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Quasilinear systems with linearizable characteristic webs

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Abstract

We classify quasilinear systems in Riemann invariants whose characteristic webs are linearizable on every solution. Although the linearizability of an individual web is a rather nontrivial differential constraint, the requirement of linearizability of characteristic webs on *all* solutions imposes simple second-order constraints for the characteristic speeds of the system. It is demonstrated that every such system with $n > 3$ components can be transformed by a reciprocal transformation to n uncoupled Hopf equations. All our considerations are local.

MSC: 35L40, 53A60

Keywords: hyperbolic systems, characteristics, webs, linearization problem, reciprocal transformations.

1 Introduction

In this paper we investigate the geometry of characteristics of quasilinear systems in Riemann invariants

$$R^i_t = \lambda^i(R) R^i_x, \tag{1}$$

$i = 1, \dots, n$ (no summation over i). Systems of this form govern a wide range of problems in pure and applied mathematics. Let us recall the basic concepts needed to state our main results.

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Semi-Hamiltonian property. System (1) is called *semi-Hamiltonian* if its characteristic speeds λ^i satisfy the constraints

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right),$$

here $\partial_i = \partial/\partial R^i$. It was shown by Tsarev [22] that this property is equivalent to integrability: semi-Hamiltonian systems (1) possess infinitely many conservation laws and commuting flows, and can be solved by the generalised hodograph method.

Linear degeneracy. System (1) is said to be *linearly degenerate* if its characteristic speeds satisfy the conditions

$$\partial_i \lambda^i = 0,$$

no summation, $i = 1, \dots, n$. Linear degeneracy is known to prevent the breakdown of smooth initial data, which is typical for genuinely nonlinear systems of type (1), and has been thoroughly investigated in the literature, see e.g. [19, 16, 21].

Reciprocal transformations. There exists a natural class of *reciprocal transformations* acting on systems of type (1). These are non-local changes of the independent variables, $(x, t) \to (\tilde{x}, \tilde{t})$, defined as

$$d\tilde{x} = A dt + B dx, \quad d\tilde{t} = M dt + N dx, \tag{2}$$

where the right-hand sides are two conservation laws of system (1), that is, two 1-forms that are closed on every solution (A, B, M, N are functions of R's). The transformed system reads

$$R^i_{\tilde{t}} = \tilde{\lambda}^i(R) R^i_{\tilde{x}},$$

where the transformed characteristic speeds are given by

$$\tilde{\lambda}^i = \frac{\lambda^i B - A}{M - \lambda^i N}. \tag{3}$$

Reciprocal transformations are known to preserve both the semi-Hamiltonian property and the linear degeneracy [8].

Characteristics. *Characteristic curves* of the i-th family are defined by the equation $dx + \lambda^i dt = 0$. Altogether, characteristics form an n-web (that is, n one-parameter families of curves) on every solution. We refer to [3, 4] for an introduction to the web geometry.

Parallelizable webs. An n-web is said to be *parallelizable* if it is *locally diffeomorphic* to n families of parallel lines (parallelizable 3-webs are also known as *hexagonal*). The following result provides a link between the above concepts:

Theorem 1 *[9, 10] The following conditions are equivalent:*
(a) System (1) has a parallelizable characteristic web on every solution.
(b) System (1) can be linearized by a reciprocal transformation.
(c) System (1) satisfies the following conditions:

- *semi-Hamiltonian property;*
- *linear degeneracy;*

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- *every quadruple by characteristic speeds has a constant cross-ratio (for $n = 3$ this condition becomes redundant).*

Linearizable webs. An n-web is said to be *linearizable* (rectifiable) if it is *locally diffeomorphic* to n families of lines, not necessarily parallel. We emphasize that the condition of linearizability is far more subtle than that of parallelizability, see [5, 6, 3, 15, 13, 2, 12, 1] and references therein for a discussion of the linearizability problem.

The aim of this paper is to establish an analogue of Theorem 1 for systems (1) whose characteristic webs are linearizable on every solution. Our main observation is that, although the linearizability of an individual web is a rather nontrivial differential constraint, the requirement of linearizability of characteristic webs on *all* solutions leads to simple second-order differential constraints for the characteristic speeds, see Theorem 3 below. In particular, any such system is reciprocally related to n uncoupled Hopf equations, $R^i_t = f^i(R^i)R^i_x$.

2 Linearizability of a planar web

Any projective transformation takes a linear web into a linear web. Thus, the linearizability of a web can be expressed in terms of projective differential invariants of the linearizing map, namely, its Schwarzian derivative [20]. This approach can be traced back to the pioneering work of Bol [5, 6]. The following form of the linearizability criterion will be most convenient for our purposes:

Theorem 2 (Hénaut, [15]) *A planar n-web formed by integral curves of vector fields*

$$\{V_i = \partial_t - \lambda^i(t,x)\partial_x\},$$

is linearizable if and only if there exists a solution E, F, G, H of the following system of PDEs,

$$\begin{aligned} &2G_{tx} + H_{xx} + F_{tt} - 6GG_x + 2HE_t + EH_t + 3FH_x - 3GF_t + 3HF_x = 0,\\ &G_{xx} + E_{tt} + 2F_{tx} + 3EG_t - 3FG_x + 3GE_t + HE_x + 2EH_x - 6FF_t = 0, \end{aligned} \tag{4}$$

subject to the constraints

$$E(\lambda^i)^3 + 3F(\lambda^i)^2 + 3G\lambda^i + H = V_i(\lambda^i), \quad i = 1, ..., n. \tag{5}$$

Remark. The functions E, F, G, H are linearly independent components of the 2-dimensional Schwarzian derivative of the linearizing map [1]. For $n \geq 4$, relations (5) uniquely define E, F, G, H, and equations (4) lead to explicit second-order constraints for λ^i. On the contrary, for $n = 3$, relations (5) are only sufficient to determine, say, E, F, G in terms of H, so that relations (4) give an over-determined second-order system for H. The analysis of this system is quite involved, in particular, differential constraints for λ^i appear at differential order six and higher. This explains why we treat the cases $n \geq 4$ and $n = 3$ separately.

3 $n \times n$ systems with linearizable characteristics ($n \geq 4$)

The main result of this paper is the following Theorem.

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Theorem 3 *For $n \geq 4$ the following conditions are equivalent:*

(a) System (1) has a linearizable characteristic web on every solution.

(b) System (1) can be transformed by a reciprocal transformation to n uncoupled Hopf equations.

(c) Characteristic speeds of system (1) satisfy the following conditions:

- *For every quadruple of pairwise distinct indices i, j, k, l one has a relation*

$$a_{ij}(\lambda^k - \lambda^l) + a_{kj}(\lambda^l - \lambda^i) + a_{lj}(\lambda^i - \lambda^k) = 0, \tag{6}$$

  here $a_{ij} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$. This allows one to introduce the parametrization $a_{ij} = p_j \lambda^i + q_j$.

- *The 1-forms*

$$\omega_{11} = \sum_{i=1}^{n} p_i \lambda^i dR^i, \quad \omega_{12} = \sum_{i=1}^{n} q_i \lambda^i dR^i, \quad \omega_{21} = \sum_{i=1}^{n} p_i dR^i, \quad \omega_{22} = \sum_{i=1}^{n} q_i dR^i, \tag{7}$$

  satisfy the $gl(2)$ structure equations,

$$d\omega_{ab} = \sum_{c=1}^{2} \omega_{ac} \wedge \omega_{cb}, \tag{8}$$

  $a, b = 1, 2$, which are equivalent to $\partial_j p_i = a_{ij} p_i$, $\partial_j q_i = a_{ij} q_i$.

A system satisfying either of the equivalent conditions (a), (b) or (c) is automatically semi-Hamiltonian.

Proof:

The equivalence of (b) and (c) can be seen as follows. Consider reciprocal transformation (2), note the relations $\partial_i A = \lambda^i \partial_i B$, $\partial_i M = \lambda^i \partial_i N$. Requiring that the transformed characteristic speed (3) depends on the variable R^i only, namely $\partial_j \tilde{\lambda}^i = 0$ for every $j \neq i$, we obtain all first-order partial derivatives of A, B, M, N. Comparing the relations $\partial_j \tilde{\lambda}^i = \partial_j \tilde{\lambda}^k = \partial_j \tilde{\lambda}^l = 0$ we obtain first-order relations (6). This allows one to set $a_{ij} = p_j \lambda^i + q_j$, leading to

$$dA = A\omega_{11} + B\omega_{12}, \quad dB = A\omega_{21} + B\omega_{22},$$

$$dM = M\omega_{11} + N\omega_{12}, \quad dN = M\omega_{21} + N\omega_{22},$$

where ω_{ab} are as in (7). The compatibility conditions of these relations are nothing but $gl(2)$ structure equations (8). A direct calculation shows that they are equivalent to $\partial_j p_i = a_{ij} p_i$, $\partial_j q_i = a_{ij} q_i$. By construction, in the new independent variables $\tilde{x}$, $\tilde{t}$ the system reduces to n uncoupled Hopf equations, $R^i_{\tilde{t}} = f^i(R^i) R^i_{\tilde{x}}$, where f^i are arbitrary functions (possibly, constants). Note that non-constant f^i can be reduced to R^i via a reparametrisation of Riemann invariants.

Since characteristics of a Hopf equation are straight lines, and reciprocal transformations are nothing but non-local changes of variables depending on a solution, this also establishes the implication (b) $\Longrightarrow$ (a) (equivalently, (c) $\Longrightarrow$ (a)).

As for the less elementary implication, (a) $\Longrightarrow$ (c), let us first consider the case $n = 4$. Equations (5) give E, F, G, H as functions of the characteristic speeds λ^i and their derivatives, $\lambda^i_x = \sum_k \lambda^i_k R^k_x$, $\lambda^i_t = \sum_k \lambda^i_k \lambda^k R^k_x$ (we fix a solution so that R^i and λ^i become functions of

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x, t). Substituting these expressions into (1) we obtain polynomial equations in the differential variables $R^i_x, R^i_{xx}, R^i_{xxx}$. Since the characteristic 4-web is required to be linearizable on *every* solution, the equations split with respect to these variables. Thus, equating to zero coefficients at R^i_{xxx} we get first-order relations (6). Further, taking coefficients at $R^i_{xx}R^j_x$ and differentiating relations (6) with respect to R^k yields all second-order relations for λ^i that are equivalent to $gl(2)$ structure equations (8).

Now the general case $n > 4$ readily follows. Let as fix four pairwise distinct indices i, j, k, l, say $1, 2, 3, 4$. Consider special solutions to system (1) such that $R^s = const,\ s > 4$. This reduces system (1) to a 4-component system for $R^1, \dots, R^4$, with linearizable characteristic 4-webs. Therefore we have all relations (6), as well as all other necessary conditions involving indices $1, 2, 3, 4$. The rest follows from the fact that every linearizability condition involves maximum 4 distinct indices.

Finally, the semi-Hamiltonian property follows from the fact that a system of uncoupled Hopf equations is automatically semi-Hamiltonian, and reciprocal transformations preserve the semi-Hamiltonian property.

□

Remark. Note that the equations for A, B and M, N uncouple and the forms ω_{ab} verify $gl(2)$ structure equations. Therefore (A, B) and (M, N) can be interpreted as sections of a flat 2-dimensional vector bundle over the hodograph space.

Example. The simplest example of an n-component system with linearizable characteristics is provided by n uncoupled Euler equations,

$$R^i_t = R^i R^i_x.$$

Remarkably, it appears as the modulational Whitham system for the Benjamin-Ono equation [7]. Applying to this system reciprocal transformations one obtains generic systems with linearizable characteristics. Explicitly, let us take two conservation laws,

$$Adt + Bdx = [\sum((f^k)'R^k - f^k)]dt + [\sum(f^k)']dx,$$

$$Mdt + Ndx = [\sum((g^k)'R^k - g^k)]dt + [\sum(g^k)']dx,$$

where $f^k(R^k)$ and $g^k(R^k)$ are arbitrary functions, and prime indicates derivative. The transformed characteristic speeds take the form

$$\tilde{\lambda}^i = \frac{\lambda^i B - A}{M - \lambda^i N} = -\frac{\sum[(f^k)'(R^i - R^k) + f^k]}{\sum[(g^k)'(R^i - R^k) + g^k]}.$$

This formula gives characteristic speeds of *generic* systems with linearizable characteristics. Examples of this type appeared as hydrodynamic reductions of integrable hydrodynamic chains in [18]. Degenerations can be obtained by replacing some of the Euler equations by linear equations, $R^i_t = c^i R^i_x$, $c^i = const$. In particular, starting with n linear equations and applying reciprocal transformations one obtains

$$\tilde{\lambda}^i = -\frac{\sum(c^i - c^k)f^k(R^k)}{\sum(c^i - c^k)g^k(R^k)},$$

which is a general formula for characteristic speeds of linearly degenerate semi-Hamiltonian systems with constant cross-ratios [11]. In this limiting case the characteristic web is parallelizable on every solution.

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4 3 × 3 systems with linearizable characteristics

Even though for $n = 3$ relations (6) are vacuous, we can always represent a_{ij} in the form $a_{ij} = p_j\lambda^i + q_j$. Explicitly,

$$p_j = \frac{a_{ij} - a_{kj}}{\lambda^i - \lambda^k}, \quad q_j = \frac{a_{kj}\lambda^i - a_{ij}\lambda^k}{\lambda^i - \lambda^k},$$

$i, k \neq j$. Let us introduce the forms ω_{ab} by formulae (7), where the summation is from 1 to 3. We have

$$\omega_{11} = \frac{a_{31} - a_{21}}{\lambda^3 - \lambda^2}\lambda^1 dR^1 + \frac{a_{12} - a_{32}}{\lambda^1 - \lambda^3}\lambda^2 dR^2 + \frac{a_{23} - a_{13}}{\lambda^2 - \lambda^1}\lambda^3 dR^3,$$

$$\omega_{12} = \frac{a_{21}\lambda^3 - a_{31}\lambda^2}{\lambda^3 - \lambda^2}\lambda^1 dR^1 + \frac{a_{32}\lambda^1 - a_{12}\lambda^3}{\lambda^1 - \lambda^3}\lambda^2 dR^2 + \frac{a_{13}\lambda^2 - a_{23}\lambda^1}{\lambda^2 - \lambda^1}\lambda^3 dR^3,$$

$$\omega_{21} = \frac{a_{31} - a_{21}}{\lambda^3 - \lambda^2} dR^1 + \frac{a_{12} - a_{32}}{\lambda^1 - \lambda^3} dR^2 + \frac{a_{23} - a_{13}}{\lambda^2 - \lambda^1} dR^3,$$

$$\omega_{22} = \frac{a_{21}\lambda^3 - a_{31}\lambda^2}{\lambda^3 - \lambda^2} dR^1 + \frac{a_{32}\lambda^1 - a_{12}\lambda^3}{\lambda^1 - \lambda^3} dR^2 + \frac{a_{13}\lambda^2 - a_{23}\lambda^1}{\lambda^2 - \lambda^1} dR^3.$$

Proposition 4 *System (1) can be transformed by a reciprocal transformation to 3 uncoupled Hopf equations if and only if the forms ω_{ab} satisfy $gl(2)$ structure equations (8).*

Proof:

This proposition claim can be proved exactly as in Theorem (3). Namely, requiring $\partial_j\tilde{\lambda}^i = 0$ for any $j \neq i$, we obtain all first-order partial derivatives of A, B, M, N in the form

$$dA = A\omega_{11} + B\omega_{12}, \quad dB = A\omega_{21} + B\omega_{22},$$

$$dM = M\omega_{11} + N\omega_{12}, \quad dN = M\omega_{21} + N\omega_{22},$$

where ω_{ab} are as above. The compatibility conditions of these relations are the $gl(2)$ structure equations.

□

Due to the complexity of linearizability conditions for 3-webs, we were unable to prove the analogue of Theorem (3) for $n = 3$. Thus, we can only formulate the following conjecture.

Conjecture 5 *For 3-component system (1), the following conditions are equivalent:*
(a) System (1) has a linearizable characteristic 3-web on every solution.
(b) System (1) can be transformed by a reciprocal transformation to 3 uncoupled Hopf equations.
(c) The forms ω_{ab} satisfy $gl(2)$ structure equations (8).
Any system satisfying either of the above conditions is automatically semi-Hamiltonian.

Discussion:

The equivalence of (b) and (c) is proved in Proposition 4.

The implication (b) $\Longrightarrow$ (a), equivalently, (c) $\Longrightarrow$ (a), is also straightforward: characteristics of a Hopf equation are straight lines, and reciprocal transformations are non-local changes of variables depending on a solution.

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The main problem is the converse implication, (a) $\Longrightarrow$ (c), which is a highly nontrivial computational challenge due to the complexity of linearizability conditions for 3-webs. This calculation seems to be out of reach for the modern computer algebra systems. □

Remark. The Gronwall conjecture [14] states that, modulo projective equivalence, a linearizable non-hexagonal 3-web has a unique linear representation. While this is still open at the level of individual webs (for partial results see [6, 23, 1]), our results show that the natural analogue of this statement holds at the level of systems. Namely, if a system can be decoupled by a reciprocal transformation, then the transformation is unique up to linear changes of x and t.

5 Concluding remarks

It remains a considerable computational challenge to establish the implication $(a) \implies (c)$ of Conjecture 5. The difference with the case $n > 3$ can be explained geometrically as follows. Any 4-subweb of a planar n-web defines a unique projective connection∇. The 4-subweb is linearizable if and only if all web leaves are geodesics of ∇, and the curvature of the connection vanishes. In fact, the functions E, F, G, H defined by equations (5) (where i now runs over the indices of the 4-subweb) are, up to constant factors, the so-called Thomas coefficients of ∇ (see [17]). The flatness of ∇ manifests itself in equations (4).

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