X_T^{t,x} = X_s^{t,x} + \int_s^T b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_s^T \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r.$$

Note that the forward SDE is from s to T . We apply Itô's formula to $(|X_r^{t,x}|^2)^{\frac{p}{2}}$ and $(|Y_r^{t,x}|^2)^{\frac{p}{2}}$ from s to T , and use a similar approach as in the proof in Lemma 3.1 to obtain

$$\begin{aligned} &\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p + \mathbb{E} \int_s^T |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 dr \\ &\leq C_{p,L} \left(1 + |x|^p + \mathbb{E}|X_T^{t,x}|^p + \mathbb{E} \int_s^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right). \end{aligned}$$

To estimate $\mathbb{E}|X_T^{t,x}|^p$, following (3.9) we have that

$$\mathbb{E} e^{-KT} |X_T^{t,x}|^p \leq C_{p,L,\mu} e^{-Kt} |x|^p + C_{p,L,\mu} \int_t^T e^{-Kr} dr,$$

which leads to

$$\mathbb{E}|X_T^{t,x}|^p \leq C_{p,L,\mu} e^{-K(t-T)} |x|^p + C_{p,L,\mu} e^{KT} \int_t^T e^{-Kr} dr \leq C_{L,\mu} (1 + |x|^p).$$

Therefore we have

$$\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L,\mu,T} \left(1 + |x|^p + \mathbb{E} \int_s^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr \right).$$

By the Gronwall inequality, we have

$$\mathbb{E}|X_s^{t,x}|^p + \mathbb{E}|Y_s^{t,x}|^p \leq C_{p,L,\mu,T}(1 + |x|^p).$$

And the rest of the proof is exactly the same as that in Lemma 3.1.

Lemma 3.3. For any $t, t' \in [0, T]$, $x, x' \in \mathbb{R}^d$, let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ and $(X_s^{t',x'}, Y_s^{t',x'}, Z_s^{t',x'})_{0 \leq s \leq T}$ stand for the solutions of (1.3) associated to the initial conditions (t, x) and (t', x') . Then under conditions (B.1) (or (B.2)) and (B.3), there exist a constant $C_{p,L,\mu,T} > 0$ only depending on p, L, μ and T such that

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \\ & \leq C_{p,L,\mu,T} |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}}. \end{aligned} \quad (3.11)$$

Proof. For $t \leq t' \leq s \leq T$,

$$\left\{ \begin{aligned} X_s^{t,x} - X_s^{t',x'} &= x - x' + \int_t^{t'} b(r, X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t'} \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \\ &\quad + \int_{t'}^s \left(b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t',x'}, Y_r^{t',x'}) \right) dr \\ &\quad + \int_{t'}^s \left(\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'}) \right) dW_r, \\ Y_s^{t,x} - Y_s^{t',x'} &= h(X_T^{t,x}) - h(X_T^{t',x'}) - \int_s^T (Z_r^{t,x} - Z_r^{t',x'}) dW_r \\ &\quad + \int_s^T \left(f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'}) \right) dr. \end{aligned} \right.$$

We apply Itô's formula to $(|X_r^{t,x} - X_r^{t',x'}|^2)^{\frac{p}{2}}$ then we have

$$\begin{aligned} & |X_s^{t,x} - X_s^{t',x'}|^p \\ &= |x - x'|^p + p \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) \right\rangle dr \\ &\quad + p \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, b(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t',x'}, Y_r^{t',x'}) \right\rangle dr \\ &\quad + \frac{p}{2} (p-1) \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \\ &\quad + \frac{p}{2} (p-1) \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'})\|^2 dr \\ &\quad + p \int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r \right\rangle \\ &\quad + p \int_{t'}^s |X_r^{t,x} - X_r^{t',x'}|^{p-2} \left\langle X_r^{t,x} - X_r^{t',x'}, \left(\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t',x'}, Y_r^{t',x'}) \right) dW_r \right\rangle \\ &=: G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7. \end{aligned} \quad (3.12)$$

By using Hölder's inequality, Young's inequality and the Burkholder-Davis-Gundy inequality, the third, the fifth and the seventh terms on the RHS of (3.12) can be estimated as follows,

$$\begin{aligned} \mathbb{E}[G_3 + G_5] + \mathbb{E} \sup_{t \leq s \leq T} [G_7] &\leq C_{p,L} \mathbb{E} \int_t^T \left(|X_r^{t,x} - X_r^{t',x'}|^p + |Y_r^{t,x} - Y_r^{t',x'}|^p \right) dr \\ &\quad + \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{L,\mu,T} (1 + |x|^p + |x'|^p). \end{aligned}$$

For the the second, the fourth and the sixth terms on the RHS in (3.12), we need the following estimates. By Cauchy-Schwarz inequality and Lemma 3.1, we have that

$$\begin{aligned} &\mathbb{E} \left(\int_t^{t'} \|\sigma(r, X_r^{t,x}, Y_r^{t,x})\|^2 dr \right)^{\frac{p}{2}} \\ &\leq \mathbb{E} \left(\int_t^{t'} 2 \|\sigma(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, 0, 0)\|^2 dr + \int_t^{t'} 2 \|\sigma(r, 0, 0)\|^2 dr \right)^{\frac{p}{2}} \\ &\leq C_{p,L} \mathbb{E} \left(1 + \sup_{t \leq s \leq T} |X_s^{t,x}|^2 + \sup_{t \leq s \leq T} |Y_s^{t,x}|^2 \right)^{\frac{p}{2}} \left(\int_t^{t'} dr \right)^{\frac{p}{2}} \\ &\leq C_{p,L,\mu,T} (1 + |x|^p) |t - t'|^{\frac{p}{2}}. \end{aligned} \tag{3.13}$$

Similarly,

$$\mathbb{E} \left(\int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right)^p \leq C_{p,L,\mu,T} (1 + |x|^p) |t - t'|^{\frac{p}{2}}. \tag{3.14}$$

Now we consider the second, the fourth and the sixth terms on the RHS in (3.12). By Young's inequality and (3.14), the second term can be estimated as

$$\begin{aligned} \mathbb{E}[G_2] &\leq p \mathbb{E} \left(\int_t^{t'} |X_r^{t,x} - X_r^{t',x'}|^{p-1} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right) \\ &\leq p \mathbb{E} \left(\left[\frac{1}{N} \sup_{t \leq r \leq T} |X_r^{t,x} - X_r^{t',x'}|^{p-1} \right] \left[N \int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right] \right) \\ &\leq (p-1) \left(\frac{1}{N} \right)^{\frac{p}{p-1}} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + N^p \mathbb{E} \left(\int_t^{t'} |b(r, X_r^{t,x}, Y_r^{t,x})| dr \right)^p \\ &\leq \frac{1}{8} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{p,L,\mu,T} (1 + |x|^p) |t - t'|^{\frac{p}{2}}. \end{aligned}$$

Here we can choose N big enough such that $(p-1) \left(\frac{1}{N} \right)^{\frac{p}{p-1}} < \frac{1}{8}$. Similarly, by (3.13), the fourth and the sixth terms are

$$\mathbb{E}[G_4] + \mathbb{E} \sup_{t \leq s \leq T} [G_6] \leq \frac{1}{4} \mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + C_{p,L,\mu,T} (1 + |x|^p) |t - t'|^{\frac{p}{2}}.$$

Therefore, from (3.12) we have

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p \leq 2|x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}} \tag{3.15}$$

$$+C_{p,L}\mathbb{E}\int_t^T\left(|X_r^{t,x}-X_r^{t',x'}|^p+|Y_r^{t,x}-Y_r^{t',x'}|^p\right)dr.$$

Next, we apply Itô's formula to $(|Y_r^{t,x}-Y_r^{t',x'}|^2)^{\frac{p}{2}}$. Since the procedure is almost the same as the proof of Lemma 3.1, we will not give any detail here. But we have that

$$\begin{aligned}\mathbb{E}\sup_{t\leq s\leq T}|Y_s^{t,x}-Y_s^{t',x'}|^p &\leq C_{p,L}|x-x'|^p+C_{L,\mu,T}(1+|x|^p+|x'|^p)|t-t'|^{\frac{p}{2}} \\ &+C_{p,L}\mathbb{E}\int_t^T\left(|X_r^{t,x}-X_r^{t',x'}|^p+|Y_r^{t,x}-Y_r^{t',x'}|^p\right)dr.\end{aligned}\quad (3.16)$$

Following the similar procedure as in Step 2 of the proof of Lemma 3.1, we can find a constant $p\in(2,\infty)$ such that

$$\begin{aligned}&\mathbb{E}\sup_{0\leq s\leq T}|X_s^{t,x}-X_s^{t',x'}|^p+\mathbb{E}\sup_{0\leq s\leq T}|Y_s^{t,x}-Y_s^{t',x'}|^p \\ &\leq C_{p,L,\mu,T}|x-x'|^p+C_{p,L,\mu,T}(1+|x|^p+|x'|^p)|t-t'|^{\frac{p}{2}},\end{aligned}$$

and

$$\mathbb{E}\left(\int_0^T\|Z_r^{t,x}-Z_r^{t',x'}\|^2dr\right)^{\frac{p}{2}}\leq C_{p,L,\mu,T}|x-x'|^p+C_{p,L,\mu,T}(1+|x|^p+|x'|^p)|t-t'|^{\frac{p}{2}}.$$

So (3.11) follows. \square

Now we study the regularity of $Y_t^{t,x}$ with respect to x , including the continuity with respect to t and differentiability with respect to x .

Theorem 3.4. *Under conditions (B.0), (B.1) (or (B.2)) and (B.3), $\{X_s^{t,x}; (s,t)\in[0,T]^2, x\in\mathbb{R}^d\}$ and $\{Y_s^{t,x}; (s,t)\in[0,T]^2, x\in\mathbb{R}^d\}$ have versions in $C^{0,0,2}([0,T]^2\times\mathbb{R}^d)$.*

Proof. First, by Lemma 3.3 and Kolmogorov's continuity theorem, we have $(t,x)\rightarrow X_s^{t,x}$ is a.s. continuous for $t\in[0,T], x\in\mathbb{R}^d$. Moreover, Since $X_s^{t,x}\in S^2([0,T];\mathbb{R}^k)$, so $s\rightarrow X_s^{t,x}$ is a.s. continuous for $s\in[0,T], x\in\mathbb{R}^d$. So we conclude that $\{X_s^{t,x}; s,t\in[0,T]^2, x\in\mathbb{R}^d\}$ has an a.s. continuous versions. The continuity of $Y_s^{t,x}$ follows.

Next, we will consider the continuity of $\nabla Y_s^{t,x}$ w.r.t. x . Without losing generality, in the following proof, we assume $t'\geq t$. Set

$$\Sigma_{r,\lambda}^{t,x;t',x'}=\left(r,X_r^{t',x'}+\lambda(X_r^{t,x}-X_r^{t',x'}),Y_r^{t',x'}+\lambda(Y_r^{t,x}-Y_r^{t',x'}),Z_r^{t',x'}+\lambda(Z_r^{t,x}-Z_r^{t',x'})\right).$$

and

$$\Delta_l^i X_s^{t,x}\triangleq\frac{X_s^{t,x+le_i}-X_s^{t,x}}{l},$$

where $l \in \mathbb{R} \setminus \{0\}$, $\{e_1, e_2, \dots, e_d\}$ is an orthonormal basis of \mathbb{R}^d . And $\Delta_l^i Y_s^{t,x}$ and $\Delta_l^i Z_s^{t,x}$ can be defined similarly. Then by the mean value theorem, we have for $t \leq s \leq T$

$$\left\{ \begin{array}{l} \Delta_l^i X_s^{t,x} = e_i + \int_t^s \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x}) \right) d\lambda dr \\ \quad + \int_t^s \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (\sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x}) \right) d\lambda dW_r \\ \Delta_l^i Y_s^{t,x} = \int_0^1 h' \left(X_T^{t,x} + \lambda l \Delta_l^i X_T^{t,x} \right) \Delta_l^i X_T^{t,x} d\lambda - \int_s^T \Delta_l^i Z_r^{t,x} dW_r \\ \quad + \int_s^T \int_0^1 \left(f'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + (f'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} + (f'_z(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Z_r^{t,x}) \right) d\lambda dr, \end{array} \right. \quad (3.17)$$

where $\Sigma_{r,\lambda}^{t,x,l} = \left(r, X_r^{t,x} + \lambda l \Delta_l^i X_r^{t,x}, Y_r^{t,x} + \lambda l \Delta_l^i Y_r^{t,x}, Z_r^{t,x} + \lambda l \Delta_l^i Z_r^{t,x} \right)$.

Now we investigate this new type of Δ_l^i FBDSEs (3.17). Note **(B.0)** and **(B.1)** imply that Δ_l^i FBDSEs (3.17) satisfies the corresponding monotone-Lipschitz assumptions. By using a similar method as in the proof of Theorem 2.6, (3.17) has a unique solution $(\Delta_l^i X_s^{t,x}, \Delta_l^i Y_s^{t,x}, \Delta_l^i Z_s^{t,x})_{0 \leq s \leq T}$. And by Lemma 3.3 we have that, there exists $C > 0$ only depending on p, L, μ and T such that

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|\Delta_l^i Z_s^{t,x}\|^2 ds \right)^{\frac{p}{2}} \\ &= |l|^{-p} \mathbb{E} \sup_{0 \leq s \leq T} \left(|X_s^{t,x+le_i} - X_s^{t,x}|^p + |Y_s^{t,x+le_i} - Y_s^{t,x}|^p \right) + \mathbb{E} \left(\int_0^T \|Z_s^{t,x+le_i} - Z_s^{t,x}\|^2 ds \right)^{\frac{p}{2}} \\ &\leq |l|^{-p} (C_{L,\mu,T} |x + le_i - x|^p) \\ &\leq C. \end{aligned} \quad (3.18)$$

Finally, we consider

$$\begin{aligned} \Delta_l^i X_s^{t,x} - \Delta_{l'}^i X_s^{t',x'} &= \int_{t'}^s \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\ &\quad + \int_{t'}^s \int_0^1 \left(b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\ &\quad + \int_t^{t'} \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + b'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right) d\lambda dr \\ &\quad + \int_{t'}^s \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \\ &\quad + \int_{t'}^s \int_0^1 \left(\sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - \sigma'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dW_r \\ &\quad + \int_t^{t'} \int_0^1 \left(\sigma'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} + \sigma'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i Y_r^{t,x} \right) d\lambda dW_r, \end{aligned}$$

and

$$\Delta_l^i Y_s^{t,x} - \Delta_{l'}^i Y_s^{t',x'} = \int_0^1 h' \left(X_T^{t,x} + \lambda l \Delta_l^i X_T^{t,x} \right) \Delta_l^i X_T^{t,x} d\lambda$$

$$\begin{aligned}
& - \int_0^1 h' \left(X_T^{t',x'} + \lambda l' \Delta_l^i X_T^{t',x'} \right) \Delta_{l'}^i X_T^{t',x'} d\lambda \\
& + \int_s^T \int_0^1 \left(f'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
& + \int_s^T \int_0^1 \left(f'_y(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_y(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
& + \int_s^T \int_0^1 \left(f'_z(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - f'_z(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda dr \\
& - \int_s^T \left(\Delta_l^i Z_r^{t,x} - \Delta_{l'}^i Z_r^{t',x'} \right) dW_r.
\end{aligned}$$

By a similar procedure of Lemma 3.3, there exists $C > 0$ only depending on p, L, μ and T such that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i X_s^{t,x} - \Delta_{l'}^i X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |\Delta_l^i Y_s^{t,x} - \Delta_{l'}^i Y_s^{t',x'}|^p \\
& + \mathbb{E} \left(\int_{t \wedge t'}^T \|\Delta_l^i Z_s^{t,x} - \Delta_{l'}^i Z_s^{t',x'}\|^2 ds \right)^{\frac{p}{2}} \\
& \leq C|x - x'|^p + C|l - l'|^p + C(1 + |x|^p + |x'|^p + |l|^p + |l'|^p)|t - t'|^{\frac{p}{2}}. \tag{3.19}
\end{aligned}$$

Here we only calculate the following term, others can be calculated similarly,

$$\begin{aligned}
& p\mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^{p-2} \left\langle \Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}, \right. \\
& \quad \left. \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) \Delta_l^i X_r^{t,x} - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \Delta_{l'}^i X_r^{t',x'} \right) d\lambda \right\rangle dr \\
& \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr + C_p \mathbb{E} \int_{t'}^s \left| \int_0^1 b'_x(\Sigma_{r,\lambda}^{t,x,l}) (\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}) d\lambda \right|^p dr \\
& \quad + C_p \mathbb{E} \int_{t'}^s \left| \int_0^1 \left(b'_x(\Sigma_{r,\lambda}^{t,x,l}) - b'_x(\Sigma_{r,\lambda}^{t',x',l'}) \right) \Delta_{l'}^i X_r^{t',x'} d\lambda \right|^p dr \\
& \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr + C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
& \quad + C_p \sqrt{\mathbb{E} \int_{t'}^s \left(\int_0^1 |\Sigma_{r,\lambda}^{t,x,l} - \Sigma_{r,\lambda}^{t',x',l'}|^2 d\lambda \right)^p} dr \sqrt{\mathbb{E} \int_{t'}^s |\Delta_{l'}^i X_r^{t',x'}|^{2p} dr} \\
& \leq C_p \mathbb{E} \int_{t'}^s |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
& \quad + C_p \mathbb{E} \sup_{t' \leq r \leq s} |X_r^{t,x} - X_r^{t',x'}|^p + C_p \mathbb{E} \sup_{t' \leq r \leq s} |X_r^{t,x+le_i} - X_r^{t',x'+l'e_i}|^p \\
& \quad + C_p \mathbb{E} \sup_{t' \leq r \leq s} |Y_r^{t,x} - Y_r^{t',x'}|^p + C_p \mathbb{E} \sup_{t' \leq r \leq s} |Y_r^{t,x+le_i} - Y_r^{t',x'+l'e_i}|^p \\
& \leq C_p \mathbb{E} \int_t^T |\Delta_l^i X_r^{t,x} - \Delta_{l'}^i X_r^{t',x'}|^p dr \\
& \quad + C|x - x'|^p + C|l - l'|^p + C(1 + |x|^p + |x'|^p + |l|^p + |l'|^p)|t - t'|^{\frac{p}{2}}.
\end{aligned}$$

Now using Kolmogorov's continuity theorem, it immediately follows from (3.19) that for any $t, s \in [0, T]^2$, $x \in \mathbb{R}^d$, the mapping $x \rightarrow X_s^{t,x}$ is a.s. differentiable, and the partial derivatives with respect to x , denoted by $\frac{\partial X_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i X_s^{t,x}$, has a version which are a.s. continuous with respect to (s, t, x) . The continuity of the first order derivative of $Y_s^{t,x}$ with respect to x follows. By a similar procedure, the existence of continuous second order derivatives of $X_s^{t,x}$ and $Y_s^{t,x}$ with respect to x can be proved. \square

Corollary 3.5. *Under conditions (B.0), (B.1) (or (B.2)) and (B.3), for any $t \in [0, T]$, the mappings $x \rightarrow X_t^{t,x}$ and $x \rightarrow Y_t^{t,x}$, are of class C^2 a.s., i.e. the functions and their derivatives of order one and two are continuous in (t, x) a.s..*

Corollary 3.6. *Under conditions (B.0), (B.1) (resp. (B.2)) and (B.3), $(\nabla X_s^{t,x} = \frac{\partial X_s^{t,x}}{\partial x}, \nabla Y_s^{t,x} = \frac{\partial Y_s^{t,x}}{\partial x}, \nabla Z_s^{t,x} = \frac{\partial Z_s^{t,x}}{\partial x})_{0 \leq s \leq T}$ is the unique solution of the following ∇ FBDSDEs,*

$$\left\{ \begin{array}{l} \nabla X_s^{t,x} = 1 + \int_t^s \sigma'_x(r, X_r^{t,x}, Y_r^{t,x}) \nabla X_r^{t,x} dW_r + \int_t^s \sigma'_y(r, X_r^{t,x}, Y_r^{t,x}) \nabla Y_r^{t,x} dW_r \\ \quad + \int_t^s b'_x(r, X_r^{t,x}, Y_r^{t,x}) \nabla X_r^{t,x} dr + \int_t^s b'_y(r, X_r^{t,x}, Y_r^{t,x}) \nabla Y_r^{t,x} dr \\ \nabla Y_s^{t,x} = h'(X_T^{t,x}) \nabla X_T^{t,x} - \int_s^T \nabla Z_r^{t,x} dW_r \\ \quad + \int_s^T f'_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla X_r^{t,x} dr + f'_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Y_r^{t,x} dr \\ \quad + \int_s^T f'_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Z_r^{t,x} dr. \end{array} \right. \quad (3.20)$$

Proof. The Corollary follows easily from the result of Theorem 3.4 and the definition of partial derivatives,

$$\frac{\partial X_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i X_s^{t,x}, \frac{\partial Y_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i Y_s^{t,x}, \frac{\partial Z_s^{t,x}}{\partial x_i} = \lim_{l \rightarrow 0} \Delta_l^i Z_s^{t,x}.$$

It is easy to check that (3.20) satisfies the corresponding monotone-Lipschitz assumptions., Therefore $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ is the unique solution. \square

Next we use the Malliavin calculus to express Z as the Malliavin derivative of Y . Then we compare the ∇ FBDSDEs (3.20) with the Malliavin differential form (3.21) to give a formula relating Z with the gradients of Y and X . Let us recall the notion of the derivation on the Wiener space. Denote by \mathbb{H} the set of random variables ξ of the form: $\xi = \varphi(W(h_1), \dots, W(h_n))$, where $\varphi \in C_b^\infty(\mathbb{R}^n)$ is a polynomial function, $h_1, \dots, h_n \in L^2([0, T], \mathbb{R}^d)$ and $W(h_i) \triangleq \int_0^T \langle h_i(t), dW_t \rangle$. The random variable ξ has a derivative $\{\mathcal{D}_r \xi; r \in [0, T]\}$ defined as

$$\mathcal{D}_r \xi = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i(t), \quad 0 \leq r \leq T.$$

For such a ξ , we define its 1,2-norm as

$$\|\xi\|_{1,2}^2 = \mathbb{E}(\xi^2) + \mathbb{E} \int_0^T |\mathcal{D}_r \xi|^2 dr.$$

And we define the Sobolev space: $\mathbb{D}^{1,2} \triangleq \bar{\mathbb{H}}^{\|\cdot\|_{1,2}}$ as the completion of \mathbb{H} under the norm $\|\cdot\|_{1,2}$. From [17], we know that the “derivation operator” \mathcal{D} extends as an operator from $\mathbb{D}^{1,2}$ into $L^2(\Omega; L^2([0, T], \mathbb{R}^d))$. It turns out that the components $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ take values in $\mathbb{D}^{1,2}$ under the assumptions in this section.

Proposition 3.7. *Under conditions (B.0), (B.1) (or (B.2)) and (B.3), for any $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$, $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$, and a version of $\{\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x}; t \leq r \leq T, t \leq s \leq T\}$ is given by*

- (i) $\mathcal{D}_r X_s^{t,x} = 0, \mathcal{D}_r Y_s^{t,x} = 0, \mathcal{D}_r Z_s^{t,x} = 0, r \in [0, T] \setminus (t, s];$
- (ii) *For any $t < r \leq T$, $\{\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x}; r \leq s \leq T\}$ is the unique solution of the following differential form of FBSDEs with respect to Wiener process.*

$$\left\{ \begin{array}{l} \mathcal{D}_r X_s^{t,x} = \sigma(r, X_r^{t,x}, Y_r^{t,x}) + \int_r^s b'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} d\tau + \int_r^s b'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} d\tau \\ \quad + \int_r^s \sigma'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} dW_\tau + \int_r^s \sigma'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} dW_\tau \\ \mathcal{D}_r Y_s^{t,x} = h'(X_T^{t,x}) \mathcal{D}_r X_T^{t,x} - \int_s^T \mathcal{D}_r Z_\tau^{t,x} dW_\tau + \int_s^T f'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} d\tau \\ \quad + \int_s^T f'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} d\tau + \int_s^T f'_z(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Z_\tau^{t,x} d\tau. \end{array} \right. \quad (3.21)$$

Moreover, $\{\mathcal{D}_s Y_s^{t,x}, t \leq s \leq T\}$ is a version of $\{Z_s^{t,x}, t \leq s \leq T\}$.

Proof. First, we will show that $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$. Recall the iteration procedure for FBSDEs

$$\left\{ \begin{array}{l} X_s^{t,x,N} = x + \int_t^s b(r, X_r^{t,x,N}, Y_r^{t,x,N-1}) dr + \int_t^s \sigma(r, X_r^{t,x,N}, Y_r^{t,x,N-1}) dW_r \\ Y_s^{t,x,N} = h(X_T^{t,x,N}) + \int_s^T f(r, X_r^{t,x,N}, Y_r^{t,x,N}, Z_r^{t,x,N}) dr - \int_s^T Z_r^{t,x,N} dW_r. \end{array} \right.$$

When $N=1$, we let $Y_s^{t,x,0} = 0$, then the above FBSDEs becomes a BSDE in [21]. From results in [21] and [22], $(X^{t,x,1}, Y^{t,x,1}, Z^{t,x,1}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$ and (3.21) holds. For $N=2$, we can subject $(X^{t,x,1}, Y^{t,x,1}, Z^{t,x,1})$ into above FBSDEs, and show $(X^{t,x,2}, Y^{t,x,2}, Z^{t,x,2}) \in L^2([t, T]; (\mathbb{D}^{1,2})^d) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^k) \otimes L^2([t, T]; (\mathbb{D}^{1,2})^{k \times d})$ and (3.21) holds. By this iterative procedure and boundedness of the derivatives of functions, we can easily show that $(\mathcal{D}_r X_s^{t,x,N}, \mathcal{D}_r Y_s^{t,x,N}, \mathcal{D}_r Z_s^{t,x,N})$ is a Cauchy sequence in L^2 sense, and its limit denoted by $(\mathcal{D}_r X_s^{t,x}, \mathcal{D}_r Y_s^{t,x}, \mathcal{D}_r Z_s^{t,x})$ satisfies (3.21) for any $r \leq s \leq T$.

Finally, we consider the following equation

$$Y_s^{t,x} = Y_t^{t,x} - \int_t^s f(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) d\mu + \int_t^s Z_\tau^{t,x} dW_\tau.$$

For $t \leq r \leq s \leq T$, we have

$$\mathcal{D}_r Y_s^{t,x} = Z_r^{t,x} - \int_r^s f'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r X_\tau^{t,x} d\tau - \int_r^s f'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Y_\tau^{t,x} d\tau$$

$$- \int_r^s f'_z(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x}) \mathcal{D}_r Z_\tau^{t,x} d\tau + \int_r^s \mathcal{D}_r Z_\tau^{t,x} dW_\tau.$$

It is easy to see that $\mathcal{D}_r Y_s^{t,x} = Z_s^{t,x}$ a.s. at $r = s$. This means that

$$\mathcal{D}_s Y_s^{t,x} \triangleq \lim_{r \rightarrow s} \mathcal{D}_r Y_s^{t,x} = Z_s^{t,x}, \quad \text{a.s..}$$

□

Proposition 3.8. *Under conditions (B.0), (B.1) (or (B.2)) and (B.3), $\{Z_s^{t,x}; 0 \leq t \leq s \leq T, x \in \mathbb{R}^d\}$ has an a.s. continuous version which is given by:*

$$Z_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(s, X_s^{t,x}, Y_s^{t,x}).$$

In particular $Z_t^{t,x} = \nabla Y_t^{t,x} \sigma(t, x, Y_t^{t,x})$.

Proof. First, we will show that $\{\mathcal{D}_s Y_s^{t,x}\}$ processes an a.s. continuous version. For this, recall Corollary 3.6, we have $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})_{0 \leq s \leq T}$ solves ∇ FBDSEs (3.20) of which the forward equation can be written as

$$\begin{aligned} \nabla X_s^{t,x} = & \nabla X_r^{t,x} + \int_r^s \sigma'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla X_\tau^{t,x} dW_\tau + \int_r^s \sigma'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla Y_\tau^{t,x} dW_\tau \\ & + \int_r^s b'_x(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla X_\tau^{t,x} d\tau + \int_r^s b'_y(\tau, X_\tau^{t,x}, Y_\tau^{t,x}) \nabla Y_\tau^{t,x} d\tau \end{aligned} \quad (3.22)$$

Comparing (3.21), (3.20) and (3.22), by the uniqueness of solution of (3.21) and the linearity of the equation, we have

$$\mathcal{D}_r X_s^{t,x} = \nabla X_s^{t,x} (\nabla X_r^{t,x})^{-1} \sigma(r, X_r^{t,x}, Y_r^{t,x}),$$

and

$$\mathcal{D}_r Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_r^{t,x})^{-1} \sigma(r, X_r^{t,x}, Y_r^{t,x}), \quad t \leq r \leq s \leq T. \quad (3.23)$$

Thus the continuity of $\mathcal{D}_s Y_s^{t,x}$ follows from that of $\nabla Y_s^{t,x}$, $\nabla X_s^{t,x}$, $X_s^{t,x}$ and $Y_s^{t,x}$. Finally, using the Proposition 3.7 and (3.23), we have $Z_s^{t,x} = \mathcal{D}_s Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(s, X_s^{t,x}, Y_s^{t,x})$. The continuity follows from the continuity of $\{\mathcal{D}_s Y_s^{t,x}; t \leq s \leq T\}$. This gives the first part of the proposition. The second part easily follows when $s = t$. □

Proposition 3.9. *Under conditions (B.1) (or (B.2)) and (B.3), (1.3) has a unique solution $(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})_{t \leq r \leq T}$, then for any $t \leq s \leq T$, $X_r^{s, X_s^{t,x}} = X_r^{t,x}$, $Y_r^{s, X_s^{t,x}} = Y_r^{t,x}$ and $Z_r^{s, X_s^{t,x}} = Z_r^{t,x}$ for any $r \in [s, T]$ a.s..*

Proof. Note if (1.3) has a unique solution $(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})_{t \leq r \leq T}$, then for $t \leq s \leq r \leq T$, it is easy to check that $(X_r^{s, X_s^{t,x}}, Y_r^{s, X_s^{t,x}}, Z_r^{s, X_s^{t,x}})$ is the solution of the following equations

$$\begin{cases} X_r^{s, X_s^{t,x}} = X_s^{t,x} + \int_s^r b(u, X_u^{s, X_s^{t,x}}, Y_u^{s, X_s^{t,x}}) du + \int_s^r \sigma(u, X_u^{s, X_s^{t,x}}, Y_u^{s, X_s^{t,x}}) dW_u, \\ Y_r^{s, X_s^{t,x}} = h(X_T^{s, X_s^{t,x}}) + \int_r^T f(u, X_u^{s, X_s^{t,x}}, Y_u^{s, X_s^{t,x}}, Z_u^{s, X_s^{t,x}}) du - \int_r^T Z_u^{s, X_s^{t,x}} dW_u. \end{cases} \quad (3.24)$$

By the uniqueness of the solutions of FBSDEs, it follows from comparing with (1.3) that for any $s \in [t, T]$, $X_r^{s, X_s^{t,x}} = X_r^{t,x}$, $Y_r^{s, X_s^{t,x}} = Y_r^{t,x}$ and $Z_r^{s, X_s^{t,x}} = Z_r^{t,x}$ for any $r \in [s, T]$ a.s.. □

Now we can link FBSDE (1.3) with the classical solution of quasi-linear PDE (1.1). The idea follows from [21] for BSDEs and the classical solution of semi-linear PDEs. We nevertheless include a complete proof for the convenience of reader. First, we give the probabilistic representation of solution of quasi-linear parabolic PDEs in terms of FBSDEs.

Theorem 3.10. *Under conditions (B.1) (or (B.2)) and (B.3), if $u \in C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ solves PDE (1.1), then $u(t, x) = Y_t^{t,x}$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{0 \leq s \leq T}$ is the unique solution of FBSDE (1.3).*

Proof. It suffices to show that $(X_s^{t,x}, u(t, X_s^{t,x}), \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}); t \leq s \leq T)$ solves FBSDE (1.3). Let $t = t_0 < t_1 < t_2 < \dots < t_n = T$,

$$\begin{aligned} & \sum_{i=0}^{n-1} \left[u(t_i, X_{t_i}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x}) \right] \\ &= \sum_{i=0}^{n-1} \left[u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x}) \right] + \sum_{i=0}^{n-1} \left[u(t_i, X_{t_{i+1}}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x}) \right] \\ &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathcal{L}u(t_i, X_s^{t,x}) ds - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sigma^*(t_i, X_s^{t,x}, u(t_i, X_s^{t,x})) \nabla u(t_i, X_s^{t,x}) dW_s \\ & \quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[\mathcal{L}u(s, X_{t_{i+1}}^{t,x}) \right. \\ & \quad \left. + f\left(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}), \sigma^*\left(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x})\right) \nabla u(s, X_{t_{i+1}}^{t,x})\right) \right] ds. \end{aligned}$$

Here we applied Itô's formula to $u(t_i, \cdot)$ to calculate $u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x})$ (note the fact that $u(t_i, \cdot) \in C^2(\mathbb{R}^d; \mathbb{R}^k)$), and compute $u(t_i, X_{t_{i+1}}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})$ from the PDE (1.1). Finally, by the fact that $u \in C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ and the monotone-Lipschitz assumptions, we let the mesh size go to zero to obtain

$$\begin{aligned} u(s, X_s^{t,x}) - h(X_T^{t,x}) &= \int_s^T f(r, X_r^{t,x}, u(r, X_r^{t,x}), \sigma^*(r, X_r^{t,x}, u(r, X_r^{t,x})) \nabla u(r, X_r^{t,x})) dr \\ & \quad - \int_s^T \sigma^*(r, X_r^{t,x}, u(r, X_r^{t,x})) \nabla u(r, X_r^{t,x}) dW_r, \end{aligned}$$

where $(X_s^{t,x}, u(s, X_s^{t,x}), \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}))$ solves the FBSDE (1.3). By the uniqueness of solutions of FBSDEs, $(u(s, X_s^{t,x}), \sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x})) = (Y_s^{t,x}, Z_s^{t,x})$. In particular, $u(t, x) = Y_t^{t,x}$. \square

We can also prove the converse part to Theorem 3.10, which means the solutions of FBSDEs give the unique classical solutions of a quasi-linear parabolic PDEs.

Theorem 3.11. *Under conditions (B.0), (B.1) (or (B.2)) and (B.3), $\{u(t, x) \triangleq Y_t^{t,x}; 0 \leq t \leq T, x \in \mathbb{R}^d\}$ is of class $C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$, and solves the PDE (1.1).*

Proof. From Theorem 3.4, $u(t, x) \in C^{0,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$, where $u(t, x) = Y_t^{t,x}$. Let $h > 0$ be such that $t + h \leq T$. By the flow property in Proposition 3.9, $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$. Hence

$$\begin{aligned} u(t+h, x) - u(t, x) &= u(t+h, x) - u(t+h, X_{t+h}^{t,x}) + u(t+h, X_{t+h}^{t,x}) - u(t, x) \\ &= - \int_t^{t+h} \mathcal{L}u(t+h, X_s^{t,x}) ds \\ &\quad - \int_t^{t+h} \sigma^*(t+h, X_s^{t,x}, u(t+h, X_s^{t,x})) \nabla u(t+h, X_s^{t,x}) dW_s \\ &\quad - \int_t^{t+h} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + \int_t^{t+h} Z_s^{t,x} dW_s. \end{aligned}$$

Here we applied Itô's formula to $u(t+h, \cdot)$ to calculate $u(t+h, x) - u(t+h, X_{t+h}^{t,x})$. Note here $u(t_i, \cdot) \in C^2(\mathbb{R}^d; \mathbb{R}^k)$. Moreover, $u(t+h, X_{t+h}^{t,x}) - u(t, x) = Y_{t+h}^{t+h, X_{t+h}^{t,x}} - Y_t^{t,x} = Y_{t+h}^{t,x} - Y_t^{t,x}$ satisfies FBSDE (1.3). Now let $t = t_0 < t_1 < \dots < t_n = T$. We have

$$\begin{aligned} u(T, x) - u(t, x) &= - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\mathcal{L}u(t_{i+1}, X_s^{t_i,x}) + f(s, X_s^{t_i,x}, Y_s^{t_i,x}, Z_s^{t_i,x})) ds \\ &\quad + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (Z_s^{t_i,x} - \sigma^*(t_{i+1}, X_s^{t_i,x}, u(t_{i+1}, X_s^{t_i,x})) \nabla u(t_{i+1}, X_s^{t_i,x})) dW_s. \end{aligned}$$

We take a sequence $t = t_0^n < t_1^n < \dots < t_n^n = T$ such that $\lim_{n \rightarrow \infty} \sup_{i \leq n-1} (t_{i+1}^n - t_i^n) = 0$. Proposition 3.8 and the fact that $Y_s^{t,x}$ and $\nabla Y_s^{t,x}$ are uniformly continuous w.r.t. (s, t, x) a.s. suggest that

$$u(t, x) = h(x) + \int_t^T [\mathcal{L}u(s, x) + f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x))] ds.$$

Hence $u(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$ and satisfies PDE (1.1). \square

Remark 3.12. For the existence and uniqueness, we can allow b and σ involving z although in the paper we only deal with the case when b involving z . Our method still works for σ involving z , but the Lipschitz constant has to be small. For the regularity of the solutions, we can not deal with the case when b and σ involving z . The main difficulty arise in the L^p estimate in the proof of regularity. In particular, if b and σ involve z , e.g. b and σ are Lipschitz continuous in z as well, then the estimation (3.3) becomes:

$$\begin{aligned} &\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p \\ &\leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr + \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |Z_r|^2 dr \right). \end{aligned}$$

Therefore (3.6) will become

$$\mathbb{E} \sup_{t \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{t \leq s \leq T} |Y_s^{t,x}|^p$$

$$\leq C_{p,L} \left(1 + |x|^p + \mathbb{E} \int_t^T (|X_r^{t,x}|^p + |Y_r^{t,x}|^p) dr + \mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |Z_r|^2 dr \right).$$

The extra term essentially can only be estimated by

$$\mathbb{E} \int_t^T |X_r^{t,x}|^{p-2} |Z_r|^2 dr \leq C(\mathbb{E} \int_t^T |X_r^{t,x}|^p dr + \mathbb{E} \int_t^T |Z_r|^p dr).$$

But we can not get the estimate for $\mathbb{E} \int_t^T |Z_r|^p dr$. That is the reason that we did not involve z in b and σ .

4 Weak solutions for quasi-linear parabolic PDEs

For smooth coefficients, the PDE (1.1) has a unique classical solution (see Theorem 3.11). In this section, we relax our assumptions and study the weak solution of PDE (1.1) in the Sobolev space. First we define the solution in the Sobolev space:

Definition 4.1. A process u is called a weak solution (solution in $L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)$) of PDE (1.1) if $(u, \sigma^* \nabla u) \in M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ and for an arbitrary $\Psi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u(s, x) \partial_s \Psi(s, x) dx ds + \int_{\mathbb{R}^d} u(t, x) \Psi(t, x) dx - \int_{\mathbb{R}^d} u(T, x) \Psi(T, x) dx \\ & + \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \sigma^*(s, x, u(s, x)) \nabla u(s, x) \sigma^*(s, x, u(s, x)) \nabla \Psi(s, x) dx ds \\ & + \int_t^T \int_{\mathbb{R}^d} u(s, x) \operatorname{div} \left((b - \tilde{A}) \Psi(s, x) \right) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x)) \Psi(s, x) dx ds. \end{aligned} \quad (4.1)$$

Here $\tilde{A}_j \triangleq \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma \sigma^*)_{i,j}(s, x, u(s, x))$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

Note this definition can be easily understood if we note the following integration by parts formula: for $\varphi_1, \varphi_2 \in C_c^2(\mathbb{R}^d)$,

$$- \int_{\mathbb{R}^d} \mathcal{L} \varphi_1(x) \varphi_2(x) dx = \frac{1}{2} \int_{\mathbb{R}^d} (\sigma^* \nabla \varphi_1)(x) (\sigma^* \nabla \varphi_2)(x) dx + \int_{\mathbb{R}^d} \varphi_1(x) \operatorname{div} \left((b - \tilde{A}) \varphi_2 \right)(x) dx.$$

We assume:

(C.0): For any $s \in [0, T]$, $b(s, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $f(s, \cdot, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $h \in C_{l,b}^{1,\alpha}(\mathbb{R}^d; \mathbb{R}^k)$; $\sigma(s, \cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$ for some $\alpha \in (0, 1)$, where $C_{l,b}^{1,\alpha}$ denote the set of $C_{l,b}^1$ -functions whose first derivative is Hölder continuous of order α . Assume σ is bounded.

(C.1): Condition **(B.1)**, but $2\mu > K + C_L$, $K > C'_L$ is changed to $q\mu > K + C_{q,L}$, $K > C'_{q,L}$, where $C_{q,L}, C'_{q,L}$ only depending on q and L . Here $q \geq 2$ is the power of the weight function ρ (see Remark 2.2).

(C.2): Condition (B.2), but $2\mu > K + C_L$, $K > C'_L$ is changed to $q\mu > K + C_{q,L}$, $K > C'_{q,L}$, where $C_{q,L}, C'_{q,L}$ only depending on q and L .

(C.3): For some $p \in (2, 2q - 1)$,

$$\int_0^T (|b(s, 0, 0)|^p + \|\sigma(s, 0, 0)\|^p + |f(s, 0, 0, 0)|^p) ds < \infty.$$

The following norm equivalence result (Lemma 4.5) is key to link the weak solution of quasi-linear PDE (1.1) with the solution of FBSDE (1.3). Relevant works for flows generated by SDEs, when BSDEs are not involved, were obtained in [13], [5], [31]. We extend their results to the FBSDEs case. For this, we need following lemmas.

Lemma 4.2. *Under conditions (C.1) (or (C.2)) and (C.3), for any $p \in [2, \infty)$, there exists a constant $C_{p,L,\mu,T} > 0$ only depending on p , L , μ and T such that the solutions of FBSDE (1.3) satisfies*

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x}\|^2 dr \right)^{\frac{p}{2}} \leq C_{p,L,\mu,T} (1 + |x|^p)$$

and

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t',x'}|^p + \mathbb{E} \left(\int_0^T \|Z_r^{t,x} - Z_r^{t',x'}\|^2 dr \right)^{\frac{p}{2}} \\ & \leq C_{p,L,\mu,T} |x - x'|^p + C_{p,L,\mu,T} (1 + |x|^p + |x'|^p) |t - t'|^{\frac{p}{2}}. \end{aligned}$$

Proof. The proof follows from Lemma 3.1 and Lemma 3.3. Note from Lemma 3.1, we can find a constant $p \in (2, \infty)$ such that (3.1) and (3.11) hold. This is enough for the regularity properties in Section 3. But in this section, we need an estimation of the weighted function $\rho(\hat{X}) := (1 + |\hat{X}_s^{t,y}|^2)^q$, $q \geq 2$ in Lemma 4.5. Therefore, we have to strengthen our assumption for μ in (C.1) and (C.2) such that the constants $C_{q,L}$ and $C'_{q,L}$ are not only depend on L but also on q . \square

Lemma 4.3. *Under conditions (C.0), (C.1) (or (C.2)) and (C.3), there exists a constant $C_{L,\mu,T} > 0$ only depending on L , μ and T such that*

$$\mathbb{E} \sup_{0 \leq s \leq T} \|\nabla X_s^{t,x}\|^2 + \mathbb{E} \sup_{0 \leq s \leq T} \|\nabla Y_s^{t,x}\|^2 + \mathbb{E} \int_0^T \|\nabla Z_r^{t,x}\|^2 dr \leq C_{L,\lambda,T}. \quad (4.2)$$

Proof. The proof follows from Corollary 3.6 and (3.18). \square

In order to prove the norm equivalence result, we have to estimate the determinant of the Jacobian matrix of $\hat{X}_s^{t,y}$, the inverse flow of $X_s^{t,x}$. We show the existence of $\hat{X}_s^{t,y}$ first.

Theorem 4.4. *Under conditions (C.0), (C.1) (or (C.2)) and (C.3), $X_s^{t,x}$ is the solution defined in the forward equation in FBSDE (1.3), then the map $X_s^{t,\cdot} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is homeomorphism a.s.. This is to say that the map $X_s^{t,\cdot}$ is one-to-one and onto, so its inverse map exists. Moreover, the inverse map, denoted by $\hat{X}_s^{t,\cdot} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, is also continuous a.s..*

Proof. We will first consider the one-to-one property of map $X_s^{t,\cdot}$. For this we need some estimates in the following. For any negative p , there exists a constant $C_{p,L,\mu,T} > 0$ only depending on p, L, μ and T such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X_s^{t,x} - X_s^{t,x'}|^p + \mathbb{E} \sup_{0 \leq s \leq T} |Y_s^{t,x} - Y_s^{t,x'}|^p \leq C_{p,L,\mu,T} |x - x'|^p. \quad (4.3)$$

To prove this, let us recall Step 2 in the proof of Lemma 3.1. We apply Itô's formula to $e^{-Kr} |X_r^{t,x} - X_r^{t,x'}|^p$ and $e^{-Kr} |Y_r^{t,x} - Y_r^{t,x'}|^p$ to get

$$\begin{aligned} & \gamma \mathbb{E} \int_t^T e^{-Kr} |X_r^{t,x} - X_r^{t,x'}|^p dr + \beta \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x} - Y_r^{t,x'}|^p dr \\ & + \left(\frac{1}{2} p(p-1) - \frac{p}{16} \right) \mathbb{E} \int_t^T e^{-Kr} |Y_r^{t,x} - Y_r^{t,x'}|^{p-2} \|Z_r^{t,x} - Z_r^{t,x'}\|^2 dr \\ & + \mathbb{E} e^{-KT} |X_T^{t,x} - X_T^{t,x'}|^p + \mathbb{E} e^{-Kt} |Y_t^{t,x} - Y_t^{t,x'}|^p \\ & \leq C_{p,L} e^{-Kt} |x - x'|^p + C_{p,L} \int_t^T e^{-Kr} dr, \end{aligned}$$

where γ and β are defined in (3.9). For any negative p , it is easy to see that γ, β and $\frac{1}{2}p(p-1) - \frac{p}{16}$ are positive. So following the rest of the proof of Lemma 3.1, we can show that (4.3) holds for any negative p . Set $\Gamma_s^{t,x,y} = \frac{1}{|X_s^{t,x} - X_s^{t,y}|}$. Using the estimates (4.2), (4.3) and following the proof of Lemma 4.1 (see [12], pp. 224-225), then there exists a constant $C'_{p,L,\mu,T} > 0$ only depending on p, L, μ and T such that for any $\delta > 0$

$$\begin{aligned} \mathbb{E} |\Gamma_s^{t,x,y} - \Gamma_{s'}^{t',x',y'}|^p & \leq C'_{p,L,\mu,T} \delta^{-2p} \left(|x - x'|^p + |y - y'|^p \right. \\ & \quad \left. + (|x|^p + |x'|^p + |y|^p + |y'|^p) (|t - t'|^\frac{p}{2} + |s - s'|^\frac{p}{2}) \right) \end{aligned} \quad (4.4)$$

holds for all $x, y, x', y' \in \mathbb{R}^d$ such that $|x - y| \geq \delta$ and $|x' - y'| \geq \delta$. As a result, we can use (4.4) and the idea of [12] (pp. 225) to show the one-to-one property of map $X_s^{t,\cdot}$.

We will next consider the onto property of the map $X_s^{t,\cdot}$. For this, let $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ be the one point compactification of \mathbb{R}^d . Set $\hat{x} = |x|^{-2}$ and $\tilde{\Gamma}_s^{t,\hat{x}} = \frac{1}{1 + |X_s^{t,x}|}$ if $x \neq 0$, and $\tilde{\Gamma}_s^{t,\hat{x}} = 0$ if $x = 0$. Then there is $C''_{p,L,\mu,T} > 0$ only depending on p, L, μ and T such that

$$\mathbb{E} |\tilde{\Gamma}_s^{t,\hat{x}} - \tilde{\Gamma}_{s'}^{t',\hat{x}'}|^p \leq C''_{p,L,\mu,T} \delta^{-2p} \left(|\hat{x} - \hat{x}'|^p + |t - t'|^\frac{p}{2} + |s - s'|^\frac{p}{2} \right). \quad (4.5)$$

The proof follows from estimates (4.2), (4.2) and Lemma 4.2 (see [12], pp. 225-226). Using (4.5) and the idea of [12] (pp. 226-227) we can show the onto property of map $X_s^{t,\cdot}$.

So far, we show that the map $X_s^{t,\cdot}$ is one-to-one and onto *a.s.* Consequently, its inverse map $\hat{X}_s^{t,\cdot}$ exists. Moreover, due to the fact that $X_s^{t,\cdot}$ is one-to-one and continuous, the continuity of the inverse map $\hat{X}_s^{t,\cdot}$ easily follows. \square

Lemma 4.5. (*Norm Equivalence Principle*) Assume conditions (C.0), (C.1) (or (C.2)) and (C.3). Let $X_s^{t,x}$ be the solution of forward equation in FBSDE (1.3), ρ be a weighted function. Then there exist constants $c, C > 0$ such that for every $s \in [t, T]$, $\varphi \in L^1_\rho(\mathbb{R}^d; \mathbb{R}^k)$,

$$c \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx \leq \mathbb{E} \left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho(x) dx \right] \leq C \int_{\mathbb{R}^d} |\varphi(x)| \rho(x) dx, \quad (4.6)$$

and for every $\Psi \in L^1_\rho([t, T] \otimes \mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{aligned} c \int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho(x) dx ds &\leq \mathbb{E} \left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s, X_s^{t,x})| \rho(x) dx ds \right] \\ &\leq C \int_t^T \int_{\mathbb{R}^d} |\Psi(s, x)| \rho(x) dx ds. \end{aligned} \quad (4.7)$$

Here c and C depend on T, L, μ, ρ and the bounds of the first order derivatives of b, σ, h and f , but do not depend on the initial value x .

Proof. First, we take $\rho(x) := (1 + |x|^2)^q$, $q \geq 2$. We claim that there exist constants $c, C > 0$ such that

$$c \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(x)} \right] \leq C, \quad \forall y \in \mathbb{R}^d, \quad t \leq s \leq T. \quad (4.8)$$

Here $\hat{X}_s^{t,y}$ is the inverse flow of $X_s^{t,x}$, $J(\hat{X}_s^{t,y}) := \det \nabla \hat{X}_s^{t,y}$ is the determinant of the Jacobian matrix of $\hat{X}_s^{t,y}$. The existence of $\hat{X}_s^{t,y}$ is given in Theorem 4.4.

Now we prove (4.8). Assume that $T - h \leq t \leq T$ for some small $h > 0$. We substitute $x = \hat{X}_s^{t,y}$ into FBSDE (1.3) (see [12], pp. 234-237), with $X_s^{t, \hat{X}_s^{t,y}} = X_s^{t, \cdot} \circ \hat{X}_s^{t,y} = y$, then

$$\begin{cases} \hat{X}_s^{t,y} = y - \int_t^s b(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) dr - \int_t^s \sigma(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) dW_r, \\ Y_s^{t, \hat{X}_s^{t,y}} = h(X_T^{t, \hat{X}_s^{t,y}}) + \int_s^T f(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}, Z_r^{t, \hat{X}_s^{t,y}}) dr - \int_s^T Z_r^{t, \hat{X}_s^{t,y}} dW_r. \end{cases} \quad (4.9)$$

Here $\int_t^s \sigma(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) dW_r := \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r|_{x=\hat{X}_s^{t,y}}$. Others can be treated similarly. We differentiate with respect to y in (4.9) in order to get

$$\begin{aligned} \nabla \hat{X}_s^{t,y} &= I - \int_t^s b'_x(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) \nabla X_r^{t, \hat{X}_s^{t,y}} dr - \int_t^s b'_y(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) \nabla Y_r^{t, \hat{X}_s^{t,y}} dr \\ &\quad - \int_t^s \sigma'_x(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) \nabla X_r^{t, \hat{X}_s^{t,y}} dW_r - \int_t^s \sigma'_y(r, X_r^{t, \hat{X}_s^{t,y}}, Y_r^{t, \hat{X}_s^{t,y}}) \nabla Y_r^{t, \hat{X}_s^{t,y}} dW_r \\ &=: I + J_s^t(y). \end{aligned} \quad (4.10)$$

When we consider the upper bound, we can use the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[\frac{J(\hat{X}_s^{t,y}) \rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq \sqrt{\mathbb{E} |J(\hat{X}_s^{t,y})|^2} \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2} \leq \sqrt{C + C \mathbb{E} \|J_s^t(y)\|^2} \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2}.$$

When $s - t$ are small enough,

$$\begin{aligned} |J(\hat{X}_s^{t,y})|^2 &= |\det(I + J_s^t(y))|^2 \\ &\leq |1 + \text{Tr}(J_s^t(y)) + o(\|J_s^t(y)\|)|^2 \\ &\leq 3 \left(1 + |\text{Tr}(J_s^t(y))|^2 + o(\|J_s^t(y)\|^2) \right) \end{aligned}$$

$$\leq C \left(1 + \text{Tr}[J_s^t(y)(J_s^t(y))^*] \right).$$

For the lower bound, we note $J(\hat{X}_s^{t,y}) \geq 1 - c\|J_s^t(y)\|$. Similarly,

$$\mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \geq \mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] - c \sqrt{\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2} \sqrt{\mathbb{E}\|J_s^t(y)\|^2}.$$

It is obvious that both the upper and lower bounds relate on the estimates of $\mathbb{E}\|J_s^t(y)\|^2$, $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2$ and $\mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right]$. First calculate $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,x})}{\rho(x)} \right|^2$ and $\mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,x})}{\rho(x)} \right]$. From (4.9), applying Itô's formula to $(1 + |\hat{X}_s^{t,y}|^2)^q$, we have

$$\begin{aligned} (1 + |\hat{X}_s^{t,y}|^2)^q &= (1 + |y|^2)^q - 2q \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{q-1} X_r^{t,\hat{X}_s^{t,y}} b(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dr \\ &\quad - 2q \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{q-1} X_r^{t,\hat{X}_s^{t,y}} \sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}}) dW_r \\ &\quad - q(2q-1) \int_t^s (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^{q-2} |X_r^{t,\hat{X}_s^{t,y}}|^2 \|\sigma(r, X_r^{t,\hat{X}_s^{t,y}}, Y_r^{t,\hat{X}_s^{t,y}})\|^2 dr \\ &=: (1 + |y|^2)^q + S_s^t(y). \end{aligned} \quad (4.11)$$

It turns out that

$$1 - \frac{|S_s^t(y)|}{(1 + |y|^2)^q} \leq \frac{(1 + |\hat{X}_s^{t,y}|^2)^q}{(1 + |y|^2)^q} \leq 1 + \frac{|S_s^t(y)|}{(1 + |y|^2)^q}. \quad (4.12)$$

From (4.9), using a similar method as in the proof of Lemma 4.2, for $q \geq 2$, $r \in [s, T]$, there exist positive constants c , c_1 and c_2 only depending on q , L , μ and T such that

$$\mathbb{E} \sup_{s \leq r \leq T} (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^q + \mathbb{E} \sup_{s \leq r \leq T} (1 + |Y_r^{t,\hat{X}_s^{t,y}}|^2)^q \leq c(1 + |y|^2)^q.$$

Similarly

$$\mathbb{E} \sup_{t \leq r \leq s} (1 + |X_r^{t,\hat{X}_s^{t,y}}|^2)^q + \mathbb{E} \sup_{t \leq r \leq s} (1 + |Y_r^{t,\hat{X}_s^{t,y}}|^2)^q \leq c(1 + |y|^2)^q + \mathbb{E}|Y_s^{t,\hat{X}_s^{t,y}}|^{2q} \leq 2c(1 + |y|^2)^q.$$

So

$$\mathbb{E}|S_s^t(y)| \leq c_1 \mathbb{E} \int_t^s \left(1 + |X_r^{t,\hat{X}_s^{t,y}}|^2 + |Y_r^{t,\hat{X}_s^{t,y}}|^2 \right)^q dr \leq (s-t)c_1(1 + |y|^2)^q.$$

Therefore (4.12) leads to

$$1 - c_1(s-t) \leq \mathbb{E} \left[\frac{(1 + |\hat{X}_s^{t,y}|^2)^q}{(1 + |y|^2)^q} \right] = \mathbb{E} \left[\frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq 1 + c_1(s-t). \quad (4.13)$$

Using similar estimates, $\mathbb{E} \left| \frac{\rho(\hat{X}_s^{\tau,x})}{\rho(x)} \right|^2$ can be easily estimated as

$$1 - c_2(s-t) \leq \mathbb{E} \left[\frac{(1 + |\hat{X}_s^{t,y}|^2)^{2q}}{(1 + |y|^2)^{2q}} \right] = \mathbb{E} \left| \frac{\rho(\hat{X}_s^{t,y})}{\rho(y)} \right|^2 \leq 1 + c_2(s-t). \quad (4.14)$$

For $\mathbb{E}\|J_t^s(x)\|^2$, we consider (4.10), apply Itô's formula and use a similar method as in the proof of Lemma 4.3. Then there exists a constant $c_3 > 0$ only depending on L, μ, T and the bounds of the first order derivatives of b, σ, h and f such that

$$\mathbb{E} \sup_{t \leq r \leq T} (\|\nabla X_r^{t, \hat{X}_s^{t,y}}\|^2 + \|\nabla Y_r^{t, \hat{X}_s^{t,y}}\|^2) \leq c_3.$$

So

$$\mathbb{E}\|J_s^t(y)\|^2 \leq c_3(s-t). \quad (4.15)$$

From the result of (4.13), (4.14) and (4.15), the upper bound and the lower bound can be estimated as

$$B_{low} \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq B_{up}.$$

Here $B_{low} = 1 - c_1(s-t) - c\sqrt{c_2(s-t)}\sqrt{1+c_3(s-t)}$ and $B_{up} = \sqrt{C+c_3(s-t)}\sqrt{1+c_2(s-t)}$. If $s-t$ small enough, the lower bound $1 - c_1(s-t) - c\sqrt{c_2(s-t)}\sqrt{1+c_3(s-t)} > 0$. Therefore, we can take h small enough such that (4.8) holds for $T-h \leq s \leq T$. Note that c and C does not depend on the initial value y . So we use the flow property $\hat{X}_s^{t,y} = \hat{X}_r^{t,\cdot} \circ \hat{X}_s^{r,y}, \forall t \leq r \leq s \leq T$ (using Proposition 3.9) in order to drop the restriction $T-h \leq t \leq T$ and so extend the inequality (4.8) to the whole of $[t, T]$.

Finally, we prove (4.6), using the change of variable $y = X_s^{t,x}$, conditional expectation with respect to $\mathcal{F}_{t,s}$, and noting that $\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)}$ is $\mathcal{F}_{t,s}$ measurable, we get

$$\begin{aligned} \mathbb{E} \left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})| \rho(x) dx \right] &= \int_{\mathbb{R}^d} \mathbb{E} \left[\mathbb{E} \left[|\varphi(y)| \rho(y) \frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \middle| \mathcal{F}_{t,s} \right] \right] dy \\ &= \int_{\mathbb{R}^d} |\varphi(y)| \rho(y) \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] dy. \end{aligned}$$

By (4.8), $c \leq \mathbb{E} \left[\frac{J(\hat{X}_s^{t,y})\rho(\hat{X}_s^{t,y})}{\rho(y)} \right] \leq C, \quad \forall x \in \mathbb{R}^d, \quad t \leq s \leq T$ for any $y \in \mathbb{R}^d, s \in [t, T]$, we prove (4.6). Moreover, for function $(s, x) \mapsto \Psi(s, x)$ we consider $x \mapsto \Psi(s, x)$ by the same way as above, integrate with respect to $s \in [t, T]$ to get (4.7). So the lemma is proved. \square

Next, we will use the idea of [5], [31] to give a unique weak solution of PDE (1.1) via the solution of FBSDE (1.3). The outline of the proof is as follows: firstly, we construct a smotherized FBSDE (4.16) with C^∞ functions $(b^m, \sigma^m, f^m, h^m) \rightarrow (b, \sigma, f, h)$ as $m \rightarrow \infty$, and their solution $(X_{\cdot,m}^{t,\cdot}, Y_{\cdot,m}^{t,\cdot}, Z_{\cdot,m}^{t,\cdot}) \rightarrow (X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$. Secondly, by Theorem 3.11, $u^m(t, x) = Y_{t,m}^{t,x}$ is the classical solution of the corresponding smotherized PDE (4.17). Meanwhile, $u^m(t, x)$ also satisfies the weak formulation of smotherized PDE (4.19). Finally, by the norm equivalence result (Lemma 4.5) and the convergence of $Y_{\cdot,m}^{t,\cdot}$ to $Y^{t,\cdot}$ as $m \rightarrow \infty$ we can show that the weak formulation of smotherized PDE (4.19) converges to the weak formulation of PDE (4.1), and $u(t, x)$ is the weak solution of PDE (1.1).

Let the mollifier K_d be defined as: $K_d(x) := C_d \exp\left(\frac{-1}{1-|x|^2}\right)$ when $|x| < 1$ and $K_d(x) = 0$, when $|x| \geq 1$, where C_d is chosen so that $\int_{\mathbb{R}^d} K_d(x) dx = 1$. Denote $K_d^m(x) := m^d K_d(mx)$. Suppose that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Hölder-continuous function with exponent $\gamma \in (0, 1)$ and let us define for each $m > 0$,

$$\phi^m(x) := \int_{\mathbb{R}^d} K_d^m(x - x') \phi(x') dx'.$$

As a result, ϕ^m is a C^∞ function, and Hölder-continuous with exponent γ . Moreover, $\phi^m \rightarrow \phi$ uniformly on \mathbb{R} as $m \rightarrow \infty$. Similarly, we define

$$\begin{aligned} h^m(x) &= \int_{\mathbb{R}^d} K_d^m(x - x') h(x') dx', \\ \sigma^m(r, x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^k} K_d^m(x - x') K_k^m(y - y') \sigma(r, x', y') dx' dy', \\ b^m(r, x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^k} K_d^m(x - x') K_k^m(y - y') b(r, x', y') dx' dy', \\ f^m(r, x, y, z) &= \int_{\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}} K_d^m(x - x') K_k^m(y - y') K_{k \times d}^m(z - z') f(r, x', y', z') dx' dy' dz'. \end{aligned}$$

It is easy to see that, $(b^m, \sigma^m, f^m, h^m)_{m \in \mathbb{N}}$ are C^∞ smooth functions such that for any $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$, $(b^m, \sigma^m, f^m, h^m)(t, x, y, z) \rightarrow (b, \sigma, f, h)(t, x, y, z)$ as $m \rightarrow \infty$. From the definition, one can easily check that h^m, b^m, σ^m and f^m also satisfy the monotone-Lipschitz condition which is independent of m . From Theorem 2.5 or Theorem 2.6, the smotherized FBSDEs

$$\begin{cases} X_{s,m}^{t,x} = x + \int_t^s b^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) dr + \int_t^s \sigma^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}) dW_r, \\ Y_{s,m}^{t,x} = h^m(X_{T,m}^{t,x}) + \int_s^T f^m(r, X_{r,m}^{t,x}, Y_{r,m}^{t,x}, Z_{r,m}^{t,x}) dr - \int_s^T Z_{r,m}^{t,x} dW_r, \end{cases} \quad (4.16)$$

has a unique solution $(X_{\cdot,m}^{t,\cdot}, Y_{\cdot,m}^{t,\cdot}, Z_{\cdot,m}^{t,\cdot})_{t \leq s \leq T} \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Remark 4.6. In (4.16), the functions h^m, b^m, σ^m and f^m satisfy the monotone-Lipschitz condition, in which the monotone-Lipschitz constants are independent of m . We can easily check that the corresponding estimates in Lemma 4.2 hold, in which the constants are also independent of m . Moreover, from Lemma 4.3 we can verify $\mathbb{E} \sup_{0 \leq s \leq T} \|\nabla Y_{s,m}^{t,x}\|^2 \leq C_{L,\lambda,T}$, where $C_{L,\lambda,T}$ is independent of m . Therefore $\mathbb{E} \|\nabla Y_{s,m}^{t,x}\|_{L_\rho^2}^2$ is uniformly bounded.

Lemma 4.7. Under conditions (C.1) (or (C.2)) and (C.3), $(X_{\cdot,m}^{t,\cdot}, Y_{\cdot,m}^{t,\cdot}, Z_{\cdot,m}^{t,\cdot}) \rightarrow (X_{\cdot}^{t,\cdot}, Y_{\cdot}^{t,\cdot}, Z_{\cdot}^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$.

Proof. Applying Itô's formula to $e^{-Ks} |X_{s,m}^{t,x} - X_s^{t,x}|^2$ and $e^{-Ks} |Y_{s,m}^{t,x} - Y_s^{t,x}|^2$, using a similar estimate as in the proof of Theorem 2.5 and the fact that $(b^m, \sigma^m, f^m, h^m) \rightarrow (b, \sigma, f, h)$ as $m \rightarrow \infty$, we have

$$(K - 2L^3 - L^2 - 5L - \frac{19}{20} - \frac{1}{N_1} - \frac{1}{N_2} - \frac{1}{N_4}) \|X_{r,m}^{t,x} - X_r^{t,x}\|_{M^{2,-K}([0,T]; L_\rho^2)}^2$$

$$\begin{aligned}
& + (2\mu - K - L^2 - 6L - \frac{3}{4} - \frac{1}{N_1} - \frac{1}{N_3} - \frac{1}{N_4}) \|Y_{r,m}^{t,x} - Y_r^{t,x}\|_{M^{2,-K}([0,T];L_\rho^2)}^2 \\
& + (\frac{4}{5} - \frac{1}{N_1}) \|Z_{r,m}^{t,x} - Z_r^{t,x}\|_{M^{2,-K}([0,T];L_\rho^2)}^2 \\
\leq & N_1 \|h^m(X_T^{t,x}) - h(X_T^{t,x})\|_{L_\rho^2}^2 \\
& + N_2 \|b^m(r, X_r^{t,x}, Y_r^{t,x}) - b(r, X_r^{t,x}, Y_r^{t,x})\|_{M^{2,-K}([0,T];L_\rho^2)}^2 \\
& + N_3 \|f^m(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\|_{M^{2,-K}([0,T];L_\rho^2)}^2 \\
& + N_4 \|\sigma^m(r, X_r^{t,x}, Y_r^{t,x}) - \sigma(r, X_r^{t,x}, Y_r^{t,x})\|_{M^{2,-K}([0,T];L_\rho^2)}^2 \\
\rightarrow & 0 \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Here we can choose $\frac{1}{N_2}, \frac{1}{N_3}, \frac{1}{N_4}, \frac{1}{N_5}$ small enough such that $\frac{1}{N_5} \leq \frac{1}{20}$, $\frac{1}{N_3} + \frac{1}{N_4} + \frac{1}{N_5} \leq \frac{1}{4}$ and $\frac{1}{N_2} + \frac{1}{N_4} + \frac{1}{N_5} \leq \frac{1}{20}$. Eventually, we have $(X_{\cdot,m}^{t,\cdot,n}, Y_{\cdot,m}^{t,\cdot,n}, Z_{\cdot,m}^{t,\cdot,n}) \rightarrow (X_{\cdot}^{t,\cdot,n}, Y_{\cdot}^{t,\cdot,n}, Z_{\cdot}^{t,\cdot,n})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$. \square

Lemma 4.8. *Under conditions (C.0), (C.1) (or (C.2)) and (C.3), $(\nabla X_{\cdot,m}^{t,\cdot}, \nabla Y_{\cdot,m}^{t,\cdot}, \nabla Z_{\cdot,m}^{t,\cdot}) \rightarrow (\nabla X_{\cdot}^{t,\cdot}, \nabla Y_{\cdot}^{t,\cdot}, \nabla Z_{\cdot}^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$, where $(\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ is the solution of ∇ FBSDE (3.20) and $(\nabla X_{s,m}^{t,x}, \nabla Y_{s,m}^{t,x}, \nabla Z_{s,m}^{t,x})$ is the solution of the smotherized ∇ FBSDEs (with coefficients $(b^m, \sigma^m, f^m, h^m)$).*

Proof. From Corollary 3.6, it is easy to check that $(\nabla X_{s,m}^{t,x}, \nabla Y_{s,m}^{t,x}, \nabla Z_{s,m}^{t,x})$ is the unique solution of the smotherized ∇ FBSDEs. Following the same procedure of the proof of Lemma 4.7, we can also show that $(\nabla X_{\cdot,m}^{t,\cdot}, \nabla Y_{\cdot,m}^{t,\cdot}, \nabla Z_{\cdot,m}^{t,\cdot}) \rightarrow (\nabla X_{\cdot}^{t,\cdot}, \nabla Y_{\cdot}^{t,\cdot}, \nabla Z_{\cdot}^{t,\cdot})$ in $M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ as $m \rightarrow \infty$. \square

Theorem 4.9. *Assume conditions (C.0), (C.1) (or (C.2)) and (C.3). Let $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ be the solution of FBSDE (1.3). If we define $u(t, x) = Y_t^{t,x}$, then $\sigma^*(t, x, u(t, x)) \nabla u(t, x)$ exists for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$, and $u(s, X_s^{t,x}) = Y_s^{t,x}$, $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s..*

Proof. Thanks to the structure of smotherized FBSDE (4.16), $(X_{s,m}^{t,x}, Y_{s,m}^{t,x}, Z_{s,m}^{t,x})$ is the unique solution of (4.16) and f^m, b^m, σ^m, h^m are C^∞ functions. From Theorem 3.11, the following smotherized PDEs has a unique solution $u^m(t, x) = Y_{t,m}^{t,x}$:

$$\begin{cases} \partial_t u^m(t, x) + \mathcal{L}^m u^m(t, x) + f^m(t, x, u^m(t, x), (\sigma^m)^*(t, x, u^m(t, x))) \nabla u^m(t, x) = 0, \\ u^m(T, x) = h^m(x), \end{cases} \quad (4.17)$$

where

$$\mathcal{L}^m = \frac{1}{2} \sum_{i,j=1}^d (\sigma^m (\sigma^m)^*)_{ij}(t, x, u^m(t, x)) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^m(t, x, u^m(t, x)) \frac{\partial}{\partial x_i}.$$

And also from Proposition 3.8 and Proposition 3.9, we have

$$u^m(s, X_{s,m}^{t,x}) = Y_{s,m}^{t,x}, \quad (\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x}) = Z_{s,m}^{t,x}. \quad (4.18)$$

Moreover, $u^m(t, x)$ also satisfies the following weak formulation of PDE (4.19): for any smooth test function $\Psi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{aligned} & \int_t^T \int_{\mathbb{R}^d} u^m(s, x) \partial_s \Psi(s, x) dx ds + \int_{\mathbb{R}^d} u^m(t, x) \Psi(t, x) dx - \int_{\mathbb{R}^d} h^m(x) \Psi(T, x) dx \\ & + \frac{1}{2} \int_t^T \int_{\mathbb{R}^d} \sigma^*(s, x, u^m(s, x)) \nabla \Psi(s, x) dx ds \\ & + \int_t^T \int_{\mathbb{R}^d} u^m(s, x) \operatorname{div} \left((b^m - \tilde{A}^m) \Psi(s, x) \right) dx ds \\ & = \int_t^T \int_{\mathbb{R}^d} f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x)) \Psi(s, x) dx ds. \end{aligned} \quad (4.19)$$

Here $\tilde{A}_j^m \triangleq \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma^m(\sigma^m)^*)_{i,j}(s, x, u(s, x))$, and $\tilde{A}^m = (\tilde{A}_1^m, \tilde{A}_2^m, \dots, \tilde{A}_d^m)^*$. Note that $u_m \in C^{1,2}$ (see Theorem 3.11), and it is also not difficult to prove that $\nabla u^m(s, \cdot)$ and the second order derivative $\nabla^2 u^m(s, \cdot)$ are bounded uniformly in m .

For any $m_1, m_2 \in \mathbb{N}$, by Lemma 4.5, we have

$$\begin{aligned} & \int_0^T \|u^{m_1}(s, x) - u^{m_2}(s, x)\|_{L_\rho^2} ds \\ & \leq C \int_0^T \mathbb{E} \|u^{m_1}(s, X_{s,m_1}^{0,x}) - u^{m_2}(s, X_{s,m_1}^{0,x})\|_{L_\rho^2} ds \\ & \leq C \mathbb{E} \int_0^T \left(\|u^{m_1}(s, X_{s,m_1}^{0,x}) - u^{m_2}(s, X_{s,m_2}^{0,x})\|_{L_\rho^2} + \|u^{m_2}(s, X_{s,m_2}^{0,x}) - u^{m_2}(s, X_{s,m_1}^{0,x})\|_{L_\rho^2} \right) ds \\ & \leq C \mathbb{E} \int_0^T \left(\|Y_{s,m_1}^{0,x} - Y_{s,m_2}^{0,x}\|_{L_\rho^2} + \|X_{s,m_1}^{0,x} - X_{s,m_2}^{0,x}\|_{L_\rho^2} \right) ds \\ & \leq C \mathbb{E} \int_0^T \left(\|Y_{s,m_1}^{0,x} - Y_s^{0,x}\|_{L_\rho^2} + \|Y_{s,m_2}^{0,x} - Y_s^{0,x}\|_{L_\rho^2} + \|X_{s,m_1}^{0,x} - X_s^{0,x}\|_{L_\rho^2} + \|X_{s,m_2}^{0,x} - X_s^{0,x}\|_{L_\rho^2} \right) ds \\ & \leq CT^{\frac{1}{2}} \mathbb{E} \int_0^T \left(\|Y_{s,m_1}^{0,x} - Y_s^{0,x}\|_{L_\rho^2}^2 + \|Y_{s,m_2}^{0,x} - Y_s^{0,x}\|_{L_\rho^2}^2 \right. \\ & \quad \left. + \|X_{s,m_1}^{0,x} - X_s^{0,x}\|_{L_\rho^2}^2 + \|X_{s,m_2}^{0,x} - X_s^{0,x}\|_{L_\rho^2}^2 \right) ds \\ & \rightarrow 0, \end{aligned} \quad (4.20)$$

when $m_1, m_2 \rightarrow \infty$, where C is a generic constant. Therefore $u^m(\cdot, \cdot)$ is a Cauchy sequence in $M^1([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$, denoted by $\hat{u}(\cdot, \cdot) \in M^1([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$ its limit. So

$$\int_0^T \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} ds \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (4.21)$$

Define $u(t, x) = Y_t^{t,x}$. Then by Proposition 3.9 and Fubini theorem, we have $u(s, X_s^{t,x}) = Y_s^{s, X_s^{t,x}} = Y_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.. By (4.20), (4.18), Lemma 4.7 and (4.21), we have

$$\int_0^T \|u(s, x) - \hat{u}(s, x)\|_{L_\rho^2} ds$$

$$\begin{aligned}
&\leq 2 \int_0^T \left(\|u(s, x) - u^m(s, x)\|_{L_\rho^2} + \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} \right) ds \\
&\leq C \mathbb{E} \int_0^T \left(\|u(s, X_s^{0,x}) - u^m(s, X_s^{0,x})\|_{L_\rho^2} + \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} \right) ds \\
&\leq C \mathbb{E} \int_0^T \left(\|u(s, X_s^{0,x}) - u^m(s, X_{s,m}^{0,x})\|_{L_\rho^2} + \|u^m(s, X_{s,m}^{0,x}) - u(s, X_s^{0,x})\|_{L_\rho^2} \right. \\
&\quad \left. + \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} \right) ds \\
&\leq C \mathbb{E} \int_0^T \left(\|u(s, X_s^{0,x}) - u^m(s, X_{s,m}^{0,x})\|_{L_\rho^2} + \|X_s^{0,x} - X_{s,m}^{0,x}\|_{L_\rho^2} + \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} \right) ds \\
&= C \mathbb{E} \int_0^T \left(\|Y_s^{0,x} - Y_{s,m}^{0,x}\|_{L_\rho^2} + \|X_s^{0,x} - X_{s,m}^{0,x}\|_{L_\rho^2} + \|u^m(s, x) - \hat{u}(s, x)\|_{L_\rho^2} \right) ds \\
&\rightarrow 0,
\end{aligned} \tag{4.22}$$

as $m \rightarrow \infty$. Hence $u(t, x) = \hat{u}(t, x)$ for a.e. $t \in [0, T]$, $x \in \mathbb{R}^d$.

Similarly, we can show that $(\sigma^m)^*(t, x, u^m(t, x)) \nabla u^m(t, x)$ is a Cauchy sequence in $M^1([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$. For $m_1, m_2 \in \mathbb{N}$, by Lemma 4.5, $\sigma^m(t, \cdot, \cdot)$ is bounded uniformly in m and Lipschitz continuous, the first and second order derivative of $u^m(t, \cdot)$ are bounded uniformly in m , we have

$$\begin{aligned}
&\int_0^T \|(\sigma^{m_1})^*(s, x, u^{m_1}(s, x)) \nabla u^{m_1}(s, x) - (\sigma^{m_2})^*(s, x, u^{m_2}(s, x)) \nabla u^{m_2}(s, x)\|_{L_\rho^2} ds \\
&\leq \int_0^T \|(\sigma^{m_1})^*(s, X_{s,m_1}^{t,x}, u^{m_1}(s, X_{s,m_1}^{t,x})) \nabla u^{m_1}(s, X_{s,m_1}^{t,x}) \\
&\quad - (\sigma^{m_2})^*(s, X_{s,m_1}^{t,x}, u^{m_2}(s, X_{s,m_1}^{t,x})) \nabla u^{m_2}(s, X_{s,m_1}^{t,x})\|_{L_\rho^2} ds \\
&\leq \int_0^T \left(\|(\sigma^{m_1})^*(s, X_{s,m_1}^{t,x}, u^{m_1}(s, X_{s,m_1}^{t,x})) \nabla u^{m_1}(s, X_{s,m_1}^{t,x}) \right. \\
&\quad \left. - (\sigma^{m_2})^*(s, X_{s,m_2}^{t,x}, u^{m_2}(s, X_{s,m_2}^{t,x})) \nabla u^{m_2}(s, X_{s,m_2}^{t,x})\|_{L_\rho^2} \right. \\
&\quad + \|(\sigma^{m_2})^*(s, X_{s,m_2}^{t,x}, u^{m_2}(s, X_{s,m_2}^{t,x})) \nabla u^{m_2}(s, X_{s,m_2}^{t,x}) \\
&\quad \left. - (\sigma^{m_2})^*(s, X_{s,m_2}^{t,x}, u^{m_2}(s, X_{s,m_2}^{t,x})) \nabla u^{m_2}(s, X_{s,m_1}^{t,x})\|_{L_\rho^2} \right. \\
&\quad \left. + \|(\sigma^{m_2})^*(s, X_{s,m_2}^{t,x}, u^{m_2}(s, X_{s,m_2}^{t,x})) \nabla u^{m_2}(s, X_{s,m_1}^{t,x}) \right. \\
&\quad \left. - (\sigma^{m_2})^*(s, X_{s,m_1}^{t,x}, u^{m_2}(s, X_{s,m_1}^{t,x})) \nabla u^{m_2}(s, X_{s,m_1}^{t,x})\|_{L_\rho^2} \right) ds \\
&\leq C \int_0^T \left(\|Z_{s,m_1}^{t,x} - Z_{s,m_2}^{t,x}\|_{L_\rho^2} + \|\nabla u^{m_2}(s, X_{s,m_2}^{t,x}) - \nabla u^{m_2}(s, X_{s,m_1}^{t,x})\|_{L_\rho^2} \right. \\
&\quad \left. + \|X_{s,m_1}^{t,x} - X_{s,m_2}^{t,x}\|_{L_\rho^2} + \|u^{m_2}(s, X_{s,m_2}^{t,x}) - u^{m_2}(s, X_{s,m_1}^{t,x})\|_{L_\rho^2} \right) ds \\
&\leq C \int_0^T \left(\|Z_{s,m_1}^{t,x} - Z_{s,m_2}^{t,x}\|_{L_\rho^2} + \|X_{s,m_1}^{t,x} - X_{s,m_2}^{t,x}\|_{L_\rho^2} \right) ds \\
&\leq C \int_0^T \left(\|Z_{s,m_1}^{t,x} - Z_s^{t,x}\|_{L_\rho^2} + \|Z_s^{t,x} - Z_{s,m_2}^{t,x}\|_{L_\rho^2} + \|X_{s,m_1}^{t,x} - X_s^{t,x}\|_{L_\rho^2} + \|X_s^{t,x} - X_{s,m_2}^{t,x}\|_{L_\rho^2} \right) ds \\
&\rightarrow 0,
\end{aligned}$$

when $m_1, m_2 \rightarrow \infty$ and C is a generic constant. So there exists a limit in $M^1([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$

of $(\sigma^m)^*(t, x, u^m(t, x)) \nabla u^m(t, x)$ and similarly to (4.22) and the above calculation, the limit is $\sigma^* \nabla u$,

$$\int_0^T \|(\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x) - \sigma^*(s, x, u(s, x)) \nabla u(s, x)\|_{L_\rho^2} ds \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Finally, we show that $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s.. This can be proved by the following

$$\begin{aligned} & \mathbb{E} \int_t^T \|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) - Z_s^{t,x}\|_{L_\rho^2} ds \\ & \leq \mathbb{E} \int_t^T \left(\|\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) - (\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x})\|_{L_\rho^2} \right. \\ & \quad \left. + \|(\sigma^m)^*(s, X_{s,m}^{t,x}, u^m(s, X_{s,m}^{t,x})) \nabla u^m(s, X_{s,m}^{t,x}) - Z_{s,m}^{t,x}\|_{L_\rho^2} + \|Z_{s,m}^{t,x} - Z_s^{t,x}\|_{L_\rho^2} \right) ds \\ & \rightarrow 0, \quad \text{as } m \rightarrow \infty. \end{aligned}$$

□

Theorem 4.10. Assume conditions (C.0), (C.1) (or (C.2)) and (C.3). Define $u(t, x) = Y_t^{t,x}$, where $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ is the solution of FBSDE (1.3). Then $u(t, x)$ is the unique weak solution of PDE (1.1) with $u(T, x) = h(x)$. Moreover, $u(s, X_s^{t,x}) = Y_s^{t,x}$, $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t, T]$, $x \in \mathbb{R}^d$ a.s..

Proof. From Theorem 4.9, we only need to verify that u is the unique weak solution of PDE (1.1) with $u(T, x) = h(x)$. By Lemma 4.5,

$$\begin{aligned} & \int_0^T \left(\|u(s, x)\|_{L_\rho^2}^2 + \|\sigma^*(s, x, u(s, x)) \nabla u(s, x)\|_{L_\rho^2}^2 \right) ds \\ & \leq C \mathbb{E} \int_0^T \left(\|u(s, X_s^{0,x})\|_{L_\rho^2}^2 + \|\sigma^*(s, X_s^{0,x}, u(s, X_s^{0,x})) \nabla u(s, X_s^{0,x})\|_{L_\rho^2}^2 \right) ds \\ & = C \mathbb{E} \int_0^T \left(\|Y_s^{0,x}\|_{L_\rho^2}^2 + \|Z_s^{0,x}\|_{L_\rho^2}^2 \right) ds < \infty. \end{aligned}$$

So $(u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x)) \in M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^2([0, T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. Now we verify that $u(t, x)$ satisfies (4.1) with $u(T, x) = h(x)$ by passing the limit in $L_\rho^2(dx ds)$ to (4.19). We only show the convergence of the last term. By Lipschitz condition, the fact that $f^m(t, x, y, z) \rightarrow f(t, x, y, z)$ in L_ρ^2 sense as $m \rightarrow \infty$, and the convergences in Theorem 4.9, for any $\Psi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$

$$\begin{aligned} & \left| \int_t^T \int_{\mathbb{R}^d} f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x)) \Psi(s, x) dx ds \right. \\ & \quad \left. - \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x)) \Psi(s, x) dx ds \right|^2 \\ & \leq C_p \int_t^T \|f^m(s, x, u^m(s, x), (\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x)) \\ & \quad - f(s, x, u(s, x), \sigma^*(s, x, u(s, x)) \nabla u(s, x))\|_{L_\rho^2}^2 ds \end{aligned}$$

$$\begin{aligned}
& +C_p \int_t^T \|f^m(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x) \\
& \quad - f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)\|_{L_\rho^2}^2 ds \\
\leq & C_{p,L} \int_t^T \left(\|u^m(s, x) - u(s, x)\|_{L_\rho^2}^2 \right. \\
& \quad \left. + \|(\sigma^m)^*(s, x, u^m(s, x)) \nabla u^m(s, x) - \sigma^*(s, x, u(s, x)) \nabla u(s, x)\|_{L_\rho^2}^2 \right) ds \\
& +C_p \int_t^T \|f^m(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x) \\
& \quad - f(s, x, u(s, x), \sigma^*(s, x, u(s, x))) \nabla u(s, x)\|_{L_\rho^2}^2 ds \rightarrow 0, \quad \text{as } m \rightarrow \infty.
\end{aligned}$$

Therefore $u(t, x)$ satisfies (4.1), so is a weak solution of (1.1) with $u(T, x) = h(x)$. Conversely, it is easy to prove that if PDE (1.1) has a weak solution $u(t, x)$, then $u(s, X_s^{t,x}) = Y_s^{t,x}$, $\sigma^*(s, X_s^{t,x}, u(s, X_s^{t,x})) \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ is the solution of FBSDE (1.3) by similar approximation method as we know that the smooth systems have such a relation. Thus we have the one to one correspondence between the solutions of the PDE and the FBSDE and the uniqueness of PDEs follows from the uniqueness of solutions of FBSDEs. \square

5 Infinite horizon FBSDEs and quasi-linear elliptic PDEs

In this section, we study the unique weak solution of elliptic type PDE (1.6) through the stationary solution of infinite horizon FBSDE (1.5). First, we consider a more general infinite horizon FBSDEs,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x}) dW_r, \\ e^{-Ks} Y_s^{t,x} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r, \end{cases} \quad (5.1)$$

for $s \geq t$. Here the functions $b : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^d$, $\sigma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^{d \times d}$, $f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$. We also assume that b , σ and f are measurable functions with respect to the Borelian σ -fields. We assume:

(D.1): Condition (A.1), $t \in [0, T]$ is changed to $t \geq 0$, and $2\mu - K - 2L^2 - 7L - 1 > 0$ with $K > 2L^3 + L^2 + 5L + 1$ is changed to $2\mu > K + 8\frac{1}{5}L + 1$ with $K > 5L + 1$.

(D.2): Integrability conditions:

$$\int_0^\infty e^{-Ks} (|b(s, 0, 0, 0)|^2 + \|\sigma(s, 0, 0)\|^2 + |f(s, 0, 0, 0)|^2) ds < \infty.$$

Before we study FBSDE (5.1), let us recall some results for BSDEs case.

Remark 5.1. Zhang-Zhao ([31]) considered the following finite horizon BSDEs with terminal value of Y being $h = 0$,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r, \\ Y_s^{t,x,n} = \int_s^n f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) dr - \int_s^n Z_r^{t,x,n} dW_r. \end{cases} \quad (5.2)$$

the unique solution of above finite BSDEs (5.2) for each $n \in \mathbb{N}$, is also a Cauchy sequence in the space $S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$. The limit of this sequence, denoted by $(Y_s^{t,x}, Z_s^{t,x})$, is a unique solution of following BSDEs with infinite horizon,

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r, \\ e^{-Ks} Y_s^{t,x} = \int_s^\infty e^{-Kr} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_s^\infty K e^{-Kr} Y_r^{t,x} dr - \int_s^\infty e^{-Kr} Z_r^{t,x} dW_r. \end{cases} \quad (5.3)$$

Here the functions satisfy the same conditions as those in FBSDE (5.1).

For the infinite horizon BSDEs (5.3), we have

Theorem 5.2. Under conditions (D.1) and (D.2), BSDEs (5.3) has a unique solution, i.e. there exists a unique process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying the integral form of (5.3).

Proof. Note that the SDE in (5.3) is slightly different from that in [31]. In both cases the SDE can be solved (see [18] or [12]). For the infinite horizon BSDE in (5.3), we can use a similar method as in the proof of Theorem 5.1 in [31] to prove that there exists a unique solution $(Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

It is easy to see that SDE in (5.3) has a unique solution $X^{t,\cdot} \in M^{2,-K}([0, T]; L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$. Then applying Itô's formula to $e^{-Ks}|X_s^{t,x}|^2$, taking the spatial integration $\rho^{-1}(x)dx$ on both sides and applying stochastic Fubini theorem, we have

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + (K - L) \mathbb{E} \int_t^s \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr \\ & \leq \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C \int_t^s e^{-Kr} dr. \end{aligned}$$

As $s \rightarrow \infty$, we have

$$\mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr < \infty.$$

By the B-D-G inequality,

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} |X_r^{t,x}|^2 \rho^{-1}(x) dx dr + \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C \int_t^T e^{-Kr} dr. \end{aligned}$$

As $T \rightarrow \infty$, we have

$$\mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx < \infty.$$

So $X_s^{t,x} \in S^{2,-K} \cap M^{2,-K}([t, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$. Following a similar procedure as in Step 2 of the proof of Theorem 2.5, we can extend our result from $[t, \infty)$ to $[0, \infty)$. So $X^{t,\cdot} \in S^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d))$. \square

Now we consider the infinite horizon FBSDE (5.1),

Theorem 5.3. *Under conditions (D.1) and (D.2), (5.1) has a unique solution, i.e. there exists a unique process $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ satisfying the spatial integral form of (5.1).*

Proof. We use the *Contraction Mapping Method*. Consider the map

$$\begin{aligned} \Xi & : M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})) \\ & \rightarrow M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})), \\ & (X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \mapsto (\bar{X}^{t,\cdot}, \bar{Y}^{t,\cdot}, \bar{Z}^{t,\cdot}), \end{aligned}$$

Given $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, $(\bar{X}_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})$ is defined as follows: for any $s \geq 0$

$$\bar{X}_s^{t,x} = x + \int_t^s b(r, \bar{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + \int_t^s \sigma(r, \bar{X}_r^{t,x}, Y_r^{t,x}) dW_r. \quad (5.4)$$

and

$$e^{-Ks} \bar{Y}_s^{t,x} = \int_s^\infty e^{-Kr} f(r, \bar{X}_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) dr + K \int_t^\infty e^{-Kr} \bar{Y}_r^{t,x} dr - \int_s^\infty e^{-Kr} \bar{Z}_r^{t,x} dW_r. \quad (5.5)$$

The existence and uniqueness of $(\bar{X}_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x})$ were given by Theorem 5.2. Similarly $(\bar{U}, \bar{V}, \bar{\mathcal{W}})$ can be defined in the same way as $(\bar{X}, \bar{Y}, \bar{Z})$ from (U, V, \mathcal{W}) . Now applying Itô's formula to $e^{-Ks} |\bar{X}_s^{t,x} - \bar{U}_s^{t,x}|^2$ and $e^{-Ks} |\bar{Y}_s^{t,x} - \bar{V}_s^{t,x}|^2$ and following the similar procedure as in the proof of Theorem 2.5, we have

$$\begin{aligned} & (K - 5L - \frac{9}{20}) \|\bar{X}^{t,\cdot} - \bar{U}^{t,\cdot}\|_{M^{2,-K}([t,T]; L_\rho^2(dx))}^2 + (2\mu - K - 5L) \|\bar{Y}^{t,\cdot} - \bar{V}^{t,\cdot}\|_{M^{2,-K}([t,T]; L_\rho^2(dx))}^2 \\ & + \frac{4}{5} \|\bar{Z}^{t,\cdot} - \bar{\mathcal{W}}^{t,\cdot}\|_{M^{2,-K}([t,T]; L_\rho^2(dx))}^2 + \mathbb{E} e^{-KT} \|\bar{X}_T^{t,\cdot} - \bar{U}_T^{t,\cdot}\|_{L_\rho^2(dx)}^2 + \mathbb{E} e^{-KT} \|\bar{Y}_T^{t,\cdot} - \bar{V}_T^{t,\cdot}\|_{L_\rho^2(dx)}^2 \\ & \leq (\frac{1}{4} + L) \|Y^{t,\cdot} - V^{t,\cdot}\|_{M^{2,-K}([t,T]; L_\rho^2(dx))}^2 + \frac{1}{4} \|Z^{t,\cdot} - \mathcal{W}^{t,\cdot}\|_{M^{2,-K}([t,T]; L_\rho^2(dx))}^2. \end{aligned} \quad (5.6)$$

Now let us construct the contraction mapping. We adopt the similar notation as in (2.11) with a replacement of the space $M^{2,-K}([t, T]; L_\rho^2)$ by $M^{2,-K}([t, \infty); L_\rho^2)$. Now we take the limit as $T \rightarrow \infty$ in (5.6), we have

$$(K - 5L - \frac{9}{20}) \bar{A} + (2\mu - K - 5L) \bar{B} + \frac{4}{5} \bar{C} \leq (\frac{1}{4} + L) B + \frac{1}{4} C.$$

If we assume $1 + 4L < \frac{2\mu - K - 5L}{\frac{4}{5}}$ and $K - 5L - \frac{9}{20} > 0$, then we have,

$$\left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}} \right) \bar{A} + (1 + 4L) \bar{B} + \bar{C} \leq \frac{5}{16} \left\{ \left(\frac{K - 5L - \frac{9}{20}}{\frac{4}{5}} \right) \bar{A} + (1 + 4L) B + C \right\}.$$

So the map Ξ is a contraction from $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \times M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$ into itself. Consequently, (5.1) has a unique solution $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ in $M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d}))$.

Finally, applying Itô's formula to $e^{-Ks}|X_s^{t,x}|^2$ and $e^{-Ks}|Y_s^{t,x}|^2$, taking integration $\rho^{-1}(x)dx$, applying the stochastic Fubini theorem and using the B-D-G inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^2 \rho^{-1}(x) dx \\ & \leq C_p \mathbb{E} \int_t^T \int_{\mathbb{R}^d} e^{-Kr} (|X_r^{t,x}|^2 + |Y_r^{t,x}|^2 + \|Z_r^{t,x}\|^2) \rho^{-1}(x) dx dr \\ & \quad + \int_{\mathbb{R}^d} e^{-Kt} x^2 \rho^{-1}(x) dx + C_{L,\mu} \int_t^T e^{-Kr} dr. \end{aligned}$$

As $T \rightarrow \infty$,

$$\mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |X_s^{t,x}|^2 \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq t} \int_{\mathbb{R}^d} e^{-Ks} |Y_s^{t,x}|^2 \rho^{-1}(x) dx < \infty.$$

Therefore, $(X_s^{t,x}, Y_s^{t,x}) \in S^{2,-K} \cap M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$. Following a similar proof of Theorem 2.5, we can extend this result from $[t, \infty)$ to $[0, \infty)$. \square

Now we desire to study the quasi-linear elliptic PDE (1.6) through its corresponding infinite horizon FBSDE (1.5). So far, a more general form of infinite horizon FBSDE (5.1) with time dependent functions has been studied and the existence and uniqueness result has been obtained. In the following we consider FBSDE (1.5) where coefficients are independent of time. If FBSDE (1.5) has a unique solution, then for an arbitrary T , we have

$$Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad 0 \leq s \leq T. \quad (5.7)$$

In Section 4, we deduced the following PDEs associated with FBSDE (5.7) in the weak sense

$$u(t, x) = u(T, x) + \int_t^T [\mathcal{L}u(s, x) + f(x, u(s, x), \sigma^*(x, u(s, x)) \nabla u(s, x))] ds. \quad (5.8)$$

Here $u(T, x) = Y_T^{T,x}$.

Definition 5.4. A process u is called a weak solution of a quasi-linear elliptic type PDE (1.6) if $(u, \sigma^* \nabla u) \in L_\rho^2(\mathbb{R}^d; \mathbb{R}^k) \otimes L_\rho^2(\mathbb{R}^d; \mathbb{R}^{k \times d})$ and for an arbitrary $\Psi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^k)$,

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d} \sigma^*(x, u(x)) \nabla u(x) \sigma^*(x, u(x)) \nabla \Psi(x) dx + \int_{\mathbb{R}^d} u(x) \operatorname{div} \left((b - \tilde{A}) \Psi(x) \right) dx \\ & = \int_{\mathbb{R}^d} f(x, u(x), \sigma^*(s, x, u(x)) \nabla u(x)) \Psi(x) dx. \end{aligned}$$

Here $\tilde{A}_j \triangleq \sum_{i=1}^d \frac{\partial}{\partial x_i} (\sigma \sigma^*)_{i,j}(x, u(x))$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_d)^*$.

To find the weak solution of (1.6), first we study the stationary property of the infinite horizon FBSDE (1.5). By using the connection between (5.7) and (5.8) proved in Section 4 such that $u(t, \cdot) = Y_t^{t,\cdot}$, we can transfer the stationary property from $Y_t^{t,\cdot}$ to $u(t, \cdot)$. Since $u(t, \cdot)$ is a deterministic function, together with the stationary property, immediately we have that $u(t, \cdot)$

is independent of t . Therefore, the quasi-linear parabolic PDE (5.8) turns into a quasi-linear elliptic type PDE (1.6), where u is the weak solution of such a PDE. Consider

(E.0): Functions $b(\cdot, \cdot) \in C^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)$; $f(\cdot, \cdot, \cdot) \in C^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}; \mathbb{R}^k)$; $\sigma(\cdot, \cdot) \in C_{l,b}^{1,\alpha}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^{d \times d})$ for some $\alpha \in (0, 1)$.

(E.1): Denote $g = (b, \sigma)$. Assume there exists a constant $L > 0$ such that for any $x, x_1, x_2 \in \mathbb{R}^d$, $y, y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k \times d}$,

$$\begin{aligned} |g(t, x_1, y_1) - g(t, x_2, y_2)|^2 &\leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ |f(t, x_1, y, z_1) - f(t, x_2, y, z_2)|^2 &\leq L(|x_1 - x_2|^2 + \|z_1 - z_2\|^2). \end{aligned}$$

For $q \geq 2$, where q is the exponent in the weight function ρ , there exist positive constants μ , $C_{q,L}$ and $C'_{q,L}$, where $C_{q,L}, C'_{q,L}$ only depending on q and L , satisfying $q\mu > K + C_{q,L}$ and $K > C'_{q,L}$ such that

$$\langle y_1 - y_2, f(x, y_1, z) - f(x, y_2, z) \rangle \leq -\mu|y_1 - y_2|^2, \quad |f(0, y, 0)|^2 \leq L(1 + |y|^2).$$

Theorem 5.5. Under condition **(E.1)**, (1.5) has a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$. Moreover

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |X_s^{t,x}|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx < \infty. \quad (5.9)$$

Proof. Since the condition **(E.1)** is stronger than conditions **(D.1)** and **(D.2)** in Theorem 5.3, so there exists a unique solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ of (1.5). We only need to prove $\mathbb{E}[\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} (|X_s^{t,x}|^p + |Y_s^{t,x}|^p) \rho^{-1}(x) dx] < \infty$. Applying Itô's formula to $e^{-pKr} |\bar{X}_r|^p$ and $e^{-pKr} |\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$ and following a similar procedure as in the proof of Lemma 3.3, we have

$$\begin{aligned} &\alpha' \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr + \beta' \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\ &+ \left\{ \frac{1}{2} p(p-1) - \frac{p}{16} \right\} \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\ &\leq C_p \int_{\mathbb{R}^d} (|b(0, 0)|^p + \|\sigma(0, 0)\|^p + |f(0, 0, 0)|^p) \rho^{-1}(x) dx + C_p \int_{\mathbb{R}^d} e^{-pKt} |x|^p \rho^{-1}(x) dx, \end{aligned}$$

where $\alpha' = (K - 4pL - \frac{p}{8} + \frac{1}{8} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8})$ and $\beta' = (p\mu - K - 4pL - \frac{p-2}{16} - \varepsilon - L(p-1)^2(1+\varepsilon) - \frac{1}{8})$ are positive from condition **(E.1)**. Thus there exists a constant $C_{p,L,\mu}$ only depending on p , L and μ such that

$$\begin{aligned} &\mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\ &+ \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr \\ &\leq C_{p,L,\mu} \int_{\mathbb{R}^d} (|b(0, 0)|^p + \|\sigma(0, 0)\|^p + |f(0, 0, 0)|^p) \rho^{-1}(x) dx + C_{p,L,\mu} \int_{\mathbb{R}^d} e^{-Kt} |x|^p \rho^{-1}(x) dx \\ &< \infty. \end{aligned} \quad (5.10)$$

Next, by the B-D-G inequality, the Cauchy-Schwarz inequality, the Young inequality and (5.10), we can obtain another estimation

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |X_s^{t,x}|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x}|^p \rho^{-1}(x) dx$$

$$\begin{aligned}
&\leq C_{p,L,\mu} \int_{\mathbb{R}^d} (|b(0,0)|^p + \|\sigma(0,0)\|^p + |f(0,0,0)|^p) \rho^{-1}(x) dx \\
&\quad + C_{p,L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |X_r^{t,x}|^p \rho^{-1}(x) dx dr + C_{p,L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^p \rho^{-1}(x) dx dr \\
&\quad + C_{p,L,\mu} \mathbb{E} \int_0^\infty \int_{\mathbb{R}^d} e^{-pKr} |Y_r^{t,x}|^{p-2} \|Z_r^{t,x}\|^2 \rho^{-1}(x) dx dr + C_{p,L,\mu} \int_{\mathbb{R}^d} e^{-Kt} |x|^p \rho^{-1}(x) dx \\
&< \infty.
\end{aligned}$$

The desired result is obtained. \square

Theorem 5.6. Assume conditions **(E.0)**, **(E.1)** and let $u(t, \cdot) \triangleq Y_t^{t, \cdot}$, where $(X^{t, \cdot}, Y^{t, \cdot}, Z^{t, \cdot})$ is the solution of (1.5). Then for an arbitrary T and $t \in [0, T]$, $u(t, \cdot)$ is a weak solution of (5.8). Moreover, $u(t, \cdot)$ is a.s. continuous with respect to t in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$.

Proof. Let $(Y_s^{t,x})_{s \geq 0}$, $(Y_s^{t',x})_{s \geq 0}$ be the solutions of (1.5) with t and t' respectively. First we claim that, for an arbitrary $T > 0$, $t, t' \in [0, T]$,

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t,x} - Y_s^{t',x}|^p \rho^{-1}(x) dx \leq C_p |t' - t|^{\frac{p}{2}}.$$

To see this, set

$$\begin{aligned}
\bar{X}_s &= X_s^{t',x} - X_s^{t,x}, \quad \bar{Y}_s = Y_s^{t',x} - Y_s^{t,x}, \quad \bar{Z}_s = Z_s^{t',x} - Z_s^{t,x}, \\
\bar{b}(s) &= b(X_s^{t',x}, Y_s^{t',x}) - b(X_s^{t,x}, Y_s^{t,x}), \quad \bar{\sigma}(s) = \sigma(X_s^{t',x}, Y_s^{t',x}) - \sigma(X_s^{t,x}, Y_s^{t,x}), \\
\bar{f}(s) &= f(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}) - f(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}), \quad s \geq 0.
\end{aligned}$$

From Theorem 5.5, we have (5.9). Applying Itô's formula to $e^{-pKr} |\bar{X}_r|^p$ and $e^{-pKr} |\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$ and following a similar procedure as in the proof of Lemma 3.3, we have

$$\begin{aligned}
&\mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{X}_r|^p \rho^{-1}(x) dx dr + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{Y}_r|^p \rho^{-1}(x) dx dr \\
&\quad + \mathbb{E} \int_t^\infty \int_{\mathbb{R}^d} e^{-pKr} |\bar{Y}_r|^{p-2} \|\bar{Z}_r\|^2 \rho^{-1}(x) dx dr \leq C_{L,\mu} |t' - t|^{\frac{p}{2}}.
\end{aligned}$$

Also by the B-D-G inequality, we have

$$\mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |\bar{X}_s|^p \rho^{-1}(x) dx + \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |\bar{Y}_s|^p \rho^{-1}(x) dx \leq C_{L,\mu} |t' - t|^{\frac{p}{2}}.$$

As a result, we have

$$\begin{aligned}
&\mathbb{E} \left(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{p}{2}} \\
&\leq C_{L,\mu} \mathbb{E} \sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-pKs} |Y_s^{t',x} - Y_s^{t,x}|^p \rho^{-1}(x) dx \left(\int_{\mathbb{R}^d} \rho^{-1}(x) dx \right)^{\frac{p-2}{2}} \leq C_{L,\mu} |t' - t|^{\frac{p}{2}}.
\end{aligned}$$

Noting $p > 2$, by Kolmogorov's continuity theorem, we have $t \mapsto Y_s^{t,x}$ is a.s. continuous for $t \in [0, T]$ under the norm $(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |\cdot|^2 \rho^{-1}(x) dx)^{\frac{1}{2}}$. Without losing any generality, assume that $t' > t$. Then we have

$$\lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} e^{-2Kt'} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \leq \lim_{t' \rightarrow t} \left(\sup_{s \geq 0} \int_{\mathbb{R}^d} e^{-2Ks} |Y_s^{t',x} - Y_s^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0,$$

a.s.. Notice $t' \in [0, T]$, so

$$\lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \quad (5.11)$$

Since $Y_{t'}^{t,\cdot} \in S^{2,-K} \cap M^{2,-K}([t, \infty); L_\rho^2(\mathbb{R}^d; \mathbb{R}^k))$, $Y_{t'}^{t,\cdot}$ is continuous w.r.t. t' in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^k)$. This is to say for each t ,

$$\lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \quad (5.12)$$

By (5.11) and (5.12)

$$\begin{aligned} & \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} \\ & \leq \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t',x} - Y_{t'}^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} + \lim_{t' \rightarrow t} \left(\int_{\mathbb{R}^d} |Y_{t'}^{t,x} - Y_t^{t,x}|^2 \rho^{-1}(x) dx \right)^{\frac{1}{2}} = 0 \quad a.s.. \end{aligned}$$

Therefore, for an arbitrary $T > 0$, $0 \leq t \leq T$, define $u(t, \cdot) = Y_t^{t,\cdot}$, then $u(t, \cdot)$ is a.s. continuous w.r.t. t in $L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)$. Moreover, recall the results in Section 3 and Section 4, it is easy to check that $u(t, x)$ is a weak solution for (5.8). \square

We now construct the measurable metric dynamical system through defining a measurable and measure preserving shift. Let $\theta_t : \Omega \rightarrow \Omega, t \geq 0$, be a measurable mapping on (Ω, \mathcal{F}, P) , defined by $\theta_t \circ W_s = W_{s+t} - W_t$. Then for any $s, t \geq 0$, (i) $P \circ \theta_t^{-1} = P$; (ii) $\theta_0 = I$, where I is the identity transformation on Ω ; (iii) $\theta_s \circ \theta_t = \theta_{s+t}$. Also for an arbitrary \mathcal{F} -measurable $\phi : \Omega \rightarrow H$, where H is a Hilbert space, set

$$\theta \circ \phi(\omega) = \phi(\theta(\omega)).$$

Theorem 5.7. Assume conditions **(E.0)** and **(E.1)**. Let $u(t, \cdot) \triangleq Y_t^{t,\cdot}$, where $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot})$ is the solution of (1.5). Then $u(t, \cdot)$ is the weak solution of quasi-linear elliptic PDE (1.6).

Proof. The main idea of the proof follows [31]. Note that the backward equations of (1.5) is equivalent to

$$Y_s^{t,x} = Y_T^{t,x} + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r, \quad \lim_{T \rightarrow \infty} e^{-KT} Y_T = 0 \quad a.s.. \quad (5.13)$$

First we will prove that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{s \geq 0}$ is a "perfect" stationary solution of (5.13), i.e.

$$\theta_r \circ X_s^{t,x} = X_{s+r}^{t+r,x}, \quad \theta_r \circ Y_s^{t,x} = Y_{s+r}^{t+r,x}, \quad \theta_r \circ Z_s^{t,x} = Z_{s+r}^{t+r,x}.$$

Here the integral w.r.t. Brownian motion W is a standard Itô's integral, θ_t is a shift with respect to W . Note for any $\{\tilde{h}(s, \cdot)\}_{s \geq 0}$ being an \mathcal{F}_s -measurable and locally square integrable stochastic process with values in $L^2_\rho(\mathbb{R}^d; \mathbb{R}^l)$, for an arbitrary $T > 0$ and $0 \leq t \leq T$,

$$\theta_r \circ \int_t^T \tilde{h}(s, \cdot) dW_s = \int_{t+r}^{T+r} \theta_r \circ \tilde{h}(s-r, \cdot) dW_s, \quad (5.14)$$

and

$$\theta_r \circ \int_t^T \tilde{h}(s, \cdot) ds = \int_{t+r}^{T+r} \theta_r \circ \tilde{h}(s-r, \cdot) ds. \quad (5.15)$$

From **(E.1)** and $(X^{t,\cdot}, Y^{t,\cdot}, Z^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, it is easy to see that b, σ, f are locally square integrable. Now applying θ_r on both sides of (5.13), by (5.14) and (5.15), we know that $(\theta_r \circ X_s^{t,x}, \theta_r \circ Y_s^{t,x}, \theta_r \circ Z_s^{t,x})$ satisfies

$$\begin{cases} \theta_r \circ X_s^{t,x} = x + \int_{t+r}^{s+r} b(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}) du + \int_{t+r}^{s+r} \sigma(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}) dW_u, \\ \theta_r \circ Y_s^{t,x} = \theta_r \circ Y_{T+r}^{t,x} + \int_{s+r}^{T+r} f(\theta_r \circ X_{u-r}^{t,x}, \theta_r \circ Y_{u-r}^{t,x}, \theta_r \circ Z_{u-r}^{t,x}) du - \int_{s+r}^{T+r} \theta_r \circ Z_{u-r}^{t,x} dW_u \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} \theta_r \circ Y_{T+r}^{t,x} = 0 \quad a.s. \end{cases} \quad (5.16)$$

On the other hand, from (5.13), it follows that

$$\begin{cases} X_{s+r}^{t+r,x} = x + \int_{t+r}^{s+r} b(X_{u-r}^{t+r,x}, Y_{u-r}^{t+r,x}) du + \int_{t+r}^{s+r} \sigma(X_{u-r}^{t+r,x}, Y_{u-r}^{t+r,x}) dW_u, \\ Y_{s+r}^{t+r,x} = Y_{T+r}^{t+r,x} + \int_{s+r}^{T+r} f(X_{u-r}^{t+r,x}, Y_{u-r}^{t+r,x}, Z_{u-r}^{t+r,x}) du - \int_{s+r}^{T+r} Z_{u-r}^{t+r,x} dW_u, \\ \lim_{T \rightarrow \infty} e^{-K(T+r)} Y_{T+r}^{t+r,x} = 0 \quad a.s. \end{cases} \quad (5.17)$$

By the uniqueness of the solution of (5.13) in the space $S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)) \otimes S^{2,-K} \cap M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^k)) \otimes M^{2,-K}([0, \infty); L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d}))$, it follows from comparing (5.16) and (5.17) that for any $r \geq 0$ and $t \geq 0$, in the space $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$, for all $s \geq t$,

$$\theta_r \circ X_s^{t,\cdot} = X_{s+r}^{t+r,\cdot}, \quad \theta_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \theta_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad a.s. \quad (5.18)$$

By the perfection procedure ([2],[3]), we can prove above identities (5.18) are true for all $s \geq t$, $r \geq 0$, but fixed $t \geq 0$ a.s. In particular, for any $t \geq 0$, in the space $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$

$$\theta_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot}, \quad \text{for all } r \geq 0 \quad a.s. \quad (5.19)$$

So we get from (5.19) that in the space $L^2_\rho(\mathbb{R}^d; \mathbb{R}^d) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^k) \otimes L^2_\rho(\mathbb{R}^d; \mathbb{R}^{k \times d})$

$$\theta_r \circ u(t, \cdot) = u(t+r, \cdot), \quad \text{for all } r \geq 0, \quad t \geq 0, \quad a.s.$$

From Theorem 5.6, we know that $u(t, \cdot) \triangleq Y_t^{t, \cdot}$ is the continuous weak solution of (5.8). But u is now deterministic. This means $u(t+r, \cdot) = u(t, \cdot)$ for all $t, r \geq 0$. So $u(t, \cdot)$ is independent of time t and (5.8) immediately turns to be (1.6). Therefore u is the weak solution of the quasi-linear elliptic PDE (1.6). \square

Remark 5.8. *As for the uniqueness of the solution of elliptic PDE (1.6), the idea is to show that a solution of the PDE is also the solution of infinite horizon FBSDE (1.5). This one-to-one correspondence will give the uniqueness of elliptic PDE following the uniqueness of the solution of the infinite horizon FBSDE. For this, we need to verify the regularity of the solution of BSDE (1.5) for sufficiently smooth coefficients following the idea of Section 3. In this case, the one-to-one correspondence follows from Itô's formula. Then we use the approximations by FBSDEs and PDEs with smooth coefficients and prove the desired convergence following the procedure of Section 4. However, due to the length of the paper, we will not include the full argument here. For the viscosity and classical solutions of the semilinear elliptic PDEs and related BSDEs, we refer to [7] and [19].*

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