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On derivative bounds for the rational quadratic Bézier paths

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Abstract

New derivative bounds for the rational quadratic Bézier paths are obtained, both for particular weight vectors and for classes of equivalent parametrisations. A comprehensive analysis of our bounds against existing bounds is made.

Key words: Derivative bounds, Quadratic Bézier paths, Rational parametrisation, Optimal parametrisation.

1 Introduction

Let $V = \{(v_0, v_1, v_2) : v_i \in \mathbb{R}^d\}$ and $\Omega = \{(w_0, w_1, w_2) : w_i \in \mathbb{R}^+\}$, where d is a natural number and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. The rational quadratic Bézier path with vertices $v \in V$ and weights $w \in \Omega$ may be written as $\sigma[v, w]$ where

$$\sigma[v, w](t) = \frac{w_0(1-t)^2 v_0 + 2w_1 t(1-t)v_1 + w_2 t^2 v_2}{w_0(1-t)^2 + 2w_1 t(1-t) + w_2 t^2}$$

for $t \in [0, 1]$. In terms of derivative bounds, most previous work, and that of this paper, is concerned with uniform bounds on $\sigma'[v, w]$ of the tensor product form

$$|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Phi(w), \tag{1}$$

for all $(v, w) \in V \times \Omega$ and $t \in [0, 1]$. Here, $\Delta_1(v) = \max_{0 \leq j \leq 1} |v_j - v_{j+1}|$, $\Delta_2(v) = \max_{0 \leq i,j \leq 2} |v_i - v_j|$ and $\tau \in \{1, 2\}$. A fundamental problem is to obtain such bounds with the weight function $\Phi : \Omega \to \mathbb{R}^+$ as small as possible. We shall refer to (1) as a *pointwise* bound.

For a given pair $(v, w) \in V \times \Omega$ it is natural to determine the *invariant* bound $2\Delta_\tau(v)\widetilde{\Phi}(w)$, associated with the pointwise bound $2\Delta_\tau(v)\Phi(w)$, where

$$\widetilde{\Phi}(w) = \min\{\Phi(\widetilde{w}) : \widetilde{w} \sim w\}.$$

Here, $\widetilde{w} \sim w$ if there exists $\lambda \in \mathbb{R}^+$ such that

$$\widetilde{w} = \mathrm{diag}(1, \lambda, \lambda^2)w = (w_0, \lambda w_1, \lambda^2 w_2). \tag{2}$$

It is well-known that the paths $\sigma[v, w]$ and $\sigma[v, \widetilde{w}]$ parametrise the same curve if $\widetilde{w} \sim w$. Writing

$$\mathcal{I}(w) = \frac{w_1}{(w_0 w_2)^{1/2}}$$

we have $\mathcal{I}(\widetilde{w}) = \mathcal{I}(w)$ whenever $\widetilde{w} \sim w$, and $\mathcal{I}$ therefore parametrises the space of equivalence classes.

The purpose of this paper is to obtain new pointwise and invariant bounds for quadratic Bézier paths. Importantly, we also compare our bounds with existing bounds in a comprehensive way, clarifying the merits of each approach that has been taken, with a view to new developments in higher degree cases.

Derivative bounds for the rational Bézier paths were first obtained in Floater (1992); this work inspired a number of recent papers in which improvements on the bounds were sought (see Wang et al (1997) for corresponding results for the rectangular Bézier surface patches). Improved bounds for Bézier paths were obtained by Hermann (1999), for the important quadratic and cubic cases and when $\tau = 1$. Hermann's approach is to employ a Möbius transformation, to normalise the path form, and capitalise on the induced symmetry to efficiently analyse the maximum value of the derivative of the normalised form using elementary calculus. Several authors have made use of an alternative, more algebraic, approach of degree-elevation and convexity, possibly combined with the de Casteljau algorithm. Such an approach was used in Selimovic (2005) for paths of arbitrary degree and $\tau \in \{1, 2\}$. Selimovic's bounds were improved upon by Zhang and Ma (2006), for degree less than seven when $\tau = 1$, and for arbitrary degree when $\tau = 2$.

In certain cases, we provide definitive comparisons of pointwise bounds. In particular, we shall prove that the pointwise bounds obtained by Zhang and Ma, for the quadratic case and $\tau \in \{1, 2\}$, are improved upon by our bounds obtained by degree-elevating to the quartic case (the lowest possible degree). We also show that Hermann's pointwise bound is superior to the bound given by Zhang and Ma in the quadratic case when $\tau = 1$. See the forthcoming Theorems 4.1 and 4.3. In most cases, however, it is not possible to find a simple characterisation of the weight space where one pointwise bound is superior to

another – but in several such cases, we provide a definitive comparison of the associated invariant bounds. To summarise in an over-simplified manner, we shall see that the invariant bounds obtained from Hermann's approach are not easy to lower using the algebraic arguments mentioned above; see Theorems 4.2 and 4.4 for some precise statements. In order to be in a position to make such comparisons, we follow the approach of Hermann to obtain a new pointwise bound for the quadratic case when $\tau = 2$ (see Theorem 2.1). It is also necessary to provide a theorem which establishes the full scope of the degree-elevation and convexity approach; in Section 3, we prove such a general theorem which confirms that bounds obtained from degree-elevating and convexity improve as the degree increases, as one would expect. This general result is applied in the quadratic case; furthermore, we obtain the associated invariant bounds when the degree is elevated to four and five, albeit for "small" values of the invariant $\mathcal{I}$ in the latter case; see Theorem 3.3. This is a pertinent case to consider because all existing bounds analysed in this paper are "sharp" for "large" values of the invariant $\mathcal{I}$. In Section 4 we make this precise and moreover conclude that the invariant bounds arising from Hermann's approach are essentially sharp as the invariant $\mathcal{I}$ approaches zero; these observations are also novel.

2 Bounds from Hermann's approach

A direct computation yields

$$\tfrac{1}{2}\sigma'[v,w](t) = \frac{w_0w_1(1-t)^2(v_1-v_0) + w_0w_2t(1-t)(v_2-v_0) + w_1w_2t^2(v_2-v_1)}{(w_0(1-t)^2 + 2w_1t(1-t) + w_2t^2)^2}. \tag{3}$$

For $\mu \in \mathbb{R}^+$ let $M_\mu : [0,1] \to [0,1]$ denote the Möbius transformation

$$M_\mu(t) = (\mu + (1-\mu)t)^{-1}t.$$

Then $\sigma[v,w](M_\mu(t)) = \sigma[v,\widetilde{w}](t)$ where $\widetilde{w} = (w_0, \mu^{-1}w_1, \mu^{-2}w_2) \sim w$. Choosing $\mu = (w_2/w_0)^{1/2}$ and homogeneity has the effect of normalising the weight $w \mapsto (1, \mathcal{I}(w), 1)$. Using (3) and the triangle inequality we find that

$$\frac{|\sigma'[v,\widetilde{w}](t)|}{2\Delta_\tau(v)} \leq \frac{\mathcal{I}(w)(1-t)^2 + (3-\tau)t(1-t) + \mathcal{I}(w)t^2}{((1-t)^2 + 2\mathcal{I}(w)t(1-t) + t^2)^2}. \tag{4}$$

The effect of this symmetrisation of the weight is that one may now easily compute *exactly* the maximum value over $[0,1]$ of the rational function on the right-hand side of (4). Indeed, we may immediately restrict our attention to $t \in [0, \frac{1}{2}]$ by invariance under $t \mapsto 1-t$, and furthermore

$$\frac{\mathcal{I}(w)(1-t)^2 + (3-\tau)t(1-t) + \mathcal{I}(w)t^2}{((1-t)^2 + 2\mathcal{I}(w)t(1-t) + t^2)^2} = \frac{\mathcal{I}(w)(1-2s) + (3-\tau)s}{(1-2s+2\mathcal{I}(w)s)^2},$$

where $s = t(1-t) \in [0, \frac{1}{4}]$. It is precisely this reduction in the degree of the variable that permits a straightforward exact computation of the maximum value. Elementary considerations using calculus show that

$$\max_{s\in[0,1/4]} \left| \frac{\mathcal{I}(w)(1-2s) + (3-\tau)s}{(1-2s+2\mathcal{I}(w)s)^2} \right| = \mathrm{H}_\tau(\mathcal{I}(w)), \tag{5}$$

where $\mathrm{H}_1(\mathcal{I}) = \max\{\mathcal{I}, 2(1+\mathcal{I})^{-1}\}$ and

$$\mathrm{H}_2(\mathcal{I}) = \begin{cases} (1+2\mathcal{I})(1+\mathcal{I})^{-2} & \text{for} \quad \mathcal{I} \in (0, C_0) \\ \frac{1}{8}(1-2\mathcal{I})^2(\mathcal{I}-1)^{-1}(1-2\mathcal{I}^2)^{-1} & \text{for} \quad \mathcal{I} \in [C_0, C_1] \\ \mathcal{I} & \text{for} \quad \mathcal{I} \in (C_1, \infty). \end{cases}$$

Here, $C_0 = \frac{1}{12}(1+\sqrt{73})$, $C_1 = \frac{1}{4}(1+\sqrt{5})$ and note that $0 < C_0 < C_1 < 1$. Setting

$$\Theta_\tau(w) = \mathrm{H}_\tau(\mathcal{I}(w)) \max\{(\tfrac{w_2}{w_0})^{\frac{1}{2}}, (\tfrac{w_0}{w_2})^{\frac{1}{2}}\},$$

via the chain rule, we have shown the following, due to Hermann (1999) when $\tau = 1$ using the above, and a new pointwise bound when $\tau = 2$.

Theorem 2.1. *For each* $(v, w) \in V \times \Omega$, $\tau \in \{1, 2\}$ *and* $t \in [0, 1]$,

$$|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Theta_\tau(w).$$

It is clear from (5) that $\Theta_2(w) \leq \Theta_1(w)$ for each $w \in \Omega$. Thus, our new bound in Theorem 2.1 for $\tau = 2$ is a strict improvement on what one obtains from the $\tau = 1$ bound obtained by Hermann combined with the triviality $\Delta_1(v) \leq \Delta_2(v)$. An obvious advantage of the above approach, where a Möbius mapping is used to transform an arbitrary $w \in \Omega$ into a weight vector whose components are functions of the invariant $\mathcal{I}(w)$, is that the corresponding invariant bound is trivial to compute. In the case of the pointwise bounds in Theorem 2.1, we have $\widetilde{\Theta}_\tau(w) = \mathrm{H}_\tau(\mathcal{I}(w))$ since, trivially,

$$\min_{\lambda\in\mathbb{R}^+} \max\{\lambda(\tfrac{w_2}{w_0})^{\frac{1}{2}}, \lambda^{-1}(\tfrac{w_0}{w_2})^{\frac{1}{2}}\} = 1$$

which is attained at $\lambda = (w_0/w_2)^{1/2}$. As we shall see in the sequel, the invariant bounds $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ are not easy to lower. [1]

[1] Of course, there is the potential cost of estimating the maximum value of a product of functions by the product of the maximum values of each function. This provides scope for improvement for "small" values of $\mathcal{I}(w)$.

3 New bounds from degree elevation and convexity

If $n \in \mathbb{N}$ and $\alpha_j, \beta_j \in \mathbb{R}^+$ for each $0 \leq j \leq n$, then we have the following convexity inequality

$$\frac{\sum_{j=0}^n \alpha_j}{\sum_{j=0}^n \beta_j} \leq \max_{0\leq j\leq n} \frac{\alpha_j}{\beta_j}, \tag{6}$$

which has been used by a number of authors in obtaining certain pointwise bounds. We use (6) in our subsequent theorem, which verifies the expected fact that the bounds obtained by degree-elevation improve as the degree increases. Although the focus of the current paper is the quadratic case, we state our result in general since we have not been able to find this in the literature. We use the notation $B_j^{(n)}(t) = (1-t)^{n-j}t^j$ and adopt the convention that $\binom{n}{j} = 0$ for $j < 0$ and $j > n$.

Theorem 3.1. *Suppose $a_0, \ldots, a_\ell, b_0, \ldots, b_m \in \mathbb{R}^+$, where $\ell, m \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ with $n \geq \max\{\ell, m\}$,*

$$\frac{\sum_{j=0}^\ell a_j B_j^{(\ell)}(t)}{\sum_{j=0}^m b_j B_j^{(m)}(t)} = \frac{\sum_{j=0}^n \alpha_j^{(n)} B_j^{(n)}(t)}{\sum_{j=0}^n \beta_j^{(n)} B_j^{(n)}(t)} \leq \max_{0\leq j\leq n} \left\{\frac{\alpha_j^{(n)}}{\beta_j^{(n)}}\right\}, \tag{7}$$

where, for each $0 \leq j \leq n$, $\alpha_j^{(n)} = \sum_{k=0}^\ell \binom{n-\ell}{j-k} a_k$ and $\beta_j^{(n)} = \sum_{k=0}^m \binom{n-m}{j-k} b_k$. Furthermore, the sequence $(\Phi_n)_{n\geq \max\{\ell,m\}}$ given by $\Phi_n = \max_{0\leq j\leq n}\{\alpha_j^{(n)}/\beta_j^{(n)}\}$ is non-increasing and hence convergent.

Proof. The formulae for $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ are well-known and the bound in (7) follows from (6). To see that (Φ_n) is non-increasing, first note that $\alpha_0^{(n+1)}/\beta_0^{(n+1)} = \alpha_0^{(n)}/\beta_0^{(n)}$ and $\alpha_{n+1}^{(n+1)}/\beta_{n+1}^{(n+1)} = \alpha_n^{(n)}/\beta_n^{(n)}$. Now fix $1 \leq j \leq n$. Then

$$\alpha_j^{(n+1)} = \sum_{k=0}^{\ell} \binom{n+1-\ell}{j-k} a_k = \sum_{k=0}^{\ell} \binom{n-\ell}{j-k} a_k + \sum_{k=0}^{\ell} \binom{n-\ell}{j-k-1} a_k$$

and similarly

$$\beta_j^{(n+1)} = \sum_{k=0}^{m} \binom{n-m}{j-k} b_k + \sum_{k=0}^{m} \binom{n-m}{j-k-1} b_k.$$

Using (6) it follows that

$$\frac{\alpha_j^{(n+1)}}{\beta_j^{(n+1)}} \leq \max\left\{\frac{\sum_{k=0}^\ell \binom{n-\ell}{j-k} a_k}{\sum_{k=0}^m \binom{n-m}{j-k} b_k}, \frac{\sum_{k=0}^\ell \binom{n-\ell}{j-k-1} a_k}{\sum_{k=0}^m \binom{n-m}{j-k-1} b_k}\right\} = \max\left\{\frac{\alpha_j^{(n)}}{\beta_j^{(n)}}, \frac{\alpha_{j-1}^{(n)}}{\beta_{j-1}^{(n)}}\right\} \leq \Phi_n.$$

Taking a maximum over j, it follows that $\Phi_{n+1} \leq \Phi_n$ as claimed. □

Using (3), the triangle inequality and Theorem 3.1 we obtain new pointwise bounds. In particular, with $(\ell, m) = (2, 4)$, and the inputs

$$(a_0, a_1, a_2) = (w_0 w_1, (3-\tau) w_0 w_2, w_1 w_2) \tag{8}$$

and

$$(b_0, b_1, b_2, b_3, b_4) = (w_0^2, 4w_0 w_1, 4w_1^2 + 2w_0 w_2, 4w_1 w_2, w_2^2),$$

we obtain the following decreasing sequences of pointwise bounds.

Theorem 3.2. *For each $(v, w) \in V \times \Omega, \tau \in \{1, 2\}, t \in [0, 1]$ and $n \geq 4$,*

$$|\sigma'[v, w](t)| \leq 2\Delta_\tau(v)\Phi_{\tau,n}(w),$$

where $\Phi_{\tau,n}(w)$ is equal to

$$\max_{0 \leq j \leq n} \frac{\binom{n-2}{j} w_0 w_1 + (3-\tau)\binom{n-2}{j-1} w_0 w_2 + \binom{n-2}{j-2} w_1 w_2}{\binom{n-4}{j} w_0^2 + 4\binom{n-4}{j-1} w_0 w_1 + 2\binom{n-4}{j-2}(2w_1^2 + w_0 w_2) + 4\binom{n-4}{j-3} w_1 w_2 + \binom{n-4}{j-4} w_2^2}$$

and satisfies $\Phi_{\tau,n+1}(w) \leq \Phi_{\tau,n}(w)$.

The decreasing sequences of bounds, of Theorems 3.1 and 3.2, provide a means of investigating the limits of the degree-elevation approach to the determination of bounds. However, for reasons that will become clear, we consider the cases $n \in \{4, 5\}$ separately and derive the corresponding invariant bounds (for "small" values of the invariant $\mathcal{I}(w)$ only in the latter case) which are new.

Theorem 3.3. *For each $w \in \Omega$, $\tau \in \{1, 2\}$,*

$$\widetilde{\Phi}_{\tau,4}(w) = \max\left\{\mathcal{I}(w), \frac{3-\tau+2\mathcal{I}(w)}{4\mathcal{I}(w)}, \frac{3-\tau+\mathcal{I}(w)}{1+2\mathcal{I}(w)^2}\right\}$$

and, for $\mathcal{I}(w)^2 < \frac{1}{6}(5 - 3\tau + \sqrt{9\tau^2 - 48\tau + 70})$,

$$\widetilde{\Phi}_{\tau,5}(w) = \max\left\{\mathcal{I}(w), \frac{3-\tau+3\mathcal{I}(w)}{1+4\mathcal{I}(w)}, \frac{3(3-\tau)+4\mathcal{I}(w)}{2+4\mathcal{I}(w)+4\mathcal{I}(w)^2}\right\}.$$

Proof. For $n \in \{4, 5\}$, $0 \leq j \leq n$, $x_0, x_1 \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$, let $\phi_{\tau,n,j}(x_0, x_1, \lambda)$ be given by

$$\frac{\binom{n-2}{j}\frac{1}{\lambda x_1} + (3-\tau)\binom{n-2}{j-1} + \binom{n-2}{j-2}\lambda x_0}{\binom{n-4}{j}\frac{1}{\lambda^2 x_0 x_1} + 4\binom{n-4}{j-1}\frac{1}{\lambda x_1} + 2\binom{n-4}{j-2}(2\frac{x_0}{x_1} + 1) + 4\binom{n-4}{j-3}\lambda x_0 + \binom{n-4}{j-4}\lambda^2 x_0 x_1}. \tag{9}$$

Since $\Phi_{\tau,n}(\widetilde{w}) = \max_{0 \leq j \leq n} \phi_{\tau,n,j}(\frac{w_1}{w_0}, \frac{w_2}{w_1}, \lambda)$, where $\widetilde{w} = (w_0, \lambda w_1, \lambda^2 w_2)$, we wish to calculate

$$\min_{\lambda \in \mathbb{R}^+} \max_{0 \leq j \leq n} \phi_{\tau,n,j}(x_0, x_1, \lambda). \tag{10}$$

Note that we have the following useful symmetry property

$$\phi_{\tau,n,j}(x_0,x_1,\lambda)=\phi_{\tau,n,n-j}(\tfrac{1}{x_1},\tfrac{1}{x_0},\tfrac{1}{\lambda}). \tag{11}$$

Moreover, when $\lambda=(x_0x_1)^{-1/2}$ we have $\phi_{\tau,n,j}(x_0,x_1,\lambda)=\phi_{\tau,n,n-j}(x_0,x_1,\lambda)$. It is easy to prove that for any $x_0,x_1\in\mathbb{R}^+$, $\phi_{\tau,n,j}(x_0,x_1,\cdot)$ is increasing for $j\in\{0,1\}$, or equivalently using (11), $\phi_{\tau,n,j}(x_0,x_1,\cdot)$ is decreasing for $j\in\{n,n-1\}$. Therefore, if $j\in\{0,1\}$ then

$$\max\{\phi_{\tau,n,j}(x_0,x_1,\lambda),\phi_{\tau,n,n-j}(x_0,x_1,\lambda)\}$$

is decreasing for $\lambda\in(0,(x_0x_1)^{-1/2})$ and increasing for $\lambda\in((x_0x_1)^{-1/2},\infty)$. When $n=4$, the remaining function $\phi_{\tau,4,2}(x_0,x_1,\lambda)$ is also decreasing for $\lambda\in(0,(x_0x_1)^{-1/2})$ and increasing for $\lambda\in((x_0x_1)^{-1/2},\infty)$. The claimed expression for $\widetilde{\Phi}_{\tau,4}$ follows. When $n=5$, the restriction $\frac{x_0}{x_1}<\frac{1}{6}(5-3\tau+\sqrt{9\tau^2-48\tau+70})$ implies $\phi_{\tau,5,2}(x_0,x_1,\cdot)$ is increasing, and consequently $\phi_{\tau,5,3}(x_0,x_1,\cdot)$ is decreasing. The claimed formula for $\widetilde{\Phi}_{\tau,5}$ now follows as above for $n=4$. □

The proof above demonstrates that for $n\in\{4,5\}$ the quantity in (10) is attained at $\lambda=(x_0x_1)^{-1/2}$. However for $n\geq 6$ this is not necessarily the case because of the following.

Proposition 3.4. *Suppose $w\in\Omega$, $\tau\in\{1,2\}$ and n is an even integer greater than or equal to 6. Then there exists a neighbourhood $\mathcal{N}_{\tau,n}$ of zero such that whenever $\mathcal{I}(w)\in\mathcal{N}_{\tau,n}$ the mapping $\lambda\mapsto\Phi_{\tau,n}(w_0,\lambda w_1,\lambda^2w_2)$ is not minimised at $\lambda=(w_2/w_0)^{1/2}$.*

We do not give a full proof of Proposition 3.4 here and simply indicate why it is true. Firstly, as in the above, let $\phi_{\tau,n,j}(x_0,x_1,\lambda)$ be given by the expression in (9). A direct argument using calculus shows that $\phi_{\tau,n,n/2}(x_0,x_1,\lambda)$ has a global maximum, as function of λ, which is uniquely attained at $\lambda=(x_0x_1)^{-1/2}$, provided that $n\geq 6$ and x_0/x_1 is sufficiently small. Now for each j, with $x_0=w_1/w_0$ and $x_1=w_2/w_1$, we have that $\phi_{\tau,n,j}(x_0,x_1,\frac{1}{(x_0x_1)^{1/2}})$ is equal to

$$\frac{\binom{n-2}{j}\mathcal{I}(w)+(3-\tau)\binom{n-2}{j-1}+\binom{n-2}{j-2}\mathcal{I}(w)}{\binom{n-4}{j}+4\binom{n-4}{j-1}\mathcal{I}(w)+2\binom{n-4}{j-2}(2\mathcal{I}(w)^2+1)+4\binom{n-4}{j-3}\mathcal{I}(w)+\binom{n-4}{j-4}}.$$

If $\mathcal{I}(w)\in\mathcal{N}_{\tau,n}$ is sufficiently small, by continuity and since

$$\max_{0\leq j\leq n}\left[\binom{n-4}{j}+2\binom{n-4}{j-2}+\binom{n-4}{j-4}\right]^{-1}\binom{n-2}{j-1}.$$

is uniquely attained at $j=n/2$, it follows that $\phi_{\tau,n,j}(x_0,x_1,\frac{1}{(x_0x_1)^{1/2}})$ is uniquely

maximised when $j = n/2$. Since

$$\Phi_{\tau,n}(w_0, \lambda w_1, \lambda^2 w_2) = \max_{0 \le j \le n} \phi_{\tau,n,j}(x_0, x_1, \lambda)$$

it follows, again by continuity, that this cannot be minimised when $\lambda = (x_0 x_1)^{-1/2}$, as claimed.

We remark that when the invariant $\mathcal{I}(w)$ is "large" (i.e. the complementary case to Proposition 3.4), all of the invariant bounds considered in this paper cannot be improved as we demonstrate at the end of the subsequent section.

4 A comparison and evaluation of bounds

We begin with $\tau = 1$ and note that, for each $(v, w) \in V \times \Omega$ and $t \in [0, 1]$, the bound $|\sigma'[v, w](t)| \le 2\Delta_1(v)\Lambda_1(w)$ was proved in Zhang and Ma (2006), where [2] $\Lambda_1(w) = \max\{\frac{w_0}{w_1}, \frac{w_1}{w_2}, \frac{w_1}{w_0}, \frac{w_2}{w_1}\}$. The following theorem shows that the pointwise bounds Θ_1 (due to Hermann) and $\Phi_{1,4}$ (of Theorem 3.2) are superior to Λ_1.

Theorem 4.1. *If $w \in \Omega$ then $\Theta_1(w) \le \Lambda_1(w)$ and $\Phi_{1,4}(w) \le \Lambda_1(w)$.*

Before offering some remarks and proof of this, we also note the following comparison of the invariant bounds determined by $\Theta_1, \Phi_{1,4}, \Phi_{1,5}$ and Λ_1.

Theorem 4.2. *If $w \in \Omega$ then $\widetilde{\Lambda}_1(w) = \max\{\mathcal{I}(w), \mathcal{I}(w)^{-1}\}$, and*

$$\widetilde{\Theta}_1(w) \le \widetilde{\Phi}_{1,4}(w) \le \widetilde{\Lambda}_1(w).$$

If, in addition $\mathcal{I}(w)^2 < \frac{1}{6}(2 + \sqrt{31})$, then we have

$$\widetilde{\Theta}_1(w) \le \widetilde{\Phi}_{1,5}(w) \le \widetilde{\Phi}_{1,4}(w) \le \widetilde{\Lambda}_1(w).$$

Given Theorem 4.1 it is natural to compare Θ_1 and $\Phi_{1,4}$ (and, of course, $\Phi_{1,n}$ for $n \ge 5$ in light of Theorem 3.2). We note that it is possible to find weights $w \in \Omega$ for which $\Phi_{1,4}(w) < \Theta_1(w)$, however, a full characterisation of such weights is not easy to describe. We also remark that Theorem 4.2 highlights that the invariant bounds are significantly easier to compare. As $\mathcal{I}(w)$ approaches zero, notice that the invariant bounds $\widetilde{\Lambda}_1(w)$ and $\widetilde{\Phi}_{1,4}(w)$ blow-up to infinity. Elevating the degree once more removes this singularity from the invariant bound – as is evident from the expression for $\widetilde{\Phi}_{1,5}(w)$ in Theorem 3.3.

[2] The commonly used notation in the literature for $\Lambda_1(w)$ is $\max\{\omega, \omega^{-1}\}$, where $\omega = \max_i \frac{w_i}{w_{i+1}}$; this notation is misleading because the two quantities are different.

Proof of Theorem 4.1. Observe that

$$\Lambda_1(w) = \max\{\mathcal{I}(w), \mathcal{I}(w)^{-1}\} \max\{(\tfrac{w_2}{w_0})^{\frac{1}{2}}, (\tfrac{w_0}{w_2})^{\frac{1}{2}}\},$$

and since $2(1+\mathcal{I})^{-1} \leq \mathcal{I}^{-1}$ for $\mathcal{I} \in (0,1]$, it follows that $\Theta_1(w) \leq \Lambda_1(w)$. To see that $\Phi_{1,4}(w) \leq \Lambda_1(w)$, note that

$$\Phi_{1,4}(w) = \max\left\{\frac{w_1}{w_0}, \frac{w_1}{w_2}, \frac{1}{2}\left(1+\frac{w_2}{w_1}\right), \frac{1}{2}\left(1+\frac{w_0}{w_1}\right), \frac{w_0w_1+4w_0w_2+w_1w_2}{4w_1^2+2w_0w_2}\right\}$$

and so it clearly suffices to check that

$$\frac{w_0w_1+4w_0w_2+w_1w_2}{4w_1^2+2w_0w_2} \leq \max\left\{\frac{w_0}{w_1}, \frac{w_1}{w_2}, \frac{w_1}{w_0}, \frac{w_2}{w_1}\right\}.$$

Equivalently, by setting $x_0 = w_1/w_0$ and $x_1 = w_2/w_1$, we show that

$$\frac{1}{x_1} + 4 + x_0 \leq 2\left(2\frac{x_0}{x_1}+1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\} \tag{12}$$

for all $x_0, x_1 \in \mathbb{R}^+$. Note that (12) is obvious when $x_0 \geq x_1$. When $1 \leq x_0 \leq x_1$ we have

$$\frac{1}{x_1} + 4 + x_0 \leq 5 + x_0 \leq 4x_0 + 2x_1 = 2\left(2\frac{x_0}{x_1}+1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\}$$

and using the symmetry $(x_0, x_1) \mapsto (1/x_1, 1/x_0)$ it follows that (12) holds for $x_0 \leq x_1 \leq 1$ as well. For the remaining case $x_0 \leq 1 \leq x_1$, first assume $x_1 \leq 1/x_0$. Since $x_0 + 1/x_0 \geq 2$ it follows that

$$\frac{1}{x_1} + 4 + x_0 \leq \frac{1}{x_1} + \left(\frac{3}{x_1} + \frac{2}{x_0}\right) = 2\left(2\frac{x_0}{x_1}+1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\}.$$

Similarly, when $x_1 \geq 1/x_0$ we use $x_1 + 1/x_1 \geq 2$ to obtain

$$\frac{1}{x_1} + 4 + x_0 \leq (3x_0 + 2x_1) + x_0 = 2\left(2\frac{x_0}{x_1}+1\right) \max\left\{x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}\right\},$$

which completes our proof of (12). □

Proof of Theorem 4.2. It is straightforward to check that

$$\min_{\lambda \in \mathbb{R}^+} \max\{\lambda x_0, \lambda x_1, (\lambda x_0)^{-1}, (\lambda x_1)^{-1}\} = \max\{(\tfrac{x_0}{x_1})^{\frac{1}{2}}, (\tfrac{x_1}{x_0})^{\frac{1}{2}}\}$$

which is attained when $\lambda = (x_0x_1)^{-1/2}$. Consequently we obtain the claimed formula for $\widetilde{\Lambda}_1(w)$ by taking $x_0 = w_1/w_0$ and $x_1 = w_2/w_1$. By Theorems 3.2

and 4.1, it remains to show that $\widetilde{\Theta}_1(w) \leq \widetilde{\Phi}_{1,5}(w)$ whenever $\mathcal{I}(w) < 1$; that is,

$$\frac{2}{1+\mathcal{I}} \leq \max\left\{\frac{2+3\mathcal{I}}{1+4\mathcal{I}}, \frac{3+2\mathcal{I}}{1+2\mathcal{I}+2\mathcal{I}^2}\right\}$$

whenever $\mathcal{I} < 1$. This follows because the quadratic $\mathcal{I} \mapsto 2\mathcal{I}^2 - \mathcal{I} - 1$ is negative for $\mathcal{I} \in (0,1)$. □

We conclude this section with some comparisons regarding the case $\tau = 2$. The best pointwise bounds appear to be due to Zhang and Ma (2006), who followed a number of authors in using the de Casteljau algorithm for rational Bézier curves, due to Farin (1983). In the quadratic case, the bound in Zhang and Ma (2006) follows from

$$|\sigma'[v,w](t)| \leq 2\Delta_2(v)\frac{(w_0(1-t)+w_1t)(w_1(1-t)+w_2t)}{(w_0(1-t)^2+2w_1t(1-t)+w_2t^2)^2}. \tag{13}$$

In particular, considering separately the cases

$$w_1(1-t)+w_2t \geq w_0(1-t)+w_1t \quad \text{and} \quad w_1(1-t)+w_2t \leq w_0(1-t)+w_1t \tag{14}$$

and using Theorem 3.1, one obtains $|\sigma'[v,w](t)| \leq 2\Delta_2(v)\Lambda_2(w)$, which is due to Zhang and Ma (2006). Here, $\Lambda_2(w) = \max\{\frac{w_1}{w_0}, \frac{w_1}{w_2}, \frac{1}{2}(1+\frac{w_2}{w_1}), \frac{1}{2}(1+\frac{w_0}{w_1})\}$. We note that a better bound is easily obtained by slightly modifying the above argument. In particular, bypassing considerations like (14), and using Theorem 3.1 on the right-hand side of (13) already improves the result in Zhang and Ma (2006). However, this may be bettered still; from this approach, the input coefficient vector for the numerator would be

$$(w_0w_1, w_0w_2 + w_1^2, w_1w_2)$$

which has first and third components equal, and second component *greater* than, the respective components of the input in (8) which led to Theorem 3.2 (the denominators are, of course, the same). We conclude that the approach based on the de Casteljau algorithm in Zhang and Ma (2006) does not appear to yield better results than Theorem 3.2. In particular, we have shown the following.

Theorem 4.3. *For each* $w \in \Omega$, $\Phi_{2,4}(w) \leq \Lambda_2(w)$.

We remark that $\Lambda_2(w)$ (and hence $\Phi_{2,n}(w)$ for each $n \geq 4$) beats the bound $\Theta_2(w)$, of Theorem 2.1, for weights w in a non-trivial, but difficult to describe, subset of Ω. At the level of the invariant bounds, the picture is clearer and we have the following analogue of Theorem 4.2.

Theorem 4.4. *Let $w \in \Omega$. Then $\widetilde{\Lambda}_2(w) = \max\{\mathcal{I}(w), \frac{1}{2}(1+\mathcal{I}(w)^{-1})\}$ and*

$$\widetilde{\Theta}_2(w) \leq \widetilde{\Phi}_{2,4}(w) \leq \widetilde{\Lambda}_2(w).$$

If, in addition $\mathcal{I}(w)^2 < \frac{1}{6}(\sqrt{10}-1)$, then we have

$$\widetilde{\Theta}_2(w) \leq \widetilde{\Phi}_{2,5}(w) \leq \widetilde{\Phi}_{2,4}(w) \leq \widetilde{\Lambda}_2(w).$$

Proof. One can compute $\widetilde{\Lambda}_2$ in a similar way that we computed $\widetilde{\Lambda}_1$ in the proof of Theorem 4.2; we omit the details. Next, straightforward considerations yield $\widetilde{\Theta}_2(w) = \widetilde{\Phi}_{2,4}(w) = \mathcal{I}(w)$ whenever $\mathcal{I}(w) \in [C_1, \infty)$. For $\mathcal{I}(w) \in (0, C_0]$, we have $\widetilde{\Theta}_2(w) \leq \widetilde{\Phi}_{2,4}(w)$ since $(2\mathcal{I}+1)(1+\mathcal{I})^{-2} \leq \frac{1}{2} + \frac{1}{4}\mathcal{I}^{-1}$ for all $\mathcal{I} \in (0,1)$. Finally, for $\mathcal{I} \in (C_0, C_1)$ we have $\frac{1}{8}(2\mathcal{I}-1)^2(1-\mathcal{I})^{-1}(2\mathcal{I}^2-1)^{-1} \leq \frac{1}{2} + \frac{1}{4}\mathcal{I}^{-1}$ because the quartic $\mathcal{I} \mapsto -8\mathcal{I}^4 + 12\mathcal{I}^2 - 3\mathcal{I} - 2$ is positive on (C_0, C_1). Therefore $\widetilde{\Theta}_2(w) \leq \widetilde{\Phi}_{2,4}(w)$ whenever $\mathcal{I}(w) \in (C_0, C_1)$, and consequently for all weights. It remains to show that whenever $\mathcal{I}(w)^2 < \frac{1}{6}(\sqrt{10}-1)$ we have $\widetilde{\Theta}_2(w) \leq \widetilde{\Phi}_{2,5}(w)$. One can easily check that for such weights w,

$$\widetilde{\Theta}_2(w) = \frac{1+2\mathcal{I}(w)}{(1+\mathcal{I}(w))^2} \quad \text{and} \quad \widetilde{\Phi}_{2,5}(w) = \frac{3+4\mathcal{I}(w)}{2+4\mathcal{I}(w)+4\mathcal{I}(w)^2}$$

in which case the desired inequality holds because the cubic $\mathcal{I} \mapsto 4\mathcal{I}^3+\mathcal{I}^2-2\mathcal{I}-1$ is negative for $\mathcal{I} > 0$ with $\mathcal{I}^2 < \frac{1}{6}(\sqrt{10}-1)$). □

We note that invariant bounds have also been obtained in Zheng (2005) which correspond to the uniform pointwise bounds $|\sigma'[v,w](t)| \leq 2\Delta_\tau(v)\Upsilon_\tau(w)$, where $\Upsilon_\tau(w) = (\max_i w_i / \min_j w_j)^{3-\tau}$, obtained by Floater (1992). Zheng proved that $\widetilde{\Upsilon}_\tau(w) = \max\{\mathcal{I}(w)^{3-\tau}, \mathcal{I}(w)^{\tau-3}\}$. These invariant bounds are outperformed by the Zhang and Ma bounds, $\widetilde{\Lambda}_\tau(w)$, $\tau \in \{1,2\}$, and hence by all other invariant bounds of their respective type considered in this paper. Zheng (2005) also considered the invariant bounds from the general degree bounds obtained in Floater (1992), however, except for the quadratic case, Zheng does not provide an explicit formula for these invariant bounds. We point out that in Bez and Bez (2012/2) we establish certain invariant bounds which are explicit and improve upon Zheng's bounds in the general degree case.

We conclude this section by providing certain sharpness considerations in the case $\tau = 1$. Analogous conclusions are possible when $\tau = 2$; we leave the details to the reader. Observe that we always have $|\sigma'[v,w](0)| = 2\frac{w_1}{w_0}|v_1 - v_0|$ and $|\sigma'[v,w](1)| = 2\frac{w_1}{w_2}|v_2 - v_1|$, so if $|v_1 - v_0| = |v_2 - v_1|$ then

$$\max_{t\in[0,1]} |\sigma'[v,w](t)| \geq 2\Delta_1(v)\frac{w_1}{\min\{w_0, w_2\}}. \tag{15}$$

When $\mathcal{I}(w) \geq 1$ each of the upper bounds $\Theta_1(w)$, $\Phi_{1,n}(w)$, $\Lambda_1(w)$ coincides with $\frac{w_1}{\min\{w_0,w_2\}}$, which shows sharpness in this pointwise sense. Moreover, $\frac{w_1}{\min\{w_0,w_2\}} \geq \mathcal{I}(w)$ and therefore, by (15), for weights with $\mathcal{I}(w) \geq 1$, the invariant bounds $\widetilde{\Theta}_1(w), \widetilde{\Phi}_{1,n}(w), \widetilde{\Lambda}_1(w)$ coinciding with $\mathcal{I}(w)$, are sharp. Importantly, the invariant bound $\widetilde{\Theta}_1(w)$ is also sharp at $\mathcal{I}(w) = 0$; to show this we require the following lemma.

Lemma 4.5. *We have* $\min_{a\in(0,\infty)} \max_{t\in[0,1]} \phi(a,t) = 1$*, where*

$$\phi(a,t) = at(1-t)((1-t)^2 + at^2)^{-2}.$$

Proof. Use $\partial_j\phi$ to denote the jth partial derivative of ϕ for $j \in \{1,2\}$. For each $a > 0$ there exists a unique point $t(a) \in (0,1)$ such that $\partial_2\phi(a,t(a)) = 0$. Note that $t(1) = \frac{1}{2}$, $a \mapsto t(a)$ is decreasing and $\max_{t\in[0,1]} \phi(a,t) = \phi(a,t(a))$. Thus, it suffices to prove that $\min_{a\in(0,\infty)} \phi(a,t(a)) = \phi(1,t(1))$. This follows from the mean value theorem and the fact that $\partial_1\phi(a,t(a))$ is positive for $a > 1$ and negative for $a < 1$. □

At $\mathcal{I}(w) = 0$ we[3] have $w = (w_0, 0, w_2)$. If v is such that $v_1 - v_0$ and $v_2 - v_1$ are parallel unit vectors, then $\Delta_1(v) = 1$ and the invariant bound $\widetilde{\Theta}_1(w)$ gives $\min_{\widetilde{w}\sim w} \max_{t\in[0,1]} |\sigma'[v,w](t)| \leq 4$. Writing $a = w_2/w_0$ we have from (3) that

$$|\sigma'[v,w](t)| = \frac{2at(1-t)}{((1-t)^2+at^2)^2}\,|v_2 - v_0| = \frac{4at(1-t)}{((1-t)^2+at^2)^2},$$

and from Lemma 4.5 it follows that $\min_{\widetilde{w}\sim w} \max_{t\in[0,1]} |\sigma'[v.w](t)| = 4$, which occurs at $\widetilde{w} = (1,0,1)$. Hence $\widetilde{\Theta}_1(w)$ is sharp at $\mathcal{I}(w) = 0$.

5 Conclusions

In this paper, we have established clear and comprehensive comparisons of both pointwise and invariant bounds for quadratic rational Bézier paths. A new bound Θ_2 is derived, following the approach in Hermann (1999); moreover, we demonstrated that the invariant bounds $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ are currently the best known and are difficult to beat using the approach of degree raising and convexity. This is because after the initial normalisation of the weights, the arguments leading to the bounds Θ_1 and Θ_2 cannot possibly be improved. However, in the cubic case, after an initial normalisation of the weights, the argument in Hermann (1999) is less tight. Indeed, in significant regions of the

[3] Strictly speaking, the weight vector $(w_0, 0, w_2)$ does not belong to the weight space Ω under analysis in this paper; however, for such sharpness considerations in the limiting case of the invariant, it makes sense to allow such weights.

invariant space, we have shown that the associated invariant bound arising from

$$\max\left\{\frac{w_0}{w_1},\frac{w_1}{w_2},\frac{w_2}{w_3},\frac{w_1}{w_0},\frac{w_2}{w_1},\frac{w_3}{w_2}\right\}$$

(which comes from a degree elevation and convexity argument) is smaller than that obtained from Hermann's cubic bound (see Bez and Bez (2012/1)). This highlights the potential for obtaining further improvements on the invariant bounds for rational Bézier paths of degree 3 and above using degree elevation and convexity; progress in this direction has already been made in Bez and Bez (2012/2).

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