Rigorous numerical approximation of escape rates

RIGOROUS NUMERICAL APPROXIMATION
OF ESCAPE RATES

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ABSTRACT. An interval map with holes is a mathematical model which is used in the study of nonequilibrium statistical mechanics. We use Ulam's method to approximate the escape rate for an interval map with holes and find a bound on the approximation error.

1. INTRODUCTION

Consider an infinite set of identical particles evolving independently in a box I according to a stochastic process T which throws some of the particles outside of I. If a particle escapes from I under the process T, it will never come back again. Such systems are of central interest in the study of nonequilibrium statistical mechanics [6, 7, 15].

A physical quantity associated to this problem is the escape rate of the particles from the box. In [6], a theoretic escape rate formula is found. However, it is often impossible to compute the exact escape rate using this formula.

Interval maps with holes are one possible mathematical model of such systems. In this paper we deal with Lasota-Yorke maps with holes [2, 12]. These maps, which were first introduced in [14], are defined as follows:

Let $T : \mathbb{R} \to \mathbb{R}$ be a piecewise monotonic and piecewise C^2 nonsingular transformation (with respect to Lebesgue measure) and piecewise expanding. Let H be a finite union of disjoint open intervals, $H = \bigcup_{j=1}^{L} H_j$, $H \subset [0,1]$, $T^n(H) \cap [0,1] = \emptyset$ for all $n \in \mathbb{N}$, and $T([0,1] \setminus H) = [0,1]$. Define $T_H : [0,1] \setminus H \to [0,1]$ to be the restriction of T to $[0,1] \setminus H$.

This model was studied in [2] where existence of an absolutely continuous conditionally invariant measure (see Definition 3.1) was proved under certain conditions. Our goal is to present a numerical technique to approximate the escape rate for an interval map with holes and find a bound on the approximation error. This technique is based on Ulam's scheme [16] and it relies on the abstract perturbation result of [9] and its numerical counterpart [11].

In Section 2 we present the abstract stability theorem of [9]. In fact, we need the statement of the this result since the constants involved in it are needed in

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our numerical computations. Section 3 is devoted to numerical approximation of the escape rate. In particular, we prove, in Lemma 3.10, a result which enables us to use the computer and estimate the BV norm of the resolvent of a finite rank operator. Our main result in this section is obtaining an algorithm which can be implemented to compute the approximate escape rate. In Section 4, we obtain rates of convergence of the approximation scheme of Section 3. Our main result in this section is Theorem 4.6. In Section 5, we show that the algorithm of Section 3 is actually feasible to implement. In particular, we use the algorithm to give rigorous approximation of the escape rate of a certain Lasota-Yorke map with holes.

2. Abstract perturbation Result

Let $(I, \mathfrak{B}, \lambda)$ be the measure space where $I = [0, 1]$, $\mathfrak{B}$ is the Borel σ-algebra and λ is Lebesgue measure. Let $L^1 = L^1(I, \mathfrak{B}, \lambda)$. For $f \in L^1$, we define

$$Vf = \inf_{\overline{f}} \{ \text{var} \overline{f} : f = \overline{f} \text{ a.e.} \},$$

where

$$\text{var} \overline{f} = \sup \{ \sum_{i=0}^{l-1} |\overline{f}(x_{i+1}) - \overline{f}(x_i)| : 0 = x_0 < x_1 < \cdots < x_l = 1 \}.$$

We denote by BV the space of functions of bounded variation on I equipped with the norm $|| \cdot ||_{BV} = V(\cdot) + || \cdot ||_1$ [5].

Let $P_i : BV(I) \to BV(I)$ be two bounded linear operators, i=1,2. We assume that: For $f \in L^1$

(2.1) $$||P_i f||_1 \leq ||f||_1,$$

and $\exists \alpha \in (0, 1)$, $A > 0$ and $B \geq 0$ such that

(2.2) $$||P_i^n f||_{BV} \leq A \alpha^n ||f||_{BV} + B ||f||_1 \quad \forall n \in \mathbb{N} \ \forall f \in BV(I).$$

Further, we introduce the mixed operator norm:

$$|||P_i||| = \sup_{||f||_{BV} \leq 1} ||P_i f||_1.$$

For any bounded linear operator $P : BV \to BV$ with spectrum $\sigma(P)$, consider the set

$$V_{\delta,r}(P) = \{ z \in \mathbb{C} : |z| \leq r \text{ or } \text{dist}(z, \sigma(P)) \leq \delta \}.$$

Since the complement of $V_{\delta,r}(P)$ belongs to the resolvent of P, it follows that ([5] Lemma 11, VII.6.10)

$$H_{\delta,r}(P) = \sup_{z} \{ ||(z - P)^{-1}||_{BV} : z \in \mathbb{C} \setminus V_{\delta,r} \} < \infty.$$

Remark 2.1. α in (2.2) is an upper bound on the essential spectral radius of P_i [1].

Theorem 2.2. [9, 11] *Consider two operators $P_i : BV(I) \to BV(I)$ which satisfy (2.1) and (2.2). For $r \in (\alpha, 1)$, let*

$$n_1 = \lceil \frac{\ln 2A}{\ln r/\alpha} \rceil$$

$$C = r^{-n_1}; \qquad D = A(A + B + 2)$$

$$n_2 = \lceil \frac{\ln 8BDCH_{\delta,r}(P_1)}{\ln r/\alpha} \rceil.$$

If

$$|||P_1 - P_2||| \leq \frac{r^{n_1+n_2}}{8B(H_{\delta,r}(P_1)B + (1-r)^{-1})} \overset{def}{=} \varepsilon_1(P_1, r, \delta)$$

then for each $z \in \mathbb{C} \setminus V_{\delta,r}(P_1)$, we have

$$||(z - P_2)^{-1}f||_{BV} \leq \frac{4(A+B)}{1-r}r^{-n_1}||f||_{BV} + \frac{1}{2\varepsilon_1}||f||_1.$$

Set

$$\gamma = \frac{\ln(r/\alpha)}{\ln(1/\alpha)},$$

$$a = \frac{8[2A(A+B) + (1-r)^{-1}](A+B)^2 r^{-n_1} + 1}{1-r}$$

and

$$b = 2[(4(A+B)^2(D+B) + B)(1-r)^{-1}r^{-n_1} + B].$$

If

(2.3)

$$|||P_1 - P_2||| \leq \min\{\varepsilon_1(P_1, r, \delta), \left[\frac{r^{n_1}}{4B\left(H_{\delta,r}(P_1)(D+B) + 2A(A+B) + (1-r)^{-1}\right)}\right]^\gamma\}$$

$$\overset{def}{=} \varepsilon_0(P_1, r, \delta)$$

then for each $z \in \mathbb{C} \setminus V_{\delta,r}(P_1)$, we have

(2.4)

$$|||(z - P_2)^{-1} - (z - P_1)^{-1}||| \leq |||P_1 - P_2|||^\gamma (a||(z - P_1)^{-1}||_{BV} + b||(z - P_1)^{-1}||_{BV}^2).$$

Corollary 2.3. [9, 11] *If $|||P_1 - P_2||| \leq \varepsilon_1(P_1, r, \delta)$ then $\sigma(P_2) \subset V_{\delta,r}(P_1)$. In addition, if $|||P_1 - P_2||| \leq \varepsilon_0(P_1, r, \delta)$, then in each connected component of $V_{\delta,r}(P_1)$ that does not contain 0 both $\sigma(P_1)$ and $\sigma(P_2)$ have the same multiplicity; i.e., the associated spectral projections have the same rank.*

3. ULAM'S METHOD FOR ESCAPE RATES

Let T be an interval map with holes and denote by P the Perron-Frobenius operator associated with T [1]. Let $f \geq 0$, $\int_I f d\lambda = 1$ and denote by supp(f) the support of f. Suppose for a moment that a.e. x eventually escapes from I, then for some $n \geq 1$, $\lambda(T^{-n}(I)) < \lambda(\text{supp}(f))$ and we have

$$||P^n f||_1 = \int_I P^n f d\lambda = \int_{T^{-n}I} f d\lambda < \int_I f d\lambda = ||f||_1.$$

Thus, P eventually decreases the norm of functions. Consequently T cannot preserve a measure which is absolutely continuous with respect to Lebesgue measure. In this case, the measure which describes the dynamics is given by the following definition.

Definition 3.1. A probability measure μ on $[0, 1]$, $d\mu = g d\lambda$, is said to be a T-absolutely continuous conditionally invariant measure if there exists $0 < \rho < 1$ such that $Pg = \rho g$.

For results on the existence of absolutely continuous conditionally invariant measures we refer the reader to [2, 3, 12] and to the recent survey article [4].

The escape rate of the points through the holes under the iteration of T is defined by $-\ln \rho$ where ρ is the dominant eigenvalue of the Perron-Frobenius operator associated with T. In this section, we present a numerical method for estimating ρ. We assume:

(A1) $\exists \; \alpha \in (0,1)$, and $B_0 \geq 0$ such that $\forall f \in BV(I)$

$$VPf \leq \alpha Vf + B_0||f||_1;$$

(A2) P, as operator on $BV(I)$, has a dominant eigenvalue ρ such that:

$$\alpha < \rho < 1.$$

Remark 3.2. Since α is an upper bound on the essential spectral radius of P, any eigenvalue of P with modulus bigger than α is an isolated eigenvalue of finite multiplicity. Thus, $\exists \; r_0 > \alpha$ and $\delta_0 > 0$ such that any eigenvalue ρ_i of P, with $|\rho_i| > r_0$, satisfies:

  (1) $B(\rho_i, 2\delta_0) \cap B(0, r_0) = \emptyset$;
  (2) $B(\rho_i, 2\delta_0) \cap B(\rho_j, 2\delta_0) = \emptyset$ for $i \neq j$.

Remark 3.3. Given a Lasota-Yorke map with holes, α and B_0 in the inequality of (A1) are computable.

We now present Ulam's approximation of P. Let η be a finite partition of I into intervals. Let $\text{mesh}(\eta)$ be the mesh size of η; i.e, the maximum length of an interval in η, and let $\mathfrak{B}_\eta$ be the finite σ-algebra associated with η. For $f \in L^1$, let

$$\Pi_\eta f = \mathbb{E}(f|\mathfrak{B}_\eta),$$

where $\mathbb{E}(\cdot|\mathfrak{B}_\eta)$ denotes the conditional expectation with respect to $\mathfrak{B}_\eta$. Then, if $x \in I_\eta \in \eta$

$$(\Pi_\eta f)(x) = \frac{1}{\lambda(I_\eta)} \int_{I_\eta} f d\lambda.$$

Define

$$P_\eta = \Pi_\eta \circ P \circ \Pi_\eta.$$

P_η is called Ulam's approximation of P. Using the basis $\{\frac{1}{\lambda(I_\eta)}\chi_{I_\eta}\}$ in L^1, P_η can be represented by a (row) substochastic matrix acting on vectors from $\mathbb{R}^{d(\eta)}$ by left multiplication. The entries of Ulam's matrix are given by:

$$P_{I_\eta J_\eta} = \frac{\lambda(I_\eta \cap T^{-1}J_\eta)}{\lambda(I_\eta)}.$$

Lemma 3.4. *P_η has a positive dominant eigenvalue $\rho_\eta \leq 1$.*

Proof. By definition P_η is a non-negative matrix. Therefore, the existence of the positive dominant eigenvalue ρ_η follows from the Perron-Frobenius Theorem on non-negative matrices [10]. Moreover, for $f \in L^1$, we have

$$||P_\eta f||_1 = ||(\Pi_\eta \circ P \circ \Pi_\eta)f||_1 \leq ||P\Pi_\eta f||_1 \leq ||\Pi_\eta f||_1 \leq ||f||_1.$$

Thus, $\rho_\eta \leq 1$. $\square$

The following lemma mainly states that the projection operator Π_η does not increase variation when η is defined as above. This is a well known fact [13]. We will denote the identity operator by $\mathbf{1}$.

Lemma 3.5. *For $f \in BV(I)$ and $\varepsilon = mesh(\eta)$ we have*

    (1) $V(\Pi_\eta f) \le V(f)$;

    (2) $||f - \Pi_\eta f||_1 \le \varepsilon V(f)$.

Lemma 3.6.

    (1) P_η and P satisfy (2.2) with the same constants;

    (2) $|||P_\eta - P||| \le \Gamma\varepsilon$, where $\Gamma = \max\{\alpha + 1, B_0\}$ and $\varepsilon = mesh(\eta)$.

Proof. Firstly, for $f \in BV(I)$, by (A1), for any $n \in \mathbb{N}$

$$V P^n f \le \alpha^n V f + B_1 ||f||_1,$$

where $B_1 = \frac{B_0}{1-\alpha}$. Thus, P satisfies (2.2) with A=1, and $B = B_1 + 1$. By the first statement of Lemma 3.5, we have

$$V P_\eta f = V((\Pi_\eta \circ P \circ \Pi_\eta)f) \le V(P\Pi_\eta f) \le \alpha V f + B_0 ||f||_1.$$

Thus P_η satisfies (2.2) with A=1, and $B = B_1 + 1$. This completes the proof of the first part of the lemma. By the second statement of Lemma 3.5 we obtain

$$
\begin{aligned}
||(P_\eta - P)f||_1 &\le ||(\Pi_\eta \circ P \circ \Pi_\eta - P \circ \Pi_\eta)f||_1 + ||(P \circ \Pi_\eta - P)f||_1 \\
&= ||(\Pi_\eta - \mathbf{1})P\Pi_\eta f||_1 + ||P(\Pi_\eta - \mathbf{1})f||_1 \\
&\le \varepsilon V(P\Pi_\eta f) + \varepsilon V f \\
&\le \varepsilon \left((\alpha + 1)V f + B_0 ||f||_1 \right) \\
&\le \varepsilon \Gamma ||f||_{BV}.
\end{aligned}
$$

$\square$

Remark 3.7. Our goal is to approximate the dominant eigenvalue of P by the dominant eigenvalue of P_η for a suitable η. However, the constants in Theorem 2.2 depend on the norm of the resolvent of the operator. Since all these constants are computable for P_η (we will show this later in this section), we use Theorem 2.2 with P_η as the first operator and P as the second operator.

First, we need the following lemma to show that we can really use Theorem 2.2 with P_η as the first operator and P as the second operator. The proof is based on the proof of Lemma 4.2 of [11].

Lemma 3.8. [11] *Given P, $\delta > 0$ and $r > \alpha$, there exists $\varepsilon_2 > 0$ such that for each η with $0 < mesh(\eta) \le \varepsilon_2$, we have*

(3.1) $$mesh(\eta) \le (2\Gamma)^{-1} \varepsilon_0(P_\eta, r, \delta),$$

and

(3.2) $$|||P_\eta - P||| \le \frac{1}{2} \varepsilon_0(P_\eta, r, \delta).$$

Proof. Since (3.2) follows from (3.1) by Lemma 3.6, we only prove (3.1). Let P, P_η be P_1, P_2 of Theorem 2.2 respectively and $\varepsilon = mesh(\eta)$. There exist a δ_0 and an r_0, $0 < \delta_0 \le \delta$ and $\alpha < r_0 \le r$, such that for any eigenvalue ρ_i of P, with $|\rho_i| > \alpha$, we have:

(1) $B(\rho_i, 2\delta_0) \cap B(0, r_0) = \emptyset$;

(2) $B(\rho_i, 2\delta_0) \cap B(\rho_j, 2\delta_0) = \emptyset$ for $i \neq j$.

If $2\Gamma\varepsilon \leq \varepsilon_0(P, r_0, \delta_0/2)$ ($\varepsilon_0 \leq \varepsilon_1(P, r_0, \delta_0/2)$), then there exists a constant c_* such that

$$\sup_{z \in \mathbb{C} \setminus V_{\delta_0/2, r_0}(P)} ||(z - P_\eta)^{-1}||_{BV} \leq c_*.$$

In particular, $\sigma(P_\eta) \subset V_{\delta_0/2, r_0}(P)$. Let $\rho \in \sigma(P)$ with $|\rho| > r_0$. Then, by Corollary 2.3, there exists a $\rho_\eta \in \sigma(P_\eta)$ such that $|\rho - \rho_\eta| \leq \delta_0/2$. Thus, if $|z - \rho_\eta| \geq \delta_0$ we obtain

$$|z - \rho| \geq |\rho_\eta - z| - |\rho - \rho_\eta| \geq \delta_0/2.$$

Thus, $V_{\delta_0/2, r_0}(P) \subset V_{\delta_0, r_0}(P_\eta) \subseteq V_{\delta, r}(P_\eta)$; i.e.,

$$\mathbb{C} \setminus V_{\delta, r}(P_\eta) \subseteq \mathbb{C} \setminus V_{\delta_0/2, r_0}(P).$$

Therefore,

(3.3) $$\sup_{z \in \mathbb{C} \setminus V_{\delta, r}(P_\eta)} ||(z - P_\eta)^{-1}||_{BV} \leq \sup_{z \in \mathbb{C} \setminus V_{\delta_0/2, r_0}(P)} ||(z - P_\eta)^{-1}||_{BV} \leq c_*.$$

Thus, $H_{\delta, r}(P_\eta) \leq c_*$ and for fixed r and δ, $\varepsilon_0(P_\eta, r, \delta)$ is uniformly bounded from below. Set

$$\varepsilon_2 = \inf\{(2\Gamma)^{-1}\varepsilon_0(P_\eta, r, \delta) : \varepsilon = \text{mesh}(\eta) \leq (2\Gamma)^{-1}\varepsilon_0(P, r_0, \delta_0/2)\} > 0.$$

$\square$

Remark 3.9. The main difficulty in applying the above lemma is the computation of c_*. In fact $(z - P_\eta)^{-1}$ acts on the whole of BV and consequently, its norm cannot be found using the computer. In the next lemma, we show how we deal with this difficulty.

Lemma 3.10. *For $f \in BV(I)$ with $||f||_{BV} = 1$ and $z \in \mathbb{C} \setminus V_{\delta, r}(P_\eta)$ we have:*

(1) $||(z - P_\eta)^{-1}f||_{BV} \leq (\frac{B_0}{r - \alpha} + 1)||(z - P_\eta)^{-1}\Pi_\eta f||_1 + \frac{1}{r - \alpha} + \frac{3}{r}$;

(2) $||(z - P_\eta)^{-1}\Pi_\eta f||_1 \leq ||(z - P_\eta)^{-1}f||_{BV} + \frac{3}{r}$.

Proof. Let $f \in BV(I)$, $||f||_{BV} = 1$. Firstly, using the fact $P_\eta^i f = P_\eta^i \Pi_\eta f$ for $i \geq 1$ and the identity

$$(z - P_\eta)^{-1} = z^{-1}(z - P_\eta)^{-1}P_\eta + z^{-1} \cdot \mathbf{1},$$

we have

(3.4)
$$||(z - P_\eta)^{-1}(f - \Pi_\eta f)||_{BV} = ||(z^{-1}(z - P_\eta)^{-1}P_\eta + z^{-1} \cdot \mathbf{1})(f - \Pi_\eta f)||_{BV}$$
$$= ||\frac{f}{z} - \frac{\Pi_\eta f}{z}||_{BV} \leq \frac{1}{|z|}||(\Pi_\eta - \mathbf{1})f||_{BV} \leq \frac{3}{r}.$$

Now, write $(z - P_\eta)^{-1}\Pi_\eta f = h$, then $h = \frac{1}{z}(P_\eta h + \Pi_\eta f)$. Therefore, we have

(3.5)
$$Vh \leq \frac{1}{|z|}(VP_\eta h + 1)$$
$$\leq \frac{1}{r}(\alpha Vh + B_0||h||_1 + 1).$$

By (3.5), we obtain

$$Vh \leq \frac{B_0||h||_1 + 1}{r - \alpha},$$

and consequently,

$$(3.6) \qquad ||(z - P_\eta)^{-1}\Pi_\eta f||_{BV} \leq (\frac{B_0}{r - \alpha} + 1)||(z - P_\eta)^{-1}\Pi_\eta f||_1 + \frac{1}{r - \alpha}.$$

By (3.4) and (3.6) the first part of the lemma follows. For the second part, by (3.4), we have

$$(3.7) \qquad \begin{aligned} ||(z - P_\eta)^{-1}\Pi_\eta f||_1 &\leq ||(z - P_\eta)^{-1}\Pi_\eta f||_{BV} \\ &\leq ||(z - P_\eta)^{-1}f||_{BV} + ||(z - P_\eta)^{-1}(\Pi_\eta f - f)||_{BV} \\ &\leq ||(z - P_\eta)^{-1}f||_{BV} + \frac{3}{r}. \end{aligned}$$

$\square$

Remark 3.11. One can use the matrix representation of P_η and the computer to find $||(z - P_\eta)^{-1}\Pi_\eta f||_1$ since $\Pi_\eta f$ is just a vector.

Let

$$H^*_{\delta,r}(P_\eta) \stackrel{\text{def}}{=} \sup\{(\frac{B_0}{r - \alpha} + 1)||(z - P_\eta)^{-1}\Pi_\eta f||_1 + \frac{1}{r - \alpha} + \frac{3}{r} : ||f||_{BV} = 1, z \in \mathbb{C}\backslash V_{\delta,r}(P_\eta)\},$$

and

$$\varepsilon^*_0(P_\eta, \delta, r) = \min\{\frac{r^{n_1 + \lceil \frac{\ln 8BDCH^*_{\delta,r}(P_\eta)}{\ln r/\alpha} \rceil}}{8B(H^*_{\delta,r}(P_\eta)B + (1 - r)^{-1})},$$

$$\left[\frac{r^{n_1}}{4B\left(H^*_{\delta,r}(P_\eta)(D + B) + 2(1 + B) + (1 - r)^{-1}\right)}\right]^\gamma \}.$$

Lemma 3.12.
  (1) $\varepsilon^*_0(P_\eta, \delta, r)$ *is uniformly bounded below;*
  (2) $\varepsilon^*_0(P_\eta, \delta, r) \leq \varepsilon_0(P_\eta, \delta, r);$
  (3) $mesh(\eta) \leq (2\Gamma)^{-1}\varepsilon^*_0(P_\eta, \delta, r)$ *implies* $mesh(\eta)$ *satisfies (3.1).*

Proof. By the proof of Lemma 3.8, $\varepsilon_0(P_\eta, \delta, r)$ is uniformly bounded below by a positive number. Then using part 2 of Lemma 3.10, we see that $\varepsilon^*_0(P_\eta, \delta, r)$ is uniformly bounded below by a positive number. The second part follows from part (1) of lemma 3.10. The last part of the lemma is a consequence of the second part. $\square$

Based on the fact that $\varepsilon^*_0(P_\eta, \delta, r)$ is really computable, we present an algorithm which can be implemented to approximate ρ with a specified tolerance, τ, on the approximation error. This could not have been achieved using $\varepsilon_0(P_\eta, \delta, r)$.

Algorithm I
  (1) Given an interval map with holes whose P satisfies conditions (A1) and (A2). Then, α and B_0 are given. Let $\tau > 0$ be a given tolerance on the error.
  (2) Pick $\delta = \frac{1}{2k}$ and $r = \alpha + \frac{1}{k}$ for $k \in \mathbb{N}$ such that $\delta \leq \tau$ and $r < 1$.
  (3) Feed in a partition of I into intervals. Call it η.
  (4) Compute ε the mesh size of η.

(5) Find $P_\eta = (P_{I_\eta J_\eta})$ where

$$P_{I_\eta J_\eta} = \frac{\lambda(I_\eta \cap T^{-1} J_\eta)}{\lambda(I_\eta)}.$$

(6) Compute the following: $H_{\delta,r}^*(P_\eta)$, $n_1 = \lceil \frac{\ln 2}{\ln r/\alpha} \rceil$, $C = r^{-n_1}$, $D = 3 + B$, $n_2 = \lceil \frac{\ln 8BDCH_{\delta,r}^*(P_\eta)}{\ln r/\alpha} \rceil$, $\gamma = \frac{\ln(r/\alpha)}{\ln(1/\alpha)}$, $B = \frac{B_0}{1-r} + 1$, $\Gamma = \max\{1 + \alpha, B_0\}$.

(7) Check if $\varepsilon \le (2\Gamma)^{-1} \varepsilon_0^*(P_\eta, \delta, r)$.
    If (7) is not satisfied, feed in a new η with a smaller mesh size and repeat (3)-(7); otherwise, continue.

(8) List the eigenvalues of P_η whose modulus is bigger than r: $\rho_{\eta,i}$, $i = 1, \ldots, d$.
    (a) $P_\eta = QR$ decomposition, where Q is an orthogonal matrix and R is an upper triangular matrix.
    (b) $RQ = Q^T P_\eta Q$.
    (c) $P_{\eta,1} = P_\eta$, $P_{\eta,k} = Q_k R_k$, $P_{\eta,k+1} = Q_k^T P_{\eta,k} Q_k$.
    This iterative method is called the QR method [17]. Most mathematical software uses the QR method to compute eigenvalues.

(9) Find ρ_η the dominant eigenvalue of P_η and define:

$CL = \{$ all the eigenvalues from the list which are in $B(\rho_\eta, \delta)\}$.

(10) Check:
    (a) The list in (8) is not empty.
    (b) $\overline{B}(\rho_\eta, \delta) \cap \overline{B}(0, r) = \emptyset$;
    (c) If $\rho_{\eta,i} \notin CL$, then $\overline{B}(\rho_{\eta,i}, \delta) \cap \overline{B}(\rho_\eta, \delta) = \emptyset$.

(11) If (10) is satisfied, report $|\rho - \rho_\eta| < \delta$; otherwise, multiply k by 2 and repeat steps (2)-(11) starting with the last η that satisfied (7).

Proposition 3.13. *The above algorithm stops after finitely many steps.*

Proof. By Lemma 3.12, for each $\delta > 0$ and $r > \alpha$ $\exists$ $\varepsilon = \text{mesh}(\eta) > 0$ such that

$$\varepsilon < (2\Gamma)^{-1} \varepsilon_0^*(P_\eta, r, \delta).$$

Therefore, the internal loop of the algorithm (2)-(7) stops after finitely many steps. We now prove that the outer loop of the algorithm stops after finitely many steps. There exist a $K \in \mathbb{N}$, $K < +\infty$, such that $\delta = \frac{1}{2K} < \delta_0$, $r = \alpha + \frac{1}{K} < r_0$ and η with $\varepsilon = \text{mesh}(\eta) > 0$ such that

   (1) $\varepsilon < (2\Gamma)^{-1} \varepsilon_0^*(P_\eta, \alpha + \frac{1}{K}, \frac{1}{2K})$;
   (2) $\varepsilon < (2\Gamma)^{-1} \varepsilon_0(P, \alpha + \frac{1}{K}, \frac{1}{2K})$.

(1) implies $\sigma(P) \subset V_{\delta,r}(P_\eta)$ and (2) implies $\sigma(P_\eta) \subset V_{\delta,r}(P) \subset V_{\delta_0,r_0}(P)$. Thus, $\rho \in B(\rho_\eta, \delta)$, $CL \subset B(\rho, \delta)$, and any P_η eigenvalue which is not in CL is contained in $V_{\delta_0,r_0}(P) \setminus B(\rho, \delta)$. By Remark 3.2, (11) of the algorithm is satisfied for this K. $\square$

4. RATES OF CONVERGENCE

In this section, in addition to assumptions (A1) and (A2), we assume:

(A3) ρ, the dominant eigenvalue of P, is simple.

We first present an algorithm which enables us to find a P_η whose dominant eigenvalue, ρ_η, is simple and within a specified distance from ρ.

Algorithm II

(1) Given an interval map with holes whose P satisfies conditions (A1), (A2) and (A3). Then, α and B_0 are given. Let $\tau > 0$ be a given tolerance on the error.
(2)-(9) are the same as in Algorithm I.
(10) (a) Check that $CL = \{\rho_\eta\}$.
    (b) $B(\rho_\eta, 2\delta) \cap B(0, r) = \emptyset$;
    (c) If $\rho_{\eta,i} \notin CL$, then $B(\rho_{\eta,i}, 2\delta) \cap B(\rho_\eta, 2\delta) = \emptyset$.
(11) is the same as in Algorithm I.

Proposition 4.1. *Algorithm II stops after finitely many steps.*

Proof. The internal loop (2)-(7) of Algorithm II stops after finitely many steps for the same reasons as in Proposition 3.13. We now prove that the outer loop stops after finitely many steps. There exist a $K \in \mathbb{N}$, $K < +\infty$, such that $\delta = \frac{1}{2K} < \delta_0$, $r = \alpha + \frac{1}{K} < r_0$ and η with $\varepsilon = \text{mesh}(\eta) > 0$ such that

    (1) $\varepsilon < (2\Gamma)^{-1}\varepsilon_0^*(P_\eta, \alpha + \frac{1}{K}, \frac{1}{2K})$;
    (2) $\varepsilon < (2\Gamma)^{-1}\varepsilon_0(P, \alpha + \frac{1}{K}, \frac{1}{2K})$.

(1) implies $\sigma(P) \subset V_{\delta,r}(P_\eta)$ and (2) implies $\sigma(P_\eta) \subset V_{\delta,r}(P) \subset V_{\delta_0,r_0}(P)$. Thus, $\rho \in B(\rho_\eta, \delta)$ and $\rho_\eta \in B(\rho, \delta)$. Since ρ is simple, then by Corollary 2.3, ρ_η is the only eigenvalue of P_η in $B(\rho, \delta)$ and any non dominant P_η eigenvalue is contained in $V_{\delta_0,r_0}(P) \setminus B(\rho, \delta)$. By Remark 3.2, (11) of the algorithm is satisfied for this K. $\square$

Let $\varepsilon_k = \text{mesh}(\eta_k), \delta_k$ and r_k, such that $\varepsilon_k < (2\Gamma)^{-1}\varepsilon_0(P_\eta, \delta_k, r_k)$ and $|\rho_{\eta_k} - \rho| < \delta_k$, where ρ_{η_k} is the dominant eigenvalue of P_{η_k}, be given from Algorithm II.

Lemma 4.2. *Let $\varepsilon = \text{mesh}(\eta) < \varepsilon_k$, then for $z \in \mathbb{C} \setminus V_{\delta_k,r}(P_{\eta_k})$ we have*

$$\|(z - P_\eta)^{-1}\|_{BV} \le \left(\frac{4(A+B)}{1-r_k}r_k^{n_1} + \frac{1}{2\varepsilon_1} \right),$$

where $\varepsilon_1 = \frac{r_k^{n_1+n_2}}{8B(H_{\delta_k,r_k}(P_{\eta_k})B+(1-r_k)^{-1})}$.

Proof. We have

$$\||P_{\eta_k} - P_\eta|\| \le \||P_{\eta_k} - P|\| + \||P - P_\eta|\| \le 2\varepsilon_k\Gamma \le \varepsilon_0(P_{\eta_k}, r_k, \delta_k).$$

Therefore, we can use Theorem 2.2 with $P_{\eta_k} = P_1$ and $P_\eta = P_2$. Thus, we obtain

$$\|(z - P_\eta)^{-1}f\|_{BV} \le \frac{4(A+B)}{1-r_k}r_k^{n_1}\|f\|_{BV} + \frac{1}{2\varepsilon_1}\|f\|_1$$

(4.1)

$$\le \left(\frac{4(A+B)}{1-r_k}r_k^{n_1} + \frac{1}{2\varepsilon_1} \right)\|f\|_{BV}.$$

$\square$

Remark 4.3. We are now in a situation where $|\rho_{\eta_k} - \rho_\eta| < \delta_k$, $|\rho_{\eta_k} - \rho| < \delta_k$ and $|\rho_\eta - \rho| < \delta_k$ and ρ is at a distance of at least $2\delta_k$ from any non dominant eigenvalue of P, P_{η_k} and P_η.

Now, we can define the spectral projection [8] associated with ρ_η by

$$\Pi_{\rho_\eta} = \frac{1}{2\pi i} \int_{\partial B(\rho, 2\delta_k)} (z - P_\eta)^{-1} dz.$$

Lemma 4.4. *Let $\varepsilon = mesh(\eta) < \varepsilon_k$. We have*

(4.2) $|||\Pi_{\rho_\eta} - \Pi_\rho||| \le C_0 \varepsilon^\gamma,$

where

$$C_0 = 2\delta_k \Gamma^\gamma \left(a \left(\frac{4(A+B)}{1-r_k} r_k^{n_1} + \frac{1}{2\varepsilon_1^*} \right) + b \left(\frac{4(A+B)}{1-r_k} r_k^{n_1} + \frac{1}{2\varepsilon_1^*} \right)^2 \right),$$

and

$$\varepsilon_1^* = \frac{r_k^{n_1 + \lceil \frac{\ln 8BDCH_{\delta_k, r_k}^*(P_{\eta_k})}{\ln r/\alpha} \rceil}}{8B(H_{\delta_k, r_k}^*(P_{\eta_k})B + (1-r_k)^{-1})}.$$

Proof. Theorem 2.2 implies

(4.3)

$$|||\Pi_{\rho_\eta} - \Pi_\rho||| \le \frac{1}{2\pi} \int_{\{z \in \mathbb{C}: |z-\rho|=2\delta_k\}} |||(z-P_\eta)^{-1} - (z-P)^{-1}||| \, |dz|$$

$$\le \sup_{\{z \in \mathbb{C}: |z-\rho|=2\delta_k\}} \left(a||(z-P_\eta)^{-1}||_{BV} + b(||(z-P_\eta)^{-1}||_{BV})^2 \right) \cdot 2\delta_k \Gamma^\gamma \varepsilon^\gamma.$$

Since $z \in \mathbb{C} \setminus V_{\delta_k, r_k}(P_{\eta_k})$ and $\varepsilon_1 \ge \varepsilon_1^*$, Lemma 4.2 implies

$$\sup_{\{z \in \mathbb{C}: |z-\rho|=2\delta_k\}} ||(z-P_\eta)^{-1}||_{BV} \le \frac{4(A+B)}{1-r_k} r_k^{n_1} + \frac{1}{2\varepsilon_1^*}.$$

This completes the proof.   □

Remark 4.5. The constant C_0 in Lemma 4.4 is computable.

Theorem 4.6. *Let $\varepsilon = mesh(\eta) < \varepsilon_k$. Let g and g_η be the normalized eigenvectors associated with ρ and ρ_η respectively. Then:*

(1) $||g - g_\eta||_1 \le \frac{2\overline{C}}{1-\overline{C}\varepsilon^\gamma} \varepsilon^\gamma,$

(2) $|\rho - \rho_\eta| \le \frac{2\overline{C}}{1-\overline{C}\varepsilon^\gamma} \varepsilon^\gamma,$

where $\overline{C} = C_0(1 + \frac{B_0}{r_k - \alpha})$.

Proof. Let g be the normalized eigenvector corresponding to ρ; i.e., $g \ge 0$ and $||g||_1 = 1$. Define $\overline{g}_\eta \stackrel{\text{def}}{=} \Pi_{\rho_\eta} g$. Firstly, we estimate the BV norm of g. We have, by (A1),

$$\rho(Vg) = VPg \le \alpha Vg + B_0.$$

Therefore,

$$Vg \le \frac{B_0}{r_k - \alpha}$$

and

$$||g||_{BV} \le \frac{B_0}{r_k - \alpha} + 1.$$

We now use this and Lemma 4.4 to get an estimate on the L^1 norm of $\overline{g}_\eta$. We have

$$
\begin{aligned}
||g||_1 - ||\overline{g}_\eta||_1 &\leq ||g - \overline{g}_\eta||_1 \\
&= ||\Pi_\rho g - \Pi_{\rho_\eta} g||_1 \\
&\leq |||\Pi_\rho - \Pi_{\rho_\eta}||| \cdot ||g||_{BV} \\
&\leq C_0 \varepsilon^\gamma (\frac{B_0}{r_k - \alpha} + 1) = \overline{C} \varepsilon^\gamma.
\end{aligned}
$$

Thus, $||\overline{g}_\eta||_1 \in \left(1 - \overline{C}\varepsilon^\gamma, 1 + \overline{C}\varepsilon^\gamma\right)$. Let $g_\eta \overset{\text{def}}{=} \frac{\overline{g}_\eta}{||\overline{g}_\eta||_1}$. Then

$$
\begin{aligned}
||g_\eta - g||_1 &\leq ||\frac{\overline{g}_\eta}{||\overline{g}_\eta||_1} - \frac{g}{||\overline{g}_\eta||_1}||_1 + ||\frac{g}{||\overline{g}_\eta||_1} - g||_1 \\
&\leq \frac{1}{||\overline{g}_\eta||_1}||\overline{g}_\eta - g||_1 + \frac{|1 - ||\overline{g}_\eta||_1|}{||\overline{g}_\eta||_1}||g||_1 \\
&\leq \frac{\overline{C}\varepsilon^\gamma}{1 - \overline{C}\varepsilon^\gamma} + \frac{\overline{C}\varepsilon^\gamma}{1 - \overline{C}\varepsilon^\gamma} = \frac{2\overline{C}}{1 - \overline{C}\varepsilon^\gamma}\varepsilon^\gamma.
\end{aligned}
$$

Finally, we can obtain an estimate on the distance between ρ and ρ_η:

$$
\begin{aligned}
|\rho_\eta - \rho| &= |\,||P_\eta g_\eta||_1 - ||Pg||_1\,| \\
&= |\,||(\Pi_\eta \circ P \circ \Pi_\eta)g_\eta||_1 - ||Pg||_1\,| = |\,||Pg_\eta||_1 - ||Pg||_1\,| \\
&\leq ||Pg_\eta - Pg||_1 \leq ||g_\eta - g||_1 \leq \frac{2\overline{C}}{1 - \overline{C}\varepsilon^\gamma}\varepsilon^\gamma.
\end{aligned}
$$

$\square$

5. EXAMPLE

In this section we use Algorithm I to approximate the escape rate of a Lasota-Yorke map with holes. We analyse a simple example to demonstrate the performance and feasiblity of the algorithm.

Example 5.1. We consider the transformation

$$
T(x) = \left\{ \begin{array}{ll} 2.08x & \text{for } x < \frac{1}{2} \\ 2 - 2x & \text{for } x \geq \frac{1}{2} \end{array} \right. .
$$

Remark 5.2. We chose a piecewise linear example so that the escape rate may be computed directly, and thereafter compared to the estimate derived from the algorithm. In fact, for $U \in \mathfrak{B}(I)$, we have

$$
\lambda(T^{-1}(U)) = \frac{1}{2.08}\lambda(U) + \frac{1}{2}\lambda(U) = \frac{4.08}{4.16}\lambda(U).
$$

Thus, the exact dominant eigenvalue of P is $\rho = \frac{4.08}{4.16}$.

Calculations for the algorithm

First we will show that the Perron-Frobenius operator P associated with T satisfies (A1) and (A2). It is easily seen that for $f \in BV(I)$ we have

(5.1) $$ VPf \leq \frac{1}{2}Vf. $$

Thus, (A1) is satisfied with $\alpha = \frac{1}{2}$ and $B_0 = 0$. Consequently,

$$||Pf||_{BV} \leq \frac{1}{2}Vf + ||f||_1.$$

We now check that (A2) is satisfied without using Remark 5.2. Thus, the following is a procedure for checking that (A2) is satisfied without knowing ρ a priori. Let $H = (\frac{1}{2.08}, \frac{1}{2})$. Then $\lambda(H) = \frac{0.08}{4.16}$, where H is the hole in $[0,1]$ from which the particles escape under the evolution of T. Let $q \in (0, \frac{1/2 - \lambda(H)}{\lambda(H)})$ and define

$$E_q = \{f \in L^1 : Vf < q, \ f \geq 0, \ \int_I f d\lambda = 1\}.$$

Then E_q is a compact convex subset of $L^1(I)$. For $f \in E_q$

(5.2)
$$||Pf||_1 = 1 - \int_H f d\lambda \geq 1 - \lambda(H) \cdot ||f||_\infty$$

$$\geq 1 - \lambda(H)(Vf + 1) > \frac{1}{2} = \alpha.$$

Let

$$\overline{P}f = \frac{Pf}{||Pf||_1}.$$

Then, for $f \in E_q$,

$$V(\overline{P}f) \leq ||Pf||_1^{-1}V(Pf) < Vf.$$

Thus $\overline{P}(E_q) \subset E_q$. The Schauder-Tychonoff Theorem [5] implies $\overline{P}$ has a fixed point in E_q; i.e.,

$$\frac{Pg}{||Pg||_1} = g,$$

or

$$Pg = \rho g,$$

where $\rho = ||Pg||_1 > \alpha$. Thus, (A2) is satisfied.

Now, we implement Algorithm I and obtain rigorous approximation of ρ. From inequality (5.1), we obtain that $\alpha = \frac{1}{2}$ and $B_0 = 0$. Let the tolerance on the approximation error be $\tau = 0.01$. Let $k = 50$ and pick $\delta = \frac{1}{2k}$ and $r = \alpha + \frac{23.5}{k}$. Notice that we have modified r from $\alpha + \frac{1}{k}$ to $\alpha + \frac{23.5}{k}$. Since the algorithm works for any decreasing sequences of r's and δ's, this modification does not affect the proofs of the previous sections. We picked up this sequence of r's because it will require the outer loop of the algorithm to run only once. Table 1 shows the output of the algorithm after running the inner loop twice. In the first run, we took the mesh size $\varepsilon = 2 \times 10^{-4}$ which failed to satisfy $\varepsilon < (2\Gamma)^{-1}\varepsilon_0^*$. Then, we reduced the mesh size to $\varepsilon = 1.25 \times 10^{-4}$ and it satisfied $\varepsilon < (2\Gamma)^{-1}\varepsilon_0^*$. Thus, we obtained a rigorous approximation of ρ. We have used MATLAB in our computations. The main difficulty in implementing the algorithm is the computation of $H_{\delta,r}^*$. In the discussion below, we will explain how we have computed $H_{\delta,r}^*$ for this particular example.

We present the computation of $H_{\delta,r}^*$ when $\varepsilon = mesh(\eta) = 1.25 \times 10^{-4}$. The computation of $H_{\delta,r}^*$ when $\varepsilon = mesh(\eta) = 2 \times 10^{-4}$ is done in exactly the same way.

δ	10^{-2}	10^{-2}
r	0.97	0.97
ε	2×10^{-4}	1.25×10^{-4}
ρ_η	0.98076929201212	0.98076931645293
$H^*_{\delta,r}$	162.9499805	165.1996625
$(2\Gamma)^{-1}\varepsilon^*_0$	$1.30392654 \times 10^{-4}$	$1.289151061 \times 10^{-4}$
Inner Loop	Reduce Mesh Size	Pass
Outer Loop	Use Same δ and r	Pass
Report	No Report	$\|\rho - 0.98076931645293\| < 10^{-2}$

<center>TABLE 1</center>

After choosing $\varepsilon = mesh(\eta) = 1.25 \times 10^{-4}$ and setting the matrix P_η, we find its dominant eigenvalue ρ_η and its second largest eigenvalue. We noticed that ρ_η is simple and that the modulus of second largest eigenvalue is smaller than α. Thus, $\sigma(P_\eta) \setminus \{\rho_\eta\} \subset B(0, \alpha)$. We have (see [8])

$$(5.3) \qquad \|(z - P_\eta)^{-1}\|_1 \le \delta^{-1}\|\Pi_{\rho_\eta}\|_1 + \|R(z)\|_1,$$

where $\|\Pi_{\rho_\eta}\|$ is the rank-one projection associated with ρ_η and $R(z)$ is the resolvent of the operator $P_\eta(\mathbf{1} - \Pi_{\rho_\eta})$. Since the modulus of the second largest eigenvalue of P_η is smaller than α and $|z| > r > \alpha$, $R(z)$ can be represented by a convergent Neumann series. We have
(5.4)

$$\|R(z)\|_1 = \|\sum_{n=0}^{\infty} \frac{\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^n}{z^{n+1}})\|_1$$

$$\le \frac{1}{r}\left(\sum_{n=0}^{17} \frac{\|\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^n\|_1}{r^n} + \sum_{n=18}^{\infty} \frac{\|\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^n\|_1}{r^n}\right)$$

$$\le \frac{1}{r}\left[\sum_{n=0}^{17} \frac{\|\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^n\|_1}{r^n}\left(1 + \sum_{m=1}^{\infty} \left(\frac{\|\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^{18}\|_1}{r^{18}}\right)^m\right)\right]$$

$$< 34.27921938.$$

We have used the fact that $\|\left(P_\eta(\mathbf{1} - \Pi_{\rho_\eta})\right)^{18}\|_1 < 0.04169901783$, to obtain a convergent geometric series $\sum_{m=1}^{\infty} \left(\frac{\|\left(P_\eta(\mathbf{1}-\Pi_{\rho_\eta})\right)^{18}\|_1}{r^{18}}\right)^m$. Computing the L^1 norm of the projection

$$\|\Pi_{\rho_\eta}\|_1 = 1.257,$$

and using the definition of $H^*_{\delta,r}$ and inequality (5.3), we obtain that

$$H^*_{\delta,r} \le 165.1996625.$$

Therefore,

$$(2\Gamma)^{-1}\varepsilon^*_0 \ge 1.289151061 \times 10^{-4} > 1.25 \times 10^{-4} = \varepsilon.$$

Thus, the inner loop of the algorithm is satisfied. The outer loop is also satisfied since $\overline{B}(0, r) \cap \overline{B}(\rho_\eta, \delta) = \emptyset$. Thus, $0.9707 < \rho < 0.9908$.

Remark 5.3. The interval $(0.9707, 0.9908)$ does indeed contain the exact dominant eigenvalue of P which was calculated in Remark 5.2.

Remarks on estimating $H_{\delta,r}^*$

(1) It is clear that the main numerical difficulty in implementing the algorithm is the rigorous estimation (from above) of the quantity $H_{\delta,r}^*$. The calculation in the example above shows one way of doing this whenever P_η has a single, simple dominant eigenvalue outside of the disc $B(0,\alpha)$.

(2) In cases when the spectrum of P_η (and hence P) is more complicated on $(B(0,\alpha))^c$, the *rigorous* estimation of $H_{\delta,r}^*$ is an interesting and challenging problem.

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